CS3200 – Assignment 2 Written Report
Skylar Shyu

Question 1:

The first question asked us to generate $\{w_i, x_i\}$ weighted pairs using 6 different N-values (17, 33, 65, 129, 257, 513) for all 3 Newton-Cotes methods:

1. Composite Midpoint rule (QuadSchemeMidpoint.m)
2. Composite Trapezoid rule (QuadSchemeTrapezoid.m)
3. Composite Simpson formula (QuadSchemeSimpson.m)

Implementation of the functions to get w_i, x_i values were more-or-less the same for all three methods. Vectors of size-N were pre-generated to hold the w_i, x_i, then Δx was calculated. Finally, a loop was used to handle generating $\{w_i, x_i\}$ pairs with respect to N-number of partitions. Edge cases were handled outside the loop. The equations that were used to calculate the values for said pairings were obtained from the Numerical Integration slides presented in class.

This question did not ask for results and figures to be generated.

Question 2:

The Gaussian-Legendre was created via mapping the $\{w_i, x_i\}$ pairs that were generated for [-1 1] to [a b]. The pairings that were produced were hard-coded and the value of N determined which sets of pairing was used via a switch statement. Then, like the methods in Question 1, the formula for Gauss was borrowed from the slides and implemented.

As with question 1, this question did not ask for results and figures to be generated.

Question 3:

Many methods were at work in Question 3, as the question itself had three major components. To create the requested convergence plots for the 3 Newton-Cotes methods for N = 17, 33, 65, 129, 257, 513 (representing the number of subintervals) and determine which method converged the quickest as N grew. The N-values 1-dimension vectors were placed in a vector, and other 1-dimensional vectors were initialized with zeroes to hold the composite integration results and subsequent error calculations. The function f(x) given in the problem was passed into a function handler, and the range of a and b were hard-coded.

A for-loop was used to iterate each of the N values in the vector. During each loop, $\{w_i, x_i\}$ pairs were calculated using the three methods implemented in the first question (Composite Midpoint Rule, Composite Trapezoid Rule, and Composite Simpson's Formula), thereby yielding three sets of $\{w_i, x_i\}$ pairs.

All three of the methods share a common integral summation, $\sum w_i f(x_i)$, so I made a helper function called weightPairSum to assist with the 'piecewise' integral summation. This function essentially took the $\{w_i, x_i\}$ pairs for each of the three methods and the f(x) function, returning the composite integration calculation for each respective method.

