PSTAT 120A, Summer 2022: Practice Problems 4: Midterm Review

Week 2

Conceptual Review

(a) Review the conceptual questions from the previous Discussion Worksheets!

Problem 1: Stamps!

Morgan is an avid philatelist[1], who is trying to get their hands on a very rare stamp. They know that only 2% of stores in their hometown sell this stamp, so they go from store to store trying to find this stamp. Out of desperation, Morgan will sometimes visit the same store twice.

(a) On average, how many stores (including the final store) does Morgan need to visit in order to collect 3 copies of this rare stamp?

> **Solution:** Let X denote the number of stores, including the final store, Morgan has to visit in order to collect their third stamp; then $X \sim \text{NegBin}(3, 0.02)$ and
>
> $$\mathbb{E}[X] = \frac{3}{0.02} = \boxed{150 \text{ stores}}$$

(b) What is the probability that among the first 10 stores Morgan visits none of them have the rare stamp?

> **Solution:** There are two ways to do this problem.
>
> **(1)** Let Y denote the number of rare stamps Morgan finds among the first 10 stores they visit; then $Y \sim \text{Bin}(10, 0.02)$ and
>
> $$\mathbb{P}(Y = 0) = \binom{10}{0}(0.02)^0(0.98)^{10} = \boxed{(0.98)^{10}}$$
>
> **(2)** Alternatively, we could have let Z denote the number of stores, including the final store, Morgan needs to visit in order to obtain a copy of the rare stamp; then $Z \sim \text{Geom}(0.02)$ and
>
> $$\begin{aligned}\mathbb{P}(Z > 10) = \mathbb{P}(Z \geq 11) &= \sum_{k=11}^{\infty} (0.98)^{k-1}(0.02) \\ &= \frac{0.02}{0.98} \times \frac{(0.98)^{11}}{(0.02)} = \boxed{(0.98)^{10}}\end{aligned}$$

[1] A philatelist is someone who collects stamps

Problem 2: Poisson Predicament

In *GauchoVille*, the number of earthquakes follows a Poisson Process with rate 2 per year. Meteorologists are interested in tracking the number of earthquakes over time.

(a) What is the probability that *GauchoVille* will experience exactly 4 earthquakes in the next two years?

Solution: Let X denote the number of earthquakes occurring over the next two years; then $X \sim \text{Pois}(2 \cdot 2)$ and so

$$\mathbb{P}(X = 4) = e^{-4} \cdot \frac{4^4}{4!}$$

(b) What is the average length of time (in years) that elapses between the 3^{rd} and 5^{th} earthquakes?

Solution: Let T denote the length of time between the 3^{rd} and 5^{th} earthquakes; then $T \sim \text{Gamma}(2, 2)$ and so

$$\mathbb{E}[T] = \frac{2}{2} = \text{1 year}$$

Problem 3: Probabilistic Perimeters

A circle of radius R is to be constructed, where R is a random variable with probability density function (p.d.f.) given by

$$f_R(r) = \begin{cases} \frac{3}{r^4} & \text{if } r \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Verify that $f_R(r)$ is a valid p.d.f.

Solution: We check the two requisite conditions: nonnegativity and integrating to unity.

(1) If $r \geq 1$ then $r^4 \geq 0$ and so $f_R(r) \geq 0$ for all $r \in \mathbb{R}$.

(2) We compute

$$\int_1^\infty \frac{3}{r^4} \, \mathrm{d}r = -\left[\frac{1}{r^3}\right]_{r=1}^{r=\infty} = 1 \checkmark$$

(b) Compute $\mathbb{P}(R \geq 3 \mid R \geq 2)$

Solution: By the definition of conditional probability,

$$\mathbb{P}(R \geq 3 \mid R \geq 2) = \frac{\mathbb{P}(\{R \geq 3\} \cap \{R \geq 2\})}{\mathbb{P}(R \geq 2)}$$

If R is greater than 3 then it is automatically greater than 2; hence $\{R \geq 3\} \subseteq \{R \geq 2\}$ and so

$$\mathbb{P}(\{R \geq 3\} \cap \{R \geq 2\}) = \mathbb{P}(R \geq 3) = \int_3^\infty \frac{3}{r^4} \, \mathrm{d}r = \frac{1}{27}$$

Additionally,

$$\mathbb{P}(R \geq 2) = \int_2^\infty \frac{3}{r^4} \, \mathrm{d}r = \frac{1}{8}$$

meaning

$$\mathbb{P}(R \geq 3 \mid R \geq 2) = \frac{\mathbb{P}(\{R \geq 3\} \cap \{R \geq 2\})}{\mathbb{P}(R \geq 2)} = \frac{\left(\frac{1}{27}\right)}{\left(\frac{1}{8}\right)} = \boxed{\frac{8}{27}}$$

(c) Find $F_R(r)$, the c.d.f. of R.

