

Topic 5: Confidence Intervals

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Outline

1. Asymptotic Confidence Intervals using the MLE
2. Non-Normal Confidence Intervals

Asymptotic Confidence Intervals using the MLE



Result

Theorem (Asymptotic MLE Result)

Given an i.i.d. sample \vec{Y} from a sample with unknown parameter θ and $\hat{\theta}_{\text{MLE}}$, the maximum likelihood estimator for θ , we have, under certain “regularity conditions,” that

$$\frac{\tau(\hat{\theta}_{\text{MLE}}) - \tau(\theta)}{\sqrt{\frac{[\tau'(\theta)]^2}{\mathcal{I}_n(\theta)}}} \rightsquigarrow \mathcal{N}(0, 1)$$



Example

Example

Let $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$ for some unknown $\theta > 0$. Find the MLE for the population variance, and use this to construct a large-sample 95% confidence interval.

Non-Normal Confidence Intervals



Leadup

- Last time, we discussed how to construct confidence intervals under the assumption of a normally-distributed population. What do we do if our population is *not* normally distributed?



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- Well, there are quite a few options available to us!



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- For example, if $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, how might we construct a $(1 - \alpha) \times 100\%$ confidence interval for θ ?
- Well, there are quite a few options available to us!
- One of the most popular ways of constructing CIs is called the pivotal method (or method of pivots).



Pivots

Definition (Pivot)

Given a sample Y_1, \dots, Y_n from a distribution with unknown parameter θ , we define a **pivot** (or **pivotal quantity**) for θ to be a function $U := g(\vec{Y}, \theta)$ whose distribution doesn't depend on θ .



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- So, for example, if $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, 1)$, then $U := \sqrt{n}(\bar{Y}_n - \mu)$ is a pivot for μ since its distribution doesn't depend on μ .



Constructing CIs using the Pivotal Method

- Consider a sample Y_1, \dots, Y_n from a distribution with unknown parameter θ , and assume $U(\vec{Y}, \theta)$ is a pivotal quantity. Here's how we use the notion of a pivotal quantity to construct a $(1 - \alpha) \times 100\%$ confidence interval for θ :

- Set up the interval $\{a \leq U(\vec{Y}, \theta) \leq b\}$, and use the distribution of $U(\vec{Y}, \theta)$ to find a and b such that

$$\mathbb{P}(a \leq U(\vec{Y}, \theta) \leq b) = 1 - \alpha$$

- Invert the interval $\{a \leq U(\vec{Y}, \theta) \leq b\}$ to be of the form $\{\hat{\theta}_L \leq \theta \leq \hat{\theta}_U\}$, for random variables $\hat{\theta}_L$ and $\hat{\theta}_U$ that depend on $U(\vec{Y}, \theta)$ and a and b .



Example

Example

Suppose we have a single observation $Y \sim \text{Exp}(\theta)$. Construct a pivotal quantity for θ , and use this to construct a 95% confidence interval for θ .



Solution

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- But hey, we've done this a million times already (back in Topic 02 of this course)!

$$U \sim \text{Exp}\left(\frac{1}{\theta} \cdot \theta\right) \sim \text{Exp}(1)$$



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$$U \sim \text{Exp}\left(\frac{1}{\theta} \cdot \theta\right) \sim \text{Exp}(1)$$

- Because the distribution of U doesn't depend on θ , it is, by definition, a pivotal quantity for θ .



Solution

- So, our 95% CI will take the form

$$\left\{ a \leq \frac{Y}{\theta} \leq b \right\}$$

for constants a and b such that

$$\mathbb{P} \left(a \leq \frac{Y}{\theta} \leq b \right) = 0.95$$



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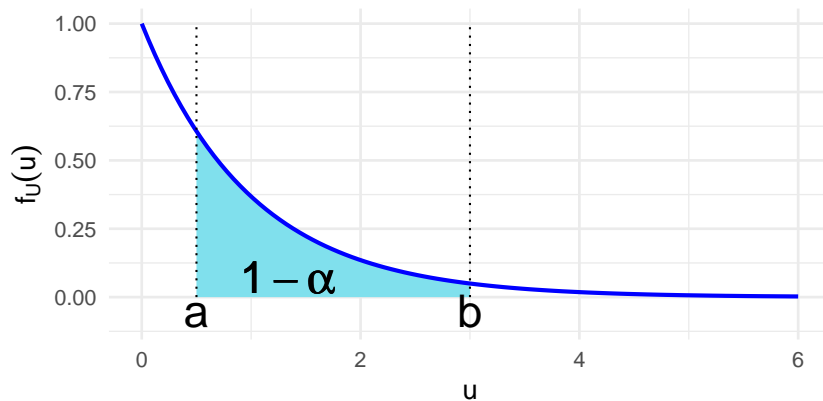
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- Let's pause, and sketch a picture.

