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The Statistics of Price Changes: An Informal Primer

If you are going to use probability to model a financial market, then you had better use the right kind of probability. Real markets are wild.

(Benoît B. Mandelbrot)

During the past 40 years, financial engineering has grown tremendously. Today, both the financial industry and its regulators rely heavily on a wide range of models to address many different phenomena on many different scales. These models serve as tools to inform trading decisions and assess risk in a diverse set of applications, including risk management, risk control, portfolio construction, derivative pricing, hedging, and even market design.

Among these models, the most widely used are those that seek to describe changes in an asset's price. Given their prominence, it is important to consider the extent to which these models really reflect empirically observed price series, because models whose assumptions are at odds with real markets are likely to produce poor output. Also, because so much of the modern financial world relies on such models so heavily, widespread application of unsuitable models can create unstable feedback loops and lead to the emergence of system-wide instabilities. For example, the severe market crash in 1987 is often attributed to the prevalence of models that assumed independent Gaussian price returns, and thereby severely underestimated the probability of large price changes. Bizarrely, financial crises can be induced by the very models designed to prevent them.

Market crashes serve as a wake-up call to reject idealistic simplifications and to move towards a more realistic framework that encompasses the real statistical properties of price changes observable in empirical data. Despite considerable recent effort in this direction, this goal remains elusive, due partly to the fact that many of the statistical properties of real price series are highly non-trivial and sometimes counter-intuitive. These statistical properties are called the **stylised facts** of financial price series.

The aim of this chapter is to provide an informal introduction to the most important of these stylised facts. In addition to being interesting in their own right, these stylised facts constitute a set of quantitative criteria against which to evaluate models' outputs. As we discuss throughout the chapter, a model's inability to reproduce one or more stylised facts can be used as an indicator for how it needs to be improved, or even as a reason to rule it out altogether.

Developing a unifying theory that illustrates how these non-trivial statistical properties emerge from some deeper mechanism of individual actions and interactions is a core goal that permeates the field of market microstructure. This general point of view – specifically, a "bottom-up" approach that seeks to explain the emergence of different phenomena on different scales – is natural among physicists. It is also consistent with the work of some economists, such as Richard Lyons, who writes that "microstructure implications may be long-lived" and "are relevant to macroeconomics".[1] If true, these statements are obviously relevant for regulators, who might consider altering the microstructural organisation of financial markets to improve their efficiency and stability (see Chapter 22).

2.1 The Random Walk Model

The first model of price changes dates back to a PhD thesis called *Théorie de la spéculation*, written in 1900 by Louis Bachelier. In his thesis, Bachelier proposed a theory relating Brownian motion to stock markets, five years before Einstein's celebrated paper on Brownian motion.

Bachelier's main arguments were as follows. Since each transaction in a financial market involves both a buyer and a seller, at each instant in time the number of people who think that the price is going to rise balances the number of people who think that the price is going to fall. Therefore, argues Bachelier, future price changes are *de facto* unpredictable. In more technical language, prices are martingales: the best estimate of tomorrow's price is today's price. A simple example, which Bachelier had in mind, is when price changes are independent identically distributed (IID) random variables with zero mean. Bachelier also noted that due to the Central Limit Theorem, aggregate price changes over a given time period (such as a full trading day) should be Gaussian random variables, because they consist of a very large number of small price changes resulting from individual transactions.

These observations led Bachelier to consider price series as **Gaussian random walks** (i.e. random walks whose increments are IID Gaussian random variables). Within this framework, Bachelier was able to derive a large number of interesting results, including statistics describing the first-passage time for a price to reach a

[1] Lyons, R. K. (2001). *The microstructure approach to exchange rates* (Vol. 12). MIT Press.

certain level, and even the price of options contracts – 70 years before Black and Scholes![2]

2.1.1 Bachelier's First Law

The simplest of Bachelier's results states that typical price variations $p_{t+\tau} - p_t$ grow like the square root of the lag τ. More formally, under the assumption that price changes have zero mean (which is a good approximation on short time scales), then the price **variogram**

$$\mathcal{V}(\tau) := \mathbb{E}[(p_{t+\tau} - p_t)^2] \tag{2.1}$$

grows linearly with time lag τ, such that $\mathcal{V}(\tau) = D\tau$.

Subsequent to Bachelier's thesis, many empirical studies noted that the typical size of a given stock's price change tends to be proportional to the stock's price itself. This suggests that price changes should be regarded as multiplicative rather than additive, which in turn suggests the use of *geometric Brownian motion* for price-series modelling. Indeed, geometric Brownian motion has been the standard model in the field of mathematical finance since the 1960s.

