Topic 01: Conditional Distributions and Expectations

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Outline

1. An Introductory Example
2. Conditional Distributions
3. Conditional Expectations
4. The Gamma Distribution

An Introductory Example

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Example

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Suppose I roll a fair six-sided die. Then, whatever number the die lands on, I flip that many fair coins. Let X denote the number of heads. What is the **PMF** (probability mass function) of X?

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- Now, X *sounds* binomial. But, there's a (not-so slight) problem... Can anyone tell me what that problem is? That's right; the binomial distribution requires a *fixed* number of Bernoulli trials.
  - In other words, if the number of coins I tossed remained fixed across repetitions of this experiment, then X would follow a Binomial distribution. But, because the number of coins I toss *itself* potentially changes across repetitions, we can no longer classify X as being binomially distributed.

Notation

- Let's start off (as we should with any problem) by defining some notation.

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- Specifically, it seems like I need to keep track of two things: the result of the die roll, and the number of heads in the resulting tosses of the coin.
- As such, let's assign a random variable to each of these quantities:

$$\begin{aligned} N &:= \text{result of the die roll} \\ X &:= \text{number of heads among the coin tosses} \end{aligned}$$

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Assumptions

- From the problem statement, it's safe to assume

$$N \sim \text{DiscUnif}\{1, 2, 3, 4, 5, 6\}$$

  that is, that N follows the discrete uniform distribution on the set $\{1, 2, \cdots, 6\}$.

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- Now, to reiterate what we said at the beginning of this discussion, it is **NOT** correct to simply say that X is binomially distributed!

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- Now, to reiterate what we said at the beginning of this discussion, it is **NOT** correct to simply say that X is binomially distributed!
- But, that doesn't mean we can't get at its PMF directly.

Example

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- Again, the issue is that we don't know how many coins we tossed!
- *If* we knew how many coins we tossed - say, for example, 6 - then we'd be in business! Specifically, the probability of observing 2 heads among six tosses of a fair coin is easily computed using the Binomial PMF: $\binom{6}{2}(1/2)^6$.

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- In slightly more formal language - specifically, the language of <u>**conditional probabilities**</u>, what we have just shown is that

$$\mathbb{P}(X = 2 \mid N = 6) = \binom{6}{2}\left(\frac{1}{2}\right)^6$$

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Example (cont'd)

- Let's get a bit more practice with understanding our notation! What is, say, $\mathbb{P}(X = 2 \mid N = 5)$?

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- Well, in words, this is asking us to compute the probability of observing 2 heads among 5 tosses of a fair coin.
- We can again use the Binomial formula:

$$\mathbb{P}(X = 2 \mid N = 5) = \binom{5}{2} \left(\frac{1}{2} \right)^5$$

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Example (cont'd)

- Generalizing a bit, let's see if we can find an expression for $\mathbb{P}(X = 2 \mid N = n)$, where n is an arbitrary integer in the set $\{1, 2, \cdots, 6\}$.

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- Again, in words this is asking us to compute the probability of observing 2 heads among n tosses of a fair coin.
- Once again, we use the Binomial PMF:

$$\mathbb{P}(X = 2 \mid N = n) = \binom{n}{2} \left(\frac{1}{2} \right)^n$$

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Example (cont'd)

- Generalizing one step further:

$$\mathbb{P}(X = x \mid N = n) = \binom{n}{x}\left(\frac{1}{2}\right)^n$$

where $x \in \{1, 2, \cdots, n\}$ and $n \in \{1, 2, \cdots, 6\}$.

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where $x \in \{1, 2, \cdots, n\}$ and $n \in \{1, 2, \cdots, 6\}$.

  - BTW, can anyone tell me what happens if $x > n$? Think both in terms of intuition, as well as the mathematical formula above!

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  - BTW, can anyone tell me what happens if $x > n$? Think both in terms of intuition, as well as the mathematical formula above!

- Also, don't forget:

$$\mathbb{P}(N = n) = \frac{1}{6}, \qquad \text{if } n \in \{1, 2, \cdots, 6\}$$

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Clicker Question!

