

# Velocity circulation statistics in counterflow turbulence

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## INTRODUCTION

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## METHOD

Superfluid dynamics are modelled by employing the recently developed FOUCAULT model [1]. We follow Schwarz's approach [2], by exploiting the large separation of scales between the  $^4\text{He}$  vortex core size  $a_0$  and the average inter-vortex spacing  $\ell$ . We parametrize the superfluid vortex lines as 1D space curves  $\mathbf{s}(\xi, t)$  with  $\xi$  and  $t$  the arclength and time respectively. The corresponding equation of motion is

$$\dot{\mathbf{s}}(\xi, t) = \mathbf{v}_{s\perp} + \beta \mathbf{s}' \times \mathbf{v}_{ns} + \beta' \mathbf{s}' \times [\mathbf{s}' \times \mathbf{v}_{ns}] \quad (1)$$

where  $\mathbf{s}' = \partial \mathbf{s} / \partial \xi$  is the unit tangent vector at  $\mathbf{s}$ ,  $\mathbf{v}_{ns} = \mathbf{v}_n - \mathbf{v}_s$  where  $\mathbf{v}_n$  and  $\mathbf{v}_s$  are the normal fluid and superfluid velocities at  $\mathbf{s}$ , and  $\beta, \beta'$  are temperature and Reynolds number dependent mutual friction coefficients. The superfluid velocity  $\mathbf{v}_s$  is computed via a desingularized Biot-Savart integral, accelerated using the tree algorithm [3] (see Supplementary Materials). We describe the normal fluid in a classical way, using the

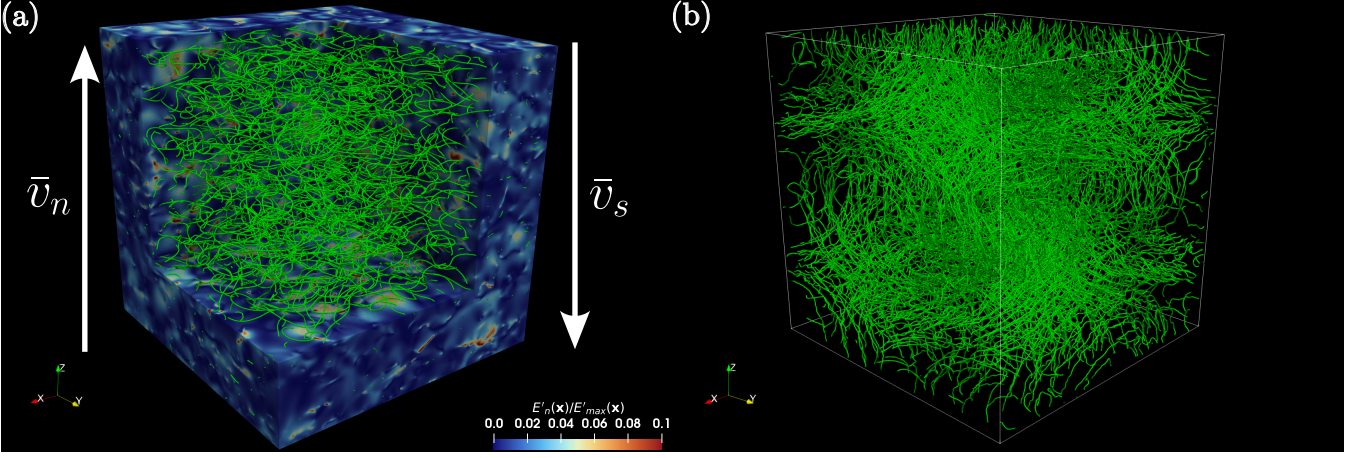


FIG. 1: Visualisation of turbulent vortex tangles. (a) Counterflow-induced turbulence generated at  $T = 2.1$  K with a counterflow velocity of  $v_{ns} = \bar{v}_n - \bar{v}_s = 0.94$  cm/s. The blue and red surface rendering shows the normalised kinetic energy density of normal fluid fluctuations  $E'_n/E'_{max}$  where  $E'_n = |\mathbf{v}_n - \bar{\mathbf{v}}_n|^2$ . (b) Turbulence generated by an initial Taylor-Green configuration (see Supplementary Materials for details). In both (a) and (b) superfluid vortex lines are represented as green tubes with exaggerated size.

