Senile Waiter Problem

Suppose that 4 people go out to dinner together and they order 4 different meals from a nice but senile waiter who remembers and orders the correct meals but then distributes them randomly. What is the probability that noone gets the meal that they ordered?

Solution

Label the meals $0, 1, 2, 3$. Consider rearranged sequences as ways the waiter can distribute the meals. For example, $(1, 0, 3, 2)$ represents the delivery where the waiter switches diner 0 and diner 1's meals and does the same for 2 and 3. There are $4 \times 3 \times 2 \times 1 = 24$ ways to rearrange $0, 1, 2, 3$. We need to count how many of these don't leave any of the original digits in their original place. In mathematical terms, we need to count $d_4 =$ the number of permutations of $4 = \{0, 1, 2, 3\}$ that have no fixed points. The swapping example above is one such permutation.

Now d_4 equals 24 minus the number of permutations that have fixed points. We count the complement by partitioning on the number of fixed points. For $n = 1, 2, 3$ let c_n be the number of permutations of 4 that have exactly n fixed points. Then the number the number of permutations of 4 with no fixed points equals

$$24 - \sum_{i=1}^{3} c_i$$

Now we compute the c_i. Consider c_1. We can count this by partitioning on the element that is fixed and recursing on n. There are $n = 4$ choices for the fixed point. The permutations that fix just one point are extensions of permutations of n - 1 = 3 with no fixed points. So if we define d_n to be the number of permutations of n that have no fixed points, then

$$c_1 = \binom{4}{1} d_3$$

$$c_2 = \binom{4}{2} d_2$$

$$c_3 = \binom{4}{3} d_1 = 0$$

$$c_4 = 1$$

In each case, we count the number of ways to choose the fixed points and multiply by the number of permutations of the smaller set with no fixed points. Now d_2 is the number of permutations of $\{0, 1\}$ that have no fixed points. There are two permutations of $\{0, 1\}$, viz., the identity and the one that swaps the pair. The swap has no fixed points, so $d_2 = 1$. You can see easily by counting that d_3 is 2 or look at it as

$$3 \times 2 = 6 - (\binom{3}{1} d_2 + 1) = 6 - (3 \times 1 + 1) = 2$$

Now we can write

$$d_4 = 24 - (c_1 + c_2 + c_3 + c_4) = 24 - (4 \times 2 + 6 \times 1 + 1) = 24 - 15 = 9$$

So the probablility of a complete "derangement" of the meals is $9/24 = 3/8$.