Random variable

$(\Omega, \mathcal{F}, P)$: prob space

In many situations, we may not be directly interested in the sample space Ω or the $\mathcal{F}$; rather we may be interested in some numerical aspect of Ω, i.e. assignment of numbers to elements of Ω.

e.g : Interested to know prop of defective items in a lot

Sample of size n is drawn

Sample space : 2^n elements of the form

$$(a_1, a_2, \ldots, a_n) ; \quad a_i = D \quad \text{If item is def}$$
$$= N \quad \text{If item is non-def}$$

$$\Omega \xrightarrow{X} \{0, 1, \ldots, n\}$$

$$X(a_1, \ldots a_n) = r \quad \text{If } r \text{ of } a_i\text{s are } D$$

e.g : fair coin tossed 2 times

$$\Omega = \{HH, HT, TH, TT\}$$

$$P(\{\omega\}) = \frac{1}{4} \quad \forall \omega \in \Omega$$

$X(\omega)$: # of heads in ω

$X : \Omega \to \mathbb{R}$

$$X(\omega) = \begin{cases} 0, & \text{If } \omega = TT \\ 1, & \text{If } \omega = TH \text{ or } HT \\ 2, & \text{If } \omega = HH \end{cases}$$

$\omega : X(\omega) = 0$

$$P(X=0) = P(TT) = \frac{1}{4}$$

$$P(X=1) = P(TH \text{ or } HT) = \frac{1}{2}$$

$$P(X=2) = P(HH) = \frac{1}{4}$$

$$P(X \in \{0, 1, 2\}) = 1$$

Def^n: $(\Omega, \mathcal{F}, P)$ be prob space

A real valued f^n $X : \Omega \to \mathbb{R}$ defined on the sample space Ω is called a random variable.

Remark: A more advanced textbook on prob would define r.v. as.

A real valued f^n $X : \Omega \to \mathbb{R}$ is called a r.v. if the inverse images under X of all Borel sets in $\mathbb{R}$ are events, i.e. if

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{B}. \quad - (*)$$

Further, to check whether a real valued f^n on $(\Omega, \mathcal{F})$ is a r.v., it is not necessary to check $(*)$ $\forall$ Borel sets $B \in \mathcal{B}$.

It suffices to verify $(*)$ for any class of subsets of $\mathbb{R}$ that generates $\mathcal{B}$; e.g. we can take the class of subsets as semiclosed intervals $(-\infty, x]$, $x \in \mathbb{R}$ or $(-\infty, x)$, $x \in \mathbb{R}$

In such a case, we would say

X is a r.v. iff $\forall x \in \mathbb{R}$

$$X^{-1}(-\infty, x] = \{\omega : X(\omega) \le x\} \in \mathcal{F}.$$

Ex: $\Omega = \{HH, TH, HT, TT\}$; $\mathcal{F}$: power set of Ω

$X(\omega)$: # of Hs in $\{\omega\}$ $\qquad X(\omega) = \begin{cases} 0, & TT \\ 1, & TH, HT \\ 2, & HH \end{cases}$

To show that X is r.v., we look at

$$X^{-1}(-\infty, x] = \{\omega : X(\omega) \le x\} = \begin{cases} \phi, & x < 0 \\ \{TT\}, & 0 \le x < 1 \\ \{TT, HT, TH\} & 1 \le x < 2 \\ \Omega & x \ge 2 \end{cases}$$

$\Rightarrow \downarrow \in \mathcal{F} \quad \forall x \in \mathbb{R}$

$\Rightarrow X$ is a r.v.

Induced probability space

$(\Omega, \mathcal{F}, P)$: prob space

$X : \Omega \to \mathbb{R}$ a r.v.

$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{B}$

Define a set f^n $P_X : \mathcal{B} \to [0,1]$

$$P_X(B) = P(\omega \in \Omega : X(\omega) \in B) = P(X^{-1}(B))$$

$$(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}, P_X)$$

This is a prob space with $P_X(.)$ as a probability measure, referred to as the induced prob measure

$(\mathbb{R}, \mathcal{B}, P_X)$ is the induced prob space, induced by X.

