TEM Waves Draft

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1 Draft

We look for a numerical solution for

$$\vec{E} = E_0 U(x,y) e^{i(\omega t - \beta z)} \hat{y}$$

in a waveguide with varying refractive indices along the $\hat{x}$ and $\hat{y}$ axes.

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \beta^2 U = -\mu\epsilon\omega^2 U$$
$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \mu\epsilon\omega^2 U = \beta^2 U$$
$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + n(x,y)^2 k_0^2 U = \beta^2 U$$

Making the above discrete gives us

$$\frac{U_{i+1,j} + U_{i-1,j} - 2U_{i,j}}{\Delta x^2} + \frac{U_{i,j+1} + U_{i,j-1} - 2U_{i,j}}{\Delta y^2} + {n_{i,j}}^2 k_0^2 U_{i,j} = \beta^2 U_{i,j}$$

$\Delta y \to \Delta x$

$$\begin{aligned} U_{i+1,j} + U_{i-1,j} - 2U_{i,j} + U_{i,j+1} + U_{i,j-1} - 2U_{i,j} + {n_{i,j}}^2 \Delta x^2 k_0^2 U_{i,j} &= \beta^2 \Delta x^2 U_{i,j} \\ U_{i+1,j} + U_{i-1,j} - 4U_{i,j} + {n_{i,j}}^2 \Delta x^2 k_0^2 U_{i,j} + U_{i,j+1} + U_{i,j-1} &= \beta^2 \Delta x^2 U_{i,j} \\ U_{i+1,j} + U_{i-1,j} + ({n_{i,j}}^2 \Delta x^2 k_0^2 - 4) U_{i,j} + U_{i,j+1} + U_{i,j-1} &= \beta^2 \Delta x^2 U_{i,j} \end{aligned}$$

Let $\alpha_{i,j} = ({n_{i,j}}^2 \Delta x^2 k_0^2 - 4)$

$$U_{i+1,j} + U_{i-1,j} + \alpha_{i,j} U_{i,j} + U_{i,j+1} + U_{i,j-1} = \beta^2 \Delta x^2 U_{i,j} \tag{1}$$

U is an $m \times n$ matrix. m rows, n columns.
$U_{i,j}$ is the i^{th} row, j^{th} column. $(1 \le i \le m), (1 \le j \le n)$.
We want to rewrite matrix U as a column vector V. $U_{l,k} \mapsto {V_{m(k-1)+l}}^*$.

$$\mathbf{U} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & & & \vdots \\ a_{31} & & & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & \dots & a_{mn} \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \\ a_{12} \\ \vdots \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix} = \mathbf{V}$$

Figure 1: Mapping U to V

The map is 1-1 because,

$$\begin{aligned} \text{Suppose } m(k'-1)+l' &= m(k-1)+l, \\ \Rightarrow m(k'-k) &= l-l', \end{aligned}$$

$$\begin{aligned} \text{then } |m(k'-k)| \leq m-1, &\text{ since } |l-l'| \leq m-1, \\ |k'-k| &\leq \frac{1}{m}-1 \\ \Rightarrow l = l' &\Rightarrow k = k'. \end{aligned}$$

Note that indices i and j of U both start at 1 impies the index of V starts at 1. Below is a tabulation of indices appearing in equation 1 to see what index of V it maps to.

Table 1: Index Table

k	l	m(l-1) + k	
i + 1	j	mi + j	m(j-1) + i + 1
i - 1	j	m(i-2) + j	m(j-1) + i - 1
i	j	m(i-1) + j	m(j-1) + i
i	j+1	m(i-1) +j + 1	m(j-1) + i + m
i	j-1	m(i-1) + j - 1	m(j-1) + i - m

Equation 1 becomes

$$V_{m(j-1)+i+1} + V_{m(j-1)+i-1} - \alpha_{i,j} V_{m(j-1)+i} + V_{m(j-1)+i+m} + V_{m(j-1)+i-m} = \beta^2 \Delta x^2 V_{m(j-1)+i}$$

rewrite in increasing index order,

$$V_{m(j-1)+i-m} + V_{m(j-1)+i-1} - \alpha_{i,j} V_{m(j-1)+i} + V_{m(j-1)+i+1} + V_{m(j-1)+i+m} = \beta^2 \Delta x^2 V_{m(j-1)+i} \tag{2}$$

The index on the first and last terms tells us that this relationship isn't valid for every entry V_k. So we must find the upper and lower indices for which this relationship is valid; and manually find the other entries. The lowest index has to be ≥ 1. The highest index $\leq mn$. By substituting $p = m(j-1)+i$,

$$V_{p-m} + V_{p-1} - \alpha_{i,j} V_p + V_{p+1} + V_{p+m} = \beta^2 \Delta x^2 V_p \tag{3}$$

we can see that $m+1 \leq p \leq m(n-1)$.

Now we manipulate inequalities to find out which (i,j)'s correspond to these limits. Note that $k = k(i,j) = m(j-1)+i$ is also an increasing function of both i and j, (i.e. $\frac{\partial k}{\partial i}, \frac{\partial k}{\partial j} > 0$), so after finding (i,j)'s satisfying the above, we will know which other one's to exclude. Equations 4 and 5 shows that (3) applies to the inner columns of matrix U.

$$\begin{aligned} m(j-1)+i-m &\geq 1 \\ m(j-2)+i &\geq 1 \\ &\boxed{i=1, j=2} \end{aligned} \tag{4}$$

$$\begin{aligned} m(j-1)+i+m &\leq mn \\ m(j-n)+i &\leq 0 \\ &\boxed{i=m, j=n-1} \end{aligned} \tag{5}$$

The outer column entries (of matrix U) equal zero because they are boundaries. We now have an eigenvalue problem of an $m(n-2) \times m(n-2)$ matrix. Equation 3 gives the following matrix.

$$\begin{bmatrix} \alpha_{21} & 1 & 0 & 0 & \dots & \dots & 1 & \dots & 0 \\ 1 & \alpha_{31} & 1 & 0 & 0 & & & & 0 \\ 0 & 1 & \alpha_{41} & 1 & 0 & 0 & & & 0 \\ 0 & 0 & 1 & \alpha_{51} & 1 & 0 & 0 & & 0 \\ \vdots & 0 & & & \ddots & & & & \vdots \\ \vdots & \vdots & & & & \ddots & & & \vdots \\ \vdots & \vdots & & & & \dots & \alpha_{n-3,m} & 1 & 0 \\ \vdots & \vdots & & & & 0 & 1 & \alpha_{n-2,m} & 1 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & \alpha_{n-1,m} \end{bmatrix} \begin{bmatrix} V_{m+1} \\ V_{m+2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ V_{m(n-1)} \end{bmatrix} = \beta^2 \Delta x^2 \begin{bmatrix} V_{m+1} \\ V_{m+2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ V_{m(n-1)} \end{bmatrix}$$

Figure 2: Eigen Matrix