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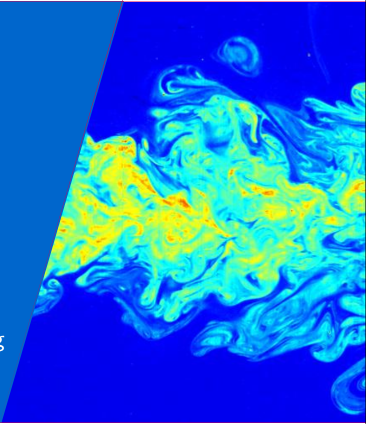
CE597D Special Topics, Fall, 2014

Chapter 7, Part 1: Turbulence Models Chap. 3 of the book



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Outline

Introduction of Turbulence

References used in preparation of this lecture notes:

- B. M. Sumer, Lecture notes on Turbulence, Technical University of Denmark
- Hrvoje Jasak, Lecture note "Complex Physical Models", 2007
- CFD-Online wiki

Introduction of Turbulence

Good films about turbulent flows:

- Part 1: http://youtu.be/oOGXEfgKttM
- Part 2: http://youtu.be/IFeSZZ49vAs
- Part 3: http://youtu.be/o53ghmaSFY8

Introduction of Turbulence

Characteristics of Turbulent Flows:

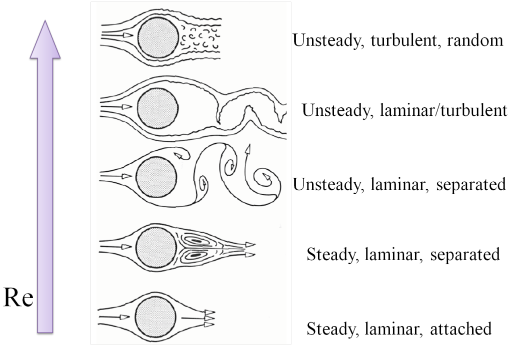
- Turbulent flow is fully governed by the Navier-Stokes equations
- One important parameter is Reynolds number

$$Re = \frac{UL}{\nu} \tag{1}$$

which measures the relative importance of advection and diffusion.

Characteristics of Turbulent Flows

- When *Re* goes over a critical value, the flow goes from **laminar** status through **transition** to **turbulent**.



Introduction of Turbulence

Characteristics of Turbulent Flows

- Example: turbulent river flow
  - Largest scale of interest is based on the scale of the river: average width and depth
  - In turbulent flows, energy is introduced into large scales and through the process of vortex stretching transferred into smaller scales. Most dissipation happens at smallest scales

source: http://www.byu.edu

Characteristics of Turbulent Flows

- The size of smallest scale depends on *Re* number: Kolmogorov length scale:

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}, \tag{2}$$

  Kolmogorov time scale:

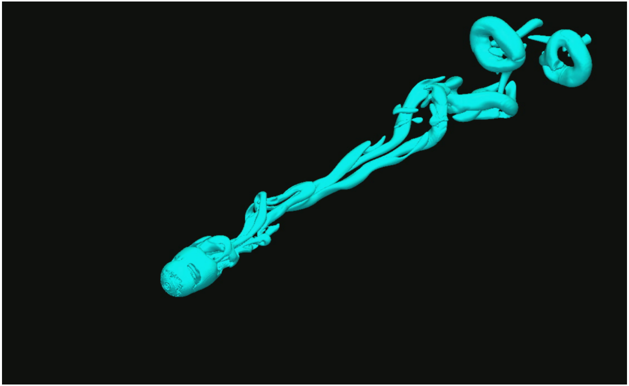
$$\tau = \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}, \tag{3}$$

- For example, if the length scale is 1 m, velocity scale is 1 m/s, and the kinematic viscosity for water is 10^{-6} m^2/s, the the estimated Kolmogorov length scale is about 30 μm, and time scale is 1 ms.
- Our computing capability does not allow the capture of all length scales in high *Re* flows.
- Instead, we need `turbulence modeling`

Introduction of Turbulence

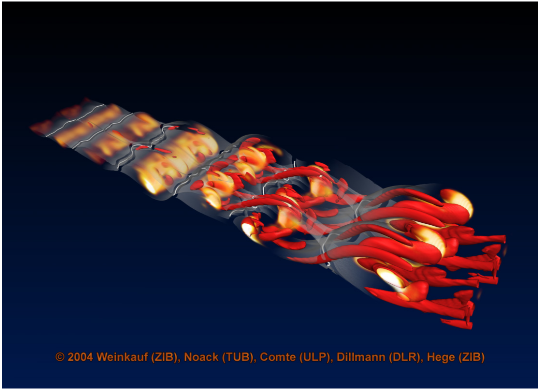
Where does the turbulence come from?

- Non-linearity of the NS equations
- Turbulent flow looks chaotic, random, however coherent structures exist.
- Due to non-linearity, steady initial and boundary conditions could evolve to unsteady (chaotic) solutions



Vortical structure at $Re_D = 1000$

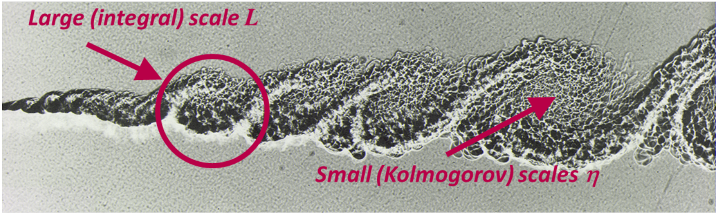
Coherent structures



source: www.courant.nyu.edu

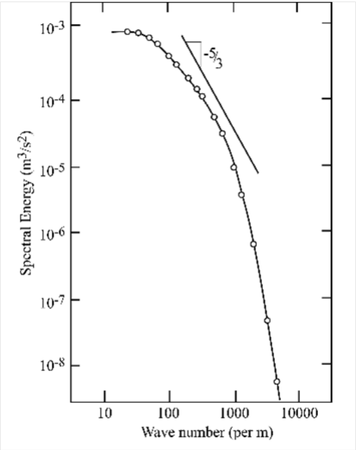
Energy and Scales in Turbulence

- Energy in turbulence is mainly contained in large scales
- Energy is transferred through the cascade of smaller scales
- Energy is dissipated into heat at Komlgorov scale (smallest scale).