incompressible ( $\nabla \cdot \mathbf{v}_n = 0$ ) Navier-Stokes equation

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n = -\frac{1}{\rho} \nabla p + \nu_n \nabla^2 \mathbf{v}_n + \frac{\mathbf{F}_{ns}}{\rho_n} \quad (2)$$

where  $\rho_n$  and  $\rho_s$  are the normal fluid and superfluid densities,  $\rho = \rho_n + \rho_s$ ,  $p$  is the pressure,  $\nu_n$  is the kinematic viscosity of the normal fluid and  $\mathbf{F}_{ns}$  is the mutual friction force per unit volume.

In this study, we prepare two distinct turbulent vortex tangles generated by (a) a  $T = 2.1$  K thermal counterflow (CF) with an average inter-vortex distance  $\ell_{CF}$  and (b) a  $T = 0$  K evolution of a tangle with a Taylor-Green initial condition (TG) with an average inter-vortex spacing  $\ell_{TG}$ . Both of these tangles are visualized in the left and right panels of Fig. 1 respectively. The velocity circulation statistics of  $\mathbf{v}_s$  are obtained by a vorticity coarse-graining process described in the Supplementary materials. Furthermore, the statistics related to the counterflow are decomposed into the normal fluid (NF) and superfluid (SF) components. Each of these components are further decomposed into the perpendicular ( $\perp$ ) and ( $\parallel$ ) components, corresponding to the directions over which statistics are averaged. Perpendicular (parallel) relates to the statistics averaged in the direction where the flux of vorticity through a square loop of size  $r$  is perpendicular (parallel) to the counterflow velocity  $\mathbf{v}_{ns}$ .

## RESULTS

Firstly, we begin by considering the simplest observable for velocity circulation statistic - the variance  $\langle |\Gamma|^2 \rangle$  for different loop sizes  $r$ . The circulation variance scalings are shown in Fig. 2, where we have identified two

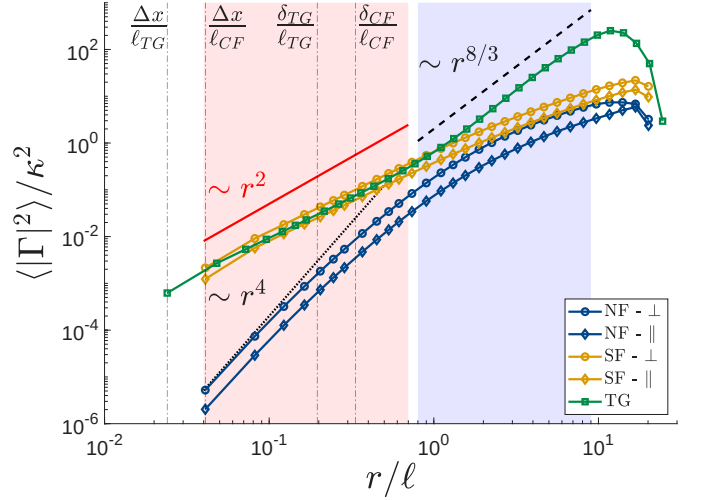


FIG. 2: Circulation variance  $\langle |\Gamma|^2 \rangle$  around square loops of size  $r$ . The blue and yellow curves show the normal fluid and super fluid simulations respectively, which are subdivided into the perpendicular (circles) and parallel (diamonds) components. The green squares show the Taylor-Green simulation.

scaling regimes, a quantum regime  $r \ll \ell$  with red shading and a classical regime  $r \gtrsim \ell$  with blue shading. Indeed in the quantum regime we find good agreement for the superfluid velocity statistics with the small-scale theory  $\langle |\Gamma|^2 \rangle \sim r^2$  as discussed in Ref. [4] using the Gross-Pitaevskii (GP) model. We observe that in this regime the scaling not only follows the GP theory, but the quantum region scaling is independent of thermal effects and the nature of the turbulent driving force. Similarly, the normal fluid also exhibits the well known viscous scal-

