Supplementary Material document to *Diaconis & Ylvisaker penalized likelihood for $p/n \to \kappa \in (0,1)$ logistic regression*

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October 30, 2023

Contents

1 Overview

`sec:intro`

This document contains further simulation results as well as the proofs of all results in *Diaconis & Ylvisaker penalized likelihood for $p/n \to \kappa \in (0,1)$ logistic regression* along with key Lemmas, which are listed in the proofs of the respective results, and minor auxiliary Lemmas, which are listed in G. To make this document self-contained, Section A recites the modelling assumptions and the general setup of the paper.

2 Further numerical results and simulation experiments

sec:more_simuls

2.1 Stochastic boundedness of the mDYPL estimator

subsec:beta_norm_simul

To investigate the behaviour of the ℓ_2-norm of the mDYPL-estimator, we simulate $\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}}(\alpha)\|_2\sqrt{n}$, for $\alpha \in \{0.9, 0.95, 0.99, 0.99\}$ over a range of (κ, γ) values and $n \in \{200, 400, \dots, 8000\}$. Covariates are have i.i.d. rows $\boldsymbol{x}_i \sim \mathcal{N}(\mathbf{0}_p, n^{-1}\boldsymbol{I}_p)$ and the signal is the same as in Sur and Candès (2019, Figure 2) scaled to achieve the desired signal strength γ. For each specification of $\kappa, \gamma, \alpha, n$, $\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}}(\alpha)\|_2\sqrt{n}$ is estimated as the mean from 50 independent draws of $\{\boldsymbol{y}, \boldsymbol{X}\}$. The results are displayed in Figure 1 below and indicate that $\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}}\|_2(\alpha)\sqrt{n}$ is bounded in probability for all (κ, γ) ranges considered even when there is almost no shrinkage present. Hence, the limitations of Theorem 3.2 appear more of theoretical relevance rather material constraint to our theory.

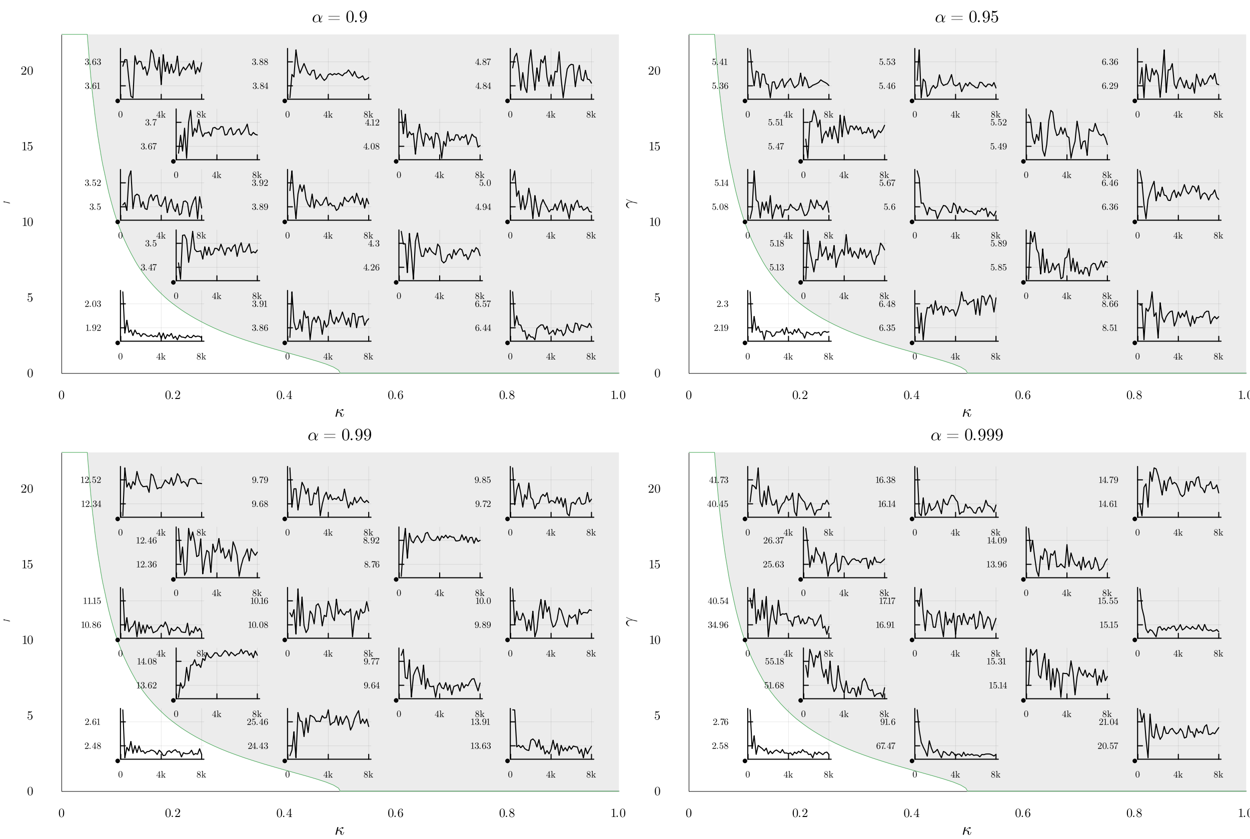


Figure 1: Estimates of $\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}}(\alpha)\|_2/\sqrt{n}$, $\alpha \in \{0.9, 0.95, 0.99, 0.999\}$ for $n \in 200, 400, \dots, 8000$, computed over 50 repetitions per n and each (κ, γ) grid point, in the setting of in Sur and Candès (2019, Figure 2). The white and gray areas indicate regions where the MLE exists (white) or does not (gray) asymptotically. Notably, $\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}}(\alpha)\|_2/\sqrt{n}$ appears bounded in probability across all considered (κ, γ) values. fig:betady_norm_alphas

2.2 Minimal MSE of unscaled estimator

subsec:min_mse_unscaled

Recall the setup of Section 4.3 of the main text. The state evolution equations (9)-(11) are solved numerically over a (κ, γ, α) grid, with $\kappa \in \{0.05, 0.0625, \dots, 0.975\}$, $\gamma \in \{0.5, 0.625, \dots, 10\}$, and $\alpha \in \{0.0125, 0.025, \dots, 1.0\}$. For each (κ, γ) combination, the α that minimizes $(1 - \mu_*)^2\gamma^2/\kappa + \sigma_*^2$, which is the almost sure limit of $\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}} - \boldsymbol{\beta}_0\|_2^2/p$, is chosen over those values for which a solution to (9)-(11) exists. Figure 2 shows the surface plot for these values.

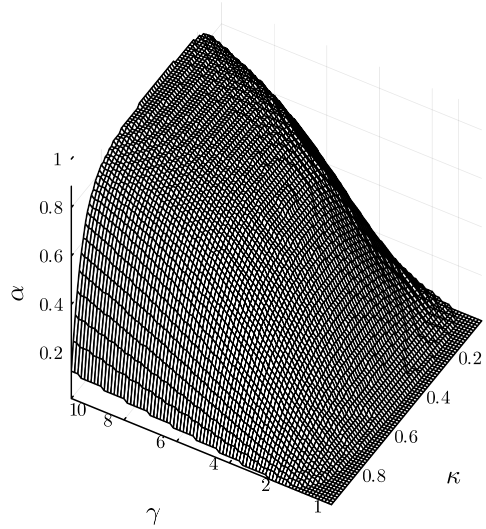
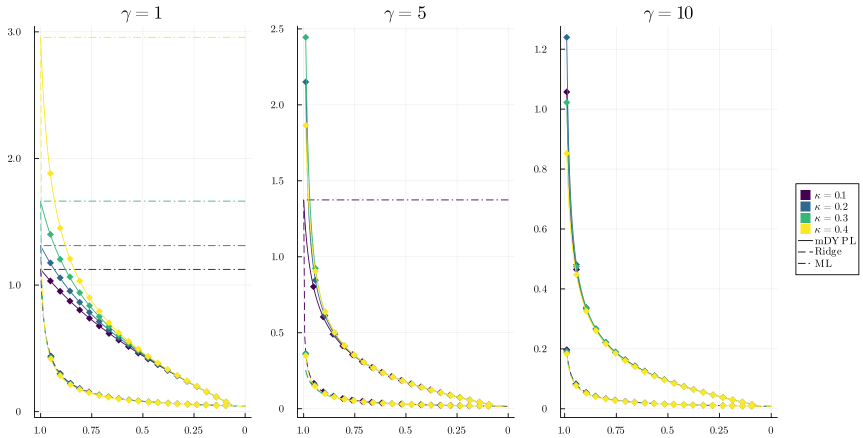


Figure 2: Surface plot of the shrinkage parameter α that minimizes the asymptotic mean squared error of the unscaled estimator, i.e. $\|\hat{\boldsymbol{\beta}}^{\text{DY}} - \boldsymbol{\beta}_0\|_2^2/p$. `fig:mse_unscaled_alpha`

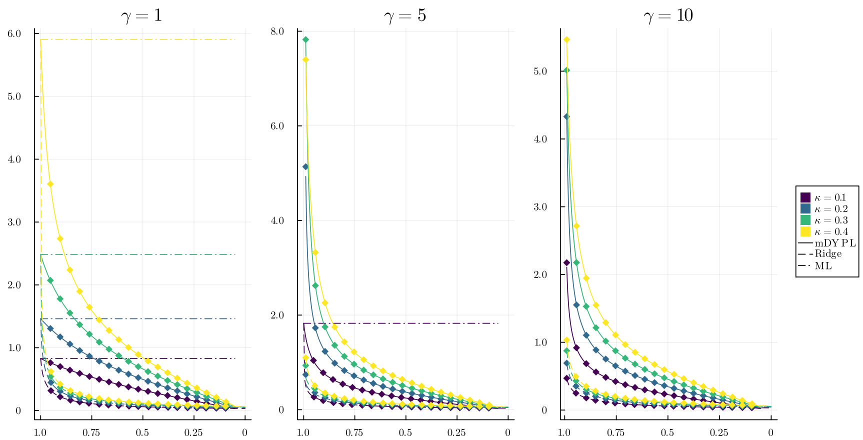
2.3 Further comparisons with the logistic Ridge estimator

`subsec:supp_ridge`

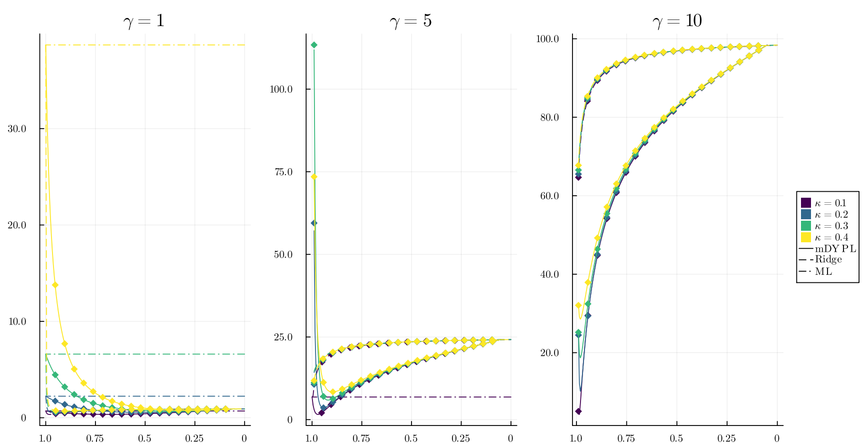
This Section provides further comparisons of the logistic Ridge estimator of Salehi et al. (2019) and the mDYPL estimator. In the setting of Section 4.4 of the main text, Figure 3 provides comparisons of the bias parameters $\mu_*, \bar{\mu}$, the variances $\kappa\sigma_*^2, \bar{\sigma}^2$ and the asymptotic MSEs of the unscaled estimators.



(a) Bias parameters $\mu_*, \bar{\mu}$



(b) Variance parameters $\sigma_*, \bar{\sigma}$



(c) Unscaled MSEs $\|\hat{\boldsymbol{\beta}}^{\text{DY}} - \boldsymbol{\beta}_0\|_2^2/p$, $\|\hat{\boldsymbol{\beta}}^{\text{R}} - \boldsymbol{\beta}_0\|_2^2/p$

Figure 3: Comparison of various metrics of the rescaled mDYPL estimator (solid lines), the ridge estimator (dashed lines) and the MLE (dash-dot lines, when available), over varying values of shrinkage parameters. The x-axis indicates the shrinkage parameter α for $\hat{\boldsymbol{\beta}}^{\text{DY}}(\alpha)$ and the transformed shrinkage parameter $1-\lambda/4.5$ for $\hat{\boldsymbol{\beta}}^{\text{R}}(\lambda)$. The analysis spans across $\gamma \in \{1, 5, 10\}$ (panels) and $\kappa \in \{0.1, 0.2, 0.3, 0.4\}$ (colours) in the setting of Salehi et al. (2019). Diamonds indicate empirical estimates of the MSEs over 1000 independent runs for $n = 2000$ and $\boldsymbol{\beta}_0$ as in Figure 2 of the main text.

`fig:rlr_supp`

A Setup

sec:setup

Consider data $\{\boldsymbol{y}, \boldsymbol{X}\}$ with responses $\boldsymbol{y} = (y_1, \ldots, y_n)^\top$, $y_j \in \{0, 1\}$, and covariate matrix $\boldsymbol{X} \in \Re^{n \times p}$ with rows $\boldsymbol{x}_j \in \Re^p$ $(j = 1, \ldots, n)$, coming from the logistic regression model, where conditional on $\boldsymbol{x}_j$, the responses y_j are independent realizations of Bernoulli random variables with mean $\zeta'(\boldsymbol{x}_j^\top \boldsymbol{\beta}_0)$ where $\zeta(x) = \log(1 + e^x)$ and $\boldsymbol{\beta}_0 \in \Re^p$ is the unknown p-dimensional vector of regression coefficients.

A.1 Maximum Diaconis & Ylvisaker penalized likelihood

subsec:dy_mape

Given hyperparameters $\tau > 0$ and $\boldsymbol{\beta}_P \in \Re^p$, the log of the Diaconis and Ylvisaker (1979) prior, bar the normalizing constant, for the logistic regression model is given by

eq:prior

$$\log p(\boldsymbol{\beta}; \boldsymbol{X}) = \tau \sum_{j=1}^{n} \left\{ \zeta'\left(\boldsymbol{x}_j^\top \boldsymbol{\beta}_P\right) \boldsymbol{x}_j^\top \boldsymbol{\beta} - \zeta\left(\boldsymbol{x}_j^\top \boldsymbol{\beta}\right) \right\} , \tag{1}$$

where $\boldsymbol{\beta}_P$ is the mode of the prior, while τ controls the variability of the prior distribution. The DY-prior penalized log-likelihood, without the normalizing constant is given by

eq:supp_posterior

$$\log p(\boldsymbol{\beta} | \boldsymbol{y}, \boldsymbol{X}) = (\tau + 1) \sum_{j=1}^{n} \left\{ \left(\frac{1}{\tau + 1} y_j + \frac{\tau}{\tau + 1} \zeta'\left(\boldsymbol{x}_j^\top \boldsymbol{\beta}_P\right) \right) \boldsymbol{x}_j^\top \boldsymbol{\beta} - \zeta\left(\boldsymbol{x}_j^\top \boldsymbol{\beta}\right) \right\} . \tag{2}$$

The maximizer of (2) is denoted by $\hat{\boldsymbol{\beta}}^{\text{DY}}$. Straightforward algebra reveals that $\hat{\boldsymbol{\beta}}^{\text{DY}}$ is the maximizer of a logistic regression model log-likelihood with pseudo-responses $\boldsymbol{y}^* = (y_1^*, \ldots, y_n^*)^\top$, $y_j^* = \alpha y_j + (1-\alpha)\zeta'(\boldsymbol{x}_j^\top \boldsymbol{\beta}_P)$, $\alpha = 1/(1+\tau) \in (0, 1)$. Indeed,

eq:supp_pseudo_loglik

$$\begin{aligned} \{\tau + 1\} \ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X}) &= \{\tau + 1\} \sum_{j=1}^{n} \left\{ y_j^* \boldsymbol{x}_j^\top \boldsymbol{\beta} - \zeta\left(\boldsymbol{x}_j^\top \boldsymbol{\beta}\right) \right\} \\ &= \{\tau + 1\} \sum_{j=1}^{n} \left\{ \left(\alpha y_j + (1-\alpha)\zeta'\left(\boldsymbol{x}_j^\top \boldsymbol{\beta}_P\right)\right) \boldsymbol{x}_j^\top \boldsymbol{\beta} - \zeta\left(\boldsymbol{x}_j^\top \boldsymbol{\beta}\right) \right\} \qquad (3) \\ &= \{\tau + 1\} \sum_{j=1}^{n} \left\{ \left(\frac{1}{\tau + 1} y_j + \frac{\tau}{\tau + 1} \zeta'\left(\boldsymbol{x}_j^\top \boldsymbol{\beta}_P\right) \right) \boldsymbol{x}_j^\top \boldsymbol{\beta} - \zeta\left(\boldsymbol{x}_j^\top \boldsymbol{\beta}\right) \right\} , \end{aligned}$$

which, aside from the scaling $\{\tau + 1\}$ coincides with the log of (2). For the remainder of this work, $\boldsymbol{\beta}_P = \boldsymbol{0}_p$ and $\hat{\boldsymbol{\beta}}^{\text{DY}}$ is understood as the maximizer of $\ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X})$.

B Proof of Theorem 3.1

appendix:thm1

Theorem 3.1 is set in the random design of Sur and Candès (2019). In particular, it is assumed that the model matrix $\boldsymbol{X}$ has independent rows $\boldsymbol{x}_i \sim \mathcal{N}(\boldsymbol{0}_p, n^{-1}\boldsymbol{I}_p)$, where $\boldsymbol{I}_p$ is the $p \times p$ identity matrix, and that the linear predictors $\boldsymbol{x}_i^\top \boldsymbol{\beta}_0$ have asymptotic variance $\lim_{n\to\infty} \text{var}(\boldsymbol{x}_i^\top \boldsymbol{\beta}_0) = \lim_{n\to\infty} \|\boldsymbol{\beta}_0\|_2^2 / n = \gamma^2$. It is further assumed that there exists a random variable $\bar{\beta} \sim \pi_{\bar{\beta}}$ with finite second moment, such that the empirical distribution of the entries of $\boldsymbol{\beta}_0$ converges weakly in distribution to $\pi_{\bar{\beta}}$ and $\lim_{n\to\infty} \frac{1}{p} \sum_{j=1}^{n} \boldsymbol{\beta}_{0,j}^2 = \text{E}(\bar{\beta}^2)$. Finally, let $\lim_{n\to\infty} p/n = \kappa$ for $\kappa \in (0, 1)$.

B.1 Asymptotic characterization via stationary state evolution

Recall the state evolution equations (4) - (6) below.

$$\mathrm{E}\left[2\zeta'(Z)Z\left\{\frac{1+\alpha}{2}-\zeta'\left(\mathrm{prox}_{b\zeta}\left(Z_*+\frac{1+\alpha}{2}b\right)\right)\right\}\right]=0 \qquad \texttt{eq:supp\_stat\_mu} \quad (4)$$

$$1-\kappa-\mathrm{E}\left[\frac{2\zeta'(Z)}{1+b\zeta''(\mathrm{prox}_{b\zeta}\left(Z_*+\frac{1+\alpha}{2}b\right))}\right]=0 \qquad \texttt{eq:supp\_stat\_b} \quad (5)$$

$$\sigma^2-\frac{b^2}{\kappa^2}\mathrm{E}\left[2\zeta'(Z)\left\{\frac{1+\alpha}{2}-\zeta'\left(\mathrm{prox}_{b\zeta}\left(Z_*+\frac{1+\alpha}{2}b\right)\right)\right\}^2\right]=0\,. \qquad \texttt{eq:supp\_stat\_sigma} \quad (6)$$

Here, $Z \sim N(0,\gamma^2)$, $Z_* = \mu Z + \kappa^{1/2}\sigma G$ for $G \sim N(0,1)$ which is independent of Z, $\zeta'(\cdot), \zeta''(\cdot)$ are the first and second derivatives of $\zeta(\cdot)$ and $\mathrm{prox}_{b_*\zeta}(x) = \arg\min_u \left\{b_*\zeta(u)+\frac{1}{2}(x-u)^2\right\}$ denotes the proximal operator. Equations (9)-(11) define the stationary points to the state evolution of the underlying AMP recursion, which can also be derived as the first order conditions of a related auxiliary optimization problem to the maximum penalized likelihood estimation (see for example Salehi et al. 2019, Appendix C).

Denote by $\boldsymbol{J}(\mu,b,\sigma)$ the Jacobian matrix of the LHS of (9)-(11) with respect to (μ,b,σ).

thm:aggregate_asymptotics

Theorem 3.1. *Assume that κ,γ,α are such that $\|\hat{\boldsymbol{\beta}}^{DY}\|_2 = \mathcal{O}(n^{1/2})$ almost surely and that (4) - (6) admit a solution (μ_*, b_*, σ_*) such that $\boldsymbol{J}(\mu_*, b_*\sigma_*)$ is nonsingular. Then for any function $\psi : \Re^2 \to \Re$ that is pseudo-Lipschitz of order 2,*

$$\frac{1}{p}\sum_{j=1}^{n}\psi(\hat{\boldsymbol{\beta}}_j^{DY}-\mu_*\boldsymbol{\beta}_{0,j},\boldsymbol{\beta}_{0,j}) \xrightarrow{a.s.} E\left[\psi(\sigma_* G,\bar{\beta})\right]\,, \quad \text{as } n\to\infty\,, \qquad \texttt{eq:thm\_eq} \quad (7)$$

where $G \sim N(0,1)$ is independent of $\bar{\beta} \sim \pi_{\bar{\beta}}$.

The proof of Theorem 3.1 closely follows the proof of Theorem 1 in Sur and Candès (2019), where the logistic indicator function $h(u,v) = \{1-\mathrm{sign}(v \le \zeta'(u))\}/2$ is simply replaced by $h(u,v) = \alpha\{1-\mathrm{sign}(v \le \zeta'(u))\}/2 + (1-\alpha)/2$ and the derivations are adjusted accordingly. Further, a smoothing argument that formally justifies the asymptotic results of the relevant AMP recursion, and which has been missing hitherto, is provided.

B.2 Proof outline for Theorem 3.1

The following steps constitute the proof of Theorem 3.1:

A.2.1) *Target AMP recursion*: As a starting point, an AMP recursion, whose stationary point coincides with $\hat{\boldsymbol{\beta}}^{\mathrm{DY}}$, is formulated.

A.2.2) *Centered target AMP recursion*: A centred AMP recursion, for which the aggregate asymptotic behaviour can be established via existing AMP results, is devised based on the target AMP recursion.

A.2.3) *Matrix-valued AMP embedding*: The centred AMP recursion is embedded into a symmetric, matrix valued AMP recursion of Javanmard and Montanari (2013) for which the topical AMP asymptotics hold. A reduction argument, which is also due to Javanmard and Montanari (2013) and which was subsequently used in Donoho and Montanari (2016) and Sur and Candès (2019), identifies the correct asymptotic behaviour for the centred AMP recursion.

A.2.4) *Smoothing argument*: As the logistic indicator function is not Lipschitz, which is a prerequisite for the application of the asymptotic result of of Javanmard and Montanari 2013, Theorem 1, it is instead established for a Lipschitz approximation parametrized by $\epsilon > 0$ to the indicator function such that $\epsilon = 0$ recovers the original recursions of A.2.1 and A.2.2.

A.2.5) *Asymptotic equivalence of target and centred AMP*: Next, it is shown that the target recursion and the centred AMP recursion, whose asymptotic behaviour is characterized through A.2.3, are asymptotically equivalent for sufficiently large iterations.

A.2.6) *Asymptotic equivalence of AMP and mDYPL*: Finally, it is shown that for $\epsilon \to 0$, the approximate AMP iterates are arbitrarily close to $\hat{\boldsymbol{\beta}}^{\mathrm{DY}}$ asymptotically if iterated for sufficiently long.

B.3 Matrix-valued AMP

For the readers' convenience, this subsection introduces the matrix valued AMP recursion of Javanmard and Montanari (2013), which is the theoretical workhorse to derive the asymptotic behaviour of the AMP recursion of interest. It is more or less a verbatim replication of Javanmard and Montanari 2013, Section 2, tailored to the specific requirements of the problem at hand.

Given integers q, N, let $\mathcal{V}_{q,N} \equiv (\Re^q)^N$, so that a vector $x \in \mathcal{V}_{q,N}$ is regarded as a N-vector with entries in $\Re^q$, i.e. $x = (\mathbf{x}_1, \dots, \mathbf{x}_q)$ where $\mathbf{x}_i \in \Re^q$ with components denoted by $\mathbf{x}_i = (\mathbf{x}_i(1), \dots, \mathbf{x}_i(q))$ $(i = 1, \dots, N)$. Given $x \in \mathcal{V}_{q,N}$, define its ℓ_2-norm as $\|x\|_2 = \left(\sum_{i=1}^N \|\mathbf{x}_i\|_2^2\right)^{1/2}$. Given a matrix $\boldsymbol{A} \in \Re^{N\times N}$, let it act on $\mathcal{V}_{q,N}$ so that for $v, v' \in \mathcal{V}_{q,N}$, $v' = \boldsymbol{A}v$ for $\mathbf{v}'_i = \sum_{j=1}^N A_{ij}\mathbf{v}_j$ for $i \in [N]$, where $[N]$ denotes the numbers integers $1, \dots, N$. That is, $\boldsymbol{A}$ is identified with the Kronecker product $\boldsymbol{A} \otimes \boldsymbol{I}_q$ when acting on elements of $\mathcal{V}_{N,q}$.

def:symmetric_AMP

Definition 1 (Symmetric AMP)**.** A symmetric AMP instance is a triple $(\boldsymbol{A}, \mathcal{F}, x^0)$, where

1) $\boldsymbol{A} = \boldsymbol{G} + \boldsymbol{G}^\top$, where $\boldsymbol{G} \in \Re^{N\times N}$ has i.i.d. $N(0, (2N)^{-1})$ entries
2) $\mathcal{F} = \left\{f^k : k \in [N]\right\}$ is a collection of mappings $f^k : \Re^q \times \mathbb{N} \to \Re^q$, $(\mathbf{x}, t) \mapsto f^k(\mathbf{x}, t)$ is locally Lipschitz in its first argument
3) $x^0 \in \mathcal{V}_{q,N}$ is an initial condition.

Given $\mathcal{F} = \left\{f^k : k \in [N]\right\}$, denote by $f(\cdot, t)$ the mapping from $\mathcal{V}_{q,N} \to \mathcal{V}_{q,N}$ such that $v' = f(v, t)$ for $\mathbf{v}'_i = f^i(\mathbf{v}_i, t)$ and all $i \in [N]$.

def:orbit

Definition 2 (AMP orbit)**.** The AMP orbit corresponding to an AMP instance $(\boldsymbol{A}, \mathcal{F}, x^0)$ is the sequence of vectors $\{x^t\}_{t\in\mathbb{N}_0}$, $x^t \in \mathcal{V}_{q,N}$ such that for $t \geq 0$,

eq:AMP_sequence

$$x^{t+1} = \boldsymbol{A}f(x^t, t) - \boldsymbol{B}_t f(x^{t-1}, t-1)\,, \tag{8}$$

where $\boldsymbol{B}_t : \mathcal{V}_{q,N} \to \mathcal{V}_{q,N}$ is the linear operator such that for $v' = \boldsymbol{B}_t v$,

$$\mathbf{v}'_i = \left(\frac{1}{N}\sum_{j=1}^{} \frac{\partial f^j}{\partial \mathbf{x}}(\mathbf{x}_j^t, t)\right)\mathbf{v}_i\,, \tag{9}$$

and where $\frac{\partial f^j}{\partial \mathbf{x}}$ denotes the Jacobian matrix of $f^j(\cdot, t) : \Re^q \to \Re^q$ given t.

Any expression with negative t-index is considered zero in the above definition.

def:con_AMP

Definition 3 (Converging AMP)**.** A sequence of AMP instances $\left\{(\boldsymbol{A}(N), \mathcal{F}_N, x^{0,N})\right\}_{N\in\mathbb{N}_0}$ is said to be converging if

i) There exist integers q, q', and
ii) a function $g : \Re^q \times \Re^q \times [q'] \times \mathbb{N} \to \Re^q$ with $g(\mathbf{x}, \mathbf{y}, a, t) = (g_1(\mathbf{x}, \mathbf{y}, a, t), \dots, g_q(\mathbf{x}, \mathbf{y}, a, t))$, such that for each $r \in [q], a \in [q'], t \in \mathbb{N}$, $g_r(\cdot, \cdot, a, t)$ is Lipschitz,
iii) q' probability measures $P_1, \dots, P_{q'}$ on $\Re^q$,
iv) for each N, a finite partition $C_1^N \cup \ldots \cup C_{q'}^N = [N]$,

v) q' positive definite matrices $\widehat{\mathbf{\Sigma}}_1^0 \ldots, \widehat{\mathbf{\Sigma}}_{q'}^0 \in \Re^{q \times q}$,

such that the following happens:

1) For each $a \in [q']$, one has $\lim_{N\to\infty} \frac{|C_a^N|}{N} = c_a \in (0,1)$,

2) There exists $y \in \mathcal{V}_{q,N}$ with $y = (\mathbf{y}_1, \ldots, \mathbf{y}_N), \mathbf{y}_i \in \Re^q \, (i = 1, \ldots, N)$, such that for each $N \in \mathbb{N}_0$, $a \in [q']$, $i \in C_a^N$, $f^i(\mathbf{x}, t) = g(\mathbf{x}, \mathbf{y}_i, a, t)$. Further, the empirical distribution of $\{\mathbf{y}_i\}_{i \in C_a^N}$, denote it by $\widehat{P}_a$, converges weakly to P_a, i.e.

$$\frac{1}{|C_a^N|} \sum_{i \in C_a^N} \delta_{\mathbf{y}_i} \xrightarrow{d} P_a \,, \tag{10}$$

3) For each $a \in [q']$, in probability,

$$\lim_{N\to\infty} \frac{1}{|C_a^N|} \sum_{i \in C_a^N} g(\mathbf{x}_i^0, \mathbf{y}_i, a, 0) g(\mathbf{x}_i^0, \mathbf{y}_i, a, 0)^\top = \widehat{\mathbf{\Sigma}}_a^0 \,. \tag{11}$$

This definition follows Remark 1 of Javanmard and Montanari (2013) and allows for a partition size q' that is distinct from q. Further note that the functions $f^i(\cdot, \cdot)$ of Definition 3, 2), depends on $\mathbf{y}_i$, but as these do not change between iterations, this dependence is suppressed for notational convenience.

def:state_evolution

Definition 4 (State evolution)**.** Given $q, q' \in \mathbb{N}_0$ and starting points $\{\widehat{\mathbf{\Sigma}}_a^0\}_{a \in [q']}$, $\widehat{\mathbf{\Sigma}}_a^0 \in \Re^{q \times q}$ positive definite, and weights $\{c_a\}_{a \in [q']}, c_a \in \Re$, the state evolution defines a sequence of positive definite matrices $\{\mathbf{\Sigma}^t\}_{t \in \mathbb{N}}$, such that for each $t \in \mathbb{N}$,

eq:state_1

$$\mathbf{\Sigma}^t = \sum_{b=1}^{q'} c_b \widehat{\mathbf{\Sigma}}_b^{t-1} \tag{12}$$

eq:state_2

$$\widehat{\mathbf{\Sigma}}_a^t = \mathrm{E}\left[g(Z_a^t, Y_a, a, t) g(Z_a^t, Y_a, a, t)^\top\right] \quad (a = 1, \ldots, q') \,, \tag{13}$$

where $Y_a \sim P_a$, $Z_a^t \sim N(\mathbf{0}_q, \mathbf{\Sigma}^t)$ and where $Y_a \perp Z_a^t$ for $a \in [q']$.

It is now possible to state Theorem 1 of Javanmard and Montanari (2013) which will serve as the blackbox result to establish the asymptotic behaviour of the mDYPL estimator. This result makes use of pseudo-Lipschitz functions, a definition of which is given for completeness.

def:pl

Definition 5 (Pseudo-Lipschitzianity)**.** For $k \geq 1$, a function $\psi : \Re^m \to \Re$ is said to be pseudo-Lipschitz of order k, denote the corresponding class of functions by $\mathrm{PL}(k)$, if there exists a constant $L > 0$ such that for all $\boldsymbol{u}, \boldsymbol{v} \in \Re^m$,

$$|\psi(\boldsymbol{u}) - \psi(\boldsymbol{v})| \leq L \left(1 + \|\boldsymbol{u}\|_2^{k-1} + \|\boldsymbol{v}\|_2^{k-1}\right) \|\boldsymbol{u} - \boldsymbol{v}\|_2 \,. \tag{14}$$

Hence, if $\psi \in \mathrm{PL}(k)$, there exists a constant L' such that for all $\boldsymbol{u} \in \Re^m$, $|\psi(\boldsymbol{u})| \leq L'(1 + \|\boldsymbol{u}\|_2^k)$. Further, note that every Lipschitz function is also pseudo Lipschitz for any $k \geq 1$.

thm:AMP

Theorem B.1 (Javanmard and Montanari (2013) Theorem 1)**.** *Let $\left\{(\boldsymbol{A}(N), \mathcal{F}_N, x^{0,N})\right\}_{N \in \mathbb{N}_0}$ be a converging sequence of AMP instances and denote by $\{x^t\}_{t \in \mathbb{N}_0}$ the corresponding AMP orbit. Suppose further that for some $k \geq 2$ and all $a \in [q']$, $E_{P_a}[\|Y_a\|^{2k-2}]$ is bounded and that $E_{\widehat{P}_a}[\|Y_a\|^{2k-2}] \to E_{P_a}[\|Y_a\|^{2k-2}]$*

as $N \to \infty$. Then for all $t \in \mathbb{N}$, $a \in [q']$ and any pseudo-Lipschitz function $\psi : \Re^q \times \Re^q \to \Re$ of order k,

$$\lim_{N\to\infty} \frac{1}{|C_a^N|} \sum_{i \in C_a^N} \psi(\mathbf{x}_i^t, \mathbf{y}_i) = E[\psi(Z_a^t, Y_a)]\,, \tag{15}$$

where $Y_a \sim P_a$, $Z_a^t \sim N(\mathbf{0}_q, \mathbf{\Sigma}^t)$ and $Y_a \perp Z_a^t$.

B.4 Proof of Theorem 3.1

B.4.1 Target AMP recursion

subsub:gen_AMP

Consider the following recursion. Given initial conditions $\hat{\boldsymbol{\beta}}^0$, $\boldsymbol{S}^0 = \boldsymbol{X}\hat{\boldsymbol{\beta}}^0$, and a sequence of constants $\{\mu_t, b_t, \sigma_t\}_{t\in\mathbb{N}_0}$, for $t \in \mathbb{N}$, set

eq:gen_AMP_beta

$$\hat{\boldsymbol{\beta}}^t = \hat{\boldsymbol{\beta}}^{t-1} + \kappa^{-1}\boldsymbol{X}^\top \Psi^{t-1}(\boldsymbol{y}^*, \boldsymbol{S}^{t-1}) \tag{16}$$

eq:gen_AMP_s

$$\boldsymbol{S}^t = \boldsymbol{X}\hat{\boldsymbol{\beta}}^t - \Psi^{t-1}(\boldsymbol{y}^*, \boldsymbol{S}^{t-1})\,, \tag{17}$$

where $\Psi^t(\boldsymbol{y}^*, \boldsymbol{S}^t)$ acts elementwise on $\boldsymbol{y}^*, \boldsymbol{S}^t$ as

eq:psi_def

$$\Psi^t(y_i^*, \boldsymbol{S}_i^t) = b_t\left\{y_i^* - \zeta'\left(\mathrm{prox}_{b_t\zeta}\left(b_t y_i^* + \boldsymbol{S}_i^t\right)\right)\right\}\,, \tag{18}$$

for $i = 1, \ldots, n$ and where $y_i^* = \alpha y_i + (1-\alpha)/2$. For $\alpha = 1$, this recursion corresponds to equations (49) and (50) of the supplementary material of Sur and Candès (2019).

The relation of this recursion to the maximizer of $\ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X})$ is shown in Sur and Candès 2019, Appendix B: If there exists a fixed points $\hat{\boldsymbol{\beta}}_*, \boldsymbol{S}_*$ along with (μ_*, b_*, σ_*), then by construction

$$\boldsymbol{X}^\top b_*\left(\boldsymbol{y}^* - \zeta'(\mathrm{prox}_{b_*\zeta}(b_*\boldsymbol{y}^* + \boldsymbol{S}_*))\right) = \mathbf{0}_p \tag{19}$$

$$(b_*\boldsymbol{y}^* + \boldsymbol{S}_*) - b_*\zeta'(\mathrm{prox}_{b_*\zeta}(b_*\boldsymbol{y}^* + \boldsymbol{S}_*)) = \boldsymbol{X}\hat{\boldsymbol{\beta}}_*\,. \tag{20}$$

Now using that $x - b\zeta'(\mathrm{prox}_{b\zeta}(x)) = \mathrm{prox}_{b\zeta}(x)$, it follows immediately that

$$\boldsymbol{X}^\top\left(\boldsymbol{y}^* - \zeta'(\boldsymbol{X}\hat{\boldsymbol{\beta}}_*)\right) = \mathbf{0}_p\,, \tag{21}$$

the left hand side of which is proportional to the gradient of of (3), and by strict concavity thereof, $\hat{\boldsymbol{\beta}}_*$ is equal to $\hat{\boldsymbol{\beta}}^{\mathrm{DY}}$.