Distribution function of a random variable

__Def^n:__ Let X be a r.v. defined on a prob space $(\Omega, \mathcal{F}, P)$ and let $(\mathbb{R}, \mathcal{B}, P_X)$ be the prob space induced by X. Define $F_X : \mathbb{R} \to \mathbb{R}$ as

$$F_X(x) = P(\omega : X(\omega) \le x) = P_X(-\infty, x]$$

$F_X(.)$ is called the cumulative dist^n f^n or just dist^n f^n of r.v. X

__Remark:__ As ~~intervals~~ class of intervals of the type $(-\infty, x]$ generates $\mathcal{B}$, c.d.f $F_X(.)$ determine the $P_X(.)$ uniquely.

Thus to study the random behavior of r.v. X it suffices to study it's c.d.f F.

Examples

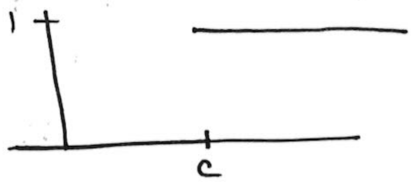
(1) $(\Omega, \mathcal{F}, P)$

$$X(\omega) = c \quad \forall\ \omega \in \Omega$$

$$P(X = c) = P(\omega : X(\omega) = c) = P(\Omega) = 1$$

$$F(x) = P(X \le x) = P(\omega : X(\omega) \le x)$$

$$= \begin{cases} 0, & x < c \\ 1, & x \ge c \end{cases}$$



Note that

$F(-\infty) = 0$; $F(\infty) = 1$
F is non-decreasing
F is right continuous
$\Big\}$ – (*)

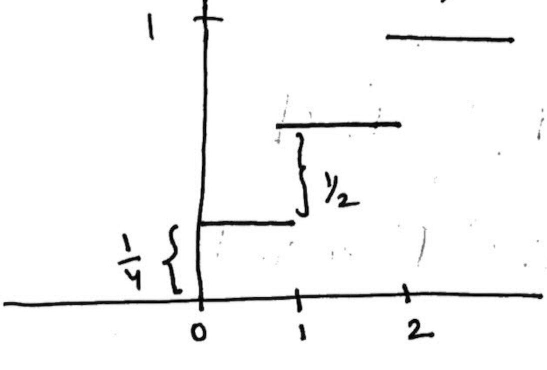
F(.) have 1 pt of jump discontinuity

(2) $\Omega = \{HH, HT, TH, TT\}$ example

$X(\omega)$: # of heads in ω

$$P(X = 0) = \frac{1}{4} ; \quad P(X = 1) = \frac{1}{2} ; \quad P(X = 2) = \frac{1}{4}$$

$$F(x) = P(X \le x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \le x < 1 \\ \frac{1}{4} + \frac{1}{2}, & 1 \le x < 2 \\ \frac{1}{4} + \frac{1}{2} + \frac{1}{4}, & x \ge 2 \end{cases} = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \le x < 1 \\ \frac{3}{4}, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$



Once again (*) is satisfied by the above F(.)

F(.) has 3 pts of jump discontinuity

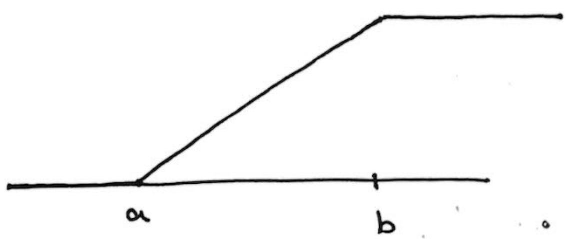
Example 3:

