Leg kinematics with toe

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1 Leg structure

A simplified leg kinematics structure is shown in Fig.1. The angular position of motors are o_1 and o_2. Let (x_{end}, y_{end}) be the position of toe. Length of linkages are l_1,l_2,l_3.

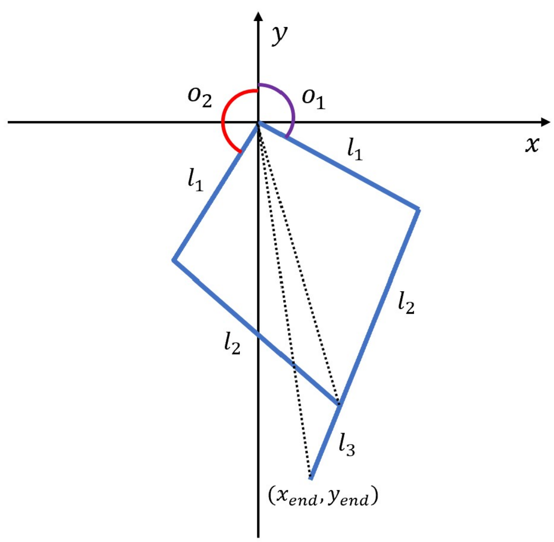

Figure 1: Leg structure.

2 FK

For leg's FK solution,

$$x_{end} = \sqrt{\mathrm{nn}_1}\,\cos\left(o_1 - \frac{\pi}{2} + \mathrm{asin}\left(\frac{\sin(n_1)\,(l_2+l_3)}{\sqrt{\mathrm{nn}_1}}\right)\right) \tag{1}$$

$$y_{end} = -\sqrt{\mathrm{nn}_1}\,\sin\left(o_1 - \frac{\pi}{2} + \mathrm{asin}\left(\frac{\sin(n_1)\,(l_2+l_3)}{\sqrt{\mathrm{nn}_1}}\right)\right) \tag{2}$$

Where,

$$nn_1 = (l_2+l_3)^2 + {l_1}^2 - 2\,l_1\,\cos(n_1)\,(l_2+l_3) \tag{3}$$

$$n_1 = \frac{o_1}{2} + \frac{o_2}{2} + \mathrm{asin}\left(\frac{l_1\,\sin\left(\frac{o_1}{2}+\frac{o_2}{2}-\pi\right)}{l_2}\right) \tag{4}$$

3 IK

For leg's IK solution,

$$o_1 = \frac{\pi}{2} - \mathrm{atan2}\left(y_{\mathrm{end}}, x_{\mathrm{end}}\right) - \mathrm{asin}\left(\frac{m_1\,\sqrt{1-\frac{{\left(-{l_1}^2-{m_1}^2+{x_{\mathrm{end}}}^2+\mathrm{mm}_1\right)}^2}{4\,{l_1}^2\,{m_1}^2}}}{\sqrt{{x_{\mathrm{end}}}^2+\mathrm{mm}_1}}\right) \tag{5}$$

Where,

$$mm_1 = {y_{\mathrm{end}}}^2 \tag{6}$$

$$m_1 = l_2 + l_3 \tag{7}$$

$$\begin{aligned} o_2 = \frac{3\,\pi}{2} + \mathrm{atan2}\left(y_{\mathrm{end}}, x_{\mathrm{end}}\right) - 2\,\mathrm{acos}\left(\frac{2\,{l_1}^2+\mathrm{mm}_2}{2\,l_1\,\sqrt{{l_1}^2+{l_2}^2+\mathrm{mm}_2}}\right) \\ + \mathrm{asin}\left(\frac{\left(l_2+l_3\right)\,\sqrt{1-\frac{{m_2}^2}{4\,{l_1}^2\,{\left(l_2+l_3\right)}^2}}}{\sqrt{{x_{\mathrm{end}}}^2+{y_{\mathrm{end}}}^2}}\right) \end{aligned} \tag{8}$$

Where,

$$mm_2 = -\frac{l_2\,m_2}{l_2+l_3} \tag{9}$$

$$m_2 = {\left(l_2+l_3\right)}^2 + {l_1}^2 - {x_{\mathrm{end}}}^2 - {y_{\mathrm{end}}}^2 \tag{10}$$

