Topic 02: Transformations

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Outline

1. Univariate Transformations

2. Method of Distribution Functions (CDF Method)

3. Method of Transformations (Change of Variable Formula)

4. Method of Moment-Generating Functions (MGF Method)

5. Multivariate Transformations

Leadup

- Recall, from PSTAT 120A, that given an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we can think of a **random variable** X as a mapping:

$$X : \Omega \to \mathbb{R}$$

- Additionally, recall the following fact from precalculus: given a mapping $f_1 : A \to B$ and another mapping $f_2 : B \to C$, then $(f_2 \circ f_1) : A \to C$.
- This means, given a function $g : \mathbb{R} \to \mathbb{R}$ and a random variable $X : \Omega \to \mathbb{R}$, we have $(g \circ X) : \Omega \to \mathbb{R}$.

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Leadup

- In this way, we can think of $(g \circ X)$ as a random variable itself!
    - For example, given a random variable X, then the quantity X^2 will also be a random variable.
- Another way of saying this: functions of random variables are themselves random variables.
- "Functions of random variables?" That sounds awfully abstract...
- But, if we think about it a bit more, this isn't as abstract as it may seem!

Leadup

- For example, let H_I denote the height of a randomly-selected individual as measured in inches, and suppose $H_I \sim \mathcal{N}(70, 2)$.
- Let H_C denote the height of a randomly-selected individual as measured in <u>centimeters</u>.
- Clearly, the random variables H_I and H_C are related: specifically, $H_C = g(H_I)$ where $g(t) = 2.54 * t$ [since this is the conversion formula between inches and centimeters].
  - So, **<u>unit conversion</u>** is a fairly simple example of one way transformations (i.e. taking functions of random variables) can be useful.

Leadup

- Transformations can also be used to **summarize** data.
- For example, consider a sequence $\{X_i\}_{i=1}^n := X_1, \cdots, X_n$ of random variables.
  - By the way, I'll be using this notation a lot: $\{X_i\}_{i=1}^n$ is a shorthand for $X_1, \cdots, X_n$.
- The **sample mean** $\overline{X}_n := n^{-1} \sum_{i=1}^n X_i$ [which you hopefully saw in PSTAT 120A!] is actually a *function* of the original sequence of random variables, and is hence an example of a transformation.

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Leadup

- Now, these two examples indicate that there are perhaps two sub-cases to consider: transformations of *single* random variables, and transformations of *multiple* random variables.
  - We often refer to a transformation of a single random variable as a **univariate transformation**, and a transformation of multiple random variables as a **multivariate transformation**.

- For simplicity's sake, let's start off with univariate transformations.
  - Specifically, given a random variable Y and a function $g : \mathbb{R} \to \mathbb{R}$, we will seek to explore properties of the random variable $U := g(Y)$.

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Univariate Transformations

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Goal

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- What do we mean by "describe" the random variable U?
- Well, there are a couple of things we could seek to do. First, we could try to compute $\mathbb{E}[U]$.

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LOTUS

- It turns out... we've already done that!
- Specifically, since $U := g(Y)$, we have that $\mathbb{E}[U] = \mathbb{E}[g(Y)]$.
- The **<u>Law of the Unconscious Statistician</u>** (LOTUS), which we saw in PSTAT 120A, tells us

$$\mathbb{E}[g(Y)] = \int_{\mathbb{R}} g(y) f_Y(y) \ \mathrm{d}y$$

- Similar considerations will allow us to compute $\mathrm{Var}(U)$.

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Distributions

- Okay, that's useful! But it's not the whole picture.
- Why don't we get a little more ambitious, and seek to find the *distribution* of U?
- First, let me be a little more clear about what I mean by "distribution".
- Sometimes, we can identify a distribution by name (e.g. "Exponential distribution with parameter $\theta = 0.5$", or "Standard normal distribution").
- But, a distribution could just as easily have been described by any of the following:
  - Its **distribution function** (i.e. CDF)
  - Its **density function** (PDF)
  - Its **MGF** (moment-generating function)

Distributions

- For example, suppose I tell you the random variable W has density function given by
$$f_W(w) = 2e^{-2w} \cdot \mathbb{1}_{\{w \geq 0\}}$$
- You would immediately be able to tell me "oh, W follows the Exponential distribution with parameter $\theta = 1/2$."
- This would, in turn, automatically tell you that W has distribution function
$$F_W(w) = \begin{cases} 1 - e^{-2w} & \text{if } w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$
and MGF
$$M_W(t) = \begin{cases} (1 - t/2)^{-1} & \text{if } t < 1/2 \\ \infty & \text{otherwise} \end{cases}$$