Introduction of Turbulence

Energy spectrum $E(k)$ of turbulence behind a grid: energy cascade.
Here $k = 2\pi/\lambda$, λ is the wavelength of the eddies (eddy length scale).
Energy peaks at low wavenumber (large eddies)



Introduction of Turbulence

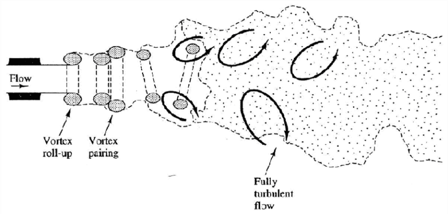
Properties of large and small eddies

- Large eddies are more anisotropic: fluctuations are different in different directions, strongly affected by problem boundaries
- Small eddies are more isotropic and homogeneous
- Kolmogorov derived the universal spectral properties of eddies of intermediate size (inertial subrange), which are sufficiently large to be unaffected by viscosity but small enough to be expressed as a function of dissipation rate: $-5/3$ law of the slope.

Introduction of Turbulence

Transition from laminar to turbulent flow

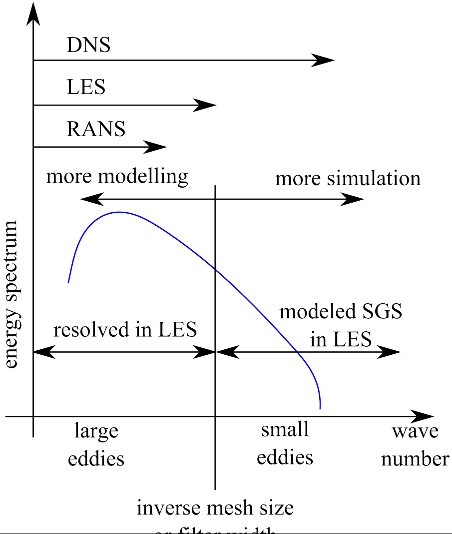
- The transition starts when any small disturbance grow
- Linear hydrodynamic stability theory: identify conditions that amplify the disturbances: For example, critical Reynolds number
- Many CFD codes ignore transition entirely and classify flows as either laminar or turbulent: good approximation if the transition region is small



Introduction of Turbulence

Simulations of Turbulent Flows:

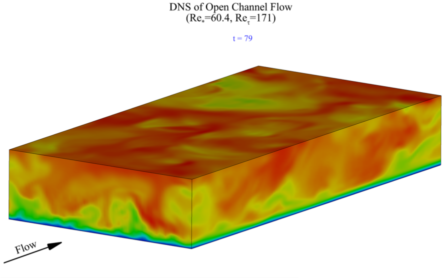
- Very challenging due to the large spectrum of scales
- Three basic approaches: DNS, LES, and RANS



Simulation of Turbulent Flows

Direct Numerical Simulation (DNS):

- Simulate all scales (length and time). No modeling at all.
  - Again, very expensive!
  - Mesh requirement $N \sim (Re)^{9/4}$ and time step requirement $\Delta t \sim (Re)^{-1/2}$
  - Only possible for low to moderate Re flows using computers today
  - Mostly used for fundamental research where geometry is simple
  - Can be used to gather data for model evaluation and fundamental understanding of turbulent interaction



Simulation of Turbulent Flows

Large Eddy Simulation (LES):

- Larger scales depends on the domain boundary
- Smaller scales have less influence from domain boundary
- Smaller scales are more isotropic and homogenous. There are easier to `model`.
- Idea of LES: resolve larger scales and model smaller scales.

Simulation of Turbulent Flows

Reynolds Averaged Navier-Stokes Equations (RANS):

- Velocity, pressure, and other solution variables are decomposed into the mean and fluctuating component

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}' \tag{4}$$

$$p = \overline{p} + p' \tag{5}$$

$$C = \overline{C} + C' \tag{6}$$

- Substituting the above into the Navier-Stokes equations and eliminating second-order terms yields the equations in terms of **averaged properties**: $\overline{\mathbf{u}}$, $\overline{p}$, $\overline{C}$,
- Extra terms appear which require ***turbulence modeling-closure problem***.

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In the following, we will derive Reynolds Averaged Navier-Stokes Equations for turbulence modeling:

- Method of averaging and Reynolds decomposition: Mean value (time averaging)

$$\overline{u_i} = \frac{1}{T}\int_{t_0}^{t_0+T} u_i dt \tag{7}$$

where t_0 is any arbitrary time, and T is the time over which the mean is taken. T should be sufficiently large to give a reliable mean value.

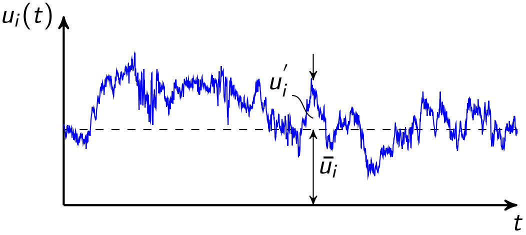

Figure: Time series of u_i

- Method of averaging and Reynolds decomposition: Reynolds decomposition:

$$u_i = \overline{u_i} + u_i^{'} \tag{8}$$

where $u_i^{'}$ the fluctuation of u_i. Reynolds decomposition can be applied to any hydrodynamic quantity.