4 Jacobian matrix

The leg's Jacobian matrix,

$$J = \begin{pmatrix} \frac{\partial x_{end}}{\partial o_1} & \frac{\partial x_{end}}{\partial o_2} \\ \frac{\partial y_{end}}{\partial o_1} & \frac{\partial y_{end}}{\partial o_2} \end{pmatrix} = \begin{pmatrix} j_1 & j_2 \\ j_3 & j_4 \end{pmatrix} \tag{11}$$

$$\begin{aligned} j_1 = \mathrm{je}_1\,\mathrm{jw}_1\,l_1\,\cos\left(o_1-\frac{\pi}{2}+\mathrm{asin}\left(\mathrm{je}_1\,\sin\left(\mathrm{jq}_1\right)\,\left(l_2+l_3\right)\right)\right)\,\sin\left(\mathrm{jq}_1\right)\,\left(l_2+l_3\right) \\ - \frac{\sin\left(o_1-\frac{\pi}{2}+\mathrm{asin}\left(\mathrm{je}_1\,\sin\left(\mathrm{jq}_1\right)\,\left(l_2+l_3\right)\right)\right)\,\left(\frac{\mathrm{je}_1\,\mathrm{jw}_1\,\cos\left(\mathrm{jq}_1\right)\,\left(l_2+l_3\right)-{\mathrm{je}_1}^3\,\mathrm{jw}_1\,l_1\,{\sin\left(\mathrm{jq}_1\right)}^2\,{\left(l_2+l_3\right)}^2}{\sqrt{1-{\mathrm{je}_1}^2\,{\sin\left(\mathrm{jq}_1\right)}^2\,{\left(l_2+l_3\right)}^2}}+1\right)}{\mathrm{je}_1} \end{aligned} \tag{12}$$

$$\begin{aligned} j_2 = \mathrm{je}_2\,\mathrm{jw}_2\,l_1\,\cos\left(o_1-\frac{\pi}{2}+\mathrm{asin}\left(\mathrm{je}_2\,\sin\left(\mathrm{jq}_2\right)\,\left(l_2+l_3\right)\right)\right)\,\sin\left(\mathrm{jq}_2\right)\,\left(l_2+l_3\right) \\ - \frac{\sin\left(o_1-\frac{\pi}{2}+\mathrm{asin}\left(\mathrm{je}_2\,\sin\left(\mathrm{jq}_2\right)\,\left(l_2+l_3\right)\right)\right)\,\left(\mathrm{je}_2\,\mathrm{jw}_2\,\cos\left(\mathrm{jq}_2\right)\,\left(l_2+l_3\right)-{\mathrm{je}_2}^3\,\mathrm{jw}_2\,l_1\,{\sin\left(\mathrm{jq}_2\right)}^2\,{\left(l_2+l_3\right)}^2\right)}{\mathrm{je}_2\,\sqrt{1-{\mathrm{je}_2}^2\,{\sin\left(\mathrm{jq}_2\right)}^2\,{\left(l_2+l_3\right)}^2}} \end{aligned} \tag{13}$$

$$\begin{aligned} j_3 = = -\frac{\cos\left(o_1-\frac{\pi}{2}+\mathrm{asin}\left(\mathrm{je}_3\,\sin\left(\mathrm{jq}_3\right)\,\left(l_2+l_3\right)\right)\right)\,\left(\frac{\mathrm{je}_3\,\mathrm{jw}_3\,\cos\left(\mathrm{jq}_3\right)\,\left(l_2+l_3\right)-{\mathrm{je}_3}^3\,\mathrm{jw}_3\,l_1\,{\sin\left(\mathrm{jq}_3\right)}^2\,{\left(l_2+l_3\right)}^2}{\sqrt{1-{\mathrm{je}_3}^2\,{\sin\left(\mathrm{jq}_3\right)}^2\,{\left(l_2+l_3\right)}^2}}+1\right)}{\mathrm{je}_3} \\ - \mathrm{je}_3\,\mathrm{jw}_3\,l_1\,\sin\left(o_1-\frac{\pi}{2}+\mathrm{asin}\left(\mathrm{je}_3\,\sin\left(\mathrm{jq}_3\right)\,\left(l_2+l_3\right)\right)\right)\,\sin\left(\mathrm{jq}_3\right)\,\left(l_2+l_3\right) \end{aligned} \tag{14}$$