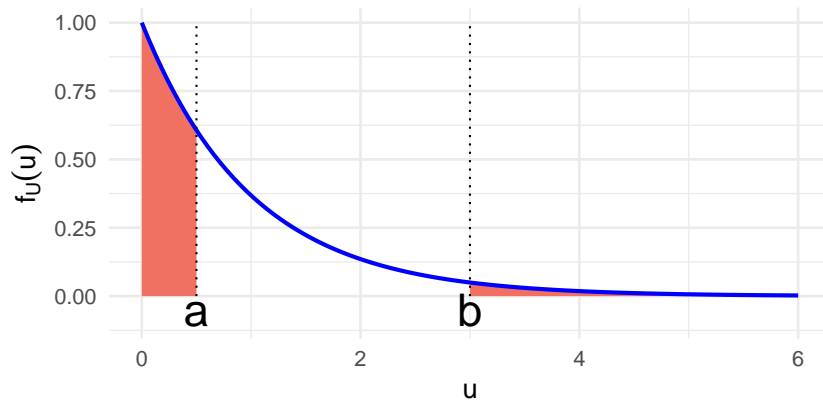


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- Well, since we know the distribution of (Y/θ) , we can compute these probabilities directly!

$$\mathbb{P}\left(\frac{Y}{\theta} < a\right)$$



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$$\mathbb{P}\left(\frac{Y}{\theta} > b\right) = \int_b^{\infty} f_U(u) \, du = \int_b^{\infty} e^{-u} \, du = e^{-b}$$

- Hence, $1 - e^{-a} = \alpha/2$ and $e^{-b} = \alpha/2$. Equivalently: $a = \ln\left(\frac{2}{2-\alpha}\right)$ and $b = \ln\left(\frac{2}{\alpha}\right)$.



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- Let me quickly explain what I mean by this. The interval we started with is

$$\left\{ a \leq \frac{Y}{\theta} \leq b \right\}$$

where a and b are given by the quantities we just found on the previous slide



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- But, this isn't the right form for a confidence interval - remember that a true confidence interval needs to have random *endpoints*. Right now, our randomness (i.e. the random variable Y) is in the *interior* of our interval.



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- But, not to fret!



Solution

$$\bullet \left\{ a \leq \frac{Y}{\theta} \right\}$$



Solution

$$\bullet \left\{ a \leq \frac{Y}{\theta} \right\} \implies \{a\theta \leq Y\}$$



Solution

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Solution

$$\begin{aligned} & \bullet \left\{ a \leq \frac{Y}{\theta} \right\} \implies \{a\theta \leq Y\} \implies \left\{ \theta \leq \frac{Y}{a} \right\} \\ & \bullet \left\{ \frac{Y}{\theta} \leq b \right\} \end{aligned}$$



Solution

- $\left\{a \leq \frac{Y}{\theta}\right\} \implies \{a\theta \leq Y\} \implies \left\{\theta \leq \frac{Y}{a}\right\}$
- $\left\{\frac{Y}{\theta} \leq b\right\} \implies \{Y \leq \theta b\}$



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- $\left\{\frac{Y}{\theta} \leq b\right\} \implies \{Y \leq \theta b\} \implies \left\{\theta \geq \frac{Y}{b}\theta\right\}$
- So, combining everything:

$$\left\{a \leq \frac{Y}{\theta} \leq b\right\} \implies \left\{\frac{Y}{b} \leq \theta \leq \frac{Y}{a}\right\}$$



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- So, combining everything:

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- This is now the right form for a CI - the randomness is in the endpoints!