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Distributions

- Similarly, if I tell you that the random variable T has MGF given by

$$M_X(t) = \exp\left\{2t + \frac{1}{2}t^2\right\}$$

you would immediately be able to say

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-2)^2\right\}$$

and

$$F_X(x) = \Phi\left(x - 2\right); \qquad \Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,\mathrm{d}z$$

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Distributions

- Now, what if we have a random variable X whose density is given by

$$f_X(x) = \cos(x) \cdot \mathbb{1}_{\{0 \leq x \leq \pi/2\}}$$

- What is the distribution of X?
- Well… it's just the density above!
- What I mean is this - the distribution of X doesn't have a name, like "Exponential" or "Gamma". But it certainly *has* a distribution!
- All of this is to say: I encourage you to get into the habit of thinking about "distributions" fairly broadly, and thinking of a distribution as either a density function, distribution function, or MGF (or all three).

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Back to Transformations

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- Now, our discussion on the previous few slides tells us that there are three approaches to achieving our goal above.
- We could go after the density function of U.
- Or we could go after the distribution function of U.
- Or we could go after the MGF of U.
- Indeed, each of these three approaches are what our textbook calls different "methods".

Support

- Before we dive into these three methods, let's talk a bit about **support**.
- Recall that the support (aka "state space") of a random variable Y is the set of all values that Y maps to: i.e. $S_Y := Y(\Omega)$. Equivalently, it's the set of all values y for which the density $f_Y(y)$ is nonzero.
- Then, given a random variable $U := g(Y)$, we have $S_U = g(S_Y)$.
  - That is, the support of a transformed random variable is the image of the original support under the transformation.

- Though this formula seems inoccuous enough, finding the support of a transformed random variable can be trickier than it first appears...

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Support

- A simple way I like to think about things is to draw a picture.
- Specifically, let's say we have an interval $[a, b]$ and a transformation $g : \mathbb{R} \to \mathbb{R}$.
- To figure out what $g([a, b])$ looks like, simply graph the function $y = g(x)$, indicate $[a, b]$ on the $x-$axis, and figure out what the corresponding values on the $y-$axis are.

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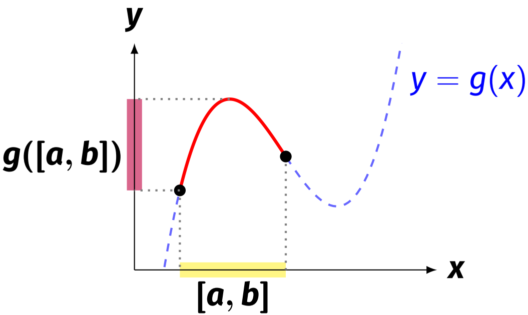
Support



- Note: in general, $g([a, b]) \neq [g(a), g(b)]$!

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Clicker Question!

Clicker Question 1

For $A = [0\ , 6]$ and $g(x) = \cos(\pi x)$, what is the correct expression for $g(A)$?

(A) $[0\ ,\ 1]$ (B) $[0\ ,\ 6]$ (C) $[-1\ , 1]$ (D) $\{0\}$
(E) None of the above

Try this On Your Own:

Example

For $A = [-1\ ,\ 1]$ and $g(x) = x^2$, what is the correct expression for $g(A)$?

Method of Distribution Functions (CDF Method)

CDF Method

- Let's consider the following rephrasing of our goal:

Goal

Given a random variable Y and a function $g()$, we seek to derive an expression for $F_U(u) := \mathbb{P}(U \leq u)$, the CDF of U.

- As a concrete example, let $Y \sim \text{Exp}(\theta)$ and let $U := cY$ for a positive constant c.
  - If it helps, you can think of this in terms of our inches-to-centimeter conversion example from the start of this lecture: Y can denote the heights in inches and U can denote the heights in cenimeters.