$$j_4 = -\frac{\cos\left(o_1 - \frac{\pi}{2} + \mathrm{asin}\left(\mathrm{je}_4\, \sin\left(\mathrm{jq}_4\right)\, \left(l_2 + l_3\right)\right)\right)\, \left(\mathrm{je}_4\, \mathrm{jw}_4\, \cos\left(\mathrm{jq}_4\right)\, \left(l_2 + l_3\right) - {\mathrm{je}_4}^3\, \mathrm{jw}_4\, l_1\, \sin\left(\mathrm{jq}_4\right)^2\, \left(l_2 + l_3\right)^2\right)}{\mathrm{je}_4\, \sqrt{1 - {\mathrm{je}_4}^2\, \sin\left(\mathrm{jq}_4\right)^2\, \left(l_2 + l_3\right)^2}}$$
$$- \mathrm{je}_4\, \mathrm{jw}_4\, l_1\, \sin\left(o_1 - \frac{\pi}{2} + \mathrm{asin}\left(\mathrm{je}_4\, \sin\left(\mathrm{jq}_4\right)\, \left(l_2 + l_3\right)\right)\right)\, \sin\left(\mathrm{jq}_4\right)\, \left(l_2 + l_3\right) \tag{15}$$

Where,

$$jq_1 = jq_2 = jq_3 = jq_4 = \frac{1}{\sqrt{\left(l_2 + l_3\right)^2 + {l_1}^2 - 2\, l_1\, \cos\left(\mathrm{jq}_4\right)\, \left(l_2 + l_3\right)}} \tag{16}$$

$$jw_1 = jw_2 = jw_3 = jw_4 = \frac{l_1\, \cos\left(\frac{o_1}{2} + \frac{o_2}{2} - \pi\right)}{2\, l_2\, \sqrt{1 - \frac{{l_1}^2\, \sin\left(\frac{o_1}{2} + \frac{o_2}{2} - \pi\right)^2}{{l_2}^2}}} + \frac{1}{2} \tag{17}$$

$$je_1 = je_2 = je_3 = je_4 = \frac{o_1}{2} + \frac{o_2}{2} + \mathrm{asin}\left(\frac{l_1\, \sin\left(\frac{o_1}{2} + \frac{o_2}{2} - \pi\right)}{l_2}\right) \tag{18}$$

So the estimated toe force

$$f = \left(J^{\mathsf{T}}\right)^{-1} \tau \tag{19}$$

5 Simulations

Leg's workspace is shown in Fig.2. Where $l_1 = 100mm$, $l_2 = 200mm$, $l_3 = 50mm$.

