# Topic 4: Sufficiency, and MVUEs

Ethan P. Marzban University of California, Santa Barbara PSTAT 120B

# Outline

1. Sufficiency

2. MVUEs

# Sufficiency


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Leadup

- Perhaps you've noticed that certain quantities arise repeatedly in the context of estimating certain parameters.

# Leadup

- Perhaps you've noticed that certain quantities arise repeatedly in the context of estimating certain parameters.
- For example, when estimating a *population* mean $\mu$ (using either the method of moments or maximum likelihood estimation), the *sample* mean $\overline{Y}_n$ appears often.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Leadup

- Perhaps you've noticed that certain quantities arise repeatedly in the context of estimating certain parameters.
- For example, when estimating a *population* mean $\mu$ (using either the method of moments or maximum likelihood estimation), the *sample* mean $\overline{Y}_n$ appears often.
- When estimating the population variance of a zero-mean distribution, the quantity $\sum_{i=1}^{n} Y_i^2$ arises frequently.

# Leadup

- Perhaps you've noticed that certain quantities arise repeatedly in the context of estimating certain parameters.
- For example, when estimating a *population* mean $\mu$ (using either the method of moments or maximum likelihood estimation), the *sample* mean $\overline{Y}_n$ appears often.
- When estimating the population variance of a zero-mean distribution, the quantity $\sum_{i=1}^{n} Y_i^2$ arises frequently.
- As such, let's take a brief break from estimation and return back to the general notion of a **statistic**.

# Statistics

**Definition (Statistic)**

Given a random sample $\vec{\boldsymbol{Y}} = \{Y_i\}_{i=1}^{n}$, a **statistic** $T$ is simply a function of $\vec{\boldsymbol{Y}}$:

$$T := T(\vec{\boldsymbol{Y}}) = T(Y_1, \cdots, Y_n)$$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Statistics

**Definition (Statistic)**

Given a random sample $\vec{\boldsymbol{Y}} = \{Y_i\}_{i=1}^n$, a **statistic** $T$ is simply a function of $\vec{\boldsymbol{Y}}$:

$$T := T(\vec{\boldsymbol{Y}}) = T(Y_1, \cdots, Y_n)$$

- Example: sample mean $\overline{Y}_n := \frac{1}{n} \sum_{i=1}^n Y_i$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Statistics

**Definition (Statistic)**

Given a random sample $\vec{\boldsymbol{Y}} = \{Y_i\}_{i=1}^n$, a <u>**statistic**</u> $T$ is simply a function of $\vec{\boldsymbol{Y}}$:

$$T := T(\vec{\boldsymbol{Y}}) = T(Y_1, \cdots, Y_n)$$

- Example: sample mean $\overline{Y}_n := \frac{1}{n}\sum_{i=1}^n Y_i$
- Example: sample variance $S_n^2 \frac{1}{n-1}\sum_{i=1}^n (Y_i - \overline{Y}_n)^2$

# Statistics

**Definition (Statistic)**

Given a random sample $\vec{\boldsymbol{Y}} = \{Y_i\}_{i=1}^{n}$, a **statistic** $T$ is simply a function of $\vec{\boldsymbol{Y}}$:

$$T := T(\vec{\boldsymbol{Y}}) = T(Y_1, \cdots, Y_n)$$

- Example: sample mean $\overline{Y}_n := \frac{1}{n}\sum_{i=1}^{n} Y_i$
- Example: sample variance $S_n^2 \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2$
- Example: sample maximum $Y_{(n)}$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

## Statistics as Data Reduction

- A statistic, inherently, is a form of *data reduction*.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Statistics as Data Reduction

- A statistic, inherently, is a form of *data reduction*.
- That is, we take a sample $\vec{\boldsymbol{Y}}$ consisting of $n$ elements (i.e. observations) and *reduce* it to a single quantity (like the mean, variance, maximum, etc.).


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Statistics as Data Reduction

- A statistic, inherently, is a form of *data reduction*.
- That is, we take a sample $\vec{Y}$ consisting of $n$ elements (i.e. observations) and *reduce* it to a single quantity (like the mean, variance, maximum, etc.).
  - Again, this is just a more heuristic way of saying that a statistic is a *function* of our sample!


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Statistics as Data Reduction

- A statistic, inherently, is a form of *data reduction*.
- That is, we take a sample $\vec{\boldsymbol{Y}}$ consisting of $n$ elements (i.e. observations) and *reduce* it to a single quantity (like the mean, variance, maximum, etc.).
  - Again, this is just a more heuristic way of saying that a statistic is a *function* of our sample!
- For this reason, statistics are sometimes referred to as **summary statistics**, as they *summarize* our sample in some way (e.g. summarize where the "center" of our sample is, summarize how "spread out" our sample is, etc.)


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Leadup

- Intuitively (as was mentioned at the beginning of this lecture), the sample mean seems like a pretty good proxy for the population mean.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Leadup

- Intuitively (as was mentioned at the beginning of this lecture), the sample mean seems like a pretty good proxy for the population mean.
- Conversely, the sample variance might not give us a lot of information about the population mean (unless we have a very specific distribution).


