

# CS221 Fall 2018 Homework [car]

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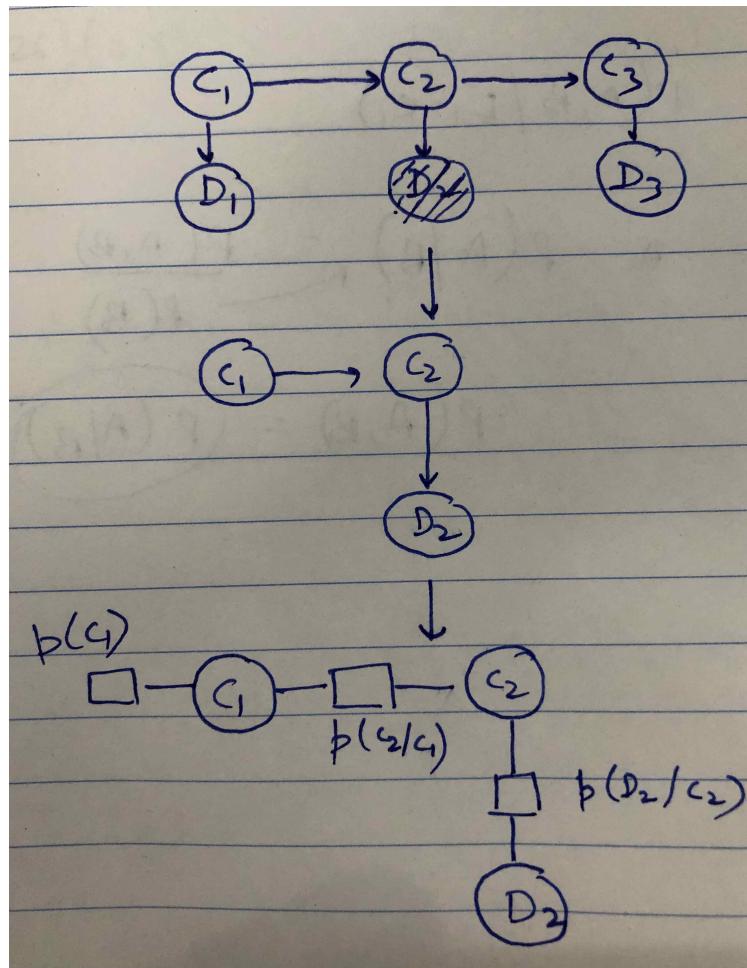
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## Problem 1

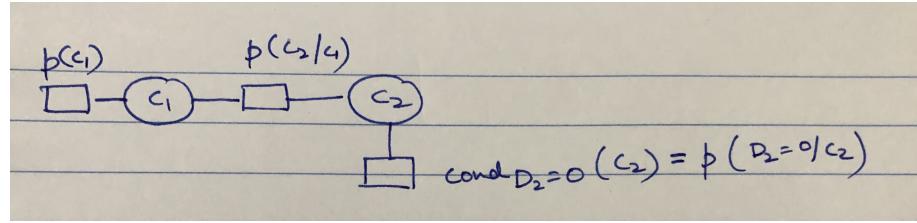
(a) Step 1: Remove variables that are not ancestors

Step 2: Converting to factor graph

Step 1 and step 2 are shown in diagram below:



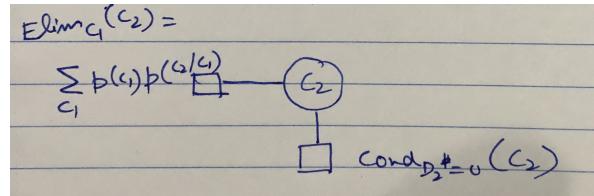
Step3: Conditioning on  $D_2 = 0$



Condition variable  $D_2$  on value  $D_2 = 0$ , replacing it with a factor  $\text{cond}_{D_2=0}(C_2)$ , we get

$$\begin{array}{ll} \text{cond}_{D_2=0}(C_2) & C_2 \\ 1 - \eta & 0 \\ \eta & 1 \end{array}$$