Solution: There are two cases to consider:

- If $x < 1$ then $F_X(x) = 0$.
- If $x \geq 1$, then we compute

$$F_X(x) = \int_1^x \frac{3}{r^4} \, \mathrm{d}r = 1 - \frac{1}{x^3}$$

So, putting everything together,

$$F_R(r) = \begin{cases} 0 & \text{if } x < 1 \\ 1 - \frac{1}{x^3} & \text{if } x \geq 1 \end{cases}$$

(d) Letting P denote the perimeter of the circle, find $F_P(p)$, the c.d.f. of P.

Recall: Perimeter is 2π times the radius

Solution: As the hint mentions, $P = 2\pi R$ meaning

$$F_P(p) := \mathbb{P}(P \leq p) = \mathbb{P}(2\pi R \leq p) = \mathbb{P}\left(R \leq \frac{p}{2\pi}\right) = F_R\left(\frac{p}{2\pi}\right)$$

In other words,

$$F_P(p) = \begin{cases} 0 & \text{if } \frac{p}{2\pi} < 1 \\ 1 - \frac{1}{\left(\frac{p}{2\pi}\right)^3} & \text{if } \frac{p}{2\pi} \geq 1 \end{cases} = \begin{cases} 0 & \text{if } p < 2\pi \\ 1 - \frac{8\pi^3}{p^3} & \text{if } p \geq 2\pi \end{cases}$$

(e) Use your answer to part (d) to find $f_P(p)$, the p.d.f. of P.

Solution: We know that in general

$$f_P(p) = \frac{\mathrm{d}}{\mathrm{d}p} F_P(p)$$

If $p < 2\pi$ then $F_P(p) = 0$ and so clearly $f_P(p) = 0$. If $p \geq 2\pi$ then

$$f_P(p) = \frac{\mathrm{d}}{\mathrm{d}p}\left[1 - \frac{8\pi^3}{p^3}\right] = \frac{24\pi^3}{p^4}$$

meaning, putting everything together,

$$f_P(p) = \begin{cases} \frac{24\pi^3}{p^4} & \text{if } p \geq 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Problem 4: Dicey Situation

Consider the experiment of tossing two far 6-sided dice.

(a) Write down a possible outcome space Ω. Be very clear and explicit about your notation!

Solution: Let (x, y) denote "first die landed x and second die landed y," and let i denote "die landed on i" for $i = 1, 2, \cdots, 6$. Then

$$\Omega = [|1 : 6|]^2$$

(b) What is the probability that the first die lands on a number strictly less than the second die?

Solution: For notational convenience, let A denote the event "the first die lands on a number strictly less than the second die." Now, there are a couple of ways to do this problem. One is to list out the elements in Ω explicitly, and identify which outcomes belong to the event A:

$$\begin{aligned}
\Omega = \{&(1,1),\ (1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6), \\
&(2,1),\ (2,2),\ (2,3),\ (2,4),\ (2,5),\ (2,6), \\
&(3,1),\ (3,2),\ (3,3),\ (3,4),\ (3,5),\ (3,6), \\
&(4,1),\ (4,2),\ (4,3),\ (4,4),\ (4,5),\ (4,6), \\
&(5,1),\ (5,2),\ (5,3),\ (5,4),\ (5,5),\ (5,6), \\
&(6,1),\ (6,2),\ (6,3),\ (6,4),\ (6,5),\ (6,6)\}
\end{aligned}$$

$$\implies \mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{15}{36} = \frac{5}{23}$$

Alternatively, we could have appealed to our counting arguments, much like we did in the first homework. In fact, we can generalize slightly:

- If the second die lands on i, for $i = 1, 2, 3, 4, 5, 6$, then there are $6 - i$ possibilities for the first die. Therefore,

$$|A| = \sum_{i=1}^{6} (6 - i) = 36 - \frac{6 \cdot 7}{2} = 15$$

  as we saw before.

