Topic 02: Transformations

Ethan P. Marzban University of California, Santa Barbara PSTAT 120B

Outline

1. Univariate Transformations
2. Method of Distribution Functions (CDF Method)
3. Method of Transformations (Change of Variable Formula)
4. Method of Moment-Generating Functions (MGF Method)
5. Multivariate Transformations

Leadup

- Recall, from PSTAT 120A, that given an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we can think of a **random variable** X as a mapping:

$$X : \Omega \to \mathbb{R}$$

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- Additionally, recall the following fact from precalculus: given a mapping $f_1 : A \to B$ and another mapping $f_2 : B \to C$, then $(f_2 \circ f_1) : A \to C$.

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$$X : \Omega \to \mathbb{R}$$

- Additionally, recall the following fact from precalculus: given a mapping $f_1 : A \to B$ and another mapping $f_2 : B \to C$, then $(f_2 \circ f_1) : A \to C$.
- This means, given a function $g : \mathbb{R} \to \mathbb{R}$ and a random variable $X : \Omega \to \mathbb{R}$, we have $(g \circ X) : \Omega \to \mathbb{R}$.

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Leadup

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    - For example, given a random variable X, then the quantity X^2 will also be a random variable.
- Another way of saying this: functions of random variables are themselves random variables.
- "Functions of random variables?" That sounds awfully abstract...
- But, if we think about it a bit more, this isn't as abstract as it may seem!

Leadup

- For example, let H_I denote the height of a randomly-selected individual as measured in inches, and suppose $H_I \sim \mathcal{N}(70, 2)$.

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- Let H_C denote the height of a randomly-selected individual as measured in <u>centimeters</u>.
- Clearly, the random variables H_I and H_C are related: specifically, $H_C = g(H_I)$ where $g(t) = 2.54 * t$ [since this is the conversion formula between inches and centimeters].

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    - So, **<u>unit conversion</u>** is a fairly simple example of one way transformations (i.e. taking functions of random variables) can be useful.

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Leadup

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  - By the way, I'll be using this notation a lot: $\{X_i\}_{i=1}^n$ is a shorthand for $X_1, \cdots, X_n$.

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- The **sample mean** $\overline{X}_n := n^{-1} \sum_{i=1}^{n} X_i$ [which you hopefully saw in PSTAT 120A!] is actually a *function* of the original sequence of random variables, and is hence an example of a transformation.

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Leadup

- Now, these two examples indicate that there are perhaps two sub-cases to consider: transformations of *single* random variables, and transformations of *multiple* random variables.

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  - We often refer to a transformation of a single random variable as a **univariate transformation**, and a transformation of multiple random variables as a **multivariate transformation**.

- For simplicity's sake, let's start off with univariate transformations.
  - Specifically, given a random variable Y and a function $g : \mathbb{R} \to \mathbb{R}$, we will seek to explore properties of the random variable $U := g(Y)$.

Univariate Transformations

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Goal

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- What do we mean by "describe" the random variable U?

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Goal

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Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- What do we mean by "describe" the random variable U?
- Well, there are a couple of things we could seek to do. First, we could try to compute $\mathbb{E}[U]$.

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LOTUS

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- The **<u>Law of the Unconscious Statistician</u>** (LOTUS), which we saw in PSTAT 120A, tells us

$$\mathbb{E}[g(Y)] = \int_{\mathbb{R}} g(y) f_Y(y) \, \mathrm{d}y$$

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- Similar considerations will allow us to compute $\mathrm{Var}(U)$.

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- Okay, that's useful! But it's not the whole picture.

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- Why don't we get a little more ambitious, and seek to find the *distribution* of U?
- First, let me be a little more clear about what I mean by "distribution".

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- Sometimes, we can identify a distribution by name (e.g. "Exponential distribution with parameter $\theta = 0.5$", or "Standard normal distribution").

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  - Its **density function** (PDF)

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- But, a distribution could just as easily have been described by any of the following:
  - Its **distribution function** (i.e. CDF)
  - Its **density function** (PDF)
  - Its **MGF** (moment-generating function)

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Distributions

- For example, suppose I tell you the random variable W has density function given by
$$f_W(w) = 2e^{-2w} \cdot \mathbb{1}_{\{w \geq 0\}}$$

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- This would, in turn, automatically tell you that W has distribution function

$$F_W(w) = \begin{cases} 1 - e^{-2w} & \text{if } w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and MGF

$$M_W(t) = \begin{cases} (1 - t/2)^{-1} & \text{if } t < 1/2 \\ \infty & \text{otherwise} \end{cases}$$

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$$M_X(t) = \exp\left\{2t + \frac{1}{2}t^2\right\}$$

you would immediately be able to say

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(x-2)^2\right\}$$

and

$$F_X(x) = \Phi\left(x - 2\right); \qquad \Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, \mathrm{d}z$$

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- All of this is to say: I encourage you to get into the habit of thinking about "distributions" fairly broadly, and thinking of a distribution as either a density function, distribution function, or MGF (or all three).

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Back to Transformations

Goal

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Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

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- We could go after the density function of U.

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- Or we could go after the distribution function of U.

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- We could go after the density function of U.
- Or we could go after the distribution function of U.
- Or we could go after the MGF of U.
- Indeed, each of these three approaches are what our textbook calls different "methods".

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Support

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- Recall that the support (aka "state space") of a random variable Y is the set of all values that Y maps to: i.e. $S_Y := Y(\Omega)$. Equivalently, it's the set of all values y for which the density $f_Y(y)$ is nonzero.

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- Then, given a random variable $U := g(Y)$, we have $S_U = g(S_Y)$.

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- Then, given a random variable $U := g(Y)$, we have $S_U = g(S_Y)$.
  - That is, the support of a transformed random variable is the image of the original support under the transformation.

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    - That is, the support of a transformed random variable is the image of the original support under the transformation.

- Though this formula seems inoccuous enough, finding the support of a transformed random variable can be trickier than it first appears...

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Support

- A simple way I like to think about things is to draw a picture.

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- Specifically, let's say we have an interval $[a, b]$ and a transformation $g : \mathbb{R} \to \mathbb{R}$.

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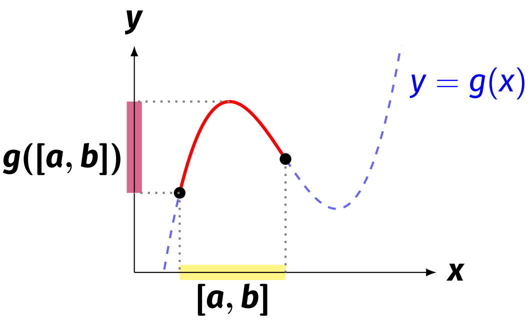
- A simple way I like to think about things is to draw a picture.
- Specifically, let's say we have an interval $[a, b]$ and a transformation $g : \mathbb{R} \to \mathbb{R}$.
- To figure out what $g([a, b])$ looks like, simply graph the function $y = g(x)$, indicate $[a, b]$ on the $x-$axis, and figure out what the corresponding values on the $y-$axis are.

Support



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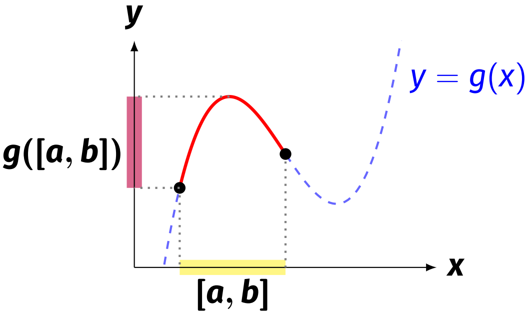
Support



- Note: in general, $g([a, b]) \neq [g(a), g(b)]$!

Clicker Question!

Clicker Question 1

For $A = [0\ , 6]$ and $g(x) = \cos(\pi x)$, what is the correct expression for $g(A)$?

