Assignment
10.5.4-2

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Question:

Q2) The sum of the third and the seventh terms of AP is 6 and their product is 8. Find the sum of first sixteen terms of the AP

Solution:

Table of Parameters

Input Variables	Input Condition
x(2)+x(6)	6
x(2).x(6)	8

Then general term x(n) of arithmetic progression is given by:

$$x(n) = x(0) + n.d \tag{1}$$

Then from (1):

$$x(2) = x(0) + 2.d \tag{2}$$

$$x(6) = x(0) + 6.d \tag{3}$$

Then from table of parameters,

$$x(2) + x(6) = 6 \tag{4}$$

$$x(2).x(6) = 8 \tag{5}$$

Then (4) in (5)

$$x(2) + x(6) = 6$$

$$x(2) = 6 - x(6) \tag{6}$$

From (6)

$$x(2).x(6) = 8$$

$$x(6).(6 - x(6)) = 8$$

$$6.x(6) - (x(6))^2 = 8$$

$$(x(6))^2 - 6.x(6) + 8 = 0$$

$$(x(6) - 2).(x(6) - 4) = 0$$

$$x(6) = 2 \ or \ 4 \tag{7}$$

Then from (4) and (7)

$$x(2) = 4 \ or \ 2 \tag{8}$$

from (2),(3) and (7),(8)
for x(2) = 2 and x(6) = 4

$$2 = x(0) + 2.d$$

$$4 = x(0) + 6.d$$

$$d = \frac{1}{2} \ and \ x(0) = 1$$

for x(2) = 4 and x(6) = 2

$$4 = x(0) + 2.d$$

$$2 = x(0) + 6.d$$

$$d = -\frac{1}{2} \ and \ x(0) = 5$$

We know that the sum of first n terms of arithmetic progression is given by:

$$S(n) = \frac{n}{2}(2.x(0) + (n-1).d) \tag{9}$$

Then from (9) let sum of first 16 terms of arithmetic progression be S_{16}:

$$S(16) = \frac{16}{2}(2.x(0) + 15d) \tag{10}$$

Hence from (10), for $x(0)$=1,d=$\frac{1}{2}$

$$S(16) = 76$$

or from (10), for $x(0)$=5,d=-$\frac{1}{2}$

$$S(16) = 20$$

The general term of AP (a_n) and sum of first n terms of AP (a_n) are given by:

$$x(n) = x(0) + n.d \quad and \quad S(n) = \frac{n}{2}(2.x(0) + (n-1).d)$$

$$x(n) = \frac{n+2}{2} \quad and \quad S(n) = \frac{n.(n+3)}{4} \quad for\,(x(0) = 1, d = \frac{1}{2})$$

$$x(n) = \frac{10-n}{2} \quad and \quad S(n) = \frac{n.(21-n)}{4} \quad for\,(x(0) = 5, d = -\frac{1}{2})$$

We know that Z-Transform of x(n) is given by:

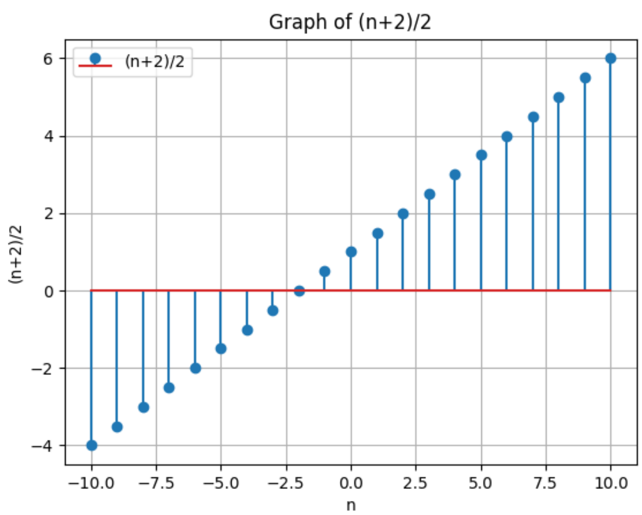
$$X(z) = \sum_{k=-\infty}^{\infty} x(k).z^{-k} \tag{11}$$

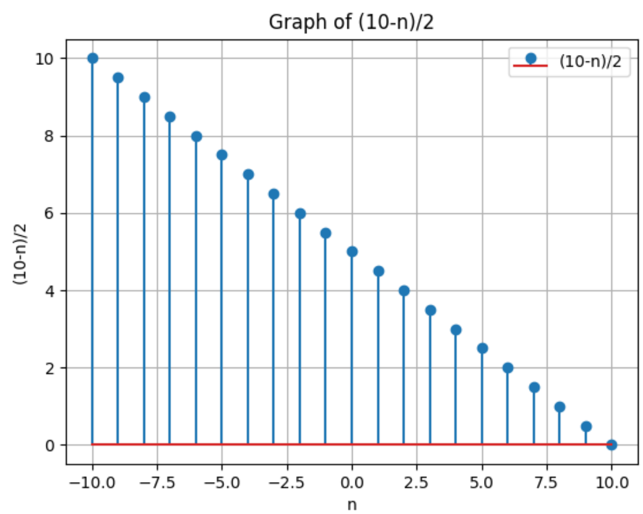
where, we assume that x(k)=0 for ($k < 0$)
Then, (11) modifiy as follows:

