6

Single-Queue Dynamics for Large-Tick Stocks

The calculus of probability can doubtless never be applied to market activity, and the dynamics of the Exchange will never be an exact science. But it is possible to study mathematically the state of the market at a given instant.

(Louis Bachelier)

In Chapter 5, we studied two simple models of a single order queue. Despite the apparent simplicity of our modelling assumptions, we saw that deriving expressions about the queue's behaviour required some relatively sophisticated machinery. In this chapter, we take the next step towards developing a stochastic model of an LOB by extending the models of the previous chapter towards more realistic situations. We focus on large-tick stocks, for which two major simplifications of the resulting queue dynamics occur.[1]

First, the bid–ask spread $s(t)$ is constrained heavily by the tick size ϑ, such that $s(t) = \vartheta$ for the vast majority of the time. Therefore, traders are unable to submit limit orders inside the spread, so price changes only occur when either the bid-queue or the ask-queue depletes to $V = 0$. Second, considerable volumes of limit orders typically accumulate at the bid-price and at the ask-price. In fact, V_b and V_a alone can correspond to a non-negligible fraction of the daily traded volume (see Table 4.1). For small-tick stocks, by contrast, V_b and V_a typically correspond to about 10^{-2} of the daily traded volume, it is common for limit orders to arrive inside the spread, and some arriving market orders cause price changes of several or even several tens of ticks. These empirical facts make modelling of LOBs much more difficult for small-tick stocks.

Because of the large volumes at the best quotes for large-tick stocks, and because price changes correspond to the times when either the bid-queue or the ask-queue deplete to 0, such events occur relatively infrequently, so that the typical daily

This chapter is not essential to the main story of the book.

[1] Recall from Chapter 4 that a large-tick stock is a stock whose tick size ϑ is a relatively large fraction of its price (typically tens of basis points or even some percentage points).

price change for a large-tick stock is usually between a few ticks and a few tens of ticks. For example, consider a stock with relative tick size $\vartheta_r \approx 0.002$ and a daily volatility of 2%. The number N of (random) daily mid-price changes satisfies

$$0.02 \approx 0.002\,\sqrt{N},$$

so $N \approx 100$. If this stock's trading day lasts eight hours, then on average the mid-price only changes value (usually by a single tick) about once every five minutes, during which a very large number of events take place at the best quotes.

Given that their depletion constitutes a relatively rare event, the dynamics of the bid- and ask-queues for a large-tick stock becomes an interesting modelling topic in its own right. As we will see in Section 6.2, we only require a few, relatively small extensions of our single-queue model from Chapter 5 to produce a useful model for large-tick stocks.

6.1 Price-Changing Events

We start our exploration of more realistic models precisely where we left off in Chapter 5: with a Fokker–Planck description of a single queue, in the continuum limit. As we will see in this chapter, this description is much more general than the specific cases that we have considered previously. The power of this approach is well known in many different disciplines, including queueing theory, where it is called the *heavy traffic limit* (see Queue Models for LOB Dynamics in Section 5.6) and statistical physics.

In the present context, the core of this approach is as follows. Provided that the change in queue size is the result of many small events that are weakly correlated in time, then the long-time dynamics of the queue are given by the general Fokker–Planck equation

$$\frac{\partial P(V,t)}{\partial t} = -\partial_V\,[F(V)P(V,t)] + \partial^2_{VV}\,[D(V)P(V,t)], \tag{6.1}$$

where $F(V)$ is the drift and $D(V)$ is the diffusion. In the Q-CIR model of Section 5.4.4, we assumed that $F(V)$ and $D(V)$ depended linearly on V, but in general this dependence can take any linear or non-linear form. In this chapter, we derive several results within this framework, then use empirical data to calibrate the drift and diffusion terms, and compare the corresponding model predictions to real queue dynamics.

Equation (6.1) models the stochastic evolution of a single queue. Assume now that this queue is either the best bid- or ask-queue in the LOB for a large-tick stock. In addition to the changes in queue volume, the dynamics of an LOB are also influenced by events that change the price of the best bid or ask. Therefore, to extend our single-queue analysis from Chapter 5 into the context of an LOB, we must also supplement our approach to account for these price-changing events.

Suppose, for concreteness, that we focus on the bid-queue, such that $V(t) = V_b(t)$ denotes the volume at the best bid at time t. Because we study large-tick stocks, throughout this chapter we assume that the spread is equal to its minimum size of one tick. Therefore, no limit orders can arrive inside the spread, so price changes can only occur when a queue depletes to $V = 0$. Whenever a queue depletes in this way, we assume that some new volume must arrive in the gap immediately (otherwise, the spread would be greater than one tick). This new volume can be either a buy limit order, in which case the bid- and ask-prices both remain the same as they were before the depletion, or a sell limit order, in which case both the bid- and ask-prices decrease by one tick from the previous values before the depletion.

More formally, we assume that:

(i) If the bid-queue volume reaches $V = 0$, then one of the following two possibilities must occur:
- With some probability ϕ_0, a new buy limit order arrives, such that the bid-queue replenishes and the bid- and ask-prices do not change. The size $\delta V > 0$ of the new buy limit order is distributed according to some distribution $\varrho(\delta V|V = 0)$.
- With probability $1 - \phi_0$, a new sell limit order arrives (with a volume distribution $\varrho(\delta V|V = 0)$) and the bid- and ask-prices both decrease by one tick. The new bid-queue is then the queue that was previously the second-best bid-queue, whose volume we assume is distributed according to some distribution $P_{\text{2-best}}(V)$.