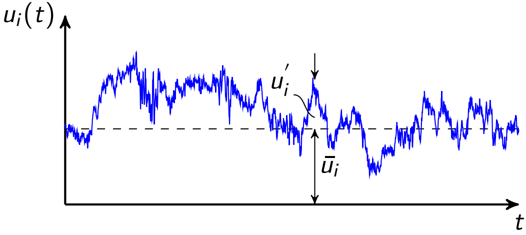

Figure: Time series of u_i

- Method of averaging and Reynolds decomposition:

Let f and g be any such quantities, the following Reynolds conditions hold:

$$\overline{f+g} = \overline{f} + \overline{g}, \quad \overline{af} = a\overline{f}, \quad \overline{a} = a \tag{9}$$

$$\overline{\frac{\partial f}{\partial t}} = \frac{\partial \overline{f}}{\partial t}, \quad \overline{\overline{f}g} = \overline{f}\,\overline{g}, \quad \overline{\overline{f}} = \overline{f} \tag{10}$$

$$\overline{f'} = 0, \quad \overline{\overline{f}\,\overline{g}} = \overline{f}\,\overline{g}, \quad \overline{\overline{f}h'} = \overline{f}\,\overline{h'} = 0 \tag{11}$$

where a is a constant.

Variance:

$$\overline{(f')^2} = \frac{1}{T}\int_{t_0}^{t_0+T} \left(f'\right)^2 dt \tag{12}$$

R.M.S:

$$f_{rms} = \sqrt{\overline{(f')^2}} \tag{13}$$

The R.M.S of velocity components are of particular interest. They can be easily measured and express the average magnitude of velocity fluctuations.

$\overline{(u')^2}$, $\overline{(v')^2}$ and $\overline{(w')^2}$ show up when we derive Reynolds averaged NS equations; they are proportional to the momentum fluxes induced by turbulent eddies (additional normal stress).

TKE (turbulence kinetic energy):

$$k = \frac{1}{2}\left(\overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2}\right) \tag{14}$$

Turbulence intensity T_i is the average R.M.S. velocity divided by a reference mean flow velocity:

$$T_i = \frac{\left(\frac{2}{3}k\right)^{1/2}}{U_{ref}} \tag{15}$$

Moments of different fluctuating variables:

$$\overline{f' g'} = \frac{1}{T} \int_{t_0}^{t_0 + T} f^{'} g^{'} dt \qquad (16)$$

It can reveal important details of the structure of the fluctuations.

If $f^{'}$ and $g^{'}$ are independent random variables, then $\overline{f' g'} = 0$.

However, in turbulence, vortical coherent structures and the induced velocity components are chaotic, but not independent. So moments such as $\overline{u' v'}$, $\overline{u' w'}$, and $\overline{v' w'}$ are not zero (Reynolds shear stress).

High order moments:

$$\text{skewness(asymmetry)}: \quad \overline{(f')^3} = \frac{1}{T}\int_{t_0}^{t_0+T} (f')^3 dt \tag{17}$$

Skewness measures the asymmetry of the fluctuations around the mean. If large fluctuations are more likely to be positive than negative, the skewness will be positive.

$$kurtosis(peakedness): \quad \overline{(f')^4} = \frac{1}{T}\int_{t_0}^{t_0+T} (f')^4 dt \tag{18}$$

Kurtosis (also flatness) measures how intermittent the fluctuations are. If most of the departure from the mean occurs in large rare excursions, then the flat is large.

Correlation functions - time and space:

Autocorrelation in time at the same point:

$$R_{f'f'}(\tau) = \overline{(f')(t)f'(t+\tau)} = \frac{1}{T}\int_{t_0}^{t_0+T} f^{'}(t)f^{'}(t+\tau)dt \qquad (19)$$

Correlation between two different points:

$$R_{f'f'}(\xi) = \overline{(f')(\mathbf{x},t)f'(\mathbf{x}+\xi,t)} = \frac{1}{T}\int_{t_0}^{t_0+T} f^{'}(\mathbf{x},t)f^{'}(\mathbf{x}+\xi,t)dt \qquad (20)$$

When the time shift τ or space distance ξ is zero, then $R_{f'f'}(0)$ is just the variance and will have the largest value since anything is perfectly correlated to itself.

We expect the fluctuations become increasingly de-correlated when the temporal or spatial separation goes to infinity, i.e., $R_{f'f'}(\tau \to \infty) \to 0$ and $R_{f'f'}(\xi \to \infty) \to 0$.

It is also possible to define cross-correlation functions $R_{f'g'}(\tau)$ and $R_{f'g'}(\xi)$.

Probability density function $P(f')$: the fraction of time that a fluctuating signal spends between $f^{'}$ and $f^{'} + df$

$$\overline{(f^{'})^{n}} = \int_{-\infty}^{\infty} (f^{'})^{n} P(f^{'}) df^{'} \tag{21}$$

It is easy to see that when $n = 1$ and 2, it is the mean and variance.

Continuity equation for incompressible fluid:

$$\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} = 0 \tag{22}$$

where the Einstein's summation convention is used, namely there is summation over the repeated indices:

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \tag{23}$$

Taking time average, we get:

$$\overline{\nabla \cdot \mathbf{u}} = \overline{\frac{\partial u_i}{\partial x_i}} = \frac{\partial \overline{u_i}}{\partial x_i} = 0 \tag{24}$$

Subtracting from preceding equation, we find:

$$\nabla \cdot \mathbf{u}^{'} = \frac{\partial u_i^{'}}{\partial x_i} = 0 \tag{25}$$

which means the divergence of the fluctuating part is also zero (divergence free).

