Velocity circulation statistics in counterflow turbulence

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INTRODUCTION

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METHOD

Superfluid dynamics are modelled by employing the recently developed FOUCAULT model [1]. We follow Schwarz's approach [2], by exploiting the large separation of scales between the ^{4}He vortex core size a_0 and the average inter-vortex spacing ℓ. We parametrize the superfluid vortex lines as 1D space curves $\mathbf{s}(\xi, t)$ with ξ and t the arclength and time respectively. The corresponding equation of motion is

$$\dot{\mathbf{s}}(\xi, t) = \mathbf{v}_{s_\perp} + \beta\mathbf{s}' \times \mathbf{v}_{ns} + \beta'\mathbf{s}' \times [\mathbf{s}' \times \mathbf{v}_{ns}] \quad (1)$$

where $\mathbf{s}' = \partial\mathbf{s}/\partial\xi$ is the unit tangent vector at $\mathbf{s}$, $\mathbf{v}_{ns} = \mathbf{v}_n - \mathbf{v}_s$ where $\mathbf{v}_n$ and $\mathbf{v}_s$ are the normal fluid and superfluid velocities at $\mathbf{s}$, and β, β' are temperature and Reynolds number dependent mutual friction coefficients. The superfluid velocity $\mathbf{v}_s$ is computed via a desingularized Biot-Savart integral, accelerated using the tree algorithm [3] (see Supplementary Materials). We describe the normal fluid in a classical way, using the incompressible ($\nabla \cdot \mathbf{v}_n = 0$) Navier-Stokes equation

$$\frac{\partial\mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n = -\frac{1}{\rho}\nabla p + \nu_n\nabla^2\mathbf{v}_n + \frac{\mathbf{F}_{ns}}{\rho_n} \quad (2)$$

where ρ_n and ρ_s are the normal fluid and superfluid densities, $\rho = \rho_n + \rho_s$, p is the pressure, ν_n is the kinematic viscosity of the normal fluid and $\mathbf{F}_{ns}$ is the mutual friction force per unit volume.

In this study, we prepare two distinct turbulent vortex tangles generated by (a) a $T = 2.1$ K thermal counterflow (CF) with an average inter-vortex distance ℓ_{CF} and (b) a $T = 0$ K evolution of a tangle with a Taylor-Green initial condition (TG) with an average inter-vortex spacing ℓ_{TG}. Both of these tangles are visualized in the left
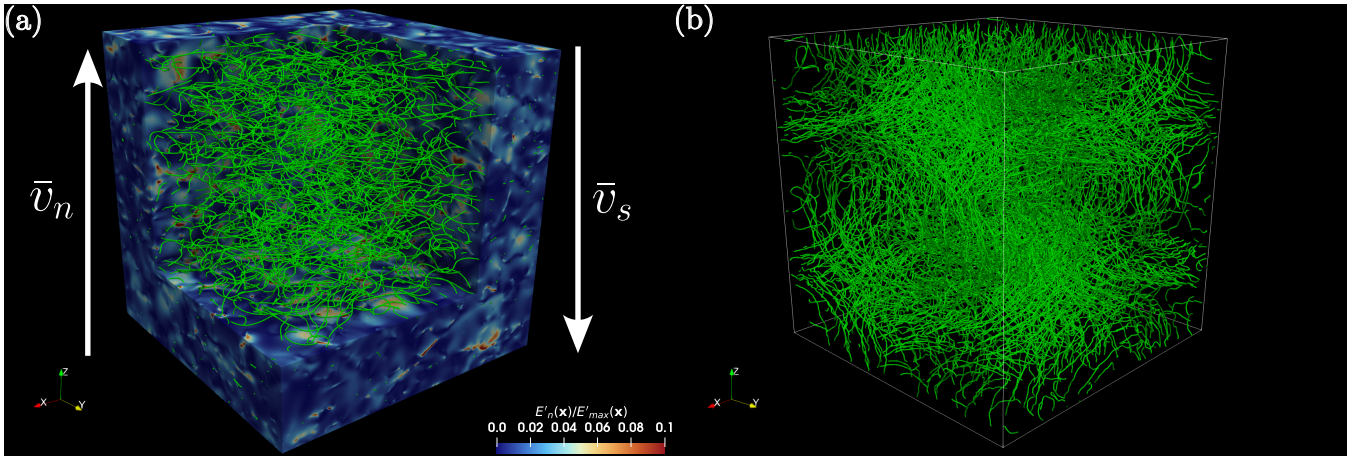
FIG. 1: Visualisation of turbulent vortex tangles. (a) Counterflow-induced turbulence generated at $T = 2.1$ K with a counterflow velocity of $v_{ns} = \bar{v}_n - \bar{v}_s = 0.94$ cm/s. The blue and red surface rendering shows the normalised kinetic energy density of normal fluid fluctuations E'_n/E'_{max} where $E'_n = |\mathbf{v}_n - \bar{\mathbf{v}}_n|^2$. ($b$) Turbulence generated by an initial Taylor-Green configuration (see Supplementary Materials for details). In both (a) and (b) superfluid vortex lines are represented as green tubes with exaggerated size.