Clicker Question 1

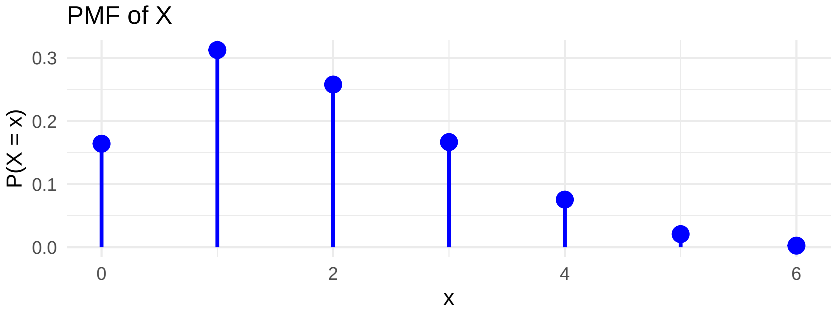
Based on the work we've done so far, which PSTAT 120A topic do you think will help us complete the calculation for $\mathbb{P}(X = x)$?

(A) The Complement Rule
(B) The Inclusion-Exclusion Principle (aka the Addition Rule)
(C) The Law of Total Probability
(D) The Central Limit Theorem
(E) None of the above

Example (cont'd)

- So, once the dust settles, we have

$$\mathbb{P}(X = x) = \frac{1}{6} \sum_{n=1}^{6} \binom{n}{x} \left(\frac{1}{2} \right)^n, \qquad \text{for } x \in \{0, 1, \cdots, 6\}$$



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Recap

- So, what did we learn?

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- Well, I hope one thing became clear: after conditioning on the result of the die roll, our considerations for the number of heads became much simpler!

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- Well, I hope one thing became clear: after conditioning on the result of the die roll, our considerations for the number of heads became much simpler!
- In a way, it's tempting to write

$$(X \mid N = n) \sim \text{Bin}(n, 1/2)$$

to indicate the fact that, if we knew the die landed on n, then X becomes binomially distributed.

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Recap

- So, what did we learn?
- Well, I hope one thing became clear: after conditioning on the result of the die roll, our considerations for the number of heads became much simpler!
- In a way, it's tempting to write

$$(X \mid N = n) \sim \text{Bin}(n, 1/2)$$

  to indicate the fact that, if we knew the die landed on n, then X becomes binomially distributed.
- Indeed, such notation *is* proper - well, it *will* be after we discuss its meaning more carefully!

Conditional Distributions

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Conditional Distributions

- Thinking back to our PSTAT 120A days (as we will often do in this class), recall the notion of a **joint probability density/mass function**.

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- Essentially, a joint PDF/PMF is a way to jointly specify/quantify the distribution of two random variables that are potentially related in some way.

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Conditional Distributions

- Thinking back to our PSTAT 120A days (as we will often do in this class), recall the notion of a **joint probability density/mass function**.
- Essentially, a joint PDF/PMF is a way to jointly specify/quantify the distribution of two random variables that are potentially related in some way.
- Let's consider (temporarily) the discrete and continuous cases separately.

Joint PMF

- Consider two random variables X and Y, both of which are discrete.

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Joint PMF

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- Then the joint PMF of X and Y, notated $p_{X,Y}(x, y)$ is defined as

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  - For example, letting X and N be defined as they were in our initial die-and-coin example, then $p_{X,N}(x, n)$ is the probability that we observed x heads and the die landed on n.

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- What happens if we divide both sides of our definition for $p_{X,Y}(x, y)$ by $p_Y(y) := \mathbb{P}(Y = y)$?

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Leadup

- Well, first things first - we need to make sure we're not dividing by zero! So, let's assume that y is such that $\mathbb{P}(Y = y) \neq 0$.

