Topic 02: Transformations

Ethan P. Marzban University of California, Santa Barbara PSTAT 120B

Outline

1. Univariate Transformations
2. Method of Distribution Functions (CDF Method)
3. Method of Transformations (Change of Variable Formula)
4. Method of Moment-Generating Functions (MGF Method)

Leadup

- Recall, from PSTAT 120A, that given an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we can think of a **random variable** X as a mapping:

$$X : \Omega \to \mathbb{R}$$

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- Additionally, recall the following fact from precalculus: given a mapping $f_1 : A \to B$ and another mapping $f_2 : B \to C$, then $(f_2 \circ f_1) : A \to C$.

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$$X : \Omega \to \mathbb{R}$$

- Additionally, recall the following fact from precalculus: given a mapping $f_1 : A \to B$ and another mapping $f_2 : B \to C$, then $(f_2 \circ f_1) : A \to C$.
- This means, given a function $g : \mathbb{R} \to \mathbb{R}$ and a random variable $X : \Omega \to \mathbb{R}$, we have $(g \circ X) : \Omega \to \mathbb{R}$.

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- "Functions of random variables?" That sounds awfully abstract...
- But, if we think about it a bit more, this isn't as abstract as it may seem!

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Leadup

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- Clearly, the random variables H_I and H_C are related: specifically, $H_C = g(H_I)$ where $g(t) = 2.54 * t$ [since this is the conversion formula between inches and centimeters].

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  - So, <u>**unit conversion**</u> is a fairly simple example of one way transformations (i.e. taking functions of random variables) can be useful.

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- The **sample mean** $\overline{X}_n := n^{-1} \sum_{i=1}^{n} X_i$ [which you hopefully saw in PSTAT 120A!] is actually a *function* of the original sequence of random variables, and is hence an example of a transformation.

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- For simplicity's sake, let's start off with univariate transformations.
  - Specifically, given a random variable Y and a function $g : \mathbb{R} \to \mathbb{R}$, we will seek to explore properties of the random variable $U := g(Y)$.

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Univariate Transformations

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- What do we mean by "describe" the random variable U?
- Well, there are a couple of things we could seek to do. First, we could try to compute $\mathbb{E}[U]$.

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- The **<u>Law of the Unconscious Statistician</u>** (LOTUS), which we saw in PSTAT 120A, tells us

$$\mathbb{E}[g(Y)] = \int_{\mathbb{R}} g(y) f_Y(y) \, \mathrm{d}y$$

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- Similar considerations will allow us to compute $\mathrm{Var}(U)$.

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  - Its **density function** (PDF)

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    - Its **distribution function** (i.e. CDF)
    - Its **density function** (PDF)
    - Its **MGF** (moment-generating function)

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- This would, in turn, automatically tell you that W has distribution function
$$F_W(w) = \begin{cases} 1 - e^{-2w} & \text{if } w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
and MGF
$$M_W(t) = \begin{cases} (1 - t/2)^{-1} & \text{if } t < 1/2 \\ \infty & \text{otherwise} \end{cases}$$

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you would immediately be able to say

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-2)^2\right\}$$

and

$$F_X(x) = \Phi\left(x - 2\right); \qquad \Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\ \mathrm{d}z$$

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- What I mean is this - the distribution of X doesn't have a name, like "Exponential" or "Gamma". But it certainly *has* a distribution!
- All of this is to say: I encourage you to get into the habit of thinking about "distributions" fairly broadly, and thinking of a distribution as either a density function, distribution function, or MGF (or all three).

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Back to Transformations

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- We could go after the density function of U.

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- We could go after the density function of U.
- Or we could go after the distribution function of U.
- Or we could go after the MGF of U.
- Indeed, each of these three approaches are what our textbook calls different "methods".

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- Recall that the support (aka "state space") of a random variable Y is the set of all values that Y maps to: i.e. $S_Y := Y(\Omega)$. Equivalently, it's the set of all values y for which the density $f_Y(y)$ is nonzero.

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- Then, given a random variable $U := g(Y)$, we have $S_U = g(S_Y)$.

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  - That is, the support of a transformed random variable is the image of the original support under the transformation.

- Though this formula seems inoccuous enough, finding the support of a transformed random variable can be trickier than it first appears...