(ii) Similarly, if the ask-queue volume reaches $V = 0$, then one of the following two possibilities must occur:
- With some probability ϕ_0, a new sell limit order arrives, such that the ask-queue replenishes and the bid- and ask-prices do not change. The size $\delta V > 0$ of the new sell limit order is distributed according to some distribution $\varrho(\delta V|V = 0)$.
- With probability $1 - \phi_0$, a new buy limit order arrives (with a volume distribution $\varrho(\delta V|V = 0)$) and the bid- and ask-prices both increase by one tick. The new ask-queue is then the queue that was previously the second-best ask-queue, whose volume we assume is distributed according to some distribution $P_{\text{2-best}}(V)$.

How can we implement these modelling assumptions into our Fokker–Planck framework? In Chapter 5, we introduced the concept of exit flux $J_0(t)$ to account for queue depletions. In that chapter, we did not consider the corresponding price changes that occur after a bid- or ask-queue depletion, but in this chapter we seek to address this effect directly. Therefore, we extend the concept of a single exit flux $J_0(t)$ to also encompass possible asymmetries in the bid- and ask-queues. Specifically, we let $J_b(t)$ denote the probability per unit time that the bid-queue empties and $J_a(t)$ denote the probability per unit time that the ask-queue empties. The mechanisms above then add some reinjection terms on the right-hand side of Equation (6.1) to govern the distribution of volume at the bid:

$$\mathcal{J}(V,t) = J_b(t)\left(\phi_0\varrho(V|0) + (1-\phi_0)P_{\text{2-best}}(V)\right) + J_a(t)(1-\phi_0)\left(\varrho(V|0) - P(V,t)\right). \tag{6.2}$$

The first two terms in Equation (6.2) correspond to the bid-queue depleting to 0, and the last two terms correspond to the ask-queue depleting to 0. Finally, the absorbing condition fixes $P(0,t) = 0$, and the probability of the bid-queue hitting zero is given by:

$$J_b(t) = \partial_V \left[D(V)P(V,t)\right]\Big|_{V=0}.$$

By symmetry of the system, a similar result holds for J_a. One can directly check that $\partial_t \int \mathrm{d}V P(V,t) = 0$, which shows that we have correctly accounted for all outgoing and incoming order flow. This fixes the complete description of the queue dynamics, which allows us to derive several results, for example concerning the stationary distribution of the bid- or the ask-queue.

Throughout this chapter, we restrict our attention to the symmetric, stationary case where $J_a(t) = J_b(t) = J_0$. Using this symmetry, we arrive at

$$\partial_V \left[F(V)P_{\text{st.}}(V)\right] - \partial^2_{VV}\left[D(V)P_{\text{st.}}(V)\right] = J_0\left[\varrho(V|0) + (1-\phi_0)\left(P_{\text{2-best}}(V) - P_{\text{st.}}(V)\right)\right], \tag{6.3}$$

where the exit flux J_0 must satisfy

$$J_0 = \partial_V \left[D(V)P_{\text{st.}}(V)\right]\Big|_{V=0}. \tag{6.4}$$

6.2 The Fokker–Planck Equation

Throughout this chapter, the **Fokker–Planck equation** (6.3) is our central focus. Given its importance, we first take a moment to consider where it comes from and why it holds.

Assume that there exists some time $\tau_c \geq 0$ such that on time scales beyond τ_c, order flow can be assumed to be uncorrelated. Let $\varrho(\delta V|V)$ denote the probability that the change in queue size in the next time interval τ_c is δV, given that the current queue length is V. Values $\delta V > 0$ correspond to limit order arrivals, and values $\delta V < 0$ correspond to limit order cancellations and market order arrivals. We assume that all of these events occur with a probability that depends on the queue volume at time t. The general master equation for the size distribution then reads

$$P(V, t+\tau_c) = \sum_{\delta V} P(V-\delta V, t)\varrho(\delta V|V-\delta V). \tag{6.5}$$

Assume for now that there is no change of the bid- or ask-prices between t and $t+\tau_c$. Assume further that within a time interval of length τ_c, the typical change of queue size is relatively small, such that $\delta V \ll V$. In this framework, one can expand the master equation (6.5) in powers of δV. The general expansion is called the **Kramers–Moyal expansion**, which when truncated to second order reads

$$\begin{aligned} P(V, t+\tau_c) \approx \sum_{\delta V} \Bigg[&\left(P(V,t) - \delta V \partial_V P(V,t) + \frac{1}{2}(\delta V)^2 \partial^2_{VV} P(V,t) \right) \\ &\times \left(\varrho(\delta V|V) - \delta V \partial_V \varrho(\delta V|V) + \frac{1}{2}(\delta V)^2 \partial^2_{VV} \varrho(\delta V|V) \right) \Bigg]. \end{aligned} \tag{6.6}$$

Regrouping the terms by order:

$$P(V, t+\tau_c) \approx P(V,t) - \partial_V \left[\sum_{\delta V} \delta V \varrho(\delta V|V) P(V,t) \right] + \frac{1}{2}\partial^2_{VV} \left[\sum_{\delta V} (\delta V)^2 \varrho(\delta V|V) P(V,t) \right]. \tag{6.7}$$