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Leadup

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- Then, we find that

$$\frac{p_{X,Y}(x, y)}{p_Y(y)} = \frac{\mathbb{P}(X = x, Y = y)}{\mathbb{P}(Y = y)}$$

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- The RHS should look familiar! Specifically, if we let $A := \{X = x\}$ and $B := \{Y = y\}$, then the RHS is simply

$$\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

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- So,

$$\frac{p_{X,Y}(x, y)}{p_Y(y)} = \mathbb{P}(X = x \mid Y = y)$$

- We can use the shorthand $p_{X|Y}(x \mid y)$ to denote the RHS. That is, we define

$$p_{X|Y}(x \mid y) := \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

and call this the **conditional PMF** of X given Y.

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$$\mathbb{P}(X = x \mid N = n) = \binom{n}{x}\left(\frac{1}{2}\right)^n$$

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  based on the setup of the problem.
- Indeed, this is just the conditional PMF of X given $N = n$:

$$p_{X|N}(x \mid n) = \binom{n}{x} \left(\frac{1}{2} \right)^n$$

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Conditional PMF

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Definition (Conditional PMF)

Given a pair of bivariate random variables (X, Y), we define the **conditional PMF** of X given $Y = y$ to be

$$p_{X|Y}(x \mid y) := \frac{p_{X,Y}(x, y)}{p_Y(y)} = \mathbb{P}(X = x \mid Y = y)$$

provided that y is such that $p_Y(y) \neq 0$. If $p_Y(y) = 0$, then $p_{X|Y}(x \mid y)$ is undefined.

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Some Notes

- Though $p_{X|Y}(x \mid y)$ does involve both x and y, we typically view it as a function of x alone.

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- I'd like to also stress the fact that $p_{X|Y}(x \mid y)$ is undefined whenever $p_Y(y) = 0$. It's not 0, or ∞ - it's just undefined.

Theorem

For any fixed value of y (such that all quantities are defined), $p_{X|Y}(x \mid y)$ is a valid PMF.

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Proof.

- Recall that to verify a given function is a valid PMF, we need to establish two things: nonnegativity, and summation to unity.

$\square$

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$$\sum_x p_{X|Y}(x \mid y) = \sum_x \frac{p_{X,Y}(x, y)}{p_Y(y)} \qquad \text{[Definition of } p_{X|Y}(x \mid y)\text{]}$$

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- For summation to unity:

$$\begin{aligned} \sum_x p_{X|Y}(x \mid y) &= \sum_x \frac{p_{X,Y}(x, y)}{p_Y(y)} && \text{[Definition of } p_{X|Y}(x \mid y)\text{]} \\ &= \frac{1}{p_Y(y)} \sum_x p_{X,Y}(x, y) && \text{[Algebra]} \end{aligned}$$

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&= \frac{1}{\cancel{p_Y(y)}} \cdot \cancel{p_Y(y)}
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&= \frac{1}{\cancel{p_Y(y)}} \cdot \cancel{p_Y(y)} = 1 && \text{[Joint PMF to Marginal PMF]}
\end{aligned}
$$

□

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Joint PDF

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Joint PDF

- Alright, let's talk about the continuous case!
- That is, consider a pair of bivariate random variables (X, Y) that are both continuous. Then, information about X and Y is jointly specified through the joint PDF

$$f_{X,Y}(x, y)$$

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- That is, consider a pair of bivariate random variables (X, Y) that are both continuous. Then, information about X and Y is jointly specified through the joint PDF

$$f_{X,Y}(x, y)$$

- Now, unlike the discrete case, recall that the values of $f_{X,Y}(x, y)$ do *not* represent probabilities - rather, <u>volumes underneath</u> $f_{X,Y}(x, y)$ represent probabilities.

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Joint PDF

- Nevertheless, motivated by our considerations in the discrete case, we can still posit the following definition:

Definition (Conditional PDF)

Given a pair of bivariate random variables (X, Y), we define the **conditional PDF** of X given $Y = y$ to be

$$f_{X|Y}(x \mid y) := \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

provided that y is such that $f_Y(y) \neq 0$. If $f_Y(y) = 0$, then $f_{X|Y}(x \mid y)$ is undefined.

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- I encourage you to try the proof of this on your own!

Chalkboard Example

Suppose (X, Y) is a continuous bivariate random vector with joint p.d.f. given by

$$f_{X,Y}(x, y) = \begin{cases} \lambda^3 x e^{-\lambda y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $f_Y(y)$, the marginal density of Y.

