1. The Model

There are two equally likely states of nature, denoted by $\omega \in \{A, B\}$, and T decision-makers, $T = \{1,2,\dots,T\}$. Nature moves first and reveals to each player a private signal $s_t \in \{a, b\}$, which is true with probability q. An agent chooses between the two states of nature and receives a pay-off of 1 if it is correct and 0 otherwise.

The action of a player t with level-k is denoted α_t^k and depends on her signal as well on the inference of the observed history, that is described by $I(x_i)$, where $x_i = \{a_1, a_2, \dots, a_{t-1}\}$.

Thus the strategy of a level-0 player is,

$$\alpha_t^0(s_t, I(x_{t-1})) = \alpha_t^0(\emptyset, \emptyset) = \begin{cases} A, with\,\mathrm{P} = 0.5, \\ B, with\,\mathrm{P} = 0.5. \end{cases}$$

The best response to this random decision-making is,

$$\alpha_t^1(s_t, I(x_{t-1})) = \alpha_t^1(s_t, \emptyset) = s_t.$$

Consequently, a level-1 type always obeys her signal.

A level-2 player believes, that all her predecessors revealed their private signals, which count as much as her own. Her strategy is simply to aggregate the historic actions and her signal. In the case when she is indifferent it is assumed, that a level-2 sticks to her own signal.

Level-3 and higher players notice that other players might be influenced by their specific history. Thus, they discount uninformative actions, in which a level-2 player would decide regardless of her own signal.

1.1. Population Belief

All players exhibit a degenerate population belief on the next lower level. Thus, a level-k believes that all other players are level-$k-1$. Theoretically, a level-3 player is the first type that updates her population belief once she observers an action which is off the equilibrium path. For instance if a cascade has emerged and there is a decision contradicting the cascade, a level-3 player recognizes that the contradicting action cannot stem from a level-2 player. However, this update of population beliefs is ignored in the further analysis.

2. A world of level-0, level-1, level-2 and level-3 players

Due to the incorporation of level-3 types in this world, we have to distinguish between different kinds of cascades. A level-3 player only counts the actions that she considers to be revealing the private signals. Let $\#A^3$ and $\#B^3$ denote the number of A and B actions which a level-3 type takes into account. Whenever $\#A - \#B \geq |2|$ and $\#A^3 - \#B^3 \geq |2|$, both types ignore their private signal and optimally follow the majority. This case will be called a "real"-cascade, since both types are disregarding their signal, and is denoted by CR_t.

A "fake"-cascade emerges, when only one type is in a cascade for the same history. If $\#A - \#B \geq |2|$ but $\#A^3 - \#B^3 < |2|$ then only level-2 types play disregarding their private signal. Contrary when $\#A - \#B < |2|$ but $\#A^3 - \#B^3 \geq |2|$ only level-3 types are in a cascade. $CF2_t$ denotes a "fake"-cascade when only level-2 are following the majority and $CF3_t$ denotes a "fake"-cascade, in which only level-3 players are cascading.

The fraction of level-k players is described by φ_k. Such that, the probability of an A choice in period t given the history and $\omega = A$ is,

$$P(a_t = A|NC_t) = {}^1\!/_2 * \varphi_0 + q * \sum_{k=1}^{3} \varphi_k,$$

$$P(a_t = A|ACR_t) = {}^1\!/_2 * \varphi_0 + q * \sum_{k=1}^{3} \varphi_k,$$

$$P(a_t = A|BCR_t) = {}^1\!/_2 * \varphi_0 + q * \varphi_1,$$

$$P(a_t = A|ACF2_t) = {}^1\!/_2 * \varphi_0 + q * (\varphi_1 + \varphi_3) + \varphi_2,$$

$$P(a_t = A|BCF2_t) = {}^1\!/_2 * \varphi_0 + q * (\varphi_1 + \varphi_3),$$

$$P(a_t = A|ACF3_t) = {}^1\!/_2 * \varphi_0 + q * (\varphi_1 + \varphi_2) + \varphi_3,$$

$$P(a_t = A|BCF3_t) = {}^1\!/_2 * \varphi_0 + q * (\varphi_1 + \varphi_2),$$

and the probability of a B choice in period t given the situation is,

$$P(a_t = B|NC_t) = {}^1\!/_2 * \varphi_0 + (1 - q) * \sum_{k=1}^{3} \varphi_k,$$

$$P(a_t = B|ACR_t) = {}^1\!/_2 * \varphi_0 + (1 - q) * \varphi_1,$$

$$P(a_t = B|BCR_t) = {}^1\!/_2 * \varphi_0 + (1-q) * \sum_{k=1}^{3} \varphi_k,$$

$$P(a_t = B|ACF2_t) = {}^1\!/_2 * \varphi_0 + (1-q) * (\varphi_1 + \varphi_3),$$

$$P(a_t = B|BCF2_t) = {}^1\!/_2 * \varphi_0 + (1-q) * (\varphi_1 + \varphi_3) + \varphi_2,$$

$$P(a_t = B|ACF3_t) = {}^1\!/_2 * \varphi_0 + (1-q) * (\varphi_1 + \varphi_2),$$

$$P(a_t = B|BCF3_t) = {}^1\!/_2 * \varphi_0 + (1-q) * (\varphi_1 + \varphi_2) + \varphi_3,$$

2.1. Efficiency

The probability of a correct choice in period t is,

$$\begin{aligned} &P(a_t = A|\omega = A) \\ &= {}^1\!/_2 * \varphi_0 + q * \varphi_1 + \varphi_2 \\ &* ((P(NC_t) + P(ACF3_t) + P(BCF3_t)) * q + P(ACF2_t) + P(ACR_t)) + \varphi_3 \\ &* \Big(\big(P(NC_t) + P(ACF2_t) + P(BCF2_t)\big) * q + P(ACF3_t) + P(ACR_t)\Big). \end{aligned}$$

whereas the probability of being in a certain situation is given by,

$$P(\Theta) = \sum_{i=1}^{n} P(x_i^{\Theta}),$$

where Θ resembles the set of all possible situations,
$\Theta \in \{NC_t, AC_t^R, BC_t^R, AC_t^{F2}, BC_t^{F2}, AC_t^{F3}, BC_t^{F3}\}$.

The probability that a particular cascade arises depends on the signal precision q, the distribution of level-k's φ_k and the time in the sequence t. The higher q, the more level-1 players make the correct choice and the higher is the probability that level-2 and level-3 players choose the correct action given they are not in a cascade.