CS221 Fall 2018 Homework [car]

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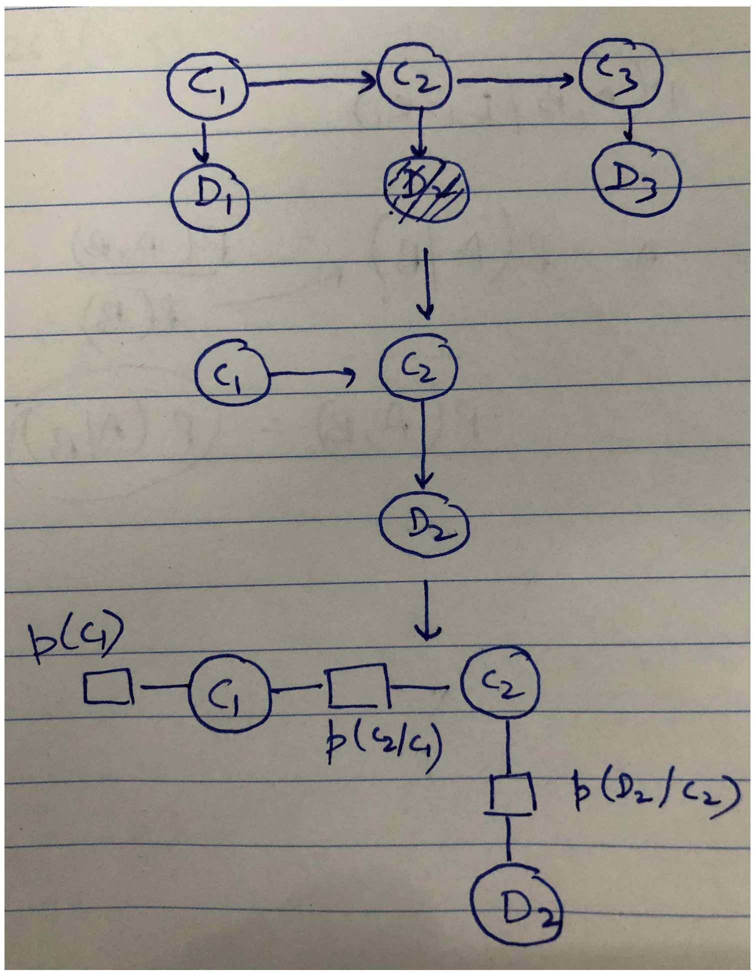
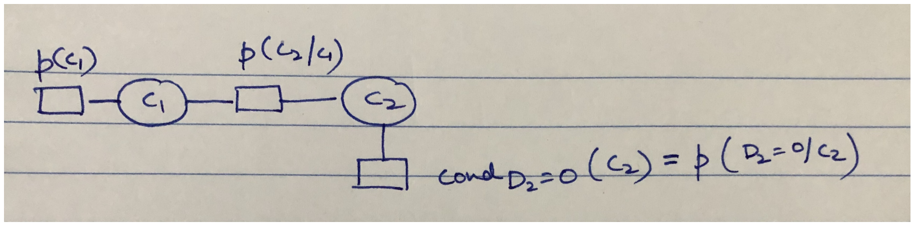
By turning in this assignment, I agree by the Stanford honor code and declare that all of this is my own work.

Problem 1

(a) **Step 1: Remove variables that are not ancestors**
Step 2: Converting to factor graph
Step 1 and step 2 are shown in diagram below:



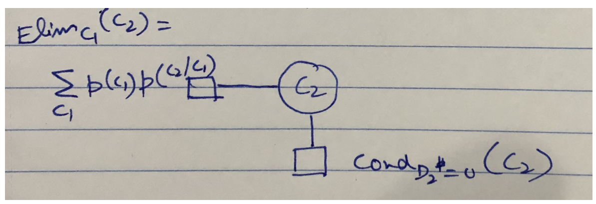
Step3: Conditioning on $D_2 = 0$



Condition variable D_2 on value $D_2 = 0$, replacing it with a factor $\text{cond}_{D_2=0}(C_2)$, we get