(b) Find $f_{X|Y}(x \mid y)$, the conditional density of $(X \mid Y = y)$

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Working With Conditional Densities

- Once we understand the idea that $f_{X|Y}(x \mid y)$ functions behaves like a PDF (because, in a way, it *is* one), the following definition becomes natural:

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Definition

Given a pair (X, Y) of continuous random variables,

$$\mathbb{P}(X \in A \mid Y = y) = \int_A f_{X|Y}(x \mid y) \, \mathrm{d}y$$

Chalkboard Example (cont'd)

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$$f_{X,Y}(x,y) = \begin{cases} \lambda^3 x e^{-\lambda y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

(c) Compute $\mathbb{P}(X \geq 1 \mid Y \geq 2)$

(d) Compute $\mathbb{P}(X \geq 1 \mid Y = 2)$

Marginal PMFs/PDFs

- Using the connection between conditional PMFs/PDFs and joint PMFs/PDFs, we can see how one can recover *marginal* PMFs/PDFs from condtitional PMFs/PDFs:

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Theorem

(1) If (X, Y) denotes a pair of continuous random variables, then

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x \mid y) f_Y(y) \, \mathrm{d}y$$

with an analogous formula for $f_Y(y)$.

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Marginal PMFs/PDFs

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Theorem

(2) If (X, Y) denotes a pair of discrete random variables, then

$$p_X(x) = \sum_y p_{X|Y}(x \mid y) p_Y(y)$$

with an analogous formula for $p_Y(y)$.

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Proof Outlines

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  - I highly encourage you to try these proofs as an exercise in reviewing some PSTAT 120A-related definitions and results!

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  - I highly encourage you to try these proofs as an exercise in reviewing some PSTAT 120A-related definitions and results!

- Now, something interesting happens when we consider the *mixed* case.

Mixed Case

- What do I mean by the “mixed” case?

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- Well, for example, consider a discrete random variable X and a continuous random variable Y. Can we define something resembling a conditional PMF/PDF?

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- The answer is, perhaps surprisingly, “*yes*”!

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- What do I mean by the “mixed” case?
- Well, for example, consider a discrete random variable X and a continuous random variable Y. Can we define something resembling a conditional PMF/PDF?
- The answer is, perhaps surprisingly, “*yes*”!
- As an example (which you will consider on your homework), suppose Y denotes the number of diseased trees in a forest (and is hence discrete), but that the *rate* of diseased trees (which is continuous) itself varies according to some distribution. Despite the fact that the number and rate of diseased trees are discrete and continuous, respectively, it still makes perfect sense to talk about the *unconditional* distribution of the number of diseased trees.

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Results

Theorem

Consider a random vector (X, Y).

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- If X is discrete and Y is continuous, then

$$p_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x \mid y) f_Y(y) \, \mathrm{d}y$$

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Consider a random vector (X, Y).

- If X is discrete and Y is continuous, then $p_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x \mid y) f_Y(y) \, \mathrm{d}y$
- If X is continuous and Y is discrete, then $f_X(x) = \sum_y f_{X|Y}(x \mid y) p_Y(y)$

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Consider a random vector (X, Y).

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- If X is continuous and Y is discrete, then
$$f_X(x) = \sum_y f_{X|Y}(x \mid y) p_Y(y)$$

- <u>Moral:</u> for mixed random vectors, integrate/sum according to the type of variable being conditioned on.

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Conditional Expectations

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Leadup

- Recall that, given a random variable X with density $f_X(x)$, the **Law of the Unconscious Statistician** (LOTUS) states

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) \, \mathrm{d}x$$

for well-behaved functions $g : \mathbb{R} \to \mathbb{R}$.

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for well-behaved functions $g : \mathbb{R} \to \mathbb{R}$.