Problem 5: Another Dicey Situation *(Pitman, 2.Rev.18)*

Seven dice are rolled. Write down unsimplified expressions for the probabilities of each of the following events:

(a) exactly three sixes;

Solution: Let X denote the number of sixes observed; then $X \sim \text{Bin}(7, 1/6)$ and

$$\mathbb{P}(X = 3) = \binom{7}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^4$$

(b) three of one kind and four of another;

Solution:

- $|\Omega| = 6^7$ (six possibilities for each of the seven dice)
- Pick the number to be the triple; $\binom{6}{1}$
- Then pick the 3 dice to be in the triple: $\binom{7}{3}$
- Pick the number (not the same as the triple) to be the quadruple: $\binom{5}{1}$
- Pick the 4 dice to be in the quadruple: $\binom{4}{4}$

$$\frac{\binom{6}{1}\binom{7}{3} \cdot \binom{5}{1}\binom{4}{4}}{6^7}$$

(c) two fours, two fives, and three sixes;

Solution: Unlike in part (b), we do not need to pick the actual numbers to be in our two doubles and one triple; thus, we only need to pick the dice:

- Pick the two dice to show the number 40: $\binom{7}{2}$
- From the remaining $7 - 2 = 5$ dice, pick two to show the number 5: $\binom{5}{2}$
- From the remaining $5 - 2 = 3$, pick all three to show the number 6: $\binom{3}{3}$

$$\frac{\binom{7}{2}\binom{5}{2}\binom{3}{3}}{6^7}$$

(d) each number appears;

Solution: Since there are only six faces on the die, and seven dice in total, if we want all six faces to be showing we need exactly one pair of dice to be showing the same number, with all of the remaining 5 dice showing a distinct number.

- First, pick the number that will be repeated: $\binom{6}{1}$
- Then, from the seven dice pick the two dice that will show a repeated number: $\binom{7}{2}$
- Finally, permute the other 5 dice in any order: $5!$. Or, you can think of it this way: $\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1}$, to explicitly imagine picking a die to show each of the 5 remaining numbers.

$$\frac{\binom{6}{1}\binom{7}{2} \times 5!}{6^7}$$

(e) the sum of the dice is 9 or more.

Solution:

$$\begin{aligned}
\mathbb{P}(\text{sum} \geq 9) &= 1 - \mathbb{P}(\text{sum} < 9) \\
&= 1 - \mathbb{P}(\text{seven 1's}) - \mathbb{P}(\text{six 1's and a 2}) \\
&= 1 - (1/6)^7 - 7(1/6)^7
\end{aligned}$$

Problem 6: Counting Cards (Again) *(Modified from ASV, 2.8)*

We shuffle a deck of cards and deal three cards (without replacement). Find the probability that the first card is a queen, the second is a king and the third is an ace. **Use methods involving Conditional Probabilities,** as opposed to simply counting favorable outcomes.

Solution: Let E_1 denote the event {the first card is a queen}, E_2 denote the event {the second card is a king}, and E_3 denote the event {the third card is an ace}. We seek $\mathbb{P}(E_1 E_2 E_3)$; using successive applications of the Multiplication Rule we see

$$\begin{aligned}
\mathbb{P}(E_1 E_2 E_3) &= \mathbb{P}(E_2 E_3 \mid E_1)\mathbb{P}(E_1) \\
&= \mathbb{P}(E_3 \mid E_1 E_2) \cdot \mathbb{P}(E_2 \mid E_1) \cdot \mathbb{P}(E_1)
\end{aligned}$$

These conditional probabilities are much easier to intuit:

$$\begin{aligned}
\mathbb{P}(E_1) &= \mathbb{P}(\text{first card queen}) = \frac{4}{52} \\
\mathbb{P}(E_2 \mid E_1) &= \mathbb{P}(\text{king after a queen}) = \frac{4}{51} \\
\mathbb{P}(E_3 \mid E_1 E_2) &= \mathbb{P}(\text{ace after a king and a queen}) = \frac{4}{50}
\end{aligned}$$

Thus, our final answer is

$$\mathbb{P}(E_1 E_2 E_3) = \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = \frac{4^3}{(50)_3} = \frac{8}{16{,}575} \approx 0.000483$$

Problem 7: Prove It!

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and two events $A, B \in \mathcal{F}$ with $A \perp B$. Prove that $A^\complement \perp B$.

Solution: We write

$$\begin{aligned} \mathbb{P}(A^\complement \cap B) &= \mathbb{P}(B) - \mathbb{P}(A \cap B) \\ &= \mathbb{P}(B) - \mathbb{P}(A) \cdot \mathbb{P}(B) \\ &= [1 - \mathbb{P}(A)] \cdot \mathbb{P}(B) = \mathbb{P}(A^\complement) \cdot \mathbb{P}(B) \end{aligned}$$

thereby completing the proof. ■

Problem 8: Crafting with Cosines

Let X be a continuous random variable with probability density function (p.d.f.) given by

$$f_X(x) = \begin{cases} c \cdot |\cos x| & \text{if } x \in (-\pi/2, \pi) \\ 0 & \text{otherwise} \end{cases}$$

where $c > 0$ is an as-of-yet undetermined constant.

(a) Find the value of c.