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Leadup

- Intuitively (as was mentioned at the beginning of this lecture), the sample mean seems like a pretty good proxy for the population mean.
- Conversely, the sample variance might not give us a lot of information about the population mean (unless we have a very specific distribution).
- So, our intuition is telling us that the sample mean is doing a better job of summarizing information about $\mu$ (the population mean) than the sample variance.

# Leadup

- Intuitively (as was mentioned at the beginning of this lecture), the sample mean seems like a pretty good proxy for the population mean.
- Conversely, the sample variance might not give us a lot of information about the population mean (unless we have a very specific distribution).
- So, our intuition is telling us that the sample mean is doing a better job of summarizing information about $\mu$ (the population mean) than the sample variance.
- Can we make this more explicit?

# Leadup

- Well, the answer is "yes" and we've actually taken some pretty good steps to making our intuition more explicit, by way of estimation!


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Leadup

- Well, the answer is "yes" and we've actually taken some pretty good steps to making our intuition more explicit, by way of estimation!
- Said differently, used as an estimator for $\mu$, $\overline{Y}_n$ possess *many* more desirable properties than, say, $S_n^2$.

# Leadup

- Well, the answer is "yes" and we've actually taken some pretty good steps to making our intuition more explicit, by way of estimation!
- Said differently, used as an estimator for $\mu$, $\overline{Y}_n$ possess *many* more desirable properties than, say, $S_n^2$.
  - For example, $\overline{Y}_n$ is an unbiased estimator for $\mu$ whereas $S_n^2$ is, in general, not.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Leadup

- Well, the answer is "yes" and we've actually taken some pretty good steps to making our intuition more explicit, by way of estimation!
- Said differently, used as an estimator for $\mu$, $\overline{Y}_n$ possess *many* more desirable properties than, say, $S_n^2$.
  - For example, $\overline{Y}_n$ is an unbiased estimator for $\mu$ whereas $S_n^2$ is, in general, not.
  - Similarly, $\overline{Y}_n$ is a consistent estimator for $\mu$ whereas $S_n^2$ is, in general, not.

# Leadup

- Well, the answer is "yes" and we've actually taken some pretty good steps to making our intuition more explicit, by way of estimation!
- Said differently, used as an estimator for $\mu$, $\overline{Y}_n$ possess *many* more desirable properties than, say, $S_n^2$.
  - For example, $\overline{Y}_n$ is an unbiased estimator for $\mu$ whereas $S_n^2$ is, in general, not.
  - Similarly, $\overline{Y}_n$ is a consistent estimator for $\mu$ whereas $S_n^2$ is, in general, not.
- But let's see if there's perhaps a *different* way to quantify our intuitions.

# Example

- This is all very abstract - let's make things more concrete.


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Example

- This is all very abstract - let's make things more concrete.
- Specifically, suppose $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Bern}(\theta)$.

# Example

- This is all very abstract - let's make things more concrete.
- Specifically, suppose $Y_1, \cdots, Y_n \stackrel{\text{i.i.d.}}{\sim} \text{Bern}(\theta)$.
  - In other words, you can imagine $Y_i$ to be the outcome of tossing a coin once and observing whether it landed on heads or tails, where $\theta$ represents the probability the coin will lands "heads" on any particular toss.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

- This is all very abstract - let's make things more concrete.
- Specifically, suppose $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Bern}(\theta)$.
  - In other words, you can imagine $Y_i$ to be the outcome of tossing a coin once and observing whether it landed on heads or tails, where $\theta$ represents the probability the coin will lands "heads" on any particular toss.
- One statistic we could consider is $U := \sum_{i=1}^{n} Y_i$.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

- This is all very abstract - let's make things more concrete.
- Specifically, suppose $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Bern}(\theta)$.
  - In other words, you can imagine $Y_i$ to be the outcome of tossing a coin once and observing whether it landed on heads or tails, where $\theta$ represents the probability the coin will lands "heads" on any particular toss.
- One statistic we could consider is $U := \sum_{i=1}^{n} Y_i$.
  - In words, $U$ denotes the number of heads in the $n$ coin tosses.

# Example

- This is all very abstract - let's make things more concrete.
- Specifically, suppose $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Bern}(\theta)$.
  - In other words, you can imagine $Y_i$ to be the outcome of tossing a coin once and observing whether it landed on heads or tails, where $\theta$ represents the probability the coin will lands "heads" on any particular toss.
- One statistic we could consider is $U := \sum_{i=1}^{n} Y_i$.
  - In words, $U$ denotes the number of heads in the $n$ coin tosses.
- Does $U$ capture the maximal amount of information about $\theta$? That is, can we gain any further information about $\theta$ by looking at other statistics?


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

- Here is one way to answer this question: let's look at the distribution of $(Y_1, \cdots, Y_n \mid U)$.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

- Here is one way to answer this question: let's look at the distribution of $(Y_1, \cdots, Y_n \mid U)$.
- Before we do, let's convince ourselves that examining this distribution is a good idea.

# Example

- Here is one way to answer this question: let's look at the distribution of $(Y_1, \cdots, Y_n \mid U)$.
- Before we do, let's convince ourselves that examining this distribution is a good idea.
- If the distribution of $(Y_1, \cdots, Y_n \mid U)$ does not depend on $\theta$, then, in essence, $U$ will have captured all of the necessary information about $\theta$.

