14

Generalised Propagator Models

When I want to understand what is happening today or try to decide what will happen tomorrow, I look back.

(Omar Khayyám)

14.1 Price Micro-Mechanics

In the previous chapter, we studied a class of propagator models that consider the change in mid-price that occurs between subsequent market order arrivals. The models that we studied in that chapter consider all market order arrivals on an equal footing, irrespective of their size and how aggressive they are.

In the present chapter, we extend those models in two different ways. First, we consider a propagator model that partitions market order arrivals according to whether or not they consume the entire opposite-side best queue upon arrival. We introduce a mathematical framework that enables us to distinguish the impact of orders partitioned in this way, and we show that this extension helps to solve some of the problems with the one-event-type propagator models from the previous chapter.

Second, we introduce a generalised propagator model that considers not only market order arrivals, but also some limit order arrivals and cancellations. In this framework, we are able to track all events that cause price changes, and we are therefore able to monitor the evolution of impact on a more microscopic scale. However, we also argue that performing such a granular analysis of order flow requires working with rather complex models, which can be difficult to calibrate. We then turn to a more intuitive formulation of these multi-event propagator models that naturally encompasses the idea of history-dependent liquidity.

14.2 Limitations of the Propagator Model

The propagator model that we introduced in Chapter 13 is a reduced-form description of LOB dynamics. The model assumes that all market orders lead

This chapter is not essential to the main story of the book.

to the same impact dynamics, characterised by the propagator $G(\ell)$, and does not explicitly track other LOB events that occur between market order arrivals. Clearly, this approach neglects many effects that could be useful for understanding or modelling price changes in real LOBs. For example, partitioning market orders according to whether or not they consume all the available liquidity at the opposite-side best quote is very useful, not only for assessing whether such market orders will cause a change in mid-price immediately upon arrival, but also for predicting the string of other LOB events that are likely to follow.

As we detailed in Section 13.2, calibrating the one-event propagator $G(\ell)$ only requires access to data describing the response function $\mathcal{R}(\ell)$ for $\ell \geq 0$ (or alternatively $S(\ell \geq 0)$) and the autocorrelation function for market order signs $C(\ell)$. This information is readily available in many empirical data sets.

Once $G(\ell)$ has been calibrated, the propagator model leads to a precise prediction for the lag-dependent volatility $\mathcal{D}(\ell) = \mathcal{V}(\ell)/\ell$ in terms of $G(\ell)$ and $C(\ell)$, with the only further adjustable parameters arising from the variance Σ^2 of the news (or other types of events) contribution ξ_t (see Equation (13.7)). Figure 13.2 shows the result of performing this analysis. As the figure reveals, the outcome is quite disappointing: the model fails to capture all the interesting structure of the signature plot $\mathcal{D}(\ell)$, and only reproduces the long-term value of $\mathcal{D}(\ell)$ when choosing a suitable value for Σ^2.

In this way, the propagator model makes a falsifiable prediction about the detailed structure of the **signature plot** (or *lag-dependent volatility* $\mathcal{D}(\ell)$). Recall that for a purely diffusive process, $\mathcal{D}(\ell)$ is lag-independent, such that volatility does not depend on the time scale chosen to measure it. If $\mathcal{D}(\ell)$ increases with lag, then returns are positively autocorrelated, so prices trend; if $\mathcal{D}(\ell)$ decreases with lag, then returns are negatively autocorrelated, so prices mean-revert (see Section 2.1.1).

Another weakness of the propagator model concerns the shape of the **negative-lag response function**, i.e. $\mathcal{R}(\ell)$ for $\ell < 0$. As emphasised above, only the positive side of $\mathcal{R}(\ell)$ is needed to calibrate $G(\ell)$. Once this is known, $\mathcal{R}(\ell < 0)$ can be predicted without any additional parameters. Figure 14.1 shows the results for the stocks in our sample. In all cases, the empirical $\mathcal{R}(\ell < 0)$ lies above the theoretical prediction of the propagator model. The discrepancy is quite important and shows that the simple propagator model fails to grasp an important aspect of the dynamics of markets.[1]

[1] The discrepancy tends to be smaller for small-tick assets than for large-tick assets; see Taranto, D. E., Bormetti, G., Bouchaud, J. P., Lillo, F., & Tóth, B. (2016). Linear models for the impact of order flow on prices I. Propagators: Transient vs. history dependent impact. https://ssrn.com/abstract=2770352.

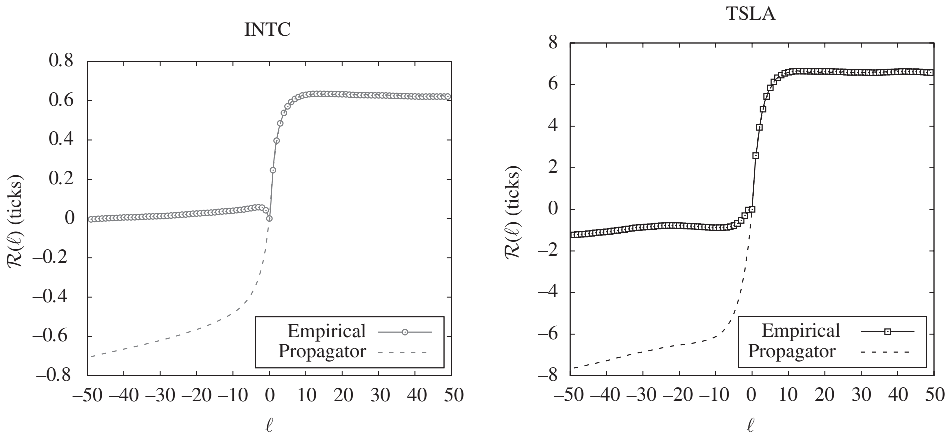

Figure 14.1. (Dashed curves) The response function calculated from the propagator model, fitted using only positive lags, for (left panel) INTC and (right panel) TSLA. The markers denote the corresponding empirical values.