Solution: We first compute

$$\begin{aligned} \int_{-\pi/2}^{\pi} |\cos(x)| \, \mathrm{d}x &= \int_{-\pi/2}^{\pi/2} \cos(x) \, \mathrm{d}x - \int_{\pi/2}^{\pi} \cos(x) \, \mathrm{d}x \\ &= \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) - \sin(\pi) + \sin\left(\frac{\pi}{2}\right) \\ &= 3 \sin\left(\frac{\pi}{2}\right) = 3 \end{aligned}$$

Thus, we have $c = 1/3$.

(b) Derive an expression for $F_X(x)$, the cumulative distribution function (c.d.f.) of X.

Hint: Your final answer should ahve 4 cases.

Solution: For $x < 0$ we have that $f_X(x) = 0$, meaning $F_X(x) = 0$; for $x > \pi$ we have that $F_X(x) = 1$. Thus, there are only two additional cases to consider:

- If $x \in (-\pi/2, \pi/2)$ then

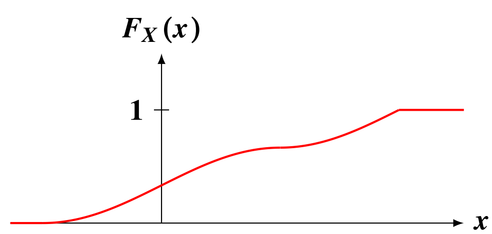
$$F_X(x) = \int_{-\pi/2}^{x} \frac{1}{3} \cos(t) \, \mathrm{d}t = \frac{1}{3} [\sin(x) + 1]$$

- If $x \in (\pi/2, \pi)$ then

$$\begin{aligned} F_X(x) &= \int_{-\pi/2}^{\pi/2} \frac{1}{3}\cos(t) - \frac{1}{3}\int_{-\pi/2}^{x} \cos(x)\ \mathrm{d}x \\ &= \frac{2}{3} - \frac{1}{3}[\sin(x) - 1] = 1 - \frac{1}{3}\sin(x) \end{aligned}$$

Thus, putting everything together:

$$F_X(x) = \begin{cases} 0 & \text{if } x \leq -\pi/2 \\ \frac{1}{3}[\sin(x) + 1] & \text{if } -\pi/2 \leq x \leq \pi/2 \\ 1 - \frac{1}{3}\sin(x) & \text{if } \pi/2 \leq x \leq \pi \\ 1 & \text{if } x \geq \pi \end{cases}$$



Problem 9: Twins *(ASV, 2.30)*

Assume that 1/3 of all twins are identical twins. You learn that Miranda is expecting twins, but you have no other information.

Note that Identical twins must be of the same gender, whereas Fraternal twins may or may not be of the same gender.

Solution: Identical twins have the same gender. We assume that identical twins are equally likely to be boys or girls. Fraternal twins are also equally likely to be boys or girls, but independently of each other. Thus fraternal twins are two girls with probability $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. Let I be the event that the twins are identical, F the event that the twins are fraternal.

(a) Find the probability that Miranda will have two girls.

Solution:

$$\mathbb{P}(\text{two girls}) = \mathbb{P}(\text{two girls} \mid I)\mathbb{P}(I) + \mathbb{P}(\text{two girls} \mid F)\mathbb{P}(F) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{3}$$

(b) You learn that Miranda gave birth to two girls. What is the probability that the girls are identical twins?

> **Solution:**
> $$\mathbb{P}(I \mid \text{two girls}) = \frac{\mathbb{P}(\text{two girls} \mid I)\mathbb{P}(I)}{\mathbb{P}(\text{two girls})} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{3}} = \frac{1}{2}$$

Explain any assumptions you make.

Problem 10: Something Fishy

The weight of a fish caught in a particular pond is well-modeled by a normal distribution with mean 6lbs and standard deviation 2lbs.

(a) What is the probability that a randomly selected fish will exceed 7lbs in weight?

> **Solution:** Let X denote the weight of a randomly selected fish from this pond; then $X \sim \mathcal{N}(6, 4)$, and
> $$\mathbb{P}(X \geq 7) = \mathbb{P}\left(\frac{X-6}{2} \geq \frac{7-6}{2}\right) = 1 - \Phi\left(\frac{1}{2}\right)$$

(b) Zeke claims to have caught a fish at the 90^{th} percentile of weights. How much does Zeke's fish weight?