Given initial conditions (μ_0, b_0, σ_0), let the state evolution of (μ_t, b_t, σ_t) be characterized by the solution to the following system of nonlinear equations

eq:bt

$$1 - \kappa = \mathrm{E}\left[\frac{1}{1 + b_t\zeta''\left(\mathrm{prox}_{b_t\zeta}(b_t Y^* + Z_t)\right)}\right] \tag{22}$$

eq:mut

$$\mu_{t+1} = \frac{b_t}{\kappa\gamma^2}\mathrm{E}\left[Z\left\{Y^* - \zeta'\left(\mathrm{prox}_{b_t\zeta}(b_t Y^* + Z_t)\right)\right\}\right] + \mu_t \tag{23}$$

eq:sigmat

$$\sigma_{t+1}^2 = \frac{b_t^2}{\kappa^2}\mathrm{E}\left[\left\{Y^* - \zeta'\left(\mathrm{prox}_{b_t\zeta}(b_t Y^* + Z_t)\right)\right\}^2\right]\,, \tag{24}$$

for independent $Z \sim N(0, \gamma^2)$, $G \sim N(0,1)$, and $\bar{\varepsilon} \sim U(0,1)$, and letting

eq:y_star

$$Y^* = \frac{\alpha}{2}\left\{1 - \mathrm{sign}\left(\bar{\varepsilon} - \zeta'(Z)\right)\right\} + \frac{1-\alpha}{2}, \quad Z_t = \mu_t Z + \kappa^{1/2}\sigma_t G\,. \tag{25}$$

The relation to equations (22)-(24) of the supplementary material of Sur and Candès (2019) and equation (118) in Feng et al. (2022) is evident.

B.4.2 Centered target AMP recursion

subsubsec:centred_AMP

As in Sur and Candès (2019) and Donoho and Montanari (2016), consider the following centred recursion with adjusted initial conditions $\boldsymbol{\nu}^0 = \hat{\boldsymbol{\beta}}^0 - \mu_0\boldsymbol{\beta}_0$, $\boldsymbol{R}^0 = \boldsymbol{S}^0$ for $\hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0 \in \Re^p$, $\mu_0 \in \Re$ and $\boldsymbol{S}^0 \in \Re^n$, for $t \in \mathbb{N}$, let

$$\boldsymbol{\nu}^t = q_{t-1}(\boldsymbol{\nu}^{t-1} + \mu_{t-1}\boldsymbol{\beta}_0) - a_t\boldsymbol{\beta}_0 + \kappa^{-1}\boldsymbol{X}^\top\Psi^{t-1}(\boldsymbol{y}^*, \boldsymbol{R}^{t-1}) \quad \text{eq:centred\_rec1} \quad (26)$$

$$\boldsymbol{R}^t = \boldsymbol{X}(\boldsymbol{\nu}^t + \mu_t\boldsymbol{\beta}_0) - \Psi^{t-1}(\boldsymbol{y}^*, \boldsymbol{R}^{t-1}), \quad \text{eq:centred\_rec2} \quad (27)$$

where $a_0 = \mu_0$ and for $t \in \mathbb{N}$,

$$q_t = -\frac{1}{\kappa n}\sum_{i=1}^n \frac{\partial\Psi^t}{\partial R}(y_i^*, R)\Big|_{R=\boldsymbol{R}_i^t} \quad \text{eq:aux\_centred\_rec1} \quad (28)$$

$$a_t = \frac{1}{\kappa n}\sum_{i=1}^n \frac{\partial\Psi^{t-1}}{\partial Z}(h(Z,\bar{\varepsilon}_i), \boldsymbol{R}_i^{t-1})\Big|_{Z=\boldsymbol{x}_i^\top\boldsymbol{\beta}_0}, \quad \text{eq:aux\_centred\_rec2} \quad (29)$$

$$(30)$$

where $\bar{\varepsilon}_i \sim U(0,1)\,(i = 1,\dots,n)$ are i.i.d. draws independent of all other random variables.

B.4.3 Matrix-valued AMP-embedding

This section embeds the centred AMP recursion of (B.4.2) in a symmetric AMP-instance, which is shown to satisfy all properties of Definitions 1-4, apart from a Lipschitzianity constraint on $g(\cdot,\cdot,a,t)$ in Definition 3, which is dealt with in Section B.4.4. Theorem B.1 then characterizes the asymptotic behaviour of said recursion. In particular, the centred recursion of (26)-(29), which is not symmetric, is embedded in a symmetric AMP recursion as defined in B.4.1. A subset of the resulting orbit then corresponds to the orbit of interest, namely (26)-(29). This approach was proposed in Javanmard and Montanari 2013, Lemma 1 and also followed in Sur and Candès 2019, Lemma 3, Donoho and Montanari 2016, Lemma C.1. The symmetric AMP instance is built up for even and odd iterates and the partition $[n] \cup [N] \setminus [n]$ of $[N]$ separately, which is then extended to a complete AMP recursion.

In what follows, iterates are indicated by the letters $t, t-1, t-2$ and 0, which for scalar values are used as subscripts, e.g μ_0, σ_{t-1} and are used as superscripts for functions and nonscalar elements, e.g. $\Psi^{t-1}(\boldsymbol{y}, \boldsymbol{R}^{t-1})$.

Further, since relating a recursion that is parametrized in terms of model parameters $\boldsymbol{y}, y_i, \boldsymbol{X}, \boldsymbol{x}_i$ etc. and a general AMP recursion, which is parametrized by elements such as $x, y, \mathbf{x}_i, \mathbf{y}_i, Y_a$, may lead to confusion, an effort is made to point out the disparity. It is specifically pointed out that boldface elements pertaining to the original model use mathematical boldface (e.g $\boldsymbol{x}_i$), while boldface elements of a AMP recursion use standard boldface (e.g. $\mathbf{x}_i$). Moreover, note that Javanmard and Montanari (2013) chose symbols x, y to denote elements in $\mathcal{V}_{q,N}$, and in particular, $y = (\mathbf{y}_1, \dots, \mathbf{y}_N)$ a convention which is kept for comparability. This is not the same as $\boldsymbol{y} = (y_1, \dots, y_n)$, which is used to denote the response variables in the logistic regression model. Moreover, the response variables y_i are assumed to be realizations of random variables Y_i for $i = 1,\dots,n$. This may stand in conflict with Y_a, the random variables in $\Re^q$ as defined in the AMP setup, which are indexed by by partition cell labels $a \in [q']$. To avoid ill-definedness, the index i is exclusively used when referring to the Bernoulli random variables Y_i that give rise to observations y_i, while the subscript a is used in the context of the random variables for the AMP recursion. Finally, recall that $\boldsymbol{\beta}_0 \in \Re^p$ is the parameter vector that is to be estimated, not an initial condition.

Let q be an arbitrarily large integer, $q' = 2$, $N = n + p$, where n, p denote the rows and columns of $\boldsymbol{X}$ respectively and $\kappa = \kappa(N) = p/n$. Following Definition 1, 1), let $\boldsymbol{A}$ be the symmetric $N \times N$ matrix whose upper triangular entries are given by $\boldsymbol{A}_{ij} = \sqrt{n/N}\boldsymbol{X}_{i,j-n}$ for $i \in [n]$ and $j \in [N] \setminus [n]$ and all other entries $\boldsymbol{A}_{ij}$ $i < j$ are i.i.d. $N(0, N^{-1})$ independent of all other entries in $\boldsymbol{A}$. That is, letting $\boldsymbol{B}_1, \boldsymbol{B}_2$ be $n \times n$ matrices with i.i.d. $N(0,(2N)^{-1})$ entries and defining $\boldsymbol{C}_k = \boldsymbol{B}_k + \boldsymbol{B}_k^\top$ for $k = 1, 2$, then

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{C}_1 & \sqrt{\frac{n}{N}}\boldsymbol{X} \\ \sqrt{\frac{n}{N}}\boldsymbol{X}^\top & \boldsymbol{C}_2 \end{bmatrix} . \tag{31}$$

eq:symmetric_A

Next, the collection of functions $\mathcal{F}$ is constructed. For this, recall from Definition 3, 2), that the collections $\mathcal{F}_N$ of a converging sequence of AMP-instances with functions $f^i(\cdot,\cdot)$ may depend on $\mathbf{y}_i$ for some $y \in \mathcal{V}_{q,N}$. Hence, given $\mathbf{x}, \mathbf{y}_i \in \Re^q$, which should be thought of as entries of some $x, y \in \mathcal{V}_{q,N}$ that are concretely specified in (34) and (35), for $i \in [n]$, and odd t define $f^i(\mathbf{x}, t)$ as

$$f^i(\mathbf{x},t) = \kappa^{-1}\sqrt{\frac{N}{n}}\left[0, \Psi^0\left(h(\mathbf{x}(1),\mathbf{y}_i(1)),\mathbf{x}(3)\right), 0, \Psi^1\left(h(\mathbf{x}(1),\mathbf{y}_i(1)),\mathbf{x}(5)\right), \dots, \Psi^{\frac{t-1}{2}}\left(h(\mathbf{x}(1),\mathbf{y}_i(1)),\mathbf{x}(t+2)\right), 0, 0, \dots\right]^\top . \tag{32}$$

eq:f1

whereas for $i \in [N] \setminus [n]$ and odd t, let $f^i(\mathbf{x},t) = \mathbf{0}_q$. Further, for $i \in [n]$ and even t, $f^i(\mathbf{x},t) = \mathbf{0}_q$ and for $i \in [N] \setminus [n]$, even t, let

$$f^i(\mathbf{x},t) = \sqrt{\frac{N}{n}}\left[\mathbf{y}_i(1), 0, \mathbf{y}_i(2) + \mu_0\mathbf{y}_i(1), 0, \mathbf{x}(2) + \mu_1\mathbf{y}_i(1), 0, \mathbf{x}(4) + \mu_2\mathbf{y}_i(1), 0, \dots, \mathbf{x}(t) + \mu_{\frac{t}{2}}\mathbf{y}_i(1), 0, 0, \dots\right]^\top . \tag{33}$$

eq:f2

For well-definedness of (32), (33), which map to $\Re^q$, it must be the case that $t \le q-3$, which is of no consequence as q can always be chosen large enough to accommodate any $t \in \Re$. Hence, without loss of generality, all results are stated for $t \in [q]$, rather than $t \in [\tilde{q}]$ for $\tilde{q} = q + 3$ to simplify presentation.

It is straightforward to establish that $f^i(\cdot,t)$ is locally Lipschitz in its first argument, which is shown formally for the smoothed version of B.4.4, to which Theorem B.1 applies ultimately. To conclude the symmetric AMP-instance, let $x^0 = 0_{N,q}$ be the N-vector with elements $\mathbf{0}_q$.

With these definitions in place, it is possible to define the iterates x^t and y that give rise to the target AMP recursion. In particular, let $\boldsymbol{\eta}_0 = \boldsymbol{X}\boldsymbol{\beta}_0$ and for some integer k, $t \in [q]$, define $\mathbf{x}_i^t$ as

$$\mathbf{x}_i^t = \begin{cases} \left[\boldsymbol{\eta}_{0,i}, 0, \boldsymbol{R}_i^0, 0, \boldsymbol{R}_i^1, 0, \dots, \boldsymbol{R}_i^{\frac{t-1}{2}}, 0, 0, \dots\right]^\top & \text{if } t = 2k+1\,, i \in [n] \\ \mathbf{0}_q & \text{if } t = 2k+1\,, i \in [N]\setminus[n] \\ \mathbf{0}_q & \text{if } t = 2k\,, i \in [n] \\ \left[0, \boldsymbol{\nu}_{i-n}^1, 0, \boldsymbol{\nu}_{i-n}^2, \dots, \boldsymbol{\nu}_{i-n}^{\frac{t}{2}}, 0, 0, \dots\right]^\top & \text{if } t = 2k\,, i \in [N]\setminus[n] \end{cases} \tag{34}$$

eq:x_recursion

Finally, let $y \in \mathcal{V}_{N,q}$ have vectors $\mathbf{y}_i \in \Re^q$, $i = 1, \dots, N$, such that

$$\mathbf{y}_i(k) = \begin{cases} \bar{\varepsilon}_i & \text{if } i \in [n],\ k = 1 \\ \boldsymbol{\beta}_{0,i-n} & \text{if } i \in [N]\setminus[n],\ k = 1 \\ \boldsymbol{\nu}_{i-n}^0 & \text{if } i \in [N]\setminus[n],\ k = 2 \\ 0 & \text{else} \end{cases} \tag{35}$$

eq:y_recursion

That is, y can be seen as the $q \times N$ matrix

$$y = \begin{bmatrix} \bar{\varepsilon}_1 & \bar{\varepsilon}_2 & \dots & \bar{\varepsilon}_n & \boldsymbol{\beta}_{0,1} & \boldsymbol{\beta}_{0,2} & \dots & \boldsymbol{\beta}_{0,p} \\ 0 & 0 & \dots & 0 & \boldsymbol{\nu}_1^0 & \boldsymbol{\nu}_2^0 & \dots & \boldsymbol{\nu}_p^0 \\ 0 & & & & \cdots & & & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & & & & \cdots & & & 0 \end{bmatrix} \tag{36}$$

It is now possible to define an AMP-orbit corresponding to the above AMP instance, which is subject of the next two Lemmas. For this, note from Definition 2, equation (8), that the ith component of an AMP orbit $\{x^t\}_{t\in\mathbb{N}}$ for an AMP instance $(\boldsymbol{A}, \mathcal{F}, x^0)$ is given by

$$\mathbf{x}_i^t = \sum_{j=1}^N \boldsymbol{A}_{ij} f^j(\mathbf{x}_j^{t-1}, t-1) - \frac{1}{N}\left\{\sum_{j=1}^N \frac{\partial}{\partial \mathbf{x}} f^j(\mathbf{x}, t-1)\Big|_{\mathbf{x}=\mathbf{x}_j^{t-1}}\right\} f^i(\mathbf{x}_i^{t-2}, t-2) . \tag{37}$$

eq:AMP_sequence_component

lemma:partial_AMP

Lemma B.1. *Consider the symmetric AMP instance* $(\boldsymbol{A}, \mathcal{F}, x^0)$ *defined by* (31), (32), (33) *and* $x^0 = 0_{N,q}$ *and the iterates* $\{x^t\}_{t\in[q]}$*, the entries of which are defined in* (34) *and* (26)-(29)*. Then for* $t \in [q]$ *being even, and* $i \in [N] \setminus [n]$ *and* $t \in [q]$ *odd with* $i \in [n]$*, these iterates satisfy the recursion of* (37).

Proof. The proof follows from strong induction on t for even and odd iterates and straightforward algebra.

Base case (i): $t = 1, i \in [n]$. Note that by construction

eq:AMP_base_1

$$\begin{aligned}
\sum_{j=1}^{N} \boldsymbol{A}_{ij} f^j(\mathbf{x}_j^0, 0) &= \sum_{j=n+1}^{n+p} \boldsymbol{A}_{ij} f^j(\mathbf{x}_j^0, 0) \\
&= \sqrt{\frac{n}{N}} \sum_{j=1}^{p} \boldsymbol{X}_{ij} f^{j+n}(\mathbf{x}_{j+n}^0, 0) \\
&= \sum_{j=1}^{p} \boldsymbol{X}_{ij} \left[\mathbf{y}_{j+n}(1), 0, \mathbf{y}_{j+n}(2) + \mu_0 \mathbf{y}_{j+n}(1), 0, 0, \ldots\right]^\top \\
&= \sum_{j=1}^{p} \boldsymbol{X}_{ij} \left[\boldsymbol{\beta}_{0,j}, 0, \boldsymbol{\nu}_j^0 + \mu_0 \boldsymbol{\beta}_{0,j}, 0, 0, \ldots\right]^\top \\
&= \sum_{j=1}^{p} \boldsymbol{X}_{ij} \left[\boldsymbol{\beta}_{0,j}, 0, \hat{\boldsymbol{\beta}}_j^0 - \mu_0 \boldsymbol{\beta}_{0,j} + \mu_0 \boldsymbol{\beta}_{0,j}, 0, 0, \ldots\right]^\top \\
&= \left[\textstyle\sum_{j=1}^{p} \boldsymbol{X}_{ij} \boldsymbol{\beta}_{0,j}, 0, \sum_{j=1}^{p} \boldsymbol{X}_{ij} \hat{\boldsymbol{\beta}}_j^0, 0, 0, \ldots\right]^\top \\
&= \left[\boldsymbol{\eta}_{0,i}, 0, \boldsymbol{R}_i^0, 0, 0, \ldots\right]^\top \\
&= \mathbf{x}_i^1 \, ,
\end{aligned} \tag{38}$$

The first equality follows from the definition of (33), the second from the definition of (31), the third again from (33), the fourth from (35), the fifth and seventh from the definition of $\boldsymbol{\nu}^0, \boldsymbol{R}^0$ above (26) and since $\boldsymbol{\eta}_0 = \boldsymbol{X}\boldsymbol{\beta}_0$ and the last from (34). Concordance with the full recursion of (37) follows since $f^i(\mathbf{x}_i^{-1}, -1) = \mathbf{0}_q$ by definition.

Base case (ii): $t = 2, i \in [N] \setminus [n]$. Again it is easily checked that

eq:AMP_base_2

$$\begin{aligned}
\sum_{j=1}^{N} \boldsymbol{A}_{ij} f^j(\mathbf{x}_j^1, 1) = \sum_{j=1}^{n} \boldsymbol{A}_{ij} f^j(\mathbf{x}_j^1, 1) \\
&= \sqrt{\frac{N}{n}} \sum_{j=1}^{n} \boldsymbol{X}_{ji} f^j(\mathbf{x}_j^1, 1) \\
&= \kappa^{-1} \sum_{j=1}^{n} \boldsymbol{X}_{ji} \left[0, \Psi^0(h(\mathbf{x}_j^1(1), y_j(1)), \mathbf{x}_j^1(3)), 0, 0 \ldots\right]^\top \\
&= \kappa^{-1} \sum_{j=1}^{n} \boldsymbol{X}_{ji} \left[0, \Psi^0(h(\boldsymbol{\eta}_{0,j}, \bar{\varepsilon}_j), \boldsymbol{R}_j^0), 0, 0 \ldots\right]^\top \, ,
\end{aligned} \tag{39}$$

where the first equality comes from (32), the second and third from (31), the fourth again from (32) and the last from (38). Now recalling that $y_i^* = h(\boldsymbol{\eta}_{0,i}, \bar{\varepsilon}_i)$ for $h(Z, \bar{\varepsilon}) = \alpha \left\{1 - \text{sign}(\bar{\varepsilon} - \zeta'(Z))\right\}/2 + (1-\alpha)/2$, it immediately follows that the nonzero element in the last line of (39) corresponds to the ith entry of $\kappa^{-1} \boldsymbol{X}^\top \Psi^0(\boldsymbol{y}^*, \boldsymbol{R}^0)$.

Next consider

$$\sum_{j=1}^{N} \frac{\partial}{\partial \mathbf{x}} f^j(\mathbf{x},1)\Big|_{\mathbf{x}=\mathbf{x}_j^1} = \sum_{j=1}^{n} \frac{\partial}{\partial \mathbf{x}} f^j(\mathbf{x},1)\Big|_{\mathbf{x}=\mathbf{x}_j^1}$$
$$= \kappa^{-1}\sqrt{\frac{N}{n}} \sum_{j=1}^{n} \left[0, \tfrac{\partial}{\partial \mathbf{x}} \Psi^0(h(\mathbf{x}(1),\mathbf{y}(1)),\mathbf{x}(3)), 0, 0, \dots\right]^\top \Big|_{\mathbf{x}=\mathbf{x}_j^1}$$
$$= \kappa^{-1}\sqrt{\frac{N}{n}} \sum_{j=1}^{n} \begin{bmatrix} 0 & 0 & \dots & & & \\ \frac{\partial}{\partial Z}\Psi^0(h(Z,\bar{\varepsilon}_j),\boldsymbol{R}_j^0)\Big|_{Z=\boldsymbol{\eta}_{0,j}} & 0 & \frac{\partial}{\partial R}\Psi^0(y_j^*,R)\Big|_{R=\boldsymbol{R}_j^0} & 0 & 0 & \dots \\ 0 & 0 & \dots & & & \\ \vdots & \ddots & & & & \end{bmatrix}, \tag{40}$$

eq:AMP_base_2_2

where the first two equalities follow from (32) and the last from straightforward differentiation and $\mathbf{x}^1$ from (38).

Finally, it immediately follows from (33) that

$$f^i(\mathbf{x}_i^0,0) = \sqrt{\frac{N}{n}}\left[\mathbf{y}_i(1), 0, \mathbf{y}_i(2)+\mu_0\mathbf{y}_i(1), 0, 0\dots\right]^\top$$
$$= \sqrt{\frac{N}{n}}\left[\boldsymbol{\beta}_{0,i-n}, 0, \boldsymbol{\nu}_{i-n}^0+\mu_0\boldsymbol{\beta}_{0,i-n}, 0, 0\dots\right]^\top \tag{41}$$

eq:AMP_base_2_3

so that (40) and (41) yield

$$-\frac{1}{N}\left\{\sum_{j=1}^{N}\frac{\partial}{\partial \mathbf{x}} f^j(\mathbf{x},1)\Big|_{\mathbf{x}=\mathbf{x}_j^1}\right\} f^i(\mathbf{x}_i^0,0)$$
$$= -\frac{1}{n\kappa}\begin{bmatrix} 0 & 0 & \dots & & & \\ \sum_{j=1}^n \frac{\partial}{\partial Z}\Psi^0(h(Z,\bar{\varepsilon}_j),\boldsymbol{R}_j^0)\Big|_{Z=\boldsymbol{\eta}_{0,j}} & 0 & \sum_{j=1}^n \frac{\partial}{\partial R}\Psi^0(y_j^*,R)\Big|_{R=\boldsymbol{R}_j^0} & 0 & 0 & \dots \\ 0 & 0 & \dots & & & \\ \vdots & \ddots & & & & \end{bmatrix}\begin{bmatrix}\boldsymbol{\beta}_{0,i-n} \\ 0 \\ \boldsymbol{\nu}_{i-n}^0+\mu_0\boldsymbol{\beta}_{0,i-n} \\ 0 \\ 0 \\ \vdots\end{bmatrix}$$
$$= \left[0, \left\{-\tfrac{1}{n\kappa}\sum_{j=1}^n \tfrac{\partial}{\partial Z}\Psi^0(h(Z,\bar{\varepsilon}_j),\boldsymbol{R}_j^0)\Big|_{Z=\boldsymbol{\eta}_{0,j}}\right\}\boldsymbol{\beta}_{0,i-n} + \left\{-\tfrac{1}{n\kappa}\sum_{j=1}^n \tfrac{\partial}{\partial R}\Psi^0(y_j^*,R)\Big|_{R=\boldsymbol{R}_j^0}\right\}\left(\boldsymbol{\nu}_{i-n}^0+\mu_0\boldsymbol{\beta}_{0,i-n}\right), 0, 0, \dots\right]^\top$$
$$= \left[0, -a_1\boldsymbol{\beta}_{0,i-n} + q_0\left(\boldsymbol{\nu}_{i-n}^0+\mu_0\boldsymbol{\beta}_{0,i-n}\right), 0, 0, \dots\right]^\top, \tag{42}$$

eq:AMP_base_2_4

Hence, from (39), (42) and (26), (28), (29), it follows that the second entry of (37) equals

$$\kappa^{-1}\sum_{j=1}^{n}\boldsymbol{X}_{ji}\Psi^0(h(\boldsymbol{\eta}_{0,j},\bar{\varepsilon}_j),\boldsymbol{R}_j^0) - a_1\boldsymbol{\beta}_{0,i-n} + q_0\left(\boldsymbol{\nu}_{i-n}^0+\mu_0\boldsymbol{\beta}_{0,i-n}\right) = \boldsymbol{\nu}_{i-n}^1\,, \tag{43}$$

while all other entries are zero. Hence, by construction of (34) the second base case holds.

Induction hypothesis: Assume that there exists some $t<q$, $t\geq 2$ such that for all $t'\leq t$ if t' is even, and $i\in[N]\setminus[n]$ or if t' is odd with $i\in[n]$, the iterates of (34) satisfy the recursion of (37).

Induction step (i): $t+1$ odd, $i\in[n]$. Most derivations are analogous to Base case (i), so that details are skipped on repetitive steps. First note that due to the nested nature of the iterates $\mathbf{x}_i^t$ in (34) and the induction hypothesis, the induction step must only be established for the last nonzero entry in $\mathbf{x}_j^{t+1}$.

For this, note that

eq:induction_step_1_first_part

$$\sum_{j=1}^{N} \boldsymbol{A}_{ij} f^j(\mathbf{x}_j^t, t) = \sum_{j=1}^{p} \boldsymbol{X}_{ij} \left[\boldsymbol{\beta}_{0,j}, 0, \boldsymbol{\nu}_j^0 + \mu_0 \boldsymbol{\beta}_{0,j}, 0, \boldsymbol{\nu}_j^1 + \mu_1 \boldsymbol{\beta}_{0,j}, 0, \dots, \boldsymbol{\nu}_j^{\frac{t}{2}} + \mu_{\frac{t}{2}} \boldsymbol{\beta}_{0,j}, 0, 0, \dots\right]^\top . \tag{44}$$

Next, it is easily seen that

eq:induction_step_1_jacobian

$$\begin{aligned}
\sqrt{\frac{n}{N}} \sum_{j=1}^{N} \frac{\partial}{\partial \mathbf{x}} f^j(\mathbf{x}, t)\Big|_{\mathbf{x}=\mathbf{x}_j^t} &= \sqrt{\frac{n}{N}} \sum_{j=1}^{p} \frac{\partial}{\partial \mathbf{x}} f^{n+j}(\mathbf{x}, t)\Big|_{\mathbf{x}=\mathbf{x}_{n+j}^t} \\
= \sum_{j=1}^{p} \frac{\partial}{\partial \mathbf{x}} &\left[\mathbf{y}_i(1), 0, \mathbf{y}_i(2) + \mu_0 \mathbf{y}_i(1), 0, \mathbf{x}(2) + \mu_1 \mathbf{y}_i(1), 0, \mathbf{x}(4) + \mu_2 \mathbf{y}_i(1), 0, \dots, \mathbf{x}(t) + \mu_{\frac{t}{2}} \mathbf{y}_i(1), 0, \dots\right]^\top \Big|_{\mathbf{x}=\mathbf{x}_{n+j}^t} \\
&= \sum_{j=1}^{p} \tilde{I}_{q\times q} = p \tilde{I}_{q\times q} ,
\end{aligned} \tag{45}$$

where $\tilde{I}_{q\times q}$ whose odd rows with row index $5 \le i \le t+3$ have the $(i-3)$rd element equal to one, while all other elements of this matrix are equal to zero.

Further, note that for $t+1$ odd, $i \in [n]$,

eq:induction_step_1_rem

$$\kappa \sqrt{\frac{n}{N}} f^i(\mathbf{x}_i^{t-1}, t-1) = \begin{bmatrix} 0 \\ \Psi^0\left(h(\mathbf{x}_i^{t-1}(1), \mathbf{y}_i(1)), \mathbf{x}_i^{t-1}(3)\right) \\ 0 \\ \Psi^1\left(h(\mathbf{x}_i^{t-1}(1), \mathbf{y}_i(1)), \mathbf{x}_i^{t-1}(5)\right) \\ 0 \\ \vdots \\ \Psi^{\frac{t-2}{2}}\left(h(\mathbf{x}_i^{t-1}(1), \mathbf{y}_i(1)), \mathbf{x}_i^{t-1}(t+1)\right) \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \Psi^0\left(h(\boldsymbol{\eta}_{0,i}, \bar{\varepsilon}_i), \boldsymbol{R}_i^0\right) \\ 0 \\ \Psi^1\left(h(\boldsymbol{\eta}_{0,i}, \bar{\varepsilon}_i), \boldsymbol{R}_i^1\right) \\ 0 \\ \vdots \\ \Psi^{\frac{t}{2}-1}\left(h(\boldsymbol{\eta}_{0,i}, \bar{\varepsilon}_i), \boldsymbol{R}^{\frac{t}{2}-1}\right) \\ 0 \\ \vdots \end{bmatrix} , \tag{46}$$

by definition of (33), (34) and (35). Hence, from (45) and (46), it follows that

eq:induction_step_1_last_part

$$-\frac{1}{N} \left\{\sum_{j=1}^{N} \frac{\partial}{\partial \mathbf{x}} f^j(\mathbf{x}, t)\Big|_{\mathbf{x}=\mathbf{x}_j^t}\right\} f^i(\mathbf{x}_i^{t-1}, t-1) = -\kappa^{-1} \frac{p}{n} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \Psi^0\left(h(\boldsymbol{\eta}_{0,i}, \bar{\varepsilon}_i), \boldsymbol{R}_i^0\right) \\ 0 \\ \Psi^1\left(h(\boldsymbol{\eta}_{0,i}, \bar{\varepsilon}_i), \boldsymbol{R}_i^1\right) \\ 0 \\ \vdots \\ \Psi^{\frac{t}{2}-1}\left(h(\boldsymbol{\eta}_{0,i}, \bar{\varepsilon}_i), \boldsymbol{R}^{\frac{t}{2}-1}\right) \\ 0 \\ \vdots \end{bmatrix} , \tag{47}$$

Hence, it follows from equations (44) and (44), that the last nonzero entry of

$$\sum_{j=1}^{N} \boldsymbol{A}_{ij} f^j(\mathbf{x}_j^t, t) - \frac{1}{N}\left\{\sum_{j=1}^{N} \frac{\partial}{\partial \mathbf{x}} f^j(\mathbf{x}, t)\Big|_{\mathbf{x}=\mathbf{x}_j^t}\right\} f^i(\mathbf{x}_i^{t-1}, t-1) \tag{48}$$

is given by

$$\sum_{j=1}^{p} \boldsymbol{X}_{ij}\left(\boldsymbol{\nu}_j^{\frac{t}{2}} + \mu_{\frac{t}{2}}\boldsymbol{\beta}_{0,j}\right) - \Psi^{\frac{t}{2}-1}\left(h(\boldsymbol{\eta}_{0,i}, \bar{\varepsilon}_i), \boldsymbol{R}^{\frac{t}{2}-1}\right), \tag{49}$$

which by (27) corresponds to the ith entry of $\boldsymbol{R}^{\frac{t}{2}}$ and that corresponds precisely to the last nonzero entry of $\mathbf{x}_i^{t+1}$ as defined in (34). This establishes the first part of the induction step.

Induction step (ii): $t+1$ even, $i \in [N] \setminus [n]$. This step is completely analogous to base case (ii) and therefore omitted.

By strong induction, the claim of the Lemma holds for any $t \in [q]$. □

With this result it is possible to fill up the missing parts the incomplete recursion of Lemma B.1. Hence, define iterate $\tilde{x}^t$ as follows. Let $\tilde{x}^0 = 0_{N,q}$ and for $t \in [q]$ odd and $i \in [n]$ or if t is even and $i \in [N] \setminus [n]$, set $\tilde{\mathbf{x}}_i^t = \mathbf{x}_i^t$. Otherwise, let

$$\tilde{\mathbf{x}}_i^t = \sum_{j=1}^{N} \boldsymbol{A}_{ij} f^j(\tilde{\mathbf{x}}_j^{t-1}, t-1) - \frac{1}{N}\left\{\sum_{j=1}^{N} \frac{\partial}{\partial \mathbf{x}} f^j(\tilde{\mathbf{x}}, t-1)\Big|_{\tilde{\mathbf{x}}=\tilde{\mathbf{x}}_j^{t-1}}\right\} f^i(\tilde{\mathbf{x}}_i^{t-2}, t-2), \tag{50}$$

eq:full_AMP

again with the convention that terms involving negative t-indices are zero. The Lemma below establishes that this recursion is indeed a AMP orbit corresponding to instance $(\boldsymbol{A}, \mathcal{F}, \tilde{x}^0)$ and that the orbit of interest, i.e. the iterates of Lemma B.1, are preserved.

lemma:full_AMP

Lemma B.2. *Given the AMP instance* $(\boldsymbol{A}, \mathcal{F}, \tilde{x}^0)$ *of Lemma B.1, the recursion of* (50) *is an orbit according to Definition 2. Further, for* $t \in [q]$ *odd and* $i \in [n]$ *or if* t *is even and* $i \in [N] \setminus [n]$*, the iterates* $\tilde{\mathbf{x}}_i^t$ *correspond to those in Lemma B.1.*

Proof. The statement of the Lemma is shown by strong induction on t.

Base case (i): $t = 1$. Since $\tilde{x}^0 = x^0 = 0_{n,q}$, by base case (i) of Lemma B.1, the second part of the Lemma holds for $t = 1$. For $i \in [n]$, the first part of the claim follows from base case (i) of Lemma B.1 too, while for $i \in [N] \setminus [n]$ it follows trivially from (50).

Base case (ii) $t = 2$. Again, note that by the base case (ii) in Lemma B.1, it the claim of the Lemma immediately follows for $i \in [N] \setminus [n]$. For $i \in [n]$, the the second part of the Lemma follows from (50).

Induction hypothesis: The claim of the Lemma holds for all t' up to some $t < q$, $t \geq 2$.

Induction step (i): $t+1$ odd. Consider $i \in [n]$. From the induction hypothesis, one hast that $\tilde{\mathbf{x}}_j^t = \mathbf{x}_j^t$ for $i \in [N] \setminus [n]$ and similarly $\tilde{\mathbf{x}}_j^{t-1} = \mathbf{x}_j^{t-1}$ for $i \in [n]$. Hence again, the corresponding induction step of Lemma B.1 shows that the recursion an orbit according to Definition 2 and that the iterates $\tilde{\mathbf{x}}_i^t$ correspond to those in Lemma B.1. Now consider $i \in [N] \setminus [n]$. The induction step holds trivially by construction of (50).

Induction step (ii): $t+1$ even. This case is handled similarly to induction step (i) and is therefore omitted.

By strong induction, the Lemma is proven for all $t \in [q]$. □

Hence, if $(\boldsymbol{A}, \mathcal{F}, \tilde{x}^0)$ were shown to be a converging AMP instance according to Definition 3, then Theorem B.1 along with the state evolution of Definition 4, would give the large sample behaviour of the corresponding orbit of interest. However, as remarked in Feng et al. (2022), Section 4.7, the function g in Definition 3 cannot be Lipschitz, which is owed to the indicator function $h(Z, \bar{\varepsilon})$, so that Theorem B.1 does not apply. The next section deals with this technicality.

B.4.4 A smooth approximation

subsubsec:smooth_AMP

This section introduces a smooth approximation to the original recursions (16)-(17) and to (26)-(29), the latter of which can be used in a matrix-valued AMP embedding for which Theorem B.1 applies. A limiting argument then shows that the theorem still applies in the limit where the smooth approximation approaches the nonsmooth original recursion.

In particular, the step function $\{1 - \text{sign}(x)\}/2$ can, amongst others, be approximated by $\left\{1 - \frac{x}{|x|+\epsilon}\right\}/2$ for some positive constant ϵ. Note that defining the Bernoulli response $Y = \{1 - \text{sign}\left(\bar{\varepsilon} - \zeta'(Z)\right)\}/2$ rather than the more conventional $Y = \mathbb{1}\{\bar{\varepsilon} < \zeta'(Z)\}$ comes without loss of generality, as per the modelling assumption, $\boldsymbol{\eta}_{0,i}$ are independent $N(0, \gamma^2)$ normal random variables and $\bar{\varepsilon}_i$ are independent of $\boldsymbol{\eta}_0$ and uniformly distributed on the unit interval so that the event $\{\exists i \in [n] : \bar{\varepsilon}_i = \zeta'(\boldsymbol{\eta}_{0,i})\}$ has measure zero.