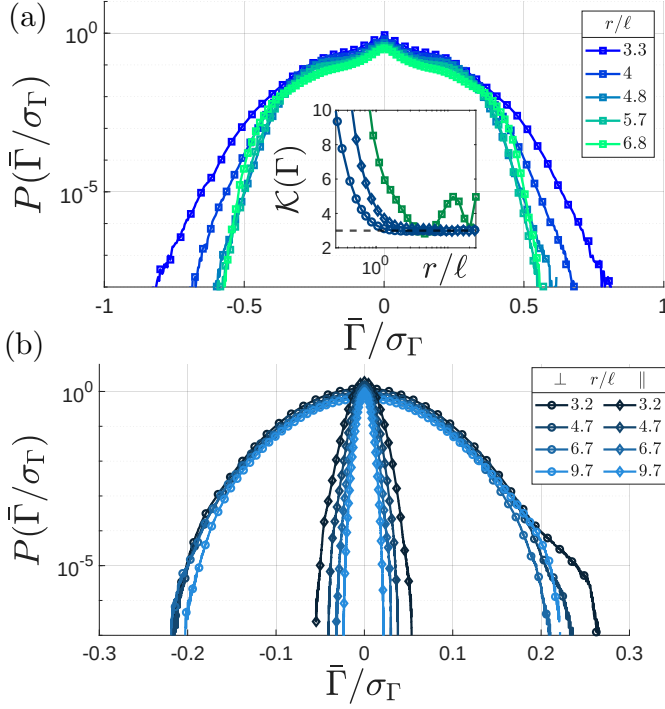


FIG. 3: PDFs of the velocity circulation for different loop sizes  $r/\ell > 1$ , scaled by the standard deviation  $\sigma_\Gamma$ . (a) shows the Taylor-Green simulation, while (b) shows both of the normal fluid parallel (diamonds) and perpendicular (circles) components. The inset of (a) shows the flatness  $\mathcal{K}(\Gamma) = \langle |\Gamma|^4 \rangle / \sigma_\Gamma^4$ , with the same color scheme as in Fig. 2. The black dashed line marks the Gaussian flatness of  $\mathcal{K} = 3$ .

ing of  $\langle |\Gamma|^2 | \rangle \sim r^4$  for small loop sizes  $r$ , as a result of smoothness at very small scales [5].

The cross-over from the quantum  $r \ll \ell$  to the classical  $r \gtrsim \ell$  marks a stark contrast in the superfluid circulation variance scaling. While the TG scales in accordance with a partial polarization of vortices where  $r^{8/3}$  [6], the CF scaling, appears to follow a scaling of  $\sim r^{1.58}$ , a weaker scaling than for a fully random tangle of vortex lines which exhibits  $\sim r^2$  scaling in the classical range. Remarkably both components of the SF and NF statistics of CF appear to adhere to this scaling law for loops  $r \gtrsim \ell$ .

For the remainder of this paper, we will only focus on the classical range where the loop sizes  $r > \ell$ . In fully-developed isotropic classical turbulence, velocity circulation statistics adhere to a strictly non-Gaussian distribution with long tails [7]. Deviations from Gaussian statistics are the hallmark of intermittency, and represent a deviation from self-similar statistics which are evident in the TG simulations shown in Fig. 3a.

