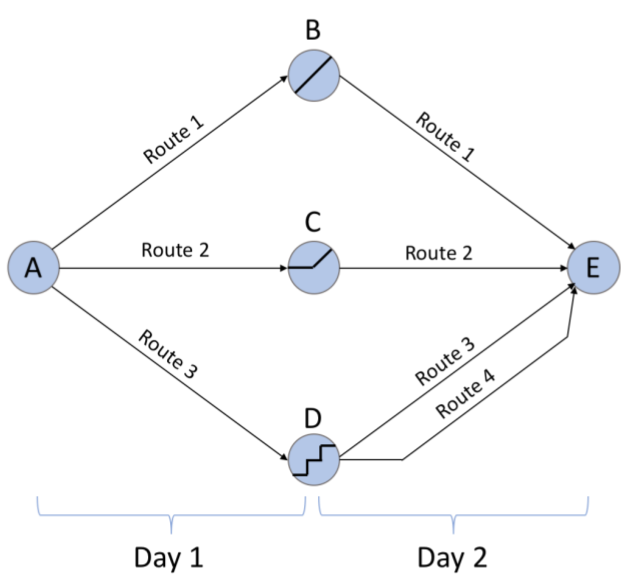
Motivation

CVXPY is a very flexible modelling language for solving convex optimization problems in Python. Its API offers users the ability to model mathematical optimization problems very intuitively, and supports numerous solvers that can handle LP, QP, SOCP, SDP, EXP and MIP programs. The challenge for the user is to ensure that their program, if possible, is formulated such that it follows DCP (Disciplined Convex Programming) rules. Many problems in practice can be formulated in convex form at the outset, while others do not lend themselves to such an obvious formulation. Here we consider a problem of MINLP form, demonstrate its reformulation into MILP form through the use of variable relaxations and constraint additions. We then model the program in CVXPY and call the open source GLPK_MI solver. In particular, the non-linearity comes from the product of continuous and binary variables $J = X \times Z$ which must be reformulated in the form $J' = X' + Z'$ to satisfy DCP rules. In addition, we deal with Boolean variables Y which are implicitly determined by the values of the continuous decision variables $Y \coloneqq f(X)$.

Problem Description

A gravel company wishes to stock one of its local distribution centres $(node\ E)$ with some combination of its ten product offerings, each with corresponding price per tonne $(p_i\ i = 1, \ldots, 10)$. The company orders it from the supplier $(node\ A)$ and it takes two days for the products to be shipped to the distribution centre, thus the products must be stored overnight in some storage facility(ies) along the way $(nodes\ s = B, C, D)$. There are three routes available for shipping out of the supply centre on the first day $(routes\ j = 1,2,3)$ each leading to the storage facilities $(nodes\ s = B, C, D)$ respectively. On the second day, four routes $(routes\ j = 1,2,3,(4))$ are available for shipping out of the storage facilities to the distribution centre; routes 1 and 2 lead from storage facilities B and C respectively, and routes 3 and 4 both lead from storage facility D. Due to the various forms of transportation available along the different routes, each route has a maximum weight transport capacity $(T_j^t\ j = 1,2,3,(4)\ t = 1,2)$ and linear transportation cost per tonne $(c_j^t\ j = 1,2,3,(4)\ t = 1,2)$ associated with it. Each of the storage facilities have maximum storage capacities $(S_s\ s = B, C, D)$ and each offer different storage cost structures. Facility B offers a simple linear cost which charges α_B per tonne, facility C offers a fixed cost α_C^L up until a lower bound of storage is reached, W_C tonnes, after which a linear cost of α_C^H per tonne is imposed on any incremental product that exceeds L_C tonnes. Facility D offers a stepwise cost structure, α_D^L is charged for any amount of storage below W_D^L tonnes, α_D^M is charged for any amount of incremental storage between W_D^L and W_D^M tonnes, and α_D^H is charged for any amount of incremental storage above W_D^M tonnes. The objective is to choose a combination of products to ship from A to E that maximize total potential revenue while minimizing storage and transportation costs. Due to demands for different products, there is a minimum and maximum amount of weight, w_i^{min}, w_i^{max}, that can be shipped of each product otherwise none of it must be shipped, and the distribution centre E has a total storage capacity of S_E. Lastly due to packaging costs, each product can only be shipped in one container (ie. you cannot ship fractional amounts of product i along different routes). Assume supplier A can provide any amount necessary.