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- Specifically, let's say we have an interval $[a, b]$ and a transformation $g : \mathbb{R} \to \mathbb{R}$.
- To figure out what $g([a, b])$ looks like, simply graph the function $y = g(x)$, indicate $[a, b]$ on the $x-$axis, and figure out what the corresponding values on the $y-$axis are.

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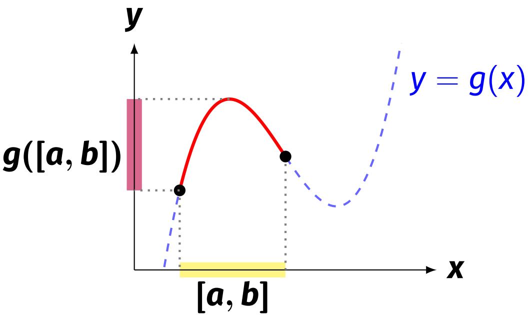
Support



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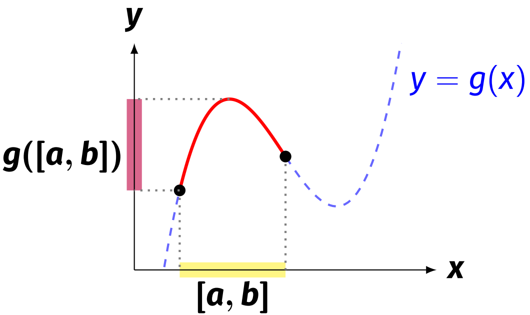

- Note: in general, $g([a, b]) \neq [g(a), g(b)]$!

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Clicker Question!

Clicker Question 1

For $A = [0\ , 6]$ and $g(x) = \cos(\pi x)$, what is the correct expression for $g(A)$?

(A) $[0\ ,\ 1]$ (B) $[0\ ,\ 6]$ (C) $[-1\ , 1]$ (D) $\{0\}$
(E) None of the above

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Try this On Your Own:

Example

For $A = [-1\,,\,1]$ and $g(x) = x^2$, what is the correct expression for $g(A)$?

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Method of Distribution Functions (CDF Method)

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CDF Method

- Let's consider the following rephrasing of our goal:

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Goal

Given a random variable Y and a function $g()$, we seek to derive an expression for $F_U(u) := \mathbb{P}(U \leq u)$, the CDF of U.

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- As a concrete example, let $Y \sim \text{Exp}(\theta)$ and let $U := cY$ for a positive constant c.

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- As a concrete example, let $Y \sim \text{Exp}(\theta)$ and let $U := cY$ for a positive constant c.
  - If it helps, you can think of this in terms of our inches-to-centimeter conversion example from the start of this lecture: Y can denote the heights in inches and U can denote the heights in cenimeters.

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- Note:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(cY \leq u)$$

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- Note:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(cY \leq u)$$

- Divide through by c:

$$F_U(u) = \mathbb{P}\left(Y \leq \frac{u}{c}\right) = F_Y\left(\frac{u}{c}\right)$$

CDF Method

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$$= \boxed{\begin{cases} 1 - e^{u/(c\theta)} & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}}$$

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&= \boxed{\begin{cases} 1 - e^{u/(c\theta)} & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}}
\end{aligned}
$$

- And we're done! We've accomplished our goal, and found an expression for $F_U(u)$, the CDF of U.

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Going Further

- Now, in this particular case, we can take things a step further.

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Theorem (Closure of Exponential Distribution under Multiplication)

Given $Y \sim \text{Exp}(\theta)$ and a positive constant c, then $(cY) \sim \text{Exp}(c\theta)$.

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Theorem (Closure of Exponential Distribution under Multiplication)

Given $Y \sim \text{Exp}(\theta)$ and a positive constant c, then $(cY) \sim \text{Exp}(c\theta)$.

- We're going to use this result a **LOT**!

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Interpretation

- I know this might seem a little abstract - what does it mean to "multiply the exponential distribution by a constant?"

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Interpretation

- I know this might seem a little abstract - what does it mean to "multiply the exponential distribution by a constant?"
- Again, if it helps, you can always think in terms of our inches-to-centimeter problem from the start of these slides.
- If $Y \sim \text{Exp}(\theta)$ denotes the height of a randomly selected person in inches, then the distribution of heights in centimeters will *also* be exponential, this time with mean 2.54θ.

Example

- Let's do another example together.