$\Omega = [a, b]$

For every $I \subset \Omega$; $P(I) = \dfrac{\text{length of } I}{b-a}$

Define. $X(\omega) = \omega$; $\omega \in \Omega$

$$F_X(x) = P(X \le x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x < b \\ \dfrac{b-a}{b-a} = 1, & x \ge b \end{cases}$$



$F(.)$ satisfies (*)

$F(.)$ is continuous everywhere

Result: Let $F(.)$ be the d.f. of a r.v. X defined on a prob space $(\Omega, \mathcal{F}, P)$. Then

(i) F is non-decreasing

(ii) F is right continuous

(iii) $F(-\infty) = \lim_{n \uparrow \infty} F(-n) = 0$ and

$F(\infty) = \lim_{n \uparrow \infty} F(n) = 1$

Pf:

(a) Let $-\infty < x < y < \infty$, then

$(-\infty, x] \subseteq (-\infty, y]$

$\Rightarrow P_X(-\infty, x] \le P_X(-\infty, y]$

i.e $P(\omega : X(\omega) \le x) \le P(\omega : X(\omega) \le y)$

$\Rightarrow F(x) \le F(y)$

$\Rightarrow F(.)$ is non-decreasing

(b) $$F(x+) = \lim_{h \downarrow 0} F(x+h)$$

$$= \lim_{n \to \infty} F\left(x + \frac{1}{n}\right)$$

$$= \lim_{n \to \infty} P_X\left(\left(-\infty, x + \frac{1}{n}\right]\right)$$

Realize that $A_n = \left(-\infty, x + \frac{1}{n}\right]$, $n = 1, 2, \ldots$ is $\ni A_n \downarrow$

and $$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \left(-\infty, x + \frac{1}{n}\right] = (-\infty, x]$$

$$\Rightarrow \lim_{n \to \infty} P_X\left(\left(-\infty, x + \frac{1}{n}\right]\right) = P_X\left(\lim_{n \to \infty} A_n\right)$$

$$= P_X\left(\bigcap_{1}^{\infty} A_n\right)$$

$$= P_X\left(\bigcap_{1}^{\infty} \left(-\infty, x + \frac{1}{n}\right]\right)$$

$$= P_X\left((-\infty, x]\right) = F(x)$$

$\Rightarrow F(x+) = F(x)$; i.e. $F(.)$ is right continuous

(b) $$F(x+) = \lim_{h \downarrow 0} F(x+h)$$
$$= \lim_{n \to \infty} F\left(x + \frac{1}{n}\right)$$
$$= \lim_{n \to \infty} P_X\left(\left(-\infty, x + \frac{1}{n}\right]\right)$$

Realize that $A_n = \left(-\infty, x + \frac{1}{n}\right]$, $n = 1, 2, \ldots$ is $\ni A_n \downarrow$

and $\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \left(-\infty, x + \frac{1}{n}\right] = (-\infty, x]$

$$\Rightarrow \lim_{n \to \infty} P_X\left(\left(-\infty, x + \frac{1}{n}\right]\right) = P_X\left(\lim_{n \to \infty} A_n\right)$$
$$= P_X\left(\bigcap_{1}^{\infty} A_n\right)$$
$$= P_X\left(\bigcap_{1}^{\infty} \left(-\infty, x + \frac{1}{n}\right]\right)$$
$$= P_X\left((-\infty, x]\right) = F(x)$$

$\Rightarrow F(x+) = F(x)$; i.e. $F(.)$ is right continuous

(c) $$F(-\infty) = \lim_{n \to \infty} F(-n)$$
$$= \lim_{n \to \infty} P_X\left((-\infty, -n]\right)$$
$$= P_X\left(\bigcap_{1}^{\infty} (-\infty, -n]\right) \quad \left((-\infty, -n] \downarrow\right)$$
$$= P_X(\phi) = 0 \qquad \bigcap_{1}^{\infty} (-\infty, -n] = \phi$$

$$F(+\infty) = \lim_{n \to \infty} F(n) \qquad \left((-\infty, n] \uparrow\right).$$
$$= \lim_{n \to \infty} P_X\left((-\infty, n]\right)$$
$$= P_X\left(\lim_{n \to \infty} (-\infty, n]\right) = P_X\left(\bigcup_{1}^{\infty} (-\infty, n]\right)$$