Why does this discrepancy occur? To address this question, we first rewrite the response function at negative lags:

$$\begin{aligned}\mathcal{R}(-\ell) &= \langle \varepsilon_t \cdot (m_{t-\ell} - m_t) \rangle, \\ &= -\langle (m_{t+\ell} - m_t) \cdot \varepsilon_{t+\ell} \rangle,\end{aligned}$$

where the second line is obtained by shifting t to $t+\ell$. The negative-lag response function therefore tells us how the price change between t and $t+\ell$ is correlated with the sign of the market order arriving at time $t+\ell$. Intuitively, one would expect that price changes themselves directly influence the sign of future trades (e.g. that a price increase motivates more sell orders, everything else being equal, contributing positively to $\mathcal{R}(-\ell)$, and vice-versa). However, this mechanism is not present in the simple propagator model, which considers order flow to have no reaction to price changes.[2] As we show below, most of the effect can simply be captured by treating the non-price-changing market orders MO^0 and price-changing market orders MO^1 separately.

14.3 Two Types of Market Orders

As an initial step towards addressing the full complexity of LOB dynamics, we first extend the propagator model to distinguish between the set of market orders MO^1 (i.e. those that cause a non-zero lag-1 price change) and MO^0 (i.e. those that do not).

[2] This mechanism can however be included within Hasbrouck's VAR model (see Section 13.4.1), and corresponds to a negative kernel $\mathcal{H}_{rv}$.

In this section, we still work in transaction time (such that each time a market order arrives, we increment t by a single unit), and we use the notation $\pi_t \in \{\mathrm{MO}^0, \mathrm{MO}^1\}$ to indicate the market order type at time t. The generalisation that we introduce now is to allow the propagator function to be different for events of type MO^0 than for events of type MO^1, such that the mid-price dynamics obey

$$m_t = m_{t_0} + \sum_{t_0 \le t' < t} \left(\sum_{\pi = \mathrm{MO}^0, \mathrm{MO}^1} \mathbb{I}(\pi_{t'} = \pi) G_\pi(t - t') \varepsilon_{t'} + \xi_{t'} \right), \tag{14.1}$$

where $\mathbb{I}$ is an indicator function, such that for each t', one selects the propagator $G_{\pi_{t'}}$ corresponding to the particular event type that occurred at that time.

To calibrate this model, we simply generalise the response function $\mathcal{R}(\ell)$ to account for whether a given market order belongs to MO^0 or MO^1:

$$\mathcal{R}_\pi(\ell) = \langle \varepsilon_t \cdot (m_{t+\ell} - m_t) | \pi_t = \pi \rangle := \frac{\langle \varepsilon_t \mathbb{I}(\pi_t = \pi) \cdot (m_{t+\ell} - m_t) \rangle}{\mathbb{P}(\pi)}. \tag{14.2}$$

The function $\mathcal{R}_\pi(\ell)$ measures the lagged covariance (measured in market order time) between the signed event $\varepsilon_t \mathbb{I}(\pi_t = \pi)$ at time t and the mid-price change between times t and $t + \ell$, normalised by the stationary probability of the event π,

$$\mathbb{P}(\pi) = \langle \mathbb{I}(\pi_t = \pi) \rangle.$$

The normalised response function in Equation (14.2) gives the expected (signed) price change after an event π.

Figure 14.2 shows the behaviour of $\mathcal{R}_\pi(\ell)$ for $\pi = \mathrm{MO}^0$ and $\pi = \mathrm{MO}^1$. The figure paints a more detailed picture of the same phenomenon as we reported for lag $\ell = 1$ in Table 11.1, where we saw that $\mathcal{R}^1(1)$ (which is equal to $\mathcal{R}_{\mathrm{MO}^1}(1)$) is larger than $\mathcal{R}^0(1)$ (which is equal to $\mathcal{R}_{\mathrm{MO}^0}(1)$). Figure 14.2 shows that $\mathcal{R}^0(\ell)$ grows with ℓ for both small-tick and large-tick assets, whereas $\mathcal{R}^1(\ell)$ converges almost immediately to a plateau value for large-tick assets.

We also define the **signed-event correlation function** as

$$C_{\pi,\pi'}(\ell) = \frac{\langle \mathbb{I}(\pi_t = \pi) \varepsilon_t \mathbb{I}(\pi_{t+\ell} = \pi') \varepsilon_{t+\ell} \rangle}{\mathbb{P}(\pi)\mathbb{P}(\pi')}, \tag{14.3}$$

where the event π occurs before the event π'. In general, there is no reason to expect **time-reversal symmetry** (i.e. that $C_{\pi,\pi'}(\ell)$ is equal to $C_{\pi',\pi}(\ell) \equiv C_{\pi,\pi'}(-\ell)$). We have already discussed this extended correlation matrix in Section 10.2; see Figure 10.2.