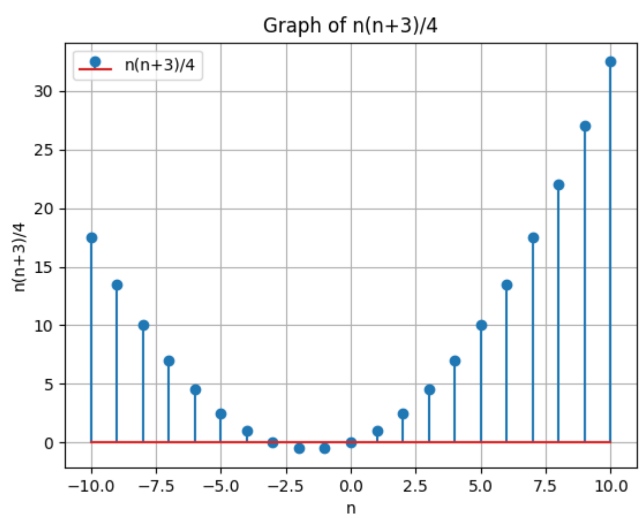
$$X(z) = \sum_{k=0}^{\infty} x(k).z^{-k} \tag{12}$$

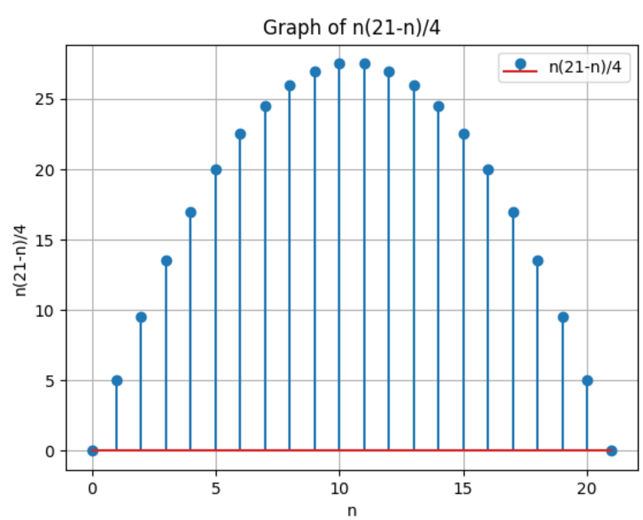
$$X(z) = \sum_{k=0}^{\infty} \left(\frac{k+2}{2}\right).z^{-k}$$

$$X(z) = \frac{1}{2}\left(\sum_{k=0}^{\infty} k.z^{-k}\right) + \sum_{k=0}^{\infty} z^{-k} \tag{13}$$









Then consider sum, say S:

$$S = \left(\sum_{k=0}^{\infty} k.z^{-k} \right)$$

$$S = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} ....\infty \tag{14}$$

for $Z \neq 0$

$$\frac{S}{z} = 0 + \frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} ....\infty \tag{15}$$

Then from (15)-(11)

$$S\left(1 - z^{-1}\right) = \sum_{k=1}^{\infty} z^{-k}$$

$$S = \frac{\sum_{k=0}^{\infty} z^{-k} - 1}{1 - z^{-1}} = \frac{Z}{(Z-1)^2} \; for \; \left(|Z^{-1}| < 1\right) \tag{16}$$

So,(16) in (13) Then,

$$X(Z) = \frac{1}{2}\left(\frac{\left(\sum_{k=0}^{\infty} z^{-k}\right) - 1}{1 - z^{-1}} + \sum_{k=0}^{\infty} z^{-k} \right)$$

$$X(Z) = \left(\lim_{n\to\infty} \sum_{k=0}^{n} z^{-k} \left(\frac{1}{2.\left(1 - z^{-1}\right)} + 1 \right) \right) - \frac{1}{2\left(1 - Z^{-1}\right)}$$

$$X(Z) = \left(\frac{1}{2.\left(1 - z^{-1}\right)} + 1 \right) . \left(\lim_{n\to\infty} \sum_{k=0}^{n} z^{-k} \right) - \frac{1}{2\left(1 - Z^{-1}\right)}$$

$$X(Z) = \left(\frac{1}{2.\left(1 - z^{-1}\right)} + 1 \right) . \lim_{n\to\infty} \left(\frac{1 - \left(z^{-1}\right)^{n+1}}{1 - z^{-1}} \right) - \frac{1}{2\left(1 - Z^{-1}\right)}$$