$= P_X(\mathbb{R})$ $\quad\quad\quad\quad$ $\left(\bigcup_{n=1}^{\infty}(-\infty, n] = \mathbb{R}\right)$

$= 1$

Remark: Converse of the prev result is true, i.e. If $F(.)$ be a function $F : \mathbb{R} \to [0, 1] \ni$

(i) $F(.)$ is non-decreasing

(ii) $F(.)$ is right continuous

(iii) $F(-\infty) = 0$ and $F(\infty) = 1$

then $F(.)$ is d.f. of some appropriate random variable

Remark: For a d.f. $F(.)$, both $F(x+)$ and $F(x-)$ exist $\forall x \in \mathbb{R}$ as $F(.)$ is monotone, bdd below and bdd above.

Remark: A d.f. is continuous at $a \in \mathbb{R}$ iff $F(a) = F(a-)$

Remark: For any $a \in \mathbb{R}$

$$P(X = a) = P(X \leq a) - P(X < a) = F(a) - F(a-)$$

If d.f. is continuous at a, then $P(X = a) = 0$

Remark: For $-\infty < a < b < \infty$

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F(b) - F(a)$$

$$P(a < X < b) = P(X < b) - P(X \leq a) = F(b-) - F(a)$$

$$P(a \leq X < b) = P(X < b) - P(X < a) = F(b-) - F(a-)$$

$$P(a \leq X \leq b) = P(X \leq b) - P(X < a) = F(b) - F(a-)$$

If $F(.)$ is continuous at a and b, then all the above are equal and equal to $F(b) - F(a)$

Remark: $F(.)$ can only have points of jump discontinuities and can have atmost countable number of such jump discontinuities

Discrete random variable

$(\Omega, \mathcal{F}, P)$: prob space

$X : \Omega \to \mathbb{R}$ be a r.v.

$(\mathbb{R}, \mathcal{B}, P_X)$: induced prob space (induced by X)

$F(.)$: d.f. of X

Defn: Random variable X is said to be a discrete r.v.

If $\exists$ a countable set $D \subset \mathbb{R}$ $\ni$

$$P(X = x) = F(x) - F(x-) > 0 \quad \forall\, x \in D$$

and $P(X \in D) = 1$

" D is called the support of the r.v. X
D is the set of all discontinuity pts of $F(.)$

Defn: Let $D = \{x_1, x_2, \ldots\}$ ← countable (finite or infinite)

$$P(X = x_i) = p_i \text{, say}, \quad p_i > 0 \ \forall\, i$$

$$P(X \in D) = \sum_i p_i = 1$$

The collection $\{p_1, p_2, \ldots\}$ is called the probability mass f^n of r.v. X.

i.e. $f_X(x) = P(X = x) \quad x \in S$ is the p.m.f. of X

$= F(x) - F(x-)$; $f(x) > 0 \ \forall\, x \in D$

$\sum_{x \in D} f(x) = 1$

Remark: (i) d.f. of a discrete r.v. increases only by jumps

(ii) number of jump discontinuities are atmost countable

(iii) d.f. determines the p.m.f uniquely and vice-versa

Example :

(1) X : r.v.

