Topic 6: Hypothesis Testing

Ethan P. Marzban University of California, Santa Barbara PSTAT 120B

Outline

1. Power of a Test

Power of a Test

UC SANTA BARBARA
Department of Statistics
and Applied Probability

Power

- Recall that α (the significance level) denotes the probability of committing a Type I error, and β denotes the probability of comitting a Type II error.

Power

- Recall that α (the significance level) denotes the probability of committing a Type I error, and β denotes the probability of comitting a Type II error.
- We can analogously define a quantity that represents the probability that a given test will lead to rejection of the null:

UC SANTA BARBARA
Department of Statistics and Applied Probability

Power

Definition (Power)

Suppose that W is the test statistic and $\mathcal{R}$ is the rejection region for a test of a hypothesis involving the value of a parameter θ. Then the power of the test, denoted by power(θ), is the probability that the test will lead to rejection of H_0 when the actual parameter value is θ. That is,

$$\text{power}(\theta) = \mathbb{P}(W \in \mathcal{R} \text{ when the parameter value is } \theta)$$

UC SANTA BARBARA
Department of Statistics
and Applied Probability

Power

Theorem (Relationship between Power and β)

If θ_A is a value of θ in the alternative hypothesis H_A, then

$$\mathrm{power}(\theta_A) = 1 - \beta(\theta_A)$$

where $\beta(\theta_A)$ denotes the probability of committing a Type II error when the true value of θ is θ_A.

Power

- As the notation suggests, we typically view power as a function of the true value of θ_A.

UC SANTA BARBARA
Department of Statistics and Applied Probability

Power

- As the notation suggests, we typically view power as a function of the true value of θ_A.
- Plotting the power of a given test at a series of specified values in the alternative space yields a so-called **power curve**.

UC SANTA BARBARA
Department of Statistics and Applied Probability

Power

- As the notation suggests, we typically view power as a function of the true value of θ_A.
- Plotting the power of a given test at a series of specified values in the alternative space yields a so-called **power curve**.
- Let's think through what the "ideal" power curve looks like.

UC SANTA BARBARA
Department of Statistics and Applied Probability

Power

- As the notation suggests, we typically view power as a function of the true value of θ_A.
- Plotting the power of a given test at a series of specified values in the alternative space yields a so-called **power curve**.
- Let's think through what the "ideal" power curve looks like.
- What would we like power(θ_0) to be?

Power

- As the notation suggests, we typically view power as a function of the true value of θ_A.
- Plotting the power of a given test at a series of specified values in the alternative space yields a so-called **power curve**.
- Let's think through what the "ideal" power curve looks like.
- What would we like power(θ_0) to be?
- Well, since power(θ_A) is, by definition and for any point θ_A, the probability of rejecting $H_0 : \theta = \theta_0$ when the true value of θ is θ_A, we'd like power$(\theta_0) = 0$.

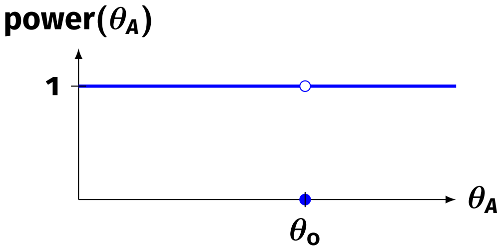
Power

- Similarly, for any $\theta_A \neq \theta_0$, we'd like $\text{power}(\theta_A) = 1$.

UC SANTA BARBARA
Department of Statistics
and Applied Probability

Power

- Similarly, for any $\theta_A \neq \theta_0$, we'd like $\text{power}(\theta_A) = 1$.
- So, the ideal power curve for a test would look like



UC SANTA BARBARA
Department of Statistics and Applied Probability

Power

- Now, keep in mind that all tests are performed at a fixed α level of significance.

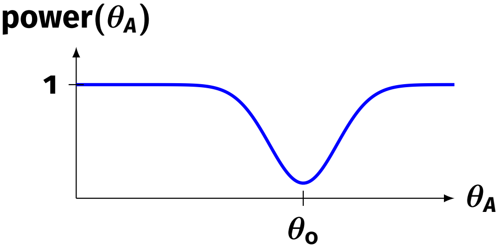
UC SANTA BARBARA
Department of Statistics
and Applied Probability

Power

- Now, keep in mind that all tests are performed at a fixed α level of significance.
- As we discussed before, it's impossible to simultaneously minimize α and β - hence, it's impossible to get a power of exactly zero.

Power

- Now, keep in mind that all tests are performed at a fixed α level of significance.
- As we discussed before, it's impossible to simultaneously minimize α and β - hence, it's impossible to get a power of exactly zero.
- A more realistic power curve for a test of $H_0 : \theta = \theta_0$ vs $H_A : \theta \neq \theta_0$ might look like



Example

Example

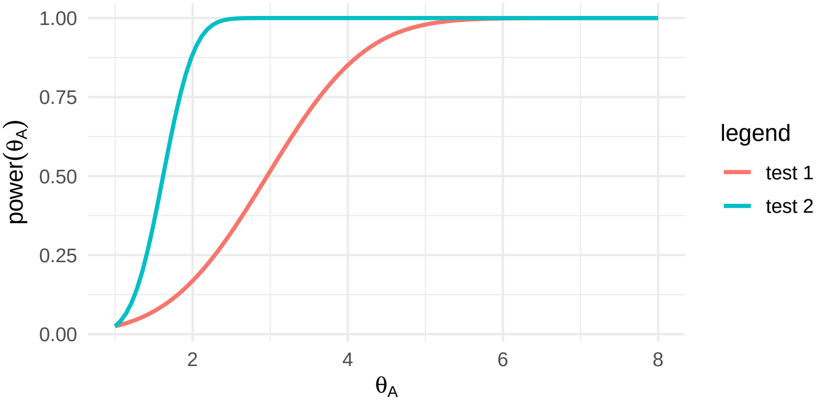
Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\mu, 1)$ for some unknown $\mu \in \mathbb{R}$, and suppose we wish to conduct a test of $H_0 : \mu = \mu_0$ vs $H_A : \mu > \mu_0$ at an $\alpha = 0.05$ level of significance. We propose two tests:

$$\textbf{Test 1: } \text{Reject } H_0 \text{ when } Y_1 - \mu_0 > \Phi^{-1}(0.975)$$

$$\textbf{Test 2: } \text{Reject } H_0 \text{ when } \frac{\overline{Y}_n - \mu_0}{1/\sqrt{n}} > \Phi^{-1}(0.975)$$

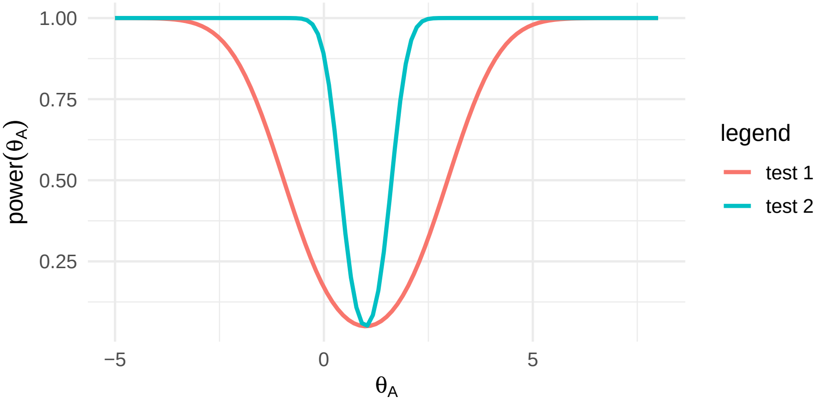
Derive expressions for the power functions for these two tests, and use this to determine if one test outperforms the other in terms of power for *all* values of θ in the alternative.

Power



UC SANTA BARBARA
Department of Statistics
and Applied Probability

Power



Power

- Since we want the power of our test to be 1 nearly everywhere, we often seek **uniformly most powerful tests**.

UC SANTA BARBARA
Department of Statistics and Applied Probability

Power

- Since we want the power of our test to be 1 nearly everywhere, we often seek **uniformly most powerful tests**.
- In general, finding such tests is very challenging (and, indeed, such tests don't always exist).

UC SANTA BARBARA
Department of Statistics and Applied Probability

Power

- Since we want the power of our test to be 1 nearly everywhere, we often seek **uniformly most powerful tests**.
- In general, finding such tests is very challenging (and, indeed, such tests don't always exist).
- However, if we restrict ourselves to a *simple-vs-simple* test, we actually *can* construct a most powerful test at a level α, using what is known as the **Neyman-Pearson Lemma**.

Neyman-Pearson Lemma

- Since we want the power of our test to be 1 nearly everywhere, we often seek **uniformly most powerful tests**.

UC SANTA BARBARA
Department of Statistics and Applied Probability

Neyman-Pearson Lemma

- Since we want the power of our test to be 1 nearly everywhere, we often seek **uniformly most powerful tests**.
- In general, finding such tests is very challenging (and, indeed, such tests don't always exist).

UC SANTA BARBARA
Department of Statistics and Applied Probability

Neyman-Pearson Lemma

- Since we want the power of our test to be 1 nearly everywhere, we often seek **uniformly most powerful tests**.
- In general, finding such tests is very challenging (and, indeed, such tests don't always exist).
- However, if we restrict ourselves to a *simple-vs-simple* test, we actually *can* construct a most powerful test at a level α, using what is known as the **Neyman-Pearson Lemma**.

Neyman-Pearson Lemma

THEOREM **10.1**

The Neyman–Pearson Lemma Suppose that we wish to test the simple null hypothesis $H_0 : \theta = \theta_0$ versus the simple alternative hypothesis $H_a : \theta = \theta_a$, based on a random sample $Y_1, Y_2, \ldots, Y_n$ from a distribution with parameter θ. Let $L(\theta)$ denote the likelihood of the sample when the value of the parameter is θ. Then, for a given α, the test that maximizes the power at θ_a has a rejection region, RR, determined by

$$\frac{L(\theta_0)}{L(\theta_a)} < k.$$

The value of k is chosen so that the test has the desired value for α. Such a test is a most powerful α-level test for H_0 versus H_a.

Likelihood Ratio Test

Definition (Likelihood Ratio Test)

Consider hypotheses $H_0 : \Theta \in \Omega_0$ and $H_A : \Theta \in \Omega_A$. Define

$$\Lambda := \frac{\mathcal{L}(\hat{\Omega}_0)}{\mathcal{L}(\hat{\Omega})} = \frac{\max_{\Theta \in \Omega_0} \mathcal{L}_{\vec{\mathbf{Y}}}(\Theta)}{\max_{\Theta \in \Omega_0 \cup \Omega_A} \mathcal{L}_{\vec{\mathbf{Y}}}(\Theta)}$$

A **likelihood ratio test** (named as such because we call Λ a **likelihood ratio**) rejects H_0 whenever $\{\Lambda < k\}$.

UC SANTA BARBARA
Department of Statistics
and Applied Probability

Example

Example

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$. Construct the likelihood ratio test for $H_0 : \theta = \theta_0$ vs $H_A : \theta \neq \theta_0$, using an α level of significance. You do not need to explicitly solve for constants; just derive the general form for the LRT.