- Plug $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$ and $p = \overline{p} + p'$ into the momentum equations, we get

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}}\,\overline{\mathbf{u}}) - \nabla \cdot (\nu \nabla \overline{\mathbf{u}}) - \nabla \cdot (\overline{\mathbf{u}'\mathbf{u}'}) = -\nabla \overline{p}$$

  This equation is also called *Reynolds equation*.
  In the derivation, the nonlinear terms ($\overline{u_i u_j}$) is simplified to

$$\overline{u_i u_j} = \overline{(\bar{u}_i + u_i')\left(\bar{u}_j + u_j'\right)} = \overline{\bar{u}_i \bar{u}_j + \bar{u}_i u_j' + u_i' \bar{u}_j + u_i' u_j'} = \bar{u}_i \bar{u}_j + \overline{u_i' u_j'}$$

- $\overline{u_i' u_j'}$: Reynolds stress term (extra term, need model). It represents the (instantaneous) convective effect.
- $\overline{u_i' u_j'}$ is a second-rank tensor with nine components.
- $\overline{u_i' u_j'}$ is also symmetrical. Thus there are only six distinct components.

$$-\overline{u_i' u_j'} = \begin{bmatrix} -\overline{u_1' u_1'} & -\overline{u_1' u_2'} & -\overline{u_1' u_3'} \\ -\overline{u_2' u_1'} & -\overline{u_2' u_2'} & -\overline{u_2' u_3'} \\ -\overline{u_3' u_1'} & -\overline{u_3' u_2'} & -\overline{u_3' u_3'} \end{bmatrix}$$

- So far, we have ten unknowns (namely, three components of velocity, $\overline{u_i}$, the pressure, $\overline{p}$, and six components of the Reynolds stress, $-\overline{u_i' u_j'}$.
- But we only have four equations, namely, the continuity equation and three momentum equations. The system is not closed.
- This is called the closure problem of turbulence.
- Idea of most turbulence models: relate Reynolds stress to mean flow quantities
- At least two modeling approaches:
    1. Boussinesq approximation: $\overline{u_i' u_j'}$ modeled as function of $\nabla \mathbf{u}$.
    2. Reynolds stress transport models: Derive and solve the transport equation for $\overline{u_i' u_j'}$. However, this approach does not solve the closure problem. In fact, it introduce more extra terms!

Category of turbulence models

Turbulence models: There are several kinds of turbulence models, most important of which are:

- Algebraic models
- Turbulence-energy equation models
- Simulation techniques such as DNS, LES and DES (detached eddy simulations).

Algebraic models

Algebraic models:

- also knows as zero-equation models, are simplest.
- the Reynolds stress is expressed as the product of a turbulence viscosity (eddy viscosity) and the mean strain rate. This approach is called Boussinesq eddy-viscosity approximation:

$$\overline{\mathbf{u}'\mathbf{u}'} = \nu_t \frac{1}{2}\left[\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T\right]$$

  where ν_t is the **turbulent eddy viscosity**
- Eddy viscosity ν_t is often modelled.
- They do not require the solution of any additional equations, and are calculated directly from the flow variables.
- Drawbacks: two simple to be used for complex flows; no history effect.
- The two most well known zero equation models are the Baldwin-Lomax model and the Cebeci-Smith model.
- OpenFOAM® does not implement any of the algebraic models

Turbulence-energy equations models

Turbulence-energy equations models:

- Again, the Boussinesq approximation is used, i.e., Reynolds stress is expressed as the product of eddy viscosity and the mean strain rate.
- The eddy viscosity comes from the turbulence kinetic energy $k = \frac{1}{2}\overline{\mathbf{u}' \cdot \mathbf{u}'}$
- Two kinds of turbulence-energy equation models:
  - One-equation models: Prandtl's one-equation model, Spalart-Allmaras model, etc.
  - Two-equation models: k-ϵ, k-ω, etc.

One-equation models

One-equation models:

- To address the drawback of zero-equation models, i.e., history effect.
- One equation turbulence models solve one turbulent transport equation, usually the turbulent kinetic energy k.

$$\frac{\partial k}{\partial t} + \nabla \cdot (\overline{\mathbf{u}}\, k) = -\overline{u_i' u_i'} \frac{\partial \overline{u_i}}{\partial x_j} - \frac{\partial}{\partial x_i} \left(\frac{1}{2} \overline{u_j' u_j' u_i'} + \overline{p' u_i'} \right) + \nu \frac{\partial^2 k}{\partial x_i \partial x_i} - \epsilon$$

$$\frac{1}{2} \overline{u_j' u_j' u_i'} + \overline{p' u_i'} \approx -\frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_i}$$

$$\epsilon \approx C^* \frac{k^{3/2}}{l}$$

One-equation models

One-equation models:

- Then $\nu_t = \frac{\sqrt{k}}{l}$.

$$\frac{\partial k}{\partial t} + \nabla \cdot (\overline{\mathbf{u}}\, k) = -\overline{u_i' u_i'}\frac{\partial \overline{u_i}}{\partial x_j} - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\overline{u_j' u_j' u_i'} + \overline{p' u_i'}\right) + \nu \frac{\partial^2 k}{\partial x_i \partial x_i} - \epsilon$$

$$\frac{1}{2}\overline{u_j' u_j' u_i'} + \overline{p' u_i'} \approx -\frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}$$

$$\epsilon \approx C^* \frac{k^{3/2}}{l}$$

- The original one-equation model is Prandtl's one-equation model.
- Other common one-equation models are: Baldwin-Barth model, Spalart-Allmaras model, Rahman-Agarwal-Siikonen Model
- Drawback (again) of one-equation models: need to specify an appropriate length scale l.

The Spalart-Allmaras Model

The Spalart-Allmaras model solves a transport equation for a viscosity-like variable $\tilde{\nu}$, which may be referred to as the Spalart-Allmaras variable.

$$\nu_t = \tilde{\nu} f_{v1}, \quad f_{v1} = \frac{\chi^3}{\chi^3 + C_{v1}^3}, \quad \chi := \frac{\tilde{\nu}}{\nu} \tag{26}$$

$$\begin{aligned} \frac{\partial \tilde{\nu}}{\partial t} + u_j \frac{\partial \tilde{\nu}}{\partial x_j} \quad = \quad & C_{b1}[1 - f_{t2}]\tilde{S}\tilde{\nu} + \frac{1}{\sigma}\{\nabla \cdot [(\nu + \tilde{\nu})\nabla\tilde{\nu}] + C_{b2}|\nabla\nu|^2\} - \\ & \left[C_{w1} f_w - \frac{C_{b1}}{\kappa^2} f_{t2}\right]\left(\frac{\tilde{\nu}}{d}\right)^2 + f_{t1}\Delta U^2 \end{aligned} \tag{27}$$

$$\tilde{S} \equiv S + \frac{\tilde{\nu}}{\kappa^2 d^2} f_{v2}, \quad f_{v2} = 1 - \frac{\chi}{1 + \chi f_{v1}} \tag{28}$$

$$S \equiv \sqrt{2\Omega_{ij}\Omega_{ij}} \tag{29}$$

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The Spalart-Allmaras Model

The Spalart-Allmaras model solves a transport equation for a viscosity-like variable $\tilde{\nu}$, which may be referred to as the Spalart-Allmaras variable.