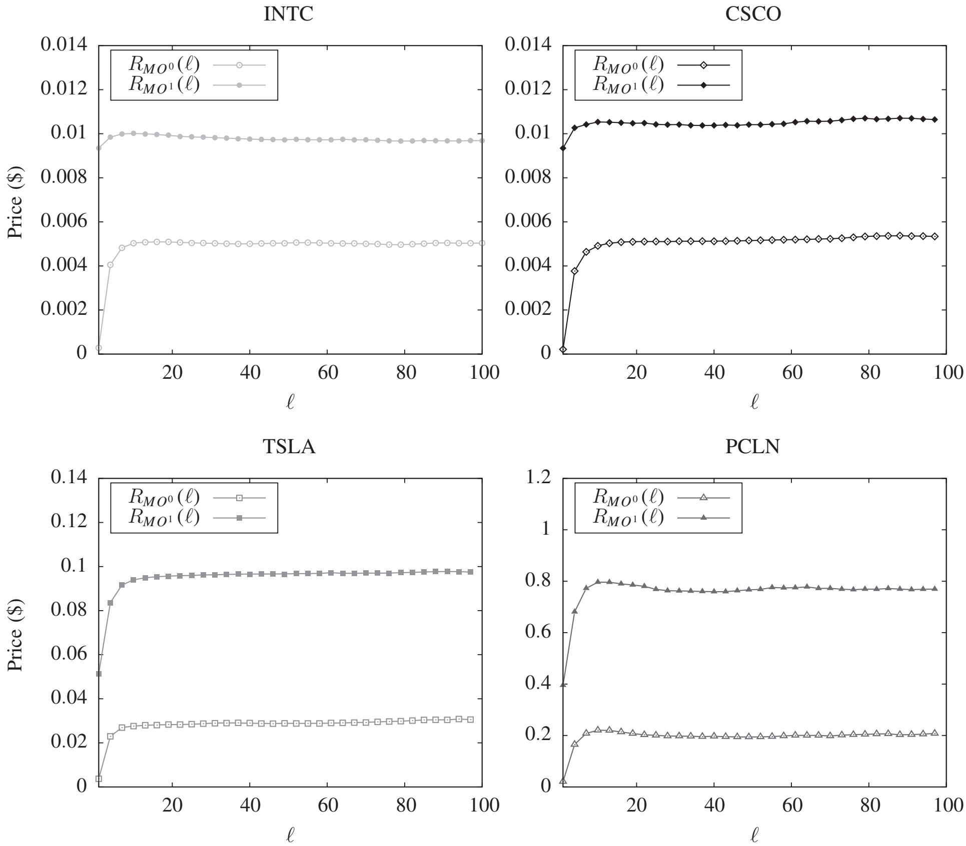

Figure 14.2. The response function of a (filled markers) price-changing and (hollow markers) non-price-changing market order for (top left) INTC, (top right) CSCO, (bottom left) TSLA and (bottom right) PCLN.

Using some straightforward algebra, the return response function in Equation (14.2) can be expressed as a generalisation of Equation (13.14):

$$S_\pi(\ell) = \sum_{\pi'} \mathbb{P}(\pi') \left[\sum_{0 \le n \le \ell} C_{\pi,\pi'}(\ell - n) K_{\pi'}(n) + \sum_{n > \ell} C_{\pi',\pi}(n - \ell) K_{\pi'}(n) \right], \qquad (14.4)$$

where $K_\pi(\ell) := G_\pi(\ell + 1) - G_\pi(\ell)$.

Similarly to Section 13.2.1, one can invert the system of equations in (14.4) to evaluate the (un-observable) G_π in terms of the (observable) S_π and $C_{\pi,\pi'}$. Figure 14.3 shows the G_π-propagators for INTC and TSLA. The price-changing propagator is, as expected, larger in amplitude than the non-price-changing propagator, but reveals a substantially stronger impact decay.

It is also possible to generalise the calculation of the price variogram $\mathcal{V}(\ell)$ to this framework, but the calculation is messy, so we present it in Appendix A.4. In any case, once the $G_\pi(\ell)$ are known, one can again (as in Equation (13.16))

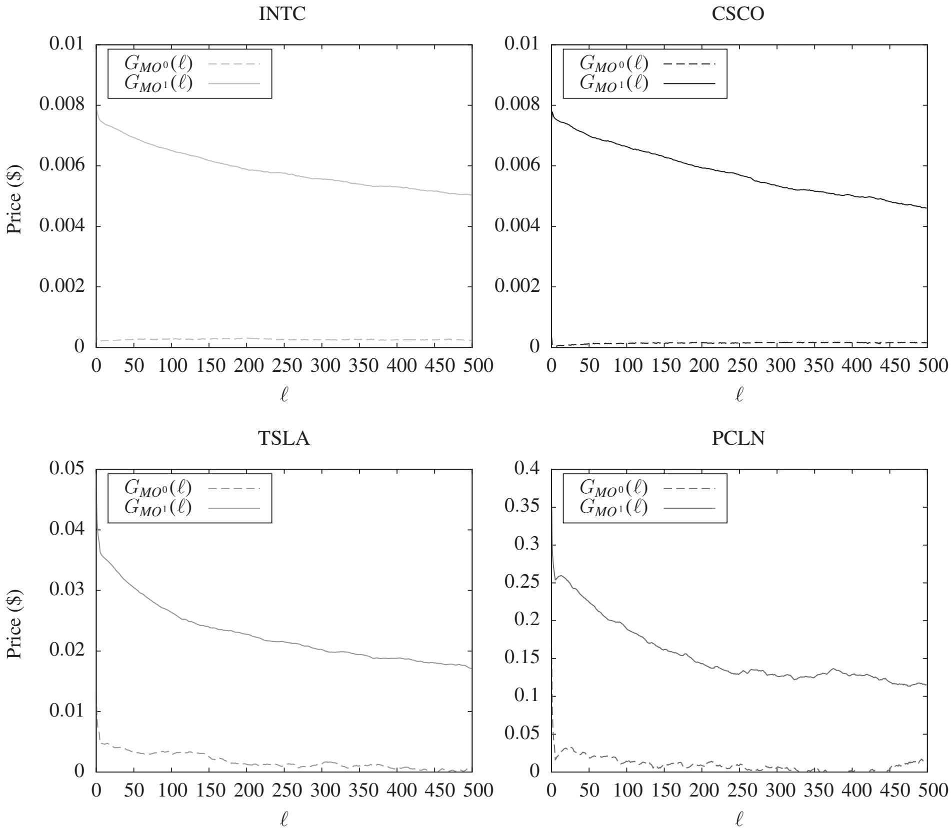

Figure 14.3. Propagators of (solid curve) price-changing market orders and (dashed curve) non-price-changing market orders for (top left) INTC, (top right) CSCO, (bottom left) TSLA and (bottom right) PCLN.

predict $\mathcal{D}(\ell) = \mathcal{V}(\ell)/\ell$ up to an additive constant Σ^2, which corresponds to the contribution of all events other than market orders. Figures 14.4 and 14.5 show the results of the two-state propagator model for the negative-lag response function $\mathcal{R}(\ell) = \sum_\pi \mathbb{P}(\pi)\mathcal{R}_\pi(\ell)$ and the signature plot $\mathcal{D}(\ell)$. The discrepancies noted for the one-state propagator are clearly reduced. Of particular interest is the fact that most of the long-term volatility is now explained solely in terms of market order impact.

