QUIZ 01

PSTAT 120B: Mathematical Statistics, I
Summer Session A, 2024 with Instructor: Ethan P. Marzban

Your Information

Your Name: ____
(*First and Last*)

Your NetID: ____
(*NOT Perm Number*)

Your Section: **2pm (Hyuk-Jean)** **3pm (Hyuk-Jean)** **4pm (Minwoo)** **5pm (Minwoo)**
(*Circle One*)

Instructions

- **PLEASE DO NOT REMOVE ANY PAGES FROM THIS QUIZ.**
- You have **20 minutes** to complete this quiz.
- The use of **calculators** is permitted, but the use of any other aids (including, but not limited to, notes, phones, laptops, etc.) is prohibited.
- Ensure you show **all your work**; correct answers with no supporting justification will not receive full credit.
- Kindly note that any instances of academic misconduct (including, but not limited to: unauthorized collaboration, or the use of unauthorized resources) will be dealt with in the strictest manner possible.
- Please refrain from discussing the contents of this quiz with **ANYONE**, until given the all-clear by the Instructor.
- **Good Luck!**

This quiz contains 15 points in total

The Quiz Begins on the Next Page

Problem 1. Jack and Jill have both taken a PSTAT 120B midterm. Because they studied together, their scores are correlated: specifically, if Y_1 denotes the (percentage) score Jack receives on the exam and Y_2 denotes the (percentage) score Jill receives on the exam, the joint distribution of Y_1 and Y_2 is given by

$$f_{Y_1,Y_2}(y_1, y_2) = 8y_1y_2 \cdot \mathbb{1}_{\{0 \le y_1 \le y_2 \le 1\}}$$

(a) (4 points) Find the marginal density function for Y_2. For full credit, make sure to clearly specify the support of Y_2.

Solution: We integrate out y_1 from the joint density:

$$\begin{aligned} f_{Y_2}(y_2) &= \int_{\mathbb{R}} f_{Y_1,Y_2}(y_1, y_2)\, \mathrm{d}y_1 \\ &= \int_{\mathbb{R}} 8y_1y_2 \cdot \mathbb{1}_{\{0 \le y_1 \le y_2\}} \cdot \mathbb{1}_{\{0 \le y_2 \le 1\}}\, \mathrm{d}y_1 \\ &= 8y_2 \cdot \mathbb{1}_{\{0 \le y_2 \le 1\}} \cdot \int_0^{y_2} y_1\, \mathrm{d}y_1 \\ &= 4y_2^3 \cdot \mathbb{1}_{\{0 \le y_2 \le 1\}} \end{aligned}$$

(b) (6 points) Given that Jack scored exactly a 50%, what is the probability that Jill scores above 75%? You may use, without proof, the fact that the marginal density of Y_1 is

$$f_{Y_1}(y_1) = 4y_1(1 - y_1^2) \cdot \mathbb{1}_{\{0 \le y_1 \le 1\}}$$

Solution: We seek to compute $\mathbb{P}(Y_2 \ge 3/4 \mid Y_1 = 1/2)$. Since we are conditioning on an event with zero probability, we cannot use the definition of conditional probabilities. Instead, we must integrate the conditional density:

$$\mathbb{P}(Y_2 \ge 3/4 \mid Y_1 = 1/2) = \int_{3/4}^{\infty} f_{Y_2|Y_1}(y_2 \mid 1/2)\, \mathrm{d}y_2$$

To find the conditional density $f_{Y_2|Y_1}(y_2 \mid y_1)$, we use the definition:

$$\begin{aligned} f_{Y_2|Y_1}(y_2 \mid y_1) &= \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_1}(y_1)} \\ &= \frac{8\cancel{y_1}y_2 \cdot \cancel{\mathbb{1}_{\{0 \le y_1 \le 1\}}} \cdot \mathbb{1}_{\{y_1 \le y_2 \le 1\}}}{4\cancel{y_1}(1 - y_1^2) \cdot \cancel{\mathbb{1}_{\{0 \le y_1 \le 1\}}}} \\ &= \frac{2y_2}{1 - y_1^2} \cdot \mathbb{1}_{\{y_1 \le y_2 \le 1\}} \end{aligned}$$

Hence, plugging in $y_1 = 1/2$ we have

$$f_{Y_2|Y_1}(y_2 \mid y_1) = \frac{8}{3} \cdot y_2 \cdot \mathbb{1}_{\{1/2 \le y_2 \le 1\}}$$

and so

$$\mathbb{P}(Y_2 \ge 3/4 \mid Y_1 = 1/2) = \int_{3/4}^{\infty} f_{Y_2|Y_1}(y_2 \mid 1/2)\, \mathrm{d}y_2$$

$$
\begin{aligned}
&= \int_{3/4}^{\infty} \frac{8}{3} \cdot y_2 \cdot \mathbb{1}_{\{1/2 \leq y_2 \leq 1\}} \, \mathrm{d}y_2 \\
&= \frac{8}{3} \cdot \int_{3/4}^{1} y_2 \, \mathrm{d}y_2 \\
&= \frac{4}{3} \cdot \left(1 - \frac{9}{16}\right) = \boxed{\frac{7}{12} \approx 58.3\%}
\end{aligned}
$$

Problem 2. The amount of time (in minutes) Pam spends on the phone is uniformly distributed between 0 minutes and 10 minutes. The amount of time (in minutes) Phyllis spends on the phone is uniformly distributed between 0 minutes and however long Pam was on the phone. Let N denote the amount of time (in minutes) Pam spends on the phone, and let X denote the amount of time (in minutes) Phyllis spends on the phone.