# Example

- Here is one way to answer this question: let's look at the distribution of $(Y_1, \cdots, Y_n \mid U)$.
- Before we do, let's convince ourselves that examining this distribution is a good idea.
- If the distribution of $(Y_1, \cdots, Y_n \mid U)$ does not depend on $\theta$, then, in essence, $U$ will have captured all of the necessary information about $\theta$.
    - Remember that the distribution of $(X \mid Y)$ can be interpreted as our beliefs on $X$ after knowing $Y$.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

- Here is one way to answer this question: let's look at the distribution of $(Y_1, \cdots, Y_n \mid U)$.
- Before we do, let's convince ourselves that examining this distribution is a good idea.
- If the distribution of $(Y_1, \cdots, Y_n \mid U)$ does not depend on $\theta$, then, in essence, $U$ will have captured all of the necessary information about $\theta$.
  - Remember that the distribution of $(X \mid Y)$ can be interpreted as our beliefs on $X$ after knowing $Y$.
  - Saying that the distribution of $(Y_1, \cdots, Y_n \mid U)$ doesn't depend on $\theta$ means, after knowing $U$, our beliefs on $(Y_1, \cdots, Y_n)$ no longer depend on $\theta$.

# Example

- Alright, let's go!


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Example

- Alright, let's go!
- Specifically, we examine $\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n \mid U = u)$.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

- Alright, let's go!
- Specifically, we examine $\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n \mid U = u)$.
- We're conditioning on an event with nonzero probability, meaning we can invoke the definition of conditional probability to write

$$\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n \mid U = u) = \frac{\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n, U = u)}{\mathbb{P}(U = u)}$$

# Example

- Alright, let's go!
- Specifically, we examine $\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n \mid U = u)$.
- We're conditioning on an event with nonzero probability, meaning we can invoke the definition of conditional probability to write

$$\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n \mid U = u) = \frac{\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n, U = u)}{\mathbb{P}(U = u)}$$

- Since $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Bern}(\theta)$, we know that $U := (\sum_{i=1}^{n} Y_i) \sim \text{Bin}(n, \theta)$, meaning

$$\mathbb{P}(U = u) = \binom{n}{u} \theta^u (1 - \theta)^{n-u}$$

# Example

- What about the numerator, $\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n, U = u)$?


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Example

- What about the numerator, $\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n, U = u)$?
- Well, if $\sum_{i=1}^{n} y_i \neq u$, the probability is zero.


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Example

- What about the numerator, $\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n, U = u)$?
- Well, if $\sum_{i=1}^{n} y_i \neq u$, the probability is zero.
    - Here's how we can think through this: say $n = 3$, and $y_1 = 1,\ y_2 = 0,\ y_3 = 0$. (That is, the first coin landed heads, the second landed tails, and the third landed tails).


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Example

- What about the numerator, $\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n, U = u)$?
- Well, if $\sum_{i=1}^{n} y_i \neq u$, the probability is zero.
  - Here's how we can think through this: say $n = 3$, and $y_1 = 1,\ y_2 = 0,\ y_3 = 0$. (That is, the first coin landed heads, the second landed tails, and the third landed tails).
  - What's the probability of the first coin landing heads, the second landing tails, the third landing tails, and observing a total number of heads that is not equal to 1 (i.e. $1 + 0 + 0$)?


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Example

- What about the numerator, $\mathbb{P}(Y_1 = y_1, \cdots, Y_1 = y_n, U = u)$?
- Well, if $\sum_{i=1}^{n} y_i \neq u$, the probability is zero.
    - Here's how we can think through this: say $n = 3$, and $y_1 = 1,\ y_2 = 0,\ y_3 = 0$. (That is, the first coin landed heads, the second landed tails, and the third landed tails).
    - What's the probability of the first coin landing heads, the second landing tails, the third landing tails, <u>and</u> observing a total number of heads that is not equal to 1 (i.e. $1 + 0 + 0$)?
    - The answer is zero!


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

- If $\sum_{i=1}^{n} y_i = u$, the event we're taking the probability of is

$$\{Y_1 = y_1, \cdots, Y_n = y_n, U = u\}$$

  which is just the probability of an independent sequences of zeros and ones with a total of $u$ ones and $(n - u)$ zeroes.

# Example

- If $\sum_{i=1}^{n} y_i = u$, the event we're taking the probability of is

$$\{Y_1 = y_1, \cdots, Y_n = y_n, U = u\}$$

  which is just the probability of an independent sequences of zeros and ones with a total of $u$ ones and $(n - u)$ zeroes.
- That is,

$$\mathbb{P}(Y_1 = y_1, \cdots, Y_n = y_n, U = u) = \theta^u (1 - \theta)^{n-u}$$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Example

- If $\sum_{i=1}^{n} y_i = u$, the event we're taking the probability of is

$$\{Y_1 = y_1, \cdots, Y_n = y_n, U = u\}$$

  which is just the probability of an independent sequences of zeros and ones with a total of $u$ ones and $(n - u)$ zeroes.
- That is,

$$\mathbb{P}(Y_1 = y_1, \cdots, Y_n = y_n, U = u) = \theta^u (1 - \theta)^{n-u}$$