Solution

- So, all in all, our $(1 - \alpha) \times 100\%$ confidence interval for θ takes the form

$$\left[\frac{Y}{b}, \frac{Y}{a} \right]$$

or, substituting in our expressions for a and b :

$$\left[\frac{Y}{\ln\left(\frac{2}{\alpha}\right)}, \frac{Y}{\ln\left(\frac{2}{2-\alpha}\right)} \right]$$



Example

Example

The wait time of a randomly-selected person at *The Arbor* is assumed to follow an exponential distribution with parameter θ . If a single person was selected at random, and it was observed that they spent 3 minutes waiting in line at *The Arbor*, construct a 95% confidence interval for θ , the true average wait times at *The Arbor*.



Solution

- All we need to do is plug into our CI from the previous example!

$$\left[\frac{3}{\ln\left(\frac{2}{0.05}\right)}, \frac{3}{\ln\left(\frac{2}{2-0.05}\right)} \right] = [0.813, 118.494]$$



Solution

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$$\left[\frac{3}{\ln\left(\frac{2}{0.05}\right)}, \frac{3}{\ln\left(\frac{2}{2-0.05}\right)} \right] = [0.813, 118.494]$$

- If that looks like a really wide interval.. it's because it is! But, remember that we only had a single sample. So, intuitively, it perhaps makes sense that our CI should be pretty wide.



Example

Example

Suppose $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, where $\theta > 0$ is an unknown parameter.

- (a) Propose a pivotal quantity for θ that is a function of $\sum_{i=1}^n Y_i$, and show that your proposed quantity is in fact a pivot for θ .
- (b) Use your pivot from part (a) to construct a $(1 - \alpha) \times 100\%$ confidence interval for θ .



Solution

- We know that $(\sum_{i=1}^n Y_i) \sim \text{Gamma}(n, \theta)$.



Solution

- We know that $(\sum_{i=1}^n Y_i) \sim \text{Gamma}(n, \theta)$.
- Therefore, $(1/\theta)(\sum_{i=1}^n Y_i) = [\sum_{i=1}^n (Y_i/\theta)] \sim \text{Gamma}(n, 1)$, which doesn't depend on θ .



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- We know that $(\sum_{i=1}^n Y_i) \sim \text{Gamma}(n, \theta)$.
- Therefore, $(1/\theta)(\sum_{i=1}^n Y_i) = [\sum_{i=1}^n (Y_i/\theta)] \sim \text{Gamma}(n, 1)$, which doesn't depend on θ .
- Hence, a pivotal quantity for θ involving $\sum_{i=1}^n Y_i$ is

$$U := \sum_{i=1}^n \left(\frac{Y_i}{\theta} \right)$$



Solution

- Is this the only pivotal quantity for θ that involves $\sum_{i=1}^n Y_i$?



Solution

- Is this the only pivotal quantity for θ that involves $\sum_{i=1}^n Y_i$? No! In fact, we can multiply U by any positive constant [that doesn't depend on θ] and get another pivotal quantity for θ .
- For example,

$$U_2 := 2 \sum_{i=1}^n \left(\frac{Y_i}{\theta} \right) \sim \text{Gamma}(n, 2) \sim \chi_{2n}^2$$

thereby showing that U_2 is *also* a pivotal quantity for θ .



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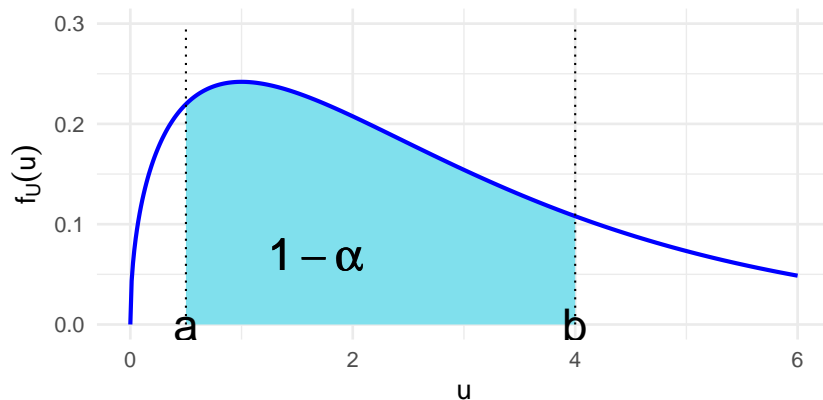
$$\left\{ a \leq 2 \sum_{i=1}^n \left(\frac{Y_i}{\theta} \right) \leq b \right\}$$