#### Step4: Eliminate $C_1$



$$\begin{aligned} \text{elim}_{C_1}(C_2) &= \sum_{C_1} p(C_1)p(C_2|C_1) \\ &= 0.5 \sum_{C_1} p(C_2|C_1) \end{aligned}$$

This is given from the below table:

$$\begin{array}{ll} \text{elim}_{C_1}(C_2) & C_2 \\ 0.5(1 - \epsilon + \epsilon) = 0.5 & 0 \\ 0.5(\epsilon + 1 - \epsilon) = 0.5 & 1 \end{array}$$

Therefore, now that we know  $\text{elim}_{C_1}(C_2)$  and  $\text{cond}_{D_2=0}(C_2)$ ,

$$p(C_2/D_2 = 0) = \text{elim}_{C_1}(C_2) * \text{cond}_{D_2=0}(C_2)$$

$$\begin{array}{ll} p(C_2/D_2 = 0) & C_2 \\ 0.5(1 - \eta) & 0 \\ 0.5\eta & 1 \end{array}$$

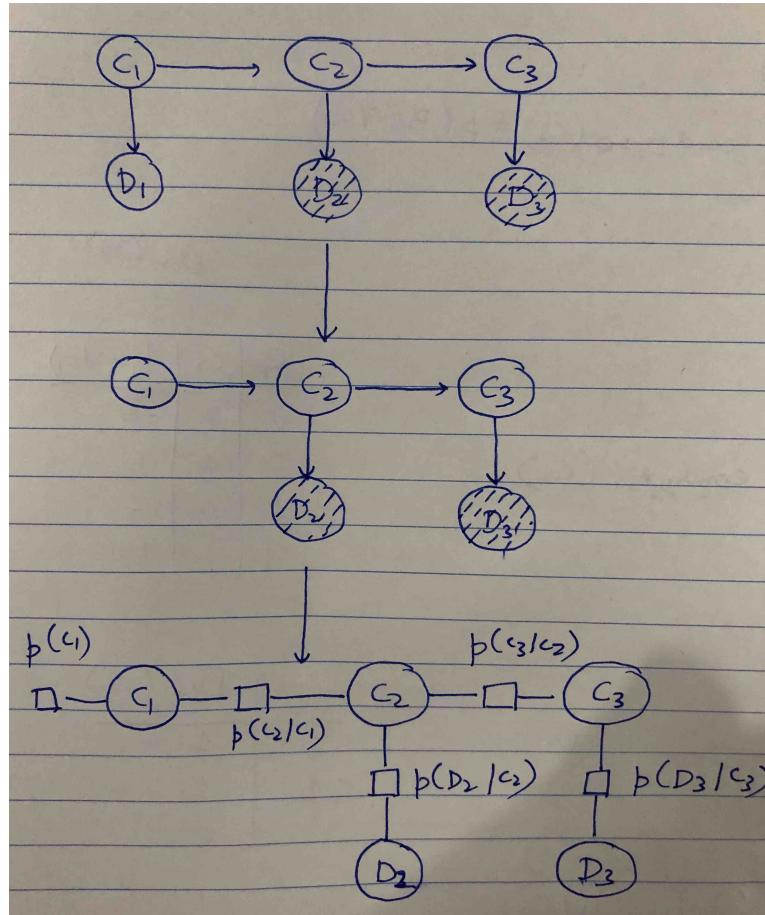
Hence, the given query,

$$\begin{aligned} p(C_2 = 1 / D_2 = 0) &= \frac{0.5\eta}{0.5\eta + 0.5(1 - \eta)} \\ &= \eta \end{aligned}$$