14.4 A Six-Event Propagator Model

Compared to the simple propagator model from Chapter 13, the extended model in Section 14.3 fares much better at reproducing both $\mathcal{D}(\ell)$ and the negative part of the response function $\mathcal{R}(\ell)$. However, this extended model is still imperfect and suffers from the problem of only considering market orders, while treating limit order arrivals and cancellations in an indirect, implicit way.

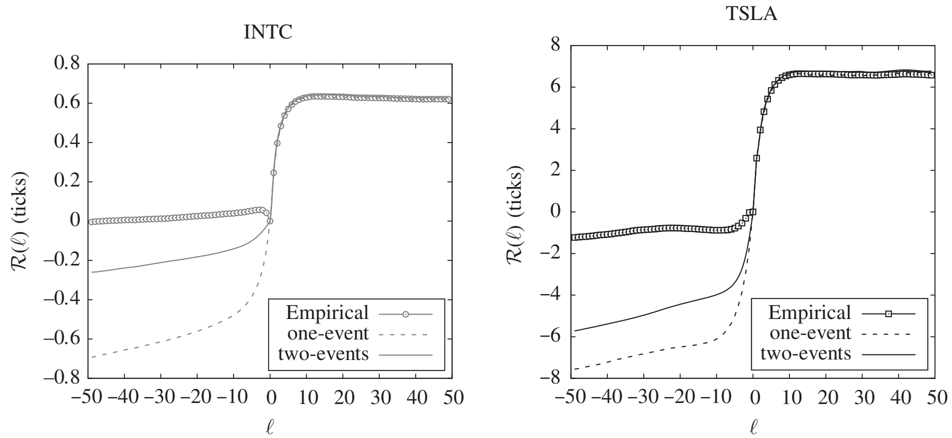

Figure 14.4. Predictions of the response function calculated from the propagator model, fitted by using only positive lags, for (left panel) INTC and (right panel) TSLA. The markers depict the empirical values, the dashed curves show the results from the one-event propagator model, and the solid curves show the results from the two-event propagator model.

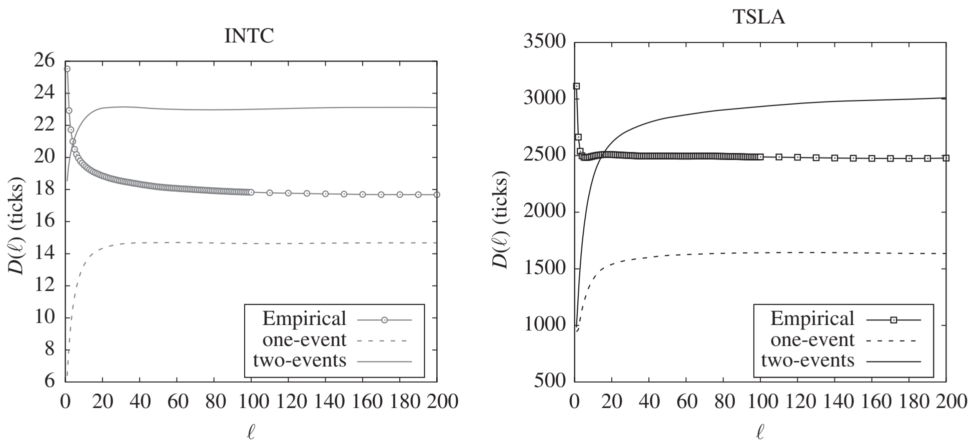

Figure 14.5. Predictions of the variogram calculated from the propagator model, fitted using only positive lags, for (left panel) INTC and (right panel) TSLA. The markers depict the empirical values, the dashed curves show the results from the one-event propagator model, and the solid curves show the results from the two-event propagator model.

In this section, we turn to a more granular formulation of the propagator model. If desired, we could formulate a version of the propagator model that considers all LOB events at all prices. However, such a model would involve processing a huge number of events, some of which occur very far away from the best quotes. Instead, we restrict our attention to LOB events that occur at or inside the best quotes, because only those event types can cause a change in the mid-price. Such events could be market order arrivals, cancellations at the best quotes, or limit order arrivals at the existing best quotes or inside the spread.

In this extended propagator model, we advance the clock by one unit whenever any of these events occurs. We again use a superscript 0 to indicate events that do not create an immediate (in this updated version of event-time) change in the mid-price, and we use a superscript 1 to indicate events that lead to an immediate change in the mid-price. We therefore partition the events that we consider into six different types of events: $\pi \in \{\mathrm{MO}^0, \mathrm{CA}^0, \mathrm{LO}^0, \mathrm{MO}^1, \mathrm{CA}^1, \mathrm{LO}^1\}$. For market order arrivals and limit order arrivals, we write $\varepsilon = +1$ to denote buy orders and $\varepsilon = -1$ to denote sell orders. For cancellations, we use the *opposite* signs, such that we write $\varepsilon = -1$ for the cancellation of a buy order and $\varepsilon = +1$ for the cancellation of a sell order. For a price-changing event, we call the associated change in mid-price the **step size** Δ_t. We provide a detailed description of these event types and definitions in Table 14.1.

It can also be useful to introduce another sign, $\varepsilon^\dagger = \pm 1$, which identifies the side of the LOB where each event occurs. We write $\varepsilon^\dagger = +1$ for any event happening on the ask side (i.e. buy market order arrivals, sell limit order arrivals and sell limit order cancellations) and $\varepsilon^\dagger = -1$ for any event happening on the bid side (i.e. sell market order arrivals, buy limit order arrivals and buy limit order cancellations).