Over short time horizons, however, there is empirical evidence that price changes are closer to being additive than multiplicative. One important reason for this is that most markets enforce resolution parameters (such as the tick size – see Section 3.1.5) that dictate the minimum allowable price change. Therefore, at the short time scales that we will focus on in this book, it is usually more appropriate to consider additive models for price changes.

Still, given the prevalence of multiplicative models for price changes on longer time scales, it has become customary to define the **volatility** σ_r in relative terms (even for short time scales), according to the equation

$$D := \sigma_r^2 \bar{p}^2, \tag{2.2}$$

where $\bar{p}$ is either the current price or some medium-term average.

2.1.2 Correlated Returns

How can we extend Bachelier's first law to the case where price changes are not independent? Assume that a price series is described by

$$p_t = p_0 + \bar{p} \sum_{t'=0}^{t-1} r_{t'}, \tag{2.3}$$

[2] Technically, Bachelier missed the hedging strategy and its associated P&L. However, when the drift can be neglected, as assumed by Bachelier, his fair-pricing argument is indeed correct. See, e.g., Bouchaud, J. P., & Potters, M. (2003). *Theory of financial risk and derivative pricing: From statistical physics to risk management*. Cambridge University Press.

where the return series r_t is time-stationary with mean

$$\mathbb{E}[r_t] = 0 \tag{2.4}$$

and variance

$$\mathbb{E}[r_t^2] - \mathbb{E}[r_t]^2 = \sigma_r^2. \tag{2.5}$$

One way to quantify the dependence between two entries $r_{t'}$ and $r_{t''}$ in the return series is via their covariance

$$\mathrm{Cov}\,(r_{t'}, r_{t''}) := \mathbb{E}[r_{t'} r_{t''}] - \mathbb{E}[r_{t'}]\mathbb{E}[r_{t''}] = \mathbb{E}[r_{t'} r_{t''}].$$

Given that the return series is time-stationary, this covariance depends only on the time lag $|t' - t''|$ between the two observations. Therefore, a common way to consider the dependence between $r_{t'}$ and $r_{t''}$ is via the **autocorrelation function** (ACF)

$$C_r(\tau) := \frac{\mathrm{Cov}\,(r_t, r_{t+\tau})}{\sigma_r^2}. \tag{2.6}$$

The case of a random walk with uncorrelated price returns corresponds to $C_r(\tau) = \delta_{\tau,0}$, where $\delta_{\tau,0}$ is the Kronecker delta function

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

For $\tau > 0$, a trending random walk has $C_r(\tau) > 0$ and a mean-reverting random walk has $C_r(\tau) < 0$. How does this affect Bachelier's first law?

One important consideration is that the volatility observed by sampling the price series on a given time scale is itself dependent on that time scale. More precisely, if we sample the p_t series once every τ seconds, then the volatility of our sampled series is given by

$$\sigma^2(\tau) := \frac{\mathcal{V}(\tau)}{\tau \bar{p}^2}, \tag{2.8}$$

where $\mathcal{V}(\tau)$ is the variogram given by Equation (2.1), and by definition $\sigma^2(1) := \sigma_r^2$.

By plugging Equation (2.3) into the definition of $\mathcal{V}(\tau)$ and expanding the square, we can derive a general formula for $\sigma^2(\tau)$ in terms of $C_r(\cdot)$:

$$\sigma^2(\tau) = \sigma_r^2 \left[1 + 2 \sum_{u=1}^{\tau} \left(1 - \frac{u}{\tau} \right) C_r(u) \right]. \tag{2.9}$$

A plot of $\sigma(\tau)$ versus τ is called a volatility **signature plot**. Figure 2.1 shows an example volatility signature plot for the simple case of a return series with

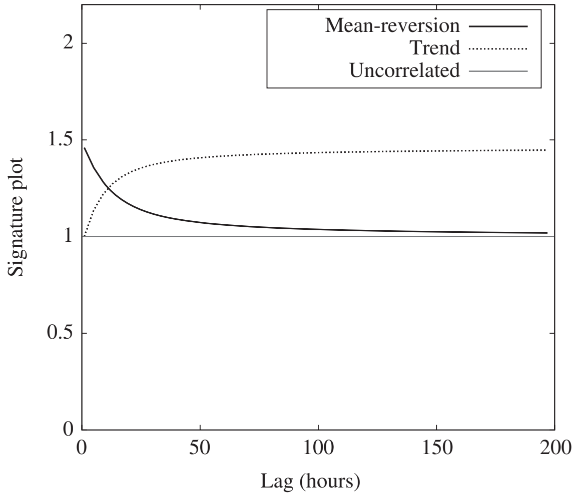

Figure 2.1. Example volatility signature plot for the simple case of a return series with an exponentially decaying ACF $C_r(u) = \rho^u$. An uncorrelated random walk (i.e. with $\rho = 0$) leads to a constant volatility. Positive correlations lead to an increasing $\sigma(\tau)$ whereas negative correlations lead to a decreasing $\sigma(\tau)$. The signature plot is flat only for time series that are neither trending nor mean-reverting on all scales.

