On Computational Poisson Geometry

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Lake Arrowhead, Geometry & learning from data in 3D and beyond

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Siméon Denis Poisson

MÉMOIRE

Sur la Variation des Constantes arbitraires dans les questions de Mécanique,

Lu à l'Institut le 16 Octobre 1809;

Par M. POISSON.

constante a ni la constante b; dans d'autres cas elle ne contiendra aucune constante arbitraire, et se réduira à une constante déterminée; mais, afin de rappeler l'origine de cette quantité, qui représente une certaine combinaison des différences partielles des valeurs de a et b, nous ferons usage de cette notation (b, a), pour la désigner; de manière que nous aurons généralement

$$\frac{db}{ds}\cdot\frac{da}{d\varphi}-\frac{da}{ds}\cdot\frac{db}{d\varphi}+\frac{db}{du}\cdot\frac{da}{d\psi}-\frac{da}{du}\cdot\frac{db}{d\psi}+\frac{db}{dv}\cdot\frac{da}{d\theta}-\frac{da}{dv}\cdot\frac{db}{d\theta}=(b, a).$$

Hamiltonian Systems in $\mathbb{R}^{2n} = \{(q_1, \ldots, q_n, p_1, \ldots, p_n)\}$

Given $H \in C^\infty_{\mathbb{R}^{2n}}$:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

- Define $\{,\} : C^\infty_{\mathbb{R}^{2n}} \times C^\infty_{\mathbb{R}^{2n}} \to C^\infty_{\mathbb{R}^{2n}}$ por

$$\{f, g\} := \sum_{i=1}^{n} \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}$$

- Then,

$$\dot{q}_i = \{H, q_i\}, \qquad \dot{p}_i = \{H, p_i\}$$

Poisson Bracket $\{,\}: C^\infty_M \times C^\infty_M \longrightarrow C^\infty_M$

- $\mathbb{R}$-lineal.
- Antisymmetric: $\{f, g\} = -\{g, h\}$
- Jacobi Identity:

$$\{f, \{g, h\}\} \;=\; \{\{f, g\}, h\} + \{g, \{f, h\}\}$$

- Leibniz Rule:

$$\{f, gh\} = g\,\{f, h\} + h\,\{f, g\}$$

Example

In $\mathbb{R}^3_x$, given

$$\psi : \mathbb{R}^3 \to \mathbb{R}^3, \qquad \psi(x) = \big(\psi_1(x), \psi_2(x), \psi_3(x)\big)^\top$$

- Define
$$\{f, g\}_\psi = \langle \psi, \nabla f \times \nabla g \rangle$$

- *Jacobi Identity:*
$$\langle \psi, \operatorname{rot} \psi \rangle = 0$$

Note: $\{f, g\}_\psi = \nabla f^\top \Pi_\psi, \nabla g$, where

$$\Pi_\psi = \begin{pmatrix} 0 & \psi_3 & -\psi_2 \\ -\psi_3 & 0 & \psi_1 \\ \psi_2 & -\psi_1 & 0 \end{pmatrix}$$

Poisson Tensors

$$\Gamma \wedge^2 \mathsf{T}M \ni \Pi \qquad \text{tq.} \qquad [\![\Pi, \Pi]\!]_{\mathrm{SN}} = 0.$$

Jacobi identity in $\mathbb{R}^n = \{x = (x^1, \ldots, x^n)\}$:

$$\Pi^{is} \frac{\partial \Pi^{jk}}{\partial x^s} + \Pi^{js} \frac{\partial \Pi^{ki}}{\partial x^s} + \Pi^{ks} \frac{\partial \Pi^{ij}}{\partial x^s} = 0,$$

$\Pi = \frac{1}{2} \Pi^{ab} \frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b}.$