Below is a flowchart illustrating the problem and cost structures, and a table defining the parameters and decision variables.



Input Parameters	Decision Variables
$\boldsymbol{p_i}$: unit price of product i (\$/tonne) $\boldsymbol{w_i^{min}}$: minimum amount of product i that is to be shipped if product i is chosen for stock (tonnes) $\boldsymbol{w_i^{max}}$: maximum amount of product i that is to be shipped if product i is chosen for stock (tonnes) $\boldsymbol{s}$: storage facility and distribution centre index (B, C, D, E) $\boldsymbol{S_s}$: maximum storage capacity of node s (tonnes) $\boldsymbol{\alpha_B}$: cost of storage at node B (\$/tonne) $\boldsymbol{\alpha_C^L}$: fixed cost of storage at node C up to W_C tonnes (\$) $\boldsymbol{W_C}$: weight threshold in node C cost structure (tonnes) $\boldsymbol{\alpha_C^H}$: cost of incremental storage at node C above W_C tonnes (\$/tonne) $\boldsymbol{\alpha_D^L}$: fixed cost of storage at node D up to W_D^L tonnes (\$) $\boldsymbol{W_D^L}$: lower weight threshold in node D cost structure (tonnes) $\boldsymbol{\alpha_D^M}$: fixed cost of incremental storage at node D between W_D^L and W_D^M tonnes (\$) $\boldsymbol{W_D^M}$: medium weight threshold in node D cost structure (tonnes) $\boldsymbol{\alpha_D^H}$: fixed cost of incremental storage at node D above W_D^M tonnes (\$) $\boldsymbol{j}$: transport route index (1, 2, 3, 4) $\boldsymbol{t}$: day index (1, 2) $\boldsymbol{T_j^t}$: maximum transportation capacity of route j on day t (tonnes) $\boldsymbol{c_j^t}$: transportation cost of route j on day t (\$/tonne)	$\boldsymbol{w_i}$: weight of product $\boldsymbol{i}$ that is being shipped $\boldsymbol{z_{ij}^t}$: Binary indicator if product $\boldsymbol{i}$ is shipped along route $\boldsymbol{j}$ on day $\boldsymbol{t}$ $\boldsymbol{y_k}$: Boolean indicating if total product weight being stored at facility D falls within range $\boldsymbol{k}$

MINLP Formulation

$$\textit{minimize}_{w,z,y} \quad -\sum_i \sum_j w_i z_{ij}^1 p_i \tag{1}$$

$$+\alpha_B \sum_i w_i z_{i1}^1 \tag{2}$$

$$+\alpha_C^H \left[\sum_i w_i z_{i2}^1 - W_c\right]^+ + \alpha_C^L \sum_i z_{i2}^1 \tag{3}$$

$$+(\alpha_D^L + \alpha_D^M + \alpha_D^H) y_H + (\alpha_D^L + \alpha_D^M) y_M + \alpha_D^L y_L + \epsilon y_0 \tag{4}$$

$$+\sum_i \sum_j w_i z_{ij}^1 c_j^1 \tag{5}$$

$$+\sum_i \sum_{j=1}^{2} w_i z_{ij}^1 c_j^2 + \sum_i \sum_{j=3}^{4} w_i z_{ij}^2 c_j^2 \tag{6}$$

subject to:

$$0 \le \sum_i w_i z_{ij}^1 \le min\{T_j^1, T_j^2\} \qquad j = 1,2 \tag{7}$$

$$0 \le \sum_i w_i z_{i3}^1 \le min\{T_3^1, T_3^2 + T_4^2\} \tag{8}$$

$$0 \le \sum_i w_i z_{ij}^2 \le T_j^2 \qquad j = 3,4 \tag{9}$$

$$0 \le \sum_j z_{ij}^t \le 1 \qquad \forall i, t \tag{10}$$

$$0 \le \sum_i w_i z_{ij}^1 \le S_s \qquad \forall \{s, j\} = \{(B,1),(C,2),(D,3)\} \tag{11}$$