Finally, substituting the approximation $P(V, t+\tau_c) \approx P(V,t) + \tau_c \partial_t P(V,t)$ into Equation (6.7) produces the Fokker–Planck equation (6.1), where

$$F(V) = \frac{1}{\tau_c} \sum_{\delta V} \delta V \varrho(\delta V|V); \qquad D(V) = \frac{1}{2\tau_c} \sum_{\delta V} (\delta V)^2 \varrho(\delta V|V) \tag{6.8}$$

are the first two moments of the conditional distribution of size changes, $\varrho(\delta V|V)$. The special Q-CIR model in Section 5.4.4 corresponds to the case

$$\begin{aligned} \tau_c &= \mathrm{d}t, \\ \varrho(\delta V = +1|V) &= \lambda \mathrm{d}t, \\ \varrho(\delta V = -1|V) &= (\mu + \nu V)\mathrm{d}t, \\ \varrho(\delta V = 0|V) &= 1 - (\lambda + \mu + \nu V)\mathrm{d}t, \end{aligned}$$

from which it again follows that

$$\begin{aligned} F(V) &= \lambda - \mu - \nu V, \\ D(V) &= \frac{\lambda + \mu + \nu V}{2}. \end{aligned}$$

The Fokker–Planck formalism allows for a much more general specification. In particular, one can remove the assumption that all events are of unit volume, and one can accommodate any dependence of the market order arrival and limit order arrival/cancellation rates on the volume V.

The general conditions under which the Fokker–Planck approximation is valid are discussed in many books.[2] Intuitively, $P(V,t)$ must vary slowly, both in V (on the scale of typical values of δV) and in time (on the correlation time scale τ_c). It is important to check on a case-by-case basis that these conditions hold.

6.2.1 The Boltzmann–Gibbs Measure

Because Equation (6.3) is a linear second-order ODE, one can in principle write down its explicit solution in full generality. However, as we saw in Chapter 5, this solution can be quite complicated, and is therefore not very useful for developing intuition about the system. A better approach is to follow a similar path to that in Section 5.4.5, and to look for cases where the effective potential

$$\mathcal{W}(V) = -\int_0^V \mathrm{d}V' \frac{F(V')}{D(V')} \tag{6.9}$$

has a minimum at $V = V^*$ sufficiently far away from $V = 0$ and sufficiently deep to keep the queue size in the vicinity of V^* for a substantial amount of time.

Analysis of this system is further complicated by the fact that the reinjection term in Equation (6.3) is non-zero. This makes finding general solutions particularly difficult. However, there are two situations in which we can avoid this difficulty. The first is where $\mathcal{W}(V^*)$ is large, so J_0 is very small (recall that J_0 decays exponentially in $\mathcal{W}(V^*)$). In this case, we can simply approximate the reinjection term in Equation (6.3) as 0. The second is where the best and second-best queues have similar stationary profiles (such that $P_{\text{2-best}}(V) \approx P_{\text{st.}}(V)$) and where the size of refilled queues is equal to a small elementary volume υ_0 (such that $\varrho(V|0) \approx \delta(V - \upsilon_0)$). In these cases, the equation for the stationary distribution when $V > 0$ takes the simpler form

$$F(V)P_{\text{st.}}(V) - \partial_V [D(V)P_{\text{st.}}(V)] = 0. \tag{6.10}$$

As can be verified by inspection, the expression

$$P_{\text{st.}}(V) = \frac{A}{D(V)} e^{-\mathcal{W}(V)} \tag{6.11}$$

is an explicit solution for Equation (6.10). The value of A is fixed by the normalisation of $P_{\text{st.}}(V)$. Equation (6.11) is called the **Boltzmann–Gibbs measure**.

By comparing $P_{\text{st.}}(0)$ to $P_{\text{st.}}(V^*)$, we can again guess that the exit flux J_0 is indeed of the order of $e^{\mathcal{W}(V^*)}$, because J_0 is itself proportional to $P_{\text{st.}}(0)$. Therefore, J_0 is very small as soon as the potential depth $|\mathcal{W}(V^*)|$ is larger than about 5. This corresponds to the regime of large, long-lived queues. In this regime, the equilibrium in the effective potential $\mathcal{W}(V)$ is reached much faster than the time needed for the queue to empty.

[2] See for example: Gardiner, C. W. (1985). *Stochastic methods*. Springer, Berlin-Heidelberg and Section 6.6.

The equilibration time (i.e. the time required to reach the stationary state) can be approximated by considering the first-order expansion of $F(V)$ in the vicinity of V^*,

$$F(V \approx V^*) \approx 0 + \kappa(V^* - V) + O((V^* - V)^2),$$

with $\kappa > 0$. Locally, the process is thus an **Ornstein–Uhlenbeck** process[3] with relaxation time κ^{-1}. In the Q-CIR model of Chapter 5, κ is simply equal to ν.

In the regime $J_0\kappa^{-1} \ll 1$ (i.e. where the relaxation is so fast that the exit flux can be neglected), the stationary distribution $P_{\mathrm{st.}}(V)$ is nearly independent of the reinjection mechanism, as we found in Chapter 5. In particular, deriving $P_{\mathrm{st.}}(V)$ does not require knowledge of either $P_{2\text{-best}}(V)$ or $\varrho(V|0)$. This remarkable property, which we discussed qualitatively in Section 5.4.4, will enable us to test empirically the relevance of the Fokker–Planck framework for describing the dynamics of bid–ask-queues. We present these empirical results in Section 6.4.