Note: The uniform distribution in this problem is the *continuous* uniform distribution.

(a) (2 points) Compute $\mathbb{E}[X \mid N]$. Show all your steps!

Solution: From the problem statement, we see that

$$
\begin{aligned}
N &\sim \text{Unif}[0, 10] \\
(X \mid N = n) &\sim \text{Unif}[0, n]
\end{aligned}
$$

To compute $\mathbb{E}[X \mid N]$, we use our two-step procedure: first we compute $\mathbb{E}[X \mid N = n]$, and then we substitute N in place of n. Since $(X \mid N = n) \sim \text{Unif}[0, n]$, we can use the formula for the expectation of a continuous uniform distribution to conclude

$$
\mathbb{E}[X \mid N = n] = \frac{n}{2}
$$

which means

$$
\boxed{\mathbb{E}[X \mid N] = N}
$$

(b) (3 points) Compute $\mathbb{E}[X]$. Again, show all your steps!

Solution: We use the **Law of Total Probability**,

$$
\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid N]] = \mathbb{E}\left[\frac{N}{2}\right] = \frac{1}{2} \cdot \mathbb{E}[N] = \frac{1}{2} \cdot 5 = \boxed{\frac{5}{2}}
$$

where we have (again) utiliuzed the formula for the expectation of a continuous uniform distribution to compute $\mathbb{E}[N]$.

END OF QUIZ

Selected Discrete Distributions

Distribution	PMF	Expec./Var.	MGF
$X \sim \text{Bin}(n,p)$	$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k \in \{0, \cdots, n\}$	$\mathbb{E}[X] = np$ $\text{Var}(X) = np(1-p)$	$M_X(t) = (1-p+pe^t)^n$
$X \sim \text{Geom}(p)$	$p_X(k) = p(1-p)^{k-1}$, $k \in \{1, 2, \cdots\}$	$\mathbb{E}[X] = \frac{1}{p}$ $\text{Var}(X) = \frac{1-p}{p^2}$	$M_X(t) = \frac{pe^t}{1-(1-p)e^t}$, $t < -\ln(1-p)$
$X \sim \text{Geom}(p)$	$p_X(k) = p(1-p)^{k}$, $k \in \{0, 1, \cdots\}$	$\mathbb{E}[X] = \frac{1-p}{p}$ $\text{Var}(X) = \frac{1-p}{p^2}$	$M_X(t) = \frac{p}{1-(1-p)e^t}$, $t < -\ln(1-p)$
$X \sim \text{NegBin}(r,p)$	$p_X(k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$, $k \in \{r, r+1, \cdots\}$	$\mathbb{E}[X] = \frac{r}{p}$ $\text{Var}(X) = \frac{r(1-p)}{p^2}$	$M_X(t) = \left[\frac{pe^t}{1-(1-p)e^t}\right]^r$, $t < -\ln(1-p)$
$X \sim \text{Pois}(\lambda)$	$p_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$, $k \in \{0, 1, \cdots\}$	$\mathbb{E}[X] = \lambda$ $\text{Var}(X) = \lambda$	$M_X(t) = \exp\left\{\lambda(e^t - 1)\right\}$

Selected Continuous Distributions

Distribution	PDF	Expec./Var.	MGF
$X \sim \text{Unif}[a,b]$	$f_X(x) = \dfrac{1}{b-a} \cdot \mathbb{1}_{\{x \in [a,b]\}}$	$\mathbb{E}[X] = \frac{a+b}{2}$ $\text{Var}(X) = \frac{(b-a)^2}{12}$	$M_X(t) = \begin{cases} \frac{e^{tb}-e^{ta}}{t(b-a)} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$
$X \sim \text{Exp}(\beta)$	$f_X(x) = \dfrac{1}{\beta} e^{-x/\beta} \cdot \mathbb{1}_{\{x \geq 0\}}$	$\mathbb{E}[X] = \beta$ $\text{Var}(X) = \beta^2$	$M_X(t) = (1-\beta t)^{-1},\ t < 1/\beta$
$X \sim \text{Gamma}(\alpha, \beta)$	$f_X(x) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \cdot \mathbb{1}_{\{x \geq 0\}}$	$\mathbb{E}[X] = \alpha\beta$ $\text{Var}(X) = \alpha\beta^2$	$M_X(t) = (1-\beta t)^{-\alpha},\ t < 1/\beta$
$X \sim \mathcal{N}(\mu, \sigma^2)$	$f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\dfrac{(x-\mu)^2}{2\sigma^2}\right\}$	$\mathbb{E}[X] = \mu$ $\text{Var}(X) = \sigma^2$	$M_X(t) = \exp\left\{\mu t + \frac{\sigma^2}{2} t\right\}$