> **Solution:** Let X be defined as in part (a); also, let c denote the weight of Zeke's fish. Then we know
> $$\mathbb{P}(X \leq c) = 0.9$$
> But, we also know
> $$\mathbb{P}(X \leq c) = \mathbb{P}\left(\frac{X-7}{2} \leq \frac{c-7}{2}\right) = \Phi\left(\frac{c-7}{2}\right)$$
> meaning
> $$\Phi\left(\frac{c-7}{2}\right) = 0.9 \implies c = 2\Phi^{-1}(0.9) + 7 \approx 8.63$$

Problem 11: A Poisson Calculation

(a) Prove the identity

$$\sum_{\substack{k=0 \\ \text{even}}}^{\infty} \frac{x^k}{k!} = \cosh(x)$$

Hint: Consider the expansions of e^x and e^{-x}, and see what happens when you sum these expansions together

Solution: Recall that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

Replacing x with $(-x)$ yields

$$\sum_{k=0}^{\infty} \frac{(-x)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^k}{k!} = e^{-x}$$

Hence,

$$\begin{aligned} e^x + e^{-x} &= \left(\sum_{k=0}^{\infty} \frac{x^k}{k!} \right) + \left(\sum_{k=0}^{\infty} (-1)^k \cdot \frac{x^k}{k!} \right) \\ &= \sum_{k=0}^{\infty} \left[1 + (-1)^k \right] \cdot \frac{x^k}{k!} \end{aligned}$$

As with problem 15 in the Week 1 Problem Bank, we see that

$$[1 + (-1)^k] = \begin{cases} 2 & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

Hence, what we have derived is

$$\begin{aligned} e^x + e^{-x} &= \sum_{k=0}^{\infty} \left[1 + (-1)^k \right] \cdot \frac{x^k}{k!} \\ &= 2 \sum_{\substack{k=0 \\ \text{even}}}^{\infty} \frac{x^k}{k!} \end{aligned}$$

meaning, dividing both sides by 2,

$$\sum_{\substack{k=0 \\ \text{even}}}^{\infty} \frac{x^k}{k!} = \frac{e^x + e^{-x}}{2} =: \cosh(x)$$

(b) If $X \sim \text{Pois}(\lambda)$, find $\mathbb{P}(X \text{ is even})$.

Solution: We have that the p.m.f. of X is given by

$$\mathbb{P}(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Hence,

$$\begin{aligned}
\mathbb{P}(X \text{ is even}) &= \sum_{\substack{k=0 \\ \text{even}}}^{\infty} \mathbb{P}(X = k) \\
&= e^{-\lambda} \sum_{\substack{k=0 \\ \text{even}}}^{\infty} \frac{\lambda^k}{k!} \\
&= e^{-\lambda} \cdot \cosh(\lambda) \\
&= e^{-\lambda} \cdot \frac{e^{\lambda} + e^{-\lambda}}{2} = \frac{e^{\lambda} + 3e^{-\lambda}}{2}
\end{aligned}$$

Problem 12: A Geometric Calculation

(a) If $X \sim \text{Geom}(p)$, derive a simple-closed form expression for $\mathbb{P}(X \geq k)$

Solution:

$$\begin{aligned}
\mathbb{P}(X \geq k) &= \sum_{x=k}^{\infty} (1-p)^{x-1} p = \frac{p}{1-p} \sum_{x=k}^{\infty} (1-p)^x \\
&= \frac{p}{1-p} \times \frac{(1-p)^k}{1 - [1-p]} = (1-p)^{k-1}
\end{aligned}$$

(b) Use the Tail-Sum Formula (which you proved on Homework 3) to re-derive the result $\mathbb{E}[X] = 1/p$.

Solution:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{k=1}^{\infty} \mathbb{P}(X \geq k) \\
&= \sum_{k=1}^{\infty} (1-p)^{k-1} = \frac{1}{1-p} \times \frac{(1-p)}{1-(1-p)} = \frac{1}{p}
\end{aligned}$$

Problem 13: Subsets *(Modified from PL 2.9.1)*

Consider the set $A = \{1, 2, \dots, n\}$, and suppose all subsets of A are equally likely to be chosen. A subset of A is randomly chosen.

(a) How many elements are in the outcome space associated with this experiment?

Solution: The outcome space consists of all subsets of A; in other words, $\Omega = \mathcal{P}(A)$. To count the number of subsets, we utilize the following logic:

- There is one way to construct a zero-element subset of A (i.e. $\varnothing$); equivalently, there are $\binom{n}{0}$ ways to construct a $0-$length subset of A.
- There are $\binom{n}{k}$ ways to construct a $k-$length subset of A, for $k = 1, \cdots, n$

Therefore,

$$|\Omega| = \sum_{k=0}^{n} \binom{n}{k} = \boxed{2^n}$$

(b) Compute the probability that the selected subset contains the number 1.