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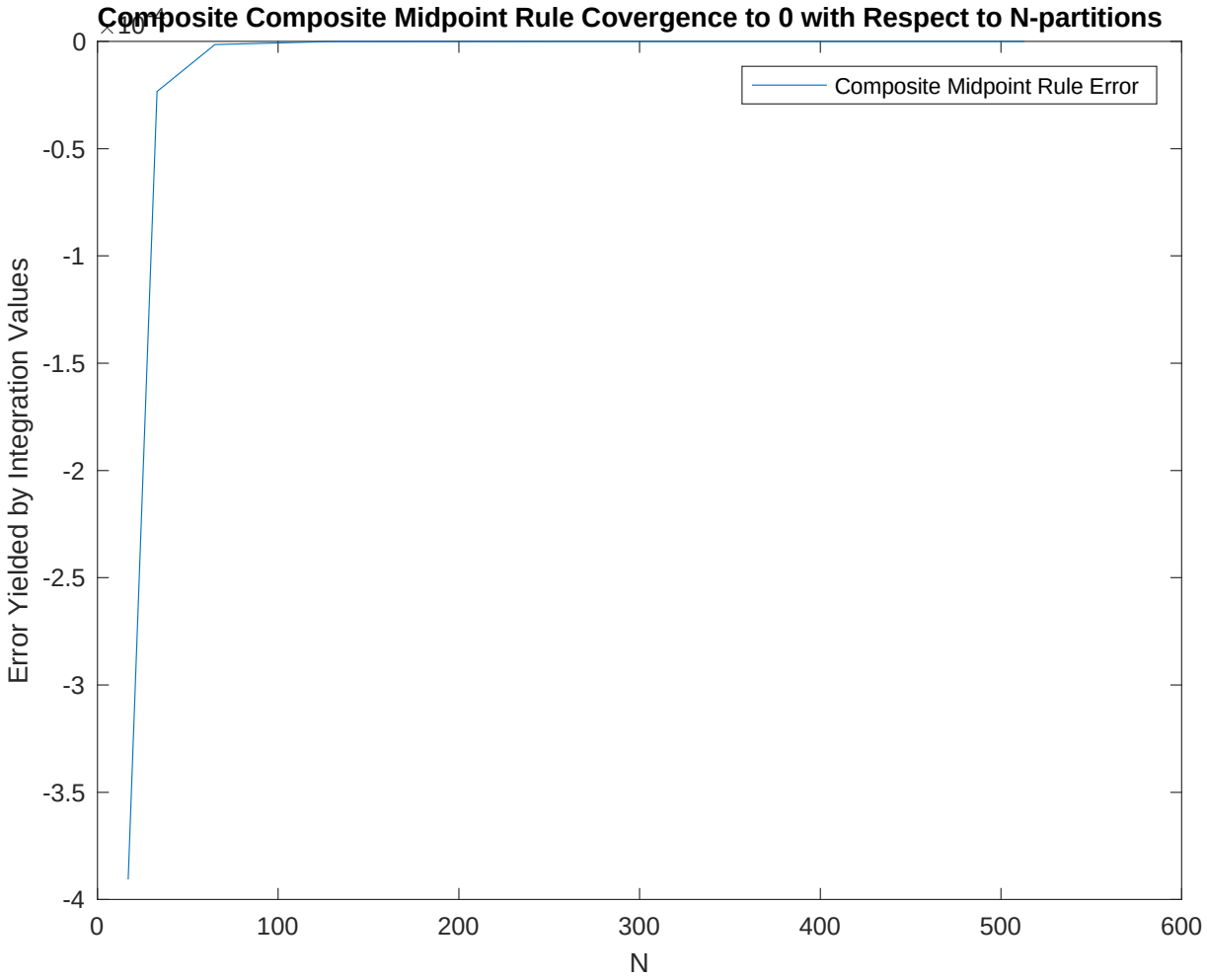
To determine how accurate each Newton-Cotes integration method was, they needed to be compared to the exact integration of f(x). Thus, the exact integral of f(x) over [a b] was calculated and placed in an N-spaced vector; N does not affect the exact integral value, but we need all vectors to span over the same N for plotting, thus the same integral value was duplicated number of N-times to fit a vector of appropriate length. Errors for each Newton-Cotes was calculated by subtracting the integration obtained by the method from the true integral value for every N-value.

