MAT 4170: Modern Algebra I **Name:** **Panya Sukphranee**

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**Homework 6**

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1. Let $G$ be a cyclic group generated by some $g \in G$. Let $a, b \in G$. Then

$$a = g^m \text{ and } b = g^n$$

for some $m, n \in \mathbb{Z}$. We have that

$$ab = g^m g^n = g^{m+n} = g^{n+m} = g^n g^m = ba.$$

Since $a, b \in G$ are arbitrary, $G$ is Abelian. We only used the fact that $G$ was cyclic to show it was Abelian. Therefore, any cyclic group is Abelian.

2. a) A generator for the subgroup of order 8 of $\mathbb{Z}_4$ is 3 by Corrollary of Thm 4.3. That is $|<3>| = 8$. We can find other generators of order 8 by taking "powers" of 3 up to 7. In the context of addition, powers means multiplication. So the generators of order 8 are

$$\{3, 6, 9, 12, 15, 18, 21\}$$

or

$$<3>=<6>=<9>=<12>=<15>=<18>=<21>.$$

b) Suppose $G =< a >$ and $|a| = 24$. By Theorem 4.3, $a^3$ generates the subgroup $< a^3 >$ which is of order 8. By Theorem 4.2, we have that

$$a^3, (a^3)^2, (a^3)^3, (a^3)^4, (a^3)^5, (a^3)^6, (a^3)^7$$

also generate subgroups of order 8. i.e.

$$< a^3 >, < a^6 >, < a^9 >, < a^{12} >, < a^{15} >, < a^{18} >, < a^{21} >$$

are all of order 8.

3. Let $G$ be a group that has no proper non-trivial subgroup. Suppose $a \in G$ s.t. $a \neq e$. We can assume that such an $a$ exists because if $G = \{e\}$, then we'd have a only the trivial subgroup $\{e\}$. Now, using Theorem 3.4, $< a >$ is a subgroup of $G$ generated by $a$. We know that $< a >$ is not the trivial subgroup because $a \in < a >$. Since $< a >$ is not a proper subgroup $< a >$ must be $G$. Thus $< a >= G$ and $G$ is therefore cyclic.

4. Let $G$ be a group of order 3 and let $e$, $a$, and $b$ be the three elements of $G$ where $e$ is the identity of the group.

a) Being a group, the element $ab$ is also in $G$. But $ab \neq a$ because that would imply $b$ is the identity and contradict the fact that $G$ has 3 elements. For the same reason, $ab \neq b$. It follows that $ab = e$ and also $ba = e$. Thus $a$ and $b$ are inverses of each other.

b) By part (a), we know that for a group of order 3, the two non-identity elements are inverses of each other. Since the inverse is unique, a non-identity element $a \in G$ can't be it's own inverse. i.e.

$$a^2 \neq e,$$

for all $a \neq e$. Therefore, the order of a non-identity element $a$ is greater than 2. It follows that the order of $< a >$ is greater than 2. But being a subgroup of $G$, the order of $< a >$ is $\leq 3$,

$$2 < | < a > | \leq 3.$$

Thus $< a >$ must be of order 3 and must be equal to $G$, i.e. $< a >= G$. Therefore, $G$ is cyclic.

5. a) (1453) Order 4.
   b) (156)(234) Order 3.
   c) (1246) Order 4.
   d) (1462) Order 4.

6. Using the table on page 105, the subgroups of $A_4$ are

$$\begin{aligned}
H_9 &= \{1\} \\
H_8 &= \{1, 2\} \\
H_7 &= \{1, 3\} \\
H_6 &= \{1, 4\} \\
H_1 &= \{1, 2, 3, 4\} \\
H_2 &= \{1, 5, 9\} \\
H_3 &= \{1, 6, 11\} \\
H_4 &= \{1, 7, 12\} \\
H_5 &= \{1, 8, 10\} \\
H_0 &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}
\end{aligned}$$