(A) $[0\ ,\ 1]$ (B) $[0\ ,\ 6]$ (C) $[-1\ , 1]$ (D) $\{0\}$
(E) None of the above

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Try this On Your Own:

Example

For $A = [-1\ ,\ 1]$ and $g(x) = x^2$, what is the correct expression for $g(A)$?

Method of Distribution Functions (CDF Method)

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CDF Method

- Let's consider the following rephrasing of our goal:

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CDF Method

- Let's consider the following rephrasing of our goal:

Goal

Given a random variable Y and a function $g()$, we seek to derive an expression for $F_U(u) := \mathbb{P}(U \leq u)$, the CDF of U.

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- As a concrete example, let $Y \sim \text{Exp}(\theta)$ and let $U := cY$ for a positive constant c.

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- As a concrete example, let $Y \sim \text{Exp}(\theta)$ and let $U := cY$ for a positive constant c.
  - If it helps, you can think of this in terms of our inches-to-centimeter conversion example from the start of this lecture: Y can denote the heights in inches and U can denote the heights in cenimeters.

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CDF Method

- Now, we know everything we could want to know about Y.

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CDF Method

- Now, we know everything we could want to know about Y.
- Specifically, we have the CDF of Y:

$$F_Y(y) = \begin{cases} 1 - e^{-y/\theta} & \text{if } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

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- So, if we can relate $F_U(u)$ to $F_Y(y)$, we'd be done.

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- So, if we can relate $F_U(u)$ to $F_Y(y)$, we'd be done.
- Note:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(cY \leq u)$$

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- Note:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(cY \leq u)$$

- Divide through by c:

$$F_U(u) = \mathbb{P}\left(Y \leq \frac{u}{c}\right) = F_Y\left(\frac{u}{c}\right)$$

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CDF Method

- So, plugging into our expression for $F_Y(y)$, we have:

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$$
\begin{aligned}
F_U(u) &= F_Y\left(\frac{u}{c}\right) \\
&= \begin{cases} 1 - e^{(u/c)/\theta} & \text{if } (u/c) \geq 0 \\ 0 & \text{otherwise} \end{cases}
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CDF Method

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\end{aligned}
$$

- And we're done! We've accomplished our goal, and found an expression for $F_U(u)$, the CDF of U.

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Going Further

- Now, in this particular case, we can take things a step further.

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Theorem (Closure of Exponential Distribution under Multiplication)

Given $Y \sim \text{Exp}(\theta)$ and a positive constant c, then $(cY) \sim \text{Exp}(c\theta)$.

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Theorem (Closure of Exponential Distribution under Multiplication)

Given $Y \sim \text{Exp}(\theta)$ and a positive constant c, then $(cY) \sim \text{Exp}(c\theta)$.

- We're going to use this result a **LOT**!

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Interpretation

- I know this might seem a little abstract - what does it mean to “multiply the exponential distribution by a constant?”

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Interpretation

- I know this might seem a little abstract - what does it mean to "multiply the exponential distribution by a constant?"
- Again, if it helps, you can always think in terms of our inches-to-centimeter problem from the start of these slides.
- If $Y \sim \text{Exp}(\theta)$ denotes the height of a randomly selected person in inches, then the distribution of heights in centimeters will *also* be exponential, this time with mean 2.54θ.

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Example

- Let's do another example together.

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- Suppose Y has density function given by

$$f_Y(y) = 2y^2 \cdot \mathbb{1}_{\{0 \le y \le 1\}}$$

and again define $U := cY$ for a positive constant c.

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  and again define $U := cY$ for a positive constant c.
- Now, before we got lucky because we immediately knew what the CDF of Y was.
- But, even though we can't *immediately* recognize the CDF of Y in this example, we can still derive it!

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Example

- By definition, for a $y \in [0, 1]$,

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\ \mathrm{d}t$$

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\begin{aligned}
F_Y(y) &= \int_{-\infty}^{y} f_Y(t) \, \mathrm{d}t \\
&= \int_{-\infty}^{y} 2t \cdot \mathbb{1}_{\{0 \leq t \leq 1\}} \, \mathrm{d}t
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- Clearly, for $y < 0$ we have $F_Y(y) = \mathbb{P}(Y \leq y) = 0$ and for $y > 1$ we have $\mathbb{P}(Y \leq y) = 1$, meaning

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ y^2 & \text{if } 0 \leq y < 1 \\ 1 & \text{if } y \geq 1 \end{cases}$$

Example

- And now we're in the same position as before!

$$\mathbb{P}(U \leq u) = \mathbb{P}(cY \leq u) = \mathbb{P}\left(Y \leq \frac{u}{x}\right)$$

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$$\begin{aligned}\mathbb{P}(U \leq u) = \mathbb{P}(cY \leq u) = \mathbb{P}\left(Y \leq \frac{u}{x}\right) \\ = F_Y\left(\frac{u}{c}\right)\end{aligned}$$

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$$
\begin{aligned}
\mathbb{P}(U \le u) = \mathbb{P}(cY \le u) &= \mathbb{P}\left(Y \le \frac{u}{X}\right) \\
&= F_Y\left(\frac{u}{c}\right) \\
&= \begin{cases} 0 & \text{if } (u/c) < 0 \\ (u/c)^2 & \text{if } 0 \le (u/c) < 1 \\ 1 & \text{if } (u/c) \ge 1 \end{cases}
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$$= F_Y\left(\frac{u}{c}\right)$$

$$= \begin{cases} 0 & \text{if } (u/c) < 0 \\ (u/c)^2 & \text{if } 0 \leq (u/c) < 1 \\ 1 & \text{if } (u/c) \geq 1 \end{cases} = \boxed{\begin{cases} 0 & \text{if } u < 0 \\ u^2/c^2 & \text{if } 0 \leq u < c \\ 1 & \text{if } u \geq c \end{cases}}$$

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Example

- One more example before we summarize.

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Example

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- Let $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$.

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- Let $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$.
- A quick sketch (see chalkboard) reveals that $S_U = [0, \infty)$. So, $F_U(u) = 0$ whenever $u < 0$.

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Example

- One more example before we summarize.
- Let $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$.
- A quick sketch (see chalkboard) reveals that $S_U = [0, \infty)$. So, $F_U(u) = 0$ whenever $u < 0$.
- Additionally, we (again) have the CDF of Y: $F_Y(y) = \Phi(y)$, where $\Phi(\cdot)$ denotes the standard normal CDF.

Example

- So, let's try and proceed like we did before! For a fixed $u \geq 0$,

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(Y^2 \leq u)$$

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$$F_U(u) = \mathbb{P}(Y^2 \leq u) = \mathbb{P}(Y \leq \sqrt{u})$$

This is, however, <u>INCORRECT</u>.

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- Let's understand why.

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Example

- There are a couple of ways to understand why the above is incorrect.

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Example

- There are a couple of ways to understand why the above is incorrect.
- One is to recall a fact from algebra/precalculus that you might have forgotten: $\sqrt{\cdot}$ means the *principal* square root, and so, for any real number x, we have $\sqrt{x^2} = |x|$.

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  - Remember, both -3 and 3 have squares equal to 9! But, when we write $\sqrt{9}$, we implicitly mean the principal square root which is why we write $\sqrt{9} = 3$.

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- So, what we really have is:

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Example

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- One is to recall a fact from algebra/precalculus that you might have forgotten: $\sqrt{\cdot}$ means the *principal* square root, and so, for any real number x, we have $\sqrt{x^2} = |x|$.
  - Remember, both -3 and 3 have squares equal to 9! But, when we write $\sqrt{9}$, we implicitly mean the principal square root which is why we write $\sqrt{9} = 3$.
- So, what we really have is:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(Y^2 \leq u) = \mathbb{P}(|Y| \leq \sqrt{u}) = \mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u})$$

Example (cont'd)

- Now, there's another way to see how to get from $\mathbb{P}(Y^2 \leq u)$ to $\mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u})$; one that doens't require us to dig into our memory banks and dredge up something from algebra/precalculus, and instead uses pictures.