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Example

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- Suppose Y has density function given by

$$f_Y(y) = 2y \cdot \mathbb{1}_{\{0 \leq y \leq 1\}}$$

and again define $U := cY$ for a positive constant c.

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  and again define $U := cY$ for a positive constant c.
- Now, before we got lucky because we immediately knew what the CDF of Y was.
- But, even though we can't *immediately* recognize the CDF of Y in this example, we can still derive it!

Example

- By definition, for a $y \in [0, 1]$,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\ \mathrm{d}t$$

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- Clearly, for $y < 0$ we have $F_Y(y) = \mathbb{P}(Y \le y) = 0$ and for $y > 1$ we have $\mathbb{P}(Y \le y) = 1$, meaning

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ y^2 & \text{if } 0 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

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Example

- And now we're in the same position as before!

$$\mathbb{P}(U \leq u) = \mathbb{P}(cY \leq u) = \mathbb{P}\left(Y \leq \frac{u}{c}\right)$$

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\begin{aligned}
\mathbb{P}(U \leq u) = \mathbb{P}(cY \leq u) = \mathbb{P}\left(Y \leq \frac{u}{c}\right) \\
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- A quick sketch (see chalkboard) reveals that $S_U = [0, \infty)$. So, $F_U(u) = 0$ whenever $u < 0$.

Example

- One more example before we summarize.
- Let $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$.
- A quick sketch (see chalkboard) reveals that $S_U = [0, \infty)$. So, $F_U(u) = 0$ whenever $u < 0$.
- Additionally, we (again) have the CDF of Y: $F_Y(y) = \Phi(y)$, where $\Phi(\cdot)$ denotes the standard normal CDF.

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Example

- So, let's try and proceed like we did before! For a fixed $u \geq 0$,

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(Y^2 \leq u)$$

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    - Remember, both -3 and 3 have squares equal to 9! But, when we write $\sqrt{9}$, we implicitly mean the principal square root which is why we write $\sqrt{9} = 3$.

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  - Remember, both -3 and 3 have squares equal to 9! But, when we write $\sqrt{9}$, we implicitly mean the principal square root which is why we write $\sqrt{9} = 3$.
- So, what we really have is:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(Y^2 \leq u) = \mathbb{P}(|Y| \leq \sqrt{u}) = \mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u})$$

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Example (cont'd)

- Now, there's another way to see how to get from $\mathbb{P}(Y^2 \leq u)$ to $\mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u})$; one that doens't require us to dig into our memory banks and dredge up something from algebra/precalculus, and instead uses pictures.

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Video

https://www.youtube.com/watch?v=HtzqjHfoRbw

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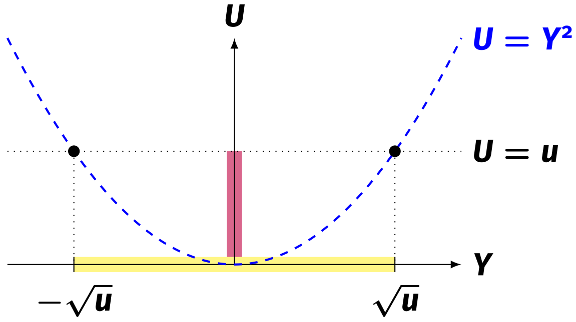
Static Image



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- So, let's finish up our example!

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$$\begin{aligned} F_U(u) &= \cdots = \mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u}) \\ &= F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) \end{aligned}$$

Example (cont'd)

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$$
\begin{aligned}
F_U(u) &= \cdots = \mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u}) \\
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\end{aligned}
$$

- That's a bit anticlimactic... Let's differentiate wrt. u and obtain the PDF of U:

Example (cont'd)

$$f_U(u) = \frac{\mathrm{d}}{\mathrm{d}u} F_U(u)$$

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Example (cont'd)

$$\begin{aligned} f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\ &= \frac{\mathrm{d}}{\mathrm{d}u} [2\Phi(\sqrt{u}) - 1] \end{aligned}$$

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Example (cont'd)

$$\begin{aligned} f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\ &= \frac{\mathrm{d}}{\mathrm{d}u} [2\Phi(\sqrt{u}) - 1] \\ &= 2 \cdot \frac{1}{2\sqrt{u}} \cdot \phi(\sqrt{u}) = \frac{1}{\sqrt{u}} \phi(\sqrt{u}) \end{aligned}$$

Example (cont'd)