$\text{cond}_{D_2=0}(C_2)$	C_2
$1 - \eta$	0
η	1

Step4: Eliminate C_1



$$\begin{aligned} elim_{C_1}(C_2) &= \sum_{C_1} p(C_1)p(C_2/C_1) \\ &= 0.5 \sum_{C_1} p(C_2/C_1) \end{aligned}$$

This is given from the below table:

$\text{elim}_{C_1}(C_2)$	C_2
$0.5(1 - \epsilon + \epsilon) = 0.5$	0
$0.5(\epsilon + 1 - \epsilon) = 0.5$	1

Therefore, now that we know $\text{elim}_{C_1}(C_2)$ and $\text{cond}_{D_2=0}(C_2)$,

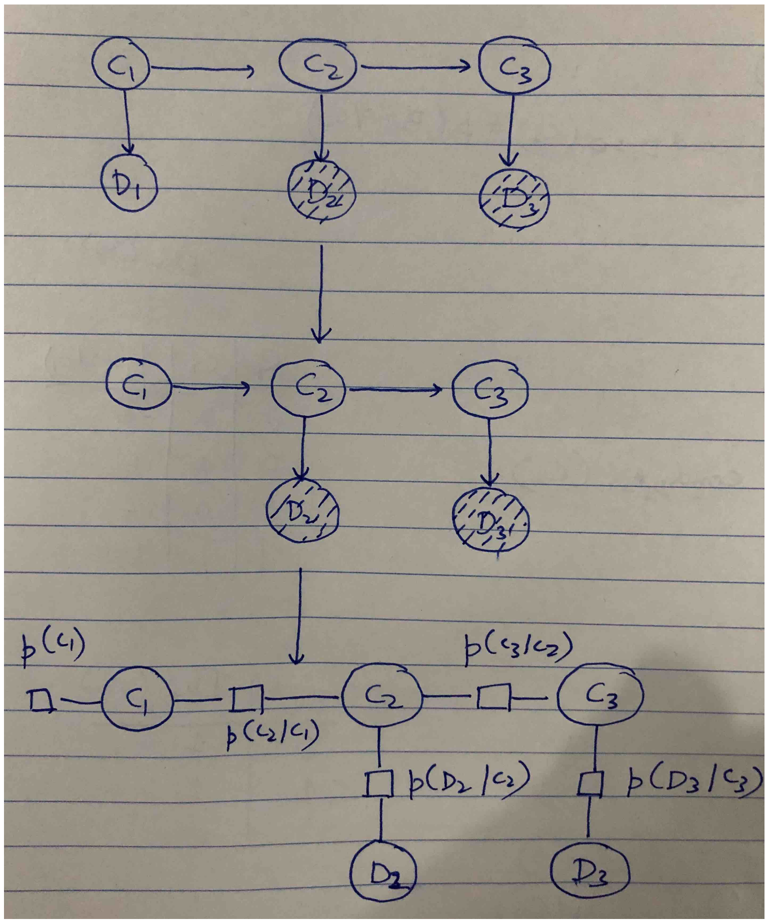
$$p(C_2/D_2 = 0) = \text{elim}_{C_1}(C_2) * \text{cond}_{D_2=0}(C_2)$$