Hence, in what follows, denote by $h_\epsilon(Z, \bar{\varepsilon})$ the function

eq:smooth_approx

$$h_\epsilon(Z, \bar{\varepsilon}) = \frac{\alpha}{2}\left\{1 - \frac{\bar{\varepsilon} - \zeta'(Z)}{|\bar{\varepsilon} - \zeta'(Z)| + \epsilon}\right\} + \frac{1-\alpha}{2}\,. \tag{51}$$

Consequently, define new sets of recursions analogous to (16)-(17) and (26)-(29) where all appearances of $h(\cdot, \cdot)$ are replaced by $h_\epsilon(\cdot, \cdot)$. In particular, define the approximate target AMP recursion with initial conditions $\hat{\boldsymbol{\beta}}^0$, $\boldsymbol{S}^0 = \boldsymbol{X}\hat{\boldsymbol{\beta}}^0$, and a sequence of constants $\{{}_\epsilon\mu_t, {}_\epsilon b_t, {}_\epsilon\sigma_t\}_{t\in\mathbb{N}_0}$, for $t \in \mathbb{N}$, as

eq:approx_gen_AMP_beta

$${}_\epsilon\hat{\boldsymbol{\beta}}^t = {}_\epsilon\hat{\boldsymbol{\beta}}^{t-1} + \kappa^{-1}\boldsymbol{X}^\top\Psi^{t-1}({}_\epsilon\boldsymbol{y}^*, {}_\epsilon\boldsymbol{S}^{t-1}) \tag{52}$$

eq:approx_gen_AMP_s

$${}_\epsilon\boldsymbol{S}^t = \boldsymbol{X}{}_\epsilon\hat{\boldsymbol{\beta}}^t - \Psi^{t-1}({}_\epsilon\boldsymbol{y}^*, {}_\epsilon\boldsymbol{S}^{t-1})\,, \tag{53}$$

where $\Psi^t({}_\epsilon\boldsymbol{y}^*, {}_\epsilon\boldsymbol{S}^t)$ acts elementwise on ${}_\epsilon\boldsymbol{y}^*, {}_\epsilon\boldsymbol{S}^t$ as

eq:approx_psi_def

$$\Psi^t({}_\epsilon y_i^*, {}_\epsilon\boldsymbol{S}_i^t) = {}_\epsilon b_t\left\{{}_\epsilon y_i^* - \zeta'\left(\text{prox}_{{}_\epsilon b_t\zeta}\left({}_\epsilon b_{t\,\epsilon} y_i^* + {}_\epsilon\boldsymbol{S}_i^t\right)\right)\right\}\,, \tag{54}$$

for $i = 1, \dots, n$ and where ${}_\epsilon y_i^* = \alpha h_\epsilon(\boldsymbol{\eta}_{0,i}, \bar{\varepsilon}_i) + (1-\alpha)/2$. Further, the state evolution is determined by the system of nonlinear equations

eq:approx_bt

$$1 - \kappa = \text{E}\left[\frac{1}{1 + {}_\epsilon b_t\zeta''\left(\text{prox}_{{}_\epsilon b_t\zeta}\left({}_\epsilon b_{t\,\epsilon}Y^* + {}_\epsilon Z_t\right)\right)}\right] \tag{55}$$

eq:approx_mut

$${}_\epsilon\mu_{t+1} = \frac{{}_\epsilon b_t}{\kappa\gamma^2}\text{E}\left[Z\left\{{}_\epsilon Y^* - \zeta'\left(\text{prox}_{{}_\epsilon b_t\zeta}\left({}_\epsilon b_{t\,\epsilon}Y^* + {}_\epsilon Z_t\right)\right)\right\}\right] + {}_\epsilon\mu_t \tag{56}$$

eq:approx_sigmat

$${}_\epsilon\sigma_{t+1}^2 = \frac{{}_\epsilon b_t^2}{\kappa^2}\text{E}\left[\left\{{}_\epsilon Y^* - \zeta'\left(\text{prox}_{{}_\epsilon b_t\zeta}\left({}_\epsilon b_{t\,\epsilon}Y^* + {}_\epsilon Z_t\right)\right)\right\}^2\right]\,, \tag{57}$$

for independent $Z, \tilde{Z} \sim N(0, \gamma^2)$, $G \sim N(0,1)$, and $\bar{\varepsilon} \sim U(0,1)$, and letting

eq:approx_y_star

$${}_\epsilon Y^* = h_\epsilon(Z, \bar{\varepsilon}), \quad {}_\epsilon Z_t = {}_\epsilon\mu_t Z + \kappa^{1/2}{}_\epsilon\sigma_t G\,. \tag{58}$$

Given initial conditions $\boldsymbol{\nu}^0 = \hat{\boldsymbol{\beta}}^0 - \mu_0\boldsymbol{\beta}_0$, $\boldsymbol{R}^0 = \boldsymbol{S}^0$ for $\hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0 \in \Re^p$, $\mu_0 \in \Re$ and $\boldsymbol{S}^0 \in \Re^n$, for $t \in \mathbb{N}$, the approximate centred recursion of (26)-(29) is defined as

eq:approx_centred_rec1

$${}_\epsilon\boldsymbol{\nu}^t = {}_\epsilon q_{t-1}({}_\epsilon\boldsymbol{\nu}^{t-1} + {}_\epsilon\mu_{t-1}\boldsymbol{\beta}_0) - {}_\epsilon a_t\boldsymbol{\beta}_0 + \kappa^{-1}\boldsymbol{X}^\top\Psi^{t-1}({}_\epsilon\boldsymbol{y}^*, {}_\epsilon\boldsymbol{R}^{t-1}) \tag{59}$$

eq:approx_centred_rec2

$${}_\epsilon\boldsymbol{R}^t = \boldsymbol{X}({}_\epsilon\boldsymbol{\nu}^t + {}_\epsilon\mu_t\boldsymbol{\beta}_0) - \Psi^{t-1}({}_\epsilon\boldsymbol{y}^*, {}_\epsilon\boldsymbol{R}^{t-1})\,, \tag{60}$$

where $a_0 = \mu_0$ and for $t \in \mathbb{N}$,

eq:approx_aux_centred_rec1

$${}_\epsilon q_t = -\frac{1}{\kappa n}\sum_{i=1}^n \frac{\partial\Psi^t}{\partial R}({}_\epsilon y_i^*, R)\Big|_{R={}_\epsilon\boldsymbol{R}_i^t} \tag{61}$$

eq:approx_aux_centred_rec2

$${}_\epsilon a_t = \frac{1}{\kappa n}\sum_{i=1}^n \frac{\partial\Psi^{t-1}}{\partial Z}(h_\epsilon(Z, \bar{\varepsilon}_i), {}_\epsilon\boldsymbol{R}_i^{t-1})\Big|_{Z=\boldsymbol{x}_i^\top\boldsymbol{\beta}_0}\,, \tag{62}$$

(63)

where $\bar{\varepsilon}_i \sim U(0,1)\,(i=1,\ldots,n)$ are i.i.d. draws independent of all other random variables.

Similarly, define $f_\epsilon^i(\mathbf{x},t)$ as functions (32) and (33) with $h(Z,\bar{\varepsilon})$ replaced by $h_\epsilon(Z,\bar{\varepsilon})$ and denote by ${}_\epsilon\mathbf{x}_i^t$ the orbit of (34) whose definition is unchanged but whose entries now depend on ϵ through the replacement of $h(\cdot,\cdot)$ by $h_\epsilon(\cdot,\cdot)$ in (26)-(29). Finally, denote by ${}_\epsilon\tilde{\mathbf{x}}_i^t$ the corresponding full AMP recursion as defined in (50) with $\mathbf{x}_i^t$ replaced by ${}_\epsilon\mathbf{x}_i^t$. Completely analogous to Lemmas B.1 and B.2, it can be shown that ${}_\epsilon\mathbf{x}_i^t$ is a AMP orbit to the AMP instance $(\boldsymbol{A},\mathcal{F}_\epsilon,\tilde{x}^0)$, where the definition of $\boldsymbol{A}$ remains unchanged, $\mathcal{F}_\epsilon$ is the collection of functions f_ϵ^i and $\tilde{x}^0=\mathbf{0}_{q,N}$ as before. This is easily verified by noting that the proofs of these Lemmas do not depend on the form of the indicator function but are concerned with the general structure of the recursion.

It remains to construct a converging AMP instance following Definition 3 to motivate Theorem B.1. Following the structure of Definition 3,

i) Let q be some arbitrarily large integer, and $q'=2$

ii) Define $g_\epsilon : \Re^q \times \Re^q \times [q'] \times \mathbb{N} \to \Re^q$ as follows. Given $k \in \mathbb{N}_0$,

$$\begin{aligned} &g_\epsilon(\mathbf{x},\mathbf{y},1,2k+1)\\ &= \kappa^{-1}\sqrt{\frac{N}{n}}\left[0,\Psi^0\left(h_\epsilon(\mathbf{x}(1),\mathbf{y}(1)),\mathbf{x}(3)\right),0,\Psi^1\left(h_\epsilon(\mathbf{x}(1),\mathbf{y}(1)),\mathbf{x}(5)\right),\ldots,\Psi^{\frac{t-1}{2}}\left(h_\epsilon(\mathbf{x}(1),\mathbf{y}(1)),\mathbf{x}(t+2)\right),0,0,\ldots\right]^\top, \end{aligned} \tag{64}$$

eq:g1

and

$$\begin{aligned} &g_\epsilon(\mathbf{x},\mathbf{y},2,2k)\\ &= \sqrt{\frac{N}{n}}\left[\mathbf{y}(1),0,\mathbf{y}(2)+\mu_0\mathbf{y}(1),0,\mathbf{x}(2)+\mu_1\mathbf{y}(1),0,\mathbf{x}(4)+\mu_2\mathbf{y}(1),0,\ldots,\mathbf{x}(t)+\mu_{\frac{t}{2}}\mathbf{y}(1),0,0,\ldots\right]^\top, \end{aligned} \tag{65}$$

eq:g2

while for all other configurations of t,a, $g_\epsilon(\mathbf{x},\mathbf{y},a,t)=\mathbf{0}_q$.

iii) Let $\pi_{\bar{\beta}}$ be the probability distribution such that the empirical distribution of $\boldsymbol{\beta}_0$ converges weakly to $\pi_{\bar{\beta}}$, which is assumed to exist. Further, given initial conditions $\hat{\boldsymbol{\beta}}^0,\mu_0$, let $\pi_{\bar{\nu}}$ be the probability distribution such that the empirical distribution of $\boldsymbol{\nu}^0=\hat{\boldsymbol{\beta}}^0-\mu_0\boldsymbol{\beta}_0$ converges weakly to, which is also assumed to exist, such as the joint distribution $\pi_{\bar{\beta},\bar{\nu}}$ which is the limit of the joint distribution of $\boldsymbol{\beta}_0,\boldsymbol{\nu}^0$. Finally, let $\pi_{\bar{\varepsilon}}$ be the distribution of a $U(0,1)$ random variable. Then let P_1 be the distribution of q independent random variables whose marginals are $\pi_{\bar{\varepsilon}}$ for the first coordinate and point masses at zero for all others. Further, let P_2 be the distribution of q random variables whose first two coordinates are distributed according to $\pi_{\bar{\beta},\bar{\nu}}$ and are independent of all other coordinates, whose distributions are independent of another and point masses at zero.

iv) For $N=n+p$, partition $[N]$ into $C_1^N=[n]$ and $C_2^N=[N]\setminus[n]$

v) For $a=1,2$, let

$$\widehat{\boldsymbol{\Sigma}}_a^0 = \operatorname*{plim}_{N\to\infty}\frac{1}{|C_a^N|}\sum_{i\in C_a^N} g_\epsilon(\mathbf{0}_q,\mathbf{y}_i,a,0)g_\epsilon(\mathbf{0}_q,\mathbf{y}_i,a,0)^\top, \tag{66}$$

which is assumed to exist. Here, $c_1=\lim_{N\to\infty} n/(n+p)=1/(1+\kappa)$ and $c_2=\lim_{N\to\infty} p/(n+p)=\kappa/(1+\kappa)$ in concordance with Definition 3 1).

Condition iii) appears a bit abstract in its stated generality. For initial condition of relevance in this paper, namely the oracle initializer $\mu_*\boldsymbol{\beta}_0+\sigma_*\boldsymbol{\xi}$, where $\boldsymbol{\xi}\sim\mathcal{N}(\boldsymbol{I}_p)$ independent of everything else, it is readily seen that the condition holds whenever the empirical distribution of $\boldsymbol{\beta}_0$ converges to some distribution $\pi_{\bar{\beta}}$.

To conclude the definition, it must now be shown that for all $r\in[q]$, the rth component of $g_\epsilon(\cdot,\cdot,a,t)$ is Lipschitz in its first two arguments and that points 1)-3) of Definition 3 hold. This is subject of the Lemma below.

lemma:conv_approx

Lemma B.3. *The AMP instance defined above is a converging AMP instance according to Definition 3.*

Proof. It must be shown that 0) each component of $g_\epsilon(\cdot,\cdot,a,t)$ is Lipschitz in its first two arguments and that points 1)-3) of Definition 3 hold. Each point is shown separately.

0) Lipschitzianity of $g_\epsilon(\cdot,\cdot,a,t)$: It is shown in Lemma G.2 ii) that $\Psi^{\frac{t-1}{2}}(h_\epsilon(\mathbf{x}(1),\mathbf{y}(1)),\mathbf{x}(t{+}2))$ is Lipschitz in $[\mathbf{x}(1),\mathbf{x}(t{+}2),\mathbf{y}(1)]^\top$. Hence by (64), for any $k\in\mathbb{N}_0$, $a=1$, $r\in[q]$, the rth component of $g_\epsilon(\cdot,\cdot,1,2k+1)$ is Lipschitz. On the other hand, by (65), Lipschitzianity of $g_\epsilon(\cdot,\cdot,2,2k)$ is immediate since it is an affine transformation of its arguments. For all other combinations of t,a, $g_\epsilon(\cdot,\cdot,a,t)=\mathbf{0}_q$ and thus trivially Lipschitz.

1) Convergence of $|C_a^N|/N$: By definition of $N=n+p$, and $C_N^1=[n]$, $C_N^2=[N]\setminus[n]$. Hence, since by assumption, $\kappa(N)=p/n$ converges to some $\kappa\in(0,1)$, 1) holds.

2): Agreement of f_ϵ^i with the ith component of g_ϵ holds by construction of y in (35), f_ϵ^i in (32),(33), g_ϵ in (64) and (65) along with the partition C_N^1, C_N^2. Convergence of the empirical distribution of $\{\mathbf{y}_i\}_{i\in C_N^a}$ to P_a follows by definition of P_a.

3) Initial condition of state evolution: Follows by assumption v). □

It thus follows from Lemma B.2 applied to the smooth approximation ${}_\epsilon\tilde{\mathbf{x}}_i^t$ and Lemma B.3 that $(\boldsymbol{A},\mathcal{F}_\epsilon,\tilde{x}^0)$ along with iterates ${}_\epsilon\tilde{\mathbf{x}}_i^t$ is a converging AMP instance and Theorem B.1 applies.

thm:smooth_half

Theorem B.2. *Under assumptions (i)-(v), for any pseudo Lipschitz-function of order 2, and for all $t\in[q]$, $a=1,2$, almost surely*

$$\lim_{N\to\infty}\frac{1}{|C_a^N|}\sum_{i\in C_a^N}\psi({}_\epsilon\tilde{\mathbf{x}}_i^t,\mathbf{y})=E\left[\psi({}_\epsilon Z_a^t,Y_a)\right]\ , \tag{67}$$

where ${}_\epsilon Z_a^t\sim N(0,{}_\epsilon\boldsymbol{\Sigma}^t)$, and ${}_\epsilon\boldsymbol{\Sigma}^t$ denotes the covariance matrices from state evolution (12) *and* (13) *with function $g_\epsilon(\mathbf{x},\mathbf{y},a,t)$*

Proof. By Lemma B.3 and Theorem B.1, for any pseudo Lipschitz function ψ of order 2,

$$\lim_{N\to\infty}\frac{1}{|C_a^N|}\sum_{i\in C_a^N}\psi({}_\epsilon\tilde{\mathbf{x}}_i^t,\mathbf{y})=\mathrm{E}\left[\psi({}_\epsilon Z_a^t,Y_a)\right]\ , \tag{68}$$

where ${}_\epsilon Z_a^t\sim N(0,{}_\epsilon\boldsymbol{\Sigma}^t)$ as defined in the statement of the Theorem.

□

cor:smooth_asymptotics_subset

Corollary B.2.1. *Under assumptions (i)-(v), and initial conditions $\hat{\boldsymbol{\beta}}^0$,*

$$\begin{aligned}{}_\epsilon\mu_0&=\frac{1}{\gamma^2}\lim_{n\to\infty}\frac{\langle\hat{\boldsymbol{\beta}}^0,\boldsymbol{\beta}_0\rangle}{n}\\ {}_\epsilon\sigma_0^2&=\lim_{n\to\infty}\frac{\left\|\hat{\boldsymbol{\beta}}^0-\mu_0\boldsymbol{\beta}_0\right\|_2^2}{p}\ ,\end{aligned} \tag{69}$$

for any pseudo Lipschitz-function of order 2, and for all $t\in[q]$, almost surely

$$\lim_{N\to\infty}\frac{1}{p}\sum_{j=1}^{p}\psi\left({}_\epsilon\boldsymbol{\nu}_j^t,\boldsymbol{\beta}_{0,j}\right)=E\left[\psi({}_\epsilon\sigma_t^2G,\bar{\beta})\right]\ ,\quad \lim_{N\to\infty}\frac{1}{n}\sum_{j=1}^{n}\psi\left(\begin{bmatrix}\boldsymbol{\eta}_{0,j}\\ {}_\epsilon\boldsymbol{R}_i^t\end{bmatrix},\begin{bmatrix}\bar{\varepsilon}_j\\ 0\end{bmatrix}\right)=E\left[\psi\left(\begin{bmatrix}Z_1^t\\ {}_\epsilon Z_2^t\end{bmatrix},\begin{bmatrix}\bar{\varepsilon}\\ 0\end{bmatrix}\right)\right]\ , \tag{70}$$

where $G \sim N(0,1)$, $Z_1^t \sim N(0,\gamma^2)$, ${}_\epsilon Z_2^t = {}_\epsilon\mu_t Z_1^t + \kappa^{1/2}{}_\epsilon\sigma_t G$, $G \sim N(0,1)$, $\bar\varepsilon \sim U(0,1)$ *and where* $G \perp \bar\beta$, $G \perp Z_1^t$ *and* $(Z_1^t, {}_\epsilon Z_2^t) \perp \bar\varepsilon$.

Proof. The proof consists of two parts, which shall be shown in turn: i) For fixed $\epsilon > 0$, extract the iterates ${}_\varepsilon\boldsymbol{\nu}^t, {}_\varepsilon\boldsymbol{R}^t$ of (59) and (60) from ${}_\epsilon\tilde{\mathbf{x}}_i^t$ and obtain their asymptotic variances according to Theorem B.2, and ii) relate these variances to the state evolution of (55)-(57).

The first step is similar to the derivations in the proof of Lemma 6 in the supplementary material of Sur and Candès (2019). For $t=1$, it is easily seen from the AMP instance defined in B.4.4 and the state evolutions (12) and (13) that

$$\begin{bmatrix} {}_\epsilon\boldsymbol{\Sigma}_{1,1}^1 & {}_\epsilon\boldsymbol{\Sigma}_{1,3}^1 \\ {}_\epsilon\boldsymbol{\Sigma}_{3,1}^1 & {}_\epsilon\boldsymbol{\Sigma}_{3,3}^1 \end{bmatrix} = \begin{bmatrix} \lim\limits_{N\to\infty} \frac{\|\boldsymbol{\beta}_0\|_2^2}{n} & \lim\limits_{N\to\infty} \frac{\langle \hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0\rangle}{n} \\ \lim\limits_{N\to\infty} \frac{\langle \hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0\rangle}{n} & \lim\limits_{N\to\infty} \frac{\|\hat{\boldsymbol{\beta}}^0\|_2^2}{n} \end{bmatrix}, \qquad \texttt{eq:sigma\_one} \tag{71}$$

where ${}_\epsilon\boldsymbol{\Sigma}_{i,j}^1$ denotes the jth entry in the ith row of ${}_\epsilon\boldsymbol{\Sigma}^1$. All other entries of ${}_\epsilon\boldsymbol{\Sigma}^1$ are zero. Now from (34) it follows that for $i \in [n]$, the first and third entries of ${}_\epsilon\tilde{\mathbf{x}}_i^1$ are $\boldsymbol{\eta}_{0,i}$, ${}_\epsilon\boldsymbol{R}_i^0 = \boldsymbol{x}_i^\top\hat{\boldsymbol{\beta}}^0$. Hence, by Theorem B.2,

$$\lim_{N\to\infty} \frac{1}{n}\sum_{j=1}^n \psi\left(\begin{bmatrix} \boldsymbol{\eta}_{0,j} \\ {}_\epsilon\boldsymbol{R}_i^0 \end{bmatrix}, \begin{bmatrix} \bar\varepsilon_j \\ 0 \end{bmatrix}\right) = \mathrm{E}\left[\psi\left(\boldsymbol{Z}^1, \begin{bmatrix} \bar\varepsilon \\ 0 \end{bmatrix}\right)\right], \tag{72}$$

where $\boldsymbol{Z}^1 = [\boldsymbol{Z}_1^1, \boldsymbol{Z}_2^1]^\top \perp \bar\varepsilon$ is bivariate normal with mean zero and covariance (71).

Next, the only nonzero entry of ${}_\epsilon\boldsymbol{\Sigma}^2$ is readily computed as $\kappa^{-2}\mathrm{E}\left[\left\{\Psi^0(h_\epsilon(\boldsymbol{Z}_1^1,\bar\varepsilon), \boldsymbol{Z}_2^1)\right\}^2\right]$. By construction of ${}_\epsilon\tilde{\mathbf{x}}_i^t$ it follows from (34) and Lemma B.3 and Theorem B.1 that almost surely

$$\lim_{N\to\infty} \frac{1}{p}\sum_{j=1}^p \psi({}_\epsilon\boldsymbol{\nu}_j^1, \boldsymbol{\beta}_{0,j}) = \mathrm{E}\left[\psi({}_\epsilon\tau_1^2 G, \bar\beta)\right], \tag{73}$$

where ${}_\epsilon\tau_1^2 = \kappa^{-2}\mathrm{E}\left[\left\{\Psi^0(h_\epsilon(\boldsymbol{Z}_1^1,\bar\varepsilon), \boldsymbol{Z}_2^1)\right\}^2\right]$ and $G \sim N(0,1) \perp \bar\beta$. Next consider ${}_\epsilon\boldsymbol{\Sigma}^3$. By the nested nature of the iterates ${}_\epsilon\mathbf{x}_i^t$, it follows that the first 3×3 principal submatrix is exactly the one given in (71). Further, it is readily verified that the remaining nonzero entries are given by

$${}_\epsilon\boldsymbol{\Sigma}_{1,5}^3 = {}_\epsilon\mu_1\gamma^2, \quad {}_\epsilon\boldsymbol{\Sigma}_{3,5}^3 = {}_\epsilon\mu_1 \lim_{N\to\infty}\frac{\langle \hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0\rangle}{n}, \quad {}_\epsilon\boldsymbol{\Sigma}_{5,5}^3 = {}_\epsilon\tau_1^2\kappa + {}_\epsilon\mu_1^2\gamma^2, \tag{74}$$

where ${}_\epsilon\mu_1$ comes from recursion (56) with initial condition μ_0. Again, Lemma B.2 and Theorem B.1 yield almost surely

$$\lim_{N\to\infty} \frac{1}{n}\sum_{j=1}^n \psi\left(\begin{bmatrix} \boldsymbol{\eta}_{0,j} \\ {}_\epsilon\boldsymbol{R}_j^0 \\ {}_\epsilon\boldsymbol{R}_j^1 \end{bmatrix}, \begin{bmatrix} \bar\varepsilon_j \\ 0 \\ 0 \end{bmatrix}\right) = \mathrm{E}\left[\psi\left(\boldsymbol{Z}^3, \begin{bmatrix} \bar\varepsilon \\ 0 \\ 0 \end{bmatrix}\right)\right], \tag{75}$$

where $\boldsymbol{Z}^3 = [\boldsymbol{Z}_1^3, \boldsymbol{Z}_2^3, \boldsymbol{Z}_3^3]^\top \perp \bar\varepsilon$ is trivariate normal with mean zero and covariance matrix

$$\begin{bmatrix} {}_\epsilon\boldsymbol{\Sigma}_{1,1}^3 & {}_\epsilon\boldsymbol{\Sigma}_{1,3}^3 & {}_\epsilon\boldsymbol{\Sigma}_{1,5}^3 \\ {}_\epsilon\boldsymbol{\Sigma}_{3,1}^3 & {}_\epsilon\boldsymbol{\Sigma}_{3,3}^3 & {}_\epsilon\boldsymbol{\Sigma}_{3,5}^3 \\ {}_\epsilon\boldsymbol{\Sigma}_{5,1}^3 & {}_\epsilon\boldsymbol{\Sigma}_{5,3}^3 & {}_\epsilon\boldsymbol{\Sigma}_{5,5}^3 \end{bmatrix} = \begin{bmatrix} \lim\limits_{N\to\infty} \frac{\|\boldsymbol{\beta}_0\|_2^2}{n} & \lim\limits_{N\to\infty} \frac{\langle \hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0\rangle}{n} & {}_\epsilon\mu_1\gamma^2 \\ \lim\limits_{N\to\infty} \frac{\langle \hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0\rangle}{n} & \lim\limits_{N\to\infty} \frac{\|\hat{\boldsymbol{\beta}}^0\|_2^2}{n} & {}_\epsilon\mu_1 \lim\limits_{N\to\infty} \frac{\langle \hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0\rangle}{n} \\ {}_\epsilon\mu_1\gamma^2 & {}_\epsilon\mu_1 \lim\limits_{N\to\infty} \frac{\langle \hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0\rangle}{n} & {}_\epsilon\tau_1^2\kappa + {}_\epsilon\mu_1^2\gamma^2 \end{bmatrix}. \tag{76}$$

Calculating the nonzero entries of ${}_\epsilon\boldsymbol{\Sigma}^4$ yields that

$$\begin{bmatrix} {}_\epsilon\boldsymbol{\Sigma}^4_{2,2} & {}_\epsilon\boldsymbol{\Sigma}^4_{2,4} \\ {}_\epsilon\boldsymbol{\Sigma}^4_{4,2} & {}_\epsilon\boldsymbol{\Sigma}^4_{4,4} \end{bmatrix} = \begin{bmatrix} {}_\epsilon\tau_1^2 & {}_\epsilon\rho_{1,2} \\ {}_\epsilon\rho_{1,2} & {}_\epsilon\tau_2^2 \end{bmatrix}, \tag{77}$$

eq:sigma 4

where

$$\begin{aligned} {}_\epsilon\tau_2^2 &= \kappa^{-2}\mathrm{E}\left[\{\Psi^1(h_\epsilon(\boldsymbol{Z}_1^3,\bar{\varepsilon}),\boldsymbol{Z}_2^3)\}^2\right] \\ {}_\epsilon\rho_{1,2} &= \kappa^{-2}\mathrm{E}\left[\Psi^0(h_\epsilon(\boldsymbol{Z}_1^3,\bar{\varepsilon}),\boldsymbol{Z}_2^3)\Psi^1(h_\epsilon(\boldsymbol{Z}_1^3,\bar{\varepsilon}),\boldsymbol{Z}_3^3)\right]. \end{aligned} \tag{78}$$

Consequently, almost surely

$$\lim_{N\to\infty}\frac{1}{p}\sum_{j=1}^p \psi\left(\begin{bmatrix} {}_\epsilon\boldsymbol{\nu}_j^1 \\ {}_\epsilon\boldsymbol{\nu}_j^2 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\beta}_{0,j} \\ 0 \end{bmatrix}\right) = \mathrm{E}\left[\psi\left(\boldsymbol{Z}^4, \begin{bmatrix} \bar{\beta} \\ 0 \end{bmatrix}\right)\right], \tag{79}$$

where $\boldsymbol{Z}^4 = [\boldsymbol{Z}_1^4, \boldsymbol{Z}_2^4]^\top$ is bivariate normal with mean zero and covariance matrix (77). Iterating further,

$$\begin{bmatrix} {}_\epsilon\boldsymbol{\Sigma}^5_{1,1} & {}_\epsilon\boldsymbol{\Sigma}^5_{1,3} & {}_\epsilon\boldsymbol{\Sigma}^5_{1,5} & {}_\epsilon\boldsymbol{\Sigma}^5_{1,7} \\ {}_\epsilon\boldsymbol{\Sigma}^5_{3,1} & {}_\epsilon\boldsymbol{\Sigma}^5_{3,3} & {}_\epsilon\boldsymbol{\Sigma}^5_{3,5} & {}_\epsilon\boldsymbol{\Sigma}^5_{3,7} \\ {}_\epsilon\boldsymbol{\Sigma}^5_{5,1} & {}_\epsilon\boldsymbol{\Sigma}^5_{5,3} & {}_\epsilon\boldsymbol{\Sigma}^5_{5,5} & {}_\epsilon\boldsymbol{\Sigma}^5_{5,7} \\ {}_\epsilon\boldsymbol{\Sigma}^5_{7,1} & {}_\epsilon\boldsymbol{\Sigma}^5_{7,3} & {}_\epsilon\boldsymbol{\Sigma}^5_{7,5} & {}_\epsilon\boldsymbol{\Sigma}^5_{7,7} \end{bmatrix} = \begin{bmatrix} \lim\limits_{N\to\infty}\frac{\|\boldsymbol{\beta}_0\|_2^2}{n} & \lim\limits_{N\to\infty}\frac{\langle\hat{\boldsymbol{\beta}}^0,\boldsymbol{\beta}_0\rangle}{n} & {}_\epsilon\mu_1\gamma^2 & {}_\epsilon\mu_2\gamma^2 \\ \lim\limits_{N\to\infty}\frac{\langle\hat{\boldsymbol{\beta}}^0,\boldsymbol{\beta}_0\rangle}{n} & \lim\limits_{N\to\infty}\frac{\|\hat{\boldsymbol{\beta}}^0\|_2^2}{n} & {}_\epsilon\mu_1\lim\limits_{N\to\infty}\frac{\langle\hat{\boldsymbol{\beta}}^0,\boldsymbol{\beta}_0\rangle}{n} & {}_\epsilon\mu_2\lim\limits_{N\to\infty}\frac{\langle\hat{\boldsymbol{\beta}}^0,\boldsymbol{\beta}_0\rangle}{n} \\ {}_\epsilon\mu_1\gamma^2 & {}_\epsilon\mu_1\lim\limits_{N\to\infty}\frac{\langle\hat{\boldsymbol{\beta}}^0,\boldsymbol{\beta}_0\rangle}{n} & {}_\epsilon\tau_1^2\kappa + {}_\epsilon\mu_1^2\gamma^2 & {}_\epsilon\rho_{1,2}\kappa + {}_\epsilon\mu_1{}_\epsilon\mu_2\gamma^2 \\ {}_\epsilon\mu_2\gamma^2 & {}_\epsilon\mu_2\lim\limits_{N\to\infty}\frac{\langle\hat{\boldsymbol{\beta}}^0,\boldsymbol{\beta}_0\rangle}{n} & {}_\epsilon\rho_{1,2}\kappa + {}_\epsilon\mu_1{}_\epsilon\mu_2\gamma^2 & {}_\epsilon\tau_2^2\kappa + {}_\epsilon\mu_2^2\gamma^2 \end{bmatrix}, \tag{80}$$

eq:sigma 5

so that almost surely

$$\lim_{N\to\infty}\frac{1}{n}\sum_{j=1}^n \psi\left(\begin{bmatrix} \boldsymbol{\eta}_{0,j} \\ {}_\epsilon\boldsymbol{R}_j^0 \\ {}_\epsilon\boldsymbol{R}_j^1 \\ {}_\epsilon\boldsymbol{R}_j^2 \end{bmatrix}, \begin{bmatrix} \bar{\varepsilon}_i \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) = \mathrm{E}\left[\psi\left(\boldsymbol{Z}^5, \begin{bmatrix} \bar{\varepsilon} \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)\right], \tag{81}$$

where once again $\boldsymbol{Z}^5 = [\boldsymbol{Z}_1^5, \ldots, \boldsymbol{Z}_4^5]^\top \perp \bar{\varepsilon}$ is multivariate normal with mean zero and covariance matrix (80). Hence continuing,

$$\begin{bmatrix} {}_\epsilon\boldsymbol{\Sigma}^6_{2,2} & {}_\epsilon\boldsymbol{\Sigma}^6_{2,4} & {}_\epsilon\boldsymbol{\Sigma}^6_{2,6} \\ {}_\epsilon\boldsymbol{\Sigma}^6_{4,2} & {}_\epsilon\boldsymbol{\Sigma}^6_{4,4} & {}_\epsilon\boldsymbol{\Sigma}^6_{4,6} \\ {}_\epsilon\boldsymbol{\Sigma}^6_{6,2} & {}_\epsilon\boldsymbol{\Sigma}^6_{6,4} & {}_\epsilon\boldsymbol{\Sigma}^6_{6,6} \end{bmatrix} = \begin{bmatrix} {}_\epsilon\tau_1^2 & {}_\epsilon\rho_{1,2} & {}_\epsilon\rho_{1,3} \\ {}_\epsilon\rho_{1,2} & {}_\epsilon\tau_2^2 & {}_\epsilon\rho_{2,3} \\ {}_\epsilon\rho_{1,3} & {}_\epsilon\rho_{2,3} & {}_\epsilon\tau_3^2 \end{bmatrix}, \tag{82}$$

eq:sigma 6

where

$$\begin{aligned} {}_\epsilon\tau_3^2 &= \kappa^{-2}\mathrm{E}\left[\{\Psi^2(h_\epsilon(\boldsymbol{Z}_1^5,\bar{\varepsilon}),\boldsymbol{Z}_2^5)\}^2\right] \\ {}_\epsilon\rho_{l,m} &= \kappa^{-2}\mathrm{E}\left[\Psi^{l-1}(h_\epsilon(\boldsymbol{Z}_1^5,\bar{\varepsilon}),\boldsymbol{Z}_{l+1}^5)\Psi^{m-1}(h_\epsilon(\boldsymbol{Z}_1^5,\bar{\varepsilon}),\boldsymbol{Z}_{m+1}^5)\right]. \end{aligned} \tag{83}$$

Therefore almost surely,

$$\lim_{N\to\infty}\frac{1}{p}\sum_{j=1}^p \psi\left(\begin{bmatrix} {}_\epsilon\boldsymbol{\nu}_j^1 \\ {}_\epsilon\boldsymbol{\nu}_j^2 \\ {}_\epsilon\boldsymbol{\nu}_j^3 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\beta}_{0,j} \\ 0 \\ 0 \end{bmatrix}\right) = \mathrm{E}\left[\psi\left(\boldsymbol{Z}^6, \begin{bmatrix} \bar{\beta} \\ 0 \\ 0 \end{bmatrix}\right)\right], \tag{84}$$

with $\boldsymbol{Z}^6 = [\boldsymbol{Z}_1^6, \boldsymbol{Z}_2^6, \boldsymbol{Z}_3^6]^\top$ trivariate normal with mean zero and covariance (82). Continuing in this vein, one gets that for all $t \in [q]$, by Lemmas B.2 and Theorem B.1, almost surely

eq:epsilon_asymptotics

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{p} \sum_{j=1}^{p} \psi\left({}_{\epsilon}\boldsymbol{\nu}_j^t, \boldsymbol{\beta}_{0,j}\right) &= \mathrm{E}\left[\psi(\tau_t G, \bar{\beta})\right] \\ \lim_{N\to\infty} \frac{1}{n} \sum_{j=1}^{n} \psi\left(\begin{bmatrix} \boldsymbol{\eta}_{0,j} \\ {}_{\epsilon}\boldsymbol{R}_j^t \end{bmatrix}, \begin{bmatrix} \bar{\varepsilon}_i \\ 0 \end{bmatrix}\right) &= \mathrm{E}\left[\psi\left(\begin{bmatrix} \boldsymbol{Z}_t^{2t+1} \\ \boldsymbol{Z}_{t+2}^{2t+1} \end{bmatrix}, \begin{bmatrix} \bar{\varepsilon} \\ 0 \end{bmatrix}\right)\right], \end{aligned} \tag{85}$$

where $G \sim \mathcal{N}(0,1) \perp \bar{\beta}$, and $(\boldsymbol{Z}_t^{2t+1}, \boldsymbol{Z}_{t+2}^{2t+1}) \perp \bar{\varepsilon}$ are bivariate normal with mean zero and covariance

eq:cov_at_at

$$\begin{bmatrix} {}_{\epsilon}\boldsymbol{\Sigma}_{1,1}^{t+2} & {}_{\epsilon}\boldsymbol{\Sigma}_{1,t+2}^{t+2} \\ {}_{\epsilon}\boldsymbol{\Sigma}_{t+2,1}^{t+2} & {}_{\epsilon}\boldsymbol{\Sigma}_{t+2,t+2}^{t+2} \end{bmatrix} = \begin{bmatrix} \lim\limits_{N\to\infty} \frac{\|\boldsymbol{\beta}_0\|_2^2}{n} & {}_{\epsilon}\mu_t \gamma^2 \\ {}_{\epsilon}\mu_t \gamma^2 & {}_{\epsilon}\tau_t^2 \kappa + {}_{\epsilon}\mu_t^2 \gamma^2 \end{bmatrix}, \tag{86}$$

and

$${}_{\epsilon}\tau_t = \kappa^{-2} \mathrm{E}\left[\left\{\Psi^{t-1}(h_{\epsilon}(\boldsymbol{Z}_1^{2t-1}, \bar{\varepsilon}), \boldsymbol{Z}_{t+1}^{2t-1})\right\}\right], \tag{87}$$

and where $\boldsymbol{Z}_1^{2t-1}, \boldsymbol{Z}_{t+1}^{2t-1} \perp \bar{\varepsilon}$ are bivariate normal with the covariance structure of (86) with ${}_{\epsilon}\mu_{t-1}, {}_{\epsilon}\tau_{t-1}$ in place of ${}_{\epsilon}\mu_t, {}_{\epsilon}\tau_t$ respectively.

Next, note that with initializations ${}_{\epsilon}\mu_0 = \gamma^{-2} \lim\limits_{n\to\infty} \langle \hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0 \rangle / n$ and ${}_{\epsilon}\sigma_0^2 = \lim\limits_{n\to\infty} \|\hat{\boldsymbol{\beta}}^0 - \mu_0 \boldsymbol{\beta}_0\|_2^2 / p$, the variances of (71) correspond to

$$\begin{bmatrix} \gamma^2 & {}_{\epsilon}\mu_0 \gamma^2 \\ {}_{\epsilon}\mu_0 \gamma^2 & {}_{\epsilon}\sigma_0^2 \kappa + {}_{\epsilon}\mu_0^2 \gamma^2 {}_{\epsilon}\mu_0 \gamma^2 \end{bmatrix}, \tag{88}$$

Hence by the definition of ${}_{\epsilon}\sigma_t$ in (57), ${}_{\epsilon}\tau_1 = {}_{\epsilon}\sigma_1$. By induction it follows that ${}_{\epsilon}\tau_t = {}_{\epsilon}\sigma_t$ for all t under consideration. This concludes the proof. □

B.4.5 Asymptotic equivalance of target and centred AMP

subsubsec:relate_AMP_centred

Next it is possible to establish the asymptotic equivalence of the centred recursion (26)-(29) with the initial target recursion (16)-(17) for appropriately centred iterates. The proof closely follows along the proof of Lemma 4 of the supplementary material of Sur and Candès (2019) which in turn is an adaptation of the proof of Lemma 6.7 of Donoho and Montanari (2016).

lemma:eq_recursions

Lemma B.4. *In the setting of Corollary B.2.1, it holds that for initial conditions* ${}_{\epsilon}\boldsymbol{\nu}^0 = \hat{\boldsymbol{\beta}}^0 - \mu_0 \boldsymbol{\beta}_0$, ${}_{\epsilon}\boldsymbol{R}^0 = {}_{\epsilon}\boldsymbol{S}^0$, $t \in [q]$,

$$\lim_{N\to\infty} \frac{1}{p} \left\| {}_{\epsilon}\boldsymbol{\nu}^t - {}_{\epsilon}\hat{\boldsymbol{\beta}}^t - {}_{\epsilon}\mu_t \boldsymbol{\beta}_0 \right\|_2^2 = 0, \quad \lim_{N\to\infty} \frac{1}{n} \left\| {}_{\epsilon}\boldsymbol{R}^t - {}_{\epsilon}\boldsymbol{S}^t \right\|_2^2 = 0, \tag{89}$$

almost surely.