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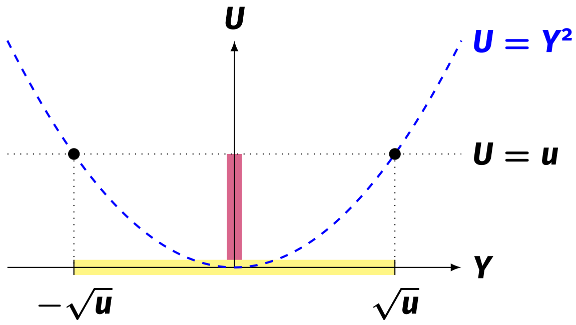
Video

https://www.youtube.com/watch?v=HtzqjHfoRbw

Static Image



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Example (cont'd)

- So, let's finish up our example!

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$$F_U(u) = \cdots = \mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u})$$

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- So, let's finish up our example!

$$
\begin{aligned}
F_U(u) = \cdots &= \mathbb{P}(-\sqrt{u} \leq Y \leq \sqrt{u}) \\
&= F_Y(\sqrt{u}) - F_Y(-\sqrt{u})
\end{aligned}
$$

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Example (cont'd)

- So, let's finish up our example!

$$\begin{aligned} F_U(u) &= \cdots = \mathbb{P}(-\sqrt{u} \le Y \le \sqrt{u}) \\ &= F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) = \Phi(\sqrt{u}) - \Phi(-\sqrt{u}) \end{aligned}$$

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Example (cont'd)

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&= \Phi(\sqrt{u}) - [1 - \Phi(\sqrt{u})] = 2\Phi(\sqrt{u}) - 1
\end{aligned}
$$

- That's a bit anticlimactic... Let's differentiate wrt. u and obtain the PDF of U:

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Example (cont'd)

$$f_U(u) = \frac{\mathrm{d}}{\mathrm{d}u} F_U(u)$$

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$$\begin{aligned} f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\ &= \frac{\mathrm{d}}{\mathrm{d}u} [2\Phi(\sqrt{u}) - 1] \end{aligned}$$

Example (cont'd)

$$\begin{aligned} f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\ &= \frac{\mathrm{d}}{\mathrm{d}u} [2\Phi(\sqrt{u}) - 1] \\ &= 2 \cdot \frac{1}{2\sqrt{u}} \cdot \phi(\sqrt{u}) = \frac{1}{\sqrt{u}} \phi(\sqrt{u}) \end{aligned}$$

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- Let's incorporate the support of U, and simplify:

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Example (cont'd)

$$f_U(u) = \frac{1}{\sqrt{u}} \phi(\sqrt{u}) \cdot \mathbb{1}_{\{u \geq 0\}}$$

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Example (cont'd)

$$\begin{aligned} f_U(u) &= \frac{1}{\sqrt{u}} \phi(\sqrt{u}) \cdot \mathbb{1}_{\{u \geq 0\}} \\ &= \frac{1}{\sqrt{u}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\sqrt{u})^2} \cdot \mathbb{1}_{\{u \geq 0\}} \end{aligned}$$

Example (cont'd)

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&= \frac{1}{\sqrt{\pi} \cdot 2^{1/2}} \cdot u^{1/2 - 1} \cdot e^{-u/2} \cdot \mathbb{1}_{\{u \geq 0\}}
\end{aligned}
$$

- One useful fact: $\Gamma(1/2) = \sqrt{\pi}$. Hence:

$$
f_U(u) = \frac{1}{\Gamma(1/2) \cdot 2^{1/2}} \cdot u^{1/2 - 1} \cdot e^{-u/2} \cdot \mathbb{1}_{\{u \geq 0\}}
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$$f_U(u) = \frac{1}{\Gamma(1/2) \cdot 2^{1/2}} \cdot u^{1/2-1} \cdot e^{-u/2} \cdot \mathbb{1}_{\{u \geq 0\}}$$

- Indeed, $U \sim \text{Gamma}(1/2\ ,\ 2) \stackrel{\text{d}}{=} \chi^2_1$!

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Theorem

- This is an ***extremely*** important result which we will use repeatedly throughout this course. Let's make it more formal by rephrasing it as a theorem:

Theorem (Square of Standard Normal)

If $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$, then $U \sim \chi^2_1$.

- The proof of this theorem is exactly the work we did on the previous slides.

Recap

- Whew- that was a lot of work! Let's recap.

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- Given a random variable Y, and $U := g(Y)$ for some function $g : \mathbb{R} \to \mathbb{R}$, we can use the **method of distribution functions** (aka the **CDF**) method to find the distribution of U.

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  (1) Writing $F_U(u)$, the CDF of U, in terms of $F_Y(y)$, the CDF of Y, by basically finding an equivalent formulation for the event $A_U := \{U \leq u\}$ that is in terms of Y

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  (2) Plugging into the CDF of Y, and simplifying as necessary.

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Method of Transformations (Change of Variable Formula)

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Leadup

- Let's, for a moment, return the example where we squared the standard normal distribution.

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- This begs the question - can we perhaps "extend" the CDF method to give us a formula for the *PDF* of U directly?

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- This begs the question - can we perhaps "extend" the CDF method to give us a formula for the *PDF* of U directly?
- The answer turns out to be "yes, under some conditions."

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Leadup

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

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Leadup

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- Let's see what happens if we try to apply the CDF method:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(g(Y) \leq u)$$

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- Let's see what happens if we try to apply the CDF method:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(g(Y) \leq u)$$

- Isn't it tempting to apply $g^{-1}(\cdot)$ to both sides of the inequality?

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Leadup

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- Let's see what happens if we try to apply the CDF method:

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(g(Y) \leq u)$$

- Isn't it tempting to apply $g^{-1}(\cdot)$ to both sides of the inequality?
- It is! <u>But we need to be careful</u>. First, remember that we don't have any guarantee that $g^{-1}(\cdot)$ even exists!

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Leadup

- Alright, then - let's add some assumption about our function $g(\cdot)$.

Leadup

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Given a random variable Y and a strictly increasing function $g()$, we seek to find $f_U(u)$, the PDF of U.

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Given a random variable Y and a strictly increasing function $g()$, we seek to find $f_U(u)$, the PDF of U.

- *Now* we are guaranteed the existence of $g^{-1}(\cdot)$.
- Furthermore, since we assumed $g(\cdot)$ itself to be strictly *increasing*, $g^{-1}(\cdot)$ will also be strictly increasing.

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Given a random variable Y and a strictly increasing function $g()$, we seek to find $f_U(u)$, the PDF of U.

- *Now* we are guaranteed the existence of $g^{-1}(\cdot)$.
- Furthermore, since we assumed $g(\cdot)$ itself to be strictly *increasing*, $g^{-1}(\cdot)$ will also be strictly increasing.
- Hence, we "preserve the direction of inequality" when applying $g^{-1}(\cdot)$ to both sides of an inequality.