an exponentially decaying ACF $C_r(u) = \rho^u$. As the plot illustrates, the case of an uncorrelated random walk (i.e. with $\rho = 0$) leads to a constant volatility. Positive correlations (which correspond to trends) lead to an increase in $\sigma(\tau)$ with increasing τ. Negative correlations (which correspond to mean reversion) lead to a decrease in $\sigma(\tau)$ with increasing τ. In fact, from Equation (2.9), one finds that in the case of an exponentially decaying ACF,

$$\lim_{\tau\to\infty} \frac{\sigma^2(\tau)}{\sigma^2(1)} = \frac{1+\rho}{1-\rho}. \tag{2.10}$$

2.1.3 High-Frequency Noise

Another interesting case occurs when the price p_t is soiled by some high-frequency noise, such as that coming from price discretisation effects, from pricing errors or data problems. In this section, we consider the case where rather than being given by Equation (2.3), p_t is instead assumed to be given by

$$p_t = p_0 + \bar{p} \sum_{t'=0}^{t-1} r_{t'} + \eta_t \tag{2.11}$$

where the noise η_t has mean zero, variance σ_η^2 and is uncorrelated with r_t, but is autocorrelated with an exponential ACF

$$C_\eta(\tau) := \frac{\mathrm{Cov}(\eta_t, \eta_{t+\tau})}{\sigma_\eta^2} = e^{-\tau/\tau_\eta}, \tag{2.12}$$

where τ_η denotes the correlation time of the noise.[3]

One possible interpretation of this model is that the market price can temporarily deviate from the true price (corresponding to $\eta_t = 0$). The mispricing is a random error term that mean-reverts over time scale τ_η (see Section 20.3 for a more detailed discussion). Another common interpretation is that of **microstructure noise**, coming from the fact that there is no unique price p_t but at best two prices, the bid b_t and the ask a_t, from which one has to come up with a proxy of the true price p_t. This necessarily generates some error term, which one can assume to be mean-reverting, leading to what is commonly known as a **bid–ask bounce** effect.

How does this noise on p_t affect the observed volatility? By replacing $C_r(\tau)$ in Equation (2.9) with the ACF $C_\eta(\tau)$, we see that compared to the volatility observed in a price series without noise, the addition of the η_t term in Equation (2.11) serves to increase the lag-τ square volatility as

$$\sigma^2(\tau) \longrightarrow \sigma^2(\tau) + \frac{2\sigma_\eta^2}{\tau}\left(1 - e^{-\tau/\tau_\eta}\right). \tag{2.13}$$

This additional term decays from $2\sigma_\eta^2/\tau_\eta$ for $\tau = 0$, to 0 for $\tau \to \infty$. The effect of this correlated high-frequency noise on a volatility signature plot is thus akin to mean-reversion, in the sense that it creates a higher short-term volatility than long-term volatility.

2.1.4 Volatility Signature Plots for Real Price Series

How well does Bachelier's first law hold for real price series? Quite remarkably, the volatility signature plots of most liquid assets are indeed almost flat for values of τ ranging from a few seconds to a few months (beyond which it becomes dubious whether the statistical assumption of stationarity still holds). To illustrate this point, Figure 2.2 shows the empirical mid-price signature plot from $\tau = 1$ second to $\tau = 10^6$ seconds (which corresponds to about 20 trading days) for the S&P500 E-mini futures contract, which is one of the most liquid contracts in the world. As Figure 2.2 shows, $\sigma(\tau)$ is almost flat over this entire range, and only decreases by about 20%, which indicates a weakly mean-reverting price. The exact form of a volatility signature plot depends on the microstructural details of the

[3] The noise η is often called an Ornstein–Uhlenbeck process, see, e.g., Gardiner, C. W. (1985). *Stochastic methods*. Springer, Berlin-Heidelberg.