Proof. To facilitate notation, let ${}_{\epsilon}\boldsymbol{\theta}^t = {}_{\epsilon}\hat{\boldsymbol{\beta}}^t - {}_{\epsilon}\mu_t \boldsymbol{\beta}_0$. Then by construction of recursions (16)-(17) and (26)-(27), and the triangle inequality it holds that

eq:centred_rec_res_diff

$$\begin{aligned} \left\| {}_{\epsilon}\boldsymbol{R}^t - {}_{\epsilon}\boldsymbol{S}^t \right\|_2 &\le \|\boldsymbol{X}\|_2 \left\| {}_{\epsilon}\boldsymbol{\nu}^t - {}_{\epsilon}\boldsymbol{\theta}^t \right\|_2 + \left\| \Psi^{t-1}({}_{\epsilon}\boldsymbol{y}^*, {}_{\epsilon}\boldsymbol{R}^{t-1}) - \Psi^{t-1}({}_{\epsilon}\boldsymbol{y}^*, {}_{\epsilon}\boldsymbol{S}^{t-1}) \right\| \\ &\le \|\boldsymbol{X}\|_2 \left\| {}_{\epsilon}\boldsymbol{\nu}^t - {}_{\epsilon}\boldsymbol{\theta}^t \right\|_2 + \left\| {}_{\epsilon}\boldsymbol{R}^{t-1} - {}_{\epsilon}\boldsymbol{S}^{t-1} \right\|_2, \end{aligned} \tag{90}$$

where the second inequality follows by Lemma G.2 iii) and where ${}_{\epsilon}\boldsymbol{y}^* = \alpha h_{\epsilon}(\boldsymbol{\eta}_0, \bar{\varepsilon}) + (1-\alpha)/2$.

Similarly,

$$\left\|{}_{\epsilon}\boldsymbol{\nu}^{t} - {}_{\epsilon}\boldsymbol{\theta}^{t}\right\|_2 \leq \left\|{}_{\epsilon}\boldsymbol{\nu}^{t-1} - {}_{\epsilon}\boldsymbol{\theta}^{t-1}\right\|_2 + |{}_{\epsilon}q_{t-1} - 1| \left\|{}_{\epsilon}\boldsymbol{\nu}^{t-1} + {}_{\epsilon}\mu_{t-1}\boldsymbol{\beta}_0\right\|_2 + \|{}_{\epsilon}\mu_t - {}_{\epsilon}a_t\| \, \|\boldsymbol{\beta}_0\| + \frac{1}{\kappa}\|\boldsymbol{X}\|_2 \left\|{}_{\epsilon}\boldsymbol{R}^{t-1} - {}_{\epsilon}\boldsymbol{S}^{t-1}\right\|_2 . \tag{91}$$

Now since ${}_{\epsilon}\boldsymbol{\nu}^0 = {}_{\epsilon}\boldsymbol{\theta}^0$, an induction argument on the number of iterations t gives that there exists a constant C, which may depend on κ, such that for each $t \in [q]$,

`eq:centred_rec_diff`

$$\left\|{}_{\epsilon}\boldsymbol{\nu}^{t} - {}_{\epsilon}\boldsymbol{\theta}^{t}\right\|_2 \leq (C\|\boldsymbol{X}\|_2)^{2t} \left(\sum_{l=1}^{t-1} |{}_{\epsilon}q_t - 1| \left\|{}_{\epsilon}\boldsymbol{\nu}^{l} + {}_{\epsilon}\mu_l\boldsymbol{\beta}_0\right\|_2 + \sum_{l=0}^{t-1} |{}_{\epsilon}\mu_l - a_l| \left\|\boldsymbol{\beta}_0\right\|_2 \right) . \tag{92}$$

Recalling that ${}_{\epsilon}q_t = -\frac{1}{\kappa n}\sum_{j=1}^{n} \frac{-b_t\zeta''(\mathrm{prox}_{b_t\zeta}(b_t h_\epsilon(\boldsymbol{\eta}_{0,j},\bar{\varepsilon}_j),\boldsymbol{R}_j^t))}{1+b_t\zeta''(\mathrm{prox}_{b_t\zeta}(b_t h_\epsilon(\boldsymbol{\eta}_{0,j},\bar{\varepsilon}_j),\boldsymbol{R}_j^t))}$ (cf. (28) and (223)), it follows from Lemma G.2 iv) and Corollary B.2.1 i), that almost surely

`eq:qt_limit`

$$\lim_{N\to\infty} {}_{\epsilon}q_t = \frac{1}{\kappa}\mathrm{E}\left[\frac{b_t\zeta''(\mathrm{prox}_{b_t\zeta}(b_t h_\epsilon(Z_1^t,\bar{\varepsilon}), {}_{\epsilon}Z_2^t))}{1 + b_t\zeta''(\mathrm{prox}_{b_t\zeta}(b_t h_\epsilon(Z_1^t,\bar{\varepsilon}), {}_{\epsilon}Z_2^t))}\right] \tag{93}$$

for $Z_1^t \sim N(0,\gamma^2)$, ${}_{\epsilon}Z_2^t = {}_{\epsilon}\mu_t Z_1^t + \kappa^{1/2}{}_{\epsilon}\sigma_t G$, $G \sim N(0,1)$, $\bar{\varepsilon} \sim U(0,1)$ and where $G \perp \bar{\beta}$, $G \perp Z_1^t$ and $(Z_1^t, {}_{\epsilon}Z_2^t) \perp \bar{\varepsilon}$. Now the right hand side of (93) can be rearranged as

$$\frac{1}{\kappa}\mathrm{E}\left[\frac{b_t\zeta''(\mathrm{prox}_{b_t\zeta}(b_t h_\epsilon(Z_1^t,\bar{\varepsilon}), {}_{\epsilon}Z_2^t))}{1 + b_t\zeta''(\mathrm{prox}_{b_t\zeta}(b_t h_\epsilon(Z_1^t,\bar{\varepsilon}), {}_{\epsilon}Z_2^t))}\right] = \frac{1}{\kappa}\mathrm{E}\left[1 - \frac{1}{b_t\zeta''(\mathrm{prox}_{b_t\zeta}(b_t h_\epsilon(Z_1^t,\bar{\varepsilon}), {}_{\epsilon}Z_2^t))}\right] = 1\,, \tag{94}$$

according to state evolution equation (55).

Again, by Corollary B.2.1 i) and $\psi(u,v) = u^2$, it follows that almost surely,

$$\lim_{N\to\infty} \sum_{j=1} ({}_{\epsilon}\boldsymbol{\nu}_j^t)^2 = \mathrm{E}\left[({}_{\epsilon}\sigma_t Z)^2\right] < \infty\,, \tag{95}$$

for $Z \sim N(0,1)$. Further recall that by assumption $\lim_{n\to\infty} \|\boldsymbol{\beta}_0\|_2^2/n = \gamma^2$ and that by Lemma G.3, $\lim_{N\to\infty} \|\boldsymbol{X}\|_2 < \infty$ almost surely. Hence it holds almost surely that

`eq:qt_terms`

$$\lim_{N\to\infty} \frac{1}{\sqrt{p}}(C\|\boldsymbol{X}\|_2)^{2t} \sum_{l=0}^{t-1} |{}_{\epsilon}q_l - 1| \left\|{}_{\epsilon}\boldsymbol{\nu}_j^l + {}_{\epsilon}\mu_l\boldsymbol{\beta}_0\right\|_2 = 0\,. \tag{96}$$

Similarly, recalling the definition of ${}_{\epsilon}a_t$ in (62) and invoking Lemma G.2 iv) and Corollary B.2.1 equation (85), almost surely

`eq:at_limit`

$$\lim_{N\to\infty} {}_{\epsilon}a_t = \frac{1}{\kappa}\mathrm{E}\left[\frac{\partial}{\partial Z}\Psi^{t-1}(h_\epsilon(Z,\bar{\varepsilon}), {}_{\epsilon}Z_2^t)\bigg|_{Z=Z_1^t}\right]\,, \tag{97}$$

for $Z_1^t \sim N(0,\gamma^2)$, ${}_{\epsilon}Z_2^t = {}_{\epsilon}\mu_t Z_1^t + \kappa^{1/2}{}_{\epsilon}\sigma_t G$, $G \sim N(0,1)$, $\bar{\varepsilon} \sim U(0,1)$ and where $G \perp \bar{\beta}$, $G \perp Z_1^t$ and $(Z_1^t, {}_{\epsilon}Z_2^t) \perp \bar{\varepsilon}$.

To compute the expectation in (97), Stein's Lemma (Stein, 1981, Lemma 1) shall prove useful. It states that for a normal random variable $z \sim N(\mu,\sigma^2)$, and a differentiable function $h(z)$, $\mathrm{E}[h(z)(z-\mu)] = \sigma^2\mathrm{E}(h'(z))$, where h is derivative of h with respect to z.

To use the Lemma, first note that ${}_{\epsilon}Z_2^t$ is a function of Z_1^t. Hence write ${}_{\epsilon}Z_2^t = t_\epsilon(Z_1^t, G_t) = {}_{\epsilon}\mu_t Z_1^t + \kappa^{1/2}{}_{\epsilon}\sigma_t G_t$. Then conditioning on both $\bar{\varepsilon}, G_t$, one gets that

$$\mathrm{E}\left[\frac{\partial}{\partial Z}\Psi^{t-1}(h_\epsilon(Z,\bar{\varepsilon}), {}_{\epsilon}Z_2^t)\bigg|_{Z=Z_1^t}\right] = \mathrm{E}_{G_t,\bar{\varepsilon}}\left\{\mathrm{E}_{Z_1^t}\left[\frac{\partial}{\partial Z}\Psi(h_\epsilon(Z,\bar{\varepsilon}), s)\bigg|_{\substack{Z=Z_1^t,\\ s=t_\epsilon(Z_1^t,G_t)}}\bigg| G_t, \bar{\varepsilon}\right]\right\}. \quad \texttt{eq:at\_cond\_exp} \tag{98}$$

Now since Z_1^t is independent of $G_t, \bar{\varepsilon}$, the conditional distribution of Z_1^t given $G_t, \bar{\varepsilon}$ is its unconditional distribution, i.e. $Z_1^t|_{G_t,\bar{\varepsilon}} \sim N(0, \gamma^2)$ and thus by Stein's Lemma,

$$\gamma^2 \mathrm{E}_{Z_1^t}\left[\frac{\partial}{\partial Z_1^t}\Psi(h_\epsilon(Z_1^t,\bar{\varepsilon}), t_\epsilon(Z_1^t, G_t))\bigg| G_t, \bar{\varepsilon}\right] = \mathrm{E}_{Z_1^t}\left[Z_1^t \Psi(h_\epsilon(Z_1^t,\bar{\varepsilon}), t_\epsilon(Z_1^t, G_t))\bigg| G_t, \bar{\varepsilon}\right]. \quad \texttt{eq:at\_stein} \tag{99}$$

Now an application of the chain rule to the left hand side of (99) yields

$$\begin{aligned}
\mathrm{E}_{Z_1^t}\left[\frac{\partial}{\partial Z_1^t}\Psi(h_\epsilon(Z_1^t,\bar{\varepsilon}), t_\epsilon(Z_1^t, G_t))\bigg| G_t, \bar{\varepsilon}\right] &= \mathrm{E}_{Z_1^t}\left[\frac{\partial}{\partial Z}\Psi(h_\epsilon(Z,\bar{\varepsilon}), s)\bigg|_{\substack{Z=Z_1^t,\\ s=t_\epsilon(Z_1^t,G_t)}}\bigg| G_t, \bar{\varepsilon}\right] \\
&\quad + {}_{\epsilon}\mu_{t-1}\mathrm{E}_{Z_1^t}\left[\frac{\partial}{\partial s}\Psi(h_\epsilon(Z,\bar{\varepsilon}), s)\bigg|_{\substack{Z=Z_1^t,\\ s=t_\epsilon(Z_1^t,G_t)}}\bigg| G_t, \bar{\varepsilon}\right]
\end{aligned} \quad \texttt{eq:at_stein2} \tag{100}$$

Combining (99) and (100) gives

$$\begin{aligned}
\mathrm{E}_{Z_1^t}\left[\frac{\partial}{\partial Z}\Psi(h_\epsilon(Z,\bar{\varepsilon}), s)\bigg|_{\substack{Z=Z_1^t,\\ s=t_\epsilon(Z_1^t,G_t)}}\bigg| G_t, \bar{\varepsilon}\right] &= \frac{1}{\gamma^2}\mathrm{E}_{Z_1^t}\left[Z_1^t \Psi(h_\epsilon(Z_1^t,\bar{\varepsilon}), t_\epsilon(Z_1^t, G_t))\bigg| G_t, \bar{\varepsilon}\right] \\
&\quad - {}_{\epsilon}\mu_{t-1}\mathrm{E}_{Z_1^t}\left[\frac{\partial}{\partial s}\Psi(h_\epsilon(Z,\bar{\varepsilon}), s)\bigg|_{\substack{Z=Z_1^t,\\ s=t_\epsilon(Z_1^t,G_t)}}\bigg| G_t, \bar{\varepsilon}\right]
\end{aligned} \tag{101}$$

Substituting back onto (98) in conjunction with (97) gives that almost surely,

$$\lim_{N\to\infty} {}_{\epsilon}a_t = \frac{1}{\kappa\gamma^2}\mathrm{E}\left[Z_1^t \Psi(h_\epsilon(Z_1^t,\bar{\varepsilon}), {}_{\epsilon}Z_2^t)\right] - \frac{{}_{\epsilon}\mu_{t-1}}{\kappa}\mathrm{E}\left[\frac{\partial}{\partial s}\Psi(h_\epsilon(Z,\bar{\varepsilon}), s)\bigg|_{\substack{Z=Z_1^t,\\ s={}_{\epsilon}Z_2^t}}\right]. \tag{102}$$

Now using (55) and (223), it follows that

$$\begin{aligned}
-\frac{1}{\kappa}\mathrm{E}\left[\frac{\partial}{\partial s}\Psi(h_\epsilon(Z,\bar{\varepsilon}), s)\bigg|_{\substack{Z=Z_1^t,\\ s={}_{\epsilon}Z_2^t}}\right] &= \frac{1}{\kappa}\mathrm{E}\left[\frac{b\zeta''(\mathrm{prox}_{b\zeta}(bh_\epsilon(Z,\bar{\varepsilon}) + {}_{\epsilon}Z_2^t))}{1 + b\zeta''(\mathrm{prox}_{b\zeta}(bh_\epsilon(Z,\bar{\varepsilon} + {}_{\epsilon}Z_2^t))}\right] \\
&= \frac{1}{\kappa}\mathrm{E}\left[1 - \frac{1}{1 + b\zeta''(\mathrm{prox}_{b\zeta}(bh_\epsilon(Z,\bar{\varepsilon} + {}_{\epsilon}Z_2^t))}\right] \\
&= 1\,.
\end{aligned} \tag{103}$$

Hence, the state evolution of ${}_\epsilon\mu_t$ as specified in (56) gives that almost surely

$$\begin{aligned}\lim_{N\to\infty} {}_\epsilon a_t &= \frac{1}{\kappa\gamma^2}\mathrm{E}\left[Z_1^t \Psi(h_\epsilon(Z_1^t,\bar{\varepsilon}), {}_\epsilon Z_2^t)\right] - \frac{{}_\epsilon\mu_{t-1}}{\kappa}\mathrm{E}\left[\frac{\partial}{\partial s}\Psi(h_\epsilon(Z,\bar{\varepsilon}),s)\Big|_{\substack{Z=Z_1^t,\\ s={}_\epsilon Z_2^t}}\right] \\ &= \frac{1}{\kappa\gamma^2}\mathrm{E}\left[Z_1^t \Psi(h_\epsilon(Z_1^t,\bar{\varepsilon}), {}_\epsilon Z_2^t)\right] + {}_\epsilon\mu_{t-1} \\ &= {}_\epsilon\mu_t \,. \end{aligned} \tag{104}$$

Hence, for any $l \in \mathbb{N}_0$, $\lim_{N\to\infty} |{}_\epsilon a_t - {}_\epsilon\mu_t| = 0$ almost surely. By almost sure boundedness of $\boldsymbol{X}$ and since $\|\boldsymbol{\beta}_0\|_2^2/n \to \gamma^2$ as $n\to\infty$, it follows that almost surely

eq:at_terms

$$\lim_{N\to\infty} \frac{1}{\sqrt{p}}\sum_{l=0}^{t-1} |{}_\epsilon a_t - {}_\epsilon\mu_t|\, \|\boldsymbol{\beta}_0\|_2 = 0\,. \tag{105}$$

In conjunction with (92) and (96) and (105), this yields that almost surely,

eq:centred_rec_equiv

$$\left\|{}_\epsilon\boldsymbol{\nu}^t - {}_\epsilon\boldsymbol{\theta}^t\right\|_2 = 0\,. \tag{106}$$

Almost sure convergence of $\|{}_\epsilon\boldsymbol{R}^t - {}_\epsilon\boldsymbol{S}^t\|_2$ then follows by an induction argument on t, (90), (106) and since ${}_\epsilon\boldsymbol{S}^0 = {}_\epsilon\boldsymbol{R}^0$. This concludes the proof. □

B.4.6 Asymptotic equivalence of AMP and mDYPL

subsubsec:asym_equiv

thm:equivalence_mle

Theorem B.3. *In the setting of Lemma B.4 and assuming that there exists a stationary solution* $({}_\epsilon\mu_*, {}_\epsilon b_*, {}_\epsilon\sigma_*)$ *to the state evolution* (56)-(57). *Then, if* $\|\hat{\boldsymbol{\beta}}^{\text{DY}}\|_2/\sqrt{n} = \mathcal{O}(1)$ *almost surely,*

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\lim_{N\to\infty} \frac{1}{p}\left\|{}_\epsilon\hat{\boldsymbol{\beta}}^t - \hat{\boldsymbol{\beta}}^{\text{DY}}\right\|_2^2 = 0 \tag{107}$$

almost surely.

Proof. First, a second order Taylor approximation of $\ell(\hat{\boldsymbol{\beta}}^{\text{DY}})$ around ${}_\epsilon\hat{\boldsymbol{\beta}}^t$ gives

eq:supp_posterior_taylor

$$\ell(\hat{\boldsymbol{\beta}}^{\text{DY}}) = \ell({}_\epsilon\hat{\boldsymbol{\beta}}^t) + \nabla\ell({}_\epsilon\hat{\boldsymbol{\beta}}^t)^\top(\hat{\boldsymbol{\beta}}^{\text{DY}} - {}_\epsilon\hat{\boldsymbol{\beta}}^t) + \frac{1}{2}\boldsymbol{r}\,, \tag{108}$$

where the remainder term $\boldsymbol{r}$ is, equal to

$$\boldsymbol{r} = (\hat{\boldsymbol{\beta}}^{\text{DY}} - {}_\epsilon\hat{\boldsymbol{\beta}}^t)^\top \nabla\nabla^\top \ell(\hat{\boldsymbol{\beta}}^{\text{DY}} - c({}_\epsilon\hat{\boldsymbol{\beta}}^t - \hat{\boldsymbol{\beta}}^{\text{DY}}))(\hat{\boldsymbol{\beta}}^{\text{DY}} - {}_\epsilon\hat{\boldsymbol{\beta}}^t)\,, \tag{109}$$

for some constant $c \in [0,1]$. Now by Sur et al. 2019, Lemma 7, with probability at least $1 - c_1\exp\{-c_2 n\}$ for absolute constants $c_1, c_2 > 0$, it holds that for all $\boldsymbol{\beta} \in \Re^p$

$$\nabla\nabla^\top\ell(\boldsymbol{\beta}) \succ \omega\left(\frac{\|\boldsymbol{\beta}\|_2}{\sqrt{n}}\right)\boldsymbol{I}_p\,, \tag{110}$$

where $\boldsymbol{A} \succ \boldsymbol{B}$ means that $\boldsymbol{A} - \boldsymbol{B}$ is positive definite. Here, $\omega(\cdot) \in (0,1)$ is a nonincreasing function independent of n. Clearly,

$$\left\|\hat{\boldsymbol{\beta}}^{\text{DY}} - c(\hat{\boldsymbol{\beta}}^{\text{DY}} - {}_\epsilon\hat{\boldsymbol{\beta}}^t)\right\|_2 \le \max\left\{\left\|\hat{\boldsymbol{\beta}}^{\text{DY}}\right\|_2, \left\|{}_\epsilon\hat{\boldsymbol{\beta}}^t\right\|_2\right\}\,, \tag{111}$$

so that with probability at least $1 - c_1 \exp\{-c_2 n\}$

$$
\begin{aligned}
\boldsymbol{r} &\geq \left\|\hat{\boldsymbol{\beta}}^{\text{DY}} - {}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right\|_2^2 \omega\left(\frac{\left\|\hat{\boldsymbol{\beta}}^{\text{DY}} - c(\hat{\boldsymbol{\beta}}^{\text{DY}} - {}_{\epsilon}\hat{\boldsymbol{\beta}}^t)\right\|_2}{\sqrt{n}}\right) \\
&\geq \left\|\hat{\boldsymbol{\beta}}^{\text{DY}} - {}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right\|_2^2 \omega\left(\max\left\{\frac{\left\|\hat{\boldsymbol{\beta}}^{\text{DY}}\right\|_2}{\sqrt{n}}, \frac{\left\|{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right\|_2}{\sqrt{n}}\right\}\right).
\end{aligned}
\tag{112}
$$

eq:remainder

Using (112) and the Cauchy-Schwarz inequality, (108) can be rearranged as

$$
\left\|\hat{\boldsymbol{\beta}}^{\text{DY}} - {}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right\|_2 \leq \frac{2}{\omega\left(\max\left\{\frac{\|\hat{\boldsymbol{\beta}}^{\text{DY}}\|_2}{\sqrt{n}}, \frac{\|{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\|_2}{\sqrt{n}}\right\}\right)} \left\|\nabla \ell({}_{\epsilon}\hat{\boldsymbol{\beta}}^t)\right\|_2, \tag{113}
$$

so that it remains to control $\|\hat{\boldsymbol{\beta}}^{\text{DY}}\|_2, \|{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\|_2$ and $\|\nabla \ell({}_{\epsilon}\hat{\boldsymbol{\beta}}^t)\|_2$. Towards bounding $\|\nabla \ell({}_{\epsilon}\hat{\boldsymbol{\beta}}^t)\|_2$, recall that by (52) and (53), any by the definition of the function $\Psi^t(\cdot,\cdot)$, and the identity $z - \zeta'(\text{prox}_{b\zeta}(z)) = \text{prox}_{b\zeta}(z)$,

$$
-\text{prox}_{{}_{\epsilon}b_t\zeta}\left({}_{\epsilon}b_t{}_{\epsilon}\boldsymbol{y}^* + {}_{\epsilon}\boldsymbol{S}^{t-1}\right) = {}_{\epsilon}\boldsymbol{S}^t - {}_{\epsilon}\boldsymbol{S}^{t-1} + \boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t, \tag{114}
$$

and thus

$$
{}_{\epsilon}\hat{\boldsymbol{\beta}}^t - {}_{\epsilon}\hat{\boldsymbol{\beta}}^{t-1} = \frac{{}_{\epsilon}b_t}{\kappa}\boldsymbol{X}^\top\left\{{}_{\epsilon}\boldsymbol{y}^* - \zeta'\left({}_{\epsilon}\boldsymbol{S}^t - {}_{\epsilon}\boldsymbol{S}^{t-1} + \boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right)\right\}. \tag{115}
$$

Therefore,

$$
\begin{aligned}
\nabla \ell({}_{\epsilon}\hat{\boldsymbol{\beta}}^t) &= \boldsymbol{X}^\top\left\{\boldsymbol{y} - \zeta'(\boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t)\right\} \\
&= \boldsymbol{X}^\top\{\boldsymbol{y} - {}_{\epsilon}\boldsymbol{y}\} \\
&+ \boldsymbol{X}^\top\left\{{}_{\epsilon}\boldsymbol{y} - \zeta'\left({}_{\epsilon}\boldsymbol{S}^t - {}_{\epsilon}\boldsymbol{S}^{t-1} - \boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right)\right\} \\
&+ \boldsymbol{X}^\top\left\{\zeta'\left({}_{\epsilon}\boldsymbol{S}^t - {}_{\epsilon}\boldsymbol{S}^{t-1} - \boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right) - \zeta'\left(\boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right)\right\}.
\end{aligned}
\tag{116}
$$

almost surely. Further, using that $\zeta'(x)$ is Lipschitz with Lipschitz constant $1/4$,

$$
\begin{aligned}
\left\|\boldsymbol{X}^\top\left\{\zeta'\left({}_{\epsilon}\boldsymbol{S}^t - {}_{\epsilon}\boldsymbol{S}^{t-1} - \boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right) - \zeta'\left(\boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right)\right\}\right\|_2 &\leq |||\boldsymbol{X}|||_2\left\|\zeta'\left({}_{\epsilon}\boldsymbol{S}^t - {}_{\epsilon}\boldsymbol{S}^{t-1} - \boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right) - \zeta'\left(\boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right)\right\|_2 \\
&\leq \frac{1}{4}|||\boldsymbol{X}|||_2\left\|{}_{\epsilon}\boldsymbol{S}^t - {}_{\epsilon}\boldsymbol{S}^{t-1} + \boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t - \boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right\|_2 \\
&= \frac{1}{4}|||\boldsymbol{X}|||_2\left\|{}_{\epsilon}\boldsymbol{S}^t - {}_{\epsilon}\boldsymbol{S}^{t-1}\right\|_2
\end{aligned}
\tag{117}
$$

eq:res1

Using (52), it immediately follows that

$$
\left\|\boldsymbol{X}^\top\left\{{}_{\epsilon}\boldsymbol{y} - \zeta'\left({}_{\epsilon}\boldsymbol{S}^t - {}_{\epsilon}\boldsymbol{S}^{t-1} - \boldsymbol{X}{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right)\right\}\right\|_2 = \frac{\kappa}{{}_{\epsilon}b^t}\left\|{}_{\epsilon}\hat{\boldsymbol{\beta}}^t - {}_{\epsilon}\hat{\boldsymbol{\beta}}^{t-1}\right\|_2. \tag{118}
$$

eq:res2

Therefore, with probability at least $1 - c_1 \exp\{-c_2 n\}$,

$$
\left\|\hat{\boldsymbol{\beta}}^{\text{DY}} - {}_{\epsilon}\hat{\boldsymbol{\beta}}^t\right\|_2 \leq \frac{2}{\omega\left(\max\left\{\frac{\|\hat{\boldsymbol{\beta}}^{\text{DY}}\|_2}{\sqrt{n}}, \frac{\|{}_{\epsilon}\hat{\boldsymbol{\beta}}^t\|_2}{\sqrt{n}}\right\}\right)}\left\{\frac{\kappa}{{}_{\epsilon}b^t}\left\|{}_{\epsilon}\hat{\boldsymbol{\beta}}^t - {}_{\epsilon}\hat{\boldsymbol{\beta}}^{t-1}\right\|_2 + \frac{1}{4}|||\boldsymbol{X}|||_2\left\|{}_{\epsilon}\boldsymbol{S}^t - {}_{\epsilon}\boldsymbol{S}^{t-1}\right\|_2 + |||\boldsymbol{X}|||_2\left\|\boldsymbol{y} - {}_{\epsilon}\boldsymbol{y}\right\|_2\right\}. \tag{119}
$$

eq:big_ineq

To finish the proof initialize the AMP recursion at the stationary solution, i.e. $({}_{\epsilon}\mu_0, {}_{\epsilon}b_0, {}_{\epsilon}\sigma_0) = ({}_{\epsilon}\mu_*, {}_{\epsilon}b_*, {}_{\epsilon}\sigma_*)$. Then it follows immediately from (85) with $\psi(x,y) = x^2$, in conjunction with

Lemma B.4 and the stationarity condition that almost surely

$$\lim_{N\to\infty}\frac{\left\|{}_\epsilon\hat{\boldsymbol{\beta}}^t\right\|_2}{\sqrt{p}} \leq \lim_{N\to\infty}\frac{\left\|{}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_0\right\|_2}{\sqrt{p}} + \lim_{n\to\infty}\frac{\|{}_\epsilon\mu_*\boldsymbol{\beta}_0\|_2}{\sqrt{p}} = {}_\epsilon\sigma_* + \frac{\gamma}{{}_\epsilon\mu_*\kappa}\,. \quad \texttt{eq:bound\_AMP} \tag{120}$$

Further, $\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}}\|_2/\sqrt{n} = \mathcal{O}(1)$ almost surely by assumption. Hence, by the Cauchy property of ${}_\epsilon\boldsymbol{\nu}^t, {}_\epsilon\boldsymbol{R}^t$, as shown in Lemma G.8 along with the equivalence of recursions of Lemma B.4, (120) and Lemma G.4 and Lemma G.5, it follows that with probability at least $1 - c_1\exp\{-c_n\}$

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{\sqrt{p}}\left\|{}_\epsilon\hat{\boldsymbol{\beta}}^t - \hat{\boldsymbol{\beta}}^{\mathrm{DY}}\right\|_2 = 0\,. \tag{121}$$

The almost sure claim comes from the first Borel-Cantelli Lemma (e.g. Shiryaev 2016, p. 308). □

It is now possible to finish the proof of Theorem 3.1. First of all, note that by Lemma G.6, there exists a unique stationary solution $({}_\epsilon\mu_*, {}_\epsilon b_*, {}_\epsilon\sigma_*)$ to the state evolution (56)-(57) that is continuous in ϵ such that $\lim_{\epsilon\to 0}({}_\epsilon\mu_*, {}_\epsilon b_*, {}_\epsilon\sigma_*) = (\mu_*, b_*, \sigma_*)$, where (μ_*, b_*, σ_*) are a stationary solution to (22)-(24).

Now, let ψ be a pseudo Lipschitz function of order 2. Then

$$\begin{aligned}
&\left|\frac{1}{p}\sum_{j=1}^p\psi\left(\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}} - \mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right) - \frac{1}{p}\sum_{j=1}^p\psi\left({}_\epsilon\boldsymbol{\nu}^t, \boldsymbol{\beta}_{0,j}\right)\right| \\
&\qquad\leq \left|\frac{1}{p}\sum_{j=1}^p\psi\left(\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}} - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right) - \frac{1}{p}\sum_{j=1}^p\psi\left({}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right)\right| \\
&\qquad\qquad + \left|\frac{1}{p}\sum_{j=1}^p\psi\left({}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right) - \frac{1}{p}\sum_{j=1}^p\psi\left({}_\epsilon\boldsymbol{\nu}^t, \boldsymbol{\beta}_{0,j}\right)\right|. \quad \texttt{eq:thm_base_ineq}
\end{aligned} \tag{122}$$

Now

$$\begin{aligned}
&\sum_{j=1}^p\left|\psi\left(\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}} - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right) - \psi\left({}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right)\right| \\
&\qquad\leq \sum_{j=1}^p\left|\psi\left(\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}} - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right) - \psi\left({}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right)\right| \\
&\qquad\leq C\sum_{j=1}^p\left(1 + \left\|\begin{bmatrix}\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}} - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j} \\ \boldsymbol{\beta}_{0,j}\end{bmatrix}\right\|_2 + \left\|\begin{bmatrix}{}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j} \\ \boldsymbol{\beta}_{0,j}\end{bmatrix}\right\|_2\right)\left\|\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}} - {}_\epsilon\hat{\boldsymbol{\beta}}_j^t\right\|_2 \\
&\qquad\leq C\sqrt{\sum_{j=1}^p\left(1 + \left\|\begin{bmatrix}\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}} - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j} \\ \boldsymbol{\beta}_{0,j}\end{bmatrix}\right\|_2 + \left\|\begin{bmatrix}{}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j} \\ \boldsymbol{\beta}_{0,j}\end{bmatrix}\right\|_2\right)^2}\left\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}} - {}_\epsilon\hat{\boldsymbol{\beta}}^t\right\|_2, \quad \texttt{eq:pseudo_l_bound}
\end{aligned} \tag{123}$$

where the first inequality follows from the triangle inequality, the second by virtue of ψ being pseudo Lipschitz of order 2 and the last by the Cauchy-Schwarz inequality.

Further, letting $\boldsymbol{a}_j = [a_{j1}, a_{j2}]^\top = [\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}} - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}]^\top$ and $\boldsymbol{b}_j = [b_{j1}, a_{j2}]^\top = [\hat{\boldsymbol{\beta}}_j^t - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}]^\top$, simple manipulations yield

$$
\begin{aligned}
\sum_{j=1}^{p}\left(1+\|\boldsymbol{a}_j\|_2+\|\boldsymbol{b}_j\|_2\right)^2 &= \sum_{j=1}^{p} 1+2\left(\|\boldsymbol{a}_j\|_2+\|\boldsymbol{b}_j\|_2\right)+\left(\|\boldsymbol{a}_j\|_2+\|\boldsymbol{b}_j\|_2\right)^2 \\
&= p+2\sum_{j=1}^{p}\|\boldsymbol{a}_j\|_2+2\sum_{j=1}^{p}\|\boldsymbol{b}_j\|_2+\sum_{j=1}^{p}\|\boldsymbol{a}_j\|_2^2+\sum_{j=1}^{p}\|\boldsymbol{b}_j\|_2^2+2\sum_{j=1}^{p}\|\boldsymbol{a}_j\|_2\|\boldsymbol{b}_j\|_2 \\
&= p+\|\boldsymbol{a}_{\bullet 1}\|_2^2+\|\boldsymbol{b}_{\bullet 1}\|_2^2+2\|\boldsymbol{a}_{\bullet 2}\|_2^2+2\sum_{j=1}^{p}\|\boldsymbol{a}_j\|_2+2\sum_{j=1}^{p}\|\boldsymbol{b}_j\|_2+2\sum_{j=1}^{p}\|\boldsymbol{a}_j\|_2\|\boldsymbol{b}_j\|_2 \\
&\le p+\|\boldsymbol{a}_{\bullet 1}\|_2^2+\|\boldsymbol{b}_{\bullet 1}\|_2^2+2\|\boldsymbol{a}_{\bullet 2}\|_2^2+2\sum_{j=1}^{p}|a_{j1}|+2\sum_{j=1}^{p}|b_{j1}|+4\sum_{j=1}^{p}|a_{j2}|+2\sum_{j=1}^{p}\|\boldsymbol{a}_j\|_2\|\boldsymbol{b}_j\|_2 \\
&= p+\|\boldsymbol{a}_{\bullet 1}\|_2^2+\|\boldsymbol{b}_{\bullet 1}\|_2^2+2\|\boldsymbol{a}_{\bullet 2}\|_2^2+2\|\boldsymbol{a}_{\bullet 1}\|_1+2\|\boldsymbol{b}_{\bullet 1}\|_1+4\|\boldsymbol{a}_{\bullet 2}\|_1+2\sum_{j=1}^{p}\|\boldsymbol{a}_j\|_2\|\boldsymbol{b}_j\|_2 \\
&\le p+\|\boldsymbol{a}_{\bullet 1}\|_2^2+\|\boldsymbol{b}_{\bullet 1}\|_2^2+2\|\boldsymbol{a}_{\bullet 2}\|_2^2+2\sqrt{p}\|\boldsymbol{a}_{\bullet 1}\|_2+2\sqrt{p}\|\boldsymbol{b}_{\bullet 1}\|_2+4\sqrt{p}\|\boldsymbol{a}_{\bullet 2}\|_2+2\sum_{j=1}^{p}\|\boldsymbol{a}_j\|_2\|\boldsymbol{b}_j\|_2 \\
&\le p+\|\boldsymbol{a}_{\bullet 1}\|_2^2+\|\boldsymbol{b}_{\bullet 1}\|_2^2+2\|\boldsymbol{a}_{\bullet 2}\|_2^2+2\sqrt{p}\|\boldsymbol{a}_{\bullet 1}\|_2+2\sqrt{p}\|\boldsymbol{b}_{\bullet 1}\|_2+4\sqrt{p}\|\boldsymbol{a}_{\bullet 2}\|_2 \\
&\quad +2\sqrt{\sum_{j=1}^{p}a_{j1}^2+a_{j2}^2}\sqrt{\sum_{j=1}^{p}b_{j1}^2+a_{j2}^2} \\
&\le p+\|\boldsymbol{a}_{\bullet 1}\|_2^2+\|\boldsymbol{b}_{\bullet 1}\|_2^2+2\|\boldsymbol{a}_{\bullet 2}\|_2^2+2\sqrt{p}\|\boldsymbol{a}_{\bullet 1}\|_2+2\sqrt{p}\|\boldsymbol{b}_{\bullet 1}\|_2+4\sqrt{p}\|\boldsymbol{a}_{\bullet 2}\|_2 \\
&\quad +2(\|\boldsymbol{a}_{\bullet 1}\|_2+\|\boldsymbol{a}_{\bullet 2}\|_2)(\|\boldsymbol{b}_{\bullet 1}\|_2+\|\boldsymbol{a}_{\bullet 2}\|_2)\,,
\end{aligned}
\tag{124}
$$

eq:many_ineqs

where $\boldsymbol{a}_{\bullet i}=(a_{1i},\ldots,a_{pi})$, $(i=1,2)$ and a similar convention applies to $\boldsymbol{b}_{\bullet 1}$. The fourth line follows since $\sqrt{x^2+y^2}\le\sqrt{x^2}+\sqrt{y^2}=|x|+|y|$, the sixth by the Cauchy-Schwarz inequality and since $\|\boldsymbol{v}\|_1=|\langle\boldsymbol{1},|\boldsymbol{v}|\rangle|$, where $\boldsymbol{1}$ is a p-vector of ones and the absolute value acts elementwise on $\boldsymbol{v}$. The seventh line follows again by Cauchy-Schwarz, and the last again since $\sqrt{x+y}\le\sqrt{x}+\sqrt{y}$. Now, since ${}_{\epsilon}\mu_*<\infty$, and by assumption on $\hat{\boldsymbol{\beta}}^{\mathrm{DY}},\boldsymbol{\beta}_0$, almost surely,

$$
\begin{aligned}
&\lim_{N\to\infty}\frac{1}{\sqrt{p}}\|\boldsymbol{a}_{\bullet 1}\|_2=\lim_{N\to\infty}\frac{1}{\sqrt{p}}\left\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}}-{}_{\epsilon}\mu_*\boldsymbol{\beta}_0\right\|_2\le\lim_{N\to\infty}\frac{1}{\sqrt{p}}\left\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}}\right\|+{}_{\epsilon}\mu_*\frac{1}{\sqrt{p}}\|\boldsymbol{\beta}_0\|_2\quad<C+{}_{\epsilon}\mu_*\gamma \\
&\lim_{N\to\infty}\frac{1}{\sqrt{p}}\|\boldsymbol{a}_{\bullet 2}\|_2=\lim_{N\to\infty}\frac{1}{\sqrt{p}}\|\boldsymbol{\beta}_0\|_2=\gamma\,,
\end{aligned}
\tag{125}
$$

for some constant $C>0$, which is indpendent of t,ϵ. Further, by (120), almost surely

$$
\lim_{N\to\infty}\frac{1}{\sqrt{p}}\|\boldsymbol{b}_{\bullet 1}\|_2=\lim_{N\to\infty}\frac{1}{\sqrt{p}}\left\|\hat{\boldsymbol{\beta}}^{t}-{}_{\epsilon}\mu_*\boldsymbol{\beta}_0\right\|_2={}_{\epsilon}\sigma_*^2\,. \tag{126}
$$