6.2.2 First-Hitting Times

Recall from Section 5.4.5 that the mean first-hitting time $\mathbb{E}[T_1(V)]$ obeys Equation (5.49). As can be verified by inspection, the general expression

$$\mathbb{E}[T_1(V)] = \int_0^V \mathrm{d}V' \int_{V'}^{\infty} \mathrm{d}V'' \frac{1}{D(V'')} e^{\mathcal{W}(V') - \mathcal{W}(V'')} \tag{6.12}$$

is a solution for Equation (5.49).

In the limit $|\mathcal{W}(V^*)| \gg 1$, one can carry out an approximate Laplace estimation of this integral to conclude that for a large region of initial conditions, the mean first-hitting time is to a very good approximation independent of the starting point V, and is given by:

$$\mathbb{E}[T_1(V)] \approx \frac{1}{D(V^*)|\mathcal{W}'(0)|} \sqrt{\frac{2\pi}{\mathcal{W}''(V^*)}} e^{|\mathcal{W}(V^*)|}.$$

Therefore, $\mathbb{E}[T_1(V)]$ grows exponentially in the barrier height. Recalling that

$$J_0 \approx \frac{1}{\mathbb{E}[T_1(V^*)]},$$

we verify that J_0 decays exponentially in $\mathcal{W}(V^*)$ and can therefore indeed be neglected in Equation (6.3).

Note also that using the quasi-independence of $\mathbb{E}[T_1(V)]$ on V (see Section 5.4.3) and following the calculation presented for the discrete case in Section 5.4.3, one can establish that the hitting process is also Poissonian for the general Fokker–Planck equation (6.3) in the limit of large barrier heights.

6.3 Sweeping Market Orders

As we discussed in Section 6.2, the Fokker–Planck approach requires individual changes in V to be small. For large-tick assets, this is indeed the case for a large

[3] An Ornstein–Uhlenbeck process is described by a linear stochastic differential equation: $\mathrm{d}V = -\kappa(V - V^*)\mathrm{d}t + \mathrm{d}W_t$, where W_t is a Brownian motion. See, e.g., Gardiner (1985).

fraction of events. Sometimes, however, very large market orders consume all the remaining volume in the best bid- or ask-queue. Due to their large size, such market orders fall outside the scope of the standard Fokker–Planck formalism for computing the first-hitting time. As we illustrate in Section 6.4, such large market orders are not particularly frequent, especially when queues are large, but are still important since they lead to immediate depletion. In this section, we introduce a way to account for these events.

6.3.1 An Extended Fokker–Planck Description

Let $\Pi(V)$ denote the probability per unit time of the arrival of a market order that consumes the entire queue under consideration, and let $\Pi^{\dagger}(V)$ denote the probability per unit time of the arrival of a market order that consumes the entire opposite-side best queue. Using this notation, we can extend Equation (6.3) to account for these events by re-writing $P(V,t)$ as

$$\begin{aligned}\frac{\partial P(V,t)}{\partial t} = &-\partial_V\left[F(V)P(V,t)\right] + \partial^2_{VV}\left[D(V)P(V,t)\right] + \mathcal{J}(V,t)\\ &-\Pi(V)P(V,t) + \bar{\Pi}(t)\left(\phi_0\varrho(V|0) + (1-\phi_0)P_{2\text{-best}}(V)\right)\\ &-(1-\phi_0)\Pi^{\dagger}(V)P(V,t) + (1-\phi_0)\bar{\Pi}^{\dagger}(t)\varrho(V|0),\end{aligned} \tag{6.13}$$

where $\mathcal{J}$ is given by Equation (6.2) and

$$\bar{\Pi}(t) = \int \mathrm{d}V \Pi(V) P(V,t),$$

and similarly for $\bar{\Pi}^{\dagger}(t)$. The second line in Equation (6.13) says that if the queue is instantaneously depleted to zero (which occurs with probability $\bar{\Pi}(t)$), then it either bounces back (with probability ϕ_0) or is replaced by the second-best queue (with probability $1-\phi_0$). The third line describes what happens when the opposite queue is instantaneously depleted to zero (which occurs with probability $\bar{\Pi}^{\dagger}(t)$). With probability $1-\phi_0$, the queue we are looking at becomes the second-best queue and is replaced by a new queue with volume taken from the distribution $\varrho(V|0)$; otherwise (with probability ϕ_0), it bounces back so nothing changes.

As we saw in Chapter 5, diffusion processes typically produce an extremely small exit flux. Depending on the frequency with which they occur, large market order arrivals can be a much more efficient mechanism than standard diffusions for emptying the queue. In other words, the $\Pi(V)$ and $\Pi^{\dagger}(V)$ terms in Equation (6.13) can cause a considerable impact on queue dynamics.

Let us define

$$\Pi^* := \max\left[\Pi(V^*), \Pi^{\dagger}(V^*)\right]. \tag{6.14}$$

Three situations can occur:

(i) $\Pi^* \ll J_0 \ll \kappa$. In this case, the probability of large market orders is so small that their effect can be neglected.

(ii) $J_0 \ll \Pi^* \ll \kappa$. In this case, large market order arrivals are the primary mechanism driving queue depletions. However, since the typical time between these large market order arrivals (which is of order $1/\Pi^*$) is still very large compared to the equilibration time $1/\kappa$, the stationary distribution $P_{\text{st.}}(V)$ is still approximately equal to the Boltzmann–Gibbs measure in Equation (6.11).