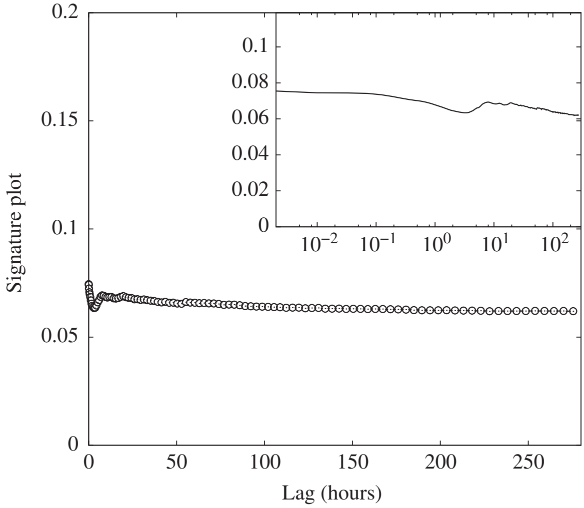

Figure 2.2. Mid-price signature plot for the S&P500 E-mini futures contract, for time lags up to approximately 280 hours (which corresponds to 20 days of trading). The signature plot is almost flat, which implies only weak mean-reversion.

underlying asset, but most liquid contracts have a similar volatility signature plot to that shown in Figure 2.2.

The important conclusion from this empirical result is that long-term volatility is almost identical to the volatility at the shortest time scales, where price formation takes place. Depending on how we view this result, it is either trivial (a simple random walk has this property) or extremely non-intuitive. In fact, as we discuss below, one should expect a rather large fundamental uncertainty about the price of an asset, which would translate into substantially larger high-frequency volatility, as with the η noise described in Section 2.1.3. Although Figure 2.2 shows that high-frequency volatility is slightly larger than low-frequency volatility, the size of this effect is small. This indicates that empirical price series exhibit only weak violations of Bachelier's first law (see Section 2.3.2), and suggests that phenomena happening on short time scales may be relevant for the long-term dynamics of prices.

2.2 Jumps and Intermittency in Financial Markets

2.2.1 Heavy Tails

Many financial models assume that returns follow a **Gaussian distribution**. However, an overwhelming body of empirical evidence from a vast array of financial instruments (including stocks, currencies, interest rates, commodities, and even implied volatility) shows this not to be the case. Instead, the unconditional

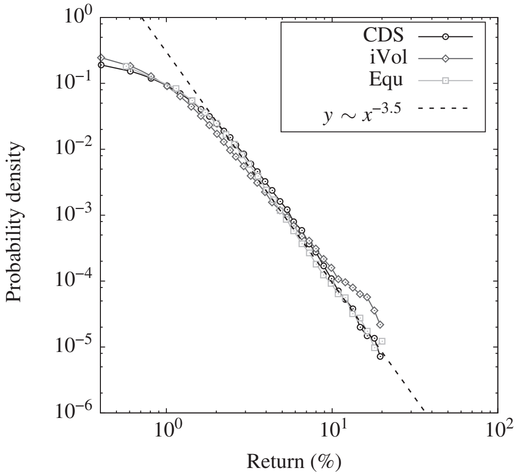

Figure 2.3. Empirical distribution of normalised daily returns of (squares) a family of US stocks, (circles) the spread of the credit default swaps (CDS) on the same stocks, and (diamonds) the average implied volatility of vanilla options again on the same stocks, all during the period 2011–2013. Returns have been normalised by their own volatility before aggregation. The tails of these distributions follow a power-law, with an exponent approximately equal to 3.5.

distribution of returns has **fat tails**, which decay as a power law for large arguments and are much heavier than the corresponding tails of the Gaussian distribution.

On short time scales (between about a minute and a few hours), the empirical density function of returns can be fit reasonably well by a **Student's *t* distribution**, whose probability density function is given by:[4]

$$f_r(r) = \frac{1}{\sqrt{\pi}} \frac{\Gamma[\frac{1+\mu}{2}]}{\Gamma[\frac{\mu}{2}]} \frac{a^\mu}{(r^2+a^2)^{\frac{1+\mu}{2}}} \underset{|r|\gg a}{\longrightarrow} \frac{a^\mu}{|r|^{1+\mu}}, \tag{2.14}$$

where μ is the **tail parameter** and a is related to the variance of the distribution through $\sigma^2 = a^2/(\mu-2)$. Empirically, the tail parameter μ is consistently found to be about 3 for a wide variety of different markets, which suggests some kind of **universality** in the mechanism leading to extreme returns (see Figure 2.3).[5] As we discuss below, this universality hints at the fact that fundamental factors are probably unimportant in determining the amplitude of most price jumps.

If σ^2 is fixed, then the Gaussian distribution is recovered in the limit $\mu \to \infty$. However, the difference between $\mu = 3$ and $\mu \to \infty$ is spectacular for tail events. In a Gaussian world, jumps of size larger than 10σ would only occur with negligible

[4] We assume here and in the following that returns have zero mean, which is appropriate for sufficiently short time scales (see discussion above).