- So, in all,

$$\mathbb{P}(Y_1 = y_1, \cdots, Y_n = y_n \mid U = u) = \begin{cases} \theta^u (1-\theta)^{n-u} & \text{if } \sum_{i=1}^{n} y_i = u \\ 0 & \text{otherwise} \end{cases}$$

# Example

- Therefore, dividing by $\mathbb{P}(U = u) = \binom{n}{u}\theta^u(1-\theta)^{n-u}$, we have

$$\mathbb{P}(Y_1 = y_1, \cdots, Y_n = y_n, U = u) = \begin{cases} \frac{1}{\binom{n}{u}} & \text{if } \sum_{i=1}^n y_i = u \\ 0 & \text{otherwise} \end{cases}$$


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

- Therefore, dividing by $\mathbb{P}(U = u) = \binom{n}{u}\theta^u(1-\theta)^{n-u}$, we have

$$\mathbb{P}(Y_1 = y_1, \cdots, Y_n = y_n, U = u) = \begin{cases} \frac{1}{\binom{n}{u}} & \text{if } \sum_{i=1}^n y_i = u \\ 0 & \text{otherwise} \end{cases}$$

- So, does this distribution depend on $\theta$?


UC SANTA BARBARA
Department of Statistics and Applied Probability

## Example

- Therefore, dividing by $\mathbb{P}(U = u) = \binom{n}{u}\theta^u(1-\theta)^{n-u}$, we have

$$\mathbb{P}(Y_1 = y_1, \cdots, Y_n = y_n, U = u) = \begin{cases} \frac{1}{\binom{n}{u}} & \text{if } \sum_{i=1}^n y_i = u \\ 0 & \text{otherwise} \end{cases}$$

- So, does this distribution depend on $\theta$?
- Nope! So, after conditioning on $U := \sum_{i=1}^n Y_i$, we have removed all dependency on $\theta$ - said differently, $U$ has captured all of the necessary information about $\theta$.

# Sufficiency

**Definition (Sufficiency)**

Let $Y_1, \cdots, Y_n$ denote a random sample from a distribution with parameter $\theta$. A statistic $U := g(Y_1, \cdots, Y_n)$ is said to be **sufficient** for $\theta$ if the conditional distribution $(Y_1, \cdots, Y_n \mid U)$ does not depend on $\theta$.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Sufficiency

- Now, we almost never use the definition of sufficiency.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Sufficiency

- Now, we almost never use the definition of sufficiency.
- Firstly, it only allows us to check whether a given statistic is sufficient - not how to actually *find* a sufficient statistic.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Sufficiency

- Now, we almost never use the definition of sufficiency.
- Firstly, it only allows us to check whether a given statistic is sufficient - not how to actually *find* a sufficient statistic.
- Furthermore, it requires us to find conditional distributions which are, in general, not particularly easy to find.

# Sufficiency

- Now, we almost never use the definition of sufficiency.
- Firstly, it only allows us to check whether a given statistic is sufficient - not how to actually *find* a sufficient statistic.
- Furthermore, it requires us to find conditional distributions which are, in general, not particularly easy to find.
- As such, in practice, we rely more heavily on the following theorem:


UC SANTA BARBARA
Department of Statistics
and Applied Probability

## Factorization Theorem

**Theorem (Factorization Theorem)**

Let $U$ be a statistic based on the random sample $\vec{\mathbf{Y}} = (Y_1, \cdots, Y_n)$. Then $U$ is a sufficient statistic for the estimation of a parameter $\theta$ if and only if the likelihood $\mathcal{L}_{\vec{\mathbf{Y}}}(\theta)$ factors as

$$\mathcal{L}_{\vec{\mathbf{Y}}}(\theta) = g(U, \theta) \times h(\vec{\mathbf{Y}})$$

where $g(U, \theta)$ is a function of only $U$ and $\theta$ (and possibly fundamental constants) and $h(\vec{\mathbf{Y}})$ does *not* depend on $\theta$.


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Example

## Example

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Bern}(\theta)$, where $\theta \in (0, 1)$ is an unknown parameter. Show that $U := \sum_{i=1}^{n} Y_i$ is a sufficient statistic for $\theta$.

# Example

## Example

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Bern}(\theta)$, where $\theta \in (0, 1)$ is an unknown parameter. Show that $U := \sum_{i=1}^{n} Y_i$ is a sufficient statistic for $\theta$.

- We've actually already shown this using the definition of sufficiency (at the start of today's lecture) - let's show this again, this time using the Factorization Theorem.