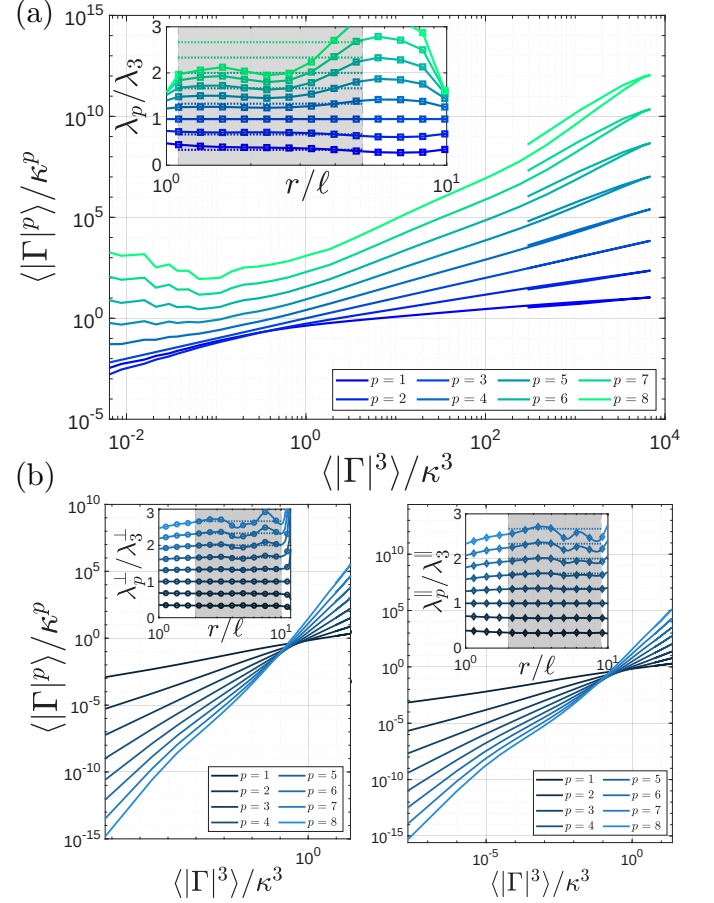


FIG. 4: Circulation moments  $\langle |\Gamma|^p \rangle$  in extended self-similarity (ESS) using  $p = 3$ . (a) shows the Taylor-Green simulation, while the left and right panels of (b) are the normal fluid perpendicular and parallel component respectively. The insets of each plot give the local slope, where  $\lambda_p/\lambda_3 = d \log \langle |\Gamma|^p \rangle / d \log \langle |\Gamma|^3 \rangle$  and the horizontal lines correspond to the K41 scalings  $\lambda_p^{K41} = 4p/3$ . The shaded regions indicate the classical region where the data used to estimate the exponents.

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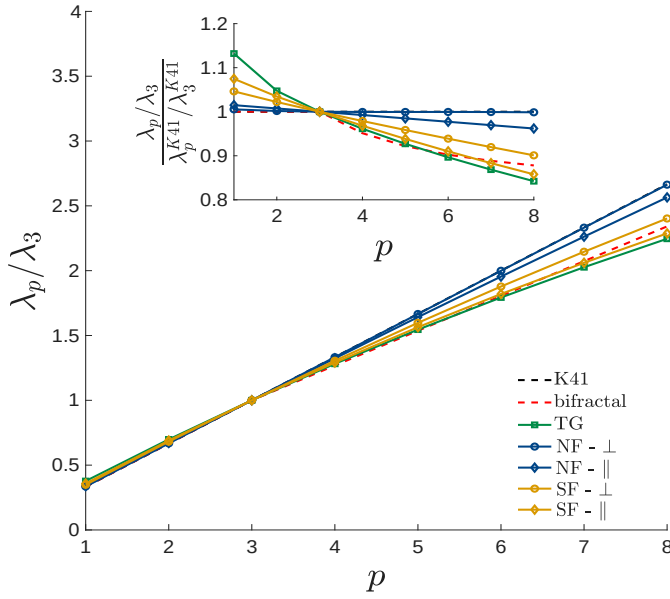


FIG. 5: Scaling exponents of velocity circulation moments, estimated within the classical range  $r > \ell$ . The colour scheme and linestyles is the same as in Fig. 2. The black dashed line represents the self-similar K41 scaling  $\lambda_p^{K41} = 4p/3$ . The inset shows the proportional deviation from K41 theory.

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