and right panels of Fig. 1 respectively. The velocity circulation statistics of $\mathbf{v}_s$ are obtained by vorticity coarse-graining process described in the Supplementary materials. Furthermore, the statistics related to the counterflow are decomposed into the normal fluid (NF) and superfluid (SF) components. Each of these components are further decomposed into the perpendicular ($\perp$) and ($\parallel$) components, corresponding to the directions over which statistics are averaged. Perpendicular (parallel) relates to the statistics averaged in the direction where the flux of vorticity through a square loop of size r is perpendicular (parallel) to the counterflow velocity $\mathbf{v}_{ns}$.

RESULTS

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[2] KW. Schwarz, Three-dimensional vortex dynamics in superfluid ^{4}He, Phys. Rev. B **38**, 2398 (1988).

[3] A. W. Baggaley and C. F. Barenghi, Tree Method for Quantum Vortex Dynamics, J. Low Temp. Phys. **166**, 3 (2012).
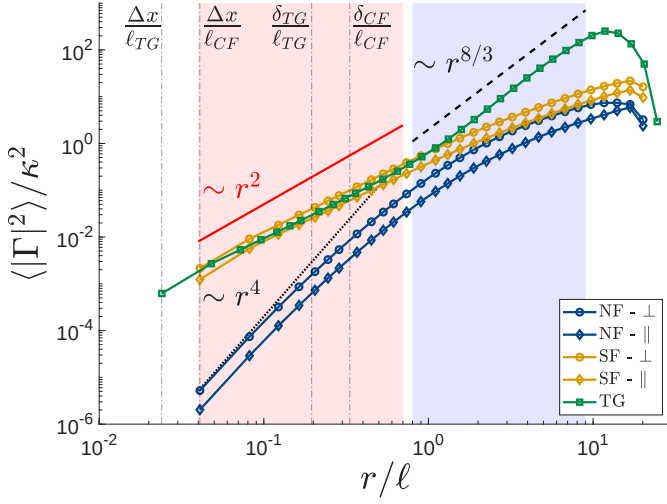
FIG. 2: Circulation variance $\langle |\Gamma|^2 \rangle$ around square loops of size r. The blue and yellow curves show the normal fluid and super fluid simulations respectively, which are subdivided into the perpendicular (circles) and parallel (diamonds) components. The green squares show the Taylor-Green simulation.
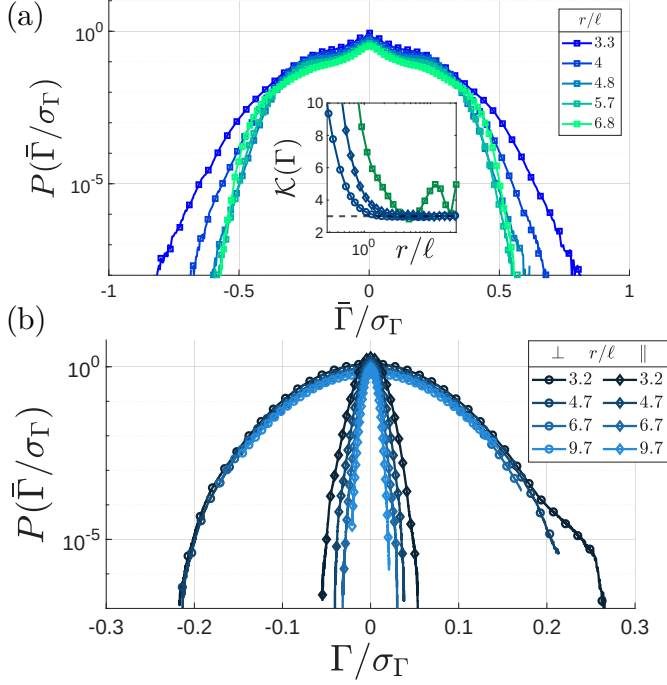