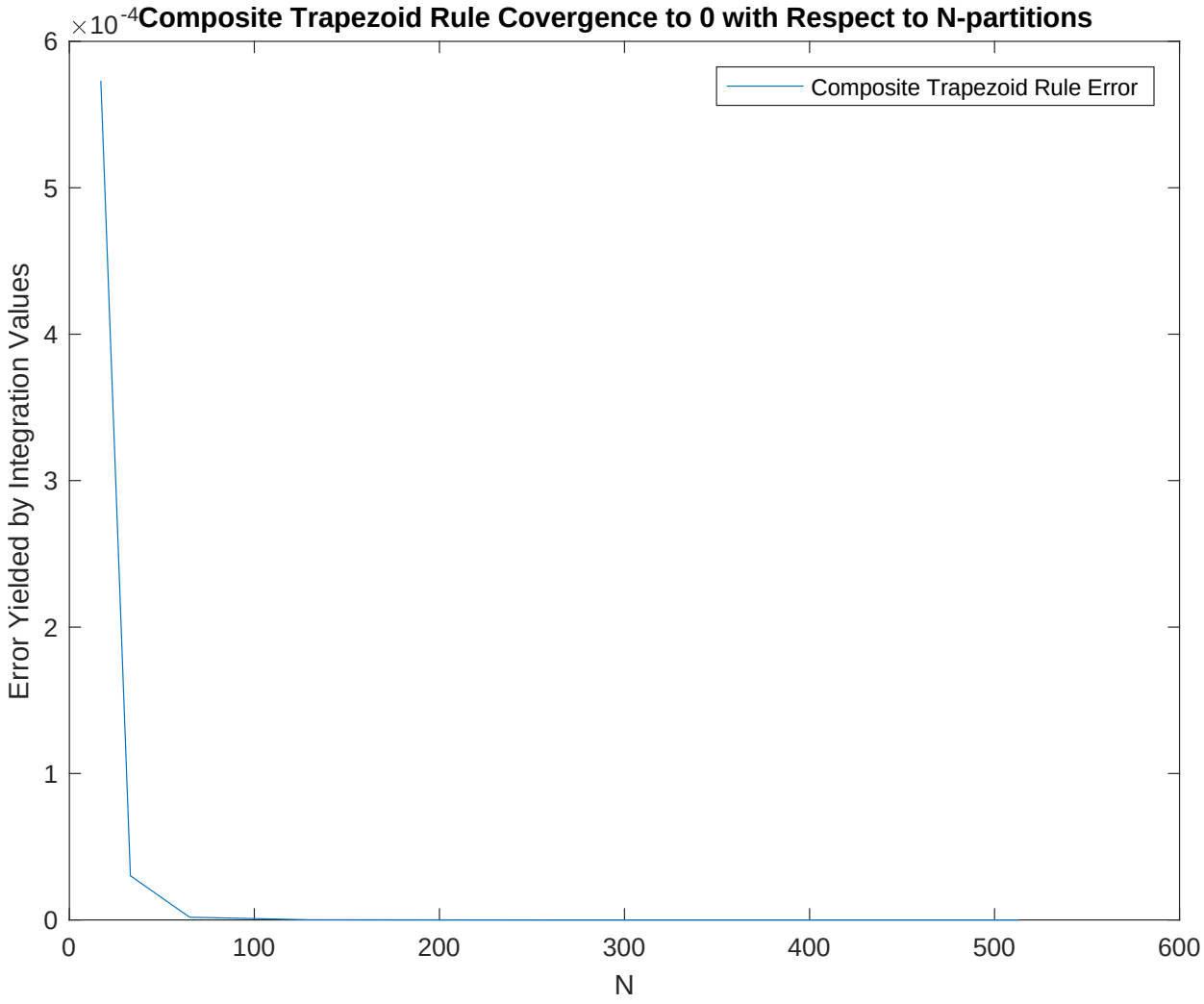
Six figures were plotted: each of the three Newton-Cotes methods were plotted against the true integration as well as their respective errors. The error plots were easier to distinguish how quickly each method converged to zero; the closer the error was to zero, the more accurate the composite integration method was.