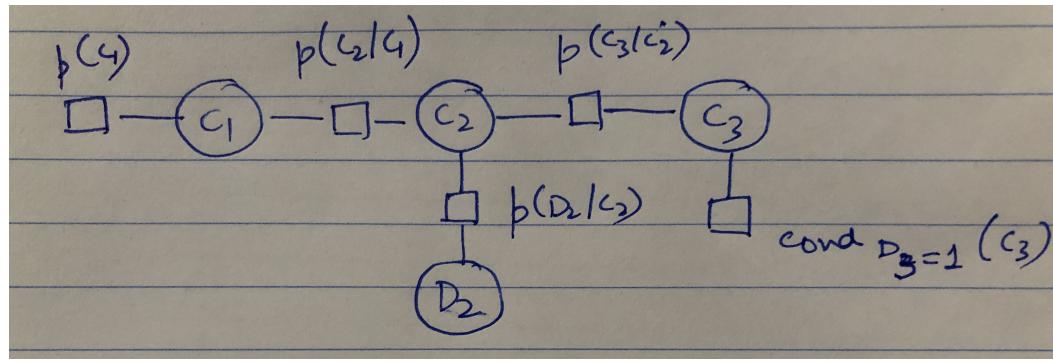
(b) **Step1:** Remove variables that are not ancestors

**Step2:** Converting to factor graph

Step 1 and step 2 are illustrated below:



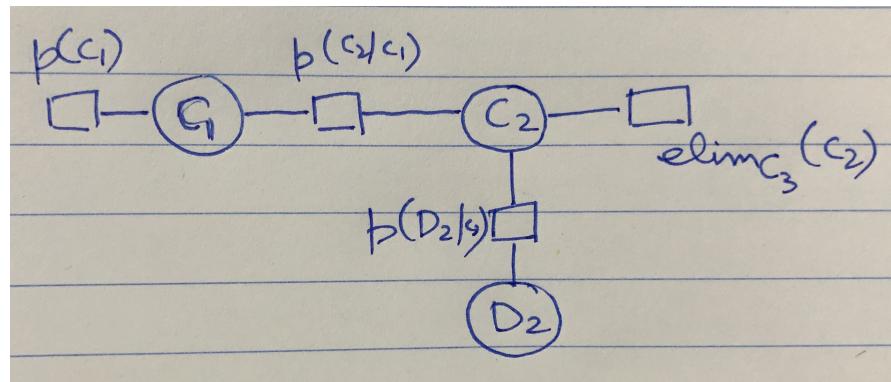
**Step3:** Conditioning on  $D_3 = 1$



Conditioning on variable  $D_3$ , and replacing it with a factor  $\text{cond}_{D_3=1}(C_3)$ , we get

$$\begin{array}{ll} \text{cond}_{D_3=1}(C_3) & C_3 \\ \eta & 0 \\ 1 - \eta & 1 \end{array}$$

#### Step4: Eliminating $C_3$



Defining function  $\text{elim}_{C_3}(C_2)$  in order to eliminate node  $C_3$  as

$$\text{elim}_{C_3}(C_2) = \sum_{C_3} \text{cond}_{D_3=1}(C_3)p(C_3|C_2)$$

The probability distribution  $p(C_3|C_2)$  is given by:

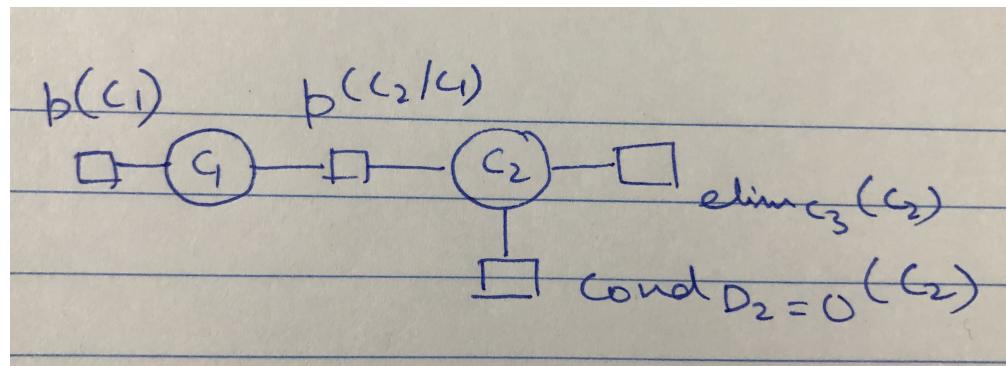
C2	C3	$p(C_3 C_2)$
0	0	$1 - \epsilon$
0	1	$\epsilon$
1	0	$\epsilon$
1	1	$1 - \epsilon$

The probability distribution  $\text{cond}_{D_3=1}(C_3)$  is defined in Step 3.