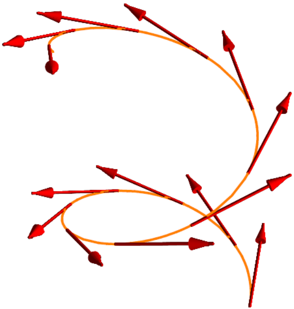
Brackets $\leftrightarrow$ Tensors

$$\text{Poisson}\{,\} \qquad \longleftrightarrow \qquad \Pi \quad \text{s.t.} \quad [\![\Pi,\Pi]\!]_{\mathrm{SN}} = 0.$$

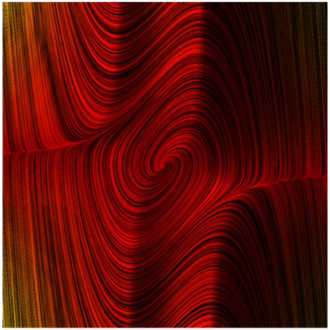
$$\{f,g\} = \Pi(\mathrm{d}f,\mathrm{d}g)$$

Foliations Induced by Vector Fields

- **Existence and Uniqueness**

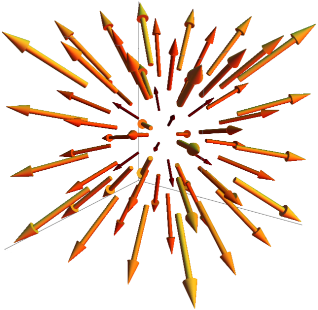


- **Foliation by Integral Curves**

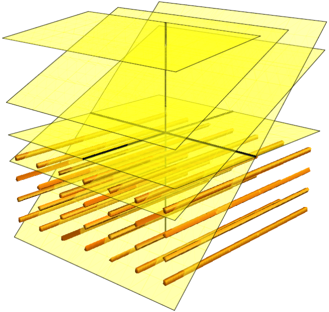


Vector Fields vs Distributions

- **Vector Field**

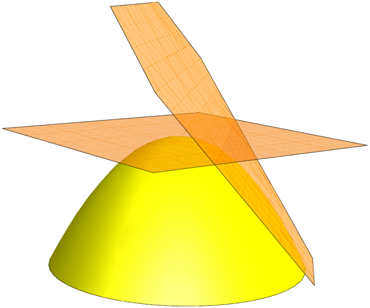


- **Distribution**

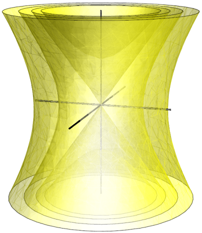


Foliations by Singular Distributions

- **Stefan-Sussman**



- **Foliation by Integral Manifolds**



$$\mathbb{R}^n \ni p \quad \mapsto \quad \mathcal{D}_p \subset \mathsf{T}_p\mathbb{R}^n,$$

$\mathcal{D}_p$ subspace of tangent vectors at p

Poisson Structure $\leftrightarrow$ Symplectic Foliation

Π Poisson tensor:

$$D^{\Pi} := \{ X_f \,|\, f \in C^{\infty}_{M} \},$$

with $X_f(g) = \{f, g\}$.

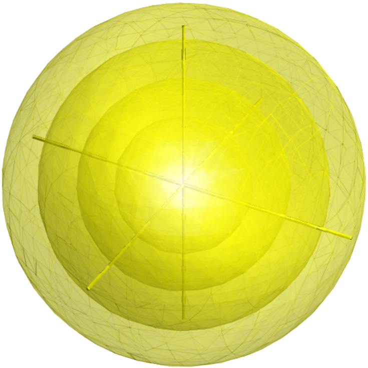
Then:

D^{Π} is integrable.

$\downarrow$

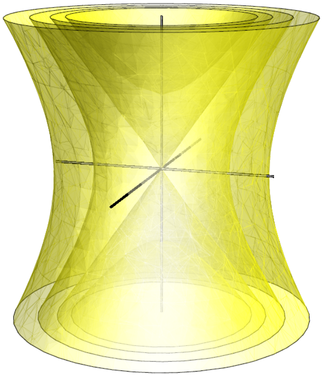
we obtain a symplectic foliation.