CDF Method

- Now, we know everything we could want to know about Y.
- Specifically, we have the CDF of Y:

$$F_Y(y) = \begin{cases} 1 - e^{-y/\theta} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

- So, if we can relate $F_U(u)$ to $F_Y(y)$, we'd be done.
- Note:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(cY \leq u)$$

- Divide through by c:

$$F_U(u) = \mathbb{P}\left(Y \leq \frac{u}{c}\right) = F_Y\left(\frac{u}{c}\right)$$

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CDF Method

- So, plugging into our expression for $F_Y(y)$, we have:

$$
\begin{aligned}
F_U(u) &= F_Y\left(\frac{u}{c}\right) \\
&= \begin{cases} 1 - e^{(u/c)/\theta} & \text{if } (u/c) \geq 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \boxed{\begin{cases} 1 - e^{u/(c\theta)} & \text{if } u \geq 0 \\ 0 & \text{otherwise} \end{cases}}
\end{aligned}
$$

- And we're done! We've accomplished our goal, and found an expression for $F_U(u)$, the CDF of U.

Going Further

- Now, in this particular case, we can take things a step further.
- Specifically, doesn't that CDF look awfully familiar?
- Indeed, it is the CDF of the $\text{Exp}(c\theta)$ distribution!
- So, what we've essentially shown is:

Theorem (Closure of Exponential Distribution under Multiplication)

Given $Y \sim \text{Exp}(\theta)$ and a positive constant c, then $(cY) \sim \text{Exp}(c\theta)$.

- We're going to use this result a **LOT**!

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Interpretation

- I know this might seem a little abstract - what does it mean to "multiply the exponential distribution by a constant?"
- Again, if it helps, you can always think in terms of our inches-to-centimeter problem from the start of these slides.
- If $Y \sim \text{Exp}(\theta)$ denotes the height of a randomly selected person in inches, then the distribution of heights in centimeters will *also* be exponential, this time with mean 2.54θ.

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Example

- Let's do another example together.
- Suppose Y has density function given by

$$f_Y(y) = 2y^2 \cdot \mathbb{1}_{\{0 \leq y \leq 1\}}$$

  and again define $U := cY$ for a positive constant c.
- Now, before we got lucky because we immediately knew what the CDF of Y was.
- But, even though we can't *immediately* recognize the CDF of Y in this example, we can still derive it!

Example

- By definition, for a $y \in [0, 1]$,

$$\begin{aligned} F_Y(y) &= \int_{-\infty}^{y} f_Y(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{y} 2t \cdot \mathbb{1}_{\{0 \le t \le 1\}}\,\mathrm{d}t = \int_0^y 2t\,\mathrm{d}t = y^2 \end{aligned}$$

- Clearly, for $y < 0$ we have $F_Y(y) = \mathbb{P}(Y \le y) = 0$ and for $y > 1$ we have $\mathbb{P}(Y \le y) = 1$, meaning

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ y^2 & \text{if } 0 \le y < 1 \\ 1 & \text{if } y \ge 1 \end{cases}$$

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Example

- And now we're in the same position as before!

$$
\begin{aligned}
\mathbb{P}(U \leq u) &= \mathbb{P}(cY \leq u) = \mathbb{P}\left(Y \leq \frac{u}{x}\right) \\
&= F_Y\left(\frac{u}{c}\right) \\
&= \begin{cases} 0 & \text{if } (u/c) < 0 \\ (u/c)^2 & \text{if } 0 \leq (u/c) < 1 \\ 1 & \text{if } (u/c) \geq 1 \end{cases} = \boxed{\begin{cases} 0 & \text{if } u < 0 \\ u^2/c^2 & \text{if } 0 \leq u < c \\ 1 & \text{if } u \geq c \end{cases}}
\end{aligned}
$$

Example

- One more example before we summarize.
- Let $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$.
- A quick sketch (see chalkboard) reveals that $S_U = [0, \infty)$. So, $F_U(u) = 0$ whenever $u < 0$.
- Additionally, we (again) have the CDF of Y: $F_Y(y) = \Phi(y)$, where $\Phi(\cdot)$ denotes the standard normal CDF.

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Example

- So, let's try and proceed like we did before! For a fixed $u \geq 0$,

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(Y^2 \leq u)$$

- Now, it's tempting to continue this as

$$F_U(u) = \mathbb{P}(Y^2 \leq u) = \mathbb{P}(Y \leq \sqrt{u})$$

This is, however, <u>INCORRECT</u>.

- Let's understand why.