$p(C_2/D_2 = 0)$	C_2
$0.5(1 - \eta)$	0
0.5η	1

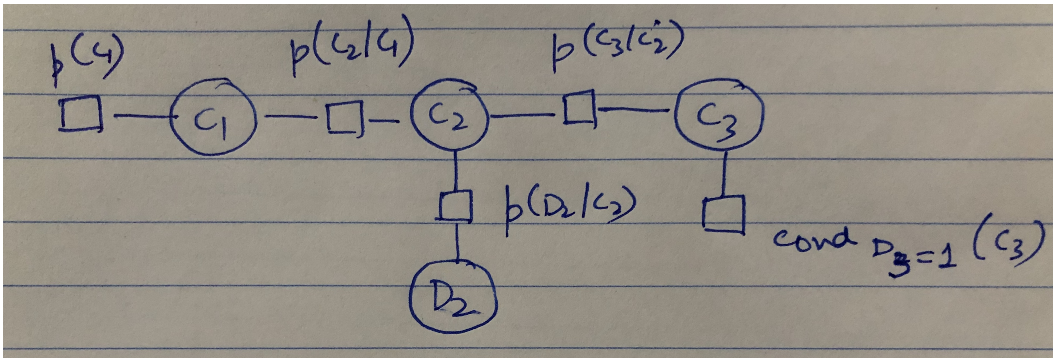
Hence, the given query,

$$p(C_2 = 1/D_2 = 0) = \frac{0.5\eta}{0.5\eta + 0.5(1-\eta)}$$
$$= \eta$$

(b) **Step1: Remove variables that are not ancestors**
Step2: Converting to factor graph
Step 1 and step 2 are illustrated below:



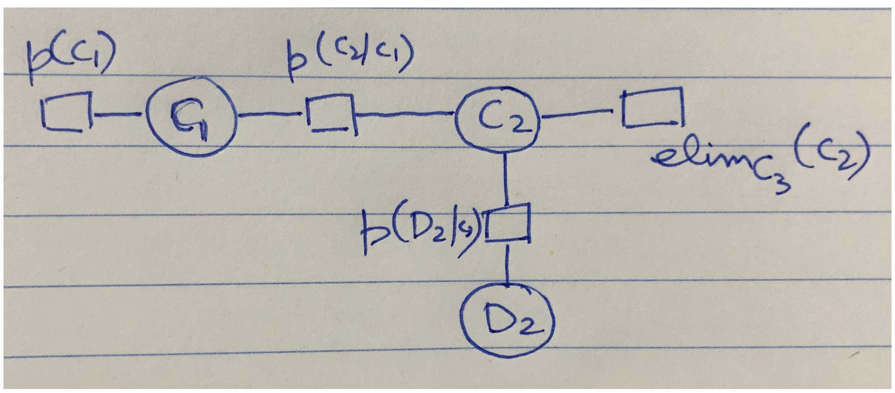
Step3: Conditioning on $D_3 = 1$



Conditioning on variable D_3, and replacing it with a factor $\text{cond}_{D_3=1}(C_3)$, we get

$\text{cond}_{D_3=1}(C_3)$	C_3
η	0
$1-\eta$	1

Step4: Eliminating C_3



Defining function $\text{elim}_{C_3}(C_2)$ in order to eliminate node C_3 as

$$\text{elim}_{C_3}(C_2) = \sum_{C_3} \text{cond}_{D_3=1}(C_3) p(C_3/C_2)$$

The probability distribution $p(C_3/C_2)$ is given by:

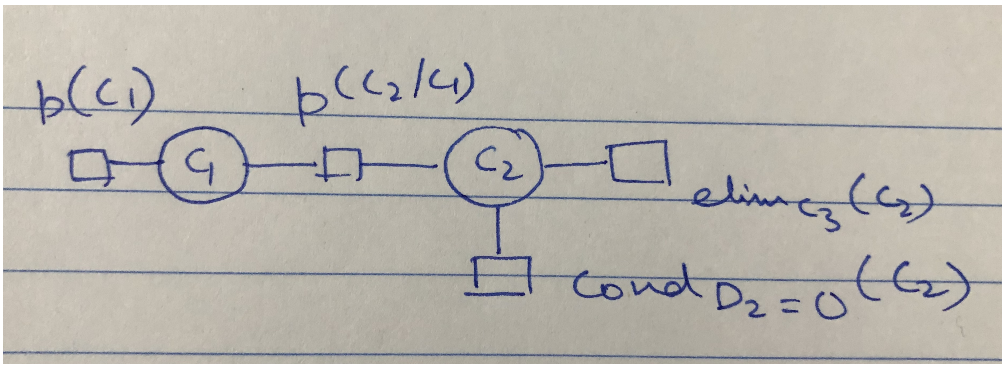
C2	C3	p(C3/C2)
0	0	$1-\epsilon$
0	1	ϵ
1	0	ϵ
1	1	$1-\epsilon$

The probability distribution $\text{cond}_{D_3=1}(C_3)$ is defined in Step 3.