[5] In fact, one can test the hypothesis that the different stocks share the same return distribution. This hypothesis cannot be rejected once volatility clustering effects are taken into account; on this point see: Chicheportiche, R., & Bouchaud, J.-P. (2011). Goodness-of-fit tests with dependent observations. *Journal of Statistical Mechanics: Theory and Experiment*. (09), PO9003.

probability ($\sim 10^{-23}$). For a Student's t distribution with $\mu = 3$, by contrast, this probability is $\sim 4 \times 10^{-4}$, which is several orders of magnitude larger.[6]

2.2.2 Volatility Clustering

Although considering the unconditional distribution of returns is informative, it is also somewhat misleading. Returns are in fact very far from being non-Gaussian IID random variables – although they are indeed nearly uncorrelated, as their flat signature plots demonstrate (see Figure 2.2). Such an IID model would predict that upon time aggregation, the distribution of returns would quickly converge to a Gaussian distribution on longer time scales (provided the second moment is finite, i.e. $\mu > 2$). Empirical data, on the other hand, indicates that returns remain substantially non-Gaussian on time scales up to weeks or even months.

The dynamics of financial markets is highly intermittent, with periods of intense activity intertwined with periods of relative calm. In intuitive terms, the volatility of financial returns is itself a dynamic variable characterised by a very broad distribution of relaxation times. In more formal terms, returns can be decomposed as the product of a time-dependent volatility component σ_t and a directional component ξ_t,

$$r_t := \sigma_t \xi_t. \tag{2.15}$$

In this representation, ξ_t are IID (but not necessarily Gaussian) random variables of unit variance and σ_t are positive random variables that are empirically found to exhibit an interesting statistical property called **long memory**, which we explore in detail in Section 10.1. More precisely, writing $\sigma_t = \sigma_0 e^{\omega_t}$, where σ_0 is a constant that sets the volatility scale, one finds that ω_t is an approximately Gaussian random variable with a variogram (see Section 2.1.1) given by

$$\mathcal{V}_\omega(\tau) = \langle(\omega_t - \omega_{t+\tau})^2\rangle \cong \chi_0^2 \ln[1 + \min(\tau, T)], \tag{2.16}$$

where T is a long cut-off time, estimated to be on the scale of years.[7] The parameter χ_0 sets the scale of the log-volatility fluctuations and is often called the volatility of the volatility (or "vol of vol"). For most assets, its value is found to be $\chi_0^2 \cong 0.05$.

[6] We have used here that for $\mu = 3$, the cumulative distribution of the Student's t distribution is:

$$1 - F_r(r) = \frac{1}{2} - \frac{1}{\pi}\left[\arctan r + \frac{r}{1+r^2}\right] \qquad (\mu = 3, a = 1, \sigma^2 = 1).$$

Another useful formula gives the partial contribution to volatility:

$$\int_0^r \mathrm{d}x\, x^2 f_r(x) = \frac{1}{\pi}\left(\arctan r - \frac{r}{1+r^2}\right) \qquad (\mu = 3, a = 1, \sigma^2 = 1).$$

[7] A recent study suggests rather that "volatility is rough" in the sense that $\mathcal{V}_\omega(\tau) \propto \tau^{2H}$ with $H \cong 0.1$, but note that the two functional forms become identical in the limit $H \to 0$. Gatheral, J., Jaisson, T., & Rosenbaum, M. (2014). Volatility is rough. arXiv:1410.3394.

The variogram in Equation (2.16) is markedly different from the one corresponding to an **Ornstein–Uhlenbeck** log–volatility process, characterised by a single relaxation time τ_ω, which would read

$$\mathcal{V}_\omega(\tau) = \chi_0^2\left(1 - e^{-\tau/\tau_\omega}\right). \tag{2.17}$$

In other words, volatility fluctuations in financial markets are *multi-time scale*: there are volatility bursts of all sizes, from seconds to years. This remarkable feature of financial time series can also be observed in several other complex physical systems, such as turbulent flows.

It is worth pointing out that volatilities σ and scaled returns ξ are not independent random variables. It is well documented that positive past returns tend to decrease future volatilities and that negative past returns tend to increase future volatilities (i.e. $\langle \xi_t \sigma_{t+\tau} \rangle < 0$ for $\tau > 0$). This is called the **leverage effect**. Importantly, however, past volatilities do not give much information on the sign of future returns (i.e. $\langle \xi_t \sigma_{t+\tau} \rangle \cong 0$ for $\tau < 0$).

2.2.3 Delayed Convergence Towards the Gaussian

Why do returns remain substantially non-Gaussian on time scales up to weeks or even months? Let us consider what happens when, as in Equation (2.3), we sum uncorrelated but dependent random variables such as those described by Equation (2.15). When the number of terms t is large, the Central Limit Theorem (CLT) holds, so the sum converges to a Gaussian random variable. The speed of convergence is a subtle issue, but the simplified picture is that in the case of symmetric IID random variables, the leading correction term (when compared to the Gaussian) scales at large t as t^{-1}.