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- Let's incorporate the support of U, and simplify:

Example (cont'd)

$$f_U(u) = \frac{1}{\sqrt{u}} \phi(\sqrt{u}) \cdot \mathbb{1}_{\{u \geq 0\}}$$

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Example (cont'd)

$$\begin{aligned} f_U(u) &= \frac{1}{\sqrt{u}} \phi(\sqrt{u}) \cdot \mathbb{1}_{\{u \geq 0\}} \\ &= \frac{1}{\sqrt{u}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{u})^2} \cdot \mathbb{1}_{\{u \geq 0\}} \end{aligned}$$

Example (cont'd)

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- One useful fact: $\Gamma(1/2) = \sqrt{\pi}$. Hence:

$$
f_U(u) = \frac{1}{\Gamma(1/2) \cdot 2^{1/2}} \cdot u^{1/2 - 1} \cdot e^{-u/2} \cdot \mathbb{1}_{\{u \geq 0\}}
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- Indeed, $U \sim \text{Gamma}(1/2\ ,\ 2) \overset{\text{d}}{=} \chi_1^2$!

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Theorem

- This is an ***extremely*** important result which we will use repeatedly throughout this course. Let's make it more formal by rephrasing it as a theorem:

Theorem (Square of Standard Normal)

If $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$, then $U \sim \chi^2_1$.

- The proof of this theorem is exactly the work we did on the previous slides.

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Recap

- Whew- that was a lot of work! Let's recap.

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- Given a random variable Y, and $U := g(Y)$ for some function $g : \mathbb{R} \to \mathbb{R}$, we can use the **method of distribution functions** (aka the **CDF**) method to find the distribution of U.

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  (1) Writing $F_U(u)$, the CDF of U, in terms of $F_Y(y)$, the CDF of Y, by basically finding an equivalent formulation for the event $A_U := \{U \leq u\}$ that is in terms of Y

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  (2) Plugging into the CDF of Y, and simplifying as necessary.

Method of Transformations (Change of Variable Formula)

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Leadup

- Let’s, for a moment, return the example where we squared the standard normal distribution.

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- This begs the question - can we perhaps "extend" the CDF method to give us a formula for the *PDF* of U directly?

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- Then, we differentiated wrt. u to obtain a formula for $f_U(u)$.
- This begs the question - can we perhaps "extend" the CDF method to give us a formula for the *PDF* of U directly?
- The answer turns out to be "yes, under some conditions."

Leadup

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

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Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- Let's see what happens if we try to apply the CDF method:

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- Isn't it tempting to apply $g^{-1}(\cdot)$ to both sides of the inequality?

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- Isn't it tempting to apply $g^{-1}(\cdot)$ to both sides of the inequality?
- It is! But we need to be careful. First, remember that we don't have any guarantee that $g^{-1}(\cdot)$ even exists!

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Leadup

- Alright, then - let's add some assumption about our function $g(\cdot)$.

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Given a random variable Y and a <u>strictly increasing</u> function $g()$, we seek to find $f_U(u)$, the PDF of U.

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Given a random variable Y and a strictly increasing function $g()$, we seek to find $f_U(u)$, the PDF of U.

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Given a random variable Y and a <u>strictly increasing</u> function $g()$, we seek to find $f_U(u)$, the PDF of U.

- *Now* we are guaranteed the existence of $g^{-1}(\cdot)$.
- Furthermore, since we assumed $g(\cdot)$ itself to be strictly *increasing*, $g^{-1}(\cdot)$ will also be strictly increasing.

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- *Now* we are guaranteed the existence of $g^{-1}(\cdot)$.
- Furthermore, since we assumed $g(\cdot)$ itself to be strictly *increasing*, $g^{-1}(\cdot)$ will also be strictly increasing.
- Hence, we "preserve the direction of inequality" when applying $g^{-1}(\cdot)$ to both sides of an inequality.