(iii) $\kappa \lesssim \Pi^*$. In this case, large market orders arrive so frequently that the queue does not have time to equilibrate between their arrivals. Therefore, the whole Fokker–Planck approach breaks down. Assets with small relative tick size (see Section 3.1.5) typically fall into this case, because the size of a typical queue for such assets is similar to the size of a typical market order.

Empirically, one finds Π^* to be smaller than but comparable to J_0 (see Table 6.1), which leaves us in an intermediate regime between cases (i) and (ii).

In the limit of small Π and $\Pi^\dagger$, and if the term $\mathcal{J}(V,t)$ is still negligible, one can still find an explicit form for the stationary solution. Assuming for simplicity that

$$P_{\text{2-best}}(V) \approx P_{\text{st.}}(V)$$

and that the incipient queues are small, i.e.

$$\varrho(V|0) \approx \delta(V - \upsilon_0),$$

the stationary equation for $V > 0$ now reads:

$$0 = \partial_V \left[F(V)P_{\text{st.}}(V)\right] - \partial^2_{VV}\left[D(V)P_{\text{st.}}(V)\right] + \\ + \epsilon\left[\Pi(V) + (1-\phi_0)(\Pi^\dagger(V) - \bar{\Pi}_{\text{st.}})\right]P_{\text{st.}}(V),$$

where $\bar{\Pi}_{\text{st.}}$ is defined as:

$$\bar{\Pi}_{\text{st.}} := \int \mathrm{d}V \Pi(V) P_{\text{st.}}(V).$$

We also introduced $\epsilon \ll 1$ as a device to highlight that all of $\Pi(V)$, $\Pi^\dagger(V)$ and $\bar{\Pi}_{\text{st.}}$ are small. This device is convenient for keeping track of the order to which we expand the solution. Looking for a stationary solution of the form

$$P_{\text{st.}}(V) = \frac{A(V)}{D(V)} e^{-\mathcal{W}(V)}, \tag{6.15}$$

one finds that $B(V) = A'(V)/A(V)$ obeys the so-called Ricatti equation

$$B^2 + B' - \mathcal{W}'B - \epsilon\widehat{\Pi} = 0; \qquad \widehat{\Pi}(V) := \frac{\Pi(V) + (1-\phi_0)(\Pi^\dagger(V) - \bar{\Pi}_{\text{st.}})}{D(V)}.$$

When the probability of large market orders is sufficiently small, one can look for a solution $B \propto \epsilon$ and neglect the B^2 term, which is of order ϵ^2. The resulting linear equation leads to the following result, valid to order ϵ:

$$\ln A(V) = -\int_0^V \mathrm{d}V' \int_{V'}^\infty \mathrm{d}V'' \widehat{\Pi}(V'') e^{\mathcal{W}(V') - \mathcal{W}(V'')}. \tag{6.16}$$

If needed, this solution can be plugged back into the Ricatti equation to find the correction to order ϵ^2.

6.3.2 When Large Market Orders Dominate

If $F(V)$ is always positive, then in the absence of large market order arrivals, the queue volume tends to infinity and the Fokker–Planck equation has no stationary state. However, the arrivals of large market orders can keep the queue length finite in this case. To build intuition, let's consider the simplest case where

$$F(V) = F_0 > 0,$$
$$D(V) = D_0 > 0,$$
$$\Pi(V) = \Pi^{\dagger}(V) = \Pi_0 > 0$$

are all strictly positive constants. We also assume that $P_{\text{2-best}}(V) = P_{\text{st.}}(V)$. In this case, the diffusive flux $\mathcal{J}(V,t)$ is very small. Therefore, in the stationary limit, Equation (6.13) becomes, for $V > 0$,

$$-F_0 \partial_V P_{\text{st.}}(V) + D_0 \partial^2_{VV} P_{\text{st.}}(V) - \Pi_0 P_{\text{st.}}(V) = 0, \qquad (6.17)$$

where $\varrho(V|0)$ is again assumed to be localised around $V = 0$.

Because Equation (6.17) is a linear, second-order ODE, we look for a solution of the form $P_{\text{st.}}(V) \propto e^{aV}$, which leads to the following second-degree equation for a:

$$D_0 a^2 - F_0 a - \Pi_0 = 0.$$

This equation has solutions

$$a_{\pm} = (F_0 \pm \sqrt{F_0^2 + 4\Pi_0 D_0})/2D_0.$$

For $P_{\text{st.}}(V)$ to be normalisable to 1, we require the solution in which $P_{\text{st.}}(V)$ tends to 0 for large V. Therefore, we require $a < 0$, so we choose $a = a_-$, and

$$P_{\text{st.}}(V) \approx |a_-| e^{-|a_-|V},$$

which decays exponentially in V. In summary, the arrival of sweeping market orders makes the probability of large queues exponentially small, even when the drift $F(V)$ is positive.

6.4 Analysing Empirical Data

6.4.1 Calibrating the Fokker–Planck Equation

We now turn our attention to fitting the Fokker–Planck model to empirical data, to see how well the results we have derived throughout the chapter are consistent with the behaviour observed in real markets.