- By the definition of coverage probability, we seek a and b such that

$$\mathbb{P} \left(a \leq 2 \sum_{i=1}^n \left(\frac{Y_i}{\theta} \right) \leq b \right) = 1 - \alpha$$

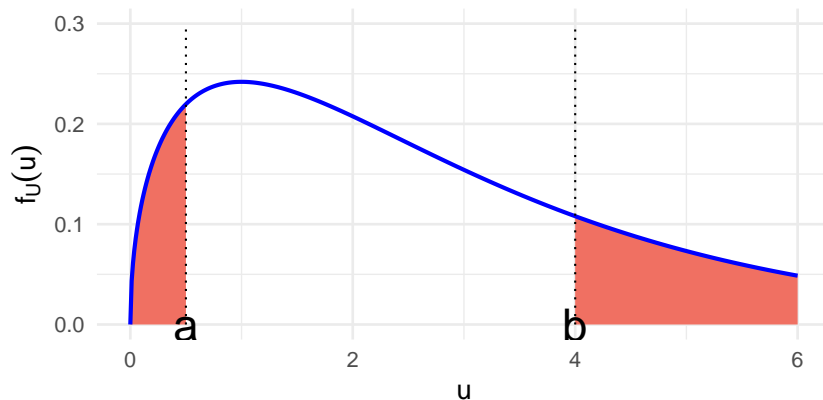


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- So, we want

$$\mathbb{P}(U_2 < a) = \frac{\alpha}{2} \quad \text{and} \quad \mathbb{P}(U_2 > b) = \frac{\alpha}{2}$$



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- Now, we don't have a simply closed-form expression for the CDF of the χ^2_{2n} distribution. But that's fine - we can still write

$$a = F_{\chi^2_{2n}}^{-1}\left(\frac{\alpha}{2}\right) \quad \text{and} \quad b = F_{\chi^2_{2n}}^{-1}\left(1 - \frac{\alpha}{2}\right)$$



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- $\left\{ \frac{2}{\theta} \sum_{i=1}^n Y_i \leq b \right\} \implies \left\{ \theta \geq \frac{2}{b} \sum_{i=1}^n Y_i \right\}$
- So, our interval is

$$\left[\frac{2 \sum_{i=1}^n Y_i}{F_{\chi_{2n}^2}^{-1} \left(1 - \frac{\alpha}{2} \right)}, \frac{2 \sum_{i=1}^n Y_i}{F_{\chi_{2n}^2}^{-1} \left(\frac{\alpha}{2} \right)} \right]$$



Solution

- Let's see what would have happened if we used our other pivotal quantity, $U_1 := (1/\theta) \sum_{i=1}^n Y_i$.



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- We'd start with

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- Ultimately, we'd find (and I encourage you to fill in these steps)

$$a = F_{\text{Gamma}(n,1)}^{-1} \left(\frac{\alpha}{2} \right); \quad b = F_{\text{Gamma}(n,1)}^{-1} \left(1 - \frac{\alpha}{2} \right)$$



Solution

- Appropriately inverting the interval (and, again, you can try this on your own), we get

$$\left[\frac{\sum_{i=1}^n Y_i}{F_{\text{Gamma}(n,1)}^{-1}\left(1 - \frac{\alpha}{2}\right)}, \frac{\sum_{i=1}^n Y_i}{F_{\text{Gamma}(n,1)}^{-1}\left(\frac{\alpha}{2}\right)} \right]$$



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- Do these give the same numerical values, given a particular observed dataset? Let's check!



Solution

- Take $n = 7$ and $\bar{y}_7 = 4.77$ (so $\sum_{i=1}^n y_i = 33.39$) and adopt a 95% coverage probability.



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- Our first CI, using the χ^2_{2n} distribution, becomes

$$\left[\frac{2(33.39)}{F_{\chi^2_{14}}^{-1}(0.975)}, \frac{2 \sum_{i=1}^n Y_i}{F_{\chi^2_{14}}^{-1}(0.025)} \right] = [2.557, 11.864]$$



Solution

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- Our first CI, using the $\text{Gamma}(n, 1)$ distribution, becomes

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Normal CI

- Finally, to tie together yesterday's lecture and today's lecture, let's see if we can actually use the pivotal method to recover the same sorts of normal-adjacent confidence intervals we constructed before.



