HOMEWORK 5

PSTAT 120A: Summer 2022

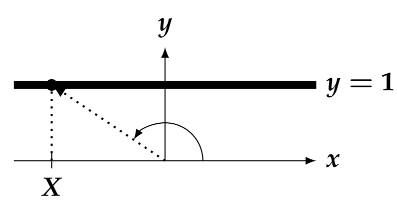
Due: 11:59pm on Wednesday, July 13
Instructor: Ethan P. Marzban

Instructions:

- Please submit your work to Gradescope by no later than the due date posted above.
- Be sure to show your work; correct answers with no supporting work will not be awarded full points.
- 2 randomly selected questions/parts will be graded, but you must still turn in your work for all problems in order to be eligible to earn full credit.

1. In each of the following parts you are supplied a random variable X with a provided probability density function (p.d.f.), along with a new random variable Y defined to be a particular function of X. Find the probability density function (p.d.f.) of Y. You may use either the c.d.f. method or the Change of Variable formula; just be sure to show all of your work. Additionally, be sure to specify the values over which your p.d.f. is nonzero.
   (a) $X \sim \text{Unif}[0, 2]$; $Y := X^2$
   (b) $X \sim \text{Unif}[-2, 2]$ $Y := X^2$
   (c) $X \sim \mathcal{N}(0, 1)$; $Y := e^X$. The distribution of Y is called the **Lognormal** distribution.
   (d) $X \sim \text{Exp}(\lambda)$; $Y := X^\beta$ for some fixed $\beta > 0$. The distribution of Y is called the **Weibull** distribution.

2. A particle is fired from the origin in a random direction pointing somewhere in the first two quadrants. The particle travels in a straight line, unobstructed, until it collides with an infinite wall located at $y = 1$. Let X denote the $x-$coordinate of the point of collision.



   (a) What is the expected value of the $x-$coordinate of the point of collision? **Do NOT first find the p.d.f. of X.**
   (b) Find $f_X(x)$, the probability density function (p.d.f.) of X
   (c) Confirm your answer to part (a) using your answer to part (b).

3. **Insurance Deductibles**. Here is a quick crash-course on how deductibles work. Suppose the insurance policy you purchased on your car comes with a \$500 deductible. Then, if you get into an accident the amount you have to pay out-of-pocket follows the following scheme: if the true cost of damages is under \$500 then you pay the full cost of damages, but if the true cost of damages is over \$500 then you only pay \$500 (and your insurance company pays the rest). So, if the true cost of damages is say \$1,000 then you only pay \$500.

   Suppose now that your deductible is m, where m is a fixed positive constant. Let X denote the true cost of damages of a particular accident, and let Y denote the amount of money you actually pay as a result of that accident. Further suppose that X is well-modeled by an $\text{Exp}(\lambda)$ distribution for some $\lambda > 0$.

(a) Express Y as a function of X. In other words, find an explicit formulation for the function $g(k)$ such that $Y = g(X)$.

(b) What is the expected amount of money you will have to pay?

(c) Find $F_Y(y)$, the cumulative distribution function (c.d.f.) of Y. **Two Hints:**

- When computing $\mathbb{P}(Y \leq y)$, use the Law of Total Probability with the partition $\{\{X \leq m\}, \{X > m\}\}$.
- Note that $\mathbb{P}(m \leq y) = \mathbb{1}_{\{y \geq m\}}$, since both y and m are deterministic. (Alternatively, you can avoid using this hint by simply considering two cases separately: $y \leq m$ and $y > m$)

(d) Is Y continuous, discrete, or neither?

4. **Double Integrals:** No plugging into WolframAlpha on this question; show **ALL** of your work!

(a) Compute $\displaystyle\int_0^1 \int_0^2 xy \, \mathrm{d}x \, \mathrm{d}y$

(b) Compute $\displaystyle\int_0^\infty \int_x^\infty e^{-y^2} \, \mathrm{d}y \, \mathrm{d}x$

(c) Compute $\displaystyle\iint_{\mathcal{R}} x^2 y^2 \, \mathrm{d}A$ where $\mathcal{R}$ is the region $\mathcal{R} := \{(x, y) : |x| + |y| \leq 1\}$