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solution

$$\mathcal{L}_{\vec{\mathbf{Y}}}(\theta) = \prod_{i=1}^{n} p(Y_i; \theta) = \prod_{i=1}^{n} \left[ \theta^{Y_i} (1 - \theta)^{1 - Y_i} \right]$$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solution

$$\mathcal{L}_{\vec{\mathbf{Y}}}(\theta) = \prod_{i=1}^{n} p(Y_i; \theta) = \prod_{i=1}^{n} \left[\theta^{Y_i}(1-\theta)^{1-Y_i}\right]$$
$$= \theta^{\sum_{i=1}^{n} Y_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} Y_i}$$

# Solution

$$
\begin{aligned}
\mathcal{L}_{\vec{\mathbf{Y}}}(\theta) &= \prod_{i=1}^{n} p(Y_i;\theta) = \prod_{i=1}^{n} \left[\theta^{Y_i}(1-\theta)^{1-Y_i}\right] \\
&= \theta^{\sum_{i=1}^{n} Y_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} Y_i} \\
&= \underbrace{\left[\theta^{\sum_{i=1}^{n} Y_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} Y_i}\right]}_{:=g(\sum_{i=1}^{n} Y_i\ ,\theta)} \times \underbrace{[1]}_{:=h(\vec{\mathbf{Y}})}
\end{aligned}
$$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solution

$$\begin{aligned}
\mathcal{L}_{\vec{\mathbf{Y}}}(\theta) &= \prod_{i=1}^{n} p(Y_i;\theta) = \prod_{i=1}^{n} \left[\theta^{Y_i}(1-\theta)^{1-Y_i}\right] \\
&= \theta^{\sum_{i=1}^{n} Y_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} Y_i} \\
&= \underbrace{\left[\theta^{\sum_{i=1}^{n} Y_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} Y_i}\right]}_{:=g(\sum_{i=1}^{n} Y_i\ ,\theta)} \times \underbrace{[1]}_{:=h(\vec{\mathbf{Y}})}
\end{aligned}$$

where $g(U,\theta) = \theta^{U} \cdot (1-\theta)^{n-U}$ and $h(\vec{\mathbf{Y}}) = 1$.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Solution

$$
\begin{aligned}
\mathcal{L}_{\vec{\mathbf{Y}}}(\theta) = \prod_{i=1}^{n} p(Y_i;\theta) &= \prod_{i=1}^{n} \left[\theta^{Y_i}(1-\theta)^{1-Y_i}\right] \\
&= \theta^{\sum_{i=1}^{n} Y_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} Y_i} \\
&= \underbrace{\left[\theta^{\sum_{i=1}^{n} Y_i} \cdot (1-\theta)^{n-\sum_{i=1}^{n} Y_i}\right]}_{:=g(\sum_{i=1}^{n} Y_i\ ,\theta)} \times \underbrace{[1]}_{:=h(\vec{\mathbf{Y}})}
\end{aligned}
$$

where $g(U,\theta) = \theta^{U} \cdot (1-\theta)^{n-U}$ and $h(\vec{\mathbf{Y}}) = 1$. Therefore, by the Factorization Theorem, $U := \sum_{i=1}^{n} Y_i$ is a sufficient statistic for $\theta$.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

## Example

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, where $\theta > 0$ is an unknown parameter. Propose a sufficient statistic for $\theta$, and show that it is sufficient.

# Example

## Example

Let $Y_1, \cdots, Y_n \stackrel{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, where $\theta > 0$ is an unknown parameter. Propose a sufficient statistic for $\theta$, and show that it is sufficient.

- We'll do this one on the board.

# Questions (to be answered together)

- **Question:** are sufficient statistics unique?


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Questions (to be answered together)

- **Question:** are sufficient statistics unique?
- **Question:** do sufficient statistics always exist?


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Questions (to be answered together)

- **Question:** are sufficient statistics unique?
- **Question:** do sufficient statistics always exist?

- Let's discuss!


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# MVUEs


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Leadup

- Alright, let's dip our toes back into the realm of estimation.

# Leadup

- Alright, let's dip our toes back into the realm of estimation.
- Recall that, a few lectures ago, I tried to convince everyone that one notion of an "ideal" estimator should be unbiased and with as little variance as possible.

# Leadup

- Alright, let's dip our toes back into the realm of estimation.
- Recall that, a few lectures ago, I tried to convince everyone that one notion of an "ideal" estimator should be unbiased and with as little variance as possible.
- Let's run with this idea a bit!


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Leadup

- Alright, let's dip our toes back into the realm of estimation.
- Recall that, a few lectures ago, I tried to convince everyone that one notion of an "ideal" estimator should be unbiased and with as little variance as possible.
- Let's run with this idea a bit!
- Indeed, we have the notion of a **Minimum Variance Unbiased Estimator** (MVUE) as a sort of "gold-standard" estimator.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Leadup

- Alright, let's dip our toes back into the realm of estimation.
- Recall that, a few lectures ago, I tried to convince everyone that one notion of an "ideal" estimator should be unbiased and with as little variance as possible.
- Let's run with this idea a bit!
- Indeed, we have the notion of a **Minimum Variance Unbiased Estimator** (MVUE) as a sort of "gold-standard" estimator.
- As the name suggests, an MVUE is an estimator that is unbiased and possesses the smallest possible variance.


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Leadup

- “Smallest possible variance.-” is it possible to get an unbiased estimator with zero variance?


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Leadup

- "Smallest possible variance.-" is it possible to get an unbiased estimator with zero variance?
- It turns out (and the reasoning behind *why* is outside the scope of this course) the answer is, in general, "no."


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Leadup

- "Smallest possible variance.-" is it possible to get an unbiased estimator with zero variance?
- It turns out (and the reasoning behind *why* is outside the scope of this course) the answer is, in general, "no."
- Indeed, there exists a lower bound for the variance of *any* unbiased estimator, called the **Cramér-Rao Lower Bound** (CRLB).