As we discussed in Section 4.2, order flows in financial markets undergo strong intra-day patterns. Attempting to fit the model without acknowledging these intra-day seasonalities is likely to produce poor estimates of the input parameters and poor outputs. To avoid this problem, we take the simple approach of only studying a period of time within which order flow is approximately stationary. As in the previous chapters, we simply discard the first and last hour of trading activity each day. This removes the strongest element of the intra-day pattern. In the following, we work in rescaled units $u := V/\bar{V}$, where $\bar{V}$ is the mean queue length, allowing the comparison between different stocks (and different time periods if volumes evolve substantially between these periods). Indeed, a remarkable property of the dynamics of queues is approximate **scale invariance**, i.e. once rescaled by the local average volume, all statistical properties of queue sizes are similar. Throughout this section, we will call the quantity u the **rescaled volume**.

Then, one can measure the drift $F(u)$ and diffusion $D(u)$ as conditional empirical averages of volume changes in a unit event-time interval $\delta n = 1$, using:[4]

$$F(u) = \frac{1}{\delta n} \langle u(n+\delta n) - u(n) \rangle |_{u(n)=u},$$
$$D(u) = \frac{1}{2\delta n} \left\langle (u(n+\delta n) - u(n))^2 \right\rangle \Big|_{u(n)=u}.$$

For simplicity, we restrict our analysis to the most natural choice $\delta n = 1$. Because we consider single-queue dynamics, event-time in this section refers to events only pertaining to the queue under consideration. Thus, each event necessarily changes the queue volume. (Note that we exclude all market order events that only execute hidden volume, which are irrelevant for the dynamics of the visible volume in the queue.)

Table 6.1 lists several summary statistics for the queue dynamics of CSCO and INTC, both of which are large-tick stocks (see Table 4.1). There are some noteworthy features:

- The rescaled volume changes per event are small ($\langle|\delta u|\rangle \cong 6\%$), which vindicates the use of a Fokker–Planck approach for the dynamics of large-tick queues.
- The initial rescaled volume of incipient queues (i.e. queues born when the spread momentarily opens) is around 10–15% on average, meaning that the choice $\varrho(V|0) = \delta(V - 0^+)$ is only a rough first approximation.
- The average number of events before depletion is large (~ 100), which suggests that queues are typically in the stationary regime of the Fokker–Planck dynamics.
- The average time to depletion is smaller for incipient queues than when second-best queues become best queues. This makes sense since the initial volumes are smaller in the former case.
- More than 20% of the depletion events lead to immediate refill, with no price change.

[4] Choosing $\delta n > 1$ is also possible, although some ambiguity is introduced in this case, because the bid- or ask-prices may have changed between n and $n + \delta n$. In particular, if the price changes and the queue is replaced by a new queue with different volume, conditioning to $u(n) = u$ makes little sense, as the volume might be very different for the last events. However, one can adopt reasonable conventions to give a meaning to u, leading in the end to very similar results for $\delta n = 10$ and $\delta n = 1$.

Table 6.1. *Summary statistics of queue-depleting events for INTC and CSCO. The initial volume is typically small for incipient queues (i.e. for queues born in an open spread), and large when the second-best queue becomes the best queue. The empirical values of the average time to depletion imply that the exit flux J_0 is of the order of* 0.01. *This is (see Figure 6.2) roughly ten times the probability of a large sweeping* MO *hitting an average queue:* $\Pi^* \cong 0.001$.

	INTC	CSCO
Number of events per queue in 5 min.	4117	3031
Average queue volume [shares]	5112	9047
Average rescaled volume change per event $\langle\lvert\delta u\rvert\rangle$	0.065	0.060
Average rescaled volume of incipient queues	0.140	0.114
Average rescaled volume of former 2nd-best queues	1.45	1.46
Probability that next event is LO	0.506	0.505
Probability that next event is CA	0.472	0.474
Probability that next event is MO	0.022	0.021
Probability that next event is a sweeping MO $\bar{\Pi}_{\text{st.}}$	0.0059	0.0046
Probability that next event depletes the queue	0.0067	0.0052
Refill probability after depletion ϕ_0	0.219	0.234
Average event-time to depletion (from incipient)	63	78
Average event-time to depletion (from 2nd-best)	180	248

- The total probability of sweeping market orders is $\cong 5 \times 10^{-3}$ per event. Most such market orders are executed when the queue is already small (see Figure 6.2).

We now turn to the shape of the drift $F(u)$ and diffusion $D(u)$ curves, shown in Figure 6.1. Interestingly, these shapes are quite similar for the two stocks considered here, and for a variety of other stocks as well.[5] One sees that $F(u)$ is positive for small queues, vanishes for some rescaled volume $u_c \cong 1.5$, and becomes negative for $u > u_c$. As one would expect, small queues grow on average and long queues shrink on average. The shape of $F(u)$ is however more complex than those assumed in the models considered in Chapter 5, where $F(u)$ was either *independent* of u and slightly negative (Section 5.3.4), or a linear function of u with a negative slope (Section 5.4.4). Empirically, $F(u)$ shows some convexity, which can be roughly fitted as $F(u) = F_0 - F_1 u^\zeta$ with F_0, F_1 constants and ζ an exponent between 0 and 1. This shape for $F(u)$ corresponds to a model intermediate between the diffusion model and the Q-CIR model, where the cancellation rate grows sublinearly with the size of the queue. Note the scale of the y-axis, which reveals

[5] On this point, see: Garèche, A., Disdier, G., Kockelkoren, J., & Bouchaud, J. P. (2013). Fokker-Planck description for the queue dynamics of large-tick stocks. *Physical Review E*, 88(3), 032809.