Combining both and substituting in equation 1, and doing summation over values of C_3, we will have probability distribution of $\text{elim}_{C_3}(C_2)$ is given by:

C_2	$\text{elim}_{C_3}(C_2)$
0	$(1-\epsilon)\eta + \epsilon(1-\eta)$
1	$\epsilon\eta + (1-\eta)(1-\epsilon)$

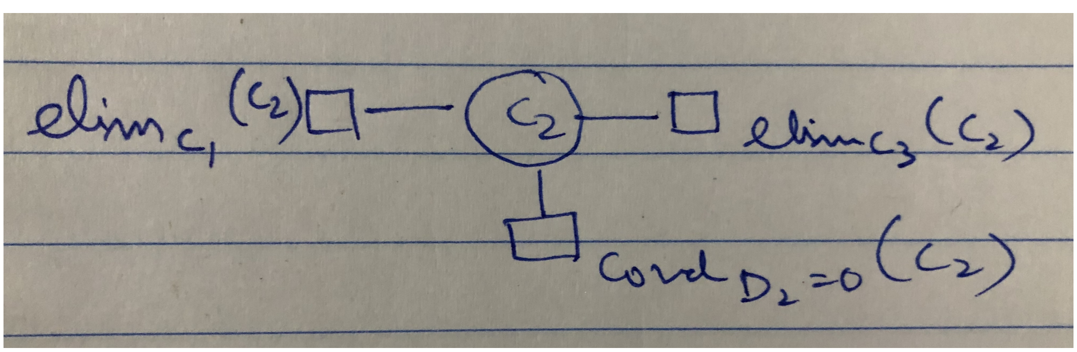
Step5: Conditioning on $D_2 = 0$



Condition variable D_2 on value $D_2 = 0$, replacing it with a factor $\text{cond}_{D_2=0}(C_2)$, we get

$\text{cond}_{D_2=0}(C_2)$	C_2
$1-\eta$	0
η	1

Step6: Eliminate C_1



$$
\begin{aligned}
elim_{C_1}(C_2) &= \sum_{C_1} p(C_1)p(C_2/C_1) \\
&= 0.5 \sum_{C_1} p(C_2/C_1)
\end{aligned}
$$

This is given from the below table:

$\text{elim}_{C_1}(C_2)$	C_2
$0.5(1-\epsilon+\epsilon) = 0.5$	0
$0.5(\epsilon+1-\epsilon) = 0.5$	1

Step7: Combining all factors of C_2

Therefore, now that we know $\text{elim}_{C_1}(C_2)$, $\text{cond}_{D_2=0}(C_2)$ and $\text{elim}_{C_3}(C_1)$,

$$
p(C_2/D_2=0, D_3=1) = \text{elim}_{C_1}(C_2) * \text{cond}_{D_2=0}(C_2) * \text{elim}_{C_3}(C_2)
$$

$p(C_2/D_2=0, D_3=1)$	C_2
$0.5((1-\epsilon)\eta + \eta(1-\epsilon))(1-\eta)$	0
$0.5(\epsilon\eta + (1-\eta)(1-\epsilon))\eta$	1

Therefore,

$$
\begin{aligned}
P(C_2=1/D_2=0, D_3=1) &= \frac{0.5(\epsilon\eta + (1-\eta)(1-\epsilon))\eta}{0.5(\epsilon\eta + (1-\eta)(1-\epsilon))\eta + 0.5((1-\epsilon)\eta + \epsilon(1-\eta))(1-\eta)} \\
&= \frac{(\epsilon\eta + (1-\eta)(1-\epsilon))\eta}{(\epsilon\eta + (1-\eta)(1-\epsilon))\eta + ((1-\epsilon)\eta + \epsilon(1-\eta))(1-\eta)}
\end{aligned}
$$