# Cramér-Rao Lower Bound

**Theorem (Cramér-Rao Lower Bound)**

Consider an i.i.d. sample $Y_1, \cdots, Y_n$ from a distribution with unknown parameter $\theta$. Under appropriate "regularity conditions", every unbiased estimator $\widehat{\theta}$ obeys the inequality

$$\mathrm{Var}(\widehat{\theta}) \geq \frac{1}{\mathcal{I}_n(\theta)}$$

where

$$\mathcal{I}_n(\theta) = \mathbb{E}\left[-\frac{\partial^2}{\partial\theta^2}\ell_{\vec{\mathbf{Y}}}(\theta)\right]$$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Some Terminology

- The Cramér-Rao Lower Bound refers to the lower bound on the variance, $[\mathcal{I}_n(\theta)]^{-1}$.

# Some Terminology

- The Cramér-Rao Lower Bound refers to the lower bound on the variance, $[\mathcal{I}_n(\theta)]^{-1}$.
- The term $\mathcal{I}_n(\theta)$ is referred to as the **Fisher Information** of the sample $\vec{\boldsymbol{Y}}$. Note that the fisher information is the expectation of the negative second-derivative of the log-likelihood of the sample.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Some Terminology

- The Cramér-Rao Lower Bound refers to the lower bound on the variance, $[\mathcal{I}_n(\theta)]^{-1}$.
- The term $\mathcal{I}_n(\theta)$ is referred to as the **Fisher Information** of the sample $\vec{\boldsymbol{Y}}$. Note that the fisher information is the expectation of the negative second-derivative of the log-likelihood of the sample.
- Note that the CRLB is not a strict inequality, meaning that certain estimators actually achieve the lower bound. An estimator that achieves the CRLB (i.e. an estimator satisfying $\mathrm{Var}(\widehat{\theta}) = [\mathcal{I}_n(\theta)]^{-1}$) is said to be a **efficient** estimator.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# A Note

- The Cramér-Rao Lower Bound only applies to *unbiased* estimators. It is possible to construct *biased* estimators that have variance smaller than the CRLB (a very popular example of such an estimator, used throughout a wide array of different disciplines, is the so-called "James-Stein estimator")

# Example

## Example

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, where $\theta > 0$ is an unknown parameter.

(a) Find the lowest attainable variance by an unbiased estimator for $\theta$.

(b) Is the estimator $\widehat{\theta}_n := \overline{Y}_n$ an efficient estimator for $\theta$?



UC SANTA BARBARA
Department of Statistics
and Applied Probability

## Solutions

- Part (a) is essentially just asking us to compute the CRLB.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Solutions

- Part (a) is essentially just asking us to compute the CRLB.
- From previous work, we have that the log-likelihood of the sample is given by

$$\ell_{\vec{\mathbf{Y}}}(\theta) = -n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} \ln \mathbb{1}_{\{Y_i \geq 0\}}$$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solutions

- Part (a) is essentially just asking us to compute the CRLB.
- From previous work, we have that the log-likelihood of the sample is given by

$$\ell_{\vec{\mathbf{Y}}}(\theta) = -n \ln(\theta) - \frac{1}{\theta} \sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} \ln \mathbb{1}_{\{Y_i \geq 0\}}$$

- We now take the first and second derivatives:


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solutions

## Solutions

$$\frac{\partial}{\partial \theta} \ell_{\vec{\mathbf{Y}}}(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} Y_i$$


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Solutions

$$\frac{\partial}{\partial \theta} \ell_{\vec{\mathbf{Y}}}(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} Y_i$$

$$\frac{\partial^2}{\partial \theta^2} \ell_{\vec{\mathbf{Y}}}(\theta) = \frac{n}{\theta^2} - \frac{2}{\theta^3} \sum_{i=1}^{n} Y_i$$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solutions

$$\frac{\partial}{\partial\theta}\ell_{\vec{\mathbf{Y}}}(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} Y_i$$

$$\frac{\partial^2}{\partial\theta^2}\ell_{\vec{\mathbf{Y}}}(\theta) = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} Y_i$$

$$-\frac{\partial^2}{\partial\theta^2}\ell_{\vec{\mathbf{Y}}}(\theta) = -\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n} Y_i$$

# Solutions

$$\frac{\partial}{\partial\theta}\ell_{\vec{\mathbf{Y}}}(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} Y_i$$

$$\frac{\partial^2}{\partial\theta^2}\ell_{\vec{\mathbf{Y}}}(\theta) = \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n} Y_i$$

$$-\frac{\partial^2}{\partial\theta^2}\ell_{\vec{\mathbf{Y}}}(\theta) = -\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n} Y_i$$

- The Fisher Information is just the expectation of this last quantity:


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solutions

# Solutions

$$\mathcal{I}_n(\theta) = \mathbb{E}\left[-\frac{\partial^2}{\partial\theta^2}\ell_{\vec{\mathbf{Y}}}(\theta)\right]$$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solutions

$$\mathcal{I}_n(\theta) = \mathbb{E}\left[ -\frac{\partial^2}{\partial \theta^2} \ell_{\vec{\mathbf{Y}}}(\theta) \right]$$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solutions

$$
\begin{aligned}
\mathcal{I}_n(\theta) &= \mathbb{E}\left[-\frac{\partial^2}{\partial\theta^2}\ell_{\vec{\mathbf{Y}}}(\theta)\right] \\
&= \mathbb{E}\left[-\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n} Y_i\right]
\end{aligned}
$$