Combining both and substituting in equation 1, and doing summation over values of  $C_3$ , we will have probability distribution of  $\text{elim}_{C_3}(C_2)$  is given by:

$$\begin{array}{ll} C_2 & \text{elim}_{C_3}(C_2) \\ 0 & (1 - \epsilon)\eta + \epsilon(1 - \eta) \\ 1 & \epsilon\eta + (1 - \eta)(1 - \epsilon) \end{array}$$

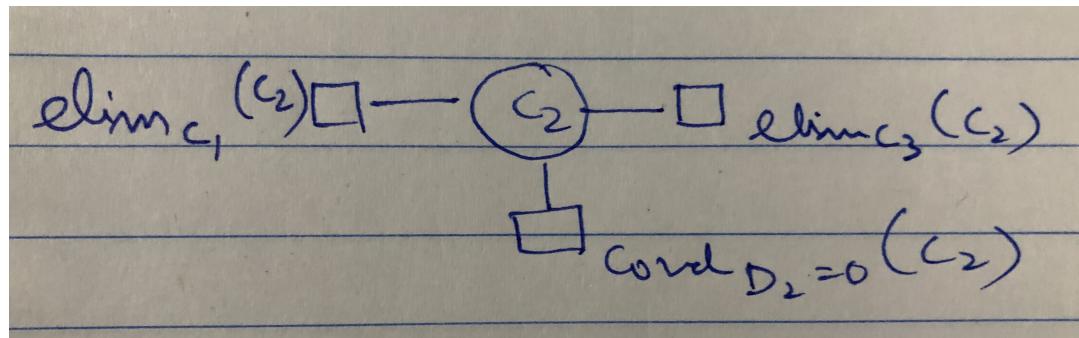
**Step5: Conditioning on  $D_2 = 0$**



Condition variable  $D_2$  on value  $D_2 = 0$ , replacing it with a factor  $\text{cond}_{D_2=0}(C_2)$ , we get

$$\begin{array}{ll} \text{cond}_{D_2=0}(C_2) & C_2 \\ 1 - \eta & 0 \\ \eta & 1 \end{array}$$

**Step6: Eliminate  $C_1$**



$$\begin{aligned} \text{elim}_{C_1}(C_2) &= \sum_{C_1} p(C_1)p(C_2/C_1) \\ &= 0.5 \sum_{C_1} p(C_2/C_1) \end{aligned}$$

This is given from the below table:

$\text{elim}_{C_1}(C_2)$	$C_2$
$0.5(1 - \epsilon + \epsilon) = 0.5$	0
$0.5(\epsilon + 1 - \epsilon) = 0.5$	1

### Step7: Combining all factors of $C_2$

Therefore, now that we know  $\text{elim}_{C_1}(C_2)$ ,  $\text{cond}_{D_2=0}(C_2)$  and  $\text{elim}_{C_3}(C_1)$ ,

$$p(C_2/D_2 = 0, D_3 = 1) = \text{elim}_{C_1}(C_2) * \text{cond}_{D_2=0}(C_2) * \text{elim}_{C_3}(C_2)$$

$p(C_2/D_2 = 0, D_3 = 1)$	$C_2$
$0.5((1 - \epsilon)\eta + \eta(1 - \epsilon))(1 - \eta)$	0
$0.5(\epsilon\eta + (1 - \eta)(1 - \epsilon))\eta$	1

Therefore,

$$\begin{aligned} P(C_2 = 1/D_2 = 0, D_3 = 1) &= \frac{0.5(\epsilon\eta + (1 - \eta)(1 - \epsilon))\eta}{0.5(\epsilon\eta + (1 - \eta)(1 - \epsilon))\eta + 0.5((1 - \epsilon)\eta + \epsilon(1 - \eta))(1 - \eta)} \\ &= \frac{(\epsilon\eta + (1 - \eta)(1 - \epsilon))\eta}{(\epsilon\eta + (1 - \eta)(1 - \epsilon))\eta + ((1 - \epsilon)\eta + \epsilon(1 - \eta))(1 - \eta)} \end{aligned}$$