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Leadup

- Then:

$$F_{(}u) := \mathbb{P}(U \leq u) = \mathbb{P}(g(Y) \leq u) = \mathbb{P}(Y \leq g^{-1}(u)) = F_Y(g^{-1}(u))$$

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Leadup

- Then:

$$F_{(}u) := \mathbb{P}(U \leq u) = \mathbb{P}(g(Y) \leq u) = \mathbb{P}(Y \leq g^{-1}(u)) = F_Y(g^{-1}(u))$$

- We can now differentiate wrt. U and apply the chain rule (from calculus; we can discuss this further on the chalkboard):

$$\begin{aligned} f_U(u) &:= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \\ &= \frac{\mathrm{d}}{\mathrm{d}u} F_Y(g^{-1}(u)) \\ &= f_Y(g^{-1}(u)) \cdot \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \end{aligned}$$

Leadup

- If we instead assume that $g(\cdot)$ is strictly decreasing, a similar computation (which I'll be asking you to complete on your homework) yields

$$f_U(u) = f_Y(g^{-1}(u)) \cdot \left[-\frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right]$$

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- So, if we instead simply assume that $g(\cdot)$ is strictly monotonic, we can summarize our findings as:

$$f_U(u) = \begin{cases} f_Y(g^{-1}(u)) \cdot \left[\frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right] & \text{if } g(\cdot) \text{ is increasing} \\ f_Y(g^{-1}(u)) \cdot \left[-\frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right] & \text{if } g(\cdot) \text{ is decreasing} \end{cases}$$

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Change of Variable Formula

- A bit of simplification (and recollections of how derivatives of increasing/decreasing functions behaves) allows us to rewrite our result above as:

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Change of Variable Formula

- A bit of simplification (and recollections of how derivatives of increasing/decreasing functions behaves) allows us to rewrite our result above as:

Theorem (Change of Variable Formula)

Given a random variable $Y \sim f_Y$ and a function $g(\cdot)$ that is strictly monotonic over the support of Y, then the random variable $U := g(Y)$ has density given by

$$f_U(u) = f_Y[g^{-1}(u)] \cdot \left| \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right|$$

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Example

- As an example, let's re-derive the closure under multiplication property of the Exponential distribution, this time using the Change of Variable formula.

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- As an example, let's re-derive the closure under multiplication property of the Exponential distribution, this time using the Change of Variable formula.
- That is: let $Y \sim \text{Exp}(\theta)$, and set $U := cY$ for some positive constant $c > 0$.

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Example

- As an example, let's re-derive the closure under multiplication property of the Exponential distribution, this time using the Change of Variable formula.
- That is: let $Y \sim \text{Exp}(\theta)$, and set $U := cY$ for some positive constant $c > 0$.
- Since the transformation $g(y) = cy$ is strictly monotonic (specifically, it's strictly increasing) it's inverse exists and is calculable as $g^{-1}(u) = u/c$. Hence:

$$\left| \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right| = \left| \frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{u}{c} \right) \right| = \left| \frac{1}{c} \right| = \frac{1}{c}$$

  where we have dropped the absolute values in the last step since we are assuming $c > 0$.

Example

- Additionally, since $Y \sim \text{Exp}(\theta)$ we know that

$$f_Y(y) = \frac{1}{\theta} \exp\left\{-\frac{y}{\theta}\right\} \cdot \mathbb{1}_{\{y \geq 0\}}$$

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$$f_Y(y) = \frac{1}{\theta} \exp\left\{-\frac{y}{\theta}\right\} \cdot \mathbb{1}_{\{y \geq 0\}}$$

- Therefore, plugging into the change of variable formula, we have

$$\begin{aligned} f_U(u) &= f_Y[g^{-1}(u)] \cdot \left| \frac{\mathrm{d}}{\mathrm{d}u} g^{-1}(u) \right| \\ &= \frac{1}{\theta} \exp\left\{ -\frac{\left(\frac{u}{c}\right)}{\theta} \right\} \cdot \mathbb{1}_{\{\frac{u}{c} \geq 0\}} \cdot \frac{1}{c} \\ &= \frac{1}{c\theta} \exp\left\{-\frac{u}{c\theta}\right\} \cdot \mathbb{1}_{\{u \geq 0\}} \end{aligned}$$

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Clicker Question!

Clicker Question 1

Given $Y \sim \text{Unif}[1, 2]$ and $U := 2X + 3$, does U also follow a Uniform Distribution?

(A) Yes; (B) No

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Change of Variable Formula

- Now, note that the only assumption we need to make about $g(\cdot)$ in order for the Change of Variable formula to hold is that it is strictly monotone *over the support of* Y.

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Change of Variable Formula

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- For example, suppose $Y \sim \text{Unif}[-1, 0]$ and take $U := Y^2$.
- Though the function $g(y) = y^2$ is not strictly monotone over $\mathbb{R}$, it *is* strictly monotone over $S_Y := [-1, 0]$ (i.e. the support of Y), and hence its inverse exists and is given by $g^{-1}(u) = -\sqrt{u}$.

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- Though the function $g(y) = y^2$ is not strictly monotone over $\mathbb{R}$, it *is* strictly monotone over $S_Y := [-1, 0]$ (i.e. the support of Y), and hence its inverse exists and is given by $g^{-1}(u) = -\sqrt{u}$.
- The Change of Variable formula can therefore safely be applied.

Change of Variable Formula

- In general, however, the Change of Variable formula does not work when we are dealing with transformations that are not strictly monotone.

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- For example, given $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$, we *cannot* directly apply the Change of Variable formula.

Change of Variable Formula

- In general, however, the Change of Variable formula does <u>not</u> work when we are dealing with transformations that are not strictly monotone.
- For example, given $Y \sim \mathcal{N}(0, 1)$ and $U := Y^2$, we *cannot* directly apply the Change of Variable formula.
  - Admittedly, there does exist a way to generalize the Change of Variable formula to work in a situation like this, but we won't cover that in PSTAT 120B. If you're curious, I'm happy to walk you through the general outline during Office Hours.

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Method of Moment-Generating Functions (MGF Method)

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Leadup

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

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Leadup

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Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- So far, we've talked about "describing" the distribution of U by both its CDF (using the CDF method) and its PDF (using the Change of Variable formula).

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Leadup

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- So far, we've talked about "describing" the distribution of U by both its CDF (using the CDF method) and its PDF (using the Change of Variable formula).
- We know that there is a third way of classifying distributions - **moment-generating functions** (MGFs).

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MGFs

Definition (MGF)

The MGF of a random variable X, notated $M_X(t)$, is defined as

$$M_X(t) := \mathbb{E}[e^{tX}]$$

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MGFs

Definition (MGF)

The MGF of a random variable X, notated $M_X(t)$, is defined as

$$M_X(t) := \mathbb{E}[e^{tX}]$$

- Recall that this expectation is computed as a sum if X is discrete and as an integral if X is continuous.

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Useful Result

Theorem

Given two random variables X and Y with MGFs $M_X(t)$ and $M_Y(t)$, respectively, that are both continuous in a small neighborhood of the origin, then $M_X(t) = M_Y(t)$ implies that X and Y have the same distribution.

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Useful Result

Theorem

Given two random variables X and Y with MGFs $M_X(t)$ and $M_Y(t)$, respectively, that are both continuous in a small neighborhood of the origin, then $M_X(t) = M_Y(t)$ implies that X and Y have the same distribution.

- This theorem is essentially just a more formal way of saying "MGFs uniquely determine random variables." For example,

$$M_X(t) = \exp\left\{2t + \frac{1}{2}t^2\right\} \iff X \sim \mathcal{N}(2, 1)$$

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Useful Result

Theorem

Given a random variable Y with MGF $M_Y(t)$, and $U := aY + b$ for constants $a, b \in \mathbb{R}$,

$$M_U(t) = e^{bt} M_Y(at)$$

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Proof.

$$M_U(t) := \mathbb{E}[e^{tU}]$$ [Definition of MGF]

$\square$

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Proof.

$$
\begin{aligned}
M_U(t) &:= \mathbb{E}[e^{tU}] && \text{[Definition of MGF]} \\
&:= \mathbb{E}[e^{t(aY+b)}] && \text{[Definition of } U\text{]}
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\end{aligned}
$$

□

- It turns out, we can use this theorem to (again) prove the closure of the exponential distribution under multiplication!

Example

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which is, as expected, the MGF of the $\text{Exp}(c\theta)$ distribution.

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Clicker Question!

Clicker Question 2

If $Y \sim \text{Pois}(\lambda)$ and $U := cY$ for some positive constant c, what is the distribution of U?