Therefore, (124) gives that almost surely

$$
\lim_{N\to\infty}\frac{1}{p}\sum_{j=1}^{p}\left(1+\|\boldsymbol{a}_j\|_2+\|\boldsymbol{b}_j\|_2\right)^2\le 1+(C+{}_{\epsilon}\mu_*\gamma)\left\{2+C+{}_{\epsilon}\mu_*\right\}+{}_{\epsilon}\sigma_*^2\left\{2+{}_{\epsilon}\sigma_*^2\right\}+4\gamma+2\left\{C+{}_{\epsilon}\mu_*\gamma+\gamma\right\}\left\{{}_{\epsilon}\sigma_*^2+\gamma\right\}\,. \tag{127}
$$

eq:pseudo_l_bound2

And thus

$$
\lim_{\epsilon\to 0}\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{p}\sum_{j=1}^{p}\left(1+\|\boldsymbol{a}_j\|_2+\|\boldsymbol{b}_j\|_2\right)^2\le 1+(C+\mu_*\gamma)\left\{2+C+\mu_*\right\}+\sigma_*^2\left\{2+\sigma_*^2\right\}+4\gamma+2\left\{C+\mu_*\gamma+\gamma\right\}\left\{\sigma_*^2+\gamma\right\}\,, \tag{128}
$$

where $\lim_{\epsilon\to 0}({}_\epsilon\mu_*, {}_\epsilon\sigma_*) = (\mu_*, \sigma_*)$ follows from Lemma G.6. In conjunction with Theorem B.3 and (123), this yields that almost surely

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{p}\sum_{j=1}^{p}\left|\psi\left(\hat{\boldsymbol{\beta}}^{\mathrm{DY}}_j - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right) - \psi\left({}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right)\right| = 0\,. \tag{129}$$

Similarly,

$$\begin{aligned}&\left|\frac{1}{p}\sum_{j=1}^{p}\psi\left({}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right) - \psi\left({}_\epsilon\boldsymbol{\nu}^t, \boldsymbol{\beta}_{0,j}\right)\right| \\ &\qquad\qquad \leq C\frac{1}{p}\sqrt{\sum_{j=1}^{p}\left(1 + \left\|\begin{bmatrix}{}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}\\ \boldsymbol{\beta}_{0,j}\end{bmatrix}\right\|_2 + \left\|\begin{bmatrix}{}_\epsilon\boldsymbol{\nu}^t\\ \boldsymbol{\beta}_{0,j}\end{bmatrix}\right\|_2\right)^2}\left\|{}_\epsilon\hat{\boldsymbol{\beta}}^t - {}_\epsilon\mu_*\boldsymbol{\beta}_0 - {}_\epsilon\boldsymbol{\nu}^t\right\|_2\,,\end{aligned} \tag{130}$$

the right hand side of which is zero almost surely as $N\to\infty$ for all $t, \epsilon > 0$, by Lemma B.4 and (127). and thus almost surely

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\lim_{N\to\infty}\left|\frac{1}{p}\sum_{j=1}^{p}\psi\left(\hat{\boldsymbol{\beta}}^{\mathrm{DY}}_j - \mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right) - \psi\left({}_\epsilon\hat{\boldsymbol{\beta}}^{\mathrm{DY}}_j - {}_\epsilon\mu_*\boldsymbol{\beta}_{0,j}, \boldsymbol{\beta}_{0,j}\right)\right| = 0\,. \tag{131}$$

Now recall that from Corollary B.2.1, and by the stationarity assumption of the phase transition parameters, $({}_\epsilon\mu_*, {}_\epsilon b_*, {}_\epsilon\sigma_*)$, almost surely

$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{p}\sum_{j=1}^{p}\psi({}_\epsilon\boldsymbol{\nu}^t_j, \boldsymbol{\beta}_{0,j}) = \lim_{N\to\infty}\frac{1}{p}\sum_{j=1}^{p}\psi({}_\epsilon\boldsymbol{\nu}^t_j, \boldsymbol{\beta}_{0,j}) = \mathrm{E}\left[\psi({}_\epsilon\sigma_* G, \bar{\beta})\right]\,, \tag{132}$$

where $G\sim N(0,1)\perp\bar{\beta}$. Since ψ is pseudo-Lipschitz of order two,

$$\begin{aligned}\left|\mathrm{E}\left[\psi({}_\epsilon\sigma_* G, \bar{\beta})\right] - \mathrm{E}\left[\psi(\sigma_* G, \bar{\beta})\right]\right| &\leq \mathrm{E}\left[\left|\psi({}_\epsilon\sigma_* G, \bar{\beta}) - \psi(\sigma_* G, \bar{\beta})\right|\right] \\ &\leq C\left|{}_\epsilon\sigma_* - \sigma_*\right|\mathrm{E}\left[\left(1 + \left\|[{}_\epsilon\sigma_* G, \bar{\beta}]^\top\right\|_2 + \left\|[\sigma_* G, \bar{\beta}]^\top\right\|_2\right)|G|\right]\,,\end{aligned} \tag{133}$$

and thus it follows that

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{p}\sum_{j=1}^{p}\psi\left({}_\epsilon\boldsymbol{\nu}^t_j, \boldsymbol{\beta}_{0,j}\right) = \mathrm{E}\left[\psi(\sigma_* G, \bar{\beta})\right]\,. \tag{134}$$

In conjunction with (122), this establishes (7). Finally, equivalence of stationary solutions (4)-(6) with the stationary solutions to (22)-(24) is shown in Lemma G.9. This concludes the proof of the Theorem.

C Proof of Theorem 3.2

thm:boundedness_betady

Theorem 3.2. *In the setting of Theorem 3.1, and if (κ, γ) are such that the ML estimator exists with probability approaching one, then almost surely*

eq:boundedness

$$\lim_{n\to\infty}\frac{\left\|\hat{\boldsymbol{\beta}}^{DY}\right\|_2}{\sqrt{n}} = \mathcal{O}(1)\,. \tag{135}$$

Further, for any pair (κ, γ), there exists an $\alpha_0 \in (0,1)$ such that for all $0 < \alpha < \alpha_0$, (135) holds almost surely for the corresponding mDYPL estimator, $\hat{\boldsymbol{\beta}}^{DY}(\alpha)$, with shrinkage parameter α.

Both parts of Theorem 3.2 are proofed in turn.

thm:boundedness_betady_partial

Theorem C.1. *Under the modelling assumption of Section A, and if* $\gamma < g_{MLE}(\kappa)$*, where* g_{MLE} *is the phase transition of Candès and Sur (2020), then almost surely*

$$\lim_{n\to\infty} \left\|\hat{\boldsymbol{\beta}}^{DY}\right\|_2 = \mathcal{O}(\sqrt{n}) \,. \tag{136}$$

Proof. The proof relies on the arguments in the proof of Theorem 4 in the supplementary material of Sur and Candès (2019). All that needs to be done is to show that the DY-prior penalty provides some slackness in the string of inequalities that guarantee boundedness of the estimator.

Write the negative log-likelihood with pseudo-responses $\boldsymbol{y}^*$ as

$$-\log \ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X}) = \alpha \sum_{i=1}^{n} \zeta\left(-\tilde{y}_i \boldsymbol{x}_i^\top \boldsymbol{\beta}\right) + (1-\alpha)\left[\sum_{i=1}^{n} -\frac{1}{2}\boldsymbol{x}_i^\top \boldsymbol{\beta} + \zeta(\boldsymbol{x}_i^\top \boldsymbol{\beta})\right] , \tag{137}$$

where $\tilde{y}_i = 2y_i - 1$ and $\zeta(x) = \log(1 + e^x)$. Assume that

i) Given a constant $c > 0$, for $\mathcal{B} = \bigcap_{n>N_0}\{s_{\min}(\boldsymbol{X}) > c\}$ be the event that there exists a N_0 such that for all $n > N_0$, $s_{\min}(\boldsymbol{X}) > c$ it:smin

ii) $-\ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X}) \leq -\ell(\boldsymbol{0}_p; \boldsymbol{y}^*, \boldsymbol{X}) = n\log(2)$, it:beta_min

iii) $\frac{\|\boldsymbol{\beta}\|_2}{\sqrt{n}} > \frac{\log(2)}{c\epsilon^2}$ it:beta_length

Let $J = \{j \in \{1, \ldots, n\} : \tilde{y}_j x_j^\top \boldsymbol{\beta} < 0\}$. Then note that ii) implies that

$$\sum_{j=1}^{n} \max\left\{-\tilde{y}_j \boldsymbol{x}_j^\top \boldsymbol{\beta}, 0\right\} \leq n\log(2) \,. \tag{138}$$

eq:rho_ineq

Indeed, consider

$$\begin{aligned}
\alpha \sum_{j=1}^{n} \max\left\{-\tilde{y}_j \boldsymbol{x}_j^\top \boldsymbol{\beta}, 0\right\} + (1-\alpha)\left[\sum_{j=1}^{n} -\frac{1}{2}\boldsymbol{x}_j^\top \boldsymbol{\beta} + \zeta(\boldsymbol{x}_j^\top \boldsymbol{\beta})\right] &= \alpha \sum_{j\in J} -\tilde{y}_j \boldsymbol{x}_j^\top \boldsymbol{\beta} + (1-\alpha)\left[\sum_{j=1}^{n} -\frac{1}{2}\boldsymbol{x}_j^\top \boldsymbol{\beta} + \zeta(\boldsymbol{x}_j^\top \boldsymbol{\beta})\right] \\
&\leq \alpha \sum_{j\in J} \zeta\left(-\tilde{y}_j \boldsymbol{x}_j^\top \boldsymbol{\beta}\right) + (1-\alpha)\left[\sum_{j=1}^{n} -\frac{1}{2}\boldsymbol{x}_j^\top \boldsymbol{\beta} + \zeta(\boldsymbol{x}_i^\top \boldsymbol{\beta})\right] \\
&\leq \alpha \sum_{j=1}^{n} \zeta\left(-\tilde{y}_j \boldsymbol{x}_j^\top \boldsymbol{\beta}\right) + (1-\alpha)\left[\sum_{j=1}^{n} -\frac{1}{2}\boldsymbol{x}_j^\top \boldsymbol{\beta} + \zeta(\boldsymbol{x}_j^\top \boldsymbol{\beta})\right] \\
&= -\ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X}) \,,
\end{aligned} \tag{139}$$

eq:max_sum

where it was used that $x < \zeta(x)$ for $x \in \Re$. Now assume that $\boldsymbol{\beta}$ is such that $-\ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X}) \leq n\log(2) = -\ell(\boldsymbol{0}_p; \boldsymbol{y}^*, \boldsymbol{X})$. Then for such $\boldsymbol{\beta}$,

$$\alpha \sum_{j\in J} \zeta\left(-\tilde{y}_j \boldsymbol{x}_j^\top \boldsymbol{\beta}\right) \leq n\log(2) - (1-\alpha)\left[\sum_{i=1}^{n} -\frac{1}{2}\boldsymbol{x}_i^\top \boldsymbol{\beta} + \zeta(\boldsymbol{x}_i^\top \boldsymbol{\beta})\right] . \tag{140}$$

eq:alpha_zeta

Now note that $(1-\alpha)\left[\sum_{i=1}^{n} -\frac{1}{2}\boldsymbol{x}_i^\top \boldsymbol{\beta} + \zeta(\boldsymbol{x}_i^\top \boldsymbol{\beta})\right]$ is always greater than zero as $\zeta(x) > x$ for $x \in \Re$

and that this expression is minimized for $\boldsymbol{\beta} = \mathbf{0}_p$. Hence, (140) can be upper bounded as

$$\begin{aligned}
\alpha \sum_{j \in J}^{n} \zeta\left(-\tilde{y}_j \boldsymbol{x}_j^\top \boldsymbol{\beta}\right) &\leq n \log(2) - (1-\alpha) \left[\sum_{i=1}^{n} -\frac{1}{2} \boldsymbol{x}_i^\top \boldsymbol{\beta} + \zeta(\boldsymbol{x}_i^\top \boldsymbol{\beta})\right] \\
&\leq n \log(2) - (1-\alpha) \left[\sum_{i=1}^{n} -\frac{1}{2} \boldsymbol{x}_i^\top \mathbf{0}_p + \zeta(\mathbf{0}_p \boldsymbol{\beta})\right] \qquad \text{eq:alpha_zeta2} \quad (141) \\
&= n \log(2) - (1-\alpha) n \log(2) \\
&= \alpha n \log(2)\,,
\end{aligned}$$

which establishes (138) since then by definition of J and as $\zeta(x) > x$,

$$\sum_{j=1}^{n} \max\left\{-\tilde{y}_j \boldsymbol{x}_j^\top \boldsymbol{\beta}, 0\right\} = \sum_{j \in J} -\tilde{y}_j \boldsymbol{x}_j^\top \boldsymbol{\beta} \leq \sum_{j \in J}^{n} \zeta\left(-\tilde{y}_j \boldsymbol{x}_j^\top \boldsymbol{\beta}\right) \leq n \log(2)\,. \qquad \text{eq:maxsum\_full} \tag{142}$$

Next, note that conditional on event $\mathcal{B}$ from assumptions i) & iii) it follows that

$$n \log(2) \leq \frac{n \log(2)}{c} s_{\min}(\boldsymbol{X}) \leq \frac{n \log(2)}{c} \frac{\|\boldsymbol{X}\boldsymbol{\beta}\|_2}{\|\boldsymbol{\beta}\|_2} = \frac{n \log(2)}{c} \frac{\|\tilde{\boldsymbol{y}} \circ \boldsymbol{X}\boldsymbol{\beta}\|_2}{\|\boldsymbol{\beta}\|_2} \leq \epsilon^2 \sqrt{n} \left\|\tilde{\boldsymbol{y}} \circ \boldsymbol{X}\boldsymbol{\beta}\right\|_2, \qquad \text{eq:nlog2} \tag{143}$$

where the first inequality holds by event $\mathcal{B}$, the second by the Courant-Fischer characterization of eigenvalues (e.g. Magnus and Neudecker 2019, Section 1.8, Theorem 7), the next equality since $\tilde{\boldsymbol{y}} \in \{-1, 1\}$ and the definition of the Hadamard product $\circ$ and the ℓ_2-norm, and the last inequality follows by iii).

Note that by definition of $\hat{\boldsymbol{\beta}}^{\text{DY}}$, it must hold that $-\ell(\hat{\boldsymbol{\beta}}^{\text{DY}}; \boldsymbol{y}^*, \boldsymbol{X}) \leq n \log(2)$. Hence, conditional on event $\mathcal{B}$, $\|\hat{\boldsymbol{\beta}}^{\text{DY}}\|_2/\sqrt{n} > \log(2)/(c\epsilon^2)$ implies that $\tilde{\boldsymbol{y}} \circ \boldsymbol{X}\hat{\boldsymbol{\beta}}^{\text{DY}}$ must fall in the cone

$$\mathcal{A} = \left\{\boldsymbol{u} \in \Re^n : \sum_{j=1}^{n} \max\left\{-u_j, 0\right\} \leq \epsilon^2 \sqrt{n} \left\|\boldsymbol{u}\right\|_2\right\}. \qquad \text{eq:cone} \tag{144}$$

Therefore

$$\Pr\left(\frac{\left\|\hat{\boldsymbol{\beta}}^{\text{DY}}\right\|_2}{\sqrt{n}} > \frac{\log(2)}{c\epsilon^2} \middle| \mathcal{B}\right) \leq \Pr\left(\tilde{\boldsymbol{y}} \circ \boldsymbol{X}\hat{\boldsymbol{\beta}}^{\text{DY}} \in \mathcal{A} \middle| \mathcal{B}\right). \tag{145}$$

Next, note that by a straightforward application of Theorem 2.6 in Rudelson and Vershynin (2010), it follows that for any κ, there exist constants $c_1, c_2 > 0$ such that

$$\Pr\left(s_{\min}(\boldsymbol{X}) > c_1\right) \geq 1 - 2\exp\left\{-c_2 n\right\}. \tag{146}$$

Therefore, letting $\mathcal{C}_n$ be the event that $s_{\min}(\boldsymbol{X}) < c_1$ for $\boldsymbol{X} \in \Re^{n \times p}$, the first Borel-Cantelli Lemma gives that the event $\Pr(\mathcal{C}_n \text{ infinitely often}) = 0$. Hence, there must exist a N_0 such that $\Pr\left(\bigcap_{n > N_0} s_{\min}(\boldsymbol{X}) > c_1\right) = 1$. Thus, letting $\mathcal{B} = \bigcap_{n > N_0} s_{\min}(\boldsymbol{X}) > c_1$, it follows that for $n > N_0$,

$\epsilon > 0$,

$$\begin{aligned}
\Pr\left(\frac{\left\|\hat{\boldsymbol{\beta}}^{\text{DY}}\right\|_2}{\sqrt{n}} > \frac{\log(2)c_1}{\epsilon^2}\right) &= \Pr\left(\frac{\left\|\hat{\boldsymbol{\beta}}^{\text{DY}}\right\|_2}{\sqrt{n}} > \frac{\log(2)c_1}{\epsilon^2}\middle|\mathcal{B}\right)\Pr(\mathcal{B}) \\
&+ \Pr\left(\frac{\left\|\hat{\boldsymbol{\beta}}^{\text{DY}}\right\|_2}{\sqrt{n}} > \frac{\log(2)c_1}{\epsilon^2}\middle|\neg\mathcal{B}\right)(1-\Pr(\mathcal{B})) \\
&= \Pr\left(\frac{\left\|\hat{\boldsymbol{\beta}}^{\text{DY}}\right\|_2}{\sqrt{n}} > \frac{\log(2)c_1}{\epsilon^2}\middle|\mathcal{B}\right)\cdot 1 + 0 \\
&= \Pr\left(\frac{\left\|\hat{\boldsymbol{\beta}}^{\text{DY}}\right\|_2}{\sqrt{n}} > \frac{\log(2)c_1}{\epsilon^2}\middle|\mathcal{B}\right)
\end{aligned} \tag{147}$$

A similar argument shows that $\Pr\left(\tilde{\boldsymbol{y}} \circ \boldsymbol{X}\hat{\boldsymbol{\beta}}^{\text{DY}} \in \mathcal{A}|\mathcal{B}\right) = \Pr\left(\tilde{\boldsymbol{y}} \circ \boldsymbol{X}\hat{\boldsymbol{\beta}}^{\text{DY}} \in \mathcal{A}\right)$. Thus, for $n > N_0$,

$$\begin{aligned}
\Pr\left(\frac{\left\|\hat{\boldsymbol{\beta}}^{\text{DY}}\right\|_2}{\sqrt{n}} > \frac{\log(2)c_1}{\epsilon^2}\right) &\leq \Pr\left(\tilde{\boldsymbol{y}} \circ \boldsymbol{X}\hat{\boldsymbol{\beta}}^{\text{DY}} \in \mathcal{A}\right) \\
&\leq \Pr\left(\{\tilde{\boldsymbol{y}} \circ \boldsymbol{X}\boldsymbol{b} : \boldsymbol{b} \in \Re^p\} \cap \mathcal{A} \neq \{\boldsymbol{0}_n\}\right)
\end{aligned} \tag{148}$$

This event is shown to have probability at most $Cn^{-\delta}$ for some constants $C > 0, \delta > 1$ in the proof of Theorem 4 in the supplementary material of Sur and Candès (2019) (see also Sur et al. 2019, Theorem 7 for more details).

The almost sure part of the claim follows again by the first Borel-Cantelli Lemma. This concludes the proof. □

Theorem C.2. *In the setting of Section A, for any pair (κ, γ), there exists an $\alpha_0 \in (0, 1)$ such that for all $0 < \alpha < \alpha_0$,*

$$\left\|\hat{\boldsymbol{\beta}}^{DY}(\alpha)\right\|_2 = \mathcal{O}(\sqrt{n}) \tag{149}$$

almost surely for the corresponding mDYPL estimator, $\hat{\boldsymbol{\beta}}^{DY}(\alpha)$, with shrinkage parameter α.

Proof. It shall be argued that for any κ, γ, there exists a radius $r_0 = c\sqrt{n}$ for some constant $c > 0$ and a cutoff $\alpha_0 \in (0, 1)$ such that for all $\alpha < \alpha_0$, the maximizer of $\ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X})$ as defined in (3) lies in the set $S = \{\boldsymbol{\beta} \in \Re^p : \|\boldsymbol{\beta}\|_2 \leq r_0\}$ eventually almost surely. Towards this end, let $p(\boldsymbol{\beta}) = \tau \sum_{i=1}^n \frac{1}{2}\boldsymbol{x}_i^\top\boldsymbol{\beta} - \zeta(\boldsymbol{x}_i^\top\boldsymbol{\beta})$ be the log of the prior (1) with $\boldsymbol{\beta}_P = \boldsymbol{0}_p$ bar the normalizing constant and fix any $\kappa \in (0, 1)$, $\gamma > 0$. By Lemma C.1, there exist $c > 0$, $\alpha_0 \in (0, 1)$ such that for all $0 < \alpha < \alpha_0$, such that

eq:sup_pen_proof

$$\sup_{\substack{\boldsymbol{\beta}\in\Re^p: \\ \|\boldsymbol{\beta}\|_2 > c\sqrt{n}}} p(\boldsymbol{\beta}) < -(\tau + 1)n\log(2)\,, \tag{150}$$

with probability at least $1 - c_1 \exp\{-c_2 n\}$ for some universal constants c_1, c_2 and $n > n_0$. Since the log-likelihood $\ell(\boldsymbol{\beta}; \boldsymbol{y}, \boldsymbol{X})$ from a logistic regression model is always bounded from above by zero, it follows that, upon the event of (150), $\ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X}) = \ell(\boldsymbol{\beta}; \boldsymbol{y}, \boldsymbol{X}) + p(\boldsymbol{\beta}) < -(\tau + 1)n\log(2) = \ell(\boldsymbol{0}_p; \boldsymbol{y}^*, \boldsymbol{X})$. On the other hand, $\ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X})$ obtains a maximum over S since it is continuous and S is compact (e.g. Rudin et al. (1976, Theorem 4.16)). Further, since for any $\boldsymbol{\beta} \notin S$, $\ell(\boldsymbol{\beta}; \boldsymbol{y}, \boldsymbol{X}) < \ell(\boldsymbol{0}_p; \boldsymbol{y}, \boldsymbol{X})$ and $\boldsymbol{0}_p \in S$, the maximizer over S is the global maximizer of $\ell(\boldsymbol{\beta}; \boldsymbol{y}, \boldsymbol{X})$, which thus must be bounded by $c\sqrt{n}$ with probability at least $1 - c_1 \exp\{-c_2 n\}$. The almost sure part of the statement follows by the Borel-Cantelli Lemma. This concludes the proof. ☐

lemma:bound_prior

Lemma C.1. *Let* $p(\boldsymbol{\beta}) = \tau \sum_{i=1}^{n} \frac{1}{2}\boldsymbol{x}_i^\top \boldsymbol{\beta} - \zeta(\boldsymbol{x}_i^\top \boldsymbol{\beta})$ *be the log of the prior (1) with* $\boldsymbol{\beta}_P = \mathbf{0}_p$ *bar the normalizing constant. Then for any* $\kappa \in (0,1)$, $\gamma > 0$, *there exist* $c > 0$, $\alpha_0 \in (0,1)$, n_0, *such that for all* $0 < \alpha < \alpha$, $n > n_0$

eq:sup_pen

$$\sup_{\substack{\boldsymbol{\beta}\in\Re^p:\\ \|\boldsymbol{\beta}\|_2 > c\sqrt{n}}} p(\boldsymbol{\beta}) < -(\tau+1)n\log(2)\,, \tag{151}$$

with probability at least $1 - c_1 \exp\{-c_2 n\}$ *for some universal constants* c_1, c_2.

Proof. First write $p(\boldsymbol{\beta}) = p(r\boldsymbol{u})$ for $r = \|\boldsymbol{\beta}\|_2, \boldsymbol{u} \in S^{p-1}$ and note that $p(r\boldsymbol{u})$ is monotonically decreasing in r. Indeed,

$$\frac{\partial}{\partial r} p(r\boldsymbol{u}) = \sum_{i=1}^{n} \boldsymbol{x}_i^\top \boldsymbol{u} \left(\frac{1}{2} - \zeta'(r\boldsymbol{x}_i^\top \boldsymbol{u})\right)\,, \tag{152}$$

and the statement is trivial for $r = 0$. Hence assume $r > 0$. If $\boldsymbol{x}_i^\top \boldsymbol{u} > 0$, then by elementary properties of $\zeta(\cdot)$, $\zeta'(r\boldsymbol{x}_i^\top \boldsymbol{u}) > 1/2$. On the other hand if $\boldsymbol{x}_i^\top \boldsymbol{u} < 0$, $\zeta'(r\boldsymbol{x}_i^\top \boldsymbol{u}) < 1/2$ and if $\boldsymbol{x}_i^\top \boldsymbol{u} = 0$, $\zeta'(r\boldsymbol{x}_i^\top \boldsymbol{u}) = 1/2$. Thus, for $r > 0$, $\frac{\partial}{\partial r} p(r\boldsymbol{u}) < 0$ except on a set of measure zero, on which it is ≤ 0. Hence it is sufficient to establish that there exists a constant $c > 0$, such that for $r = c\sqrt{n}$

eq:sup_pen_relaxed

$$\sup_{\boldsymbol{u}\in S^{p-1}} p(r\boldsymbol{u}) < -(\tau+1)n\log(2)\,, \tag{153}$$

with sufficiently high probability. A second order Taylor expansion of $\zeta(r\boldsymbol{x}_i^\top \boldsymbol{u})$ around 0 gives

eq:taylor_zeta

$$\begin{aligned}\zeta(r\boldsymbol{x}_i^\top \boldsymbol{u}) &= \zeta(0) + \zeta'(0) r\boldsymbol{x}_i^\top \boldsymbol{u} + \frac{1}{2}\zeta''(\gamma_i)(r\boldsymbol{x}_i^\top \boldsymbol{u})^2 \\ &= \log(2) + \frac{1}{2} r\boldsymbol{x}_i^\top \boldsymbol{u} + \frac{1}{2}\zeta''(\gamma_i)(r\boldsymbol{x}_i^\top \boldsymbol{u})^2\,,\end{aligned} \tag{154}$$

where γ_i lies between 0 and $r\boldsymbol{x}_i^\top \boldsymbol{u}$. Hence,

eq:pen_taylor

$$\begin{aligned}p(r\boldsymbol{u}) &= \tau \sum_{i=1}^{n} \frac{1}{2} r\boldsymbol{x}_i^\top \boldsymbol{u} - \zeta(r\boldsymbol{x}_i^\top \boldsymbol{u}) \\ &= \tau \sum_{i=1}^{n} \frac{1}{2} r\boldsymbol{x}_i^\top \boldsymbol{u} - \log(2) - \frac{1}{2} r\boldsymbol{x}_i^\top \boldsymbol{u} - \frac{1}{2}\zeta''(\gamma_i)(r\boldsymbol{x}_i^\top \boldsymbol{u})^2 \\ &= -\tau n \log(2) - \frac{\tau}{2}\zeta''(\gamma_i)(r\boldsymbol{x}_i^\top \boldsymbol{u})^2 \\ &\leq -\tau n \log(2) - \frac{\tau}{2}\sum_{i=1}^{n} \zeta''(r\boldsymbol{x}_i^\top \boldsymbol{u})(r\boldsymbol{x}_i^\top \boldsymbol{u})^2 \\ &= -\tau n \log(2) - \frac{\tau}{2}(r\boldsymbol{u})^\top \boldsymbol{X}^\top \boldsymbol{W}\{r\boldsymbol{u}\}\boldsymbol{X}(r\boldsymbol{u})\,,\end{aligned} \tag{155}$$

where $\boldsymbol{W}\{r\boldsymbol{u}\}$ is a diagonal matrix with $[\boldsymbol{W}\{r\boldsymbol{u}\}]_{ii} = \zeta''(r\boldsymbol{x}_i^\top \boldsymbol{u})$. The inequality follows since $\zeta''(\cdot)$ is positive, symmetric around zero $\zeta''(x) = \zeta''(-x)$, and strictly decreasing away from zero so that $|x| > |y|$ implies $\zeta''(y) > \zeta''(x)$. Since γ_i lies between 0 and $r\boldsymbol{x}_i^\top \boldsymbol{u}$, $|\gamma_i| \leq |r\boldsymbol{x}_i^\top \boldsymbol{u}|$ and

$\zeta''(\gamma_i) > \zeta''(r\boldsymbol{x}_i^\top \boldsymbol{u})$ as required. Hence,

$$
\begin{aligned}
\sup_{\boldsymbol{u}\in S^{p-1}} p(r\boldsymbol{u}) &\leq -\tau n \log(2) + \frac{\tau}{2} \sup_{\boldsymbol{u}\in S^{p-1}} -(r\boldsymbol{u})^\top \boldsymbol{X}^\top \boldsymbol{W}\{r\boldsymbol{u}\}\boldsymbol{X}(r\boldsymbol{u}) \\
&= -\tau n \log(2) - \frac{\tau}{2} \inf_{\boldsymbol{u}\in S^{p-1}} (r\boldsymbol{u})^\top \boldsymbol{X}^\top \boldsymbol{W}\{r\boldsymbol{u}\}\boldsymbol{X}(r\boldsymbol{u}) \\
&\leq -\tau n \log(2) - \frac{\tau}{2} \inf_{\boldsymbol{v}\in S^{p-1}} \inf_{\boldsymbol{u}\in S^{p-1}} (r\boldsymbol{u})^\top \boldsymbol{X}^\top \boldsymbol{W}\{r\boldsymbol{v}\}\boldsymbol{X}(r\boldsymbol{u}) \\
&= -\tau n \log(2) - \frac{\tau}{2} r^2 \inf_{\boldsymbol{v}\in S^{p-1}} \lambda_{\min}\left(\boldsymbol{X}^\top \boldsymbol{W}\{r\boldsymbol{v}\}\boldsymbol{X}\right),
\end{aligned}
\tag{156}
$$

and thus, to establish (153) it is sufficient that

$$
\frac{\tau}{2} r^2 \inf_{\boldsymbol{v}\in S^{p-1}} \lambda_{\min}\left(\boldsymbol{X}^\top \boldsymbol{W}\{r\boldsymbol{v}\}\boldsymbol{X}\right) > n\log(2)\,. \tag{157}
$$

By Lemma 7 of Sur et al. (2019), there exists a $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$,

eq:inf_min_eigen

$$
\inf_{\boldsymbol{v}\in S^{p-1}} \lambda_{\min}\left(\boldsymbol{X}^\top \boldsymbol{W}\{r\boldsymbol{v}\}\boldsymbol{X}\right) \geq \zeta''\left(\frac{3r}{\sqrt{\epsilon n}}\right)\left(\sqrt{1-\epsilon} - \sqrt{\frac{p}{n}} - 2\sqrt{\frac{H(\epsilon)}{1-\epsilon}}\right)^2, \tag{158}
$$

with probability at least $1-2\exp\{-nH(\epsilon)\}-2\exp\{-n/\epsilon\}$, where $H(\epsilon) = -\epsilon\log(\epsilon)-(1-\epsilon)\log(1-\epsilon)$. Hence, since $r = c\sqrt{n}$, for $0 < \epsilon \leq \epsilon_0$, we can find constants $c_1(\epsilon,\kappa), c_2(\epsilon,\kappa), C(\epsilon,\kappa) > 0$, and $n_0(\kappa,\epsilon)$ such that for all $n > n_0(\kappa,\epsilon)$,

eq:inf_min_eigen2

$$
\inf_{\boldsymbol{v}\in S^{p-1}} \lambda_{\min}\left(\boldsymbol{X}^\top \boldsymbol{W}\{r\boldsymbol{v}\}\boldsymbol{X}\right) \geq \zeta''\left(\frac{3c}{\sqrt{\epsilon}}\right) C(\epsilon,\kappa)\,, \tag{159}
$$

with probability at least $1 - c_1(\epsilon,\kappa)\exp\{-c_2(\epsilon,\kappa)n\}$. On that event, for τ large enough

$$
\frac{\tau}{2} r^2 \inf_{\boldsymbol{v}\in S^{p-1}} \lambda_{\min}\left(\boldsymbol{X}^\top \boldsymbol{W}\{r\boldsymbol{v}\}\boldsymbol{X}\right) \geq \frac{\tau}{2} c^2 n \zeta''\left(\frac{3c}{\sqrt{\epsilon}}\right) C(\kappa,\epsilon) > n\log(2)\,. \tag{160}
$$

The claim of the Lemma follows by recalling that $\alpha = \frac{1}{\tau+1}$. □

Remark 1. Note that the proof of Lemma C.1 does not depend on the specification of γ. Hence, the threshold value α_0, which guarantees boundedness of $\hat{\boldsymbol{\beta}}^{\text{DY}}$ in ℓ_2-norm also depends solely on κ. A more refined analysis might be able to produce a concrete upper bound of the cutoff value $\alpha_0(\kappa)$.

D Proof of Theorem 3.3

Recall that the setup of Theorems 3.3-3.5 is the moderately high-dimensional setting with i.i.d. Gaussian covariates with arbitrary covariance structure. In particular, $\boldsymbol{x}_i \sim \mathcal{N}(\mathbf{0}_p, \boldsymbol{\Sigma})$ for some positive definite matrix $\boldsymbol{\Sigma}$ and signal $\boldsymbol{\beta}_0 \in \Re^p$ such that $\lim_{n\to\infty} \text{var}(\boldsymbol{x}_i^\top \boldsymbol{\beta}_0) = \boldsymbol{\beta}_0^\top \boldsymbol{\Sigma}\boldsymbol{\beta}_0 = \gamma^2$ for $\gamma \in \Re^p_{>0}$ and where $\lim_{n\to\infty} p/n \to \kappa \in (0,1)$.

thm:aggregate_asymptotics2

Theorem 3.3. *Assume that (κ,γ,α) are such that the conditions of Theorem 3.1 are met. Let $\hat{\boldsymbol{\beta}}^{DY}$ be the mDYPL from a logistic regression model with i.i.d. $\mathcal{N}(\mathbf{0}_p, \boldsymbol{\Sigma})$, with $\limsup_{n\to\infty} \lambda_{\max}(\boldsymbol{\Sigma})/\lambda_{\min}(\boldsymbol{\Sigma}) < \infty$, and signal $\boldsymbol{\beta}_0$. Then for any $t \in \Re$,*

$$\frac{1}{p}\sum_{j=1}^{p}\mathbb{1}\left\{\sqrt{n}\tau_j\frac{\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}}-\mu_*\boldsymbol{\beta}_{0,j}}{\sigma_*}\leq t\right\}\xrightarrow{p}\Phi(t)\,, \tag{161}$$

where $\Phi(\cdot)$ is the CDF of a standard normal random variable, $\mathbb{1}(\cdot)$ is the indicator function and $\tau_j = var(\boldsymbol{x}_{ij}|\boldsymbol{x}_{i-j})$ is the conditional variance of the jth predictor given all others. Further, if

$$\frac{1}{p}\sum_{j=1}^{p}\delta_{\sqrt{n}\tau_j\boldsymbol{\beta}_{0,j}}\xrightarrow{d}\pi_{\bar{\beta}}\,, \tag{162}$$

for some distribution $\pi_{\bar{\beta}}$ with finite second moment and where $\delta_{(\cdot)}$ is the Dirac delta function, and

$$\frac{1}{p}\sum_{j=1}^{p}n\tau_j^2\boldsymbol{\beta}_{0,j}^2\xrightarrow{p}E(\bar{\beta}^2)\,, \tag{163}$$

for $\bar{\beta}\sim\pi_{\bar{\beta}}$, then for bivariate any pseudo-Lipschitz function $\psi(\cdot,\cdot)$ of order 2,

$$\frac{1}{p}\sum_{j=1}^{p}\psi\left(\sqrt{n}\tau_j\left(\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}}-\mu_*\boldsymbol{\beta}_{0,j}\right),\sqrt{n}\tau_j\boldsymbol{\beta}_{0,j}\right)\xrightarrow{p}E\left[\psi(\sigma_*G,\bar{\beta})\right]\,,\quad \text{as } n\to\infty\,, \tag{164}$$

for $G\sim N(0,1)$.

The proof of Theorem 3.3 is a replication of the arguments in the proof of Theorem 3.2 in Zhao et al. (2022) with the substitutions that are outlined in F. No claim of originality is made.