14.4.1 Price Changes in the Six-Event Propagator Model

As in previous chapters, let m_t denote the mid-price immediately *before* the event that occurs at time t. In this framework, the generalised propagator description of mid-price changes can be written as

$$m_t = m_{t_0} + \sum_{t_0 \le t' < t} \left(\sum_{\pi} \mathbb{I}(\pi_{t'} = \pi) G_\pi(t - t') \varepsilon_{t'} + \xi_{t'} \right), \tag{14.5}$$

where the sum over π now covers all six event types, defining six different propagators $G_\pi(\ell)$.

To build a statistical theory of price changes, and in particular to understand how a certain market event impacts the price, one needs to study the properties of the time series of events and signs. To do so, we can follow a similar approach as in Equation (14.1), except that π_t can now take six different types. Correspondingly, the correlation matrix becomes a 6×6 matrix, with each entry being a function of time.

How does this correlation matrix behave for real stocks? Plotting the full matrix graphically is difficult, so we instead summarise its salient features:[3]

[3] For a generalisation of the DAR model of Sections 10.4.2 and 13.3.2 to multi-event-time series, see Taranto D. E., Bormetti, G., Bouchaud, J. P., Lillo, F., & Tóth, B. (2016). Linear models for the impact of order flow on prices I. Propagators: Transient vs. history dependent impact. https://ssrn.com/abstract=2770352.

Table 14.1. *Summary of the* 6 *possible event types* $\pi \in \{\mathrm{MO}^0, \mathrm{CA}^0, \mathrm{LO}^0, \mathrm{MO}^1, \mathrm{CA}^1, \mathrm{LO}^1\}$, *with the corresponding event signs and step sizes.*

Type	Event	Event Signs	Step Size (Δ)
MO^0	arrival of a market order with volume less than the outstanding volume at the opposite-side best quote	$\varepsilon = \varepsilon^\dagger = 1$ for buy market orders; $\varepsilon = \varepsilon^\dagger = -1$ for sell market orders	0
CA^0	partial cancellation of the bid/ask-queue	$\varepsilon = \varepsilon^\dagger = -1$ for buy limit order cancellation; $\varepsilon = \varepsilon^\dagger = +1$ for sell limit order cancellation;	0
LO^0	arrival of a limit order at the current best bid/ask	$\varepsilon = -\varepsilon^\dagger = +1$ for buy limit order arrivals; $\varepsilon = -\varepsilon^\dagger = -1$ for sell limit order arrivals	0
MO^1	arrival of a market order with volume greater than or equal to the outstanding volume at the opposite-side best quote	$\varepsilon = \varepsilon^\dagger = +1$ for buy market orders; $\varepsilon = \varepsilon^\dagger = -1$ for sell market orders;	half of the first gap behind the ask ($\varepsilon = 1$) or bid ($\varepsilon = -1$)
CA^1	cancellation of the whole best bid/ask-queue	$\varepsilon = \varepsilon^\dagger = -1$ for buy limit order cancellation; $\varepsilon = \varepsilon^\dagger = +1$ for sell limit order cancellation	half of the first gap behind the ask ($\varepsilon = 1$) or bid ($\varepsilon = -1$)
LO^1	arrival of a limit order inside the spread	$\varepsilon = -\varepsilon^\dagger = +1$ for buy limit order arrivals; $\varepsilon = -\varepsilon^\dagger = -1$ for sell limit order arrivals	half the distance of the limit order from the previous same-side best quote

- Autocorrelations of the "side" variable $\varepsilon_t^{\dagger}$ (see Figure 14.6) decay as a power law. In other words, LOB activity persists being either bid-side or ask-side for long periods of time. This is a more general statement of the same phenomenon that we observed for market order signs in Chapter 10.
- All non-price-changing correlation functions of the form $C_{\pi^0,\pi^0}(\ell)$ (with $\pi^0 = \mathrm{MO}^0, \mathrm{LO}^0$ or CA^0) are positive and decay as a slow power law. For price-changing events ($\pi^1 = \mathrm{MO}^1, \mathrm{LO}^1$ or CA^1), this correlation becomes short-ranged for large-tick stocks.
- When considering all market order arrivals, limit order arrivals and cancellations together (i.e. mixing between the different event types), autocorrelations of ε_t decay exponentially, which indicates that this series is short-range autocorrelated. Therefore, although each of the separate order-flow sign series are (separately) long-range autocorrelated, their intertwined series is not. This is the mechanism of the "tit-for-tat" dance that makes prices diffusive, as we described in Section 13.2.
- After a market order of either type MO^0 or MO^1, the flow of limit orders and cancellations first pushes the price in the same direction for about ten events, then reverses and opposes the market order flow, particularly through LO-type events, which correspond to **liquidity refill**.
- Newly posted price-improving limit orders LO^1 attract market orders. More precisely, LO^1 events rapidly trigger a strong opposite flow of MO^1 orders and, for large-tick stocks, of MO^0 orders as well. By contrast, LO^0 events are initially followed by market orders in the same direction, before the flow of these market orders inverts.

In a nutshell, the correlation matrix is consistent with the story that market participants who seek to execute large volumes split their trades into many different orders, which they execute via a mix of both market orders and limit orders. This explains why the diagonal correlation functions are positive and long-ranged. Liquidity providers step in rather quickly to counteract the correlated flow of market orders (via liquidity refill).[4]

Endowed with the knowledge of all correlation functions $C_{\pi,\pi'}(\ell)$ and all response functions $\mathcal{R}_\pi(\ell)$, one can invert Equations (14.4) to obtain the six propagators $G_\pi(\ell)$. Using the expression in Appendix A.4, one can then reconstruct the lag-dependent volatility $\mathcal{D}(\ell)$ from $C_{\pi,\pi'}(\ell)$ and $G_\pi(\ell)$. This recipe works reasonably well for small-tick stocks, but works much less well for large-tick stocks.[5]

[4] At the very highest frequencies, the reaction of the market is actually to first reduce liquidity, before this refill occurs. This could be explained by some liquidity providers cancelling their previous limit orders and possibly replacing them slightly further from the spread, in an attempt to buy or sell at a slightly better price immediately after the trade, before the refill occurs. See Bonart, J., & Gould, M. D. (2017). Latency and liquidity provision in a limit order book. *Quantitative Finance*, 1–16.