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Example

- There are a couple of ways to understand why the above is incorrect.
- One is to recall a fact from algebra/precalculus that you might have forgotten: $\sqrt{\cdot}$ means the *principal* square root, and so, for any real number x, we have $\sqrt{x^2} = |x|$.
  - Remember, both -3 and 3 have squares equal to 9! But, when we write $\sqrt{9}$, we implicitly mean the principal square root which is why we write $\sqrt{9} = 3$.
- So, what we really have is:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(Y^2 \leq u) = \mathbb{P}(|Y| \leq \sqrt{u}) = \mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u})$$

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Example (cont'd)

- Now, there's another way to see how to get from $\mathbb{P}(Y^2 \leq u)$ to $\mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u})$; one that doens't require us to dig into our memory banks and dredge up something from algebra/precalculus, and instead uses pictures.

Video

https://www.youtube.com/watch?v=HtzqjHfoRbw

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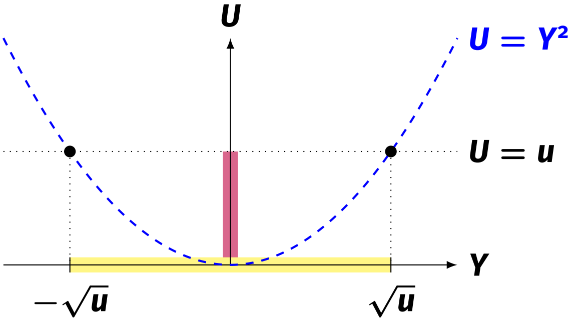
Static Image



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Example (cont'd)

- So, let's finish up our example!

$$
\begin{aligned}
F_U(u) &= \cdots = \mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u}) \\
&= F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) = \Phi(\sqrt{u}) - \Phi(-\sqrt{u}) \\
&= \Phi(\sqrt{u}) - [1 - \Phi(\sqrt{u})] = 2\Phi(\sqrt{u}) - 1
\end{aligned}
$$

- That's a bit anticlimactic... Let's differentiate wrt. u and obtain the PDF of U:

Example (cont'd)

$$
\begin{aligned}
f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\
&= \frac{\mathrm{d}}{\mathrm{d}u} [2\Phi(\sqrt{u}) - 1] \\
&= 2 \cdot \frac{1}{2\sqrt{u}} \cdot \phi(\sqrt{u}) = \frac{1}{\sqrt{u}} \phi(\sqrt{u})
\end{aligned}
$$

- Let's incorporate the support of U, and simplify:

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Example (cont'd)

$$\begin{aligned} f_U(u) &= \frac{1}{\sqrt{u}} \phi(\sqrt{u}) \cdot \mathbb{1}_{\{u \geq 0\}} \\ &= \frac{1}{\sqrt{u}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{u})^2} \cdot \mathbb{1}_{\{u \geq 0\}} \\ &= \frac{1}{\sqrt{\pi} \cdot 2^{1/2}} \cdot u^{1/2 - 1} \cdot e^{-u/2} \cdot \mathbb{1}_{\{u \geq 0\}} \end{aligned}$$

- One useful fact: $\Gamma(1/2) = \sqrt{\pi}$. Hence:

$$f_U(u) = \frac{1}{\Gamma(1/2) \cdot 2^{1/2}} \cdot u^{1/2 - 1} \cdot e^{-u/2} \cdot \mathbb{1}_{\{u \geq 0\}}$$

- Indeed, $U \sim \text{Gamma}(1/2\ ,\ 2) \stackrel{\mathrm{d}}{=} \chi_1^2$!

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Theorem

- This is an ***extremely*** important result which we will use repeatedly throughout this course. Let's make it more formal by rephrasing it as a theorem:

Theorem (Square of Standard Normal)

If $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$, then $U \sim \chi^2_1$.

- The proof of this theorem is exactly the work we did on the previous slides.

Recap

- Whew- that was a lot of work! Let's recap.
- Given a random variable Y, and $U := g(Y)$ for some function $g : \mathbb{R} \to \mathbb{R}$, we can use the **method of distribution functions** (aka the **CDF**) method to find the distribution of U.
- Specifically, this entails:
  (1) Writing $F_U(u)$, the CDF of U, in terms of $F_Y(y)$, the CDF of Y, by basically finding an equivalent formulation for the event $A_U := \{U \leq u\}$ that is in terms of Y
  (2) Plugging into the CDF of Y, and simplifying as necessary.

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