Leadup

- Then:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(g(Y) \leq u) = \mathbb{P}(Y \leq g^{-1}(u)) = F_Y(g^{-1}(u))$$

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Leadup

- Then:

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- We can now differentiate wrt. U and apply the chain rule (from calculus; we can discuss this further on the chalkboard):

$$\begin{aligned} f_U(u) &:= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\ &= \frac{\mathrm{d}}{\mathrm{d}u} F_Y(g^{-1}(u)) \\ &= f_Y(g^{-1}(u)) \cdot \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \end{aligned}$$

Leadup

- If we instead assume that $g(\cdot)$ is strictly decreasing, a similar computation (which I'll be asking you to complete on your homework) yields

$$f_U(u) = f_Y(g^{-1}(u)) \cdot \left[-\frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right]$$

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- So, if we instead simply assume that $g(\cdot)$ is strictly monotonic, we can summarize our findings as:

$$f_U(u) = \begin{cases} f_Y(g^{-1}(u)) \cdot \left[\frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right] & \text{if } g(\cdot) \text{ is increasing} \\ f_Y(g^{-1}(u)) \cdot \left[-\frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right] & \text{if } g(\cdot) \text{ is decreasing} \end{cases}$$

Change of Variable Formula

- A bit of simplification (and recollections of how derivatives of increasing/decreasing functions behaves) allows us to rewrite our result above as:

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Change of Variable Formula

- A bit of simplification (and recollections of how derivatives of increasing/decreasing functions behaves) allows us to rewrite our result above as:

Theorem (Change of Variable Formula)

Given a random variable $Y \sim f_Y$ and a function $g(\cdot)$ that is strictly monotonic over the support of Y, then the random variable $U := g(Y)$ has density given by

$$f_U(u) = f_Y[g^{-1}(u)] \cdot \left| \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right|$$

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Example

- As an example, let's re-derive the closure under multiplication property of the Exponential distribution, this time using the Change of Variable formula.

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- As an example, let's re-derive the closure under multiplication property of the Exponential distribution, this time using the Change of Variable formula.
- That is: let $Y \sim \text{Exp}(\theta)$, and set $U := cY$ for some positive constant $c > 0$.

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Example

- As an example, let's re-derive the closure under multiplication property of the Exponential distribution, this time using the Change of Variable formula.
- That is: let $Y \sim \text{Exp}(\theta)$, and set $U := cY$ for some positive constant $c > 0$.
- Since the transformation $g(y) = cy$ is strictly monotonic (specifically, it's strictly increasing) it's inverse exists and is calculable as $g^{-1}(u) = u/c$. Hence:

$$\left| \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right| = \left| \frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{u}{c} \right) \right| = \left| \frac{1}{c} \right| = \frac{1}{c}$$

  where we have dropped the absolute values in the last step since we are assuming $c > 0$.

Example

- Additionally, since $Y \sim \text{Exp}(\theta)$ we know that

$$f_Y(y) = \frac{1}{\theta} \exp\left\{-\frac{y}{\theta}\right\} \cdot \mathbb{1}_{\{y \geq 0\}}$$

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- Therefore, plugging into the change of variable formula, we have

$$\begin{aligned} f_U(u) &= f_Y[g^{-1}(u)] \cdot \left| \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right| \\ &= \frac{1}{\theta} \exp\left\{ -\frac{\left(\frac{u}{c}\right)}{\theta} \right\} \cdot \mathbb{1}_{\{\frac{u}{c} \geq 0\}} \cdot \frac{1}{c} \\ &= \frac{1}{c\theta} \exp\left\{-\frac{u}{c\theta}\right\} \cdot \mathbb{1}_{\{u \geq 0\}} \end{aligned}$$

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Clicker Question!

Clicker Question 1

Given $Y \sim \text{Unif}[1, 2]$ and $U := 2X + 3$, does U also follow a Uniform Distribution?

(A) Yes; (B) No

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Change of Variable Formula

- Now, note that the only assumption we need to make about $g(\cdot)$ in order for the Change of Variable formula to hold is that it is strictly monotone *over the support of Y*.

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- For example, suppose $Y \sim \text{Unif}[-1, 0]$ and take $U := Y^2$.
- Though the function $g(y) = y^2$ is not strictly monotone over $\mathbb{R}$, it *is* strictly monotone over $S_Y := [-1, 0]$ (i.e. the support of Y), and hence its inverse exists and is given by $g^{-1}(u) = -\sqrt{u}$.

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- The Change of Variable formula can therefore safely be applied.

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Change of Variable Formula

- In general, however, the Change of Variable formula does <u>not</u> work when we are dealing with transformations that are not strictly monotone.

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- In general, however, the Change of Variable formula does not work when we are dealing with transformations that are not strictly monotone.
- For example, given $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$, we *cannot* directly apply the Change of Variable formula.