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- So, to start, let $Y_1, \dots, Y_n \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ is unknown but $\sigma^2 > 0$ is known.



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- A pivotal quantity is

$$U := \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$$



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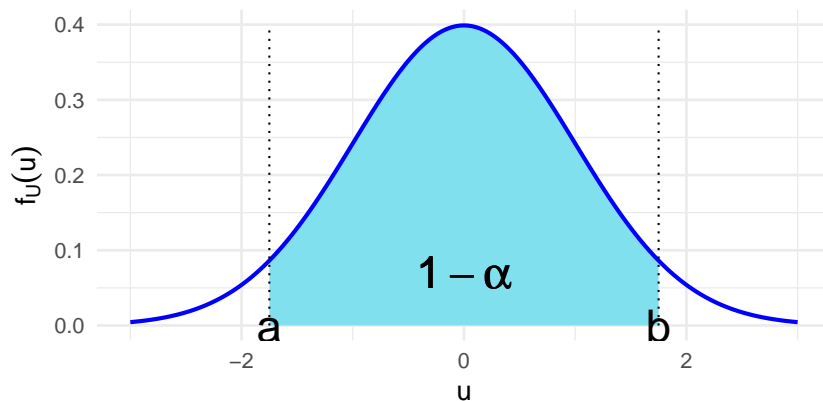
$$\left\{ a \leq \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \leq b \right\}$$

- Since we want a $(1 - \alpha) \times 100\%$ coverage probability, we get

$$\mathbb{P} \left(a \leq \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \leq b \right) = 1 - \alpha$$

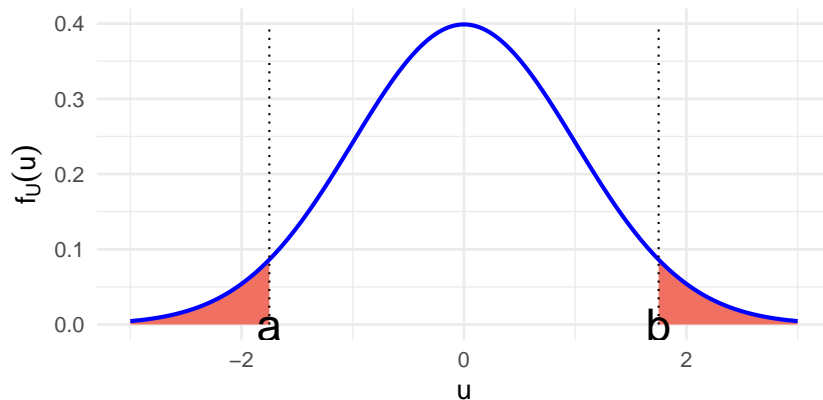


Normal CI





Normal CI





Normal CI

$$\bullet \mathbb{P} \left(\frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} < a \right) = \Phi(a) \stackrel{!}{=} \frac{\alpha}{2} \implies a = \Phi^{-1} \left(\frac{\alpha}{2} \right)$$



Normal CI

- $\mathbb{P}\left(\frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} < a\right) = \Phi(a) \stackrel{!}{=} \frac{\alpha}{2} \implies a = \Phi^{-1}\left(\frac{\alpha}{2}\right)$
- $\mathbb{P}\left(\frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} > b\right) = 1 - \Phi(b) \stackrel{!}{=} \frac{\alpha}{2} \implies b = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$



Normal CI

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- $\mathbb{P}\left(\frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} > b\right) = 1 - \Phi(b) \stackrel{!}{=} \frac{\alpha}{2} \implies b = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$
- Now, we invert our interval:



Normal CI

$$\bullet \left\{ a \leq \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \right\}$$



Normal CI

$$\bullet \left\{ a \leq \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \right\} \implies \left\{ a \cdot \frac{\sigma}{\sqrt{n}} \leq \bar{Y}_n - \mu \right\}$$



Normal CI

$$\bullet \left\{ a \leq \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \right\} \implies \left\{ a \cdot \frac{\sigma}{\sqrt{n}} \leq \bar{Y}_n - \mu \right\} \implies \left\{ \mu \leq \bar{Y}_n - a \cdot \frac{\sigma}{\sqrt{n}} \right\}$$