$F(.)$: d.f. of X

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \le x < 1 \\ \frac{1}{3}, & 1 \le x < 2 \\ \frac{3}{4}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

pts of jump discontinuities $\{0, 1, 2, 3\} = D$
(finite collection)

$P(X \in D) = 1$;

X is a discrete r.v. with support D

p.m.f.

x	$P(X = x)$
0	$F(0) - F(0-) = \frac{1}{4}$
1	$F(1) - F(1-) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$
2	$F(2) - F(2-) = \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$
3	$F(3) - F(3-) = \frac{1}{4}$

i.e. p.m.f is

$$f(x) = \begin{cases} \frac{1}{4}, & \text{If } x = 0, 3 \\ \frac{1}{12}, & \text{If } x = 1 \\ \frac{5}{12}, & \text{If } x = 2 \\ 0, & \text{o/w.} \end{cases}$$

Example :

(2) Random exp : tossing coin until head appears

$\Omega = \{H, TH, TTH, \dots\}$

X : r.v. which counts number of tosses reqd to get 1^{st} H

i.e. $X(\omega) =$ no. of T in ω + 1

possible values of X : 1, 2, 3, ...

$$P(X = i) = \frac{1}{2^i} \; ; \; i = 1, 2, \dots$$

d.f. of X

$$F(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ \frac{3}{4}, & 2 \le x < 3 \\ \vdots & \end{cases}$$

magnitude of jump at $i = \frac{1}{2^i}$ for $i = 1, 2, \dots$

$D = \{1, 2, 3, \dots\}$

countably infinite

Support

p.m.f. $\quad f(x) = \frac{1}{2^x} \; ; \; x = 1, 2, \dots$

Continuous random variable

Def^n: A random variable X is said to be a continuous r.v. if $\exists$ a non-negative, integrable function $f: \mathbb{R} \to [0, \infty)$ such that for any $x \in \mathbb{R}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$f(.)$ is called the probability density function (p.d.f.) of X.

Remark: Support of a continuous r.v is the set

$$S = \{x \in \mathbb{R} : F(x+h) - F(x-h) > 0, \ \forall h > 0\}$$

Remark: For a cont r.v. ($F(.)$ is continuous everywhere)

$$P(X=x) = F(x) - F(x-) = 0 \quad \forall x \in \mathbb{R}$$

In general, suppose $A \subset \mathbb{R}$ is any countable subset, then

$$P(X \in A) = \sum_{x \in A} P(X=x) = 0$$

Remark: p.d.f. $f(x)$, then

(i) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

and (ii) $\int_{-\infty}^{\infty} f(t)\,dt = 1$

Converse is also true

Remark: F d.f. of r.v. X, if F is differentiable, then

$$f(x) = \frac{d}{dx} F(x)$$

Remark: For a Continuous r.v.

$$P(X < x) = P(X \le x) = F(x) \quad \forall\, x \in \mathbb{R}$$

$$P(X \ge x) = 1 - P(X < x) = 1 - F(x) \quad \forall\, x \in \mathbb{R}$$

$\forall \quad -\infty < a < b < \infty$

$$P(a < X < b) = P(a \le X < b) = P(a < X \le b)$$
$$= P(a \le X \le b)$$
$$= F(b) - F(a)$$

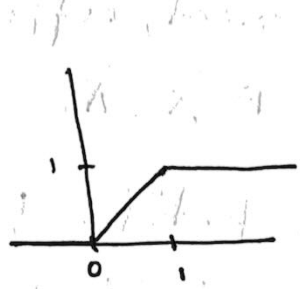
$$\& \quad F(b) - F(a) = \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$
$$= \int_{a}^{b} f(t)\,dt \,.$$

Remark: p.d.f. determines the d.f. uniquely. Converse is not true

Examples

(1)

$$F(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$



F(.) is cont everywhere

p.d.f. of X is $\quad f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & o/w \end{cases}$

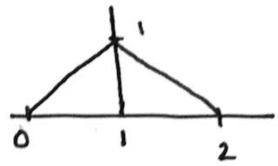
$$\frac{d}{dx} F(x)$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

(2)

p.d.f. of a r.v. X is

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & o/w \end{cases}$$



$f(x) \geq 0 \ \forall x$

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_0^1 x\,dx + \int_1^2 (2-x)\,dx$$

$$= \frac{x^2}{2}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 = \frac{1}{2} + \left(2 - \frac{3}{2}\right) = 1$$

$\Rightarrow f(\cdot)$ is a p.d.f.

