Supplementary Materials: Inverse energy transfer in finite-temperature superfluid vortex reconnections

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NUMERICAL METHOD

Using Schwarz mesoscopic model [1], vortex lines can be described as space curves $\mathbf{s}(\xi, t)$ of infinitesimal thickness, with a single quantum of circulation $\kappa = h/m_4 = 9.97 \times 10^{-8}\mathrm{m}^2/\mathrm{s}$, where h is Planck's constant, $m_4 = 6.65 \times 10^{-27}\mathrm{kg}$ is the mass of one helium atom, ξ is the natural parameterisation, arclength, and t is time. These conditions are a good approximation, since the vortex core radius of superfluid $^4\mathrm{He}(a_0 = 10^{-10}\mathrm{m})$ is much smaller than any of the length scales of interest in turbulent flows. The equation of motion is

$$\dot{\mathbf{s}}(\xi, t) = \mathbf{v}_s + \frac{\beta}{1+\beta}\left[\mathbf{v}_{ns} \cdot \mathbf{s}'\right]\mathbf{s}' + \beta \mathbf{s}' \times \mathbf{v}_{ns} + \beta' \mathbf{s}' \times \left[\mathbf{s}' \times \mathbf{v}_{ns}\right], \tag{1}$$

where $\dot{\mathbf{s}} = \partial \mathbf{s}/\partial t$, $\mathbf{s}' = \partial \mathbf{s}/\partial \xi$ is the unit tangent vector, $\mathbf{v}_{ns} = \mathbf{v}_n - \mathbf{v}_s$, $\mathbf{v}_n$ and $\mathbf{v}_s$ are the normal fluid and superfluid velocities at $\mathbf{s}$ and β,β' are temperature and Reynolds number dependent mutual fricition coefficients [2]. The superfluid velocity $\mathbf{v}_s$ at a point $\mathbf{x}$ is determined by the Biot-Savart law

$$\mathbf{v}_s(\mathbf{x}, t) = \frac{\kappa}{4\pi}\oint_{\mathcal{T}} \frac{\mathbf{s}'(\xi, t) \times [\mathbf{x} - \mathbf{s}(\xi, t)]}{|\mathbf{x} - \mathbf{s}(\xi, t)|} d\xi, \tag{2}$$

where $\mathcal{T}$ represents the entire vortex configuration. There is currently a lack of a well-defined theory of vortex reconnections in superfluid helium, like for the Gross-Pitaevskii equation [3–5]. An *ad hoc* vortex reconnection algorithm is employed to resolve the collisions of vortex lines [6].

A *two-way model* is crucial to understand the accurately interept the back-reaction effect of the normal fluid on the vortex line and vice-versa [7]. We self-consistently evolve the normal fluid $\mathbf{v}_n$ with a modified Navier-Stokes equation

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla)\mathbf{v}_n = -\nabla \frac{p}{\rho} + \nu_n \nabla^2 \mathbf{v}_n + \frac{\mathbf{F}_{ns}}{\rho_n}, \tag{3}$$

$$\mathbf{F}_{ns} = \oint_{\mathcal{T}} \mathbf{f}_{ns} \delta(\mathbf{x} - \mathbf{x}) d\xi, \quad \nabla \cdot \mathbf{v}_n = 0, \tag{4}$$

