Problem

Evaluate $lim_{x \to 0} sin(x)/x$.

Solution

This can be evaluated using l'Hospital's rule, which states that if f and g are continuously differentiable at a, and $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ then $\lim_{x \to a} f(x)/g(x) = f'(0)/g'(0)$. So $lim_{x \to 0} sin(x)/x = \lim_{x \to 0} cos(0)/1 = 1$.

But why is l'Hospital's rule true? This follows nicely from the definition of the derivative. Recall that the derivative of f at x is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

So

$$\frac{f'(0)}{g'(0)} =$$

$$\frac{\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}}{\lim_{h \to 0} \frac{g(0+h) - g(0)}{h}} =$$

$$\lim_{h \to 0} \frac{f(h) - f(0)}{g(h) - g(0)}$$

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