dist^n f^n: $F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \begin{cases} 0, & x < 0 \\ x^2/2, & 0 \le x \le 1 \\ \frac{1}{2} + \left(\left(2x - \frac{x^2}{2}\right) - \frac{3}{2}\right), & 1 \le x \le 2 \\ 1, & x \geq 2 \end{cases}$$

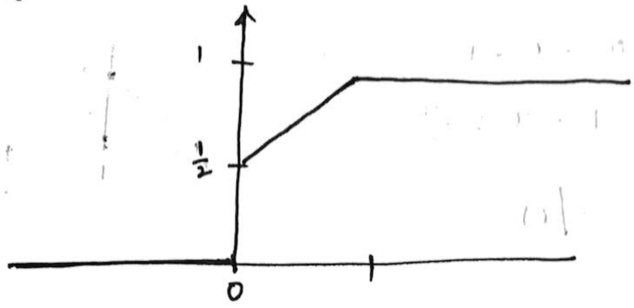
$$P\left(\frac{1}{4} \le X \le \frac{3}{2}\right) = F(3/2) - F(1/4)$$

Remark: There are random variables that are neither discrete nor continuous - random variables of mixed type

Example:

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x+1}{2}, & 0 \le x < 1 \\ 1, & x \geq 1 \end{cases}$$

It's easy to check that $F(.)$ is a d.f.



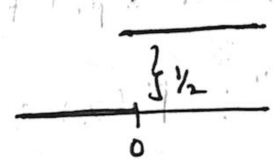
$F(.)$ has jump discontinuity at 0 - jump size $\frac{1}{2}$

$F(.)$ is continuous everywhere, except at 0

$$P(X=0) = F(0) - F(0-) = \frac{1}{2}$$

Discrete part:
(f^n which increases by jump only)

$$F_d = \begin{cases} 0, & x<0 \\ \frac{1}{2}, & x \geq 0 \end{cases}$$



Continuous part:
(increasing continuously part)

$$F_c = \begin{cases} 0, & x<0 \\ x/2, & 0 \leq x < 1 \\ \frac{1}{2}, & x \geq 1 \end{cases}$$

$$F_c = F - F_d$$

Note: F_d & F_c are not d.f.s.

Realise that

$$F(x) = \frac{1}{2} F_1(x) + \frac{1}{2} F_2(x)$$

$$F_1(x) = \begin{cases} 0, & x<0 \\ 1, & x \geq 0 \end{cases} \qquad F_2(x) = \begin{cases} 0, & x<0 \\ x, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

$F_1(x)$ & $F_2(x)$ are proper d.f.s.

d.f. of a discrete r.v.

d.f. of a continuous r.v.

Remark:

$\alpha F_1(x) + (1-\alpha) F_2(x)$ will be d.f. of mixed type

$\forall \alpha \ni 0 < \alpha < 1$

if $\alpha = 0$; X is continuous r.v.

If $\alpha = 1$. X is discrete r.v.

Remark: Any dist^n f^n $F(.)$ can be expressed as.

$$F(x) = \alpha F_d(x) + (1-\alpha) F_c(x)$$

d.f. of discrete r.v. d.f. of cont r.v.

Example: Let X be r.v. with d.f.

$$F(x) = \begin{cases} 0, & x < 0 \\ x/4, & 0 \le x < 1 \\ x/3, & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases}$$

jump discontinuities at $x = 1, 2$

$D = \{1, 2\}$; $D \neq \phi$

$\Rightarrow X$ is not cont r.v.

$$P(X \in D) = \sum_{x \in D} P(X = x)$$

$$= \sum_{x \in D} \left(F(x) - F(x-) \right)$$

$$= \sum_{x = 1,2} \left(F(x) - F(x-) \right)$$

$$= \left(\frac{1}{3} - \frac{1}{4} \right) + \left(1 - \frac{2}{3} \right) = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$$

$P(X \in D) = \frac{5}{12} < 1$

$\Rightarrow X$ is not discrete r.v.