However, the dependence between the random variables causes the convergence to occur much more slowly than for the IID case. In the presence of long-ranged volatility correlations, such as given by Equation (2.16), one can show that the leading correction to Gaussian behaviour instead decays as $t^{-\zeta}$ with $\zeta = \min(1, 4\chi_0^2)$ (i.e. as $\sim t^{-0.2}$ for the value of χ_0^2 quoted above). Hence, the corrections to the asymptotic Gaussian behaviour are very slow to disappear, and it may take months or even years before the CLT applies.[8]

2.2.4 Activity Clustering

In view of these long-range correlations of the volatility, it is interesting to study the temporal fluctuations of **market activity** itself, defined for example as the frequency of mid-price changes, or as the frequency of market order submission. Even a cursory look at the time series of mid-price changes (see Figure 2.4) suggests a strong degree of **clustering** in the activity as well. What is the relationship between the clustering of market activity and the clustering of volatility?

[8] On this specific point, see the detailed discussion in Bouchaud, J. P., & Potters, M. (2003). *Theory of financial risk and derivative pricing: From statistical physics to risk management*. Cambridge University Press.

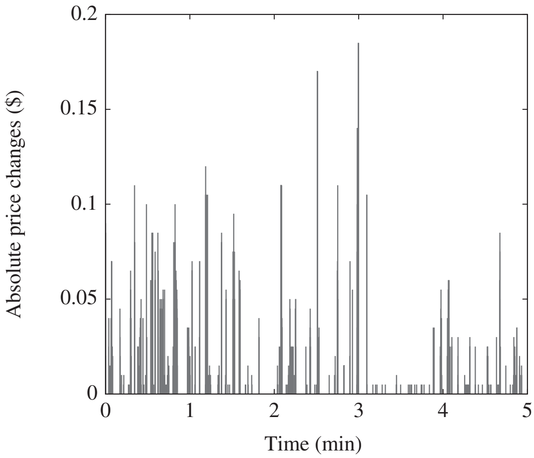

Figure 2.4. Time series of absolute changes in mid-price for TSLA, measured during a typical five-minute time interval. There are several periods with a high concentration of mid-price changes, and several other periods of relative calm.

Consider the following simple model. Assume that the price can only change by one tick ϑ at a time, either up or down, with rate φ per unit time, which quantifies the activity of the market. Assume also that when a price change occurs, up $(+\vartheta)$ and down $(-\vartheta)$ moves occur with equal probability. The price volatility in this model is then simply

$$\sigma^2 = \varphi\,\vartheta^2. \tag{2.18}$$

A more precise way to characterise this clustering property is to choose a time t and a small $\mathrm{d}t$, and count the number $\mathrm{d}N_t$ of price changes that occur during the time interval $[t, t+\mathrm{d}t]$ (i.e. count $\mathrm{d}N_t = 1$ if the mid-price changed or $\mathrm{d}N_t = 0$ if it did not). The empirical average of $\mathrm{d}N_t$ provides a way to define the average market activity $\bar{\varphi}$ as

$$\langle \mathrm{d}N_t \rangle := \bar{\varphi}\,\mathrm{d}t. \tag{2.19}$$

The covariance $\mathrm{Cov}[\frac{\mathrm{d}N_t}{\mathrm{d}t}, \frac{\mathrm{d}N_{t+\tau}}{\mathrm{d}t}]$ describes the temporal structure of the fluctuations in market activity. Figure (2.5) shows that the activity in financial markets is characterised both by long memory (the activity is autocorrelated over very long periods, of 100 days or more) and by an intricate pattern of daily and weekly periodicities.

The relationship between volatility and market activity given in Equation (2.18) however fails to address many other possible ways that prices can change. For example, it does not address the scenario where the price is mostly stable but occasionally makes very large jumps. Prices in financial markets tend to exhibit both types of volatilities: small frequent moves and rare extreme moves.

The broad distribution of price changes discussed in Section 2.2.1 describes exactly this duality between activity and jumps. The relative contribution of jumps

