

# Supplementary Materials: Inverse energy transfer in finite-temperature superfluid vortex reconnections

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## NUMERICAL METHOD

Using Schwarz mesoscopic model [1], vortex lines can be described as space curves  $\mathbf{s}(\xi, t)$  of infinitesimal thickness, with a single quantum of circulation  $\kappa = h/m_4 = 9.97 \times 10^{-8} \text{m}^2/\text{s}$ , where  $h$  is Planck’s constant,  $m_4 = 6.65 \times 10^{-27} \text{kg}$  is the mass of one helium atom,  $\xi$  is the natural parameterisation, arclength, and  $t$  is time. These conditions are a good approximation, since the vortex core radius of superfluid  $^4\text{He}$  ( $a_0 = 10^{-10} \text{m}$ ) is much smaller than any of the length scale of interest in turbulent flows. The equation of motion is

$$\dot{\mathbf{s}}(\xi, t) = \mathbf{v}_s + \frac{\beta}{1 + \beta} [\mathbf{v}_{ns} \cdot \mathbf{s}'] \mathbf{s}' + \beta \mathbf{s}' \times \mathbf{v}_{ns} + \beta' \mathbf{s}' \times [\mathbf{s}' \times \mathbf{v}_{ns}], \quad (1)$$

where  $\dot{\mathbf{s}} = \partial \mathbf{s} / \partial t$ ,  $\mathbf{s}' = \partial \mathbf{s} / \partial \xi$  is the unit tangent vector,  $\mathbf{v}_{ns} = \mathbf{v}_n - \mathbf{v}_s$ ,  $\mathbf{v}_n$  and  $\mathbf{v}_s$  are the normal fluid and superfluid velocities at  $\mathbf{s}$  and  $\beta, \beta'$  are temperature and Reynolds number dependent mutual friction coefficients [2]. The superfluid velocity  $\mathbf{v}_s$  at a point  $\mathbf{x}$  is determined by the Biot-Savart law

$$\mathbf{v}_s(\mathbf{x}, t) = \frac{\kappa}{4\pi} \oint_{\mathcal{T}} \frac{\mathbf{s}'(\xi, t) \times [\mathbf{x} - \mathbf{s}(\xi, t)]}{|\mathbf{x} - \mathbf{s}(\xi, t)|} d\xi, \quad (2)$$

where  $\mathcal{T}$  represents the entire vortex configuration. There is currently a lack of a well-defined theory of vortex reconnections in superfluid helium, like for the Gross-Pitaevskii equation [3–5]. An *ad hoc* vortex reconnection algorithm is employed to resolve the collisions of vortex lines [6].

A *two-way model* is crucial to understand the accurately intereprt the back-reaction effect of the normal fluid on the vortex line and vice-versa [7]. We self-consistently evolve the normal fluid  $\mathbf{v}_n$  with a modified Navier-Stokes equation

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n = -\nabla \frac{p}{\rho} + \nu_n \nabla^2 \mathbf{v}_n + \frac{\mathbf{F}_{ns}}{\rho_n}, \quad (3)$$

$$\mathbf{F}_{ns} = \oint_{\mathcal{T}} \mathbf{f}_{ns} \delta(\mathbf{x} - \mathbf{x}) d\xi, \quad \nabla \cdot \mathbf{v}_n = 0, \quad (4)$$

where  $\rho = \rho_n + \rho_s$  is the total density,  $\rho_n$  and  $\rho_s$  are the normal fluid and superfluid densities,  $p$  is the pressure,  $\nu_n$  is the kinematic viscosity of the normal fluid and  $\mathbf{f}_{ns}$  is the local friction per unit length [8]

$$\mathbf{f}_{ns} = -D \mathbf{s}' \times [\mathbf{s}' \times (\dot{\mathbf{s}} - \mathbf{v}_n)] - \rho_n \kappa \mathbf{s}' \times (\mathbf{v}_n - \dot{\mathbf{s}}), \quad (5)$$

where  $D$  is a coefficient dependent on the vortex Reynolds number and intrinsic properties of the normal fluid.

The results in this Letter are reported in dimensionless units, where the characteristic length scale is  $\tilde{\lambda} = D/D_0$ , where  $D^3 = (1 \times 10^{-3} \text{m})^3$  is the dimensional cube size,  $D_0^3 = (2\pi)^3$  is the non-dimensional cubic computational domain. The time scale is given by  $\tilde{\tau} = \tilde{\lambda}^2 \nu_n^0 / \nu_n$ , where the non-dimensional viscosity  $\nu_n^0$  resolves the small scales of the normal fluid. In these simulations, these quantities are  $\tilde{\lambda} = 1.59 \times 10^{-4} \text{cm}$ ,  $\nu_n^0 = 0.16$  and  $\tilde{\tau} = 0.183 \text{s}$  at  $T = 0 \text{K}$  and at  $T = 1.9 \text{K}$  and  $\tilde{\tau} = 0.242 \text{s}$  at  $T = 2.1 \text{K}$ .

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