Figure 1: Composite Midpoint Rule Error Convergence



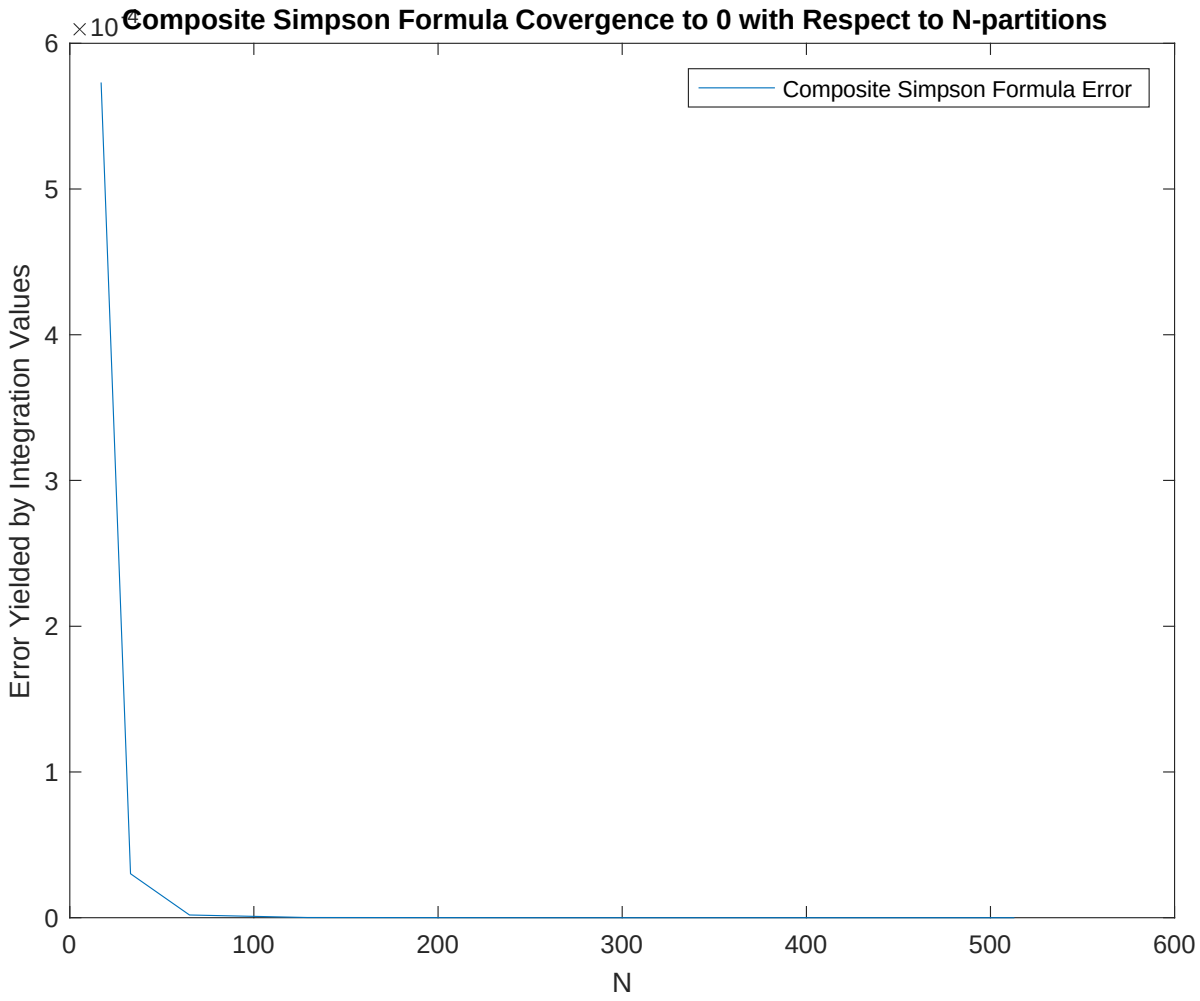
*the y-axis's scale that's clipped by the plot's title is supposed to read: 0 x 10^{-4}

Figure 2: Composite Trapezoid Rule Error Convergence



*the y-axis's scale that's clipped by the plot's title is supposed to read: 0 x 10^{-4}

Figure 3: Composite Simpson Formula Error Convergence



*the y-axis's scale that's clipped by the plot's title is supposed to read: 0 x 10^{-4}

Table 1: Error Values for Composite Newton-Cotes methods with Respect to N

N	Midpoint Error	Trapezoid Error	Simpson's Error
17	-0.000390633268067	0.000573053429959	-0.000114515002546
33	-2.34078647070e-05	3.01931351884e-05	-6.73544644502e-06
65	-1.49151675898e-06	1.81188560955e-06	-4.26933215003e-07
129	-9.51189536025e-08	1.12110846828e-07	-2.71895927995e-08
257	-6.02159477836e-09	6.98940283428e-09	-1.72065917070e-09
513	-3.79023035180e-10	4.36558345029e-10	-1.08295594714e-10

As demonstrated visually by Figures 1 to 3 (the error plots) above, Simpson's formula converged the fastest of the three Newton-Cotes methods and is in line with the theoretical error calculated in Table 1. This is probably due to the Trapezoidal rule being an average of the left and right Riemann Sums; as a result, the Composite Trapezoid Rule performs poorly when it comes to approximating very curvy equations, like ours which involves a few sine and cosines. The same idea occurs with the Composite Midpoint Rule as it samples the midpoint between two subintervals and uses that lone point as the basis of interpolation.