$$z_{i3}^1 \le \sum_{j=3}^{4} z_{ij}^2 \le z_{i3}^1 \qquad \forall i \tag{12}$$

$$0 \le \sum_i \sum_j w_i z_{ij}^1 \le S_E \tag{13}$$

$$w_i^{min} \le w_i \le w_i^{max} \qquad \forall i \tag{14}$$

$$z_{ij}^t, y_k \in \{0,1\} \qquad \forall i, j, t, k \tag{15}$$

where:

$$(y_0, y_L, y_M, y_H) = \begin{cases} (1,0,0,0) & if \quad 0 \le \sum_i w_i z_{i3}^1 < \epsilon & (16) \\ (0,1,0,0) & if \quad \epsilon \le \sum_i w_i z_{i3}^1 < W_D^L & (17) \\ (0,0,1,0) & if \quad W_D^L \le \sum_i w_i z_{i3}^1 < W_D^M & (18) \\ (0,0,0,1) & if \quad W_D^M \le \sum_i w_i z_{i3}^1 \le S_D & (19) \end{cases}$$

*Note that $[x]^+ = \max\{0, x\}$

(1): Potential revenue as a function of product weight, price and whether the product was shipped on day 1
(2): Linear storage cost of B as a function of product weight, charge, and shipping indication along route 1
(3): Rectified Linear cost of C as a function of product weight, charge, weight threshold and route 2 indication
(4): Stepwise cost of D as a function of charge and indication of which weight range is being stored
$(5)-(6)$: Linear cost of transportation as a function of product weight, shipping indication and charge
$(7)-(9)$: Ensures that product weight does not exceed capacities of any connecting routes (routes are connected if they both connect at the same storage facility)
(10): Ensures that any product can only be shipped in one package, and not fractionally along different routes
(11): Ensures that the total product weight being stored at a facility does not exceed facility capacity
(12): Ensures that the same amount of product entering facility D also leaves facility D along the two exit routes (note that this constraint is already satisfied for the B and C exit routes by the formulation of the first component in (6))
(13): Ensures that the total amount of product being shipped does not exceed capacity of distribution centre E
$(14)-(15)$: Ensures the domains of decision variables; weight constraints and binary constraints
$(16)-(18)$: Boolean logic that models the indication of which weight category is being stored at facility D

To ensure the logic of (16)-(18), we can add the following constraints

$$y_L\epsilon + y_M W_D^L + y_H W_D^M \leq \sum_i w_i z_{i3}^1 \leq y_0\epsilon + y_L W_D^L + y_M W_D^M + y_H S_D$$
$$1 \leq y_0 + y_L + y_M + y_H \leq 1$$

The second constraint ensures that the total weight being stored at facility D can only fall into one of the categories at any given choice of product combinations.

We can also see that the non-linearity comes from the product of continuous and binary variables $w_i z_{ij}^t$. To impose linearity, we can define a new set of continuous variables, X_{ij}^t, such that $\boldsymbol{X_{ij}^t = w_i z_{ij}^t}$. In order for this to work we must impose new constraints on X_{ij}^t. Recall from the problem that if product i is chosen for shipping along route j on day t, $z_{ij}^t = 1$, demand imposes that the amount shipped must be $w_i^{min} \leq w_i \leq w_i^{max}$ but if product i is not chosen for shipping along route j on day t, $z_{ij}^t = 0$, then none of the product must be shipped. As such, $X_{ij}^t = 0$ if $z_{ij}^t = 0$ but $w_i^{min} \leq X_{ij}^t \leq w_i^{max}$ if $z_{ij}^t = 1$. We also want to ensure that when $z_{ij}^t = 1$, that $X_{ij}^t = w_i$. These can be ensured with the following linear constraints on X_{ij}^t

$$0 \leq X_{ij}^t \leq w_i^{max}$$
$$z_{ij}^t w_i^{min} \leq X_{ij}^t \leq z_{ij}^t w_i^{max}$$
$$w_i - (1 - z_{ij}^t) w_i^{max} \leq X_{ij}^t \leq w_i - (1 - z_{ij}^t) w_i^{min}$$
$$X_{ij}^t \leq w_i + (1 - z_{ij}^t) w_i^{max}$$

$$\forall i, j, t$$

The first constraint ensures that X_{ij}^t can take values within the product weight ranges and 0.
The second constraint ensures:

