HOMEWORK 02

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

1. (**6.23, Modified**) Let Y be a random variable with density
$$f_Y(y) = 2(1-y) \cdot \mathbb{1}_{\{0 \leq y \leq 1\}}$$
Define:
$$\begin{aligned} U_1 &:= 2Y - 1 \\ U_2 &:= 1 - 2Y \\ U_3 &:= Y^2 \end{aligned}$$
   (a) Compute $\mathbb{E}[U_1]$, $\mathbb{E}[U_2]$, and $\mathbb{E}[U_3]$ <u>without</u> first finding the densities of $U_1,\ U_2$, and U_3. **Hint:** LOTUS.
   (b) Find $f_{U_1}(u),\ f_{U_2}(u)$, and $f_{U_3}(u)$, the densities of U_1, U_2, and U_3, using the **CDF Method**.
   (c) Find $f_{U_1}(u),\ f_{U_2}(u)$, and $f_{U_3}(u)$, the densities of U_1, U_2, and U_3, using the **Change of Variable formula**.
   (d) Recompute $\mathbb{E}[U_1]$, $\mathbb{E}[U_2]$, and $\mathbb{E}[U_3]$, now using the densities you derived in parts (b) and (c) above.

2. Let $Y \sim$ Exp(θ), and set $U := \alpha Y + \delta$ for positive constants α, β. Find the density $f_U(u)$ of u, using whichever method you like. As an aside: the distribution of U is called the **two-parameter exponential distribution**.

3. The **Rayleigh Distribution**, which admits a single parameter $\beta > 0$, is widely used throughout statistics and engineering. If $X \sim$ Ray(β), then X has density
$$f_X(x) = \frac{2x}{\beta} \cdot e^{-x^2/\beta} \cdot \mathbb{1}_{\{x \geq 0\}}$$
   (a) Let $Y \sim$ Exp(θ), and set $U := \sqrt{Y}$. Show that U follows the Rayleigh distribution, and identify its parameter.
   (b) Let $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. Show that $R := \sqrt{Y_1^2 + Y_2^2}$ follows the Rayleigh distribution, and identify its parameter. As an aside: note how this implies the Rayleigh distribution is well-suited for modeling distances! **Hint:** consider using previously-derived results from lecture, along with the result of part (a) above.

4. Consider a collection $\{X_i\}_{i=1}^n$ of random variables, and define the sample mean in the usual manner:
$$\overline{X}_n := \frac{1}{n}\sum_{i=1}^n X_i$$
   - Suppose that $X_i \sim \chi^2_{\nu_i}$ for positive integers $\{\nu_i\}_{i=1}^n$. Use the MGF method to derive the distribution of $\overline{X}_n$. (Yes, there is a way to do this using the closure property of the Gamma distribution, but I'd like you to use the MGF method for this part.)
   - Suppose that $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ for constants $\{\mu_i\}_{i=1}^n$ and positive constants $\{\sigma_i^2\}_{i=1}^n$. Derive the distribution of $\overline{X}_n$ (for this part you can use previously-derived results from lecture).