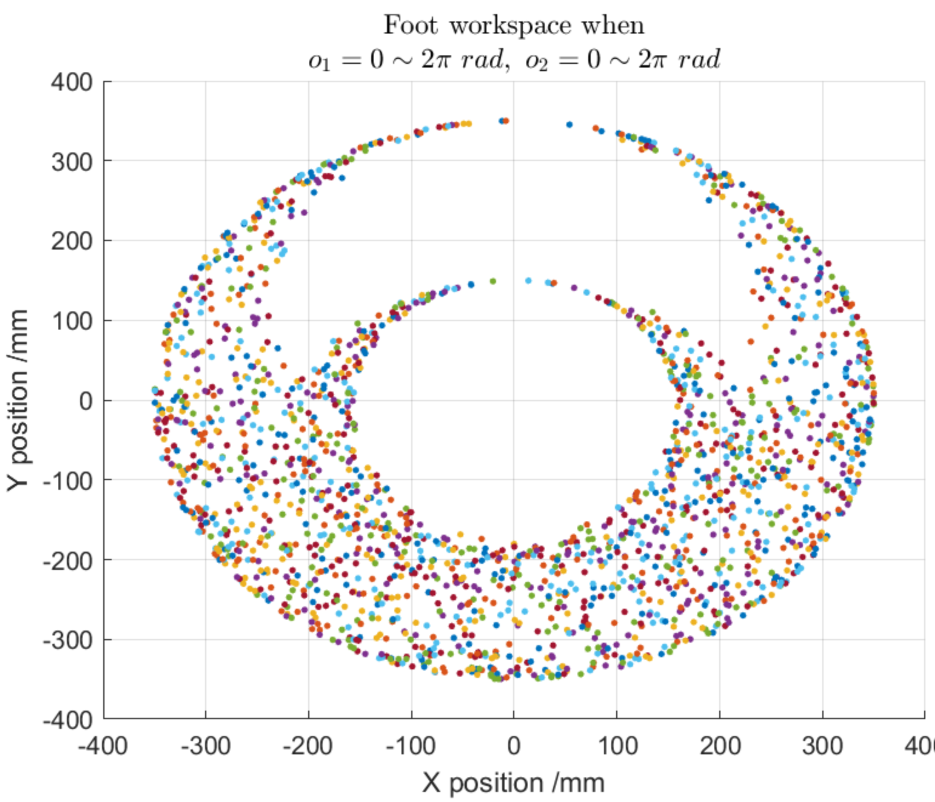

Figure 2: Leg's workspace.

Toe's maximum output force in all directions when moving horizontally is shown in Fig.3. This gives us a general idea of how the toe's position affects the linkage's efficiency.

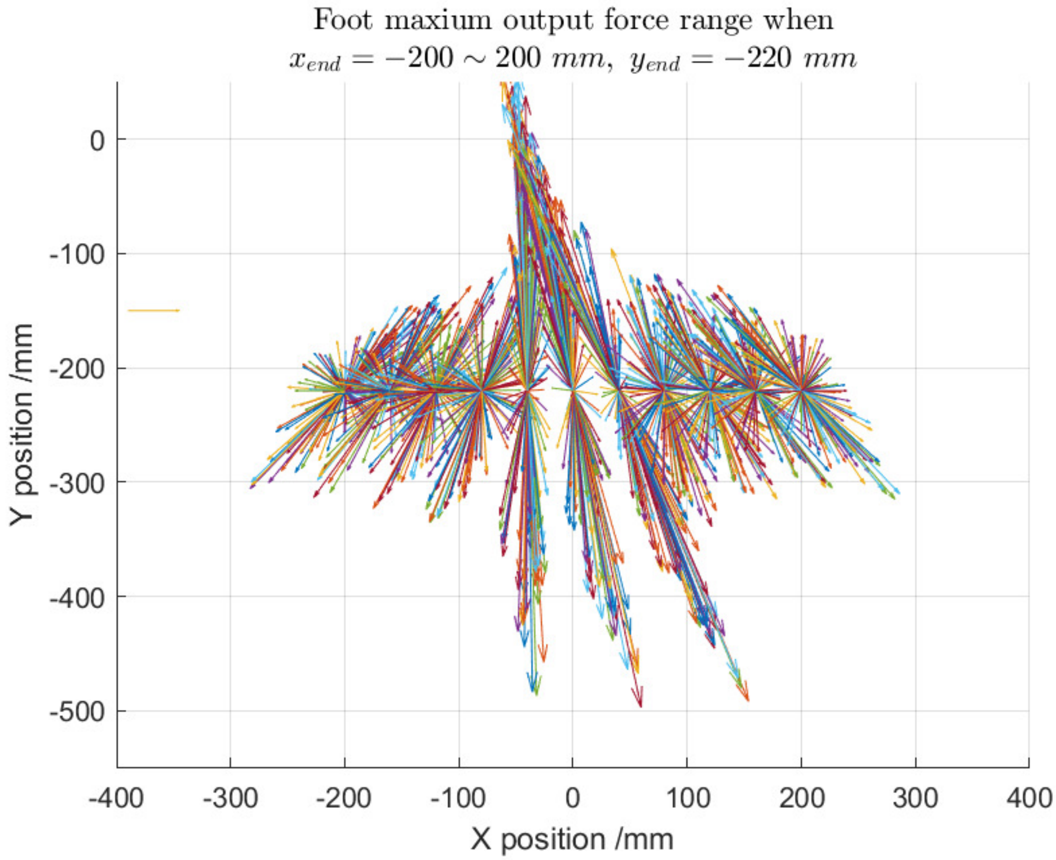

Figure 3: Toe's maximum output force sim.

The vector's length at $(-390, -150)$ represents a $10N$ toe force.

Motor's maximum torque is set to $1.66N \cdot M$, since we're using T-MOTOR U8II. The maximum torque is assumed based on the relationship between a motor's velocity constant K_V, current I_A, and torque τ. This relationship is as follows:

$$\tau \approx \frac{8.3 \cdot I_A}{K_V} \tag{20}$$