[5] On this point, see Eisler, Z., J. Kockelkoren, J., & Bouchaud, J. P. (2012). Models for the impact of all order book events. In Abergel, F. et al. (Eds.), *Market microstructure: Confronting many viewpoints*. Wiley; and Patzelt, F., & Bouchaud, J.-P. (2017). Nonlinear price impact from linear models. *Journal of Statistical Mechanics*, 12, 123404.

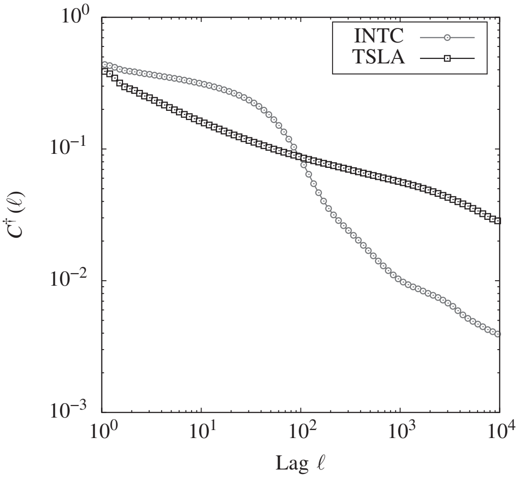

Figure 14.6. Autocorrelation function of the "side" variable $\varepsilon^\dagger$ for TSLA and INTC, plotted in doubly logarithmic coordinates. The plot suggests that the "side" process is a long-memory process.

Why is this so? In the next subsection, we argue that although generalising the transient impact model seems natural at first sight, attempting to do so naively can lead to erroneous results. Instead, we argue that an approach of history-dependent impact makes much more sense when generalised to all LOB events, and helps control the unavoidable noise that affects the determination of the propagators, especially for large-tick stocks. While the HDIM and TIM approaches are equivalent in the single-event case (see Section 13.3), they are actually distinct modelling strategies in the general case.

14.4.2 A Generalised, History-Dependent Impact Model

If we consider a model that includes all LOB events, then the dynamics of the mid-price can be *exactly* represented by the expression

$$r_t = \varepsilon_t \Delta(\pi_t, \varepsilon_t, t), \tag{14.6}$$

where $\Delta(\pi_t, \varepsilon_t, t)$ is the amplitude of the price change at time t for an event of type π_t and sign ε_t. By definition, $\Delta = 0$ for any event $\pi \in \{\mathrm{MO}^0, \mathrm{CA}^0, \mathrm{LO}^0\}$. Therefore, any price change must be due to one event $\pi \in \{\mathrm{MO}^1, \mathrm{CA}^1, \mathrm{LO}^1\}$. The definition of the non-zero Δ's depend on the type of event that occurs at time t (see Table 14.1). For example, if $\pi \in \mathrm{MO}^1$, then Δ is one-half of the gap behind the best ask (for $\varepsilon = +1$) or behind the best bid (for $\varepsilon = -1$). In contrast to Equations (13.8) and (13.19), the present model has no need for a term ξ_t to account for "news" contributions, because it already includes all LOB events.

Large-Tick Stocks: The Case of Constant Δ

We first consider Equation (14.6) as a model for large-tick stocks. For such stocks, the LOB is densely populated, and the bid–ask spread and the gap behind the best quote are usually equal to one tick ϑ. To model this case, we assume that Δ does not fluctuate at all, such that

$$\Delta(\pi^0,\varepsilon,t) = 0, \qquad \Delta(\pi^1,\varepsilon,t) = \Delta_{\pi^1} = \vartheta/2. \tag{14.7}$$

In the propagator framework, this amounts to setting

$$G_{\pi^0}(\ell) = 0; \qquad G_{\pi^1}(\ell) = \Delta_{\pi^1},$$

such that all price-changing events have constant impact. In this case, the expression for $\mathcal{R}_\pi(\ell)$ simplifies to

$$\mathcal{R}_\pi(\ell) \approx \Delta_\pi + \sum_{0<t'<\ell} \sum_{\pi^1} \Delta_{\pi^1} \mathbb{P}(\pi^1) C_{\pi,\pi^1}(t'). \tag{14.8}$$

Therefore, the price response to a given event can be understood as its own "mechanical" impact (possibly zero) plus the sum of the average impact of all future events correlated with this initial event. Within the same model, the expression for $\mathcal{V}(\ell)$ simplifies to

$$\mathcal{V}(\ell) \approx \sum_{0\le t',t''<\ell} \sum_{\pi^1} \sum_{\pi^{1\prime}} \mathbb{P}(\pi^1)\mathbb{P}(\pi^{1\prime}) C_{\pi^1,\pi^{1\prime}}(t'-t'')\Delta_{\pi^1}\Delta_{\pi^{1\prime}}. \tag{14.9}$$

For large-tick stocks, these predictions are extremely accurate. In principle, this model is equivalent to the six-event propagator model with a constrained form for G_{π^0} and G_{π^1}. The reason it fares much better than the unconstrained model is that, after matrix inversion, noisy estimates of $\mathcal{R}_\pi(\ell)$ and $C_{\pi,\pi'}(\ell)$ lead to propagators that have some non-trivial (but spurious) structure, which in turn considerably pollutes the determination of $\mathcal{V}(\ell)$.