- If $z_{ij}^t = 1$ then $w_i^{min} \leq X_{ij}^t \leq w_i^{max}$
- If $z_{ij}^t = 0$ then $X_{ij}^t = 0$

The third constraint ensures:

- If $z_{ij}^t = 1$ then $X_{ij}^t = w_i$
- If $z_{ij}^t = 0$ then $w_i - w_i^{max} \leq X_{ij}^t \leq w_i - w_i^{min}$ ***which is a range that includes 0***. The fact that this range is guaranteed to include 0 and the second constraint ensures $X_{ij}^t = 0$ if $z_{ij}^t = 0$, then X_{ij}^t is again forced to be 0

The fourth constraint again imposes an upper bound on X_{ij}^t

MILP Reformulation

$$\mathit{minimize}_{X,w,z,y} \quad -\sum_i \sum_j X_{ij}^1 p_i \tag{1'}$$

$$+\alpha_B \sum_i X_{i1}^1 \tag{2'}$$

$$+\alpha_C^H \left[\sum_i X_{i2}^1 - W_c\right]^+ + \alpha_C^L \sum_i z_{i2}^1 \tag{3'}$$

$$+(\alpha_D^L + \alpha_D^M + \alpha_D^H) y_H + (\alpha_D^L + \alpha_D^M) y_M + \alpha_D^L y_L + \epsilon y_0 \tag{4}$$

$$+\sum_i \sum_j X_{ij}^1 c_j^1 \tag{5'}$$

$$+\sum_i \sum_{j=1}^{2} X_{ij}^1 c_j^2 + \sum_i \sum_{j=3}^{4} X_{ij}^2 c_j^2 \tag{6'}$$

$\mathit{subject\ to}$:

$$0 \le \sum_i X_{ij}^1 \le \min\{T_j^1, T_j^2\} \qquad j = 1,2 \tag{7'}$$

$$0 \le \sum_i X_{i3}^1 \le \min\{T_3^1, T_3^2 + T_4^2\} \tag{8'}$$

$$0 \le \sum_i X_{ij}^2 \le T_j^2 \qquad j = 3,4 \tag{9'}$$

$$0 \le \sum_j z_{ij}^t \le 1 \qquad \forall i, t \tag{10}$$

$$0 \le \sum_i X_{ij}^1 \le S_s \qquad \{s, j\} = \{(B,1), (C,2), (D,3)\} \tag{11'}$$

$$z_{i3}^1 \le \sum_{j=3}^{4} z_{ij}^2 \le z_{i3}^1 \qquad \forall i \tag{12}$$

$$0 \le \sum_i \sum_j X_{ij}^1 \le S_E \tag{13'}$$

$$w_i^{min} \le w_i \le w_i^{max} \qquad \forall i \tag{14}$$

$$z_{ij}^t, y_k \in \{0,1\} \qquad \forall i, j, t, k \tag{15}$$

$$y_L \epsilon + y_M W_D^L + y_H W_D^M \le \sum_i X_{i3}^1 \le y_0 \epsilon + y_L W_D^L + y_M W_D^M + y_H S_D \tag{16}$$

$$1 \le y_0 + y_L + y_M + y_H \le 1 \tag{17}$$

$$0 \le X_{ij}^t \le w_i^{max} \qquad \forall i, j, t \tag{18}$$

$$z_{ij}^t w_i^{min} \le X_{ij}^t \le z_{ij}^t w_i^{max} \qquad \forall i, j, t \tag{19}$$

$$w_i - (1 - z_{ij}^t) w_i^{max} \le X_{ij}^t \le w_i - (1 - z_{ij}^t) w_i^{min} \qquad \forall i, j, t \tag{20}$$

$$X_{ij}^t \le w_i + (1 - z_{ij}^t) w_i^{max} \qquad \forall i, j, t \tag{21}$$

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