(c) i.

$$\begin{aligned} P(C_2 = 1/D_2 = 0) &= 0.2 \\ P(C_2 = 1/D_2 = 0, D_3 = 1) &= 0.4157 \end{aligned}$$

- ii. Adding second sensor reading increased the probability from 0.2 to 0.4157. Since  $D_3$  is equal to 1, it means we observed the location to be 1 at location 3. This would increase the probability of  $C_3 = 1$  since the emission probability  $p(d_t/c_t)$  favours similar values with higher probability.  $C_3 = 1$  increases the probability of  $C_2 = 1$ , since the transition probability  $p(c_t/c_{t-1})$  favours same location with higher probability.
- iii. Both the probabilities would be same when the sensor reading at  $D_3$  doesn't matter. This won't matter when the transition probabilities  $p(c_t/c_{t-1})$  are equal meaning no matter what is the value of  $c_3$  out of all the possible values, we will get constant transition probability. This would happen when  $\epsilon = 1 - \epsilon$ , therefore when  $\epsilon = 0.5$ .

## Problem 5

- (a) Simplifying the equation  $P(C_{11}, C_{12}|E_1)$  using the Bayes Theorem, we have

$$P(C_{11}, C_{12}|E_1) = P(C_{11}/E_1|C_{12}/E_1) * P(C_{12}/E_1)$$

Since the location of both the cars are independent of each other ( $C_{11}$  doesn't depend in any way on  $C_{12}$  and vice versa), therefore,  $C_{11}/E_1$  and  $C_{12}/E_1$  are independent events.

$$P(C_{11}, C_{12}|E_1) = P(C_{11}/E_1) * P(C_{12}/E_1)$$

For the given question, we have to compute  $P(C_{11}, C_{12}/E_1 = e_1)$ . This means we are given a permutation  $e_1$  out of possible domain values of  $E_1$ , which means observed distance of car 1 is  $e_{11}$  and observed distance of car 2 is  $e_{12}$ .

$$\begin{aligned} P(C_{11}, C_{12}|E_1 = e_1) &= P(C_{11}/E_1 = e_1) * P(C_{12}/E_1 = e_1) \\ &\propto p(c_{11})p_N(e_{11}, \|a_1 - c_{11}\|, \sigma^2) * p(c_{12})p_N(e_{12}, \|a_1 - c_{12}\|, \sigma^2) \end{aligned}$$

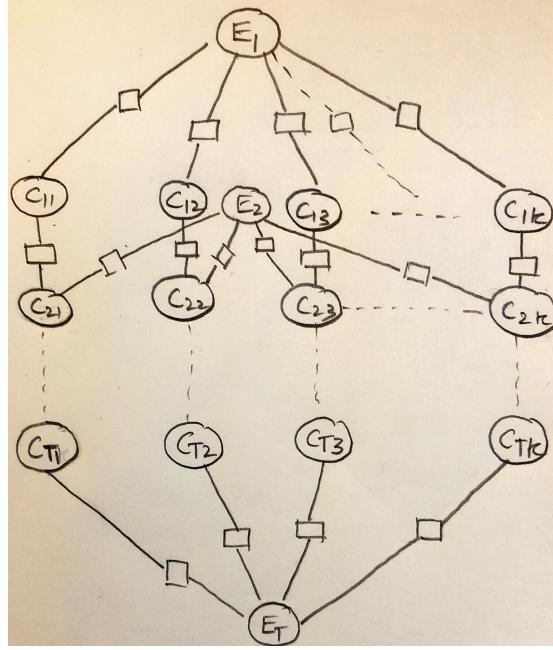
- (b) As per the solution in 5a, the joint probability would be defined as

$$\begin{aligned} P(C_{11} = c_{11}, C_{12} = c_{12} \dots C_{1K} = c_{1k}|E_1 = e_1) \\ \propto p(c_{11})p_N(e_{11}, \|a_1 - c_{11}\|, \sigma^2) * p(c_{12})p_N(e_{12}, \|a_1 - c_{12}\|, \sigma^2) \dots p(c_{1K})p_N(e_{1K}, \|a_1 - c_{1K}\|, \sigma^2) \\ \propto p(c_{1i})^K \prod_{j=1}^K p_N(e_{1j}, \|a_1 - c_{1j}\|, \sigma^2) \end{aligned}$$