Normal CI

$$\begin{aligned} & \bullet \left\{ a \leq \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \right\} \implies \left\{ a \cdot \frac{\sigma}{\sqrt{n}} \leq \bar{Y}_n - \mu \right\} \implies \left\{ \mu \leq \bar{Y}_n - a \cdot \frac{\sigma}{\sqrt{n}} \right\} \\ & \bullet \left\{ \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \leq b \right\} \end{aligned}$$



Normal CI

- $\left\{ a \leq \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \right\} \implies \left\{ a \cdot \frac{\sigma}{\sqrt{n}} \leq \bar{Y}_n - \mu \right\} \implies \left\{ \mu \leq \bar{Y}_n - a \cdot \frac{\sigma}{\sqrt{n}} \right\}$
- $\left\{ \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \leq b \right\} \implies \left\{ \bar{Y}_n - \mu \leq b \cdot \frac{\sigma}{\sqrt{n}} \right\}$



Normal CI

- $\left\{ a \leq \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \right\} \implies \left\{ a \cdot \frac{\sigma}{\sqrt{n}} \leq \bar{Y}_n - \mu \right\} \implies \left\{ \mu \leq \bar{Y}_n - a \cdot \frac{\sigma}{\sqrt{n}} \right\}$
- $\left\{ \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \leq b \right\} \implies \left\{ \bar{Y}_n - \mu \leq b \cdot \frac{\sigma}{\sqrt{n}} \right\} \implies \left\{ \mu \geq \bar{Y}_n - b \cdot \frac{\sigma}{\sqrt{n}} \right\}$



Normal CI

- $\left\{ a \leq \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \right\} \implies \left\{ a \cdot \frac{\sigma}{\sqrt{n}} \leq \bar{Y}_n - \mu \right\} \implies \left\{ \mu \leq \bar{Y}_n - a \cdot \frac{\sigma}{\sqrt{n}} \right\}$
- $\left\{ \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \leq b \right\} \implies \left\{ \bar{Y}_n - \mu \leq b \cdot \frac{\sigma}{\sqrt{n}} \right\} \implies \left\{ \mu \geq \bar{Y}_n - b \cdot \frac{\sigma}{\sqrt{n}} \right\}$
- So our interval is equivalent to

$$\left\{ \bar{Y}_n - b \cdot \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{Y}_n - a \cdot \frac{\sigma}{\sqrt{n}} \right\}$$



Normal CI

- Therefore, plugging in the values for a and b we derived earlier on, our $(1 - \alpha) \times 100\%$ CI, as derived using the method of pivotal quantities, is

$$\left[\bar{Y}_n - \Phi^{-1} \left(1 - \frac{\alpha}{2} \right) \cdot \frac{\sigma}{\sqrt{n}} , \bar{Y}_n - \Phi^{-1} \left(\frac{\alpha}{2} \right) \cdot \frac{\sigma}{\sqrt{n}} \right]$$



Normal CI

- Therefore, plugging in the values for a and b we derived earlier on, our $(1 - \alpha) \times 100\%$ CI, as derived using the method of pivotal quantities, is

$$\left[\bar{Y}_n - \Phi^{-1} \left(1 - \frac{\alpha}{2} \right) \cdot \frac{\sigma}{\sqrt{n}}, \bar{Y}_n - \Phi^{-1} \left(\frac{\alpha}{2} \right) \cdot \frac{\sigma}{\sqrt{n}} \right]$$

- Finally, note that $-\Phi^{-1} \left(\frac{\alpha}{2} \right) = \Phi^{-1} \left(1 - \frac{\alpha}{2} \right)$ to see that our interval is

$$\left[\bar{Y}_n - \Phi^{-1} \left(1 - \frac{\alpha}{2} \right) \cdot \frac{\sigma}{\sqrt{n}}, \bar{Y}_n + \Phi^{-1} \left(1 - \frac{\alpha}{2} \right) \cdot \frac{\sigma}{\sqrt{n}} \right] = \bar{Y}_n \pm \Phi^{-1} \left(1 - \frac{\alpha}{2} \right) \cdot \frac{\sigma}{\sqrt{n}}$$