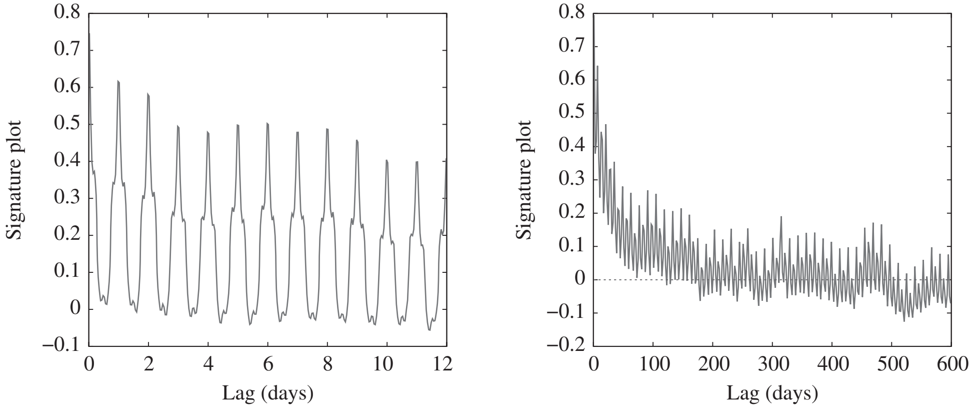

Figure 2.5. Autocorrelation of the trading activity of the S&P500 E-mini futures contract measured as the number of mid-price changes (left panel) per 30-minute interval and (right panel) per day. At the intra-day level, there are clear peaks with daily periodicity associated with the intra-day pattern. At the daily level, there is a clear weekly periodicity and still substantial correlation after 100 days.

to the total volatility depends on the precise definition of a jump, but assuming that returns follow a Student's t distribution, one finds (using the second formula of footnote 6 above) that jumps defined as events of magnitude greater than 4σ contribute about 30% of the total variance. This is quite substantial, especially given that a large fraction of these jumps appear to be unrelated to any clearly identifiable news.

Furthermore, the two types of events are intertwined in a subtle manner: an increased volatility at time t appears to trigger more activity at time $t + \tau$, much like earthquakes are followed by aftershocks. For example, a large jump is usually followed by an increased frequency of smaller price moves. More generally, some kind of **self-excitation** seems to be present in financial markets. This takes place either temporally (some events trigger more events in the future) or across different assets (the activity of one stock spills over to other correlated stocks, or even from one market to another). We present mathematical models to describe these contagion effects in Chapter 9.

2.3 Why Do Prices Move?

2.3.1 The Excess Volatility Puzzle

Why do prices behave like random walks? Often, this question is addressed via the argument that assets have a fundamental value that is known (or computed) by informed traders, who buy or sell the asset according to whether it is under- or over-priced. By the impact of making these profitable trades, the informed traders drive the price back towards its fundamental value. We develop a formal notion of price impact throughout the book, and discuss this topic in detail in Chapter 11.

In this framework, the market price of an asset can only change due to the arrival of unanticipated news (up to short-lived mispricings). The standard picture is as follows: as such news becomes available, fast market participants calculate how it affects the fundamental price, then trade accordingly. After a phase of *tâtonnement*, the price should converge to its new fundamental value, around which it makes random, high-frequency oscillations due to the trades of uninformed market participants, until the next piece of unanticipated news arrives and the whole process repeats.

This idea resides at the very heart of efficient-market theory, but is this picture correct? Can it account for the volatility observed in real markets?

Consider again the case of a typical US large-cap stock, say Apple, which has a daily turnover of around \$4–5 billion. Each second, one observes on average six transactions and of the order of 100 events for this stock alone. Compared to the typical time between news arrivals that could potentially affect the price of the company (which are on the scale of one every few days, or perhaps hours), these frequencies are extremely high, suggesting that market activity is not only driven by news.

Perhaps surprisingly, the number of large price jumps is in fact also much higher than the number of relevant news arrivals. In other words, most large price moves seem to be unrelated to news, but rather to arise endogenously from trading activity itself. As emphasised by Cutler, Poterba and Summers, "The evidence that large market moves occur on days without identifiable major news casts doubts on the view that price movements are fully explicable by news."[9]

This is a manifestation of Shiller's **excess-volatility puzzle**: the actual volatility of prices appears to be much higher than the one warranted by fluctuations of the underlying fundamental value.

2.3.2 The Flat Volatility Puzzle

Perhaps even more puzzling is the following: despite the fact that it should take some time for the market to interpret a piece of news and agree on a new price, and despite the fact that liquidity is too thin for supply and demand to be fully and instantaneously revealed, financial time series show very little under-reaction (which would create trend) or over-reaction (which would lead to mean-reversion), leading to almost flat empirical signature plots. How can this be?

[9] Cutler, D. M., Poterba, J. M., & Summers, L. H. (1989). What moves stock prices? *The Journal of Portfolio Management*, 15(3), 4–12.

If price trajectories consisted of a sequence of equilibrium values (around which prices randomly fluctuate) that are interrupted by jumps to another equilibrium price when some unanticipated news becomes available, then signature plots should show a significant decay, as in Figure 2.1. Note that relatively small fluctuations of about 0.1% around the equilibrium value, with a correlation time of 10 minutes, would lead to a high-frequency volatility contribution twice as large as the long-term volatility for the S&P500 future contract. This is clearly not observed empirically.