The following Lemma, which is a generalization of Lemma E.2 to arbitrary covariance structures, is the starting point of the analysis. Let $\langle\boldsymbol{u},\boldsymbol{v}\rangle_{\boldsymbol{\Sigma}}=\boldsymbol{u}^\top\boldsymbol{\Sigma}\boldsymbol{v}$, and $\|\boldsymbol{u}\|_{\boldsymbol{\Sigma}}^2=\langle\boldsymbol{u},\boldsymbol{u}\rangle_{\boldsymbol{\Sigma}}$. Then write $\gamma_n^2=\|\boldsymbol{\beta}_0\|_{\boldsymbol{\Sigma}}^2=\mathrm{var}(\boldsymbol{x}_i^\top\boldsymbol{\beta}_0)$ for $\boldsymbol{x}_i\sim\mathcal{N}(\boldsymbol{0}_p,\boldsymbol{\Sigma})$.

lemma:stochastic_rep2

Lemma D.1. *Let $\hat{\boldsymbol{\beta}}^{\mathrm{DY}}$ be the mDYPL estimator from $\mathcal{N}(\boldsymbol{0}_p,\boldsymbol{\Sigma})$ covariates and signal $\boldsymbol{\beta}_0$. Define*

eq:xi

$$\boldsymbol{\xi}=\boldsymbol{Z}-\left\langle\boldsymbol{Z},\frac{\boldsymbol{\beta}_0}{\gamma_n}\right\rangle_{\boldsymbol{\Sigma}}\frac{\boldsymbol{\beta}_0}{\gamma_n}\,, \tag{165}$$

for $\boldsymbol{Z}\sim\mathcal{N}(\boldsymbol{0}_p,\boldsymbol{\Sigma}^{-1})$ independent of everything else. Further, let

$$\tilde{\mu}_n=\frac{\langle\hat{\boldsymbol{\beta}}^{\mathrm{DY}},\boldsymbol{\beta}_0\rangle_{\boldsymbol{\Sigma}}}{\|\boldsymbol{\beta}_0\|_{\boldsymbol{\Sigma}}^2},\quad \tilde{\sigma}_n^2=\left\|\hat{\boldsymbol{\beta}}^{\mathrm{DY}}\right\|_{\boldsymbol{\Sigma}}^2-\frac{\langle\hat{\boldsymbol{\beta}}^{\mathrm{DY}},\boldsymbol{\beta}_0\rangle_{\boldsymbol{\Sigma}}^2}{\|\boldsymbol{\beta}_0\|_{\boldsymbol{\Sigma}}^2}\,. \tag{166}$$

Then

$$\frac{\hat{\boldsymbol{\beta}}^{\mathrm{DY}}-\tilde{\mu}_n\boldsymbol{\beta}_0}{\tilde{\sigma}_n}\overset{d}{=}\frac{\boldsymbol{\xi}}{\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}}\,. \tag{167}$$

> *Proof.* The proof is similar to the proof of Proposition B.1 in the proof of Theorem 3.2 of Zhao et al. (2022) and thus omitted. □

Lemma D.1 immediately yields the equivalence (172) below. Define $\boldsymbol{T}^{\mathrm{approx}}$ as the p-vector with elements

$$\boldsymbol{T}_j^{\mathrm{approx}}=\tau_j\sqrt{n}\frac{\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}}-\tilde{\mu}_n\boldsymbol{\beta}_{0,j}}{\tilde{\sigma}_n/\sqrt{\kappa}}\,, \tag{168}$$

which ought to approximate

$$\boldsymbol{T}_j=\tau_j\sqrt{n}\frac{\hat{\boldsymbol{\beta}}_j^{\mathrm{DY}}-\mu_*\boldsymbol{\beta}_{0,j}}{\sigma_*}\,. \tag{169}$$

Further, let $\boldsymbol{Z}_j^{\text{scaled}} = \tau_j \boldsymbol{Z}_j$, so that

$$\boldsymbol{Z}^{\text{scaled}} \sim \mathcal{N}(\mathbf{0}_p, \boldsymbol{R}), \quad \boldsymbol{R}_{ij} = \tau_i \tau_j [\boldsymbol{\Sigma}^{-1}]_{ij}\,, \tag{170}$$

and finally let $\tilde{\boldsymbol{Z}}^{\text{scaled}}$ be the p-vector with entries

$$\tilde{\boldsymbol{Z}}_j^{\text{scaled}} = \left(\boldsymbol{Z}_j^{\text{scaled}} - \left\langle \boldsymbol{Z}, \frac{\boldsymbol{\beta}_0}{\gamma_n} \right\rangle_{\boldsymbol{\Sigma}} \frac{\tau_j \boldsymbol{\beta}_{0,j}}{\gamma_n} \right) \frac{\sqrt{p}}{\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}}\,. \tag{171}$$

Then

eq:approx_equiv

$$\boldsymbol{T}^{\text{approx}} \overset{d}{=} \tilde{\boldsymbol{Z}}^{\text{scaled}}\,. \tag{172}$$

The remainder of the proof is concerned with showing that $\boldsymbol{T} \approx \boldsymbol{T}^{\text{approx}}$, $\tilde{\boldsymbol{Z}}^{\text{scaled}} \approx \boldsymbol{Z}^{\text{scaled}}$ in an adequate sense and that the empirical cdf of $\boldsymbol{Z}^{\text{scaled}}$ converges to that of a standard normal.

lemma:close_approx

Lemma D.2. *If $\limsup_{n\to\infty} \lambda_{\max}(\boldsymbol{\Sigma})/\lambda_{\min}(\boldsymbol{\Sigma}) < \infty$, then for any univariate Lipschitz function ψ,*

$$\frac{1}{p}\sum_{j=1}^{p} \left\{ \psi(\tilde{\boldsymbol{Z}}_j^{\text{scaled}}) - \psi(\boldsymbol{Z}_j^{\text{scaled}}) \right\} \xrightarrow{p} 0, \quad \frac{1}{p}\sum_{j=1}^{p} \left\{ \psi(\boldsymbol{T}_j^{\text{approx}}) - \psi(\boldsymbol{T}_j) \right\} \xrightarrow{p} 0\,. \tag{173}$$

Proof. First consider the pair $\tilde{\boldsymbol{Z}}^{\text{scaled}}, \boldsymbol{Z}^{\text{scaled}}$ and let L be the Lipschitz constant of ψ. By the triangle inequality and the definition of $\tilde{\boldsymbol{Z}}^{\text{scaled}}, \boldsymbol{Z}^{\text{scaled}}$,

eq:z_psi

$$\begin{aligned}
\left| \frac{1}{p}\sum_{j=1}^{p} \left\{ \psi(\tilde{\boldsymbol{Z}}_j^{\text{scaled}}) - \psi(\boldsymbol{Z}_j^{\text{scaled}}) \right\} \right| &\le \frac{1}{p}\sum_{j=1}^{p} \left| \psi(\tilde{\boldsymbol{Z}}_j^{\text{scaled}}) - \psi(\boldsymbol{Z}_j^{\text{scaled}}) \right| \\
&\le L \frac{1}{p}\sum_{j=1}^{p} \left| \boldsymbol{Z}_j^{\text{scaled}} \left(1 - \frac{\sqrt{p}}{\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}}\right) + \left\langle \boldsymbol{Z}, \frac{\boldsymbol{\beta}_0}{\gamma_n} \right\rangle_{\boldsymbol{\Sigma}} \frac{\tau_j \boldsymbol{\beta}_{0,j}}{\gamma_n} \frac{\sqrt{p}}{\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}} \right| \\
&\le L \left| 1 - \frac{\sqrt{p}}{\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}} \right| \frac{1}{p}\sum_{j=1}^{p} \left| \boldsymbol{Z}_j^{\text{scaled}} \right| + \left| \frac{1}{\gamma_n} \left\langle \boldsymbol{Z}, \frac{\boldsymbol{\beta}_0}{\gamma_n} \right\rangle_{\boldsymbol{\Sigma}} \right| \frac{L\sqrt{p}}{\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}} \frac{1}{p}\sum_{j=1}^{p} |\tau_j \boldsymbol{\beta}_{0,j}|\,.
\end{aligned} \tag{174}$$

Arguments similar to Lemma G.10 establish that $\sqrt{p}/\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}} \to 1$ almost surely as $n \to \infty$. Further note that $\langle \boldsymbol{Z}, \boldsymbol{\beta}_0/\gamma_n \rangle_{\boldsymbol{\Sigma}}/\gamma_n$ is a univariate Gaussian with $\mathcal{O}(1)$ variance so that it is itself $\mathcal{O}_p(1)$. Hence, it is sufficient to show that

eq:boundz_scaled

$$\frac{1}{p}\sum_{j=1}^{p} \left| \boldsymbol{Z}_j^{\text{scaled}} \right| = \mathcal{O}_p(1), \quad \frac{1}{p}\sum_{j=1}^{p} |\tau_j \boldsymbol{\beta}_{0,j}| = o(1)\,. \tag{175}$$

For the first note that $\mathrm{E}[|\boldsymbol{Z}_j^{\text{scaled}}|]^2 \le \mathrm{E}[\boldsymbol{Z}_j^{\text{scaled}\,2}] = \tau_j^2 [\boldsymbol{\Sigma}^{-1}]_{jj} \mathrm{E}[\boldsymbol{Z}_j^2] = 1$ by definition of τ_j. Now Markov's inequality (e.g. Vershynin (2018, Proposition 1.2.4)) yields that for $\epsilon > 0$,

eq:boundz_scaled2

$$\Pr\left(\frac{1}{p}\sum_{j=1}^{p} \left| \boldsymbol{Z}_j^{\text{scaled}} \right| \ge \frac{1}{\epsilon} \right) \le \epsilon \mathrm{E}\left[\frac{1}{p}\sum_{j=1}^{p} \left| \boldsymbol{Z}_j^{\text{scaled}} \right| \right] \le \epsilon\,, \tag{176}$$

so that $\frac{1}{p}\sum_{j=1}^{p} \left| \boldsymbol{Z}_j^{\text{scaled}} \right| = \mathcal{O}_p(1)$ indeed. Now, by the Cauchy-Schwarz inequality,

eq:tau_beta

$$\frac{1}{p}\sum_{j=1}^{p} |\tau_j \boldsymbol{\beta}_{0,j}| = \frac{1}{p}\sum_{j=1}^{p} |\boldsymbol{\theta}_{0,j}| \le \frac{1}{\sqrt{p}} \|\boldsymbol{\theta}_0\|_2 = \mathcal{O}(p^{-1/2})\,, \tag{177}$$

where $\boldsymbol{\theta}_0 = \boldsymbol{L}^\top \boldsymbol{\beta}_0$ for $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^\top$, and by assumption $\|\boldsymbol{\theta}_0\|_2^2 = \mathrm{var}(\boldsymbol{x}_i^\top \boldsymbol{\beta}_0) \to \gamma^2$ as $n \to \infty$, for $\boldsymbol{x}_i \sim \mathcal{N}(\mathbf{0}_p, \boldsymbol{\Sigma})$. Hence, (174) is indeed $o(1)$. Next consider $\boldsymbol{T}, \boldsymbol{T}^{\text{approx}}$.

$$
\begin{aligned}
\left| \frac{1}{p} \sum_{j=1}^{p} \psi(\boldsymbol{T}_j^{\text{approx}}) - \psi(\boldsymbol{T}_j) \right| &\leq \frac{1}{p} \sum_{j=1}^{p} \left| \psi(\boldsymbol{T}_j^{\text{approx}}) - \psi(\boldsymbol{T}_j) \right| \\
&\leq \frac{L}{\sigma_*} \sqrt{\frac{n}{p}} \frac{1}{\sqrt{p}} \sum_{j=1}^{p} \left| \tau_j \hat{\boldsymbol{\beta}}_j^{\text{DY}} \left(1 - \frac{\sqrt{\kappa}\sigma_*}{\tilde{\sigma}_n}\right) + \tau_j \boldsymbol{\beta}_{0,j} \left(\frac{\sqrt{\kappa}\sigma_*}{\tilde{\sigma}_n} \tilde{\mu}_n - \mu_* \right) \right| \\
&\leq \frac{L}{\sigma_*} \sqrt{\frac{n}{p}} \left\{ \left| 1 - \frac{\sqrt{\kappa}\sigma_*}{\tilde{\sigma}_n} \right| \frac{1}{\sqrt{p}} \sum_{j=1}^{p} \left| \tau_j \hat{\boldsymbol{\beta}}_j^{\text{DY}} \right| + \left| \frac{\sqrt{\kappa}\sigma_*}{\tilde{\sigma}_n} \tilde{\mu}_n - \mu_* \right| \frac{1}{\sqrt{p}} \sum_{j=1}^{p} |\tau_j \boldsymbol{\beta}_{0,j}| \right\} .
\end{aligned}
\tag{178}
$$

eq:t_psi

Now, it follows readily from Lemma E.1 and Lemma E.3 that $\tilde{\mu}_n \to \mu_*$ and $\tilde{\sigma}_n \to \sigma_*$ almost surely as $n \to \infty$. Indeed, note that by Lemma E.1, one can write

$$
\tilde{\mu}_n = \frac{(\hat{\boldsymbol{\beta}}^{\text{DY}})^\top \boldsymbol{\Sigma} \boldsymbol{\beta}_0}{\boldsymbol{\beta}_0^\top \boldsymbol{\Sigma} \boldsymbol{\beta}_0} = \frac{(\boldsymbol{L}^\top \hat{\boldsymbol{\beta}}^{\text{DY}})^\top (\boldsymbol{L}^\top \boldsymbol{\beta}_0)}{(\boldsymbol{L}^\top \boldsymbol{\beta}_0)^\top (\boldsymbol{L}^\top \boldsymbol{\beta}_0)} = \frac{\langle \hat{\boldsymbol{\theta}}^{\text{DY}}, \boldsymbol{\theta}_0 \rangle}{\|\boldsymbol{\theta}_0\|_2^2} , \tag{179}
$$

for $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^\top$ and where $\hat{\boldsymbol{\theta}}^{\text{DY}}$ is the mDYPL estimator from $\mathcal{N}(\mathbf{0}_p, \boldsymbol{I}_p)$ covariates and signal $\boldsymbol{\theta}_0 = \boldsymbol{L}^\top \boldsymbol{\beta}_0$. Hence, immediately by Lemma E.3, $\tilde{\mu}_n \to \mu_*$ as $n \to \infty$ almost surely and the same applies to $\tilde{\sigma}_n$. Secondly, by (177), $\frac{1}{\sqrt{p}} \sum_{j=1}^{p} |\tau_j \boldsymbol{\beta}_{0,j}| = o(1)$. Finally, recalling that $\tau_j \hat{\boldsymbol{\beta}}_j^{\text{DY}} = \hat{\boldsymbol{\theta}}_j^{\text{DY}}$, the Cauchy-Schwarz inequality yields

$$
\frac{1}{\sqrt{p}} \sum_{j=1}^{p} \left| \tau_j \hat{\boldsymbol{\beta}}_j^{\text{DY}} \right| = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} \left| \hat{\boldsymbol{\theta}}_j^{\text{DY}} \right| \leq \left\| \hat{\boldsymbol{\theta}}^{\text{DY}} \right\|_2 , \tag{180}
$$

eq:tau_betady

and by Lemma E.1 and Theorem 3.2, $\lim_{n\to\infty} \|\hat{\boldsymbol{\theta}}^{\text{DY}}\|_2 = \mathcal{O}(1)$ almost surely. Thus, (178) is $o_p(1)$ as required. □

Taken together, Lemma D.2 and (172) imply that

$$
\frac{1}{p} \sum_{j=1}^{p} \psi(\boldsymbol{T}_j) - \psi(\boldsymbol{Z}_j^{\text{scaled}}) = o_p(1) , \tag{181}
$$

so that it remains to characterize the distribution of $\boldsymbol{Z}^{\text{scaled}}$, which is fully characterized by the covariance structure $\boldsymbol{\Sigma}$. Zhao et al. (2022, Section B.2-B.4) show that if $\lambda_{\max}(\boldsymbol{\Sigma})/\lambda_{\min}(\boldsymbol{\Sigma}) = o\left(\sqrt{p}\right)$, then the empirical CDF of $\boldsymbol{Z}^{\text{scaled}}$ converges pointwise to that of a standard normal random variable in probability, which establishes the first part of the Theorem.

The proof of the second part of the Theorem follows a similar structure as the first part. In particular, for a bivariate pseudo-Lipschitz function ψ of order two, it is first shown that

$$
\frac{1}{p} \sum_{j=1}^{p} \psi\left(\sigma_* \boldsymbol{T}_j, \tau_j \sqrt{n} \boldsymbol{\beta}_{0,j}\right) - \frac{1}{p} \sum_{j=1}^{p} \psi\left(\sigma_* \boldsymbol{T}_j^{\text{approx}}, \tau_j \sqrt{n} \boldsymbol{\beta}_{0,j}\right) = o_p(1) , \tag{182}
$$

eq:t_pseudo_psi

and that

$$
\frac{1}{p} \sum_{j=1}^{p} \psi\left(\sigma_* \tilde{\boldsymbol{Z}}_j^{\text{scaled}}, \tau_j \sqrt{n} \boldsymbol{\beta}_{0,j}\right) - \frac{1}{p} \sum_{j=1}^{p} \psi\left(\sigma_* \boldsymbol{Z}_j^{\text{scaled}}, \tau_j \sqrt{n} \boldsymbol{\beta}_{0,j}\right) = o_p(1) , \tag{183}
$$

eq:z_pseudo_psi

which, in conjunction with (172) yields

$$
\frac{1}{p} \sum_{j=1}^{p} \psi\left(\sigma_* \boldsymbol{T}_j, \tau_j \sqrt{n} \boldsymbol{\beta}_{0,j}\right) - \frac{1}{p} \sum_{j=1}^{p} \psi\left(\sigma_* \boldsymbol{Z}_j^{\text{scaled}}, \tau_j \sqrt{n} \boldsymbol{\beta}_{0,j}\right) = o_p(1) , \tag{184}
$$

eq:tz_pseudo_psi

so that it then remains to analyse

$$\frac{1}{p}\sum_{j=1}^{p}\psi\left(\sigma_*\boldsymbol{Z}_j^{\text{scaled}},\tau_j\sqrt{n}\boldsymbol{\beta}_{0,j}\right)\,. \qquad \texttt{eq:zscaled\_pseudo\_psi} \tag{185}$$

Towards showing (182), consider

$$\begin{aligned}
&\left|\frac{1}{p}\sum_{j=1}^{p}\psi\left(\sigma_*\boldsymbol{T}_j,\tau_j\sqrt{n}\boldsymbol{\beta}_{0,j}\right)-\psi\left(\sigma_*\boldsymbol{T}_j^{\text{approx}},\tau_j\sqrt{n}\boldsymbol{\beta}_{0,j}\right)\right| \\
&\qquad\qquad\le\frac{1}{p}\sum_{j=1}^{p}\left|\psi\left(\sigma_*\boldsymbol{T}_j,\tau_j\sqrt{n}\boldsymbol{\beta}_{0,j}\right)-\psi\left(\sigma_*\boldsymbol{T}_j^{\text{approx}},\tau_j\sqrt{n}\boldsymbol{\beta}_{0,j}\right)\right| \\
&\le\frac{L\sigma_*}{p}\sum_{j=1}^{p}\Big(1+\left\|[\sigma_*\boldsymbol{T}_j,\tau_j\sqrt{n}\boldsymbol{\beta}_{0,j}]^\top\right\|_2+\left\|[\sigma_*\boldsymbol{T}_j^{\text{approx}},\tau_j\sqrt{n}\boldsymbol{\beta}_{0,j}]^\top\right\|_2\Big)\left|\boldsymbol{T}_j-\boldsymbol{T}_j^{\text{approx}}\right| \\
&\le\frac{L\sigma_*}{p}\left\{\sum_{j=1}^{p}\left(1+\sigma_*\left|\boldsymbol{T}_j\right|+\sigma_*\left|\boldsymbol{T}_j^{\text{approx}}\right|+2\sqrt{n}\left|\tau_j\boldsymbol{\beta}_{0,j}\right|\right)^2\right\}^{1/2}\left\{\sum_{j=1}^{p}\left(\boldsymbol{T}_j-\boldsymbol{T}_j^{\text{approx}}\right)^2\right\}^{1/2}\,,
\end{aligned} \tag{186}$$

where the last line follows by the Cauchy-Schwarz inequality and the monotonicity of ℓ_p norms. Now by construction of $\boldsymbol{T}_j,\boldsymbol{T}_j^{\text{approx}}$ and since $(a+b)^2\le 2a^2+2b^2$,

$$\begin{aligned}
\sum_{j=1}^{p}\left(\boldsymbol{T}_j-\boldsymbol{T}_j^{\text{approx}}\right)^2&=\frac{n}{\sigma_*^2}\sum_{j=1}^{p}\left(\tau_j\hat{\boldsymbol{\beta}}_j^{\text{DY}}\left(1-\frac{\sqrt{\kappa}\sigma_*}{\tilde{\sigma}_n}\right)+\tau_j\boldsymbol{\beta}_{0,j}\left(\frac{\sqrt{\kappa}\sigma_*}{\tilde{\sigma}_n}\tilde{\mu}_n-\mu_*\right)\right)^2 \\
&\le\frac{2n}{\sigma_*^2}\left\{\left(1-\frac{\sqrt{\kappa}\sigma_*}{\tilde{\sigma}_n}\right)^2\sum_{j=1}^{p}\left(\tau_j\hat{\boldsymbol{\beta}}_j^{\text{DY}}\right)^2+\left(\frac{\sqrt{\kappa}\sigma_*}{\tilde{\sigma}_n}\tilde{\mu}_n-\mu_*\right)^2\sum_{j=1}^{p}\left(\tau_j\boldsymbol{\beta}_{0,j}\right)^2\right\} \\
&=\frac{2n}{\sigma_*^2}\left\{\left(1-\frac{\sqrt{\kappa}\sigma_*}{\tilde{\sigma}_n}\right)^2\left\|\hat{\boldsymbol{\theta}}^{\text{DY}}\right\|_2^2+\left(\frac{\sqrt{\kappa}\sigma_*}{\tilde{\sigma}_n}\tilde{\mu}_n-\mu_*\right)^2\left\|\boldsymbol{\theta}_0\right\|_2\right\} \\
&=o(n)\,,
\end{aligned} \tag{187}$$

by considerations similar to those in (177)-(180). Similarly, one can bound

$$\begin{aligned}
\sum_{j=1}^{p}\left(1+\sigma_*\left|\boldsymbol{T}_j\right|+\sigma_*\left|\boldsymbol{T}_j^{\text{approx}}\right|+2\sqrt{n}\left|\tau_j\boldsymbol{\beta}_{0,j}\right|\right)^2&\le 4\sum_{j=1}^{p}1+\sigma_*^2\boldsymbol{T}_j^2+\sigma_*^2\boldsymbol{T}_j^{\text{approx}\,2}+4n\tau_j^2\boldsymbol{\beta}_{0,j}^2 \\
&=4n+4\sigma_*^2\left\|\boldsymbol{T}\right\|_2^2+4\sigma_*^2\left\|\boldsymbol{T}^{\text{approx}}\right\|_2^2+16n\left\|\boldsymbol{\theta}_0\right\|_2^2\,.
\end{aligned} \tag{188}$$

Now $\boldsymbol{T}_j=\sqrt{n}(\hat{\boldsymbol{\theta}}_j^{\text{DY}}-\mu_*\boldsymbol{\theta}_{0,j})/\sigma_*$ and by considerations analogous to Lemma E.3 and Theorem 3.1, choosing $\psi(t,u)=t^2$, yields that $\|\boldsymbol{T}\|_2^2=\mathcal{O}_p(n)$. Similarly, by Lemma E.3 and the expansion of (204), also $\|\boldsymbol{T}^{\text{approx}}\|_2^2=\mathcal{O}_p(n)$. Therefore (182) is indeed $o_p(1)$.

Next, consider the difference (183). Similarly to above,

$$\begin{aligned}
&\left|\frac{1}{p}\sum_{j=1}^{p}\psi\left(\sigma_*\tilde{\boldsymbol{Z}}_j^{\text{scaled}},\tau_j\sqrt{n}\boldsymbol{\beta}_{0,j}\right)-\psi\left(\sigma_*\boldsymbol{Z}_j^{\text{scaled}},\tau_j\sqrt{n}\boldsymbol{\beta}_{0,j}\right)\right| \\
&\qquad\le\frac{L\sigma_*}{p}\left\{\sum_{j=1}^{p}\left(1+\left|\boldsymbol{Z}_j^{\text{scaled}}\right|+\left|\tilde{\boldsymbol{Z}}_j^{\text{scaled}}\right|+2\sqrt{n}\left|\tau_h\boldsymbol{\beta}_{0,j}\right|\right)^2\right\}^{1/2}\left\{\sum_{j=1}^{p}\left(\boldsymbol{Z}_j^{\text{scaled}}-\tilde{\boldsymbol{Z}}_j^{\text{scaled}}\right)^2\right\}^{1/2}\,.
\end{aligned} \tag{189}$$

The term

$$\sum_{j=1}^{p}\left(1+\left|\boldsymbol{Z}_j^{\text{scaled}}\right|+\left|\tilde{\boldsymbol{Z}}_j^{\text{scaled}}\right|+2\sqrt{n}\left|\tau_j\boldsymbol{\beta}_{0,j}\right|\right)^2=\mathcal{O}_p(n)\,, \tag{190}$$

since $\boldsymbol{T}^{\text{approx}}\overset{d}{=}\tilde{\boldsymbol{Z}}^{\text{scaled}}$, which was shown to be bounded in the above paragraph and $\boldsymbol{Z}^{\text{scaled}}$ was appropriately bounded in (175)-(176). On the other hand,

$$\begin{aligned}\sum_{j=1}^{p}\left(\boldsymbol{Z}_j^{\text{scaled}}-\tilde{\boldsymbol{Z}}_j^{\text{scaled}}\right)^2&=\sum_{j=1}^{p}\left(\boldsymbol{Z}_j^{\text{scaled}}\left(1-\frac{\sqrt{p}}{\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}}\right)+\frac{\langle\boldsymbol{Z},\boldsymbol{\beta}_0/\gamma_n\rangle_{\boldsymbol{\Sigma}}}{\gamma_n}\tau_j\boldsymbol{\beta}_{0,j}\right)^2\\&\le 2\left(1-\frac{\sqrt{p}}{\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}}\right)^2\|\boldsymbol{Z}^{\text{scaled}}\|_2^2+\left(\frac{\langle\boldsymbol{Z},\boldsymbol{\beta}_0/\gamma_n\rangle_{\boldsymbol{\Sigma}}}{\gamma_n}\right)^2\left(\frac{\sqrt{p}}{\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}}\right)^2\|\boldsymbol{\theta}_0\|_2^2\\&=o_p(n)\,,\end{aligned} \tag{191}$$

since arguments similar to Lemma G.10 show that $\lim_{n\to\infty}\sqrt{p}/\|\boldsymbol{\xi}\|_{\boldsymbol{\Sigma}}=1$ almost surely, $\mathrm{E}[\boldsymbol{Z}_j^{\text{scaled}^2}]=1$ and thus a Markov bound similar to (175)-(176) establishes that $\|\boldsymbol{Z}^{\text{scaled}}\|_2^2=\mathcal{O}_p(p)$. The second term is bounded since additionally $\lim_{n\to\infty}\gamma_n^2=\lim_{n\to\infty}\|\boldsymbol{\theta}_0\|_2^2=\gamma^2$, and $\langle\boldsymbol{Z},\boldsymbol{\beta}_0/\gamma_n\rangle_{\boldsymbol{\Sigma}}$ is a normal random variable with $\mathcal{O}(1)$ variance.

Hence, it remains to analyse the behaviour of (185) which is done in Zhao et al. (2022, Lemma C.1). As $\boldsymbol{Z}^{\text{scaled}}$ solely depends on $\boldsymbol{\Sigma}$, this result directly applies to this setting. The conditions of said Lemma are satisfied in the current setting. Zhao et al. (2022, Lemma C.1) establishes that

$$\frac{1}{p}\sum_{j=1}^{p}\psi\left(\sigma_*\boldsymbol{Z}_j^{\text{scaled}},\tau_j\sqrt{n}\boldsymbol{\beta}_{0,j}\right)\xrightarrow{p}\mathrm{E}\left[\psi(\sigma_*G,\bar{\beta})\right]\,, \tag{192}$$

for $G\sim\mathcal{N}(0,1)\perp\bar{\beta}$. This concludes the proof the Theorem.

E Proof of Theorem 3.4

thm:z_stats

Theorem 3.4. *Assume that (κ,γ,α) are such that the conditions of Theorem 3.1 are met. Let $\hat{\boldsymbol{\beta}}^{DY}$ be the mDYPL from a logistic regression model with i.i.d. $\mathcal{N}(\boldsymbol{0}_p,\boldsymbol{\Sigma})$ covariates and signal $\boldsymbol{\beta}_0$. Then for any regression coefficient such that $\sqrt{n}\tau_j\boldsymbol{\beta}_{0,j}=\mathcal{O}(1)$,*

$$\sqrt{n}\tau_j\frac{\hat{\boldsymbol{\beta}}_j^{DY}-\mu_*\boldsymbol{\beta}_{0,j}}{\sigma_*}\xrightarrow{d}\mathcal{N}(0,1)\,, \tag{193}$$

where $\tau_j=var(\boldsymbol{x}_{ij}|\boldsymbol{x}_{i-j})$.

The proof of Theorem 3.4 is a replication of the arguments in the proof of Theorem 3.1 in Zhao et al. (2022) with the substitutions that are outlined in F and follows the general outline given in Section 3.2 of the main paper of this text. No claim of originality is made.

lemma:equivar

Lemma E.1. *For any matrix $\boldsymbol{L}\in\Re^{p\times p}$ and $\boldsymbol{\Sigma}=\boldsymbol{L}\boldsymbol{L}^\top$, let $\hat{\boldsymbol{\beta}}^{DY}$ be the mDYPL estimator from a logistic regression model with i.i.d. $\mathcal{N}(\boldsymbol{0}_p,\boldsymbol{\Sigma})$ covariates and signal $\boldsymbol{\beta}_0$. Then*

$$\hat{\boldsymbol{\theta}}^{DY}=\boldsymbol{L}^\top\hat{\boldsymbol{\beta}}^{DY} \tag{194}$$

is the mDYPL estimator from a logistic regression with $\mathcal{N}(\boldsymbol{0}_p,\boldsymbol{I}_p)$ covariates and signal $\boldsymbol{\theta}_0=\boldsymbol{L}^\top\boldsymbol{\beta}_0$.

Proof. This result follows from the invariance of the mDYPL estimator under linear transformations of the covariates (see for example Zehna (1966)) and the rotational invariance of the normal distribution. Details can be found in Zhao et al. 2022, Proposition 2.1. □

lemma:stochastic_rep

Lemma E.2. *Let $\hat{\boldsymbol{\theta}}^{DY}$ be the mDYPL estimator from a logistic regression model with i.i.d. $\mathcal{N}(\mathbf{0}_p, \boldsymbol{I}_p)$ covariates and signal $\boldsymbol{\theta}_0$. Define the random variables*

eq:sample_musig

$$\mu_n = \frac{\langle \hat{\boldsymbol{\theta}}^{DY}, \boldsymbol{\theta}_0 \rangle}{\|\boldsymbol{\theta}_0\|_2^2}, \quad \sigma_n = \left\| \boldsymbol{P}_{\boldsymbol{\theta}_0^\perp} \hat{\boldsymbol{\theta}}^{DY} \right\|_2, \tag{195}$$

where $\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}$ is the projection onto $\boldsymbol{\theta}_0^\perp$, the orthogonal complement of $\boldsymbol{\theta}_0$. Then

$$\frac{\hat{\boldsymbol{\theta}}^{DY} - \mu_n \boldsymbol{\theta}_0}{\sigma_n} \tag{196}$$

is uniformly distributed on the unit sphere lying in $\boldsymbol{\theta}_0^\perp$.

Proof. The proof is similar to Zhao et al. (2022, Lemma 2.1) and therefore omitted. □

lemma:population_pars

Lemma E.3 (Zhao et al. (2022), Lemma 3.1)**.** *Let $\hat{\boldsymbol{\beta}}^{DY}$ be the mDYPL estimator with i.i.d. covariates $\boldsymbol{x}_i \sim \mathcal{N}(\mathbf{0}_p, n^{-1}\boldsymbol{I}_p)$ and assume that the conditions of Theorem 3.1 are met. Then for μ_n, σ_n as in (195),*

$$\mu_n \xrightarrow{a.s.} \mu_*, \quad \sigma_n \xrightarrow{a.s.} \sqrt{\kappa}\sigma_*, \tag{197}$$

where (μ_*, σ_*) are the solutions corresponding to the system of equations (4) - (6).

Proof. The proof is similar to the proof of Lemma 3.1 in Zhao et al. (2022) and relies on Lemma E.1 and Theorem 3.1.

Given $\boldsymbol{\theta}_0$ such that $\lim_{n\to\infty} \|\boldsymbol{\theta}_0\|_2 = \gamma$, let $\hat{\boldsymbol{\beta}}^{\mathrm{DY}}$ be the mDYPL estimator from $\mathcal{N}(\mathbf{0}_p, n^{-1}\boldsymbol{I}_p)$ covariates and signal $\boldsymbol{\beta}_0 = \sqrt{n}\boldsymbol{U}\boldsymbol{\theta}_0$ where $\boldsymbol{U}$ is an orthogonal matrix such that $\boldsymbol{U}\boldsymbol{\theta}_0 = (\|\boldsymbol{\theta}_0\|_2, \ldots, \|\boldsymbol{\theta}_0\|_2)^\top/\sqrt{p}$, which can for example be constructed using a Householder transformation. Then by Lemma E.1, $\hat{\boldsymbol{\beta}}^{\mathrm{DY}} = \sqrt{n}\boldsymbol{U}\hat{\boldsymbol{\theta}}^{\mathrm{DY}}$ where $\hat{\boldsymbol{\theta}}^{\mathrm{DY}}$ is the mDYPL estimator with $\mathcal{N}(\mathbf{0}_p, \boldsymbol{I}_p)$ covariates and signal $\boldsymbol{\theta}_0$. Note that by construction, the empirical distribution of $\boldsymbol{\beta}_0$ converges to a point mass at $\gamma/\sqrt{k}$ and that $\lim_{n\to\infty} \frac{1}{p}\sum_{j=1}^n \boldsymbol{\beta}_{0,j}^2 = \gamma^2/\kappa$ as required. Hence, Theorem 3.1 gives that for any pseudo-Lipschitz function ψ of order two,

$$\frac{1}{p}\sum_{j=1}^p \psi(\sqrt{n}([\boldsymbol{U}\hat{\boldsymbol{\theta}}^{\mathrm{DY}}]_j - \mu_*[\boldsymbol{U}\boldsymbol{\theta}_0]_j), \sqrt{n}[\boldsymbol{U}\boldsymbol{\theta}_0]_j) \xrightarrow{a.s.} \mathrm{E}[\psi(\sigma_* G, \bar{\beta})], \tag{198}$$

where $G \sim \mathcal{N}(0,1)$, independent of $\bar{\beta} \sim \delta_{\gamma/\sqrt{\kappa}}$ and μ_*, σ_* solve (4)-(6). Then choosing $\psi(x,y) = xy$, gives that

$$\frac{1}{\kappa}\langle \boldsymbol{U}\hat{\boldsymbol{\theta}}^{\mathrm{DY}} - \mu_* \boldsymbol{U}\boldsymbol{\theta}_0, \boldsymbol{U}\boldsymbol{\theta}_0 \rangle = \frac{1}{\kappa}\langle \hat{\boldsymbol{\theta}}^{\mathrm{DY}} - \mu_*\boldsymbol{\theta}_0, \boldsymbol{\theta}_0 \rangle \xrightarrow{a.s.} 0, \tag{199}$$

from which it follows that

eq:alpha_n_lim

$$\mu_n = \frac{\langle \hat{\boldsymbol{\theta}}^{\mathrm{DY}}, \boldsymbol{\theta}_0 \rangle}{\|\boldsymbol{\theta}_0\|_2} \xrightarrow{a.s.} \mu_*. \tag{200}$$

Similarly, choosing $\psi(x,y)=x^2$, gives

$$\frac{1}{\kappa}\langle \boldsymbol{U}\hat{\boldsymbol{\theta}}^{\text{DY}} - \mu_* \boldsymbol{U}\boldsymbol{\theta}_0, \boldsymbol{U}\hat{\boldsymbol{\theta}}^{\text{DY}} - \mu_* \boldsymbol{U}\boldsymbol{\theta}_0\rangle = \frac{1}{\kappa}\langle \hat{\boldsymbol{\theta}}^{\text{DY}} - \mu_* \boldsymbol{\theta}_0, \hat{\boldsymbol{\theta}}^{\text{DY}} - \mu_* \boldsymbol{\theta}_0\rangle = \left\| \hat{\boldsymbol{\theta}}^{\text{DY}} - \mu_* \boldsymbol{\theta}_0 \right\|_2^2 \xrightarrow{a.s.} \sigma_*^2\,, \quad (201)$$

which in conjunction with (200) yields

$$\sigma_n^2 = \left\| \hat{\boldsymbol{\theta}}^{\text{DY}} - \mu_n \boldsymbol{\theta}_0 \right\|_2^2 \xrightarrow{a.s.} \kappa\sigma_*^2 \tag{202}$$

as required. □

Towards proving Theorem 3.4, note that Lemma E.1 implies that for any fixed coordinate j,

$$\sqrt{n}\tau_j \frac{\hat{\boldsymbol{\beta}}_j^{\text{DY}} - \mu_* \boldsymbol{\beta}_{0,j}}{\sigma_*} = \sqrt{n}\frac{\hat{\boldsymbol{\theta}}_j^{\text{DY}} - \mu_* \boldsymbol{\theta}_{0,j}}{\sigma_*}\,, \qquad \texttt{eq:transformed\_z} \tag{203}$$

where $\hat{\boldsymbol{\theta}}^{\text{DY}} = \boldsymbol{L}^\top \hat{\boldsymbol{\beta}}^{\text{DY}}$, is the mDYPL estimator from $\mathcal{N}(\mathbf{0}_p, \boldsymbol{I}_p)$ covariates with signal $\boldsymbol{\theta}_0 = \boldsymbol{L}^\top \boldsymbol{\beta}_0$ for $\boldsymbol{L}$ being the Cholesky factorization of $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^\top$ and where $\tau_j = \text{var}(\boldsymbol{x}_{ij}|\boldsymbol{x}_{i,-j})$ is the conditional variance of the jth covariate given all other covariates under $\boldsymbol{\Sigma}$ (see Zhao et al. (2022, Section 2.1)).

