PSTAT 120A, Summer 2022: Practice Problems 7

Week 5

Conceptual Review

(a) Why is the sum of two random variables also a random variable?

(b) What is the convolution formula?

(c) What is an indicator? How do indicators and expectations mesh?

Problem 1: Sum Useful Results

Prove each of the following results using the convolution formula.

(a) If $X \sim \text{Pois}(\lambda_X)$ and $Y \sim \text{Pois}(\lambda_Y)$ with $X \perp Y$, then $(X+Y) \sim \text{Pois}(\lambda_X+\lambda_Y)$.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

(b) If $X \sim \text{Gamma}(r_x, \lambda)$ and $Y \sim \text{Gamma}(r_y, \lambda)$ with $X \perp Y$, then $(X+Y) \sim \text{Gamma}(r_X + r_Y,\ \lambda)$.

Hint: You will need to use the so-called ***Beta Integral****:*

$$\int_0^1 x^{r-1}(1-x)^{s-1}\,\mathrm{d}x = \frac{\Gamma(r)\cdot\Gamma(s)}{\Gamma(r+s)}$$

Problem 2: The Elevator Problem

Suppose that, in a particular 10-story building, 5 people enter an elevator on the Ground Floor (let us call this "Floor 0"). We assume that people get off the elevator at a random floor, independently of all other people in the elevator. (Assume that nobody leaves on the Ground Floor) In this problem, we shall work toward answering the question: what is the expected number of floors at which the elevator will stop?

a) Let X denote the number of floors at which the elevator will stop. Define appropriate indicators $\mathbb{1}_j$ such that X can be expressed as a sum of these indicators.

Hint: We can assign indicators to people, or assign them to floors. Which will be better?

b) Using your expression from part (a), write $\mathbb{E}(X)$ in terms of $\mathbb{E}(\mathbb{1}_1), \cdots, \mathbb{E}(\mathbb{1}_{10})$. (You don't need to find the expectation of the indicators just yet; you'll do that in the next part.)

c) Now, compute $\mathbb{E}(\mathbb{1}_1), \mathbb{E}(1_2), \dots, \mathbb{E}(1_{10})$, and use this to answer the original question of "what is the expected number of floors at which the elevator will stop?"

Hint: Using symmetry, you can find an expression for $\mathbb{E}(\mathbb{1}_j)$ for an arbitrary $j = 1, 2, \dots, 10$.

Key Takeaway: This problem (hopefully) illustrates one of the many reasons why indicators are very useful, especially in the context of expectations. One can extend this logic to actually compute the *variance* of the number of floors at which the elevator will stop!

Problem 3: Poisson Predictions

(a) What is the probability that exactly 20 calls arrive in a 90-minute interval?

(b) What is the probability that the 2^{nd} and 4^{th} calls arrive within 1 hour of each other?

(c) What is the distribution of the amount of time between the 2^{nd} and 3^{rd} calls **as measured in minutes**?

(d) If T_1 measures the time in minutes until the 1^{st} call and S denotes the time in minutes between the 2^{nd} and 4^{th} calls, what is $f_{T_1,S}(t,s)$, the joint p.d.f. of (T_1, S)?

Extra Problems

Problem 4: Great Expectations

Now that we have learned a bit more about joint distributions, consider the following logic in the context of a bivariate pair (X, Y) of continuous random variables:

- On the one hand, we can integrate out y, find the marginal $f_X(x)$ of X, and then compute
$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, \mathrm{d}x$$
- On the other hand, we can also use the two-dimensional LOTUS with $g(x, y) = x$ to compute
$$\mathbb{E}[X] = \iint_{\mathbb{R}^2} x f_{X,Y}(x, y)\, \mathrm{d}A$$

A question I often get asked is: "which of these is correct?" The answer is, in fact- "both of them!" **Prove that these two formulations of $\mathbb{E}[X]$ are equivalent.**

It may be easier to start with the second formulation, and then show that it is equal to the first.

Problem 5: Hot Cross Moments

Given an $n-$dimensional random vector $\vec{X}$, we define the $\boldsymbol{k_1, \cdots, k_n}$**th cross-moment** (sometimes called a **mixed-moment**) of $\vec{X}$ to be

$$\mu_{k_1, \cdots, k_n}(\vec{X}) := \mathbb{E}\left[\prod_{i=1}^{n} X_i^{k_i}\right] = \mathbb{E}\left[X_1^{k_1} \times X_2^{k_2} \times \cdots \times X_n^{k_n}\right]$$

For example, the $(3, 5)$ cross moment of a bivariate random vector is

$$\mu_{3,5}(\vec{X}) = \mathbb{E}\left[X_1^3 \cdot X_2^5\right]$$

(a) Suppose the elements of an $n-$dimensional random vector $\vec{X}$ are independent. Additionally, let $\mu_{k_i}(X_i) := \mathbb{E}[X_i^{k_i}]$ denote the k_ith moment of X_i. Derive a relationship between $\mu_{k_1, \cdots, k_n}(\vec{X})$ and the $\mu_{k_i}(X_i)'s$.

(b) Is it true that for two $n-$dimensional random vectors $\vec{X}$ and $\vec{Y}$

$$\mu_{k_1, \cdots, k_n}(\vec{X} + \vec{Y}) = \mu_{k_1, \cdots, k_n}(\vec{X}) = +\mu_{k_1, \cdots, k_n}(\vec{Y})$$

If so, provide a brief proof. If not, explain why not.