- Given that, for a pair (X, Y) of continuous random variables, $f_{X|Y}(x \mid y)$ represents a density function [essentially of the "random variable" $(X \mid Y = y)$], it's perhaps natural to define:

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Conditional Expectation; First Pass

Definition (Conditional Expectation; First Pass)

(1) Given a continuous pair (X, Y) of random variables and a well-behaved function $g : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[g(X) \mid Y = y] := \int_{-\infty}^{\infty} g(x) f_{X|Y}(x \mid y) \, \mathrm{d}x$$

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(2) Given a discrete pair (X, Y) of random variables and a well-behaved function $g : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E}[g(X) \mid Y = y] := \sum_{y} g(x) p_{X|Y}(x \mid y)$$

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Chalkboard Example

Suppose (X, Y) is a continuous bivariate random vector with joint p.d.f. given by

$$f_{X,Y}(x, y) = \begin{cases} \lambda^3 x e^{-\lambda y} & \text{if } 0 < x < y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Compute $\mathbb{E}[X \mid Y = y]$.

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Properties of Conditional Expectations

Theorem (Properties of Conditional Expectations, I)

(I) **(Linearity)**
$\mathbb{E}[aX + bY + c \mid Z = z] = a\mathbb{E}[X \mid Z = z] + b\mathbb{E}[Y \mid Z = z] + c.$

(II) $\mathbb{E}[g(X) \mid X = x] = g(x).$

(III) If $X \perp Y$, then $\mathbb{E}[X \mid Y = y] = \mathbb{E}[X].$

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Conditional Expectation

Definition (Conditional Expectation)

Given random variables X and Y, and the function $h(y) := \mathbb{E}[X \mid Y = y]$, we define the **conditional expectation of X given Y**, notated $\mathbb{E}[X \mid Y]$, to be $h(Y)$.

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- So, in practice, here's how we compute $\mathbb{E}[X \mid Y]$:

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  (2) Substitute Y in place of y in our expression from step (1).

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  (1) Compute $h(y) := \mathbb{E}[X \mid Y = y]$ (which will be a function of y)
  (2) Substitute Y in place of y in our expression from step (1).

- Note: $\mathbb{E}[X \mid Y]$ will be a <u>random variable</u>!

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Suppose (X, Y) is a continuous bivariate random vector with joint p.d.f. given by

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Compute $\mathbb{E}[X \mid Y]$.

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Properties of Conditional Expectations

Theorem (Properties of Conditional Expectations, II)

(I) **(Linearity)** $\mathbb{E}[aX + bY + c \mid Z] = a\mathbb{E}[X \mid Z] + b\mathbb{E}[Y \mid Z] + c$.

(II) $\mathbb{E}[g(X) \mid X] = g(X)$.

(III) If $X \perp Y$, then $\mathbb{E}[X \mid Y] = X$.

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Properties of Conditional Expectations

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(III) If $X \perp Y$, then $\mathbb{E}[X \mid Y] = X$.

- Note how these follow almost directly from the theorem titled (Properties of Conditional Expectations, I), from a few slides ago.

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Leadup

- Since $\mathbb{E}[X \mid Y]$ is itself a random variable, it makes sense to take *its* expectation: $\mathbb{E}[\mathbb{E}[X \mid Y]]$.

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- It's important we understand what each of these expectations are taken with respect to.
  - The inner expectation is taken with respect to the conditional distribution $(X \mid Y)$
  - The outer expectation is taken with respect to Y.
  - Hence, it would perhaps be more accurate to write $\mathbb{E}_Y[\mathbb{E}_{X|Y}(X \mid Y)]$, but we often drop the subscripts for convenience.

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Continuous Realm

- To be explicit, assume X is a continuous random variable, and define $h(y) := \mathbb{E}[X \mid Y = y]$.