A simple expansion of the RHS of (203) gives

$$\sqrt{n}\frac{\hat{\boldsymbol{\theta}}_j^{\text{DY}} - \mu_* \boldsymbol{\theta}_{0,j}}{\sigma_*} = \sqrt{n}\frac{\hat{\boldsymbol{\theta}}_j^{\text{DY}} - \mu_n \boldsymbol{\theta}_{0,j}}{\sigma_n}\frac{\sigma_n}{\sigma_*} + \sqrt{n}\frac{(\mu_n - \mu_*)\boldsymbol{\theta}_{0,j}}{\sigma_*}\,. \qquad \texttt{eq:z\_expansion} \tag{204}$$

Now by the assumption of the Theorem, $\sqrt{n}\boldsymbol{\theta}_{0,j} = \sqrt{n}\tau_j\boldsymbol{\beta}_{0,j} = \mathcal{O}(1)$. Hence, by Lemma E.3, the second term on the RHS of (204) is $o(1)$ almost surely. Now by Lemma E.2, $(\hat{\boldsymbol{\theta}}_j^{\text{DY}} - \mu_n \boldsymbol{\theta}_{0,j})/\sigma_n = \boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}\hat{\boldsymbol{\theta}}^{\text{DY}}/\|\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}\hat{\boldsymbol{\theta}}^{\text{DY}}\|_2$ is distributed uniformly on the unit sphere lying in $\boldsymbol{\theta}_0^\perp$, where $\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}$ is the projection matrix into the orthogonal complement $\boldsymbol{\theta}_0^\perp$ of $\boldsymbol{\theta}_0$. It is a well known property of the Gaussian distribution that for $\boldsymbol{Z} \sim \mathcal{N}(\mathbf{0}_p, \boldsymbol{I}_p)$, the projection onto the unit sphere is uniformly distributed (see for example Vershynin (2018), Section 3.3.3), and thus by Lemma E.2,

$$\frac{\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}\hat{\boldsymbol{\theta}}^{\text{DY}}}{\left\|\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}\hat{\boldsymbol{\theta}}^{\text{DY}}\right\|_2} \overset{d}{=} \frac{\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}\boldsymbol{Z}}{\left\|\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}\boldsymbol{Z}\right\|_2}\,, \tag{205}$$

for $\boldsymbol{Z} \sim \mathcal{N}(\mathbf{0}_p, \boldsymbol{I}_p)$. Expanding $\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}\boldsymbol{Z}$ as

$$\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}\boldsymbol{Z} = \boldsymbol{Z} - \left\langle \boldsymbol{Z}, \frac{\boldsymbol{\theta}_0}{\|\boldsymbol{\theta}_0\|_2}\right\rangle \frac{\boldsymbol{\theta}_0}{\|\boldsymbol{\theta}_0\|_2}\,, \tag{206}$$

whose jth component is thus given by

$$Z_j\left(1 - \frac{\boldsymbol{\theta}_{0,j}^2}{\|\boldsymbol{\theta}_0\|_2^2}\right) + \xi\,, \tag{207}$$

where Z_j is the jth component of $\boldsymbol{Z}$ and ξ is a normal random variable independent of Z_j with mean zero and variance

$$\text{var}(\xi) = \frac{\|\boldsymbol{\theta}_{0,-j}\|_2^2}{\|\boldsymbol{\theta}_0\|_2^2}\frac{\boldsymbol{\theta}_{0,j}^2}{\|\boldsymbol{\theta}_0\|_2^2} = \mathcal{O}(n^{-1})\,,$$

since $\sqrt{n}\boldsymbol{\theta}_j = \mathcal{O}(1)$. Consequently $\xi = \mathcal{O}_p(n^{-1/2})$. Further, by Lemma G.10, $\|\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}\boldsymbol{Z}\|_2/\sqrt{p} \to 1$ almost surely. Hence, taken together, one gets that

$$
\begin{aligned}
\sqrt{n}\frac{\hat{\boldsymbol{\theta}}_j^{\mathrm{DY}} - \mu_* \boldsymbol{\theta}_{0,j}}{\sigma_*} &= \sqrt{n}\frac{\hat{\boldsymbol{\theta}}_j^{\mathrm{DY}} - \mu_n \boldsymbol{\theta}_{0,j}}{\sigma_n}\frac{\sigma_n}{\sigma_*} + \sqrt{n}\frac{(\mu_n - \mu_*)\boldsymbol{\theta}_{0,j}}{\sigma_*} \\
&= \sqrt{n}\frac{\hat{\boldsymbol{\theta}}_j^{\mathrm{DY}} - \mu_n \boldsymbol{\theta}_{0,j}}{\sigma_n}\frac{\sigma_n}{\sigma_*} + o_p(1) \\
&= \sqrt{n}\frac{\hat{\boldsymbol{\theta}}_j^{\mathrm{DY}} - \mu_n \boldsymbol{\theta}_{0,j}}{\sigma_n} + o_p(1) \\
&\stackrel{d}{=} \sqrt{\frac{n}{p}}\frac{\left\|\boldsymbol{P}_{\boldsymbol{\theta}_0^\perp}\boldsymbol{Z}\right\|_2}{\sqrt{p}}\left\{Z_j\left(1 - \frac{\boldsymbol{\theta}_{0,j}^2}{\|\boldsymbol{\theta}_0\|_2^2}\right) + \xi\right\} + o_p(1) \\
&= \frac{1}{\sqrt{\kappa}}\left\{Z_j\left(1 - \frac{\boldsymbol{\theta}_{0,j}^2}{\|\boldsymbol{\theta}_0\|_2^2}\right) + \xi\right\} + o_p(1) \\
&= \frac{1}{\sqrt{\kappa}}Z_j\sigma_j + o_p(1) \\
&= \frac{1}{\sqrt{\kappa}}Z_j + o_p(1)
\end{aligned} \tag{208}
$$

for $\sigma_j = 1 - \boldsymbol{\theta}_{0,j}^2/\|\boldsymbol{\theta}_0\|_2^2$ which converges to one since $\boldsymbol{\theta}_{0,j} = \mathcal{O}(n^{-1/2})$. By (203) and (204),

eq:z_distr

$$
\sqrt{n}\frac{\hat{\boldsymbol{\theta}}_j^{\mathrm{DY}} - \mu_* \boldsymbol{\theta}_{0,j}}{\sigma_*} \stackrel{d}{=} \frac{1}{\sqrt{\kappa}}Z_j + o_p(1)\,, \tag{209}
$$

and the first part of the Theorem follows. The general result follows from (209) and the rotational invariance of the Gaussian distribution in conjunction with Lemma E.1. The argument is completely similar to the one provided in the proof of Theorem 3.1 in Zhao et al. (2022) starting below (A.6) and is thus omitted.

F Proof of Theorem 3.5

sec:likl ratio

thm:likl_ratio_2

Theorem 3.5. *Assume that κ, γ, α are such that $\|\hat{\boldsymbol{\beta}}^{DY}\|_2 = \mathcal{O}(n^{1/2})$ almost surely and that (4) - (6) admit a non-singular solution (μ_*, b_*, σ_*). Then for any fixed subset of indices $I = \{i_1, \ldots, i_k\}$, the DY-prior penalized likelihood ratio test statistic*

$$
\Lambda_I = \max_{\boldsymbol{\beta} \in \Re^p} \ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X}) - \max_{\substack{\boldsymbol{\beta} \in \Re^p: \\ \boldsymbol{\beta}_j = 0,\, j \in I}} \ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X}) \tag{210}
$$

where $\boldsymbol{X}$ has rows of i.i.d. $\boldsymbol{x}_j \sim \mathcal{N}(\mathbf{0}_p, \boldsymbol{\Sigma})$ is asymptotically distributed as

$$
2\Lambda_I \xrightarrow{d} \frac{\kappa\sigma_*^2}{b_*}\chi_k^2\,, \tag{211}
$$

under the null that $\boldsymbol{\beta}_{0,i_1} = \ldots = \boldsymbol{\beta}_{0,i_k} = 0$, where χ_k^2 is a Chi-squared random variable with k degrees of freedom.

This follows immediately from Theorem F.1 and Lemma E.1 as in the discussion of Zhao et al. (2022, Section 4).

thm:likl_ratio

Theorem F.1. *Assume that κ, γ, α are such that $\|\hat{\boldsymbol{\beta}}^{DY}\|_2 = \mathcal{O}(n^{1/2})$ almost surely and that (4) - (6) admit a non-singular solution (μ_*, b_*, σ_*). Then for any fixed subset of indices $I = \{i_1, \ldots, i_k\}$, the*

DY-prior penalized likelihood ratio test statistic

$$\Lambda_I = \max_{\boldsymbol{\beta}\in\Re^p} \ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X}) - \max_{\substack{\boldsymbol{\beta}\in\Re^p:\\ \boldsymbol{\beta}_j=0,\, j\in I}} \ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X}) \tag{212}$$

where $\boldsymbol{X}$ has rows of i.i.d. $\boldsymbol{x}_j \sim \mathcal{N}(\boldsymbol{0}_p, n^{-1}\boldsymbol{I}_p)$ is asymptotically distributed as

$$2\Lambda_I \xrightarrow{d} \frac{\kappa\sigma_*^2}{b_*}\chi_k^2 , \tag{213}$$

under the null that $\boldsymbol{\beta}_{0,i_1} = \ldots = \boldsymbol{\beta}_{0,i_k} = 0$, where χ_k^2 is a Chi-squared random variable with k degrees of freedom

The proof of Theorem F.1 is analogous the proof of Theorem 3 in Sur and Candès (2019) where the MLE is replaced with the mDYPL estimator, the state evolutions in (5) are replaced by (4)-(6) and the standard logistic regression log-likelihood $\ell(\boldsymbol{\beta}; \boldsymbol{y}, \boldsymbol{X})$ is replaced by the log-likelihood with pseudo responses y^*, i.e. $\ell(\boldsymbol{\beta}; \boldsymbol{y}^*, \boldsymbol{X})$ and their Theorem 2 is replaced by Theorem 3.1. A line by line analysis reveals that the arguments in the proof of Sur and Candès (2019) go through with miniscule adjustments. As the proof is lengthy and technical, and there are no noteworthy contributions in the adaptation to the mDYPL estimator, it is omitted.

G Auxiliary Lemmas

sec:aux lemmas
lemma:approx

Lemma G.1. *Given $\epsilon > 0$, $\alpha \in [0, 1]$, consider the approximation $h_\epsilon(Z, \bar\varepsilon) = \frac{\alpha}{2}\left\{1 - \frac{\bar\varepsilon - \zeta'(Z)}{|\bar\varepsilon - \zeta'(Z)| + \epsilon}\right\} + \frac{1-\alpha}{2}$ to $h(Z, \bar\varepsilon) = \frac{\alpha}{2}\{1 - \mathrm{sign}(\bar\varepsilon - \zeta'(Z))\} + \frac{1-\alpha}{2}$. Then*

i) $\lim_{\epsilon\to 0} h_\epsilon(Z, \bar\varepsilon) = h(Z, \bar\varepsilon)$ *for all* $Z, \bar\varepsilon$

ii) $h_\epsilon(Z, \bar\varepsilon)$ *is Lipschitz in* $[Z, \bar\varepsilon]^\top$

iii) $\frac{\partial}{\partial Z} h_\epsilon(Z, \bar\varepsilon)$ *exists for all* $Z, \bar\varepsilon \in \Re$, $\epsilon > 0$ *and is Lipschitz in* $[Z, \bar\varepsilon]^\top$.

Proof. i) If $\bar\varepsilon - \zeta'(Z) = 0$, then $h_\epsilon(Z, \bar\varepsilon) = 0$ for all $\epsilon > 0$. If $\bar\varepsilon - \zeta'(Z) \neq 0$, then $\lim_{\epsilon\to 0} h_\epsilon(Z, \bar\varepsilon) = (\bar\varepsilon - \zeta'(Z))/|\bar\varepsilon - \zeta'(Z)| = \mathrm{sign}(\bar\varepsilon - \zeta'(Z))$. The claim follows.

ii) First it is shown that $f(x) = x/(|x| + \epsilon)$ is Lipschitz for any $\epsilon > 0$ with Lipschitz constant ϵ^{-1}. To that end, note that $\frac{\partial}{\partial x} f(x) = \epsilon/(|x| + \epsilon)^2 \leq \epsilon^{-1}$ and by the mean value theorem (e.g. Rudin et al. 1976, Theorem 5.10),

$$|f(x) - f(y)| \leq \sup_{z\in\Re} \left|\frac{\partial}{\partial z} f(z)\right| |x - y| \leq \frac{1}{\epsilon} |x - y| . \tag{214}$$

Further, note that $\zeta''(x)$ is positive and bounded from above by $1/4$. Hence,

$$\begin{aligned}
|h_\epsilon(Z,\bar\varepsilon) - h_\epsilon(Z',\bar\varepsilon')| &= \frac{\alpha}{2}\left|f\left(\bar\varepsilon - \zeta'(Z)\right) - f\left(\bar\varepsilon' - \zeta'(Z')\right)\right| \\
&\le \frac{\alpha}{2\epsilon}\left|\bar\varepsilon - \zeta'(Z) - \left(\bar\varepsilon' - \zeta'(Z')\right)\right| \\
&\le \frac{\alpha}{2\epsilon}\left\{|\bar\varepsilon - \bar\varepsilon'| + |\zeta'(Z) - \zeta'(Z')|\right\} \\
&\le \frac{\alpha}{2\epsilon}\left\{|\bar\varepsilon - \bar\varepsilon'| + \frac{1}{4}|Z - Z'|\right\} \\
&\le \frac{\alpha}{8\epsilon}\left\{|\bar\varepsilon - \bar\varepsilon'| + |Z - Z'|\right\} \\
&\le \frac{\alpha\sqrt{2}}{8\epsilon}\left\|\begin{bmatrix} Z \\ \bar\varepsilon\end{bmatrix} - \begin{bmatrix} Z' \\ \bar\varepsilon'\end{bmatrix}\right\|_2 ,
\end{aligned} \tag{215}$$

where the first inequality follows by Lipschitzianity of $f(\cdot)$, the second by the triangle inequality, the third by the Mean Value Theorem and since $\sup_{x\in\Re}|\zeta''(x)| \le 1/4$, and the last since $(|a|+|b|)^2 \le 2(|a|^2+|b|^2)$.

iii) Existence follows by differentiability of $f(x)$ and the the chain rule,

$$\frac{\partial}{\partial Z}h_\epsilon(Z,\bar\varepsilon) = \frac{\alpha}{2}\frac{\epsilon}{(|\bar\varepsilon - \zeta'(Z)| + \epsilon)^2}\zeta''(Z)\,. \tag{216}$$

eq:hinge_deriv

Next, it is shown that $f'(x) = \frac{\partial}{\partial x}x/(|x|+\epsilon) = \epsilon/(|x|+\epsilon)^2$ is Lipschitz in x.

Case 1: $x, y > 0$: For $x > 0$, the function simplifies to

$$f'(x) = \frac{\epsilon}{(x+\epsilon)^2}\,.$$

Differentiating with respect to x yields

$$f''(x) = \frac{-2\epsilon}{(x+\epsilon)^3}\,,$$

which exists for all $x > 0$, and it is continuous and monotonically increasing in x. Furthermore, as x approaches infinity, $f''(x)$ approaches 0, and as x approaches 0 from the right, $f'(x)$ approaches $-2/\epsilon^2$. Thus, $|f''(x)|$ is bounded by $\frac{2}{\epsilon^2}$ for all $x > 0$.

By the Mean Value Theorem it follows that

$$|f'(x) - f'(y)| \le \frac{2}{\epsilon^2}|x-y|\,.$$

Case 2: $x > 0, y < 0$: Note that

$$|f'(x) - f'(y)| = \left|\frac{\epsilon}{(x+\epsilon)^2} - \frac{\epsilon}{(-y+\epsilon)^2}\right| = \left|\frac{\epsilon}{(x+\epsilon)^2} - \frac{\epsilon}{(z+\epsilon)^2}\right| = |f'(x) - f'(z)|\,, \tag{217}$$

for $z = -y > 0$. Thus analogous to Case 1,

$$|f'(x) - f'(z)| \le \frac{2}{\epsilon^2}|x-z| < \frac{2}{\epsilon^2}|x+z| = \frac{2}{\epsilon^2}|x-y|,$$

since $z = -y > 0, x > 0$.

Case 3: $x > 0, y = 0$
The difference $|f'(x) - \epsilon^{-1}|$ simplifies to

$$\frac{x^2 + 2\varepsilon x}{x^2 + 2\epsilon x + \epsilon^2}.$$

This expression is less than Cx if and only if

$$x^2 + 2\epsilon x \leq Cx^3 + 2C\epsilon x^2 + C\epsilon^2 x.$$

A sufficient condition for this inequality is $C > 2/\epsilon$.
Case 4: $x, y < 0$
By symmetry, the function behaves the same way for $x, y < 0$ as it does for $x, y > 0$. Thus, the Lipschitz constant derived for $x, y > 0$ also applies here.

This establishes that $f'(x)$ is indeed Lipschitz in x for all $x \in \Re$. Claim (iii) follows since $\zeta''(x)$ is Lipschitz in x and $\bar{\varepsilon} - \zeta'(Z)$ is Lipschitz in both $Z, \bar{\varepsilon}$ and by (216). □

lemma:psi

Lemma G.2. *The following facts hold for the proximal operator $prox_{b\zeta}(x)$ and the function $\Psi(y, s) = b(y - prox_{b\zeta}(by + s))$:*

i) The proximal operator $prox_{b\zeta}(x)$ is nonexpansive, i.e. for $x, x' \in \Re$:

$$\left|prox_{b\zeta}(x) - prox_{b\zeta}(x')\right| \leq |x - x'|. \tag{218}$$

ii) $\Psi(h_\epsilon(Z, \bar{\varepsilon}), s) = b(h_\epsilon(Z, \bar{\varepsilon}) - \zeta'(prox_{b\zeta}(bh_\epsilon(Z, \bar{\varepsilon}) + s)))$ is Lipschitz continuous in $[Z, s, \bar{\varepsilon}]^\top$.

iii) $\Psi(y, s) = b(y - \zeta'(prox_{b\zeta}(by + s)))$ is Lipschitz continuous in s with Lipschitz constant at most 1.

iv) $\frac{\partial}{\partial Z}\Psi(h_\epsilon(Z, \bar{\varepsilon}), s)$ exists for all $Z, \bar{\varepsilon}, s \in \Re$, $\epsilon > 0$ and is Lipschitz

Proof. i) Note that $\zeta(x)$ is proper, closed and convex (see e.g. Beck 2017, Chapter 2, for the definition). Indeed, properness of $\zeta(x) : \Re \to [\infty, \infty]$ follows since it is monotonically increasing in x, $\lim_{x\to\infty} \zeta(x) = 0$ and $f(0) = 1$. Closedness follows since lower semicontinuity is equivalent to closedness (e.g. Beck 2017, Theorem 2.6) and $\zeta(x)$ is clearly continuous and therefore closed. Hence the claim follows by nonexpansivity of the proximal operator for proper, closed and convex functions (see for example Beck 2017, Theorem 6.4.2)b)), for $x, x' \in \Re$,

$$\left|\mathrm{prox}_{b\zeta}(x) - \mathrm{prox}_{b\zeta}(x')\right| \leq |x - x'|. \tag{219}$$

ii) It is shown in Lemma G.1 ii) that the approximation h_ϵ is Lipschitz.
By nonexpansivity of the proximal operator, which is shown in Lemma G.2 i),

$$\left|\mathrm{prox}_{b\zeta}(x) - \mathrm{prox}_{b\zeta}(x')\right| \leq |x - x'|. \tag{220}$$

Now letting $x = bh_\epsilon(Z, \bar{\varepsilon}) + s$, $x' = bh_\epsilon(Z', \bar{\varepsilon}') + s'$ then by Lipschtzianity of h_ϵ, and the triangle inequality and Jensen's inequality, again

$$\begin{aligned}\left|\mathrm{prox}_{b\zeta}(bh_\epsilon(Z, \bar{\varepsilon}) + s) - \mathrm{prox}_{b\zeta}(bh_\epsilon(Z', \bar{\varepsilon}') + s')\right| &\leq |b(h_\epsilon(Z, \bar{\varepsilon}) - h_\epsilon(Z', \bar{\varepsilon}')) + (s - s')| \\ &\leq bL\{|\bar{\varepsilon} - \bar{\varepsilon}'| + |Z - Z'|\} + |s - s'| \\ &\leq \max\{bL, 1\}\{|\bar{\varepsilon} - \bar{\varepsilon}'| + |Z - Z'| + |s - s'|\} \\ &\leq \sqrt{3}\max\{bL, 1\}\left\|[Z, \bar{\varepsilon}, s]^\top - [Z', \bar{\varepsilon}', s']^\top\right\|_2,\end{aligned} \tag{221}$$

for some constant L. Hence, again by boundedness of $\frac{\partial^2}{\partial x^2}\zeta(x)$, it immediately follows that for any $t \in [q]$, $\mathbf{x}, \mathbf{x}'\mathbf{y}, \mathbf{y}' \in \Re^q$, and $b > 0$, note that

$$
\begin{aligned}
&|\Psi(h_\epsilon(Z,\bar{\varepsilon}),s) - \Psi(h_\epsilon(Z',\bar{\varepsilon}'),s')| \\
&\qquad \leq b\,|h_\epsilon(Z,\bar{\varepsilon}) - h_\epsilon(Z,\bar{\varepsilon})| \\
&\qquad + b\left|\zeta'\left(\mathrm{prox}_{b\zeta}\left(bh_\epsilon(Z,\bar{\varepsilon})+s\right)\right) - \zeta'\left(\mathrm{prox}_{b\zeta}\left(bh_\epsilon(Z',\bar{\varepsilon}')+s'\right)\right)\right| \\
&\qquad \leq bL\left\{|\bar{\varepsilon}-\bar{\varepsilon}'| + |Z-Z'|\right\} + \\
&\qquad \frac{b}{4}\max\{bL,1\}\left\{|\bar{\varepsilon}-\bar{\varepsilon}'| + |Z-Z'| + |s-s'|\right\} \\
&\qquad = \left(bL + \frac{b}{4}\max\{bL,1\}\right)\left\{|\bar{\varepsilon}-\bar{\varepsilon}'| + |Z-Z'|\right\} + \frac{b}{4}\max\{bL,1\}\,|s-s'| \\
&\qquad \leq \sqrt{3}\left(bL + \frac{b}{4}\max\{bL,1\}\right)\left\|[Z,s,\bar{\varepsilon}]^\top - Z',s',\bar{\varepsilon}']^\top\right\|_2 .
\end{aligned} \tag{222}
$$

iii) From Donoho and Montanari (2016), Proposition 6.3,

$$
\frac{\partial}{\partial s}\Psi(y,s) = -\frac{b\zeta''(\mathrm{prox}_{b\zeta}\left(by+s\right))}{1+b\zeta''(\mathrm{prox}_{b\zeta}\left(by+s\right))}\,, \tag{223}
$$

eq:psi_partial_s

which is less than one in absolute value and the claim follows by the mean value theorem.

iv) By Proposition 6.3 of Donoho and Montanari (2016), the proximal operator $\mathrm{prox}_{b\zeta}(x)$ is differentiable for $b > 0$, $x \in \Re$ with partial derivative

$$
\frac{\partial}{\partial x}\mathrm{prox}_{b\zeta}(x) = \left.\frac{1}{1+b\zeta''(z)}\right|_{z=\mathrm{prox}_{b\zeta}(x)} . \tag{224}
$$

Consequently,

$$
\frac{\partial}{\partial x}\zeta'(\mathrm{prox}_{b\zeta}(x)) = \left.\frac{b\zeta''(z)}{1+b\zeta''(z)}\right|_{z=\mathrm{prox}_{b\zeta}(x)} , \tag{225}
$$

and thus using Lemma G.1 iii), and the chain rule,

$$
\begin{aligned}
\frac{\partial}{\partial Z}\Psi(h_\epsilon(Z,\bar{\varepsilon}),s) &= \left.\frac{\partial}{\partial y}\Psi(y,s)\right|_{y=h_\epsilon(Z,\bar{\varepsilon})}\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon}) \\
&= b\left(1 - \frac{b\zeta''(\mathrm{prox}_{b\zeta}\left(bh_\epsilon(Z,\bar{\varepsilon})+s\right))}{1+b\zeta''(\mathrm{prox}_{b\zeta}\left(bh_\epsilon(Z,\bar{\varepsilon})+s\right))}\right)\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon}) \\
&= \frac{b}{1+b\zeta''(\mathrm{prox}_{b\zeta}\left(bh_\epsilon(Z,\bar{\varepsilon})+s\right))}\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon})
\end{aligned} \tag{226}
$$

Thus by simple expansion and the triangle inequality,

$$
\begin{aligned}
&\left|\frac{\partial}{\partial Z}\Psi(h_\epsilon(Z,\bar{\varepsilon}),s)-\frac{\partial}{\partial Z}\Psi(h_\epsilon(Z',\bar{\varepsilon}'),s')\right| \\
&\quad = b\left|\left\{\frac{1}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z,\bar{\varepsilon})+s))}-\frac{1}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z',\bar{\varepsilon}')+s'))}\right\}\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon})\right. \\
&\qquad\qquad \left.+\frac{1}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z',\bar{\varepsilon}')+s'))}\left\{\frac{\partial}{\partial Z}h_\epsilon(Z',\bar{\varepsilon}')-\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon})\right\}\right| \\
&\quad \le b\left|\frac{b}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z,\bar{\varepsilon})+s))}-\frac{1}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z',\bar{\varepsilon}')+s'))}\right|\left|\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon})\right| \\
&\qquad\qquad +b\left|\frac{1}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z',\bar{\varepsilon}')+s'))}\right|\left|\frac{\partial}{\partial Z}h_\epsilon(Z',\bar{\varepsilon}')-\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon})\right|. \qquad (227)
\end{aligned}
$$

eq:psi_deriv_lipschitz

Now it is easily seen that the function $f(x)=1/(1+x)$ is Lipschitz over the nonnegative real line as it is differentiable and its derivative $\frac{\partial}{\partial x}f(x)=-1/(1+x)^2$ is bounded from above by 1 in absolute value for $x\ge 0$. Further, it is easily seen that $\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon})$ is bounded in absolute value. Hence, using boundedness of $\zeta'''(x)$, Lemma G.1 ii) and Lemma G.2 i), it follows immediately that

$$
\begin{aligned}
&b\left|\frac{1}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z,\bar{\varepsilon})+s))}-\frac{1}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z',\bar{\varepsilon}')+s'))}\right|\left|\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon})\right| \\
&\quad \le bC\left|\frac{1}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z,\bar{\varepsilon})+s))}-\frac{1}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z',\bar{\varepsilon}')+s'))}\right| \\
&\quad \le b^2CC'\left|\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z,\bar{\varepsilon})+s))-\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z',\bar{\varepsilon}')+s'))\right| \\
&\quad \le b^2CC'C''\left|\operatorname{prox}_{b\zeta}(bh_\epsilon(Z,\bar{\varepsilon})+s)-\operatorname{prox}_{b\zeta}(bh_\epsilon(Z',\bar{\varepsilon}')+s')\right| \\
&\quad \le b^2CC'C''C'''\left\|[Z,s,\bar{\varepsilon}]^\top-[Z',s',\bar{\varepsilon}']^\top\right\|, \qquad (228)
\end{aligned}
$$

for some positive constants C,C',C'',C'''. Similarly, by nonnegativity of ζ'' and b, as well as Lemma G.1 iii),

$$
\begin{aligned}
&b\left|\frac{1}{1+b\zeta''(\operatorname{prox}_{b\zeta}(bh_\epsilon(Z',\bar{\varepsilon}')+s'))}\right|\left|\frac{\partial}{\partial Z}h_\epsilon(Z',\bar{\varepsilon}')-\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon})\right| \\
&\quad \le bC\left|\frac{\partial}{\partial Z}h_\epsilon(Z',\bar{\varepsilon}')-\frac{\partial}{\partial Z}h_\epsilon(Z,\bar{\varepsilon})\right| \\
&\quad \le bCC'\left\|[Z,s,\bar{\varepsilon}]^\top-[Z',s',\bar{\varepsilon}']^\top\right\|, \qquad (229)
\end{aligned}
$$

for some positive constants C,C'. Along with (227) this concludes the proof. □

lemma:x_mat

Lemma G.3. *Let $\boldsymbol{X}\in\Re^{n\times p}$ have i.i.d. entries $x_{ij}\sim\mathcal{N}\left(0,n^{-1}\right)$ and let $\lim_{n\to\infty}p/n=\kappa\in(0,1)$. Then almost surely, $\left\|\boldsymbol{X}\right\|_2<\infty$.*

Proof. This follows directly from Rudelson and Vershynin 2010, Theorem 2.1. □

lemma:y_epsilon

Lemma G.4. *For $i=1,\dots,n$, let $\bar{\varepsilon}_i$ be i.i.d. $U(0,1)$ random variables and $\boldsymbol{\eta}_{0,i}$ be i.i.d. $N(0,\gamma^2)$ such that $\bar{\varepsilon}_i\perp\boldsymbol{\eta}_{0,j}$ for $i,j=1,\dots,n$. Further, let*

$$y_i^* = \alpha \mathbb{1}\left\{\bar{\varepsilon}_i < \zeta'(\boldsymbol{\eta}_{0,i})\right\} + \frac{1-\alpha}{2}, \quad {}_\epsilon y_i^* = \alpha\left\{1 - \frac{\bar{\varepsilon}_i - \zeta'(\boldsymbol{\eta}_{0,i})}{|\bar{\varepsilon}_i - \zeta'(\boldsymbol{\eta}_{0,i})| + \epsilon}\right\} + \frac{1-\alpha}{2}\,. \tag{230}$$

Then for $\boldsymbol{y}^* = (y_1^*, \ldots, y_n^*)$, ${}_\epsilon\boldsymbol{y}^* = ({}_\epsilon y_1^*, \ldots, {}_\epsilon y_n^*)$,

i) $\lim_{n\to\infty} \frac{1}{p}\|\boldsymbol{y}^* - {}_\epsilon\boldsymbol{y}^*\|_2^2 = E\left[(y_i^* - {}_\epsilon y_i^*)^2\right]$ *almost surely,*

ii) $\lim_{\epsilon\to 0} E\left[(y_i^* - {}_\epsilon y_i^*)^2\right] = 0$

Proof. (i) By definition $\frac{1}{p}\|\boldsymbol{y}^* - {}_\epsilon\boldsymbol{y}^*\|_2^2 = \frac{1}{p}\sum_{i=1}^n (y_i^* - {}_\epsilon y_i^*)^2$. By boundedness of $y_i^*, {}_\epsilon y_i^*$, $\mathrm{E}\left[(y_i^* - {}_\epsilon y_i^*)^2\right]$ exists and thus, by the strong law of large numbers (e.g. Vershynin 2018, Theorem 1.3.1), claim (i) follows. (ii) Note that since $\bar{\varepsilon}_i, \boldsymbol{\eta}_{0,i}$ are continuous independent random variables. Hence the event $\{\exists i : \bar{\varepsilon}_i = \zeta'(\boldsymbol{\eta}_{0,i})\}$ has measure zero. Hence, $\mathrm{E}[(y^* - {}_\epsilon y^*)^2] = \mathrm{E}[(h(\bar{\varepsilon}_i, \boldsymbol{\eta}_{0,i}) - {}_\epsilon h(\bar{\varepsilon}_i, \boldsymbol{\eta}_{0,i}))^2]$. The claim now follows from boundedness of $\mathrm{E}\left[(y_i^* - {}_\epsilon y_i^*)^2\right]$, the dominated convergence theorem and Lemma G.1 i). □

lemma:omega_fun

Lemma G.5. *The map $f(\boldsymbol{\beta}) = \omega\left(\|\boldsymbol{\beta}\|_2/\sqrt{n}\right)$ of Lemma 7 in Sur et al. (2019) is such that if $\lim_{\epsilon\to 0}\|\boldsymbol{\beta}(\epsilon)\|_2 = r$, for some sequence $\boldsymbol{\beta}(\epsilon) \in \Re^p$ indexed by ϵ, then $\lim_{\epsilon\to 0} f(\boldsymbol{\beta}(\epsilon)) = \omega\left(r/\sqrt{n}\right)$.*

Proof. It is readily checked that the function $\omega\left(\|\boldsymbol{\beta}\|_2/\sqrt{n}\right)$ is of the form $c_1 \inf_{z:|z|\le c_2 \frac{\|\boldsymbol{\beta}\|_2}{\sqrt{n}}} \zeta''(z)$ for some constants $c_1, c_2 > 0$ that do not depend on $\boldsymbol{\beta}$. Now $\zeta''(z)$ is symmetric and strictly decreasing in $|z|$ so that $\inf_{z:|z|\le c_2 \frac{\|\boldsymbol{\beta}\|_2}{\sqrt{n}}} \zeta''(z) = \zeta''\left(c_2\|\boldsymbol{\beta}\|_2/\sqrt{n}\right)$. The claim follows by taking limits and the continuity of the function ζ''. □

lemma:IFT

Lemma G.6. *Consider the system of equations $F(\mu, b, \sigma, \epsilon) = 0$ given by*

$$E\left[\frac{1}{1 + b\zeta''\left(prox_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)}\right] - 1 + \kappa = 0 \tag{231}$$

eq:approx_bt2

$$\frac{b}{\kappa\gamma^2} E\left[Z\left\{{}_\epsilon Y^* - \zeta'\left(prox_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}\right] = 0 \tag{232}$$

eq:approx_mut2

$$\frac{b^2}{\kappa^2} E\left[\left\{{}_\epsilon Y^* - \zeta'\left(prox_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}^2\right] - \sigma^2 = 0\,, \tag{233}$$

eq:approx_sigmat2

where $Z \sim N(0, \gamma^2)$, $G \sim N(0,1)$, and $\bar{\varepsilon} \sim U(0,1)$, all independent of each other, and ${}_\epsilon Y^* = h_\epsilon(Z, \bar{\varepsilon})$, $Z_* = \mu Z + \kappa^{1/2}\sigma G$.

Assume there exists a solution $F(\mu_*, b_*, \sigma_*, 0) = 0$ such that the Jacobian with respect to (μ_*, b_*, σ_*) of $F(\mu_*, b_*, \sigma_*, 0)$ is nonsingular.

Then, in a neighbourhood around (μ_*, b_*, σ_*), $\epsilon = 0$, there exist unique functions ${}_\epsilon\mu_*, {}_\epsilon b_*, {}_\epsilon\sigma_*$ that are continuous in ϵ such that $F({}_\epsilon\mu_*, {}_\epsilon b_*, {}_\epsilon\sigma_*, \epsilon) = 0$.

Proof. The claim follows by the Implicit Function Theorem (e.g. Rudin et al. 1976, Theorem 9.28), for which it is required that

(i) There exists a solution $F(\mu_*, b_*, \sigma_*, 0)$,

(ii) The Jacobian with respect to (μ_*, b_*, σ_*) of $F(\mu_*, b_*, \sigma_*, 0)$ is nonsingular

(iii) $F(\mu, b, \sigma, \epsilon)$ is continuously differentiable in a neighbourhood around (μ_*, b_*, σ_*), $\epsilon = 0$

(i) and (ii) are assumed so that it remains to show (iii). Hence, it is sufficient to show that the partial derivatives of (231)-(233) exist and are continuous in a neighbourhood around $(\mu_*, b_*, \sigma_*), \epsilon = 0$. Begin with (231) and consider

$$
\begin{aligned}
\frac{\partial}{\partial b}\frac{1}{1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)} &= -\frac{1}{\left[1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right]^2}\frac{\partial}{\partial b}b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\\
&= -\frac{1}{\left[1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right]^2}\Bigg\{\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\\
+b\zeta'''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)&\left[\frac{\partial}{\partial u}\operatorname{prox}_{b\zeta}(u)\Big|_{u=b_\epsilon Y^*+Z_*}\frac{\partial}{\partial b}u+\frac{\partial}{\partial b}\operatorname{prox}_{b\zeta}(u)\Big|_{u=b_\epsilon Y^*+Z_*}\right]\Bigg\}\\
&= -\frac{1}{\left[1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right]^2}\Bigg\{\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\\
+b\zeta'''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)&\left[\frac{{}_\epsilon Y^*}{1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}-\frac{\zeta'\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}{1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}\right]\Bigg\}\\
&= -\frac{1}{\left[1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right]^2}\\
\cdot\Bigg\{\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)+&\frac{b\zeta'''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}{1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}\left[{}_\epsilon Y^*-\zeta'\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right]\Bigg\},
\end{aligned}
\tag{234}
$$

eq:b_partial_b

where the first and second equality follow by the chain rule and the third uses Proposition A.1 of Donoho and Montanari (2016). Now note that $\zeta'(x) \in (0,1)$, $\zeta''(x) \in (0, \frac{1}{4}]$, $\zeta'''(x) \in \left[-\frac{1}{6\sqrt{3}}, \frac{1}{6\sqrt{3}}\right]$ and ${}_\epsilon Y^* \in [\frac{1-\alpha}{2}, \frac{1+\alpha}{2}]$. Hence (234) is bounded. Thus, by the Dominated Convergence Theorem and the Mean Value Theorem,

$$
\frac{\partial}{\partial b}\mathrm{E}\left[\frac{1}{1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}\right] = \mathrm{E}\left[\frac{\partial}{\partial b}\frac{1}{1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}\right]. \tag{235}
$$

eq:b_partial_b2

And again by dominated convergence, (235) is continuous if (234) is. By Lemma G.7, $\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)$ is continuous in $(b, {}_\epsilon Y^*, Z_*)$. Since ${}_\epsilon Y^* = \frac{\alpha}{2}\left\{1 - \frac{\bar{\varepsilon}-\zeta'(Z)}{|\bar{\varepsilon}-\zeta'(Z)|+\epsilon}\right\}$ and $Z_* = \mu Z + \kappa^{1/2}\sigma G$, $\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)$ is continuous in $(\mu, b\sigma, \epsilon)$. The claim follows by continuity of $\zeta'(\cdot), \zeta''(\cdot), \zeta'''(\cdot)$ and since (234) is the composition of continuous functions. The treatment of the other partial derivatives is similar and thus details are omitted.