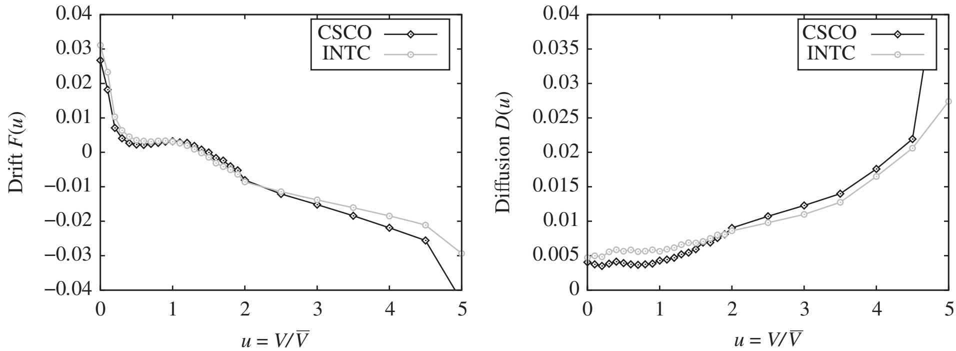

Figure 6.1. (Left panel) Drift $F(u)$ and (right panel) diffusion $D(u)$ as a function of the normalised queue volume $u = V/\bar{V}$, for INTC and CSCO, measured with $\delta n = 1$. Note that $F(u)$ changes sign at $u = u_c \cong 1.5$.

the small magnitude of the drift: a few percent of volume change on average, even for large queues with $u \sim 5$.

The diffusion coefficient $D(u)$ grows with volume, meaning that typical volume changes themselves grow with the size of the queue. This reflects the fact that the size of incoming market orders and incoming limit orders are both typically larger when the size of the queue is larger. In the simple models of Chapter 5, $D(u)$ is either flat or linear in u (see Section 5.4.4), whereas the empirically determined $D(u)$ grows faster with u.

Figure 6.2 shows the probability of sweeping market orders $\Pi(u)$ and $\Pi^\dagger(u)$. As expected, the probability of a queue disappearing by being fully consumed by a single market order is a strongly decreasing function of its volume (note that the y-axis is in log-scale). The probability that a queue is hit by a sweeping order is an increasing function of the opposite-side volume. This suggests that a large volume imbalance between the two queues tends to trigger a sweeping market order that consumes the smaller of the two volumes. We provide a detailed discussion of this phenomenon in Section 7.2.

6.4.2 Predicting the Stationary Distribution of Queue Sizes

We can now evaluate the accuracy of the Fokker–Planck formalism by comparing the predicted stationary queue size distribution (in event-time) with the corresponding empirical distribution. From the knowledge of $F(u)$, $D(u)$, $\Pi(u)$ and $\Pi^\dagger(u)$, we can use Equation (6.16) to estimate the stationary distribution of rescaled queue sizes, $P_{\text{st.}}(u)$. Figure 6.3 compares the empirically determined distribution with the theoretical Fokker–Planck prediction, Equation (6.16). The overall shape of $P_{\text{st.}}(u)$ is rather well reproduced by the Fokker–Planck prediction, in particular in the right tail ($u > u_c$) where the negative drift $F(u)$ prevents the queues from

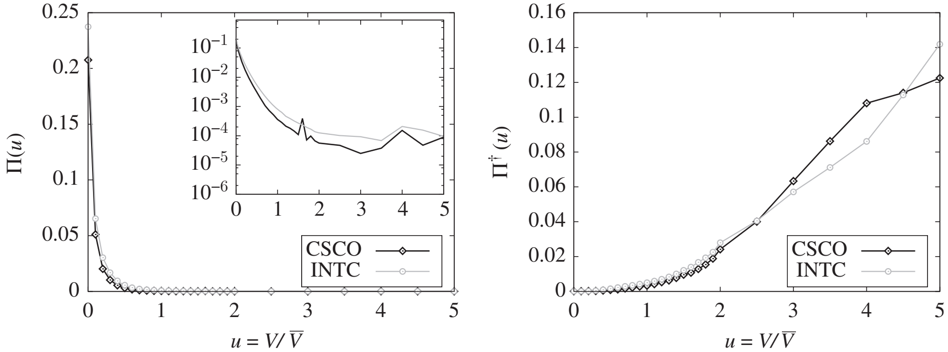

Figure 6.2. Probability of a sweeping market order, conditioned on (left panel) the same-side volume $\Pi(u)$ and (right panel) the previous opposite-side volume $\Pi^{\dagger}(u)$, as a function of the normalised queue volume $u = V/\bar{V}$, for INTC and CSCO.