$\Rightarrow X$ is neither discrete or cont

Discrete part of $F(.)$:

$$F_1(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{12}, & 1 \le x < 2 \\ \frac{5}{12} & x \geq 2 \end{cases}$$

Take $\alpha = \frac{5}{12}$; $\alpha F_d(x) = F_1(x)$

$$F_d(x) = \begin{cases} 0, & x < 1 \\ \frac{1}{5}, & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases}$$

$F_d(x)$ is d.f. of a discrete r.v. $\rightarrow$ p.m.f. $\left. \begin{array}{l} P(X_d = 1) = \frac{1}{5} \\ P(X_d = 2) = \frac{4}{5} \end{array} \right\}$.

$F_2(x)$: Continuous part of $F(.)$

$F_2(x) = F(x) - F_1(x)$

$$F_2(x) = \begin{cases} 0, & x < 0 \\ x/4, & 0 \le x < 1 \\ x/3 - \frac{1}{12}, & 1 \le x < 2 \\ \frac{7}{12} & x \geq 2 \end{cases}$$

$F_2(x) = (1-\alpha) F_c(x)$; $1 - \alpha = \frac{7}{12}$

$$F_c(x) = \begin{cases} 0, & x < 0 \\ 3x/7, & 0 \le x < 1 \\ \frac{12}{7}\left(\frac{x}{3} - \frac{1}{12}\right) = \frac{4}{7}x - \frac{1}{7}, & 1 \le x < 2 \\ 1, & x \geq 2 \end{cases}$$

$F_c(x)$ is continuous everywhere

$F_c(x)$ is the d.f. of a cont r.v.

p.d.f. of X_c

$$f_{X_c}(x) = \begin{cases} 3/7, & 0 \le x < 1 \\ 4/7, & 1 \le x < 2 \\ 0, & o/w \end{cases}$$

$$f_{X_c}(x) \geq 0 \quad \forall x$$

$$\int_{-\infty}^{\infty} f_{X_c}(x)\, dx = \int_0^1 \frac{3}{7}\, dx + \int_1^2 \frac{4}{7}\, dx = 1$$

$$F(x) = \alpha F_d(x) + (1-\alpha) F_c(x)$$

Mathematical Expectation

$(\Omega, \mathcal{F}, P) \xrightarrow{X} (\mathbb{R}, \mathcal{B}, P_X)$

$g : \mathbb{R} \xrightarrow{g} \mathbb{R}$

Expected value of $g(X)$: mathematical expectation of $g(X)$

$E(g(X))$ exists if $E|g(X)| < \infty$

Suppose, X is a discrete r.v. with p.m.f.

$X : x_1, x_2, - - -$

$P(X=x) : p_1, p_2, - - - .$

$E\,g(X)$ is said to exist and equals $\sum_{i=1}^{\infty} g(x_i)\, p_i$ provided $\sum_i |g(x_i)|\, p_i < \infty$

If X is continuous with p.d.f. $f_X(x)$, then $E(g(X))$ exists and equals $\int_{-\infty}^{\infty} g(x)\, f_X(x)\,dx$ provided

$\int_{-\infty}^{\infty} |g(x)|\, f_X(x)\,dx < \infty$

Special cases

(i) $g(X) = X$

$E\,g(X) = E\,X = \mu_1'$: mean of the distn of X

(ii) $g(X) = X^n$: n is a positive integer

$E\,g(X) = E\,X^n = \mu_n'$

n^{th} moment about origin of r.v. X

(iii) $g(X) = (X-a)^n$

$E\,g(X) = E(X-a)^n$: nth moment of X about the pt a

If $a = E(X)$, then

$E(X - E(X))^n = \mu_n$: nth order central moment of X

$n = 2$; $\mu_2 = E(X - E(X))^2 \rightarrow$ variance of X

$= \sigma^2$

$\mu_2^{1/2} = \sigma$: standard deviation of X