Problem

Find a function $f(x_0, ..., x_{n-1})$ that maps finite sequences $< x_1, ..., x_{n-1} >$ of positive integers one-to-one into the integers. From a programmer's perspective, this means find a function that takes a variable number of positive integer parameters and returns a value that is unique to the sequence. For example, the function $f(x_0, ..., x_{n-1}) = \sum_{i=0}^{n-1} x_i$ does not work, since with f defined this way $f(1, 2, 3)$ returns the same value as $f(3, 3)$.

What I am asking you to define here is essentially a *guaranteed unique* hash function.

Solution

Let $p_0, p_1, ..., p_n, p_{n+1}, ...$ be a list of the (infinitely many) prime numbers in ascending order. So $p_0 = 2, p_1 = 3$ and so on. Define

$$f(x_0, ..., x_{n-1}) = p_0^{x_0} p_1^{x_1} ... p_{n-1}^{x_{n-1}}$$

f maps $< x_1, ..., x_{n-1} >$ to the product of the first $n-1$ primes with exponents equal to the corresponding numbers in the sequence. For example, $f(3, 2, 1) = 2^3 \times 3^2 \times 5^1 = 380$. This function is 1-1 because of the uniqueness of prime factorization (the Fundamental Theorem of Arithmetic). If $f(x_0, ..., x_{n-1}) = f(y_0, ..., y_{m-1})$ then $p_0^{x_0} p_1^{x_1} ... p_{n-1}^{x_{n-1}} = p_0^{y_0} p_1^{y_1} ... p_{m-1}^{x_{m-1}}$; but this means that both of these expressions are prime factorizations of the same number. By the uniqueness of prime factorization, it follows that $n = m$ and all of the exponents are the same, which means for all $i = 0, ..., n-1, x_i = y_i$, i.e., the sequences are the same.

Remark: Above would actually be a bad hash function because it would overflow very quickly. Hash functions are not guaranteed unique partly because they in general can't be: their range is a finite subset of the positive integers.