Senile Waiter Problem

Suppose that 4 people go out to dinner together and they order 4 different meals from a nice but senile waiter who remembers and orders the correct meals but then distributes them randomly. What is the probability that noone gets the meal that they ordered?

bonus

Generalized your solution to n diners - i.e., what is the probability that a random permutation of n has no fixed points?

Solution

Label the meals $0, 1, 2, 3$. Consider rearranged sequences as ways the waiter can distribute the meals. For example, $(1, 0, 3, 2)$ represents the delivery where the waiter switches diner 0 and diner 1's meals and does the same for 2 and 3. There are $4 \times 3 \times 2 \times 1 = 24$ ways to rearrange $0, 1, 2, 3$. We need to count how many of these don't leave any of the original digits in their original place. In mathematical terms, we need to count $d_4 =$ the number of permutations of $4 = \{0, 1, 2, 3\}$ that have no fixed points. The swapping example above is one such permutation.

Now d_4 equals 24 minus the number of permutations that have fixed points. We count the complement by partitioning on the number of fixed points. For $n = 1, 2, 3$ let c_n be the number of permutations of 4 that have exactly n fixed points. Then the number the number of permutations of 4 with no fixed points equals

$$24 - \sum_{i=1}^{3} c_i$$

Now we compute the c_i. Consider c_1. We can count this by partitioning on the element that is fixed and recursing on n. There are $\binom{4}{1} = 4$ choices for the fixed point. The permutations that fix just one point are extensions of permutations of $n - 1 = 3$ with no fixed points. So if we define d_n to be the number of permutations of n that have no fixed points, then

$$c_1 = \binom{4}{1} d_3$$

$$c_2 = \binom{4}{2} d_2$$

$$c_3 = \binom{4}{3} d_1 = 4 \times 0 = 0$$

$$c_4 = 1$$

In each case, we count the number of ways to choose the fixed points and multiply by the number of permutations of the smaller set with no fixed points. Now d_2 is the number of permutations of $\{0, 1\}$ that have no fixed points. There are two permutations of $\{0, 1\}$, viz., the identity and the one that swaps the pair. The swap has no fixed points, so $d_2 = 1$. You can see easily by counting that d_3 is 2 or look at it as

$$3 \times 2 = 6 - (\binom{3}{1} d_2 + 1) = 6 - (3 \times 1 + 1) = 2$$

Now we can write

$$d_4 = 24 - (c_1 + c_2 + c_3 + c_4) = 24 - (4 \times 2 + 6 \times 1 + 0 + 1) = 24 - 15 = 9$$

So the probability of a complete "derangement" of the meals is $9/24 = 3/8$.

Bonus solution

The approach used above could be extended to provide a recursive formula. I can't see an easy way to solve the resulting recursion though. I wonder if someone else can. The standard way to solve this problem is to use inclusion-exclusion to count the number of permutations that have fixed points (i.e., the ones where at least one diner gets the right meal) and then compute 1 - P(there is a fixed point). There is a beautiful punch line below that I can't claim credit for nor can I remember where I first saw it.

Let P be the set of all permutations of $n = \{0, ..., n-1\}$ and let $A \subset P$ be those that have at least one fixed point. Then for $i = 0, ..., n-1$, define

$$A_i = \{p \in P : p(i) = i\}$$

so A_i are the permutations that fix i. It's easy to see that

$$A = \bigcup_{i=0}^{n-1} A_i$$

(nice exercise to prove this).

We count the union $A = \bigcup\limits_{i=0}^{n-1} A_i$ using inclusion / exclusion.
Inclusion-exclusion allows us to count the union of the A_i by starting with $\sum_{i=0}^{n-1} |A_i|$ and then correcting it by subtracting cardinalities of pairwise intersections, then correcting that by adding three-way intersection cardinalities, then subtracting.... all the way until we get to the intersection of all of the A_i .

$|A| = \sum_{i=0}^{n-1} |A_i| -$
$\sum_{i<j} |A_i \cap A_j| +$
$\sum_{i<j<k} |A_i \cap A_j \cap A_k| -$
$...(-1)^j \sum_{i_0<...<i_{j-1}} |A_{i_0} \cap A_{i_1} ... \cap A_{i_{j-1}}| + ...$
$(-1)^{n-1} |A_0 \cap ... \cap A_{n-1}|$

The notation above requires clarification. The $i < j$ used to define the second sum means that the sum is taken over all possible pairs of sets from the A_i. Similarly, $i < j < k$ on the next sum means all three-way intersections, and so on up to the last term, which will be ± 1 times the cardinality of the intersection of all of A_i. In the same way, $i_0 < ... < i_{j-1}$ refers to an arbitrary $j-$way intersection among the A_i and in this context j is an arbitrary integer between 3 and $n-1$ (inclusive). Using $<$ in all of the sum bounds makes sure that each distinct combination of indices is counted exactly once.

All of the A_i have the same cardinality. That cardinality is $(n-1)!$. The best way to see this is to consider that a permutation of $\{0, ..., n-1\}$ that fixes i is an extension of a permutation of the other $n-1$ elements. Each distinct permution of the elements other than i extends to exactly one permutation of n that fixes i. So the number of permutations of n that fix a given single element is the same as the number of permutations of $n-1$. Therefore,

$$\sum_{i=0}^{n-1} |A_i| = n(n-1)!$$

The sum includes n terms, each with the value $(n-1)!$

Similarly, a permutation of n that fixes a two-element subset is an extension of exactly one permutation of $n-2$, so every pairwise intersection of the A_i has cardinality $()n-2)!$. There are $\binom{n}{2}$distinct pairwise intersections among the n sets. All have cardinality $(n-2)!$. Therefore,

$$\sum_{i<j} |A_i \cap A_j| = \binom{n}{2}(n-2)!$$

In the same way,

$$\sum_{i_0<...<i_{j-1}} |A_{i_0} \cap A_{i_1} ... \cap A_{i_{j-1}}| = \binom{n}{j}(n-j)!$$

There is exactly one permutation that fixes all of the elements of n (the identity), so last term,

$$(-1)^{n-1}|A_0 \cap ... \cap A_{n-1}|$$

is either 1 or -1 depending on whether n is odd (1) or even (-1).

Putting this all together, we have

$$|A| = \sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}(n-i)! = \sum_{i=1}^{n}(-1)^{i-1}(n!/(n-i)!i!)(n-i)! = n!\sum_{i=1}^{n}(-1)^{i-1}/i!$$

The solution is therefore

$$1 - |A|/n! = 1 - (n!\sum_{i=1}^{n}(-1)^{i-1}/i)/n! = 1 - \sum_{i=1}^{n}(-1)^{i-1}/i!$$

It is interesting to consider what happens to this probability as n gets large. Does it go to 0 or 1, converge to some value between or just bounce around? A beautiful observation that I don't know who to credit for is that the solution above is the nth partial sum of the Taylor Series for e^x evaluated at $x = -1$. The Taylor series for e^x is $1 + x + x^2/2! + x^3/3! + ... + x^n/n! + ...$ So the probabilities converge to $e^{-1} = 1/e$.