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\begin{aligned}
\mathbb{E}[\mathbb{E}[X \mid Y]] &=: \mathbb{E}[h(Y)] \\
&= \int_{\mathbb{R}} h(y) f_Y(y)\ \mathrm{d}y && \text{[LOTUS]}
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\end{aligned}$$

Continuous Realm, cont'd

$$\mathbb{E}[\mathbb{E}[X \mid Y]] = \int_{\mathbb{R}} \int_{\mathbb{R}} x f_{X|Y}(x \mid y) f_Y(y) \, \mathrm{d}x \, \mathrm{d}y$$

[From prev. slide]

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&= \int_{\mathbb{R}} \int_{\mathbb{R}} x \cdot \frac{f_{X,Y}(x, y)}{\cancel{f_Y(y)}} \cdot \cancel{f_Y(y)} \, \mathrm{d}x \, \mathrm{d}y && \text{[Def of } f_{X|Y}(x \mid y)\text{]}
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Continuous Realm, cont'd

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&= \int_{\mathbb{R}} x \left(\int_{\mathbb{R}} f_{X,Y}(x,y) \, \mathrm{d}y \right) \mathrm{d}x && \text{[Rev. Order of int.]}
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&= \int_{\mathbb{R}} x f_X(x) \, \mathrm{d}x = \mathbb{E}[X] && \text{[Simplifying]}
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Law of Iterated Expectations

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Theorem (Law of Iterated Expectations)

Given random variables X and Y, we have

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provided these quantities exist.

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Theorem (Law of Iterated Expectations)

Given random variables X and Y, we have

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provided these quantities exist.

- We proved the continuous case above; I'll ask you to prove the discrete case later.

LIE and LOTUS

Theorem

Given random variables X and Y, we have

$$\mathbb{E}[\mathbb{E}[g(X) \mid Y]] = \mathbb{E}[g(X)]$$

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Theorem

Given random variables X and Y, we have

$$\mathbb{E}[\mathbb{E}[g(X) \mid Y]] = \mathbb{E}[g(X)]$$

provided these quantities exist.

- For example, $\mathbb{E}[X^2] = \mathbb{E}[\mathbb{E}[X^2 \mid Y]]$.

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Clicker Question!

Clicker Question 2

Let $(X \mid Y = y) \sim \mathrm{Bin}(y, 0.25)$ and $Y \sim \mathrm{Pois}(2)$. What is $\mathbb{E}[X]$?

(A) 0.00
(B) 0.25
(C) 0.50
(D) 2.00
(E) None of the above

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Law of Total Variance

- Since $(X \mid Y)$ is a random variable, it makes sense to ask what its variance is. Thankfully, we have a formula for that:

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Theorem (Law of Total Variance)

Given random variables X and Y, we have

$$\mathrm{Var}(X \mid Y) = \mathbb{E}[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(\mathbb{E}[X \mid Y])$$

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- As an exercise, return to the clicker question from a few slides ago and try to compute $\mathrm{Var}(X)$.

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Example Revisited

Example

Suppose I roll a fair six-sided die. Then, whatever number the die lands on, I flip that many fair coins. Let X denote the number of heads. Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.

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One More Formula

Definition (Expectation Conditional on an Event)

Given a random variable X and an event A with $\mathbb{P}(A) \neq 0$,

$$\mathbb{E}[X \mid A] = \frac{\mathbb{E}[X \cdot \mathbb{1}_A]}{\mathbb{P}(A)}$$

Example

The time that Joe spends talking on the phone is exponentially distributed with mean 5 minutes. What is the expected length of his phone call if he talks for more than 2 minutes?

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The Gamma Distribution

Gamma Function

- Hopefully, everyone is familiar with the distributions you learned in 120A (e.g. normal, exponential, binomial, etc.)

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Gamma Function

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$$\Gamma(r) := \int_0^\infty x^{r-1} e^{-x} \, dx$$

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  - recursive property: $\Gamma(r) = (r-1)\Gamma(r-1)$
  - $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$.

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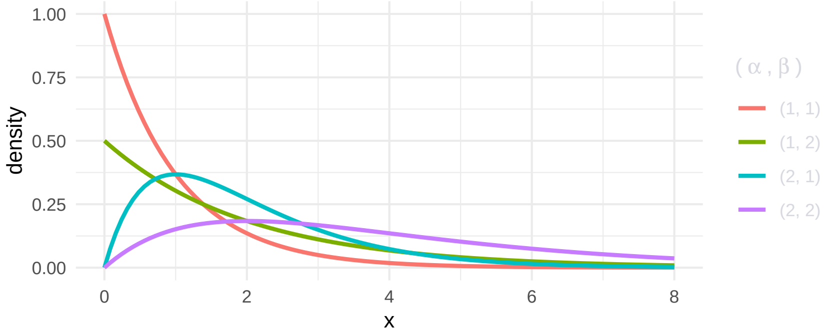
Gamma Distribution