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Solutions

$$
\begin{aligned}
\mathcal{I}_n(\theta) &= \mathbb{E}\left[ -\frac{\partial^2}{\partial \theta^2} \ell_{\vec{\mathbf{Y}}}(\theta) \right] \\
&= \mathbb{E}\left[ -\frac{n}{\theta^2} + \frac{2}{\theta^3} \sum_{i=1}^{n} Y_i \right] \\
&= -\frac{n}{\theta^2} + \frac{2}{\theta^3} \sum_{i=1}^{n} \mathbb{E}[Y_i]
\end{aligned}
$$

## Solutions

$$
\begin{aligned}
\mathcal{I}_n(\theta) &= \mathbb{E}\left[-\frac{\partial^2}{\partial\theta^2}\ell_{\vec{\mathbf{Y}}}(\theta)\right] \\
&= \mathbb{E}\left[-\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n} Y_i\right] \\
&= -\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n}\mathbb{E}[Y_i] = -\frac{n}{\theta^2} + \frac{2n}{\theta^2} = \frac{n}{\theta^2}
\end{aligned}
$$


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solutions

$$\begin{aligned}
\mathcal{I}_n(\theta) &= \mathbb{E}\left[-\frac{\partial^2}{\partial\theta^2}\ell_{\vec{\mathbf{Y}}}(\theta)\right] \\
&= \mathbb{E}\left[-\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n} Y_i\right] \\
&= -\frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n}\mathbb{E}[Y_i] = -\frac{n}{\theta^2} + \frac{2n}{\theta^2} = \frac{n}{\theta^2}
\end{aligned}$$

- The CRLB is just the reciprocal of this last quantity: $\theta^2/n$.


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solutions

- So, in other words, *any* unbiased estimator for $\theta$ (in the context of the exponential distribution) will have variance greater than or equal to $\theta^2/n$.


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solutions

- So, in other words, *any* unbiased estimator for $\theta$ (in the context of the exponential distribution) will have variance greater than or equal to $\theta^2/n$.
- To answer part (b), first note that $\widehat{\theta}_n := \overline{Y}_n$ is an unbiased estimator for $\theta$. Hence, we simply need to check whether or not its variance attains the CRLB:

$$\mathrm{Var}(\widehat{\theta}_n) = \mathrm{Var}(\overline{Y}_n) = \frac{\mathrm{Var}(Y_1)}{n} = \frac{\theta^2}{n}$$

# Solutions

- So, in other words, *any* unbiased estimator for $\theta$ (in the context of the exponential distribution) will have variance greater than or equal to $\theta^2/n$.
- To answer part (b), first note that $\widehat{\theta}_n := \overline{Y}_n$ is an unbiased estimator for $\theta$. Hence, we simply need to check whether or not its variance attains the CRLB:

$$\mathrm{Var}(\widehat{\theta}_n) = \mathrm{Var}(\overline{Y}_n) = \frac{\mathrm{Var}(Y_1)}{n} = \frac{\theta^2}{n}$$

- Since this is exactly equal to the CRLB, we conclude that $\overline{Y}_n$ is a efficient estimator for $\theta$.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Solutions

- Finally, let's try and tie the notion of efficiency back to our initial discussions on MVUEs.


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Solutions

- Finally, let's try and tie the notion of efficiency back to our initial discussions on MVUEs.
- First note: perhaps counterintuitively, it's possible that the MVUE in a given situation *won't* be efficient. We won't worry too much about why that is, for the purposes of this class.

# Solutions

- Finally, let's try and tie the notion of efficiency back to our initial discussions on MVUEs.
- First note: perhaps counterintuitively, it's possible that the MVUE in a given situation *won't* be efficient. We won't worry too much about why that is, for the purposes of this class.
- I would, however, like to stress that we would like to construct an unbiased estimator that has as low variance as possible.

# Solutions

- Finally, let's try and tie the notion of efficiency back to our initial discussions on MVUEs.
- First note: perhaps counterintuitively, it's possible that the MVUE in a given situation *won't* be efficient. We won't worry too much about why that is, for the purposes of this class.
- I would, however, like to stress that we would like to construct an unbiased estimator that has as low variance as possible.
- So, given an estimator $\widehat{\theta}_1$ for a parameter $\theta$, is it possible to "improve" (i.e. obtain a new estimator $\widehat{\theta}_2$ with a lower variance than $\widehat{\theta}_1$?)


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Solutions

- Finally, let's try and tie the notion of efficiency back to our initial discussions on MVUEs.
- First note: perhaps counterintuitively, it's possible that the MVUE in a given situation *won't* be efficient. We won't worry too much about why that is, for the purposes of this class.
- I would, however, like to stress that we would like to construct an unbiased estimator that has as low variance as possible.
- So, given an estimator $\widehat{\theta}_1$ for a parameter $\theta$, is it possible to "improve" (i.e. obtain a new estimator $\widehat{\theta}_2$ with a lower variance than $\widehat{\theta}_1$?) Yes!