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Change of Variable Formula

- In general, however, the Change of Variable formula does <u>not</u> work when we are dealing with transformations that are not strictly monotone.
- For example, given $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$, we *cannot* directly apply the Change of Variable formula.
  - Admittedly, there does exist a way to generalize the Change of Variable formula to work in a situation like this, but we won't cover that in PSTAT 120B. If you're curious, I'm happy to walk you through the general outline during Office Hours.

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Method of Moment-Generating Functions (MGF Method)

Leadup

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

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Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- So far, we've talked about "describing" the distribution of U by both its CDF (using the CDF method) and its PDF (using the Change of Variable formula).

Leadup

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- So far, we've talked about "describing" the distribution of U by both its CDF (using the CDF method) and its PDF (using the Change of Variable formula).
- We know that there is a third way of classifying distributions - **moment-generating functions** (MGFs).

MGFs

Definition (MGF)

The MGF of a random variable X, notated $M_X(t)$, is defined as

$$M_X(t) := \mathbb{E}[e^{tX}]$$

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MGFs

Definition (MGF)

The MGF of a random variable X, notated $M_X(t)$, is defined as

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- Recall that this expectation is computed as a sum if X is discrete and as an integral if X is continuous.

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Useful Result

Theorem

Given two random variables X and Y with MGFs $M_X(t)$ and $M_Y(t)$, respectively, that are both continuous in a small neighborhood of the origin, then $M_X(t) = M_Y(t)$ implies that X and Y have the same distribution.

Useful Result

Theorem

Given two random variables X and Y with MGFs $M_X(t)$ and $M_Y(t)$, respectively, that are both continuous in a small neighborhood of the origin, then $M_X(t) = M_Y(t)$ implies that X and Y have the same distribution.

- This theorem is essentially just a more formal way of saying "MGFs uniquely determine random variables." For example,

$$M_X(t) = \exp\left\{2t + \frac{1}{2}t^2\right\} \iff X \sim \mathcal{N}(2, 1)$$

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Useful Result

Theorem

Given a random variable Y with MGF $M_Y(t)$, and $U := aY + b$ for constants $a, b \in \mathbb{R}$,

$$M_U(t) = e^{bt} M_Y(at)$$

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Proof.

$$M_U(t) := \mathbb{E}[e^{tU}]$$ [Definition of MGF]

$\square$

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Proof.

$$
\begin{aligned}
M_U(t) &:= \mathbb{E}[e^{tU}] && \text{[Definition of MGF]} \\
&:= \mathbb{E}[e^{t(aY+b)}] && \text{[Definition of } U\text{]}
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$$

$\square$

- It turns out, we can use this theorem to (again) prove the closure of the exponential distribution under multiplication!

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Example

- Once again, let $Y \sim \text{Exp}(\theta)$, and let $U = cY$ for a positive constant c.

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which is, as expected, the MGF of the $\text{Exp}(c\theta)$ distribution.

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Clicker Question!

Clicker Question 2

If $Y \sim \text{Pois}(\lambda)$ and $U := cY$ for some positive constant c, what is the distribution of U?

(A) $\text{Pois}(c\lambda)$

(B) $\text{Pois}(c/\lambda)$

(C) $\text{Pois}(\lambda/c)$

(D) None of the above

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Leadup

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- However, the method of MGFs really shines when we start taking linear combinations of *multiple* random variables.
- We'll talk about multivariate transformations more after the first midterm, but let's get a quick flavor of some of them now.

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Leadup

- Suppose two friends, Jack and Jill, each enter a separate checkout lane at a grocery store.

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- The random variable $U := (Y_1 + Y_2)$ then represents the *combined* wait times of Jack and Jill (in minutes).
- If, for example, $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, then what distribution does U follow?

Leadup

- As a bit of a spoiler, we *could* try to find the distribution of U using the CDF method. (Doing so would involve computing a double integral - these are the sorts of things we'll be doing after MT01).

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Leadup

- So, plugging in the MGF of the Exp(θ) distribution, we have

$$M_U(t) = \left(\begin{cases} (1-\theta t)^{-1} & \text{if } t < 1/\theta \\ \infty & \text{otherwise} \end{cases} \right) \cdot \left(\begin{cases} (1-\theta t)^{-1} & \text{if } t < 1/\theta \\ \infty & \text{otherwise} \end{cases} \right)$$

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which is the MGF of the Gamma$(2, \theta)$ distribution!