Ejemplo : $\mathfrak{so}(3)$



- $\psi_1 = x_1, \quad \psi_2 = x_2, \quad \psi_3 = x_3$
- $\Pi_\psi = \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}, \{f,g\}_\psi = \langle \psi, \nabla f \times \nabla g \rangle$

Example : $\mathfrak{sl}(2)$



- $\psi_1 = x_1, \quad \psi_2 = x_2, \quad \psi_3 = -x_3$
- $\Pi_\psi = \begin{pmatrix} 0 & -x_3 & -x_2 \\ x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}, \{f, g\}_\psi = \langle \psi, \nabla f \times \nabla g \rangle$

Examples, 2D

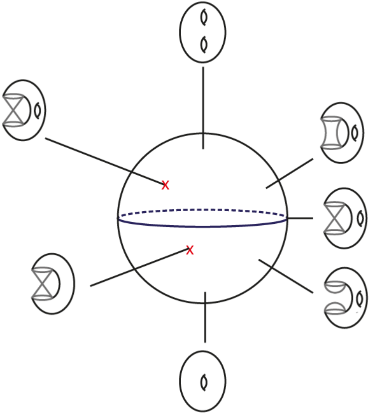
- Every closed oriented surface is symplectic, hence Poisson.
- (Radko) Classification of *topologically stable* Poisson structures on surfaces (tensor vanishes linearly on a disjoint union of simple closed curves). Explicit determination of the moduli space for 2–sphere.

Examples, 3D

- (Lickorish, Novikov, Zieschang) Every closed smooth oriented 3-manifold admits a foliation by surfaces, hence a regular rank 2 Poisson structure.
- (Evangelista-TorresOrozco-S.-Vera) Every closed smooth oriented 3-manifold admits a generic rank 2 Poisson structure with Bott-Morse singularities (circles and points).

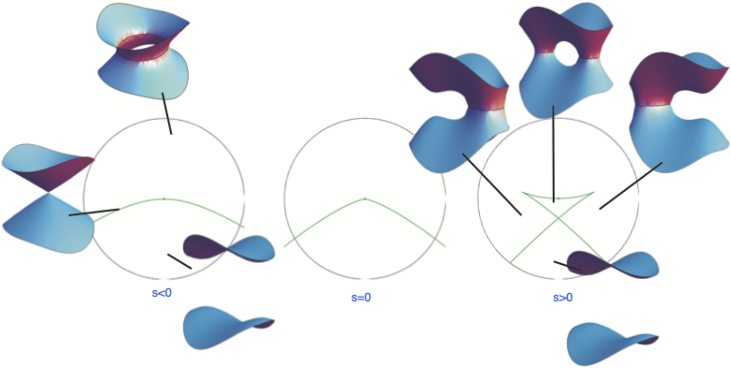
Examples, 4D

- (GarcíaNaranjo-S.-Vera) Every smooth closed orientable 4–manifold admits a generic rank 2 Poisson structure with broken Lefschetz singularities.



Examples, 4D

- (S.-TorresOrozco) Every smooth closed orientable 4–manifold admits a generic rank 2 Poisson structure with *wrinkled* singularities.



Poisson Structures

- Hamiltonian system:

$$\dot{x} = \{H, x\}, \quad x \in \mathbb{R}^n$$

- Hamiltonian fields:

$$X_h = \mathbf{i}_{\mathrm{d}h}\Pi$$

- Poisson fields:

$$\mathcal{L}_Z \Pi = 0.$$

Modular Field

- (M, Π, Ω) orientable Poisson manifold:

$$\mathcal{L}_X \Omega = \operatorname{div}_\Omega X \cdot \Omega$$

- Modular Field:

$$Z := h \longmapsto \operatorname{div}_\Omega X_h$$

$$\Downarrow$$

$$\mathcal{L}_Z \Pi = 0 \qquad \text{y} \qquad Z^{f\,\Omega} = Z^\Omega - X_{\ln|f|}$$

Definition

An orientable Poisson manifold (M, Π) is **unimodular** if it admits a volume form invariant under the flow of every Hamiltonian field.

References

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