

Problem

- a) Let x_0, \dots, x_{n-1} and M be positive real numbers. Find real numbers a_0, \dots, a_n so that $\sum_{i=0}^{n-1} x_i a_i$ is maximized subject to the constraint that $\sum_{i=0}^{n-1} a_i^2 = M$
- b) If you used the definition of the cosine between vectors as part of your solution in a), prove independently that your solution is correct (the definition depends on the result being established here).

Solution (contributed by Juan Carlos Ramirez)

Consider the equivalent problem (controlling over a): minimize $-x^T a$ subject to $a^T a = M$. Let us relax the constraint to consider the problem minimize $-x^T a$ subject to $a^T a \leq M$. If we find that the minimum of this relaxed version satisfies $a^T a = M$ then it is also a minimum of the original problem (since it contains the feasible set for the original problem). The feasible set is a closed ball, so it is compact and since $f_0(a) = -x^T a$ is linear, it is also continuous and it achieves a minimum in the closed ball (the minimum exists). Let us also note f_0 is convex and infinitely differentiable. Let $f_1(a) = a^T a - M$. Then f_1 is also convex, infinitely differentiable, and there is a point $a_s = 0$ such that $f_1(a_s) < 0$. So the problem satisfies Slater's condition, and since f_1 is convex, any local minimum of the problem will satisfy KKT conditions (In particular any global minima), i.e. (a^*, λ^*) : $f_1(a^*) \leq 0$ (Primal Feasibility constraint) $\lambda^* \geq 0$ (Dual Feasibility constraint) $\lambda^* f_1(a^*) = 0$ (Complementary slackness CS) $\nabla f_0(a^*) = -\lambda^* \nabla f_1(a^*)$ (First order condition FOC) Using the FOC, we see that λ^* cannot be 0, since then $-x = 0$, but x is a prechosen vector with all positive entries. Therefore, $\lambda^* > 0$ and by CS condition $f_1(a^*) = 0$ (this gives us that $(a^*)^T a^* = M$ and therefore both minimization problems have the same minimum). Using FOC again we get

$$-x = -\lambda^*(2a^*)$$

or

$$a^* = x/(2\lambda^*)$$

and therefore

$$(a^*)^T a^* = x^T x / (2\lambda^*)^2$$

So

$$(\lambda^*) = \sqrt{x^T x} / (2\sqrt{M})$$

and

$$a^* = \sqrt{M} x / \sqrt{x^T x}$$