Next consider,

$$
\frac{\partial}{\partial \mu}\frac{1}{1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)} = -\frac{b\zeta'''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)Z}{\left[1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right]^3}, \tag{236}
$$

eq:b_partial_mu

which is again dominated by an integrable function and thus $\frac{\partial}{\partial\mu}\mathrm{E}\left[\frac{1}{1+b\zeta''\left(\operatorname{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}\right]$ is continuous in $(\mu, b, \sigma, \epsilon)$.

Similarly,

$$\frac{\partial}{\partial\sigma}\frac{1}{1+b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}=-\frac{b\zeta'''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\kappa^{1/2}G}{\left[1+b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right]^3}\,, \tag{237}$$

eq:b_partial_sigma

and the same conclusion as for (236) holds for $\frac{\partial}{\partial\sigma}\mathrm{E}\left[\frac{1}{1+b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}\right]$.

Finally,

$$\frac{\partial}{\partial\epsilon}\frac{1}{1+b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}=-\frac{b\zeta'''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}{\left[1+b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right]^3}\frac{\alpha}{2}\frac{\bar{\varepsilon}-\zeta'(Z)}{(|\bar{\varepsilon}-\zeta'(Z)|+\epsilon)^2}\,, \tag{238}$$

eq:b_partial_eps

which is bounded and continuous in (b,μ,σ,ϵ) for $\bar{\varepsilon}\neq\zeta'(Z)$. Since the event $\bar{\varepsilon}=\zeta'(Z)$, has measure zero, as $\bar{\varepsilon}\sim U(0,1)$ and $Z\sim\mathcal{N}(0,1)$ $\bar{\varepsilon}\perp Z$, dominated convergence still applies and the claim holds.

Next, consider (232), and differentiate the integrand with respect to b,

$$\begin{aligned}\frac{\partial}{\partial b}\frac{b}{\kappa\gamma^2}Z\left\{{}_\epsilon Y^*-\zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right\}&=\frac{1}{\kappa\gamma^2}Z\left\{{}_\epsilon Y^*-\zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right\}\\&\quad-Z\frac{\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}{1+b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}\left\{{}_\epsilon Y-\zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right\}\,,\end{aligned} \tag{239}$$

of which both terms on the RHS are dominated by an integrable function and which are continuous in (μ,b,σ,ϵ). Thus $\frac{\partial}{\partial b}\frac{b}{\kappa\gamma^2}\mathrm{E}\left[Z\left\{{}_\epsilon Y^*-\zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right\}\right]$ exists and is continuous in (μ,b,σ,ϵ).

Next, differentiate with respect to μ to get

$$\frac{\partial}{\partial\mu}\frac{b}{\kappa\gamma^2}Z\left\{{}_\epsilon Y^*-\zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right\}=-\frac{b}{\kappa\gamma^2}\frac{Z^2\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}{1+b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}\,, \tag{240}$$

which is again dominated by an integrable function and continuous in (μ,b,σ,ϵ), so that

$$\frac{\partial}{\partial\mu}\frac{b}{\kappa\gamma^2}\mathrm{E}\left[Z\left\{{}_\epsilon Y^*-\zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right\}\right]$$

exists and is continuous in (μ,b,σ,ϵ).

Differentiating with respect to σ yields

$$\frac{\partial}{\partial\mu}\frac{b}{\kappa\gamma^2}Z\left\{{}_\epsilon Y^*-\zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right\}=-\frac{b}{\kappa\gamma^2}\frac{Z\kappa^{1/2}G\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}{1+b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}\,, \tag{241}$$

and the same conclusion holds for $\frac{\partial}{\partial\sigma}\frac{b}{\kappa\gamma^2}\mathrm{E}\left[Z\left\{{}_\epsilon Y^*-\zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right\}\right]$.

Finally, differentiating with respect to ϵ,

$$\frac{\partial}{\partial\mu}\frac{b}{\kappa\gamma^2}Z\left\{{}_\epsilon Y^*-\zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right\}=-\frac{b}{\kappa\gamma^2}Z\left\{1-\frac{b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}{1+b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)}\right\}\frac{\alpha}{2}\frac{\bar{\varepsilon}-\zeta'(Z)}{(|\bar{\varepsilon}-\zeta'(Z)|+\epsilon)^2}\,, \tag{242}$$

and arguing analogous to (238) yields that $\frac{\partial}{\partial\epsilon}\frac{b}{\kappa\gamma^2}\mathrm{E}\left[Z\left\{{}_\epsilon Y^*-\zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^*+Z_*\right)\right)\right\}\right]$ exists and is continuous in (μ,b,σ,ϵ).

Lastly, the same is repeated for (233).

$$\frac{\partial}{\partial b}\frac{b^2}{\kappa^2}\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}^2 - \sigma^2 = \frac{2b}{\kappa^2}\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}^2 - 1$$
$$- \frac{2b^2}{\kappa^2}\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}\frac{\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)}{1 + b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)}$$
$$\cdot\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}, \tag{243}$$

$$\frac{\partial}{\partial \mu}\frac{b^2}{\kappa^2}\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}^2 - \sigma^2 = -\frac{2b^2}{\kappa^2}\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}\frac{Z\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)}{1 + b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)}, \tag{244}$$

$$\frac{\partial}{\partial \sigma}\frac{b^2}{\kappa^2}\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}^2 - \sigma^2 = -\frac{2b^2}{\kappa^2}\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}\frac{\kappa^{1/2}G\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)}{1 + b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)}$$
$$- 2\sigma, \tag{245}$$

$$\frac{\partial}{\partial \epsilon}\frac{b^2}{\kappa^2}\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}^2 - \sigma^2 = -\frac{2b^2}{\kappa^2}\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}$$
$$\cdot\frac{1}{1 + b\zeta''\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)}\frac{\alpha}{2}\frac{\bar{\varepsilon} - \zeta'(Z)}{(|\bar{\varepsilon} - \zeta'(Z)| + \epsilon)^2}, \tag{246}$$

all of which are dominated by some integrable function and continuous almost everywhere. Thus, again by dominated convergence, all partial derivates of $\frac{b^2}{\kappa^2}\mathrm{E}\left[\left\{{}_\epsilon Y^* - \zeta'\left(\mathrm{prox}_{b\zeta}\left(b_\epsilon Y^* + Z_*\right)\right)\right\}^2\right] - \sigma^2$ exist and are continuous. Thus, $F(\mu, b, \sigma, \epsilon)$ is continuously differentiable in a neighbourhood around $(\mu_*, b_*, \sigma, 0)$ and the Implicit Function Theorem establishes the claim of the Lemma. □

lemma:cont_prox

Lemma G.7. *The proximal operator $\mathrm{prox}_{b\zeta}(by + z)$ is continuous in (b, y, z) for $b > 0$, $x, y \in \Re$ where $\zeta(x) = \log(1 + e^x)$.*

Proof. Recall that by definition of the proximal operator,

$$\mathrm{prox}_{b\zeta}(by + z) = \underset{u\in\Re}{\arg\min}\left\{b\zeta(u) + \frac{1}{2}(u - [by + z])^2\right\}, \tag{247}$$

and since $\zeta(\cdot)$ is differentiable and strictly convex, it is uniquely determined as the solution to the first order condition

$$b\zeta'(u) + u - by - z = 0. \tag{248}$$

Now note that $F(u, b, y, z) = b\zeta'(u) + u - by - z$ is continuously differentiable in (u, b, y, z) and that $\frac{\partial}{\partial u}F(u, b, y, z) = b\zeta''(u) + 1 > 0$ for all (u, b, y, z). Hence, by the Implicit Function Theorem (e.g. Rudin et al. 1976, Theorem 9.28), there exists a unique function $u(b', y', z')$ in a neighbourhood around (b, y, z) that is continuous in its arguments such that $F(u(b', y', z'), b', y', z') = 0$. Hence, the proximal operator is continuous in a neighbourhood around (b, y, z). Since the choice of (b, y, z) was arbitrary, it is continuous over its entire domain. □

lemma:cauchy_property

Lemma G.8. *In the setting of Corollary B.2.1 and assuming that a a stationary solution* $({}_\epsilon\mu_*, {}_\epsilon b_*, {}_\epsilon\sigma_*)$ *to the state evolution* (56)-(57) *exists, the AMP iterates of* (59) *and* (60) *satisfy the Cauchy property*

$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{p}\left\|{}_\epsilon\boldsymbol{\nu}^{t+1} - {}_\epsilon\boldsymbol{\nu}^t\right\|_2^2 = 0$$
$$\lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{n}\left\|{}_\epsilon\boldsymbol{R}^{t+1} - {}_\epsilon\boldsymbol{R}^t\right\|_2^2 = 0 \tag{249}$$

eq:cauchy_prop

almost surely.

Proof. The proof is a direct translation of the Lemmas 6.8-6.9 of Donoho and Montanari (2016) along the lines suggested by Sur and Candès 2019 in the proof of Theorem 7 of the supplementary material. In particular, isolating iterates ${}_\epsilon\boldsymbol{\nu}^{t-1}, {}_\epsilon\boldsymbol{\nu}^t$ in Theorem B.2, similar arguments to those in Corollary B.2.1 yield that, almost surely

$$\lim_{N\to\infty}\frac{1}{p}\sum_{j=1}\psi\left(\begin{bmatrix}{}_\epsilon\boldsymbol{\nu}_j^{t-1}\\ {}_\epsilon\boldsymbol{\nu}_j^t\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right) = \mathrm{E}\left[\psi\left(\begin{bmatrix}\boldsymbol{Z}_t^{2t}\\ \boldsymbol{Z}_{t-1}^{2t}\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right)\right], \quad \begin{bmatrix}\boldsymbol{Z}_{t-1}^{2t}\\ \boldsymbol{Z}_t^{2t}\end{bmatrix} \sim N\left(\mathbf{0}_2, \begin{bmatrix}{}_\epsilon\tau_{t-1}^2 & {}_\epsilon\rho_{t-1,t}\\ {}_\epsilon\rho_{t-1,t} & {}_\epsilon\tau_t^2\end{bmatrix}\right) \tag{250}$$

where

$$\begin{aligned}{}_\epsilon\tau_t^2 &= \kappa^{-2}\mathrm{E}\left[\{\Psi^{t-1}(h_\epsilon(\boldsymbol{Z}_1^{2t-1}, \bar{\varepsilon}), \boldsymbol{Z}_2^{2t-1})\}^2\right]\\ {}_\epsilon\rho_{t-1,t} &= \kappa^{-2}\mathrm{E}\left[\Psi^{t-1}(h_\epsilon(\boldsymbol{Z}_1^{2t-1}, \bar{\varepsilon}), \boldsymbol{Z}_{t+1}^{2t-1})\Psi^{t-2}(h_\epsilon(\boldsymbol{Z}_1^{2t-1}, \bar{\varepsilon}), \boldsymbol{Z}_t^{2t-1})\right],\end{aligned} \tag{251}$$

which in turn have covariance structure according to the state evolution (12)-(13). Now recall that given the covariance structure derived in Corollary B.2.1, it was argued that given initial conditions $\hat{\boldsymbol{\beta}}^0$,

$$\begin{aligned}{}_\epsilon\mu_0 &= \frac{1}{\gamma^2}\lim_{n\to\infty}\frac{\langle\hat{\boldsymbol{\beta}}^0, \boldsymbol{\beta}_0\rangle}{n}\\ {}_\epsilon\sigma_0^2 &= \lim_{n\to\infty}\frac{\left\|\hat{\boldsymbol{\beta}}^0 - \mu_0\boldsymbol{\beta}_0\right\|_2^2}{p},\end{aligned} \tag{252}$$

eq:inits

for all t, ${}_\epsilon\tau_t = {}_\epsilon\sigma_t$. Now, given a stationary solutions $({}_\epsilon\mu_*, {}_\epsilon b_{*\epsilon}, \sigma_*)$ to the state evolutions (55)-(57), choose the oracle initializer $\hat{\boldsymbol{\beta}}^0 = {}_\epsilon\mu_*\boldsymbol{\beta}_0 + {}_\epsilon\sigma_*\boldsymbol{\xi}$, where $\boldsymbol{\xi} \sim N(\mathbf{0}_p, \boldsymbol{I}_p)$ which is independent of $\boldsymbol{\beta}_0$. Then the initial conditions of (252) are

$$\begin{aligned}{}_\epsilon\mu_0 &= \frac{1}{\gamma^2}\lim_{n\to\infty}\frac{\langle{}_\epsilon\mu_*\boldsymbol{\beta}_0 + {}_\epsilon\sigma_*\boldsymbol{\xi}, \boldsymbol{\beta}_0\rangle}{n} \overset{a.s}{=} {}_\epsilon\mu_*\\ {}_\epsilon\sigma_0^2 &= \lim_{n\to\infty}\frac{\|\sigma_*\boldsymbol{\xi}\|_2^2}{p} \overset{a.s}{=} {}_\epsilon\sigma_*^2.\end{aligned} \tag{253}$$

eq:inits_stat

Henceforth condition on the event that ${}_\epsilon\mu_0 = {}_\epsilon\mu_*$, ${}_\epsilon\sigma_0^2 = {}_\epsilon\sigma_*^2$. Then ${}_\epsilon\tau_{t-1}^2 = {}_\epsilon\tau_t^2 = {}_\epsilon\sigma_*^2$ and letting $\psi(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{u}_1 - \boldsymbol{u}_2)^2$, for $\boldsymbol{u} = [\boldsymbol{u}_1, \boldsymbol{u}_2]^\top$, one gets that almost surely,

$$\lim_{N\to\infty}\frac{1}{p}\left\|{}_\epsilon\boldsymbol{\nu}^t - {}_\epsilon\boldsymbol{\nu}^{t-1}\right\|_2^2 = 2({}_\epsilon\sigma_*^2 - {}_\epsilon\rho_{t,t-1}). \tag{254}$$

Hence, the goal is to show that $\lim_{t\to\infty}{}_\epsilon\rho_{t,t-1} = {}_\epsilon\sigma_*^2$.

Towards that end, following the proof of Theorem 4.1 of Donoho and Montanari (2016), define the doubly infinite matrix $\boldsymbol{\Gamma} = \{\Gamma_{s,t}\}_{s,t\in\mathbb{N}_0}$ with entries

$$\Gamma_{s+1,t+1} = \frac{1}{\kappa^2}\mathrm{E}\left[\Psi^*\left(h_\epsilon(Z,\bar{\varepsilon}), {}_\epsilon Z_s\right)\Psi^*\left(h_\epsilon(Z,\bar{\varepsilon}), {}_\epsilon Z_t\right)\right] \tag{255}$$

where $\Psi^*(y,s) = {}_\epsilon b_* \left(y - \zeta'\left(\mathrm{prox}_{{}_\epsilon b_* \zeta}\left({}_\epsilon b_* y + s\right)\right)\right)$ and

$$\begin{aligned} {}_\epsilon Z_s &= {}_\epsilon\mu_* Z + \kappa^{1/2}{}_\epsilon\tilde{G}_s \\ {}_\epsilon Z_t &= {}_\epsilon\mu_* Z + \kappa^{1/2}{}_\epsilon\tilde{G}_t \end{aligned}, \quad \begin{bmatrix} {}_\epsilon\tilde{G}_s \\ {}_\epsilon\tilde{G}_t \end{bmatrix} \sim N\left(\mathbf{0}_2, \begin{bmatrix} {}_\epsilon\sigma_*^2 & \Gamma_{s,t} \\ \Gamma_{s,t} & {}_\epsilon\sigma_*^2 \end{bmatrix}\right), \tag{256}$$

where $Z \sim N(0,\gamma^2), \bar{\varepsilon}$, are all pairwise independent and independent of $\tilde{G}_s, \tilde{G}_t$. Further, take the boundary conditions $\Gamma_{0,0} = {}_\epsilon\sigma_*^2$, $\Gamma_{t,0} = \Gamma_{0,t} = 0$ so that in particular $\Gamma_{t,t} = {}_\epsilon\sigma_*^2$.

It is then readily verified that for all t, s, $\Gamma_{t,s} = {}_\epsilon\rho_{t,s}$. Next, let $q_t = \Gamma_{t+1,t}/{}_\epsilon\sigma_*^2$. Then q_t is governed by the recursion

$$\begin{aligned} q_{t+1} &= H(q_t) \\ H(q) &= \frac{1}{\kappa^2{}_\epsilon\sigma_*^2}\mathrm{E}\left[\Psi^*\left(h_\epsilon(Z,\bar{\varepsilon}), {}_\epsilon Z_1\right)\Psi^*\left(h_\epsilon(Z,\bar{\varepsilon}), {}_\epsilon Z_2\right)\right], \end{aligned} \tag{257}$$

with

$$\begin{aligned} {}_\epsilon Z_1 &= {}_\epsilon\mu_* Z + \kappa^{1/2}{}_\epsilon\sigma_* G_1 \\ {}_\epsilon Z_2 &= {}_\epsilon\mu_* Z + \kappa^{1/2}{}_\epsilon\sigma_* G_2 \end{aligned}, \quad \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \sim N\left(\mathbf{0}_2, \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}\right) \perp Z \sim N(0,\gamma^2). \tag{258}$$

Next, it is proven that:

i) $H(1) = 1$

ii) $H(q)$ is increasing for $q \in [0,1]$,

iii) $H(q)$ is strictly convex for $q \in [0,1]$

Towards i) note that

$$H(1) = \frac{1}{\kappa^2{}_\epsilon\sigma_*^2}\mathrm{E}\left[\Psi^*\left(h_\epsilon(Z,\bar{\varepsilon}), {}_\epsilon Z_1\right)^2\right], \tag{259}$$

for ${}_\epsilon Z_1 = {}_\epsilon\mu_* Z + \kappa^{1/2}{}_\epsilon\sigma_*^2{}_\epsilon G$, $G \sim N(0,1) \perp Z$. Thus $H(1) = 1$ since (57) holds for $({}_\epsilon\mu_*, {}_\epsilon b_*, {}_\epsilon\sigma_*)$.

Next, given $\bar{\varepsilon}, Z$, let $g_{Z,\bar{\varepsilon}}(G) = \Psi^*(h_\epsilon(Z,\bar{\varepsilon}), {}_\epsilon\mu_* Z + \kappa^{1/2}{}_\epsilon\sigma_* G)$ and let

$$\mathcal{H}(q) = \mathrm{E}\left[g_{Z,\bar{\varepsilon}}(G_1) g_{Z,\bar{\varepsilon}}(G_2)\middle|\bar{\varepsilon}, Z\right], \quad \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \sim N\left(\mathbf{0}_2, \begin{bmatrix} 1 & q \\ q & 1 \end{bmatrix}\right). \tag{260}$$

eq:h_cond

Then, by linearity of expectations and the tower property, it is sufficient to show that $\mathcal{H}(q)$ satisfies ii) and iii).

Following the argument in Donoho and Montanari 2016, Lemma 6.9, let $\{X_t\}_{t\geq 0}$ be the stationary Ornstein-Uhlenbeck process with covariance $\mathrm{E}(X_0 X_t) = e^{-t}$. Then

$$\mathcal{H}(q) = \mathrm{E}\left[g_{Z,\bar{\varepsilon}}(G_1) g_{Z,\bar{\varepsilon}}(G_2)\right]\Big|_{t=\log(1/q)}, \tag{261}$$

where the expectation is taken with respect to the process $\{X_t\}_{t\geq 0}$. Then by the spectral representation for $t = \log(1/q)$,

$$\mathcal{H}(q) = \sum_{l=1}^{\infty} c_l^2 e^{-lt} = \sum_{l=1}^{\infty} c_l^2 q_l,, \tag{262}$$

and the claim follows since $c_l \neq 0$ at least once for $g_{Z,\bar{\varepsilon}}$ nonlinear.

Hence, it follows that $\lim_{t\to\infty} q_t = 1$ if and only if $H'(1) \leq 1$.

Further, it holds that

$$H'(1) = \frac{1}{\kappa}\mathrm{E}\left[\left\{\frac{\partial}{\partial s}\Psi^*(h_\epsilon(\bar{\varepsilon}, Z), {}_\epsilon Z_1)\right\}^2\right]. \tag{263}$$

Indeed, again condition on $\bar{\varepsilon}, Z$ and consider $\frac{\partial}{\partial q}\mathcal{H}(q)$ as defined in (260). For notational convenience, the conditioning in the expectation of $\mathcal{H}$ is suppressed below and $g_{Z,\bar{\varepsilon}}(G)$ is simply denoted by $g(G)$.

Then, since G_2 can be written as $qG_1 + \sqrt{1-q^2}\tilde{G}$ for $\tilde{G} \sim N(0,1)$ is independent of all other random variables, it holds that

$$\begin{aligned}
\frac{\partial}{\partial q}\mathcal{H}(q) &= \frac{\partial}{\partial q}\mathrm{E}\left[g(G_1)g(G_2)\right] \\
&= \mathrm{E}\left[\frac{\partial}{\partial q}g(G_1)g(G_2)\right] \\
&= \mathrm{E}\left[\frac{\partial}{\partial q}g(G_1)g(qG_1 + \sqrt{1-q^2}\tilde{G})\right] \\
&= \mathrm{E}\left[g(G_1)g'(qG_1 + \sqrt{1-q^2}\tilde{G})\kappa^{1/2}{}_\epsilon\sigma_*\left\{G_1 - \frac{q}{\sqrt{1-q^2}}\tilde{G}\right\}\right] \\
&= \mathrm{E}\left[g(G_1)g'(qG_1 + \sqrt{1-q^2}\tilde{G})\kappa^{1/2}{}_\epsilon\sigma_* G_1\right] - \frac{q}{\sqrt{1-q^2}}\mathrm{E}\left[g(G_1)g'(qG_1 + \sqrt{1-q^2}\tilde{G})\kappa^{1/2}{}_\epsilon\sigma_*\tilde{G}\right],
\end{aligned}$$

eq:H_cond_partial_q1

(264)

where the second inequality follows by dominated convergence. Now let $F(G_1, G_2) = g(G_1)g'(G_2)$. Then note that $\frac{\partial}{\partial G_1}F(G_1,G_2) = \kappa^{1/2}{}_\epsilon\sigma_* g'(G_1)g'(G_2)$ and $\frac{\partial}{\partial G_2}F(G_1,G_2) = \kappa^{1/2}{}_\epsilon\sigma_* g(G_1)g''(G_2)$. Hence, using Gaussian integration by parts, which can be derived as a consequence of Stein's Lemma (Stein, 1981, Lemma 1),

$$\begin{aligned}
\mathrm{E}\left[g(G_1)g'(qG_1 + \sqrt{1-q^2}\tilde{G})\kappa^{1/2}{}_\epsilon\sigma_* G_1\right] &= \mathrm{E}\left[g(G_1)g'(G_2)\kappa^{1/2}{}_\epsilon\sigma_* G_1\right] \\
&= \kappa^{1/2}{}_\epsilon\sigma_*\mathrm{E}\left[G_1 F(G_1,G_2)\right] \\
&= \kappa^{1/2}{}_\epsilon\sigma_*\left\{\mathrm{cov}(G_1,G_1)\mathrm{E}\left[\frac{\partial}{\partial G_1}F(G_1,G_2)\right]\right. \\
&\left. + \mathrm{cov}(G_1,G_2)\mathrm{E}\left[\frac{\partial}{\partial G_2}F(G_1,G_2)\right]\right\} \\
&= \kappa_\epsilon\sigma_*^2\left\{\mathrm{E}\left[\frac{\partial}{\partial G_1}F(G_1,G_2)\right] + q\mathrm{E}\left[\frac{\partial}{\partial G_2}F(G_1,G_2)\right]\right\} \\
&= \kappa_\epsilon\sigma_*^2\left\{\mathrm{E}\left[g'(G_1)g'(G_2)\right] + q\mathrm{E}\left[\frac{\partial}{\partial G_2}g(G_1)g''(G_2)\right]\right\}.
\end{aligned}$$

eq:H_cond_partial_q2

(265)

Similarly,

$$\begin{aligned}
\mathrm{E}\left[g(G_1)g'(qG_1+\sqrt{1-q^2}\tilde{G})\kappa^{1/2}{}_\epsilon\sigma_*\tilde{G}\right] &= \kappa^{1/2}{}_\epsilon\sigma_*\mathrm{E}\left[F(G_1,qG_1+\sqrt{1-q^2}\tilde{G})\tilde{G})\right] \\
&= \kappa^{1/2}{}_\epsilon\sigma_*\Big\{\mathrm{cov}(G_1,\tilde{G})\mathrm{E}\left[\frac{\partial}{\partial G_1}F(G_1,qG_1+\sqrt{1-q^2}\tilde{G})\right] \\
&+\mathrm{cov}(\tilde{G},\tilde{G})\mathrm{E}\left[\frac{\partial}{\partial \tilde{G}}F(G_1,qG_1+\sqrt{1-q^2}\tilde{G})\right]\Big\} \\
&= \sqrt{1-q^2}\kappa_\epsilon\sigma_*^2\mathrm{E}\left[g(G_1)g''(qG_1+\sqrt{1-q^2}\tilde{G})\right] \\
&= \sqrt{1-q^2}\kappa_\epsilon\sigma_*^2\mathrm{E}\left[g(G_1)g''(G_2)\right]\ ,
\end{aligned} \quad \texttt{eq:H_cond_partial_q3} \tag{266}$$

where it was used that $\mathrm{cov}(G_1,\tilde{G})=0$. Thus combining (264)-(266), it follows that

$$\frac{\partial}{\partial q}\mathcal{H}(q)=\kappa_\epsilon\sigma_*^2\mathrm{E}\left[g'(G_1)g'(G_2)\right]\ . \tag{267}$$

Marginalizing over $Z,\bar{\varepsilon}$ then gives

$$\begin{aligned}
\frac{\partial}{\partial q}H(q) &= \frac{1}{\kappa^2{}_\epsilon\sigma_*^2}\frac{\partial}{\partial q}\mathrm{E}_{Z,\bar{\varepsilon}}\left[\mathrm{E}\left[g(G_1)g(G_2)\Big|Z,\bar{\varepsilon}\right]\right] \\
&= \frac{1}{\kappa^2{}_\epsilon\sigma_*^2}\mathrm{E}_{Z,\bar{\varepsilon}}\left[\mathrm{E}\left[\frac{\partial}{\partial q}g(G_1)g(G_2)\Big|Z,\bar{\varepsilon}\right]\right] \\
&= \frac{1}{\kappa}\mathrm{E}_{Z,\bar{\varepsilon}}\left[\mathrm{E}\left[g'(G_1)g'(G_2)\Big|Z,\bar{\varepsilon}\right]\right] \\
&= \frac{1}{\kappa}\mathrm{E}\left[g'(G_1)g'(G_2)\right] \\
&= \frac{1}{\kappa}\mathrm{E}\left[g'(G_1)g'(qG_1+\sqrt{1-q^2}\tilde{G})\right]\ ,
\end{aligned} \tag{268}$$

and it thus follows that $H'(1)=\frac{1}{\kappa}\mathrm{E}\left[\left\{\frac{\partial}{\partial s}\Psi^*(h_\epsilon(\bar{\varepsilon},Z),{}_\epsilon Z_1)\right\}^2\right]$ as required.

Now it is easily checked that $\frac{\partial}{\partial s}\Psi^*\in(0,1)$ and thus

$$H'(1)=\frac{1}{\kappa}\mathrm{E}\left[\left\{\frac{\partial}{\partial s}\Psi^*(h_\epsilon(\bar{\varepsilon},Z),{}_\epsilon Z_1)\right\}^2\right]\leq\frac{1}{\kappa}\mathrm{E}\left[\frac{\partial}{\partial s}\Psi^*(h_\epsilon(\bar{\varepsilon},Z),{}_\epsilon Z_1)\right]=1 \tag{269}$$

since (55) holds. Similar arguments show the Cauchy-property for $\frac{1}{n}\|{}_\epsilon\boldsymbol{R}^t-{}_\epsilon\boldsymbol{R}^{t-1}\|_2^2$. □

`lemma:eq_state_ev`

Lemma G.9. *The solution to the system of equations* (4)-(6) *is equivalent to the stationary solution of* (22)-(24)

Proof. Begin by imposing stationarity for equations (22)-(24), so that they become

$$1-\kappa=\mathrm{E}\left[\frac{1}{1+b_*\zeta''\left(\mathrm{prox}_{b_*\zeta}\left(b_*Y^*+Z_*\right)\right)}\right] \quad \texttt{eq:bt\_stat} \tag{270}$$

$$0=\frac{b_*}{\kappa\gamma^2}\mathrm{E}\left[Z\left\{Y^*-\zeta'\left(\mathrm{prox}_{b_*\zeta}\left(b_*Y^*+Z_*\right)\right)\right\}\right] \quad \texttt{eq:mut\_stat} \tag{271}$$

$$\sigma_*^2=\frac{b_*^2}{\kappa^2}\mathrm{E}\left[\left\{Y^*-\zeta'\left(\mathrm{prox}_{b_*\zeta}\left(b_*Y^*+Z_*\right)\right)\right\}^2\right]\ , \quad \texttt{eq:sigmat\_stat} \tag{272}$$

for independent $Z \sim N(0,\gamma^2)$, $G \sim N(0,1)$, and $\bar{\varepsilon} \sim U(0,1)$, and $Y^* = \alpha \mathbb{1}\{\bar{\varepsilon} \le \zeta'(Z)\} + \frac{1-\alpha}{2}$, $Z_* = \mu_* Z + \kappa^{1/2}\sigma_* G$. Hence

$$\begin{aligned}
\mathrm{E}\left[\frac{1}{1+b_*\zeta''\left(\operatorname{prox}_{b_*\zeta}\left(b_*Y^*+Z_*\right)\right)}\right] &= \mathrm{E}\left[\frac{\zeta'(Z)}{1+b_*\zeta''\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2}+Z_*\right)\right)}\right] + \mathrm{E}\left[\frac{\zeta'(-Z)}{1+b_*\zeta''\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1-\alpha}{2}+Z_*\right)\right)}\right] \\
&= \mathrm{E}\left[\frac{\zeta'(Z)}{1+b_*\zeta''\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2}+Z_*\right)\right)}\right] + \mathrm{E}\left[\frac{\zeta'(-Z)}{1+b_*\zeta''\left(-\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1-\alpha}{2}+Z_*\right)\right)}\right] \\
&= \mathrm{E}\left[\frac{\zeta'(Z)}{1+b_*\zeta''\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2}+Z_*\right)\right)}\right] + \mathrm{E}\left[\frac{\zeta'(-Z)}{1+b_*\zeta''\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2}-Z_*\right)\right)}\right] \\
&= 2\mathrm{E}\left[\frac{\zeta'(Z)}{1+b_*\zeta''\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2}+Z_*\right)\right)}\right],
\end{aligned} \tag{273}$$

where the first equality follows by conditioning on Y^*, the second follows since $\zeta''(x) = \zeta''(-x)$, the fourth since $-\operatorname{prox}_{b\zeta}(-u) = \operatorname{prox}_{b\zeta}(u+b)$ (cf. p.18 in the supplementary material of Sur and Candès (2019)), which holds since the proximal operator $x^* = \operatorname{prox}_{b\zeta}(-u)$ is the unique solution to $b\zeta'(x^*) + x^* + u = 0$ and similarly $y^* = \operatorname{prox}_{b\zeta}(u+b)$ is the unique solution to $b\zeta'(y^*) + y^* - u - b = 0$. Hence, by substitution and using that $\zeta'(-x) = 1 - \zeta'(x)$, y^* is the unique solution to $b(\zeta'(y) - \zeta'(-x^*)) + y^* - x = 0$ so that $y^* = -x^*$. Therefore, letting $-u = b_*(1-\alpha)/2 + Z_*$, $u + b_* = b_*(1+\alpha)/2 - Z_*$ as required. The last equality follows by a simple change of variable and the symmetry of the normal distribution.

Similarly,

$$\begin{aligned}
\mathrm{E}\left[Z\left\{Y^* - \zeta'\left(\operatorname{prox}_{b_*\zeta}\left(b_*Y^*+Z_*\right)\right)\right\}\right] &= \mathrm{E}\left[\zeta'(Z)Z\left\{\frac{1+\alpha}{2} - \zeta'\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2}+Z_*\right)\right)\right\}\right] \\
&\quad + \mathrm{E}\left[\zeta'(-Z)Z\left\{\frac{1-\alpha}{2} - \zeta'\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1-\alpha}{2}+Z_*\right)\right)\right\}\right] \\
&= \mathrm{E}\left[\zeta'(Z)Z\left\{\frac{1+\alpha}{2} - \zeta'\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2}+Z_*\right)\right)\right\}\right] \\
&\quad + \mathrm{E}\left[\zeta'(-Z)Z\left\{\frac{1-\alpha}{2} - \left(1-\zeta'\left(-\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1-\alpha}{2}+Z_*\right)\right)\right)\right\}\right] \\
&= \mathrm{E}\left[\zeta'(Z)Z\left\{\frac{1+\alpha}{2} - \zeta'\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2}+Z_*\right)\right)\right\}\right] \\
&\quad - \mathrm{E}\left[\zeta'(-Z)Z\left\{\frac{1+\alpha}{2} - \zeta'\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2}-Z_*\right)\right)\right\}\right] \\
&= 2\mathrm{E}\left[\zeta'(Z)Z\left\{\frac{1+\alpha}{2} - \zeta'\left(\operatorname{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2}+Z_*\right)\right)\right\}\right],
\end{aligned} \tag{274}$$

where again, the first equality comes from conditioning on Y^*, the second, from $\zeta'(-x) = 1 - \zeta'(x)$, the third by the symmetry of the proximal operator around b, i.e. $-\operatorname{prox}_{b\zeta}(-u) = \operatorname{prox}_{b\zeta}(u+b)$, and the last by change of variable and symmetry of the normal distribution.

And finally,

$$
\begin{aligned}
\mathrm{E}\left[\left\{Y^* - \zeta'\left(\mathrm{prox}_{b_*\zeta}\left(b_* Y^* + Z_*\right)\right)\right\}^2\right] &= \mathrm{E}\left[\zeta'(Z)\left\{\frac{1+\alpha}{2} - \zeta'\left(\mathrm{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2} + Z_*\right)\right)\right\}^2\right] \\
&\quad + \mathrm{E}\left[\zeta'(-Z)\left\{\frac{1-\alpha}{2} - \zeta'\left(\mathrm{prox}_{b_*\zeta}\left(b_*\frac{1-\alpha}{2} + Z_*\right)\right)\right\}^2\right] \\
&= \mathrm{E}\left[\zeta'(Z)\left\{\frac{1+\alpha}{2} - \zeta'\left(\mathrm{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2} + Z_*\right)\right)\right\}^2\right] \\
&\quad + \mathrm{E}\left[\zeta'(-Z)\left\{\frac{1+\alpha}{2} - \zeta'\left(-\mathrm{prox}_{b_*\zeta}\left(b_*\frac{1-\alpha}{2} + Z_*\right)\right)\right\}^2\right] \\
&= \mathrm{E}\left[\zeta'(Z)\left\{\frac{1+\alpha}{2} - \zeta'\left(\mathrm{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2} + Z_*\right)\right)\right\}^2\right] \\
&\quad + \mathrm{E}\left[\zeta'(-Z)\left\{\frac{1+\alpha}{2} - \zeta'\left(\mathrm{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2} - Z_*\right)\right)\right\}^2\right] \\
&= 2\mathrm{E}\left[\zeta'(Z)\left\{\frac{1+\alpha}{2} - \zeta'\left(\mathrm{prox}_{b_*\zeta}\left(b_*\frac{1+\alpha}{2} + Z_*\right)\right)\right\}^2\right] .
\end{aligned}
\tag{275}
$$

□

lemma:norm_projection

Lemma G.10. *Let $\boldsymbol{u} \in \Re^p$ be a unit vector and $\boldsymbol{P}_{\boldsymbol{u}^\perp} = \boldsymbol{I}_p - \boldsymbol{u}\boldsymbol{u}^\top$ be the projection into the subspace $\boldsymbol{u}^\perp$ that is orthogonal to $\boldsymbol{u}$. Then for $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}_p, \boldsymbol{I}_p)$, almost surely,*

$$
\lim_{p\to\infty} \frac{\|\boldsymbol{P}_{\boldsymbol{u}^\perp}\boldsymbol{Z}\|_2}{\sqrt{p}} = 1\,, \tag{276}
$$

as $p \to \infty$.

Proof. Consider $\|\boldsymbol{P}_{\boldsymbol{u}^\perp}\boldsymbol{Z}\|_2^2 = \boldsymbol{Z}^\top\boldsymbol{P}_{\boldsymbol{u}^\perp}\boldsymbol{Z}$ since $\boldsymbol{P}_{\boldsymbol{u}^\perp}$ is idempotent and note that $\mathrm{tr}\left(\boldsymbol{P}_{\boldsymbol{u}^\perp}\right) = p-1$. By the Hanson-Wright inequality (e.g. Vershynin (2018, Theorem 6.2.1)), there exists a universal constant $c > 0$ such that for any $\epsilon \in (0,1)$,

eq:hw

$$
\Pr\left(\left|\frac{\boldsymbol{Z}^\top\boldsymbol{P}_{\boldsymbol{u}^\perp}\boldsymbol{Z}}{p-1} - 1\right| > \epsilon\right) \le 2\exp\left\{-c\epsilon^2(p-1)\right\} . \tag{277}
$$

Now since for $\delta > 0$, $|z-1| \ge \delta$ implies that $|z^2-1| \ge \max\{\delta, \delta^2\}$ (see Vershynin (2018, Equation 3.2)), one gets that

eq:hw2

$$
\Pr\left(\left|\frac{\|\boldsymbol{P}_{\boldsymbol{u}^\perp}\boldsymbol{Z}\|_2}{\sqrt{p-1}} - 1\right| > \epsilon\right) \le 2\exp\left\{-c\epsilon^2(p-1)\right\} . \tag{278}
$$

By the first Borel-Cantelli Lemma, it follows that $\lim\limits_{p\to\infty} \|\boldsymbol{P}_{\boldsymbol{u}^\perp}\boldsymbol{Z}\|_2/(p-1) = 1$ almost surely and the claim follows. □

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