So,

$$X_1(Z) = \frac{2 - z^{-1}}{2.\left(1 - z^{-1}\right)^2} \quad for \; |z^{-1}| < 1$$

or,

$$X_1(Z) \;\; is\ not\ defined \qquad for\ |z^{-1}| > 1$$

From (12),

$$X(Z) = \sum_{k=0}^{\infty} \left(\frac{10-k}{2}\right).z^{-k}$$

$$X(Z) = 5.\left(\sum_{k=0}^{\infty} z^{-k}\right) - \frac{1}{2}\left(\sum_{k=0}^{\infty} k.z^{-k}\right)$$

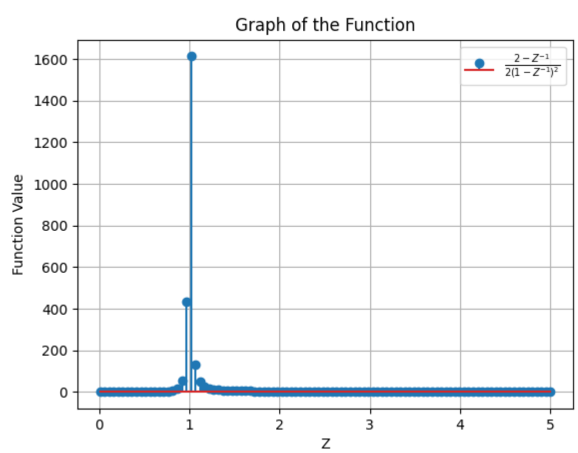
Then from (16),

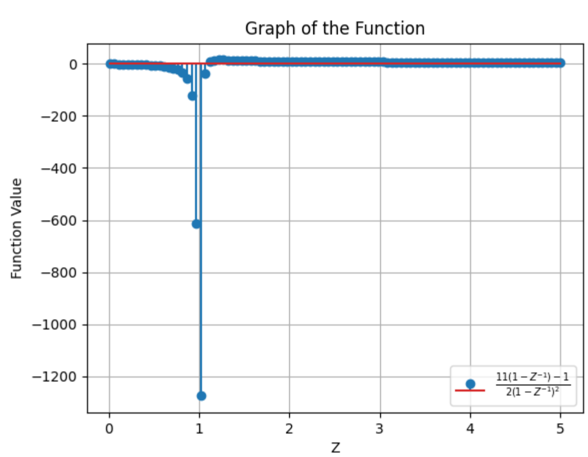
$$X(Z) = 5.\left(\sum_{k=0}^{\infty} z^{-k}\right) - \frac{1}{2}\left(\frac{\sum_{k=0}^{\infty} z^{-k} - 1}{1-z^{-1}}\right)$$

$$X(Z) = \left(\left(5 - \frac{1}{2.(1-z^{-1})}\right)\lim_{n\to\infty}\sum_{k=0}^{n} z^{-k}\right) + \frac{1}{2(1-Z^{-1})}$$

$$X_2(Z) = \frac{11.\left(1-z^{-1}\right)-1}{2.(1-z^{-1})^2} \qquad for \quad |z^{-1}| < 1$$

$$X_2(Z) \;\; is\ not\ defined \quad for \qquad |z^{-1}| > 1$$





Let, $\mathcal{Z}$ tranform of s(n) be S(Z):
from (11):

$$S(Z) = \sum_{k=-\infty}^{\infty} s(k)\, Z^{-k} = \sum_{k=0}^{\infty} s(k)\, Z^{-k} \quad (s(k) = 0\ for\ k < 0)$$

from (9)

$$S(Z) = \sum_{k=-\infty}^{\infty}\left(\left(x(0) - \frac{d}{2}\right)k + \frac{d}{2}\,k^2\right).Z^{-k}$$

$$S(Z) = \left(x(0) - \frac{d}{2}\right)\left(\sum_{k=0}^{\infty} k.Z^{-k}\right) + \frac{d}{2}.\left(\sum_{k=0}^{\infty} k^2.Z^{-k}\right) \tag{17}$$

Let,

$$\alpha = \sum_{k=0}^{\infty} k^2.Z^{-k}$$

$$\alpha = \frac{1}{Z} + \frac{4}{Z^2} + \frac{9}{Z^3} + \frac{16}{Z^4} + \frac{25}{Z^5}...\infty \tag{18}$$

$$\frac{\alpha}{Z} = \frac{1}{Z^2} + \frac{4}{Z^3} + \frac{9}{Z^4} + \frac{16}{Z^5} + \frac{25}{Z^6}...\infty \tag{19}$$