Simpson's Rule, however, places a parabola across every two subintervals on top of the fact that it also utilizes at least twice the number of subintervals compared to the other two. Generally, using more subintervals to approximate integrals yields a more accurate result, and such is the case we see here. These points explain why Simpson's Formula is superior to the Midpoint and Trapezoid Rules.

The next component of Question 3 was to estimate the explicit errors for the Trapezoid Rule and Simpson's Formula, which was done through estimating the appropriate derivatives.

To estimate the derivative $f''(\zeta)$ for the Composite Trapezoid Rule, for each N-value, the second derivative of f(x) was obtained by passing f(x) into the diff() routine and setting the n^{th} derivative to 2. Then $f''(x)$ was calculated across the domain of a to b; the maximum $f''(x)$ was obtained, and it's respective x value was verified to fall between a and b. This maximum $f''(x)$ value thus became the derivative estimation $f''(\zeta)$ for the Composite Trapezoid Rule. Obtaining $f^{(4)}(\varphi)$, the derivative estimation for Composite Simpson's, used the exact same methodology, except we were operating with the 4^{th}, not the 2^{nd}, derivative of f(x).

In our case, the derivative estimations results were:

$f''(\zeta)$ = f2_eta = 8.6612
$f^{(4)}(\varphi)$ = f4_eta = 384.7914

Δx values were calculated for both Composite Simpson's and Composite Trapezoid. The derivative estimations and Δx were then plugged into the explicit error formulas for each respective method, and the explicit error obtained.

Calculating the errors via Richardson's Extrapolation was an extension of what has already been calculated. Due to Richardson's Extrapolation using both Δx and Δx/2, we needed to calculate the composite integration of 2*N using the Simpson's and Trapezoid, as a half-step in x is equivalent to doubling the number of subintervals; we already calculated the composite integration of N in the previous component of this question. Obtaining the 2*N integration used the same methods in the previous component: calculating the

appropriate $\{w_i, x_i\}$ pairs using the functions developed in Question 1, then feeding the generated $\{w_i, x_i\}$ pairs into the weightPairSum function.

Calculating Richardson's Extrapolation error was only a matter of plugging the integral calculation using N and integral calculation of 2*N for a Newton-Cotes method into the error formula, which I've hard coded into a separate function called RichardsonExtrap().

These calculations resulted in the values found in the table below:

Table 2: Error Values for Composite Trapezoid and Composite Simpson's Formula methods using both Explicit Error and Richardson's Extrapolation Error Formulas with Respect to N

	Explicit Error		Richardson's Extrapolation	
N	**Trapezoid Rule**	**Simpson's Formula**	**Trapezoid Rule**	**Simpson's Formula**
17	0.6993538615661	0.0009790763802	0.0007285920531	-0.000144746249
33	0.1748384653915	6.895352924e-05	3.798786843e-05	-8.44533574e-06
65	0.0437096163478	4.580984607e-06	2.270955285e-06	-5.34096383e-07
129	0.0109274040870	2.952931808e-07	1.403061892e-07	-3.39939620e-08
257	0.0027318510217	1.874475488e-08	8.741642669e-09	-2.15093721e-09
513	0.0006829627554	1.180708796e-09	5.458543247e-10	-1.35366681e-10

Overall, Richardson's Extrapolation Error calculations were more accurate than the errors yielded by the Explicit Error formula. The Composite Trapezoid error benefitted more from Richardson's Error calculations than Composite Simpson's Formula did.

The final component of Question 3 examines the Gaussian Quadrature implemented back in Question 2. To confirm the question's assumption that Gaussian Quadrature performs rather poorly though plots and numerical data, we placed the values of N into a vector and pre-emptively allocated vectors to hold integral calculations, both exact and Gaussian, along with error calculations (exact subtracted by Gaussian).