Since  $e_1((e_{11}, e_{12} \dots e_{1k}))$  is one set of readings from the list of all readings in  $E_1$ , therefore, in order to find the maximum value of the above equation, the minimum number of assignments to find maximum probability would be the total number of possible permutations of different distances observed, namely  $(D_{11}, D_{12} \dots D_{1k})$  which are in  $E_1$ . For each permutation, we can compute the value of the above product and find out the permutation for which the product is the maximum. Therefore, permutations of  $(D_{11}, D_{12} \dots D_{1k})$  is:  $D_{11}$  can be assigned in  $k$  ways,  $D_{12}$  can be assigned in  $k - 1$  ways and so on. Therefore,

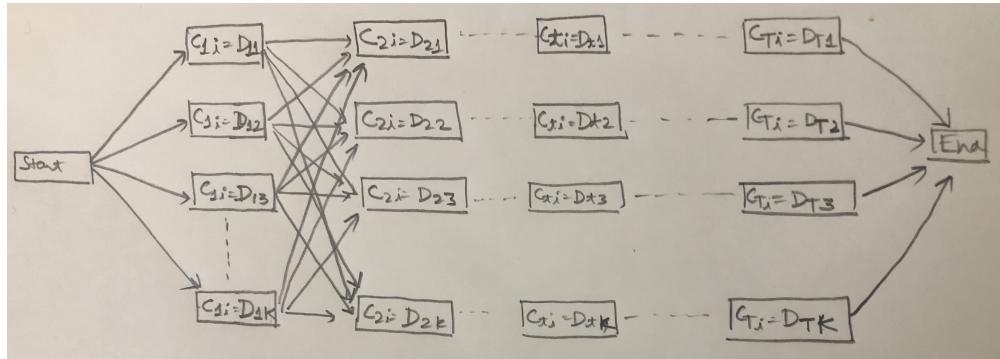
$$\begin{aligned} \text{Minimum assignments} &= k * (k - 1) * (k - 2) \dots (2)(1) \\ &= k! \end{aligned}$$

- (c) Converting to factor graph, it would look like the following for the posterior distribution over all K car locations at all T time steps conditioned on all sensor readings:



There can be two possible tree widths depending on values of T and K. If K > T

- (d) In order to compute  $p(c_{ti}/e_1, e_2, \dots, e_t)$ , this would be a smoothening query in HMM. Domain of  $C_{ti}$  would be the number of readings we get since those are exact values with equal probability of being assigned to  $C_{ti}$ , therefore has the domain  $\{D_{t1}, D_{t2}, D_{t3}, \dots, D_{tK}\}$ . Following would be the lattice representation of this scenario:



We are interested in calculating the total number of paths which will pass through a particular value of  $C_{ti}$ , let's assume  $D_{t1}$ . This can be solved through Forward-Backward algorithm.

## Forward

$$F_i(h_i) = \sum_{c_{(t-1)i}} F_{i-1}(c_{(t-1)i}) w(c_{(t-1)i}, c_{ti})$$

## Backward

$$B_i(h_i) = \sum_{c_{(t+1)i}} B_{i+1}(c_{(t+1)i})w(c_{ti}, c_{(t+1)i})$$

Finally we define  $S_i(c_{ti}) = F_i(h_i)B_i(h_i)$ , since it would be the sum of weights of paths from start to end. Finally we would normalise over the sum of the weights of all the possible paths from start to end.

The algorithm would be as follows:

Compute  $F_1, F_2, F_3, \dots, F_t$

Compute  $B_t, B_{t-1} \dots B_1$

Compute  $S_i$  for each  $i$  and normalise.