(17)-(18) gives,

$$\alpha\left(1-Z^{-1}\right) = \frac{1}{Z} + \frac{3}{Z^2} + \frac{5}{Z^3} + \frac{7}{Z^4} + \frac{9}{Z^5}...\infty \tag{20}$$

$$\alpha Z^{-1}\left(1-Z^{-1}\right) = \frac{1}{Z^2} + \frac{3}{Z^3} + \frac{5}{Z^4} + \frac{7}{Z^5} + \frac{9}{Z^6}...\infty \tag{21}$$

(19)-(20) gives,

$$\alpha\left(1-Z^{-1}\right)^2 = \frac{1}{Z} + \frac{2}{Z^2} + \frac{2}{Z^3} + \frac{2}{Z^4} + \frac{2}{Z^5} + ...\infty$$

$$\alpha\left(1-Z^{-1}\right)^2 = \frac{1}{Z} + \frac{2}{Z^2}\left(1 + \frac{1}{Z} + \frac{1}{Z^2} + \frac{1}{Z^3}...\infty\right)$$

$$\alpha\left(1-Z^{-1}\right)^2 = \frac{1}{Z} + \frac{2}{Z^2}\left(\sum_{k=0}^{\infty} Z^{-k}\right)$$

$$\alpha = \left(1-Z^{-1}\right)^{-2}.\left(\frac{1}{Z} + \frac{2}{Z^2}\left(\sum_{k=0}^{\infty} Z^{-k}\right)\right)$$

$$\alpha = \left(1-Z^{-1}\right)^{-2}.\left(\frac{1}{Z} + \frac{2}{Z^2}\lim_{n\to\infty}\left(\frac{1-\left(z^{-1}\right)^{n+1}}{1-z^{-1}}\right)\right)$$

$$\alpha = \left(1-Z^{-1}\right)^{-2}.\left(\frac{1}{Z} + \frac{2}{Z^2}\left(\frac{1}{1-z^{-1}}\right)\right) \qquad for\ |Z^{-1}| < 1$$

$$\alpha = \left(1-Z^{-1}\right)^{-2}.\left(\frac{1}{Z}\right)\left(\frac{Z+1}{Z-1}\right)$$

$$\alpha = \frac{Z(Z+1)}{(Z-1)^3} \tag{22}$$

from (16) and (22) in (17)

$$S\ (Z) = \left(x\,(0) - \frac{d}{2}\right) S\ + \frac{d}{2}\ \alpha$$

$$S\ (Z) = \left(x\,(0) - \frac{d}{2}\right) \frac{Z}{(Z-1)^2} + \frac{d}{2}\left(\frac{Z\,(Z+1)}{(Z-1)^3}\right)$$

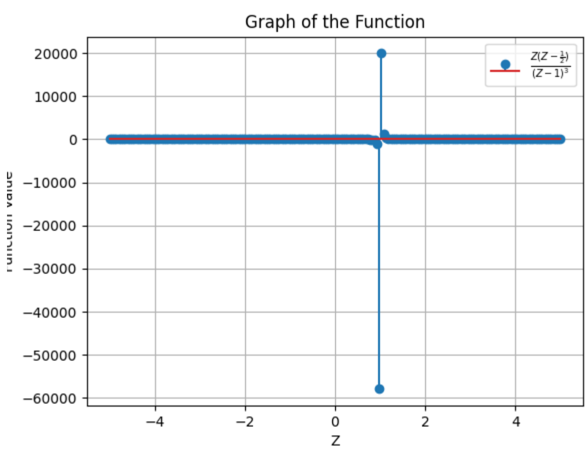
$$s\,(Z) = x\,(0)\left(\frac{Z}{(Z-1)^2}\right) - \frac{d}{2}\left(\frac{Z}{(Z-1)^2} - \frac{Z(Z+1)}{(Z-1)^3}\right)$$

$$s\,(Z) = x\,(0)\left(\frac{Z}{(Z-1)^2}\right) + d\left(\frac{Z}{(Z-1)^3}\right)$$

$$S\ (Z) = \frac{x\,(0)\,.Z.\,(Z-1) + d.Z}{(Z-1)^3}\ for\ |Z^{-1}| < 1\ and\ Z \neq 0 \quad (23)$$

from (23), for x(0)=1 and $d = \frac{1}{2}$

$$S\ (Z) = \frac{Z\left(Z - \frac{1}{2}\right)}{(Z-1)^3}$$



and from (23), for x(0)=5 and $d = -\frac{1}{2}$

$$S\ (Z) = \frac{Z\left(5.Z - \frac{11}{2}\right)}{(Z-1)^3}$$