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- So, we've shown that $(Y_1 + Y_2) \sim \text{Gamma}(2, \theta)$.

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Useful Result

Useful Result

Theorem (Important MGF Formula)

Given a collection of independent random variables $\{Y_i\}_{i=1}^n$, we have

$$M_U(t) = \prod_{i=1}^{n} M_{Y_i}(a_i t) \qquad \text{where } U := \sum_{i=1}^{n} a_i Y_i$$

Useful Result

Theorem (Closure of Gamma Distribution under Sums)

Given $\{Y_i\}_{i=1}^n$ with $Y_i \sim \text{Gamma}(\alpha_i, \beta)$ and constants $\{a_i\}_{i=1}^n$, we have

$$\left(\sum_{i=1}^n Y_i \right) \sim \text{Gamma} \left(\sum_{i=1}^n \alpha_i \ , \ \beta \right)$$

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Proof

We use the formula from the previous slide:

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Hence, plugging in, we find:

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Proof

$$M_{\sum_{i=1}^{n} a_i Y_i}(t) = \prod_{i=1}^{n} M_{Y_i}(a_i t)$$

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Proof

$$M_{\sum_{i=1}^{n} a_i Y_i}(t) = \prod_{i=1}^{n} M_{Y_i}(a_i t)$$
$$= \prod_{i=1}^{n} \left(\begin{cases} (1 - \beta t)^{-\alpha_i} & \text{if } t < 1/\beta \\ \infty & \text{otherwise} \end{cases} \right)$$

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Proof

$$
\begin{aligned}
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$$

which we recognize as the MGF of the Gamma$(\sum_{i=1}^n \alpha_i, \beta)$ distribution. Hence, we are done.

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- Specifically, I am adamant that you end with some sort of concluding *statement* - don't just leave the MGF without saying something about the underlying distribution!
    - For example, in the previous proof, notice how I ended with "which we recognize as...". Just make sure you end your MGF-related proofs with something similar!

Another Useful Result

Theorem (Closure of Normal Distribution under Linear Combinations)

Given a collection of independent random variables $\{Y_i\}_{i=1}^n$ with $Y_i \sim \mathcal{N}(\mu_i\ ,\ \sigma_i^2)$ and constants $\{a_i\}_{i=1}^n$, we have

$$U := \left(\sum_{i=1}^{n} a_i Y_i \right) \sim \mathcal{N}\left(\sum_{i=1}^{n} a_i \mu_i\ ,\ \sum_{i=1}^{n} a_i^2 \sigma_i^2 \right)$$

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Proof

- I leave the proof to you.
- One word of extreme caution: we can get the expectation and variance of U using 120A-related formulas.
- But, the *normality* of U is something that we cannot take for granted - this is why we need to use the MGF method to complete the proof!

Summary: Univariate Transformations

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

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  - The MGF Method.

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- Remember: when carrying out step 3, drawing a picture can be incredibly helpful!

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(2) Plug into the Change of Variable formula:

$$f_U(u) = f_Y[g^{-1}(y)] \cdot \left| \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right|$$

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- Remember: this method only works when the transformation $g(\cdot)$ is strictly monotonic over the support of Y!
- Also, a side note: so long as you are careful, the change of variable formula will give you the support of U. But, in some cases, it might be easier to find the support first (by drawing a picture), and then incorporating that into your answer later.

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MGF Method

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MGF Method

(1) Compute the MGF $M_U(t)$ of U by writing it in terms of the MGF $M_Y(t)$ of Y, and then recognize the resulting MGF as belonging to a particular distribution.

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- This works well for linear transformations and linear combinations of random variables, but not too well for nonlinear transformations.
- Also, the MGF method won't (typically) give you a PDF/CDF, so if you really want the PDF/CDF of U you should use a different method [unless you believe you will be able to recognize the resulting distribution as one that has a name].

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Chalkboard Example

Example

The **kinetic energy** of a particle with mass m traveling at a velocity V is given by

$$E = \frac{1}{2} m V^2$$

Consider a particle selected at random, whose velocity is a random variable V with density

$$f_V(v) = 2v^3 e^{-v^2} \cdot \mathbb{1}_{\{v > 0\}}$$

Find the distribution of the kinetic energy of this particle once using the CDF method and once using the Change of Variable formula.

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