Solution: Let E_1 denote the event {the randomly selected subset contains 1.} It will end up being easier to compute the probability of $E_1^{\complement}$, as the set of all subsets of A that do *not* contain 1 is simply $\mathcal{P}(A \setminus \{1\})$, the power set of $A \setminus \{1\} = \{2, 3, \dots, n\}$. Since $\#[\mathcal{P}(A \setminus \{1\})] = 2^{n-1}$ and all subsets are equally likely to have been chosen, we have

$$\mathbb{P}(E_1^{\complement}) = \frac{2^{n-1}}{2^n} = \frac{1}{2}$$

meaning

$$\mathbb{P}(E_1) = 1 - \mathbb{P}(E_1^{\complement}) = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$$

(c) Compute the probability that the selected subset contains both the numbers 1 and 2.

Solution: We can solve this using counting arguments: once we fix the numbers 1 and 2 to be in the selected subset, then the remaining $n-2$ numbers in A are either in or not in the selected subset. Hence, the probability in question is

$$\frac{2 \cdot 2^{n-2}}{2^n} = \boxed{\frac{1}{4}}$$

(d) Compute the probability that the selected subset contains either the number 1 or 2 (or both).

Solution: Let A_1 denote the event {the selected subset contains the number 1} and A_2 denote the event {the selected subset contains the number 2.} By part (b), $\mathbb{P}(A_1) = 1/2$ and by a similar logic $\mathbb{P}(A_2) = 1/2$. Now, we must compute $\mathbb{P}(A_1 \cap A_2)$, which is the probability that the selected subset contains both the number 1 and the number 2. This was computed in part (c) to be $3/4$ meaning, by the Addition Rule (i.e. the Inclusion-Exclusion Principle for $n = 2$ events),

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \boxed{\frac{3}{4}}$$

Alternatively, we could have used a similar argument to part (b): $(A_1 \cup A_2)^\complement = A_1^\complement \cap A_2^\complement$ denotes the event "neither 1 nor 2 were in the selected subset," meaning

$$(A_1 \cup A_2)^\complement = \mathcal{P}(A \setminus \{1, 2\}) \implies |(A_1 \cup A_2)^\complement| = 2^{n-2} \implies \mathbb{P}[(A_1 \cup A_2)^\complement] = \frac{2^{n-2}}{2^n} = \frac{1}{4}$$

Therefore, by the complement rule,

$$\mathbb{P}(A_1 \cup A_2) = 1 - \mathbb{P}[(A_1 \cup A_2)^\complement] = 1 - \frac{1}{4} = \frac{3}{4}$$

as we computed before.

Problem 14: Makin' Money Moves

A bag contains 12 red marbles and 10 blue marbles. A friend reaches in and selects 5 balls at random, with replacement. For each red marble drawn you win \$1; for each blue marble you lose \$1. Let X denote your net winnings.

(a) Find the probability mass function (p.m.f.) of X.

Solution: Let N denote the number of red marbles drawn in a lot of 5; then $N \sim \text{Bin}(5, 12/22)$. What we see is that

$$\begin{aligned}
\{N = 0\} &\iff \{X = -5\} \\
\{N = 1\} &\iff \{X = -3\} \\
\{N = 2\} &\iff \{X = -1\} \\
\{N = 3\} &\iff \{X = 1\} \\
\{N = 4\} &\iff \{X = 3\} \\
\{N = 5\} &\iff \{X = 5\}
\end{aligned}$$

This allows us to easily identify the state space of X as

$$S_X = \{-5,\ -3,\ -1,\ 1,\ 3,\ 5\}$$

Additionally,

$$\mathbb{P}(X = -5) = \mathbb{P}(N = 0) = \binom{5}{0}\left(\frac{12}{22}\right)^0\left(\frac{10}{22}\right)^5$$
$$\mathbb{P}(X = -3) = \mathbb{P}(N = 1) = \binom{5}{1}\left(\frac{12}{22}\right)^1\left(\frac{10}{22}\right)^4$$
$$\mathbb{P}(X = -1) = \mathbb{P}(N = 2) = \binom{5}{2}\left(\frac{12}{22}\right)^2\left(\frac{10}{22}\right)^3$$
$$\mathbb{P}(X = 1) = \mathbb{P}(N = 3) = \binom{5}{3}\left(\frac{12}{22}\right)^3\left(\frac{10}{22}\right)^2$$
$$\mathbb{P}(X = 3) = \mathbb{P}(N = 4) = \binom{5}{4}\left(\frac{12}{22}\right)^4\left(\frac{10}{22}\right)^1$$
$$\mathbb{P}(X = 5) = \mathbb{P}(N = 5) = \binom{5}{5}\left(\frac{12}{22}\right)^5\left(\frac{10}{22}\right)^0$$

(b) Suppose that instead of drawing with replacement your friend now draws 5 marbles *without* replacement. Find the modified p.m.f. of X, in this new situation.