(A) $\text{Pois}(c\lambda)$

(B) $\text{Pois}(c/\lambda)$

(C) $\text{Pois}(\lambda/c)$

(D) None of the above

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Leadup

- Now, it may seem a little pointless to use the method of MGFs to identify distributions of transformed random variables.

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- The last point is definitely a legitimate criticism of the method - the method of MGFs will not give you the PDF of the transformed random variable unless you're able to recognize the MGF.

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- The last point is definitely a legitimate criticism of the method - the method of MGFs will not give you the PDF of the transformed random variable unless you're able to recognize the MGF.
- However, we will see that the method of MGFs is incredibly useful in the next section of our lecture... so stay tuned!

Summary: Univariate Transformations

Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

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Summary: Univariate Transformations

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  - The CDF Method (Method of Distribution Functions)

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  - The CDF Method (Method of Distribution Functions)
  - The Change of Variable formula (Method of Transformations)
  - The MGF Method.

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CDF Method

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(1) Write $F_U(u) := \mathbb{P}(A_U)$, where $A_U := \{U \leq u\}$.

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(2) Find an equivalent expression for A_U in terms of Y; call this A_Y.

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(3) Compute $\mathbb{P}(A_Y)$ using the distribution of Y (which is known), to then find the CDF of U.

CDF Method

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(3) Compute $\mathbb{P}(A_Y)$ using the distribution of Y (which is known), to then find the CDF of U.

- Remember: when carrying out step 3, drawing a picture can be incredibly helpful!

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Change of Variable Formula

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- Remember: this method only works when the transformation $g(\cdot)$ is strictly monotonic over the support of Y!

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- Remember: this method only works when the transformation $g(\cdot)$ is strictly monotonic over the support of Y!
- Also, a side note: so long as you are careful, the change of variable formula will give you the support of U. But, in some cases, it might be easier to find the support first (by drawing a picture), and then incorporating that into your answer later.

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MGF Method

MGF Method

(1) Compute the MGF $M_U(t)$ of U by writing it in terms of the MGF $M_Y(t)$ of Y, and then recognize the resulting MGF as belonging to a particular distribution.

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MGF Method

(1) Compute the MGF $M_U(t)$ of U by writing it in terms of the MGF $M_Y(t)$ of Y, and then recognize the resulting MGF as belonging to a particular distribution.

- This works well for linear transformations, but not too well for nonlinear transformations.

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- Also, the MGF method won't (typically) give you a PDF/CDF, so if you really want the PDF/CDF of U you should use a different method.

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- Also, the MGF method won't (typically) give you a PDF/CDF, so if you really want the PDF/CDF of U you should use a different method.
- And, again, we'll see that the MGF method really shines in a slightly different context...

Chalkboard Example

Example

The **kinetic energy** of a particle with mass m traveling at a velocity V is given by

$$E = \frac{1}{2} m V^2$$

Consider a particle selected at random, whose velocity is a random variable V with density

$$f_V(v) = 2v^3 e^{-v^2} \cdot \mathbb{1}_{\{v > 0\}}$$

Find the distribution of the kinetic energy of this particle once using the CDF method and once using the Change of Variable formula.

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Multivariate Transformations

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Leadup

- Recall that our goal up until now has been:

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Goal

Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

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- As mentioned previously, this falls under the umbrella of **univariate transformations**, as we are investigating transformations of a *single* random variable Y.

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Given a random variable Y and a function $g()$, we seek to describe the random variable $U := g(Y)$.

- As mentioned previously, this falls under the umbrella of **univariate transformations**, as we are investigating transformations of a *single* random variable Y.
- In many situations, it's desired to investigate the transformation of *two or more* random variables, leading us to the realm of **multivariate transformations**.

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Bivariate Transformations

- For now, let's restrict ourselves to considering only **bivariate transformations**; i.e. transformations of two random variables.

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Given a bivariate random vector $(Y_1, Y_2) \sim f_{Y_1,Y_2}$ and a function $g : \mathbb{R}^2 \to \mathbb{R}$, we seek to describe the random variable $U := g(Y_1, Y_2)$.

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- First, note that a "bivariate random vector" is just a fancy way of saying "pair of random variables."
  - The notation $(Y_1, Y_2) \sim f_{Y_1,Y_2}$ is just a shorthand for saying that the bivariate random vector (Y_1, Y_2) has joint density given by $f_{Y_1,Y_2}(y_1, y_2)$.

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Examples

- Some examples of bivariate transformations include:

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  - More involved transformations, like $U := 2Y_1^2 - \sqrt{Y_2}$

- Notice that in each of these cases our transformation $g(\cdot)$ is a function that takes *two* inputs and returns only *one* output.

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Reference Example

- To ground ourselves even further, let's consider the following reference example: let $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, and set $U := Y_1 + Y_2$. Our goal is to find the distribution of U.

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- Well... nothing's stopping us from trying to start where we started our univariate considerations, and trying to find an expression for the CDF of U!
- Specifically,

$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(Y_1 + Y_2 \leq u)$$

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- Specifically,
$$F_U(u) := \mathbb{P}(U \leq u) = \mathbb{P}(Y_1 + Y_2 \leq u)$$
- Now, this should look **VERY** familiar to you... from PSTAT 120A!

Reference Example

- Specifically, the probability on the RHS is something we can find by double integrating the joint density $f_{Y_1,Y_2}(y_1, y_2)$ over the region

$$\mathcal{R} := \{(y_1, y_2) \in S_{Y_1,Y_2} : y_1 + y_2 \leq u\}$$

where S_{Y_1,Y_2} denotes the joint support of (Y_1, Y_2).

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where S_{Y_1,Y_2} denotes the joint support of (Y_1, Y_2).

- That is,

$$\mathbb{P}(Y_1 + Y_2 \leq u) = \iint_{\mathcal{R}} f_{Y_1,Y_2}(y_1, y_2)\, \mathrm{d}A$$

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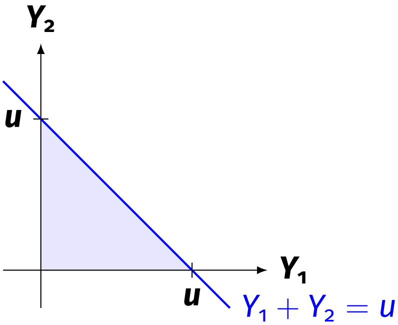
Region of Integration



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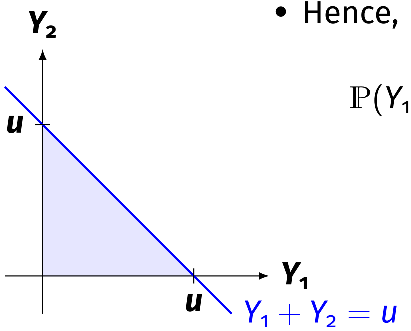

- Hence,

$$\begin{aligned}
\mathbb{P}(Y_1 + Y_2 \leq u) &= \int_0^u \int_0^{u-y_1} f_{Y_1,Y_2}(y_1, y_2)\ \mathrm{d}y_2\ \mathrm{d}y_1 \\
&= \int_0^u \int_0^{u-y_2} f_{Y_1,Y_2}(y_1, y_2)\ \mathrm{d}y_1\ \mathrm{d}y_2
\end{aligned}$$

Joint Density

- In this case, neither order of integration will be particularly easier than the other - as such, let's (somewhat arbitrarily) use the order $dy_2\, dy_1$.

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Joint Density

- In this case, neither order of integration will be particularly easier than the other - as such, let's (somewhat arbitrarily) use the order $dy_2 \, dy_1$.