- **Notation:** $X \sim \text{Gamma}(\alpha, \beta)$
- **PDF:** $f_X(x) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \cdot \mathbb{1}_{\{x \geq 0\}}$
- **Expectation and Variance:** $\mathbb{E}[X] = \alpha\beta; \quad \text{Var}(X) = \alpha\beta^2$



Special Cases of the Gamma Distribution

- Note that the Gamma$(1, \beta)$ distribution is equivalent to the Exp(β) distribution.

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  - **PLEASE NOTE:** in PSTAT 120B, we adopt the convention that the parameter of the exponential distribution is its mean. In other words, saying $X \sim \text{Exp}(\beta)$ means $\mathbb{E}[X] = \beta$ and X has a density given by $f_X(x) = (1/\beta)e^{-x/\beta} \cdot \mathbb{1}_{\{x \geq 0\}}$.

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- Note that the Gamma$(1, \beta)$ distribution is equivalent to the Exp(β) distribution.
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- Another special case of the Gamma distribution is the so-called **χ^2 distribution** (pronounced "kai-squared").

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Special Cases of the Gamma Distribution

- Note that the Gamma$(1, \beta)$ distribution is equivalent to the Exp(β) distribution.
    - **PLEASE NOTE:** in PSTAT 120B, we adopt the convention that the parameter of the exponential distribution is its mean. In other words, saying $X \sim \text{Exp}(\beta)$ means $\mathbb{E}[X] = \beta$ and X has a density given by $f_X(x) = (1/\beta)e^{-x/\beta} \cdot \mathbb{1}_{\{x \geq 0\}}$.

- Another special case of the Gamma distribution is the so-called χ^2 **distribution** (pronounced "kai-squared").
    - Specifically, the χ^2_ν distribution is equivalent to the $\mathrm{Gamma}(\nu/2\ ,\ 2)$ distribution.

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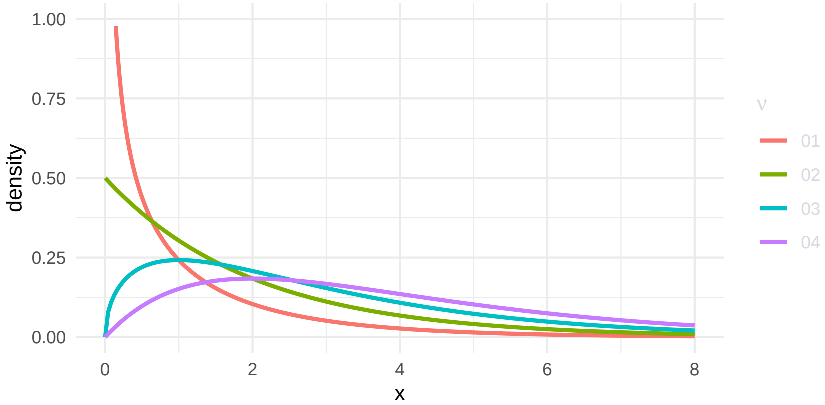
  - Specifically, the χ^2_ν distribution is equivalent to the $\mathrm{Gamma}(\nu/2\ ,\ 2)$ distribution.
  - Question for you: what are the expectation and variance of the χ^2_ν distribution?
  - Also, while we're at it, let's derive the density of the χ^2_ν distribution on the board.

χ^2_ν Distribution



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More to Come

- You'll talk a bit more about the Gamma distribution during section this week.

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- You'll talk a bit more about the Gamma distribution during section this week.
- You'll also show that if $X \sim \text{Gamma}(\alpha, \beta)$, then

$$M_X(t) = \begin{cases} (1 - \beta t)^{-\alpha} & \text{if } t < 1/\beta \\ \infty & \text{otherwise} \end{cases}$$

which will, in turn, allow you to derive the MGF of the χ^2_ν distribution.

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