(c) i.

$$
\begin{aligned}
P(C_2=1/D_2=0) &= 0.2 \\
P(C_2=1/D_2=0, D_3=1) &= 0.4157
\end{aligned}
$$

ii. Adding second sensor reading increased the probability from 0.2 to 0.4157. Since D_3 is equal to 1, it means we observed the location to be 1 at location 3. This would increase the probability of $C_3 = 1$ since the emission probability $p(d_t/c_t)$ favours similar values with higher probability. $C_3 = 1$ increases the probability of $C_2 = 1$, since the transition probability $p(c_t/c_{t-1})$ favours same location with higher probability.

iii. Both the probabilities would be same when the sensor reading at D_3 doesn't matter. This won't matter when the transition probabilities $p(c_t/c_{t-1})$ are equal meaning no matter what is the value of c_3 out of all the possible values, we will get constant transition probability. This would happen when $\epsilon = 1-\epsilon$, therefore when $\epsilon = 0.5$.

Problem 5

(a) Simplifying the equation $P(C_{11}, C_{12}|E_1)$ using the Bayes Theorem, we have

$$P(C_{11}, C_{12}|E_1) = P(C_{11}/E_1|C_{12}/E_1) * P(C_{12}/E_1)$$

Since the location of both the cars are independent of each other (C_{11} doesn't depend in any way on C_{12} and vice versa), therefore, C_{11}/E_1 and C_{12}/E_1 are independent events.

$$P(C_{11}, C_{12}|E_1) = P(C_{11}/E_1) * P(C_{12}/E_1)$$

For the given question, we have to compute $P(C_{11}, C_{12}/E_1 = e_1)$. This means we are given a permutation e_1 out of possible domain values of E_1, which means observed distance of car 1 is e_{11} and observed distance of car 2 is e_{12}.

$$\begin{aligned} P(C_{11}, C_{12}|E_1 = e_1) &= P(C_{11}/E_1 = e_1) * P(C_{12}/E_1 = e_1) \\ &\propto p(c_{11})p_N(e_{11}, ||a_1 - c_{11}||, \sigma^2) * p(c_{12})p_N(e_{12}, ||a_1 - c_{12}||, \sigma^2) \end{aligned}$$

(b) As per the solution in 5a, the joint probability would be defined as

$$\begin{aligned} &P(C_{11} = c_{11}, C_{12} = c_{12}...C_{1K} = c_{1k}|E_1 = e_1) \\ &\propto p(c_{11})p_N(e_{11}, ||a_1 - c_{11}||, \sigma^2) * p(c_{12})p_N(e_{12}, ||a_1 - c_{12}||, \sigma^2)...p(c_{1K})p_N(e_{1K}, ||a_1 - c_{1K}||, \sigma^2) \\ &\propto p(c_{1i})^K \prod_{j=1}^{K} p_N(e_{1j}, ||a_1 - c_{1j}||, \sigma^2) \end{aligned}$$

Since e_1 is a set of different readings $(e_{11}, e_{12}...e_{1k})$ and any reading can be related to any one of the cars, therefore, in order to find the maximum value of the above equation, the minimum number of assignments to find maximum probability would be the total number of possible permutations of $(e_{11}, e_{12}...e_{1k})$. For each permutation, we can compute the value of the above product and find out the permutation for which the product is the maximum. Therefore, permutations of $(e_{11}, e_{12}...e_{1k})$ is: e_{11} can be assigned in k ways, e_{12} can be assigned in $k-1$ ways and so on. Therefore,

$$\begin{aligned} \textbf{Minimum assignments} &= k * (k-1) * (k-2)...(2)(1) \\ &= k! \end{aligned}$$