Small-Tick Stocks: The Case of History-Dependent Δ

We next consider Equation (14.6) as a model for small-tick stocks. For such stocks, the LOB is sparsely populated, and the bid–ask spread and the gap behind the best quote fluctuate over time. The propagator model does not address these fluctuations, since $G_\pi(\ell)$ only depends on the event type, not on the particular state of the LOB when the event π took place. This provides motivation for considering alternative approaches to the one that we have developed so far in this chapter. One possible alternative is to consider a model in which the history of the order flow feeds back into the present values of $\Delta(\pi,\varepsilon,t)$, to reflect how past order flow affects present liquidity conditions.[6]

Assuming that all the consequences of buys are the same as those of sells (up to a sign change), we can extend the model in Equation (14.7) to

$$\Delta(\pi^1,\varepsilon,t) = \Delta_{\pi^1} + \sum_{t'<t} \sum_{\pi'} \mathbb{I}(\pi_{t'} = \pi')\kappa_{\pi',\pi^1}(t-t')\varepsilon_{t'}\varepsilon + \widetilde{\xi}_t, \tag{14.10}$$

where $\kappa_{\pi,\pi'}$ are kernels that model the dependence of gaps on past order flow, and $\widetilde{\xi}$ describes the part of the evolution of the spreads and gaps that is not explained by the past order flow. Observe that the entries whose second index corresponds to a non-price-changing event, for which $\Delta_{\pi^0} = 0$ by definition, so κ_{π,π^0} must be zero for all π. For large-tick stocks, the influence kernels $\kappa_{\pi,\pi'}$ are all extremely small and can be neglected, leading back to Equations (14.8) and (14.9).

To illustrate what this model encodes, imagine that the event taking place at time t' is a buy market order with type MO^1. This market order exerts a buy pressure on the LOB. If this pressure scares away sellers, then the gap behind the ask could increase;

[6] For more details on the content of this section, and empirical calibration, see Eisler, Z., J. Kockelkoren, J., & Bouchaud, J. P. (2012). The price impact of order book events: Market orders, limit orders and cancellations. *Quantitative Finance*, 12(9), 1395–1419; Eisler et al. (2012) as listed in Footnote 5; and Patzelt, F., & Bouchaud, J.-P. (2017). Nonlinear price impact from linear models. *Journal of Statistical Mechanics*, 12, 123404.

if this pressure attracts other sellers to submit limit orders, then the gap behind the ask could decrease. The first case would correspond to $\kappa_{\mathrm{MO}^1,\mathrm{MO}^1} > 0$ and the second case would correspond to $\kappa_{\mathrm{MO}^1,\mathrm{MO}^1} < 0$.

As we now show, the model in Equation (14.10) corresponds precisely to a generalisation of the model in Equation (13.8). By substituting Equation (14.10) into the exact evolution equation (14.6), we arrive at

$$r_t = \Delta_{\pi_t}\varepsilon_t + \sum_{t'<t} \kappa_{\pi_{t'},\pi_t}(t-t')\varepsilon_{t'} + \xi_t; \qquad (\xi_t := \widetilde{\xi}_t\varepsilon_t). \tag{14.11}$$

It is interesting to compare Equation (14.11) with the analogue for the transient impact model in Equation (14.5):

$$r_t = G_{\pi_t}(1)\varepsilon_t + \sum_{t'<t}\Big[G_{\pi_{t'}}(t-t'+1) - G_{\pi_{t'}}(t-t')\Big]\varepsilon_{t'} + \xi_t. \tag{14.12}$$

By comparing Equation (14.1) and (14.12), one sees that the two models are equivalent if and only if

$$G_\pi(1) = \Delta_\pi$$

and

$$\kappa_{\pi,\pi'}(\ell) = G_\pi(\ell+1) - G_\pi(\ell), \qquad \text{for all } \pi', \tag{14.13}$$

which can occur if and only if $\kappa_{\pi,\pi'}$ does not depend on the value of π'. This constraint cannot hold in general since when the second event is non-price-changing, $\kappa_{\pi,\pi^0} = 0$, which would imply that κ is zero in all cases. The only exception is when there is one possible type of event, in which case we recover the equivalence between transient impact models (TIM) and history-dependent impact models (HDIM), as we saw in Section 13.3.

When calibrating $\kappa_{\pi,\pi'}$ empirically, one in fact finds a significant dependence on the second event type, absent in transient impact models. The model fares quite well at reproducing the negative-lag response function and the signature plot of small-tick stocks. One furthermore finds that for ℓ large enough, and when π' is a price-changing event

$$\kappa_{\pi^1,\pi'}(\ell) \le 0, \tag{14.14}$$

Equation (14.14) recovers the **asymmetric dynamical liquidity** phenomenon in a wider context (see Section 11.3.4): past price-changing events tend to reduce the impact of future events of the same sign, and increase the impact of future events of opposite sign (recall the content of Equation (14.10)). Note however that $\kappa_{MO^1,\pi'}(\ell)$ is positive for short lags: as we mentioned in Section (14.4.1), the knee-jerk reaction of markets is first to reduce liquidity upon the arrival of aggressive market orders, before the liquidity refill phenomenon takes place.

14.5 Other Generalisations

In this section, we present other ways to tag different types of LOB events with a π_t term, different from the MO, LO, CA types discussed above. One extremely interesting case is when some trade identification is possible, allowing one to distinguish the price impact of different institutions or individual traders. Another situation where tagging is important is cross-impact, where trading one asset may impact another correlated asset in a different market.

14.5.1 Trade-Ownership Data

One very interesting case arises when the owner of each trade can be identified. For example, some data sets include an identification number for each market order, which indicates the financial institution (or even individual trader) that initiated the trade.[7] In this case, one could tag each market order with this information, and define a propagator $G_\pi(\ell)$ for each institution (or trader) π. By fitting such a model empirically, it could be possible to gain insight into how the impact of specified actions differs according to who conducted them.

14.5.2 Proprietary Trades versus Market Trades

Another situation where partial identification is possible is when a trading firm knows about its own trades, but cannot identify those of other firms. In this situation, the trading firm can tag market orders with a binary variable $\pi \in \{\text{own}, \text{other}\}$. From the empirical determination of the cross-correlation function $C_{\text{own,other}}(\ell)$, the trading firm can then determine how its own trades tend to be anticipated or followed by the rest of the market.