Solution: We can once again define N to be the number of red marbles in our sample of 5; now, however, N follows the Hypergeometric($N = 22,\ G = 12,\ n = 5$) distribution. Once again the support of N is $\{0, 1, 2, 3, 4, 5\}$ and so

$$S_X = \{-5,\ -3,\ -1,\ 1,\ 3,\ 5\}$$

Now, however, our p.m.f. computations are slightly different from those in part (a):

$$\mathbb{P}(X = -5) = \mathbb{P}(N = 0) = \frac{\binom{12}{0}\binom{10}{5}}{\binom{22}{5}}$$
$$\mathbb{P}(X = -3) = \mathbb{P}(N = 1) = \frac{\binom{12}{1}\binom{10}{4}}{\binom{22}{5}}$$
$$\mathbb{P}(X = -1) = \mathbb{P}(N = 2) = \frac{\binom{12}{2}\binom{10}{3}}{\binom{22}{5}}$$
$$\mathbb{P}(X = 1) = \mathbb{P}(N = 3) = \frac{\binom{12}{3}\binom{10}{2}}{\binom{22}{5}}$$
$$\mathbb{P}(X = 3) = \mathbb{P}(N = 4) = \frac{\binom{12}{4}\binom{10}{1}}{\binom{22}{5}}$$
$$\mathbb{P}(X = 5) = \mathbb{P}(N = 5) = \frac{\binom{12}{5}\binom{10}{0}}{\binom{22}{5}}$$

Problem 15: Inclusion-Exclusion

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and three events $A, B, C \in \mathcal{F}$, prove that

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) \\ - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) \\ + \mathbb{P}(A \cap B \cap C)\end{aligned}$$

This is sometimes referred to as the **Inclusion-Exclusion Principle for 3 events**.

Solution: We could start with the Addition Rule:

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) &= \mathbb{P}\left[(A \cup B) \cup C\right] \\ &= \mathbb{P}(A \cup B) + \mathbb{P}(C) - \mathbb{P}([A \cup B] \cap C) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(C) - \mathbb{P}([A \cap C] \cup [B \cap C]) \\ &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) + \mathbb{P}(C) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)\end{aligned}$$

Or, we could have sketched a Venn Diagram.

Problem 16: Practice with Axioms

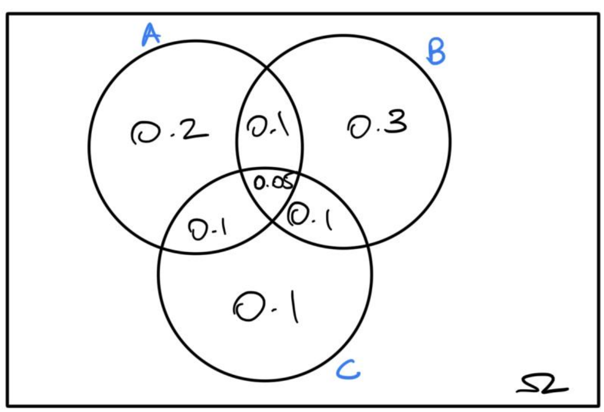
Consider three events A, B, and C such that:

- $\mathbb{P}(A) = 0.45$; $\mathbb{P}(B) = 0.55$; $\mathbb{P}(C) = 0.35$
- $\mathbb{P}(A \cap B) = 0.15$; $\mathbb{P}(A \cap C) = 0.15$; $\mathbb{P}(B \cap C) = 0.15$
- $\mathbb{P}(A \cap B \cap C) = 0.05$

Compute the following probabilities:

Hint: A Venn Diagram might be helpful here.

Solution:



(a) $\mathbb{P}[A^{\complement} \cup B^{\complement} \cup C^{\complement}]$

Solution:

$$1 - \mathbb{P}(A \cap B \cap C) = 1 - 0.05 = \boxed{0.95}$$

(b) $\mathbb{P}(A \cup B)$

Solution:
$$\mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) = 0.45 + 0.55 - 0.15 = \boxed{0.85}$$

(c) $\mathbb{P}(B \setminus C)$

Solution:
$$0.3 + 0.1 = \boxed{0.4}$$

(d) $\mathbb{P}(A \cup B \cup C)$

Solution:
$$0.2 + 0.1 + 0.1 + 0.05 + 0.1 + 0.3 + 0.1 = \boxed{0.95}$$

Problem 17: A Trip to the Movies

At the cinema, 70% of moviegoers purchase either popcorn or a drink (or both). Additionally: 50% of moviegoers purchase popcorn, and of those who purchase popcorn 20% also purchase a drink.