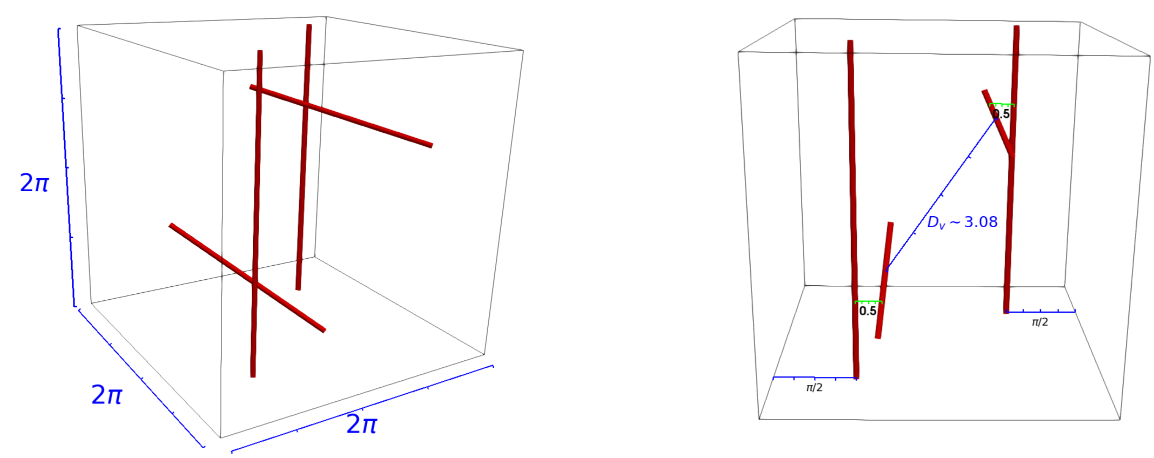

FIG. 1: Schematic diagram of the initial vortex configuration.

where $\rho = \rho_n + \rho_s$ is the total density, ρ_n and ρ_s are the normal fluid and superfluid densities, p is the pressure, ν_n is the kinematic viscosity of the normal fluid and $\mathbf{f}_{ns}$ is the local friction per unit length [8]

$$\mathbf{f}_{ns} = -\mathcal{D}\mathbf{s}' \times [\mathbf{s}' \times (\dot{\mathbf{s}} - \mathbf{v}_n)] - \rho_n \kappa \mathbf{s}' \times (\mathbf{v}_n - \dot{\mathbf{s}}), \tag{5}$$

where $\mathcal{D}$ is a coefficient dependent on the vortex Reynolds number and intrinsic properties of the normal fluid. The regularisation of the mutual fricition force onto the normal fluid grid is physically motivated by the strongly localised injection of vorticity during the momentum exchange of point-like particles and viscous flow in classical fluid dynamics [9, 10]. In short, the localised vorticity induced by the relative motion between the vortex lines and the normal fluid is diffused to discretisation of the grid spacing Δx in a time interval ϵ_R. In this way, the delta-forced fricition as defined in Eq. 5 is regularised by a Gaussian function, the fundamental solution of the diffusion equation. Further details of the method for classical fluids are contained in [9, 10] and for FOUCAULT in [2].

In this Letter, we report all results using dimensionless units, where the characteristic length scale is $\tilde{\lambda} = D/D_0$, where $D^3 = (1 \times 10^{-3}\text{m})^3$ is the dimensional cube size, $D_0^3 = (2\pi)^3$ is the non-dimensional cubic computational domain. The time scale is given by $\tilde{\tau} = \tilde{\lambda}^2 \nu_n^0 \nu_n$, where the non-dimensional viscosity ν_n^0 resolves the small scales of the normal fluid. In these simulations, these quanties are $\tilde{\lambda} = 1.59 \times 10^{-4}\text{m}$, $\nu_n^0 = 0.32$ and $\tilde{\tau} = 0.366\text{s}$ at $T = 1.9K$ and $\tilde{\tau} = 0.485\text{s}$ at $T = 2.1K$. We consider an initial configuration of two pairs of orthongal vortices, initialised as shown in the schematic of Fig. 1. The seperation between vortices in each pair d is set to be $d_v = 0.5$ in dimensionless units, and the shortest distance between pairs is $D_v = \sqrt{(\pi - d_v/2)^2 + \pi^2} \approx 3.08$, so that $d_v \ll D_v$. The Lagrangian discretisation of vortex lines is $\Delta\xi = 0.025$ (a total of 1340 discretisation points across the 4 vortex lines), using a timestep of $\Delta t_{VF} = 5.56 \times 10^{-6}$. For the normal fluid, a total of $N = 256^3$ mesh point were used, with a timestep of $\Delta t_{NS} = 45\Delta t_{VF}$.

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