

PSTAT 120A, Summer 2022: Practice Problems 2

Week 2

Conceptual Review

- (a) Intuitively, what does $\mathbb{P}(A \mid B)$ represent?
- (b) What is the definition of independence? What is the intuition behind this definition?
- (c) Does pairwise independence imply mutual independence? Does mutual independence imply pairwise independence?
- (d) What type of mathematical object is a Random Variable?

Problem 1: Proving Independence

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two events $A, B \in \mathcal{F}$. Show that if $A \perp B$, then $A^c \perp B^c$.

Problem 2: Conditional Complements

(modified from ASV 2.7)

- a) Argue that $\{A^c \cap B, A \cap B\}$ forms a partition of the event B .
- b) Show that $\mathbb{P}(A^c \mid B) = 1 - \mathbb{P}(A \mid B)$.
- c) Suppose $\mathbb{P}(A \mid B) = 0.6$ and $\mathbb{P}(B) = 0.5$. Find $\mathbb{P}(A^c \cap B)$.
- d) Suppose now that $A \subseteq B$. Find a simple formula for $\mathbb{P}(A \mid B^c)$.

Hint: You can either use mathematical arguments, or sketch a Venn Diagram.

Problem 3: Selecting Words

A word is selected at random from the sentence

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Then, a letter is selected at random from the chosen word.

- a) What is the probability that the letter “S” is selected?
- b) If X denotes the length of the chosen word, what is the PMF of X ?
- c) Continuing from part (b); what is $\mathbb{E}[X]$?
- d) Continuing from part (b); what is $\mathbb{E}\left[\frac{1}{X}\right]$?
- e) For every vowel in your selected word, you are awarded \$1; for every consonant, however, you are forced to pay \$1. Letting W denote your net gain/loss, what is $\mathbb{E}[W]$?

Hint: Try finding the PMF of W first.

Extra Problems

Problem 4: Pólya's Urn Scheme

A box contains n marbles, b of which are blue and $g := n - b$ of which are gold. A marble is drawn at random and its color is noted; the marble is then placed back into the box along with k additional marbles of the same color (so now there are $n + k + 1$ total marbles in the box). Now, another marble is drawn; find the probability that it is blue.

Problem 5: Number of Conditions

Recall that when establishing the independence of n events, there are a series of computations we must perform (i.e. the “two-way intersections,” “three-way intersections,” etc.) Show that there are a total of $2^n - n - 1$ computations involved in establishing the independence of n events.

Hint: Recall the Binomial Theorem:

$$(a + b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Problem 6: I Like to Prove It Prove It!

Prove the following:

- (a) If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.
- (b) If $A \subseteq B$, then $\mathbb{P}(B \mid A) = 1$. Provide both a mathematical proof, as well as an intuitive one.
- (c) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an event B with $\mathbb{P}(B) \neq 0$, the measure $\mathbb{P}_B(\cdot)$ defined through

$$\mathbb{P}_B(A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

is a valid probability measure.

Hint: Partition $A \cup B$; then note that $A \cup B = B$ since $A \subseteq B$.

Hint: All you need to show is that $\mathbb{P}_B(\cdot)$ satisfies the three axioms of probability. Additionally, we know that $\mathbb{P}(\cdot)$ satisfies the axioms of probability.