An alternative picture, motivated by a microstructural viewpoint, is that highly optimised execution algorithms and market-making strategies actively search for any detectable correlations or trends in return series, and implement trading strategies designed to exploit any consequent predictability in asset prices. By doing so, these algorithms and strategies iron out all irregularities in the signature plots. Expressed in terms of the ecology of financial markets (see Chapter 1), higher-frequency strategies feed on the inefficiencies generated by slower strategies, finally resulting in white-noise returns on all time scales. We will return to this important discussion in Chapter 20.

2.3.3 How Relevant is Fundamental Value?

A crucial discussion in financial economics concerns the notion of "efficiency". Are prices **fundamentally efficient** (in the sense that they are always close to some fundamental value) or merely **statistically efficient** (in the sense that all predictable patterns are exploited and removed by technical trading)? After all, purely random trades – think of a market driven by proverbial monkeys (after failing to transcribe the complete works of Shakespeare) – would generate random price changes and a flat signature plot, without reflecting any fundamental information at all. In fact, Black famously argued that prices are correct to within a factor of 2. If this is the case, the anchor to fundamental values can only be felt on a time scale T such that purely random fluctuations $\sigma\sqrt{T}$ become of the order of say 50% of the fundamental price, leading to $T = 6$ years for the stock market with a typical annual volatility of $\sigma = 20\%$.[10]

Such long time scales suggest that the notion of a fundamental price is secondary to understanding the dynamics of prices at the scale of a few seconds to a few days. These are the time scales relevant for the microstructural effects that we study in this book. Such a decoupling allows us to mostly disregard the role of fundamental

[10] This long time scale makes it very difficult to establish statistically whether mean-reversion occurs at all, although several studies have suggested that this is indeed the case, see, e.g., Bondt, W. F., & Thaler, R. (1985). Does the stock market overreact? *The Journal of Finance*, 40(3), 793–805. See also Summers, L. H. (1986). Does the stock market rationally reflect fundamental values? *The Journal of Finance*, 41(3), 591–601.

value in the following discussions, and instead focus on the notion of statistical efficiency (see, however, Chapter 20).

2.4 Summary and Outlook

The main message of this chapter is that price changes are remarkably uncorrelated over a large range of frequencies, with few signs of price adjustments or *tâtonnement* at high frequencies. The long-term volatility is already determined at the high-frequency end of the spectrum. In fact, the frequency of news that would affect the fundamental value of financial assets is much lower than the frequency of price changes. It is as if price changes *themselves* are the main source of news, and induce feedback that creates excess volatility and, most probably, price jumps that occur without any news at all. Interestingly, all quantitative volatility/activity feedback models (such as ARCH-type models,[11] or Hawkes processes, which we discuss in Chapter 9) suggest that at least 80% of the price variance is induced by self-referential effects. This adds credence to the idea that the lion's share of the short- to medium-term activity of financial markets is unrelated to any fundamental information or economic effects.

The decoupling between price and fundamental value opens the exciting prospect of building a theory of price moves that is mostly based on the endogenous, self-exciting dynamics of markets, and not on long-term fundamental effects, which are notoriously hard to model. One particularly important question is to understand the origin and mechanisms leading to price jumps, which seem to have a similar structure on all liquid markets (again indicating that fundamental factors are probably unimportant at short time scales). This is precisely the aim of a microstructural approach to financial markets: reconstructing the dynamics of prices from the bottom up and illustrating how microstructure can indeed be relevant to lower-frequency price dynamics.

Take-Home Messages

(i) Standard Gaussian random-walk models grossly underestimate extreme fluctuations in price returns. In reality, price changes follow fat-tailed, power-law distributions. Extreme events are not as rare as Gaussian models would predict.

(ii) Market activity and volatility are highly intermittent in time, with periods of intense activity intertwined with periods of relative calm.

[11] See, e.g., Bollerslev, T., Engle, R. F., & Nelson, D. B. (1994). ARCH models. In Engle, R. & McFadden, D. (Eds.), *Handbook of econometrics* (Vol. 4, pp. 2959–3028). North-Holland.

(iii) Periods of intense activity/volatility only partially overlap with the arrival of news. In fact, most activity is endogenous, and is triggered by past activity itself.
(iv) A volatility signature plot describes how the volatility of a price series varies with the lag on which it is computed. A decreasing signature plot indicates mean-reversion; an increasing signature plot indicates trending.
(v) Empirical signature plots are remarkably flat over a wide range of time scales, from seconds to months. This is a sign that markets are statistically efficient, in the sense that prices are (linearly) unpredictable.
(vi) Markets exhibit substantial excess volatility: statistical efficiency does not imply fundamental efficiency.

2.5 Further Reading

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See also Section 20.5.