FIG. 3: PDFs of the velocity circulation for different loop sizes $r/\ell > 1$, scaled by the standard deviation σ_Γ. (a) shows the Taylor-Green simulation, while (b) shows both of the normal fluid parallel (diamonds) and perpendicular (circles) components. The inset of (a) shows the flatness $\mathcal{K}(\Gamma) = \langle |\Gamma|^4 \rangle / \sigma_\Gamma^4$, with the same color scheme as in Fig. 2. The black dashed line marks the Gaussian flatness of $\mathcal{K} = 3$.
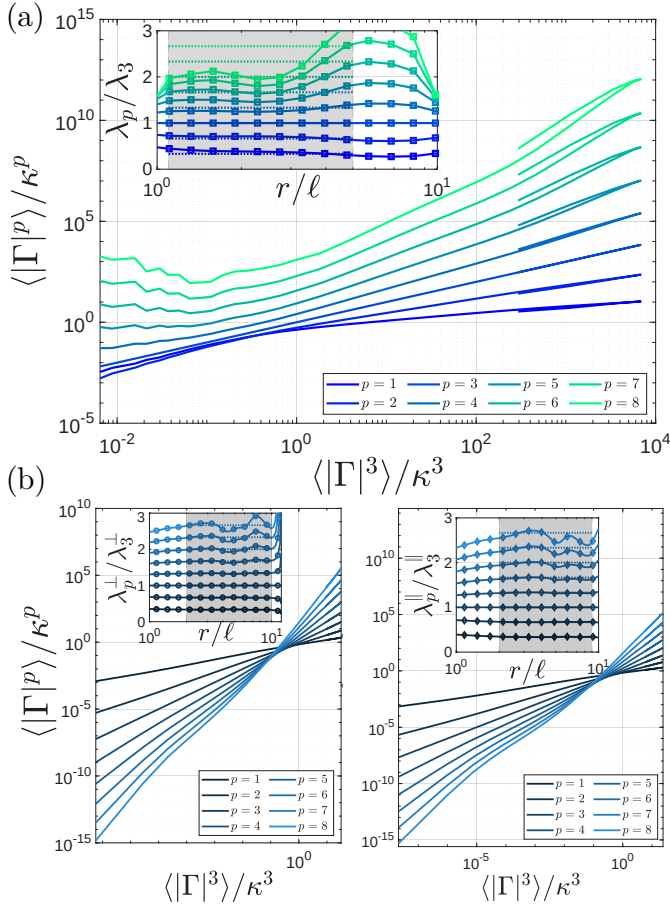
FIG. 4: Circulation moments $\langle|\Gamma|^p\rangle$ in extended self-similarity (ESS) using $p = 3$. (a) shows the Taylor-Green simulation, while the left and right panels of (b) are the normal fluid perpendicular and parallel component respectively. The insets of each plot give the local slope, where $\lambda_p/\lambda_3 = d\log\langle|\Gamma|^p\rangle/d\log\langle|\Gamma|^3\rangle$ and the horizontal lines correspond to the K41 scalings $\lambda_p^{K41} = 4p/3$. The shaded regions indicate the classical region where the data used to estimate the exponents.
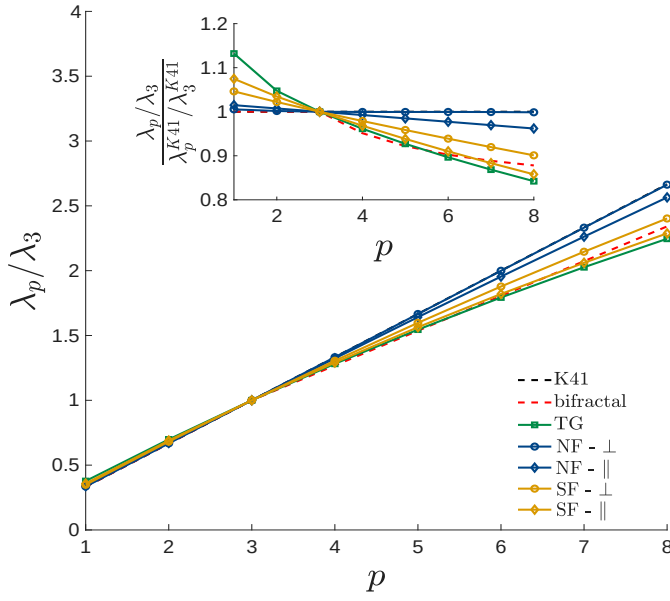
FIG. 5: Scaling exponents of velocity circulation moments, estimated within the classical range $r > \ell$. The colour scheme and linestyles is the same as in Fig. 2. The black dashed line represents the self-similar K41 scaling $\lambda_p^{K41} = 4p/3$. The inset shows the proportional deviation from K41 theory.