Solution: Let's establish some notation. Let P denote the event "a randomly selected moviegoer purchases popcorn" and D denote "a randomly selected moviegoer purchases a drink." Then we are told:
$$\mathbb{P}(P \cup D) = 0.7; \quad \mathbb{P}(P) = 0.5; \quad \mathbb{P}(D \mid P) = 0.2$$

(a) What is the probability that a randomly selected moviegoer will buy neither popcorn nor a drink?

Solution: By DeMorgan's Law
$$(P^{\complement} \cap D^{\complement}) = (P \cup D)^{\complement}$$
meaning
$$\mathbb{P}(P^{\complement} \cap D^{\complement}) = 1 - \mathbb{P}(P \cup D) = 1 - 0.7 = \boxed{0.3}$$

(b) What proportion of moviegoers purchase a drink (not necessarily *just* a drink)?

Solution: We wish to find $\mathbb{P}(D)$. Firstly, note that
$$\mathbb{P}(P \cap D) = \mathbb{P}(D \mid P) \cdot \mathbb{P}(B) = 0.2 \cdot 0.5 = 0.1$$

We also know, by the Addition Rule,

$$\begin{aligned} 0.7 = \mathbb{P}(P \cup D) &= \mathbb{P}(P) + \mathbb{P}(D) - \mathbb{P}(P \cap D) \\ &= 0.5 + \mathbb{P}(D) - 0.1 \end{aligned}$$

which, when solving for $\mathbb{P}(D)$, yields $\mathbb{P}(D) = 0.3$

Problem 18: A Simple P.M.F.

Suppose X is a random variable with p.m.f. given by

$\boldsymbol{k}$	-2.5	0.3	1.7
$\boldsymbol{p_X(k)}$	0.15	0.25	0.6

(a) Compute $\mathbb{P}(X \geq 0)$

Solution:

$$\mathbb{P}(X \geq 0) = p_X(0.3) + p_X(1.7) = 0.25 + 0.6 = \boxed{0.85}$$

(b) Compute $\mathbb{P}(|X| \leq 2)$

Solution:

$$\mathbb{P}(|X| \leq 2) = \mathbb{P}(-2 \leq X \leq 2) = p_X(0.3) + p_X(1.7) = \boxed{0.85}$$

(c) Compute $\mathbb{E}[X]$

Solution:

$$\begin{aligned} \mathbb{E}[X] &= (-2.5)p_X(-2.5) + (0.3)p_X(0.3) + (1.7)p_X(1.7) \\ &= (-2.5)(0.15) + (0.3)(0.25) + (1.7)(0.6) = \boxed{0.72} \end{aligned}$$

(d) Compute $\mathbb{E}\left[\frac{1}{X}\right]$

Solution:

$$\begin{aligned} \mathbb{E}\left[\frac{1}{X}\right] &= \frac{1}{(-2.5)}p_X(-2.5) + \frac{1}{(0.3)}p_X(0.3) + \frac{1}{(1.7)}p_X(1.7) \\ &= \frac{1}{(-2.5)}(0.15) + \frac{1}{(0.3)}(0.25) + \frac{1}{(1.7)}(0.6) \approx \boxed{1.1263} \end{aligned}$$

(e) Compute $\mathrm{Var}(X)$

Solution:

$$\begin{aligned} \mathbb{E}[X^2] &= (-2.5)^2 p_X(-2.5) + (0.3)^2 p_X(0.3) + (1.7)^2 p_X(1.7) \\ &= (-2.5)^2(0.15) + (0.3)^2(0.25) + (1.7)^2(0.6) = 2.694 \\ \mathrm{Var}(X) &= \mathbb{E}[X^2] - [\mathbb{E}(X)]^2 = 2.694 - (0.72)^2 = \boxed{2.1756} \end{aligned}$$

(f) Compute $\text{Var}(|X|)$

Solution: Note that

$$\text{Var}(|X|) = \mathbb{E}[|X|^2] - (\mathbb{E}[|X|])^2 = \mathbb{E}[X^2] - \mathbb{E}[|X|]^2$$

So, first we compute

$$\begin{aligned}\mathbb{E}[|X|] &= |(-2.5)|p_X(-2.5) + |(0.3)|p_X(0.3) + |(1.7)|p_X(1.7) \\ &= |(-2.5)|(0.15) + |(0.3)|(0.25) + |(1.7)|(0.6) = 1.47\end{aligned}$$

Therefore,

$$\text{Var}(X) = 2.694 - (1.47)^2 = \boxed{0.5331}$$

(g) Find $F_X(x)$, the cumulative distribution function (c.d.f.) of X.

Solution:

$$F_X(x) = \begin{cases} 0 & \text{if } x < -2.5 \\ 0.15 & \text{if } -2.5 \leq x < 0.3 \\ 0.4 & \text{if } 0.3 \leq x < 1.7 \\ 1 & \text{if } x \geq 1.7 \end{cases}$$