- Additionally, note that since $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$:

$$f_{Y_1,Y_2}(y_1, y_2) = f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) = \frac{1}{\theta^2} e^{-(y_1+y_2)/\theta} \cdot \mathbb{1}_{\{y_1 \geq 0, \; y_2 \geq 0\}}$$

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Integration

- So:

$$\mathbb{P}(Y_1 + Y_2 \leq u) = \int_0^u \int_0^{u-y_1} \frac{1}{\theta^2} e^{-(y_1+y_2)/\theta} \, \mathrm{d}y_2 \, \mathrm{d}y_1$$
$$= \frac{1}{\theta} \cdot \int_0^u e^{-y_1/\theta} \int_0^{u-y_1} \left[\frac{1}{\theta} e^{-y_2/\theta} \right] \mathrm{d}y_2 \, \mathrm{d}y_1$$

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Integration

- So:

$$\mathbb{P}(Y_1 + Y_2 \leq u) = \int_0^u \int_0^{u-y_1} \frac{1}{\theta^2} e^{-(y_1+y_2)/\theta} \, dy_2 \, dy_1$$
$$= \frac{1}{\theta} \cdot \int_0^u e^{-y_1/\theta} \int_0^{u-y_1} \left[\frac{1}{\theta} e^{-y_2/\theta} \right] dy_2 \, dy_1$$

- Note that the inner integral (which I've highlighted in blue) is just the CDF of the Exp(θ) distribution evaluated at $u - y_1$, and is therefore equal to $1 - e^{-(u-y_1)/\theta}$.

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Integration

- Therefore:

$$\begin{aligned}
\mathbb{P}(Y_1 + Y_2 \leq u) &= \frac{1}{\theta} \cdot \int_0^u e^{-y_1/\theta} \int_0^{u-y_1} \left[\frac{1}{\theta} e^{-y_2/\theta} \right] \, dy_2 \, dy_1 \\
&= \frac{1}{\theta} \cdot \int_0^u e^{-y_1/\theta} \left(1 - e^{-(u-y_1)/\theta}\right) \, dy_1 \\
&= \frac{1}{\theta} \cdot \int_0^u \left(e^{-y_1/\theta} - e^{-u/\theta}\right) \, dy_1 \\
&= \int_0^u \frac{1}{\theta} e^{-y_1/\theta} \, dy_1 - \frac{u}{\theta} e^{-u/\theta}
\end{aligned}$$

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Integration

- Therefore:

$$
\begin{aligned}
\mathbb{P}(Y_1 + Y_2 \leq u) &= \frac{1}{\theta} \cdot \int_0^u e^{-y_1/\theta} \int_0^{u-y_1} \left[\frac{1}{\theta} e^{-y_2/\theta} \right] \, \mathrm{d}y_2 \, \mathrm{d}y_1 \\
&= \frac{1}{\theta} \cdot \int_0^u e^{-y_1/\theta} \left(1 - e^{-(u-y_1)/\theta} \right) \, \mathrm{d}y_1 \\
&= \frac{1}{\theta} \cdot \int_0^u \left(e^{-y_1/\theta} - e^{-u/\theta} \right) \, \mathrm{d}y_1 \\
&= \int_0^u \frac{1}{\theta} e^{-y_1/\theta} \, \mathrm{d}y_1 - \frac{u}{\theta} e^{-u/\theta}
\end{aligned}
$$

- The blue integral is, once again, the CDF of the Exp(θ) distribution, this time evaluated at u.

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Integration

- Therefore:

$$\begin{aligned}\mathbb{P}(Y_1 + Y_2 \leq u) &= \int_0^u \frac{1}{\theta} e^{-y_1/\theta}\, \mathrm{d}y_1 - \frac{u}{\theta} e^{-u/\theta} \\ &= 1 - e^{-u/\theta} - \frac{u}{\theta} e^{-u/\theta}\end{aligned}$$

Integration

- Therefore:

$$\mathbb{P}(Y_1 + Y_2 \le u) = \int_0^u \frac{1}{\theta} e^{-y_1/\theta}\, dy_1 - \frac{u}{\theta} e^{-u/\theta}$$
$$= 1 - e^{-u/\theta} - \frac{u}{\theta} e^{-u/\theta}$$

- And this is the CDF of U! If we want the density, we can differentiate wrt. u:

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Differentiation

$$\begin{aligned} f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \cdot \mathbb{1}_{\{u \geq 0\}} \\ &= \frac{\mathrm{d}}{\mathrm{d}u}\left[1 - e^{-u/\theta} - \frac{u}{\theta} e^{-u/\theta}\right] \cdot \mathbb{1}_{\{u \geq 0\}} \\ &= \frac{1}{\theta} e^{-u/\theta} - \frac{1}{\theta}\left[e^{-u/\theta} - \frac{u}{\theta} e^{-u/\theta}\right] \cdot \mathbb{1}_{\{u \geq 0\}} \\ &= \boxed{\frac{u}{\theta^2} e^{-u/\theta} \cdot \mathbb{1}_{\{u \geq 0\}}} \end{aligned}$$

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Differentiation

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\begin{aligned}
f_U(u) &= \frac{\mathrm{d}}{\mathrm{d}u} F_U(u) \cdot \mathbb{1}_{\{u \geq 0\}} \\
&= \frac{\mathrm{d}}{\mathrm{d}u} \left[1 - e^{-u/\theta} - \frac{u}{\theta} e^{-u/\theta} \right] \cdot \mathbb{1}_{\{u \geq 0\}} \\
&= \frac{1}{\theta} e^{-u/\theta} - \frac{1}{\theta} \left[e^{-u/\theta} - \frac{u}{\theta} e^{-u/\theta} \right] \cdot \mathbb{1}_{\{u \geq 0\}} \\
&= \boxed{\frac{u}{\theta^2} e^{-u/\theta} \cdot \mathbb{1}_{\{u \geq 0\}}}
\end{aligned}
$$

- This is, in fact, the density of a distribution we've encountered before - specifically, it is the density of the Gamma$(2, \theta)$ distribution!

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Result

Theorem (Sum of IID Exponential Distributions)

Given $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, we have $U := (Y_1 + Y_2) \sim \text{Gamma}(2, \theta)$.

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Theorem (Sum of IID Exponential Distributions)

Given $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, we have $U := (Y_1 + Y_2) \sim \text{Gamma}(2, \theta)$.

- This is a very useful result that we will frequently use throughout this course!

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Theorem (Sum of IID Exponential Distributions)

Given $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, we have $U := (Y_1 + Y_2) \sim \text{Gamma}(2, \theta)$.

- This is a very useful result that we will frequently use throughout this course!
- Additionally, its proof (which is just the work we did over the past several slides) highlights how we can use the CDF method to find distributions of bivariate transformations.

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Chalkboard Exercise

Example

Given $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \text{Unif}[0, 1]$, find the densities of $U_1 := Y_1 Y_2$ and $U_2 := Y_1 + Y_2$.

Leadup

- Now, there is actually another way we could have proved the "Sum of IID Exponential Distributions" result.

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Leadup

- Now, there is actually another way we could have proved the "Sum of IID Exponential Distributions" result.
- Specifically, note that if $Y_1 \perp Y_2$, then

$$M_{Y_1+Y_2}(t) := \mathbb{E}[e^{t(Y_1+Y_2)}] = \mathbb{E}[e^{tY_1}] \cdot \mathbb{E}[e^{tY_2}] = M_{Y_1}(t) \cdot M_{Y_2}(t)$$

Leadup

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- Hence, given $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$,

$$\begin{aligned} M_{Y_1+Y_2}(t) &= M_{Y_1}(t) \cdot M_{Y_2}(t) = [M_{Y_1}(t)]^2 \\ &= \left(\begin{cases} (1-\theta t)^{-1} & \text{if } t < 1/\theta \\ \infty & \text{otherwise} \end{cases} \right)^2 = \begin{cases} (1-\theta t)^{-2} & \text{if } t < 1/\theta \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

which we recognize as the MGF of the Gamma$(2, \theta)$ distribution. Done!