By using a for-loop to iterate through each N-value, the Gaussian Quadrature integration was calculated; subtracting this value from the exact integration over the same [a b] domain yielded the error.

Plotting the error demonstrates what the question already confirmed: Gaussian Quadratures is a terrible method to estimate the integral of the function given in this assignment. The only N where the error is 'better' is N=4.

Figure 4: Gaussian Quadrature Error Convergence

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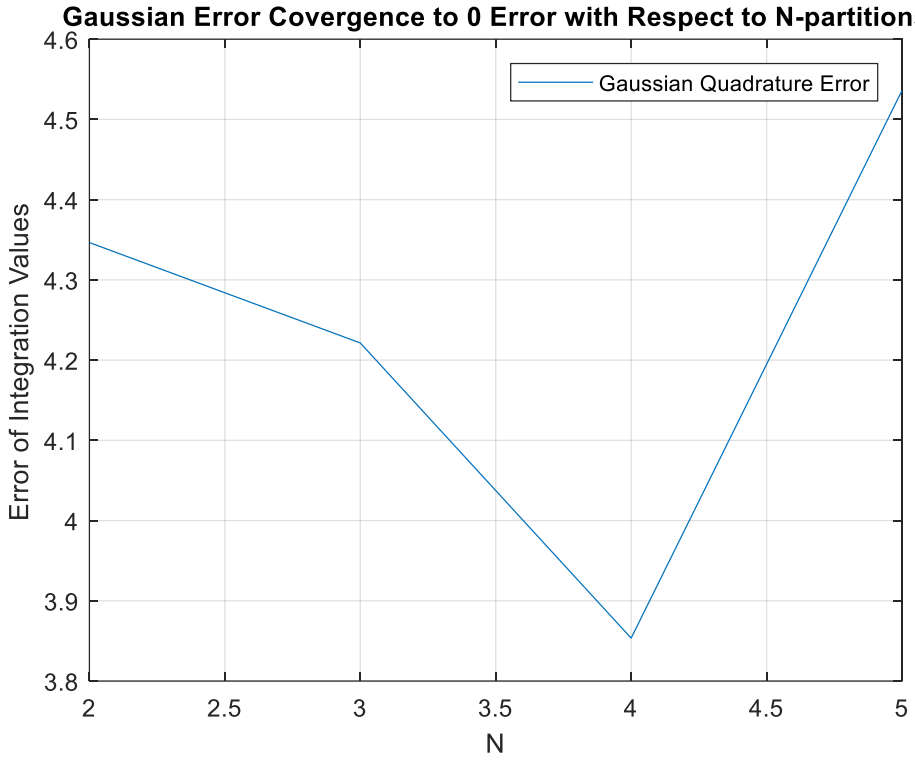

Gaussian Quadrature works poorly with our given formula f(x) = 1 + sin(x)*cos(2x/3)*sin(4x) because our $\{w_i, x_i\}$ pairs did not maximize the order of accuracy of the Gaussian Quadrature Formula. The $\{w_i, x_i\}$ pairs were determined largely by the number of points to sample (N) and the domain that we wanted to integrate over. To make the Gaussian Quadrature give better results, a and b should be chosen such that the mapped x_i should maximize the order of accuracy while w_i remains constant for each N.

Sources of Information:

Sources of information include the slides on Numerical Interpolation posted on Canvas, as well as MatLab's documentation on routines such as saveas() and figures(). Tajo, as always, was superbly helpful in his office hours in clarifying what he was looking for with our plots and the formula for Richardson's Extrapolation. Also, I did browse the Discussion forum on Canvas for any tips but didn't find them super helpful this time around. I did not collaborate with any classmates on this assignment.

As far as source codes went, I implemented all the functions that didn't already come with MatLab. For the Newton-Cotes and Gaussian quadratures, I used the formulas in the Numerical Interpolation slides on Canvas and implemented them. I also browsed the links at the end of **Numerical_Integration(a)-11.pdf** to gain further insight on the quadrature methods.