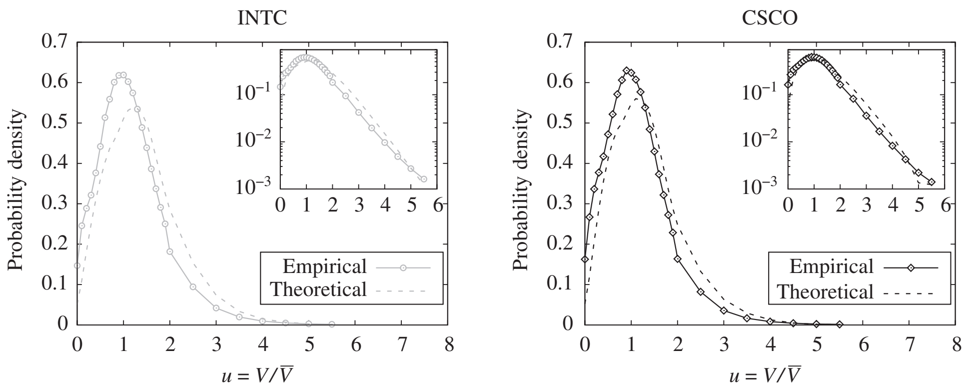

Figure 6.3. (Solid curves with markers) Empirical and (dashed curves) theoretical distribution of rescaled queue sizes as a function of $u = V/\bar{V}$, for (left panel) INTC and (right panel) CSCO.

becoming very large. This result again vindicates the use of the Fokker–Planck formalism to describe the dynamics of queues for large-tick assets.[6]

Notice that $P_{\text{st.}}(u)$ is hump-shaped, but not sharply peaked (in the sense that the ratio $P_{\text{st.}}(u^*)/P_{\text{st.}}(0)$ is not very large for these stocks). This means that the "barrier height" for depletion is not high, which is contrary to the simplifying assumption that we made in Section 6.2.2. In other words, the potential

[6] Recall that Equation (6.16) further assumes that $\varrho(u|0) = \delta(u - 0^+)$ and that $P_{\text{2-best}}(u) \approx P_{\text{st.}}(u)$, two approximations that are quite rough and that could be easily improved.

well $\mathcal{W}(u^*) = -\int^{u^*} \mathrm{d}u\, F(u)/D(u)$ is too shallow to block diffusion-induced depletion.

6.5 Conclusion

In this chapter, we have introduced an intuitive and flexible description of the queue dynamics for large-tick assets, based on the Fokker–Planck formalism. Interestingly, this description includes some dependence on the current state of the queue, through the explicit dependence of the drift and diffusion constant on the queue size. However, this approach required no restrictive assumptions on the dependence of the arrival or cancellation rates on queue size, nor on the size of queue changes. The Fokker–Planck approach can also accommodate jump events, which correspond to sudden changes of the bid or ask prices.

All of the quantities involved in this approach can be calibrated using event-resolved data that describes the order flow at the best quotes. One important observation was that the dynamical process is approximately scale invariant, in the sense that the only relevant variable is the ratio of the volume of the queue V to its average value $\bar{V}$. While the latter shows intra-day seasonalities and strong variability across stocks and time periods, the dynamics of the rescaled volumes is, to first approximation, universal.

The validity of the Fokker–Planck approximation relies on elementary volume changes δV being small. Although this assumption fails for small-tick assets, it is well justified for large-tick assets (see Table 6.1). Correspondingly, when solved to find the stationary distribution of rescaled volumes in a queue, the Fokker–Planck equation calibrated on dynamical data fares quite well at reproducing the empirical distribution.

For the two large-tick stocks that we investigated in this chapter (INTC and CSCO), the distribution of queue sizes is not strongly peaked around its maximum V^*, as the Q-CIR model would predict. This observation can be traced back to the fact that the ratio of the drift to the diffusion is *not* very large, which implies that the depletion time is large but not extremely large. The Q-CIR model, on the other hand, would predict that queues empty exceedingly rarely (see also Chapter 8).

As we will discuss in the next chapter, the fact that the depletion time is moderate allows one to predict which of the bid-queue or the ask-queue is most likely to empty first. As we will conclude in Section 7.3.3, reality seems to be best described by a model intermediate between the diffusive model (with a constant cancellation rate per queue) and the Q-CIR model (with a constant cancellation rate per order).

Take-Home Messages

(i) Under the assumptions that the best bid- and ask-queues evolve independently and via small increments, it is possible to write 1-D Fokker–Planck equations for the resulting queue dynamics.
(ii) It is straightforward to calibrate the Fokker–Planck equation on real data. The theoretical stationary queue distribution can be found using the Boltzmann–Gibbs measure, and is close to the empirical distribution.
(iii) Taking large (sweeping) market orders into account significantly improves the fit to the data. This suggests that the small-increments hypothesis central to the Fokker–Planck approach is too restrictive for modelling real order flow.

6.6 Further Reading

See also Section 5.6.

General

Feller, W. (1971). *An introduction to probability theory and its applications* (Vol. 2). John Wiley and Sons.
Van Kampen, N. G. (1983). *Stochastic processes in physics and chemistry*. North-Holland.
Risken, H. (1984). *The Fokker–Planck equation*. Springer, Berlin-Heidelberg.
Gardiner, C. W. (1985). *Stochastic methods*. Springer, Berlin-Heidelberg.
Hänggi, P., Talkner, P., & Borkovec, M. (1990). Reaction-rate theory: Fifty years after Kramers. *Reviews of Modern Physics*, 62(2), 251.
Redner, S. (2001). *A guide to first-passage processes*. Cambridge University Press.

Fokker–Planck Equations for Modelling LOBs

Cont, R., & De Larrard, A. (2012). Order book dynamics in liquid markets: Limit theorems and diffusion approximations. https://ssrn.com/abstract=1757861.
Garèche, A., Disdier, G., Kockelkoren, J., & Bouchaud, J. P. (2013). Fokker-Planck description for the queue dynamics of large-tick stocks. *Physical Review E*, 88(3), 032809.
Yang, T. W., & Zhu, L. (2016). A reduced-form model for level-1 limit order books. *Market Microstructure and Liquidity*, 2(02), 1650008.