$$\Omega_{ij} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{30}$$

$$f_w = g\left[\frac{1 + C_{w3}^6}{g^6 + C_{w3}^6}\right]^{1/6}, \quad g = r + C_{w2}(r^6 - r), \quad r \equiv \frac{\tilde{\nu}}{\tilde{S}\kappa^2 d^2} \tag{31}$$

$$f_{t1} = C_{t1} g_t \exp\left(-C_{t2}\frac{\omega_t^2}{\Delta U^2}[d^2 + g_t^2 d_t^2]\right) \tag{32}$$

$$f_{t2} = C_{t3}\exp(-C_{t4}\chi^2) \tag{33}$$

Here d is the distance to the closest surface.

The Spalart-Allmaras Model

The constants are

$$\begin{aligned}
\sigma &= 2/3 \\
C_{b1} &= 0.1355 \\
C_{b2} &= 0.622 \\
\kappa &= 0.41 \\
C_{w1} &= C_{b1}/\kappa^2 + (1 + C_{b2})/\sigma \\
C_{w2} &= 0.3 \\
C_{w3} &= 2 \\
C_{v1} &= 7.1 \\
C_{t1} &= 1 \\
C_{t2} &= 2 \\
C_{t3} &= 1.1 \\
C_{t4} &= 2
\end{aligned} \tag{34}$$

The Spalart-Allmaras Model

OpenFOAM implements a slight different version of this.

Major references:

- Spalart, P. R. and Allmaras, S. R. (1992), "A One-Equation Turbulence Model for Aerodynamic Flows", AIAA Paper 92-0439.
- Spalart, P. R. and Allmaras, S. R. (1994), "A One-Equation Turbulence Model for Aerodynamic Flows", La Recherche Aerospatiale n 1, 5-21.
- Potsdam M. and Pullian T. Turbulence modeling treatment for rotorcraft wakes. In AHS Specialist's Conference on Aeromechanics, Jan. 23-25 2008

Two-equation models

Two-equation models:

- To improve the one-equation model, i.e., to avoid the specification of a length scale.
- Introduce another turbulence quantity which allows us to determine the length scale l.
- A common choice of this second turbulence quantity is the dissipation rate ϵ, which leads to the famous k-ϵ model.
- Another common choice of the second turbulence quantity is the specific dissipation, ω, which leads to the also famous k-ω model.
- The transport equation for ϵ and ω (as well as the previous k equation) can be derived from the Navier-Stokes equation.

The $k - \epsilon$ Model

The $k - \epsilon$ Model

$$\frac{\partial k}{\partial t} + \nabla \cdot (\overline{\mathbf{u}}\, k) - \nabla \cdot (\frac{\nu_{eff}}{\alpha_k} \nabla k) = G - \epsilon$$

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot (\overline{\mathbf{u}}\, \epsilon) - \nabla \cdot (\frac{\nu_{eff}}{\alpha_\epsilon} \nabla \epsilon) = C_1 G \frac{\epsilon}{k} - C_2 \frac{\epsilon^2}{k}$$

$$\nu_t = C_\mu \frac{k^2}{\epsilon}, \quad \overline{\mathbf{u}'\mathbf{u}'} = \nu_t \frac{1}{2} \left[\nabla \overline{\mathbf{u}} + (\nabla \overline{\mathbf{u}})^T \right]$$

$$G = \nu_t \nabla \overline{\mathbf{u}} : \overline{\mathbf{u}'\mathbf{u}'}$$

Model constants:

$$C_{1\epsilon} = 1.44, \quad C_{2\epsilon} = 1.92, \quad C_{3\epsilon} = -0.33, \quad C_\mu = 0.09, \quad \sigma_k = 1.0, \quad \sigma_\epsilon = 1.3$$

The $k - \epsilon$ Model

Advantages

- Applicable over a range of flows: constant tuned for some standard situations
- Provides mean values directly. Averaging is not required and the model is capable of providing steady-state solutions
- For the cases where the mean is 2-D, a 2-D simulation is sufficient
- Moderate amount of modelling and a reasonable number of coefficients: C_1, C_2, α_k, α_ϵ and C_μ

Disadvantages

- Assumes that the Reynolds stress tensor is aligned with the velocity gradient (the same for all RANS model). This is simply not true.
- Cannot handle some specific types of flow, *e.g.* swirling or secondary flows in a square duct,
- No eddy viscosity model is capable of handling laminar to turbulent transition

The $k - \epsilon$ Model

Summary of the $k - \epsilon$ model

- In short, the $k - \epsilon$ model is a relatively old-fashioned model with extensive use in industrial applications. Simple extensions do not fix the problems but limitations are well known.
- There have been some improvement regarding the shortcomings of the original $k - \epsilon$, e.g.,
  - Realisable k-epsilon model
  - RNG k-epsilon model

The $k - \epsilon$ Model

Major references for the $k - \epsilon$ model:

- Bardina, J.E., Huang, P.G., Coakley, T.J. (1997), "Turbulence Modeling Validation, Testing, and Development", NASA Technical Memorandum 110446.
- Jones, W. P., and Launder, B. E. (1972), "The Prediction of Laminarization with a Two-Equation Model of Turbulence", International Journal of Heat and Mass Transfer, vol. 15, 1972, pp. 301-314.
- Launder, B. E., and Sharma, B. I. (1974), "Application of the Energy Dissipation Model of Turbulence to the Calculation of Flow Near a Spinning Disc", Letters in Heat and Mass Transfer, vol. 1, no. 2, pp. 131-138.
- Wilcox, David C (2006). "Turbulence Modeling for CFD". Anaheim: DCW Industries.

