EXAMPLES FROM SLIDE DECK 14

PSTAT 120A: Summer 2022 Instructor: Ethan P. Marzban

1. Suppose that I am a shopowner, and I know that the number of customers arriving at my shop follows a Poisson distribution. However, I also know that the rate at which customers arrive at my store is much lower on rainy days as opposed to dry days. Specifically, on rainy days customers arrive at a rate λ_r per day, and on dry days customers arrive at a rate λ_d per day. Further suppose that there is a $p = 10\%$ chance of rain tomorrow.

   (a) If X denotes the number of customers that will arrive at my store tomorrow, what is the p.m.f. (probability mass function) of X?

   Solution: Let R denote the event "it rains tomorrow." Then, we have

$$\begin{aligned}(X \mid R) &\sim \text{Poi}(\lambda_r)\\ (X \mid R^{\complement}) &\sim \text{Pois}(\lambda_d)\\ \mathbb{P}(R) &= 0.1\end{aligned}$$

   Therefore,

$$\begin{aligned}p_X(k) &= \mathbb{P}(X = k \mid R)\mathbb{P}(R) + \mathbb{P}(X = k \mid R^{\complement})\mathbb{P}(R^{\complement})\\ &= e^{-\lambda_r} \cdot \frac{\lambda_r^k}{k!} \cdot 0.1 + e^{-\lambda_d} \cdot \frac{\lambda_d^k}{k!} \cdot 0.9\end{aligned}$$

   (b) What $\mathbb{E}[X]$?

   Solution:

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X \mid R]\mathbb{P}(R) + \mathbb{E}[X \mid R^{\complement}]\mathbb{P}(R^{\complement})\\ &= 0.1\lambda_r + 0.9\lambda_d\end{aligned}$$

2. Suppose (X, Y) is a continuous bivariate random vector with joint p.d.f. given by

$$f_{X,Y}(x, y) = \begin{cases} \lambda^3 x e^{-\lambda y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find $f_Y(y)$, the marginal density of Y. Use this to compute $\mathbb{E}[Y]$.

Solution:

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x \\
&= \int_0^y \lambda^3 x e^{-\lambda y}\,\mathrm{d}x = \boxed{\lambda^3 \cdot \frac{1}{2} y^2 e^{-\lambda y} \cdot \mathbb{1}_{\{y \geq 0\}}}
\end{aligned}
$$

We recognize that $\boxed{Y \sim \text{Gamma}(3, \lambda)}$, meaning $\boxed{\mathbb{E}[Y] = 3/\lambda}$.

(b) Find $f_{X|Y}(x \mid y)$, the conditional density of $(X \mid Y = y)$

Solution:

$$
\begin{aligned}
f_{X|Y}(x \mid y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{\cancel{\lambda^3} x \cancel{e^{-\lambda y}} \cdot \mathbb{1}_{\{0 \leq x \leq y\}} \cdot \cancel{\mathbb{1}_{\{y \geq 0\}}}}{\cancel{\lambda^3} \cdot \frac{1}{2} y^2 \cancel{e^{-\lambda y}} \cdot \cancel{\mathbb{1}_{\{y \geq 0\}}}} = \boxed{\frac{2x}{y^2} \cdot \mathbb{1}_{\{0 \leq x \leq y\}}}
\end{aligned}
$$

(c) Compute $\mathbb{P}(X \geq 1 \mid Y = 2)$.

Solution:

$$
\begin{aligned}
\mathbb{P}(X \geq 1 \mid Y \geq 2) &= \int_1^{\infty} f_{X|Y}(x \mid 2)\,\mathrm{d}x \\
&= \int_1^{\infty} \frac{2x}{2^2} \cdot \mathbb{1}_{\{0 \leq x \leq 2\}}\,\mathrm{d}x = \frac{1}{4} \int_1^2 2x\,\mathrm{d}x = \boxed{\frac{3}{4}}
\end{aligned}
$$

(d) Compute $\mathbb{E}[X \mid Y = 1]$

Solution:

$$
\begin{aligned}
\mathbb{E}[X \mid Y = 1] &= \int_{-\infty}^{\infty} x f_{X|Y}(x \mid 1)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} x \cdot \frac{2x}{1^2} \cdot \mathbb{1}_{\{0 \leq 1 \leq y\}}\,\mathrm{d}x = \boxed{\frac{2}{3}}
\end{aligned}
$$