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Result

Theorem (Closure of Gamma Distribution under Sums)

Given an independent sequence $\{Y_i\}_{i=1}^n$ of random variables with $Y_i \sim \text{Gamma}(\alpha_i, \beta)$, then

$$U := \left(\sum_{i=1}^n Y_i \right) \sim \text{Gamma} \left(\sum_{i=1}^n \alpha_i \ , \ \beta \right)$$

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Proof.

$$M_U(t) := \mathbb{E}[e^{tU}] = \mathbb{E}\left[e^{t\sum_{i=1}^{n} Y_i}\right] \qquad \text{[Definition of MGF and } U\text{]}$$

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Proof.

$$M_U(t) := \mathbb{E}[e^{tU}] = \mathbb{E}\left[e^{t\sum_{i=1}^n Y_i}\right]$$ [Definition of MGF and U]

$$= \mathbb{E}\left[\prod_{i=1}^n e^{tY_i}\right]$$ [Exponential of Sum is Product of Exponentials]

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$$= \prod_{i=1}^{n} \mathbb{E}[e^{tY_i}]$$ [Independence]

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$$= \prod_{i=1}^n M_{Y_i}(t) \qquad \text{[Definition of MGF]}$$

$$= \prod_{i=1}^n \left(\begin{cases} (1-\beta t)^{-\alpha_i} & \text{if } t < 1/\beta \\ \infty & \text{otherwise} \end{cases} \right) \qquad \text{[MGF of } Y_i\text{]}$$

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Proof.

$$M_U(t) = \prod_{i=1}^{n} \left(\begin{cases} (1 - \beta t)^{-\alpha_i} & \text{if } t < 1/\beta \\ \infty & \text{otherwise} \end{cases} \right) \qquad \text{[MGF of } Y_i\text{]}$$

$\square$

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$$= \left(\begin{cases} (1-\beta t)^{-\sum_{i=1}^{n} \alpha_i} & \text{if } t < 1/\beta \\ \infty & \text{otherwise} \end{cases} \right) \qquad \text{[Algebra]}$$

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$$M_U(t) = \prod_{i=1}^{n} \left(\begin{cases} (1-\beta t)^{-\alpha_i} & \text{if } t < 1/\beta \\ \infty & \text{otherwise} \end{cases} \right) \qquad \text{[MGF of } Y_i\text{]}$$

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- The final step of the proof is to note that this is precisely the MGF of a Gamma($\sum_{i=1}^{n} \alpha_i \ , \ \beta$) distribution.

□

MGF Method, Revisited

- So, perhaps we can now see where the MGF method really shines - in the context of sums and linear combinations of random variables!

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MGF Method, Revisited

- So, perhaps we can now see where the MGF method really shines - in the context of sums and linear combinations of random variables!
- As an exercise, try proving the following result:

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MGF Method, Revisited

- So, perhaps we can now see where the MGF method really shines - in the context of sums and linear combinations of random variables!
- As an exercise, try proving the following result:

Theorem (MGF of Linear Combinations)

Consider an independent sequence $\{Y_i\}_{i=1}^n$ of random variables with MGFs $M_{Y_i}(t)$. Given $U := \sum_{i=1}^n a_i Y_i$ for constants $\{a_i\}_{i=1}^n$, we have

$$M_U(t) = \prod_{i=1}^n M_{Y_i}(a_i t)$$

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Closure of Normal Distribution

- After proving this result, you can verify that the following result also follows:

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Closure of Normal Distribution

- After proving this result, you can verify that the following result also follows:

Theorem (Closure of Normal Distribution under Linear Combinations)

Given an independent sequence $\{Y_i\}_{i=1}^n$ of random variables with $Y_i \sim \mathcal{N}(\mu_i\ , \sigma_i^2)$, we have

$$\left(\sum_{i=1}^n \alpha_i Y_i \right) \sim \mathcal{N}\left(\sum_{i=1}^n \alpha_i \mu_i\ , \ \sum_{i=1}^n (\alpha_i \sigma_i)^2 \right)$$

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Leadup

- So, we've seen how the CDF method can be extended to allow us to derive distributions of multivariate transformations.

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Leadup

- So, we’ve seen how the CDF method can be extended to allow us to derive distributions of multivariate transformations.
- We’ve now also seen how the MGF method can be used to help us explore the distributions of multivariate transformations.

Leadup

- So, we've seen how the CDF method can be extended to allow us to derive distributions of multivariate transformations.
- We've now also seen how the MGF method can be used to help us explore the distributions of multivariate transformations.
- A natural question arises: is there an analog of the Change of Variable formula?

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Leadup

- So, we've seen how the CDF method can be extended to allow us to derive distributions of multivariate transformations.
- We've now also seen how the MGF method can be used to help us explore the distributions of multivariate transformations.
- A natural question arises: is there an analog of the Change of Variable formula?
- The answer is, "yes - for bivariate transformations."

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- A natural question arises: is there an analog of the Change of Variable formula?
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- This method is often referred to as the **Jacobian Method**.

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- I admit that the Jacobian Method can seem a bit daunting at first.

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Leadup

- I admit that the Jacobian Method can seem a bit daunting at first.
- So, allow me to simply state the formula for you, make a few comments, and then walk through an example (which I hope should help demystify things a bit).

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Jacobian Method

Theorem (The Jacobian Method)

Consider $Y_1, Y_2 \sim f_{Y_1,Y_2}$, and random variables

$$U_1 := h_1(Y_1, Y_2); \qquad U_2 := h_2(Y_1, Y_2)$$

with inverse transformations

$$Y_1 = h_1^{-1}(U_1, U_2); \qquad Y_2 = h_2^{-1}(U_1, U_2)$$

The joint density of U_1, U_2 is given by

$$f_{U_1,U_2}(u_1, u_2) = f_{Y_1,Y_2}\left(h_1^{-1}(u_1, u_2)\ ,\ h_2^{-1}(u_1, u_2)\right) \cdot |J|; \quad J := \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial u_1} & \frac{\partial h_1^{-1}}{\partial u_2} \\ \frac{\partial h_2^{-1}}{\partial u_1} & \frac{\partial h_2^{-1}}{\partial u_2} \end{vmatrix}$$

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Some Comments

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$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \text{is the } \underline{\textbf{determinant}} \text{ of a } 2 \times 2 \text{ matrix.}$$

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- Hence,

$$J := \begin{vmatrix} \frac{\partial h_1^{-1}}{\partial u_1} & \frac{\partial h_1^{-1}}{\partial u_2} \\ \frac{\partial h_2^{-1}}{\partial u_1} & \frac{\partial h_2^{-1}}{\partial u_2} \end{vmatrix} = \left(\frac{\partial h_1^{-1}}{\partial u_1} \right) \left(\frac{\partial h_2^{-1}}{\partial u_2} \right) - \left(\frac{\partial h_1^{-1}}{\partial u_2} \right) \left(\frac{\partial h_2^{-1}}{\partial u_1} \right)$$

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  - It's just a bit more succinct to write this as a determinant, as opposed to writing it in its expanded form (like we did above).

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Example

Example

Given $Y_1, Y_2 \overset{\text{i.i.d.}}{\sim} \text{Exp}(\theta)$, find the joint density of (U_1, U_2) where

$$U_1 := \frac{Y_1}{Y_1 + Y_2}; \qquad U_2 := Y_1 + Y_2$$

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- When applying the method of Jacobian transformations, a good first step is always to find the inverse transformations. This amounts to solving the system of equations

$$\begin{cases} u_1 = h_1(y_1, y_2) \\ u_2 = h_2(y_1, y_2) \end{cases} \qquad \text{for } y_1, y_2 \text{ in terms of } u_1, u_2$$

Example

- In this example, $h_1(y_1, y_2) = y_1/(y_1 + y_2)$ and $h_2(y_1, y_2) = y_1 + y_2$. Hence, we seek to solve:

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- Substituting the second equation into the first yields

$$u_1 = \frac{y_1}{u_2} \quad \Longrightarrow \quad y_1 = u_1 u_2 =: h_1^{-1}(u_1, u_2)$$

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- Then, the second equation yields

$$y_2 = u_2 - y_1 = u_2 - u_1 u_2 = u_2(1 - u_1) =: h_2^{-1}(u_1, u_2)$$

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- So, we have $h_1^{-1}(u_1, u_2) = u_1 u_2$ and $h_2^{-1}(u_1, u_2) = u_2(1 - u_1)$.