The $k-\omega$ Model

The $k-\omega$ Model (Wilcox, 1988):

$$\nu_T = \frac{k}{\omega} \tag{35}$$

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = \tau_{ij}\frac{\partial U_i}{\partial x_j} - \beta^* k\omega + \frac{\partial}{\partial x_j}\left[(\nu + \sigma^* \nu_T)\frac{\partial k}{\partial x_j}\right] \tag{36}$$

$$\frac{\partial \omega}{\partial t} + U_j \frac{\partial \omega}{\partial x_j} = \alpha \frac{\omega}{k}\tau_{ij}\frac{\partial U_i}{\partial x_j} - \beta\omega^2 + \frac{\partial}{\partial x_j}\left[(\nu + \sigma \nu_T)\frac{\partial \omega}{\partial x_j}\right] \tag{37}$$

Model constants:

$$\alpha = \frac{5}{9}, \quad \beta = \frac{3}{40}, \quad \beta^* = \frac{9}{100}, \quad \sigma = \frac{1}{2}, \quad \sigma^* = \frac{1}{2}, \quad \varepsilon = \beta^* \omega k \tag{38}$$

Reference:

- Wilcox, D.C. (1988), "Re-assessment of the scale-determining equation for advanced turbulence models", AIAA Journal, vol. 26, no. 11, pp. 1299-1310

The $k-\omega$ SST Model

The $k-\omega$ SST Model (Menter, 1993):

- SST stands for shear stress transport model
- $k-\omega$ SST is still a two-equation eddy-viscosity model.
- The SST formulation combines eddy-viscosity model and SST model
  - $k-\omega$ formulation in the inner parts of the boundary layer makes the model directly usable all the way down to the wall through the viscous sub-layer, which makes the $k-\omega$ SST model usable as a Low-Re (LRN) turbulence model without any extra near-wall damping functions
  - The SST formulation also switches to a $k-\epsilon$ behaviour in the free-stream and thereby avoids the common $k-\omega$ problem that the model is too sensitive to the inlet free-stream turbulence properties
  - $k-\omega$ SST model has good behaviour in adverse pressure gradients and separating flows.
- The $k-\omega$ SST model does produce a bit too large turbulence levels in regions with large normal strain, like stagnation regions and regions with strong acceleration. This tendency is much less pronounced than with a normal $k-\epsilon$ model though.

The $k-\omega$ SST Model (Menter, 1993):

$$\nu_T = \frac{a_1 k}{\max(a_1\omega, SF_2)} \tag{39}$$

$$\frac{\partial k}{\partial t} + U_j \frac{\partial k}{\partial x_j} = P_k - \beta^* k\omega + \frac{\partial}{\partial x_j}\left[(\nu + \sigma_k \nu_T)\frac{\partial k}{\partial x_j}\right] \tag{40}$$

$$\frac{\partial \omega}{\partial t} + U_j \frac{\partial \omega}{\partial x_j} = \alpha S^2 - \beta\omega^2 + \frac{\partial}{\partial x_j}\left[(\nu + \sigma_\omega \nu_T)\frac{\partial \omega}{\partial x_j}\right] + 2(1-F_1)\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_i}\frac{\partial \omega}{\partial x_i} \tag{41}$$

$$F_2 = \tanh\left[\left[\max\left(\frac{2\sqrt{k}}{\beta^*\omega y}, \frac{500\nu}{y^2\omega}\right)\right]^2\right] \tag{42}$$

The $k-\omega$ SST Model (Menter, 1993):

$$P_k = \min\left(\tau_{ij}\frac{\partial U_i}{\partial x_j}, 10\beta^* k\omega\right) \tag{43}$$

$$F_1 = \tanh\left\{\left\{\min\left[\max\left(\frac{\sqrt{k}}{\beta^*\omega y}, \frac{500\nu}{y^2\omega}\right), \frac{4\sigma_{\omega 2}k}{CD_{k\omega}y^2}\right]\right\}^4\right\} \tag{44}$$

$$CD_{k\omega} = \max\left(2\rho\sigma_{\omega 2}\frac{1}{\omega}\frac{\partial k}{\partial x_i}\frac{\partial \omega}{\partial x_i}, 10^{-10}\right) \tag{45}$$

$$\phi = \phi_1 F_1 + \phi_2(1-F_1), \quad \alpha_1 = \frac{5}{9}, \alpha_2 = 0.44, \quad \beta_1 = \frac{3}{40}, \beta_2 = 0.0828 \tag{46}$$

$$\beta^* = \frac{9}{100}, \quad \sigma_{k1} = 0.85, \sigma_{k2} = 1, \quad \sigma_{\omega 1} = 0.5, \sigma_{\omega 2} = 0.856 \tag{47}$$

The $k-\omega$ SST Model

The $k-\omega$ SST Model Reference:

- Menter, F. R. (1993), "Zonal Two Equation k-ęØ Turbulence Models for Aerodynamic Flows", AIAA Paper 93-2906.
- Menter, F. R. (1994), "Two-Equation Eddy-Viscosity Turbulence Models for Engineering Applications", AIAA Journal, vol. 32, no 8. pp. 1598-1605.