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Rao-Blackwell Theorem

**Theorem (Rao-Blackwell Theorem)**

Let $\widehat{\theta}_1$ be an unbiased estimator for $\theta$ with finite variance. If $U$ is a sufficient statistic for $\theta$, define $\widehat{\theta}_2 := \mathbb{E}[\widehat{\theta}_1 \mid U]$. Then, for all $\theta$,

$$\mathbb{E}[\widehat{\theta}_2] = \theta \qquad \text{and} \qquad \mathrm{Var}(\widehat{\theta}_2) \leq \mathrm{Var}(\widehat{\theta}_1)$$

# Rao-Blackwell Theorem

**Theorem (Rao-Blackwell Theorem)**

Let $\widehat{\theta}_1$ be an unbiased estimator for $\theta$ with finite variance. If $U$ is a sufficient statistic for $\theta$, define $\widehat{\theta}_2 := \mathbb{E}[\widehat{\theta}_1 \mid U]$. Then, for all $\theta$,

$$\mathbb{E}[\widehat{\theta}_2] = \theta \qquad \text{and} \qquad \mathrm{Var}(\widehat{\theta}_2) \leq \mathrm{Var}(\widehat{\theta}_1)$$

- So, given an initial unbiased estimator $\widehat{\theta}_1$ and a sufficient statistic $U$, we can "improve" (or, at least, never do worse) by conditioning our unbiased estimator on our sufficient statistic.

# Rao-Blackwell Theorem

- Now, in practice, using the Rao-Blackwell theorem can be a bit tricky, mainly due to the intractability of some of the conditional expectations it requires us to compute.


UC SANTA BARBARA
Department of Statistics
and Applied Probability

# Rao-Blackwell Theorem

- Now, in practice, using the Rao-Blackwell theorem can be a bit tricky, mainly due to the intractability of some of the conditional expectations it requires us to compute.
    - I walk you through one particular example in problem 4 of your HW05


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Rao-Blackwell Theorem

- Now, in practice, using the Rao-Blackwell theorem can be a bit tricky, mainly due to the intractability of some of the conditional expectations it requires us to compute.
  - I walk you through one particular example in problem 4 of your HW05
- However, the Rao-Blackwell Theorem can be used to tell us that the following procedure generally gives us an MVUE:

# Rao-Blackwell Theorem

- Now, in practice, using the Rao-Blackwell theorem can be a bit tricky, mainly due to the intractability of some of the conditional expectations it requires us to compute.
  - I walk you through one particular example in problem 4 of your HW05
- However, the Rao-Blackwell Theorem can be used to tell us that the following procedure generally gives us an MVUE:
- Say we have a sufficient statistic $U$ that best summarizes our data. Additionally, say we have an estimator $\widehat{\theta} := h(U)$ that is unbiased for $\theta$. Then, typically, $\widehat{\theta}$ will be an MVUE.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Rao-Blackwell Theorem

- Of course, there are some details missing. For one, it turns out that even among sufficient statistics, some are “better” at capturing the information about a parameter than others. (These are called **minimal sufficient statistics**, which we won’t cover in this course.)


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Rao-Blackwell Theorem

- Of course, there are some details missing. For one, it turns out that even among sufficient statistics, some are "better" at capturing the information about a parameter than others. (These are called **minimal sufficient statistics**, which we won't cover in this course.)
  - So, it's really a function of a *minimal* sufficient statistic that will give us the MVUE in a given situation.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Rao-Blackwell Theorem

- Of course, there are some details missing. For one, it turns out that even among sufficient statistics, some are "better" at capturing the information about a parameter than others. (These are called **minimal sufficient statistics**, which we won't cover in this course.)
  - So, it's really a function of a *minimal* sufficient statistic that will give us the MVUE in a given situation.
  - But, again, for the purposes of this class, we won't concern ourselves with this too much.

# Rao-Blackwell Theorem

- Of course, there are some details missing. For one, it turns out that even among sufficient statistics, some are "better" at capturing the information about a parameter than others. (These are called **minimal sufficient statistics**, which we won't cover in this course.)
    - So, it's really a function of a *minimal* sufficient statistic that will give us the MVUE in a given situation.
    - But, again, for the purposes of this class, we won't concern ourselves with this too much.
- Indeed, in general, constructing MVUEs can be a pain! But, it's useful to at least know about their existence, and how sufficiency and the Rao-Blackwell theorem tie into constructing them.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

## Example

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Unif}[0, \theta]$, where $\theta > 0$ is an unknown parameter.

(a) Show that $Y_{(n)}$ is a sufficient statistic for $\theta$. (It turns out that this is a *minimal* sufficient statistic for $\theta$, but you do not need to show that.)

(b) Find an MVUE for $\theta$.


UC SANTA BARBARA
Department of Statistics and Applied Probability

# Example

## Example

Let $Y_1, \cdots, Y_n \overset{\text{i.i.d.}}{\sim} \text{Unif}[0, \theta]$, where $\theta > 0$ is an unknown parameter.

(a) Show that $Y_{(n)}$ is a sufficient statistic for $\theta$. (It turns out that this is a *minimal* sufficient statistic for $\theta$, but you do not need to show that.)

(b) Find an MVUE for $\theta$.

- Try this on your own, and feel free to ask me about it during Office Hours!