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- Now, let's take some partial derivatives:

$$\frac{\partial}{\partial u_1} h_1^{-1} = \frac{\partial}{\partial u_1}[u_1 u_2] = u_2$$
$$\frac{\partial}{\partial u_2} h_1^{-1} = \frac{\partial}{\partial u_2}[u_1 u_2] = u_1$$
$$\frac{\partial}{\partial u_1} h_2^{-1} = \frac{\partial}{\partial u_1}[u_2(1 - u_1)] = -u_2$$
$$\frac{\partial}{\partial u_2} h_2^{-1} = \frac{\partial}{\partial u_2}[u_2(1 - u_1)] = 1 - u_1$$

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Example

- The Jacobian is therefore:

$$J := \begin{vmatrix} \dfrac{\partial h_1^{-1}}{\partial u_1} & \dfrac{\partial h_1^{-1}}{\partial u_2} \\ \\ \dfrac{\partial h_2^{-1}}{\partial u_1} & \dfrac{\partial h_2^{-1}}{\partial u_2} \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_1 \\ \\ -u_2 & 1 - u_1 \end{vmatrix} = u_2(1 - u_1) + u_1 u_2 = u_2$$

Example

- Hence, by the formula provided in the theorem above:

$$
\begin{aligned}
f_{U_1,U_2}(u_1,u_2) &= f_{Y_1,Y_2}\left(h_1^{-1}(u_1,u_2)\ ,\ h_2^{-1}(u_1,u_2)\right)\cdot|J| \\
&= f_{Y_1}(u_1u_2)\cdot f_{Y_2}(u_2(1-u_1))\cdot|u_2| \\
&= \left[\frac{1}{\theta}e^{-u_1u_2/\theta}\cdot \mathbb{1}_{\{u_1u_2\geq 0\}}\right]\cdot\left[\frac{1}{\theta}e^{-u_2(1-u_1)/\theta}\cdot \mathbb{1}_{\{u_2(1-u_1)\geq 0\}}\right]\cdot u_2 \\
&= \boxed{\frac{u_2}{\theta^2}e^{-u_2/\theta}\cdot \mathbb{1}_{\{u_2\geq 0\}}\cdot \mathbb{1}_{\{0\leq u_1\leq 1\}}}
\end{aligned}
$$

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Example

- Again, what we have shown is that

$$f_{U_1,U_2}(u_1, u_2) = \frac{u_2}{\theta^2} e^{-u_2/\theta} \cdot \mathbb{1}_{\{u_2 \geq 0\}} \cdot \mathbb{1}_{\{0 \leq u_1 \leq 1\}}$$

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- Notice that this is of the form (something involving only u_2) times (something involving only u_1). This allows us to conclude that $U_1 \perp U_2$!

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- Notice that this is of the form (something involving only u_2) times (something involving only u_1). This allows us to conclude that $U_1 \perp U_2$!
- Not only that, we can simply read off the marginal densities from our joint densities: perhaps unsurprisingly, $U_2 \sim \text{Gamma}(2, \theta)$ but perhaps surprisingly

$$U_1 := \left(\frac{Y_1}{Y_1 + Y_2} \right) \sim \text{Unif}[0, 1]$$

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Marginals

- This example, in addition to revealing a very useful result, also illustrates how the Jacobian Method can be used to find distributions of multivariate transformaions.

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- This example, in addition to revealing a very useful result, also illustrates how the Jacobian Method can be used to find distributions of multivariate transformaions.
- Specifically, we found the distribution of $U_1 := Y_1/(Y_2 + Y_1)$ by first finding the joint density of (U_1, U_2) and then effectively integrating out u_2 to find the marginal density of U_1.

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- Specifically, we found the distribution of $U_1 := Y_1/(Y_2 + Y_1)$ by first finding the joint density of (U_1, U_2) and then effectively integrating out u_2 to find the marginal density of U_1.
- Let's see this in action again:

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Convolution

Theorem (Convolution Formula)

Given $(Y_1, Y_2) \sim f_{Y_1,Y_2}$, the density of $S := Y_1 + Y_2$ is given by

$$f_S(s) = \int_{-\infty}^{\infty} f_{Y_1,Y_2}(s - t\ ,\ t)\ \mathrm{d}t$$

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- Note that we could have proved this using the CDF method (and, indeed, you will do so on a homework problem).

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- Note that we could have proved this using the CDF method (and, indeed, you will do so on a homework problem).
- But, for practice, let's use the Jacobian Method.

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Proof

- When using the Jacobian Method to find the distribution of a multivariate transformation S, it's customary to introduce a **auxilliary** random variable T so that we can use the Jacobian Method to first find the joint density $f_{S,T}(s, t)$ of (S, T) and then integrate out t.

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- Now, we find the inverse transformations:

$$\begin{cases} s = y_1 + y_2 & \implies y_1 = s - y_2 = s - t =: h_1^{-1}(s,t) \\ t = y_2 & \implies y_2 = t =: h_2^{-1}(s,t) \end{cases}$$

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- So, we have $h_1^{-1}(s, t) = s - t$ and $h_2^{-1}(s, t) = t$.

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- Now, let's take some partial derivatives:

$$\begin{aligned}
\frac{\partial}{\partial s} h_1^{-1}(s,t) &= \frac{\partial}{\partial s}[s - t] = 1 \\
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- So, the Jacobian is

$$J := \begin{vmatrix} \dfrac{\partial h_1^{-1}}{\partial s} & \dfrac{\partial h_1^{-1}}{\partial t} \\ \\ \dfrac{\partial h_2^{-1}}{\partial s} & \dfrac{\partial h_2^{-1}}{\partial t} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ \\ 0 & 1 \end{vmatrix} = 1$$

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- Therefore, the joint density of (S, T) is:

$$\begin{aligned} f_{S,T}(s,t) &= f_{Y_1,Y_2}\left(h_1^{-1}(s,t)\ ,\ h_2^{-1}(s,t)\right) \cdot |J| \\ &= f_{Y_1,Y_2}(s - t\ ,\ t) \end{aligned}$$

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Proof

- Finally, we obtain the marginal density of S by integrating out t:

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f_S(s) &= \int_{-\infty}^{\infty} f_{S,T}(s, t) \, \mathrm{d}t \\
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- This completes the proof!
- A quick side note: often times, when using this formula, the bounds of integration will depend on s. There's an example of this on one of the Discussion Worksheets.

A Word of Caution

- One thing I should mention (that I did not mention before) is that, much like the Change of Variable formula, the Jacobian Method really only works when we are dealing with strictly monotone functions.

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- One thing I should mention (that I did not mention before) is that, much like the Change of Variable formula, the Jacobian Method really only works when we are dealing with strictly monotone functions.
- Exactly what that means for multivariate transformations is a bit complicated. Suffice it to say: when solving the system

$$\begin{cases} u_1 & = h_1(y_1, y_2) \\ h_2 & = h_2(y_1, y_2) \end{cases}$$

  check to ensure that that only *one* solution [for (y_1, y_2) in terms of (u_1, u_2)] exists.

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  check to ensure that that only *one* solution [for (y_1, y_2) in terms of (u_1, u_2)] exists.
    - If more than one solution exists, there *is* a way to modify the Jacobian Method to work, but for the purposes of this class you should simply resort to another method.

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Summary

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- For Bivariate Transformations (i.e. transformations of exactly two random variables), we can use either the CDF method or the Jacobian method.
- For linear combinations of any number of random variables, the MGF method is best-suited.