The same thing can be done for a generic scalar transport equation for C. Reynolds decomposition

$$C = \overline{C} + C'$$

Plugin to the scalar transport equation together with the Reynolds decomposition of velocity, we get

$$\frac{\partial \overline{C}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}}\,\overline{C}) = \nabla \cdot (\nu \nabla \overline{C}) + \nabla \cdot (\overline{\mathbf{u}' C'})$$

where the Reynolds transport tensor is a vector

$$-\overline{\mathbf{u}' C'} = \begin{bmatrix} -\overline{u_1' C'} \\ -\overline{u_2' C'} \\ -\overline{u_3' C'} \end{bmatrix}$$

Similar to the momentum equation:

- Boussinesq Approximation for scalar transport by turbulence

$$\overline{\mathbf{u}'C'} = \Gamma_t \nabla C$$

  where Γ_t is the **turbulent diffusivity**
- Need to characterise Γ_t in terms of turbulence quantities
- It is related to the turbulent viscosity ν_t using a model constant called the Schmidt number σ_t

$$\Gamma_t = \frac{\nu_t}{\sigma_t}$$

- The turbulent Schmidt number describes the ratio between the rates of turbulent transport of momentum and the turbulent transport of mass (or any passive scalar).
- σ_t is related to the turbulent Prandtl number which is concerned with turbulent heat transfer rather than turbulent mass transfer.
- The typical value of the σ_t ranges from 0.7 to 1.0

A Primer on Near-Wall Treatment

Handling Near-Wall Region

- Region near the wall contains very high gradients: thin boundary layers, rapid variation of k, ϵ, etc..
- For attached flows, near-wall behavior is well known.
- However, standard $k - \epsilon$ model does not reproduce it: need damping functions
- Near-wall gradients not particularly interesting: we wish to get the drag right

A Primer on Near-Wall Treatment

Modelling Near-Wall Turbulence

- Low-*Re* (LRN) turbulence models
  - Capture the near-wall behavior well (for fully turbulent flows)
  - Mesh resolution requirements are very large: $y^+ \approx 1$
  - Difficult to construct good meshes for 3-D geometries
- High-*Re* (HRN), aka wall functions, near-wall treatment
  - The idea is to bridge the near-wall region with an approximate solution for a fully developed boundary layer
  - Removes the need for high resolution near the wall
  - . . . but pretty bad for detached flow or in presence of pressure gradients
- Wall functions are in general use: not very good for detailed near-wall behavior but savings in mesh size are substantial

Large Eddy Simulation

According to Kolmogorov's (1941) theory:

- the large eddies of the flow are dependant on the geometry while the smaller scales more universal.
- this allows us to explicitly solve for the large eddies in a calculation and implicitly account for the small eddies by using a subgrid-scale model (SGS model).
- mathematically, it is equivilant of separating the velocity field into a resolved and sub-grid part.
- the resolved part of the field represent the "large" eddies, while the subgrid part of the velocity represent the "small scales" whose effect on the resolved field is included through the subgrid-scale model.
- formally, one may think of filtering as the convolution of a function with a filtering kernel G

$$\bar{u}_i(\vec{x}) = \int G(\vec{x} - \vec{\xi})u(\vec{\xi})d\vec{\xi}$$

- Idea: Vortices smaller than the filter scale (averaging volume size) are modeled and larger-scale turbulence and coherent structures will be resolved (simulated)

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Large Eddy Simulation

- The filtering

$$\bar{u}_i(\vec{x}) = \int G(\vec{x} - \vec{\xi})u(\vec{\xi})d\vec{\xi}$$

resulting in

$$u_i = \bar{u}_i + u_i'$$

where $\bar{u}_i$ is the resolvable scale part and u_i' is the subgrid-scale part.

- Note the fundamental difference in the decomposition above (spatial) and the Reynolds decomposition (temporal)
- Most practical implementations of LES use the grid itself as the filter (the box filter) and perform no explicit filtering.

Large Eddy Simulation

The filtered equations are developed from the incompressible Navier-Stokes equations:

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i}{\partial x_j}\right). \tag{48}$$

Substituting in the decomposition $u_i = \bar{u}_i + u_i'$ and $p = \bar{p} + p'$ and then filtering the resulting equation gives the equations of motion for the resolved field:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial \bar{u}_i}{\partial x_j}\right) + \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j}. \tag{49}$$

We have assumed that the filtering operation and the differentiation operation commute, which is not generally the case. It is thought that the errors associated with this assumption are usually small, though filters that commute with differentiation have been developed.

The extra term $\frac{\partial \tau_{ij}}{\partial x_j}$ arises from the non-linear advection terms, due to the fact that

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} \neq \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} \tag{50}$$

and hence

$$\tau_{ij} = \bar{u}_i \bar{u}_j - \overline{u_i u_j} \tag{51}$$

Similar equations can be derived for the subgrid-scale field (i.e. the residual field).

Large Eddy Simulation

Subgrid-scale turbulence models usually also employ the Boussinesq eddy viscosity assumption, and seek to calculate (the deviatoric part of) the SGS stress using:

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = -2\mu_{SGS}\bar{S}_{ij} \tag{52}$$

where $\bar{S}_{ij}$ is the rate-of-strain tensor for the resolved scale defined by

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{53}$$

and ν_{SGS} is the subgrid-scale turbulent viscosity. Substituting into the filtered Navier-Stokes equations, we then have

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j}\left(\left[\nu + \nu_{SGS}\right]\frac{\partial \bar{u}_i}{\partial x_j}\right), \tag{54}$$

where we have used the incompressibility constraint to simplify the equation and the pressure is now modified to include the trace term $\tau_{kk}\delta_{ij}/3$.

Large Eddy Simulation

Compare ν_t in RANS and ν_{SGS} in LES:

- ν_t in RANS: effect of all turbulence scales
- ν_{SGS} in LES: only due to the unresolved small-scale turbulence averaged over the filtering volume
  - Infinitely refine the mesh will drive the unresolved subgrid scales to zero, and therefore $\nu_{SGS} \to 0$
  - At the this extreme, it is in fact a DNS.
  - So how to do a DNS in OpenFOAM® ? Turn off turbulence model, refine the mesh to Kolmogorov length scale, reduce time step size to Kolmogorov time scale, and use high order numerical schemes.