Together with the two empirical response functions $\mathcal{R}_{\text{own}}(\ell)$ and $\mathcal{R}_{\text{other}}(\ell)$, it is again possible to obtain two propagators, $G_{\text{own}}(\ell)$ and $G_{\text{other}}(\ell)$, by inverting a relation similar to Equation (14.4). When this analysis is possible, one finds that the two propagators are identical, up to statistical fluctuations.[8] Since $G_{\text{other}}(\ell)$ must reflect a large fraction of noise trades, the similarity between $G_{\text{own}}(\ell)$ and $G_{\text{other}}(\ell)$ is compatible with the idea that these propagators describe the reaction impact, which is independent of the information content of the trades (which only shows up via the correlations with future orders, and on longer time scales). This is in line with the order-flow view of price formation (see the discussion in Sections 11.1 and 11.2).

14.5.3 Cross-Impact

Another natural extension of the propagator model is to consider how trades for a given asset j can (directly or indirectly) impact the price of another asset i. Restricting to market orders, and neglecting order volumes, Equation (13.8) can be generalised to describe both self- and cross-impact, as follows:[9]

$$r_{i,t} = m_{i,t+1} - m_{i,t} = \sum_{t' \le t} K_{ij}(t-n)\varepsilon_{j,t'} + \xi_{i,t}, \tag{14.15}$$

where $r_{i,t}$ is the return of asset i at time t, $\varepsilon_{j,t'}$ is the sign of the market order for asset j at time t', and $\xi_{i,t}$ are residuals that capture the component of returns not directly related to trading (and that are possibly correlated between different assets). If one considers the joint dynamics of N stocks, then the propagator is

[7] Sometimes, only the identification of the executing broker is possible, not the final buyer/seller of the asset.
[8] On this point, see Tóth, B., Eisler, Z., & Bouchaud, J. P. (2017). The short-term price impact of trades is universal. https://ssrn.com/abstract=2924029.
[9] Note that because the market order time has no reason to correspond across different assets, t is here calendar-time and not event-time.

a lag-dependent $N \times N$ matrix. The diagonal terms of this matrix correspond to **self-impact**, whereas the off-diagonal terms correspond to **cross-impact**.

This model can be calibrated by inverting a relation similar to Equation (14.4). The detailed discussion of this topic is beyond the scope of this book, so we instead refer to recent papers cited in Section 14.7. The two important empirical conclusions from these papers are:

- Both the on-diagonal and off-diagonal elements of the propagator matrix decay as a power-law of the lag, $G(\ell) \sim \ell^{-\beta}$, with $\beta < 1$, as in the single-asset case.
- Perhaps surprisingly, most of the cross-correlations between price moves are mediated by trades themselves (i.e. through a cross-impact mechanism) rather than through the cross-correlation of the residual terms $\xi_{i,t}$, which are not directly related to trading.

Yet another related situation arises in fragmented markets, when trading on different venues can impact prices differently, corresponding to different propagators.

14.6 Conclusion

We conclude our discussion of propagator models by recapping the thread of ideas that we have explored. We started our discussion in Chapter 13 with a simple picture in which we interpreted price moves in terms of the impact of market orders and an extra contribution not related to trades. In this description, we saw that the long memory of trade signs imposes constraints on how price impact must decay with time, to ensure that the mid-price remains approximately diffusive. In the present chapter, we developed our discussion towards a more complete picture of LOB dynamics, where we interpreted price moves in terms of complex, intertwined flows (of market orders, limit orders and cancellations) that statistically coordinate and respond to each other.

We have also seen that the transient nature of price impact, which is described by the decaying propagator $G(\ell)$, could equivalently be interpreted as a permanent but history-dependent impact. In this interpretation, past trades themselves shape present liquidity in a way that decreases the impact of expected market orders and increases the impact of surprising market orders (see Section 13.3).

In generalising the propagator model to describe all LOB events, it becomes apparent that transient impact models (TIM) and history-dependent impact models (HDIM) are not equivalent, but rather that transient impact models are a special subclass of HDIMs (see Equation (14.11)). The compelling idea behind HDIMs is that the current spread and liquidity in the LOB are affected by all previous LOB events. This allows one to capture in detail how past order flow shapes (on average) the LOB in the vicinity of the best quotes, and therefore how the next event is likely to impact the mid-price.

Similarly to in Section 11.3.4, we again find that liquidity is dynamically asymmetric in this extended framework. More precisely, past events tend to reduce the impact of future events of the same sign and increase the impact of future events of opposite sign, as is required if markets are to be stable and prices are to be statistically efficient. This is one of the most important messages of this chapter, and indeed of the whole book: financial markets operate in a kind of "tit-for-tat" mode, where liquidity providers react to the actions of liquidity takers, and vice-versa. During the normal-functioning of financial markets, these retroactions have a stabilising effect that allows markets to function in an orderly fashion. Any breakdown of the asymmetric liquidity mechanism may lead to crises.

Finally, we emphasise that the generalised propagator models we have presented in this chapter are still linear models. When restricted to market orders, they boil down to the effective propagator model from Chapter 13 (see Section 13.4.2). Therefore, this family of models cannot account for non-linear effects such as the square-root impact of metaorders. Reproducing these more complex effects requires genuinely new ingredients, which we turn to in Chapter 19.

Take-Home Messages

(i) Single-event propagator models are too simple to reproduce some empirical regularities of real markets. Accounting for the more complex dynamics observable in empirical data requires extending this basic framework.

(ii) One possible generalisation is to partition market orders according to whether or not they change the price. For large-tick stocks, this partitioning improves the propagator model's predictions.

(iii) Another possible generalisation is to include all LOB events. Fitting such a model requires large amounts of data and can lead to expensive computations, but allows one to measure the impact of limit orders and cancellations, on top of the impact of market orders.

(iv) In the propagator framework, there are two main approaches to including all LOB events: transient impact models (TIM), which assume that all events are characterised by a different (decaying) propagator, and history-dependent impact models (HDIM), which describe how each event changes the future impact of all other events – i.e. of future liquidity.

(v) The propagator framework can also be generalised to many other situations, including comparing the impact of different traders' actions or measuring cross-impact across different assets.

14.7 Further Reading

Generalised Propagator Models

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Impact of Limit Orders and Other Events

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