Large Eddy Simulation

Examination of the sub-grid scale stress (SGS) tensor

- Definition appears very similar to RANS: the problem reduces to the fact that $\overline{\mathbf{u}\mathbf{u}} \neq \overline{\mathbf{u}}\,\overline{\mathbf{u}}$ (where overbar means spatial filtering, not temporal averaging!)
- Sub-grid scale stress (SGS) τ

$$
\begin{aligned}
\tau &= \overline{\mathbf{u}\mathbf{u}} - \overline{\mathbf{u}}\,\overline{\mathbf{u}} \\
&= \overline{(\overline{\mathbf{u}} + \mathbf{u}')(\overline{\mathbf{u}} + \mathbf{u}')} - \overline{\mathbf{u}}\,\overline{\mathbf{u}} \\
&= (\overline{\overline{\mathbf{u}}\,\overline{\mathbf{u}}} - \overline{\mathbf{u}}\,\overline{\mathbf{u}}) + (\overline{\overline{\mathbf{u}}\mathbf{u}'} + \overline{\mathbf{u}'\overline{\mathbf{u}}}) + \overline{\mathbf{u}'\,\mathbf{u}'}
\end{aligned}
$$

- First term: Leonard stress: creation of SGS turbulence
- Second term: transfer between resolved and unresolved scales
- Third term: effect of small eddy interaction

Large Eddy Simulation

Sub-Grid Scale Models

- SGS models are of similar type as the RANS models: Boussinesq approximation is more appropriate

$$\tau = \nu_{SGS}\left[\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T\right]$$

- Zero-equation models: e.g., Smagorinsky model

$$\nu_{SGS} = (C_S\Delta)^2|\overline{\vec{S}}|$$
$$|\overline{\vec{S}}| = \left(\nabla\overline{\mathbf{u}} : \left[\nabla\overline{\mathbf{u}} + (\nabla\overline{\mathbf{u}})^T\right]\right)^{\frac{1}{2}}$$

- One-equation model: e.g., solving transport equation for k

$$\nu_{SGS} = C_k\sqrt{k}\Delta$$

- In both cases, Δ is proportional to filter width

Zero-equation models: Smagorinsky model

The Smagorinsky model could be summarised as:

$$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = -2\left(C_s\Delta\right)^2 \left|\bar{S}\right| S_{ij} \tag{55}$$

In the Smagorinsky-Lilly model, the eddy viscosity is modeled by

$$\mu_{sgs} = \rho\left(C_s\Delta\right)^2 \left|\bar{S}\right| \tag{56}$$

The filter width is usually taken to be

$$\Delta = (\text{Volume})^{\frac{1}{3}} \tag{57}$$

and

$$\bar{S} = \sqrt{2S_{ij}S_{ij}} \tag{58}$$

The effective viscosity is calculated from

$$\mu_{eff} = \mu_{mol} + \mu_{sgs} \tag{59}$$

The Smagorinsky constant usually has the value $C_s = 0.1 \sim 0.2$.
Reference: Smagorinsky, J (1963), "General circulation experiments with the primitive equations, i. the basic experiment. Monthly Weather Review", 91: pp 99-164, 1963.

Turbulence modeling in OpenFOAM

To specify turbulence model in OpenFOAM® :

- change constant/turbulenceProperties file: simulationType can be laminar, RASModel, or LESModel
- correspondingly specify RASProperties or LESProperties file

RASProperties

```
RASModel            kEpsilon;
turbulence          on;
printCoeffs         on;
```

LESProperties

```
LESModel            oneEqEddy;
delta               cubeRootVol;
printCoeffs         on;

cubeRootVolCoeffs
{
    deltaCoeff          1;
}
```

Turbulence modeling in OpenFOAM

- In many of the flow solvers in OpenFOAM® , the turbulence model (RANS or LES) is run-time selectable.
- This is implemented through the class inheritance hierarchy.

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \nabla \cdot (\overline{\mathbf{u}\mathbf{u}}) = -\nabla \overline{p} + \nabla \cdot (\nu_{eff} \nabla \overline{\mathbf{u}}) \tag{60}$$

- This effective viscosity ν_{eff} has different meaning for RANS and LES

$$\nu_{eff} = \begin{cases} \nu + \nu_t & \text{for RANS} \\ \nu + \nu_{sgs} & \text{for LES} \end{cases} \tag{61}$$

- Also note the different meaning of $\overline{\mathbf{u}}$ in RANS and LES

Turbulence modeling in OpenFOAM

In a typical solver such as `pisoFoam`, the momentum equation is written as:

```
fvVectorMatrix UEqn
(
    fvm::ddt(U)
  + fvm::div(phi, U)
  + turbulence->divDevReff(U)
);
```

where the momentum diffusion term $\nabla \cdot (\nu_{eff} \nabla \overline{\mathbf{u}})$ is represented by `turbulence->divDevReff(U)`

- `turbulence` is an object of specific turbulence model (class).
- Most of the turbulence model classes implements a member function `divDevReff(U)` which returns the momentum diffusion term.
- In fact, this functions call another one, `nuEff()` with returns the effective viscosity ν_{eff}.
- You can also call `nut()` to get ν_t if RANS, or ν_{sgs} if LES. In fact in LES models, the `nut()` calls function `nuSgs()`.

Turbulence modeling in OpenFOAM

For example in the class `kEpsilon` for RANS, we have:

```
tmp<fvVectorMatrix> kEpsilon::divDevReff(volVectorField& U) const
{
    return
    (
      - fvm::laplacian(nuEff(), U)
      - fvc::div(nuEff()*dev(T(fvc::grad(U))))
    );
}
```

In the class `GenEddyVisc` for LES, we have similar

```
tmp<fvVectorMatrix> GenEddyVisc::divDevReff(volVectorField& U) const
{
    return
    (
      - fvm::laplacian(nuEff(), U)
      - fvc::div(nuEff()*dev(T(fvc::grad(U))))
    );
}
```

A good note: http://openfoamwiki.net/index.php/BuoyantBoussinesqPisoFoam#The_Meaning_of_divDevReff.28U.29

Questions?