MAT 4170: Modern Algebra I **Name:** **Panya Sukphranee**

Homework 5

Due: Tuesday, 10/6/20

Question:	1	2	3	4	5	6	Total
Points:	3	3	3	3	3	3	18
Score:							

1. (3 points) Prove that $(n\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$ for any integer $n \in \mathbb{Z}$. Do you think there there any other subgropus of $(\mathbb{Z}, +)$ that do not have this form?

   ($\Rightarrow$) The identity element $e = 0$ is in $n\mathbb{Z}$ for all n since $0 \cdot n = 0$. So the group $(n\mathbb{Z}, +)$ is non-empty. To show $(n\mathbb{Z}, +)$ is a subgroup, we'll use the One-Step Subgroup Test. So lets suppose $a, b \in n\mathbb{Z}$. Note that $b^{-1} = -b$.

   $$ab^{-1} = a + b^{-1} = a + (-b) = a - b.$$

   Being elements of $n\mathbb{Z}$, $a = nk, b = nj$ for some $n, j \in \mathbb{Z}$. Thus,

   $$ab^{-1} = a - b = nj - nk = n(j - k).$$

   Since $j - k \in \mathbb{Z}$, ab^{-1} is an element of $n\mathbb{Z}$. Thus $(n\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$ by the One-Step Subgroup Test.

2. (3 points) Let G be an Abelian (commutative) group with identity e. Prove that the subset $H = \{x \in G \mid x^2 = e\}$ is a subgroup of G.

   ($\Rightarrow$) The identity element of G, e, is in H since $e^2 = e$. Therefore H is non-empty and we can use the One-Step Subgroup Test on it. Let $a, b \in H$. We want to show that $(ab^{-1})^2 = e$.

   $$(ab^{-1})^2 = a^2(b^{-1})^2 = a^2(b^2)^{-1} = ee^{-1} = e.$$

   The first equal sign uses a property of being Abelian. H passes the One-Step Subgroup Test and is therefore is a subgroup of G.

3. (3 points) Prove that $SL(2, \mathbb{R})$, the set of all 2×2 matrices with entries from $\mathbb{R}$ and determinant 1, is a subgroup of $GL_2(\mathbb{R})$ under multiplication.

   ($\Rightarrow$) The identity element of $GL_2(\mathbb{R})$ is

   $$e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The determinant of e being 1 implies that $e \in SL(2, \mathbb{R})$. We will use the One-Step Subgroup Test again. Let $a, b \in SL(2, \mathbb{R})$. We want to show $(ab^{-1}) \in SL(2, \mathbb{R})$ by showing its determinant is 1.

$$det(ab^{-1}) = det(a)det(b^{-1}) = det(a)\frac{1}{det(b)} = 1 \cdot 1 = 1.$$

Since $det(ab^{-1}) = 1$, it is an element of $SL(2, \mathbb{R})$ and thus $SL(2, \mathbb{R})$ is subgroup of $GL(2\mathbb{R})$.

4. (3 points) Let H_1, H_2 be subgroups of a group G.

   (a) Prove that $H_1 \cap H_2$ is a subgroup.

   ($\Rightarrow$) Being subgroups of G, H_1 and H_2 both contain e. Thus $e \in H_1 \cap H_2$. If $a, b \in H_1 \cap H_2$, $a, b \in H_1$ and $a, b \in H_2$. Being subgroups, both H_1 and H_2 contain ab^{-1}. Thus

   $$ab^{-1} \in H_1 \cap H_2.$$

   By the One-Step Subgroup Test, $H_1 \cap H_2$ is a subgroup of G.

   (b) Prove or disprove. If H_1 and H_2 are subgroups of a group G, then $H_1 \cup H_2$ is a subgroup of G.

   ($\Rightarrow$) Disprove by counterexample. Let $G = (\mathbb{Z}, +)$, $H_1 = (2\mathbb{Z}, +)$, and $H_2 = (3\mathbb{Z}, +)$. Both H_1 and H_2 are subgroups of G by Problem 1. Let $a = 2$ and $b = 3$.

   $$ab^{-1} = 2 - 3 = -1.$$

   1 is not a multiple of either 2 or 3 so it is not an element of $2\mathbb{Z} \cup 3\mathbb{Z}$.

5. (3 points) Let G be a group with the following Cayley table:

	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	8	7	6	5	4	3
3	3	4	5	6	7	8	1	2
4	4	3	2	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	4	3	2	1	8	7
7	7	8	1	2	3	4	5	6
8	8	7	6	5	4	3	2	1

   (a) Find $Z(G)$ (the center of G).

   $Z(G) = \{1, 5\}$

   (b) Find $< 3 >$ and $< 4 >$.

   $$< 3 >= \{3, 5, 7, 1\}$$

   $$< 4 >= \{4, 1\}$$

(c) A group G is called **cyclic** provided that there is an element $a \in G$ such that $G = \langle a \rangle$. Is group G in this example a cyclic group? Why or why not?

G is not a cyclic group because no element of G generates it. We come to this conclusion by evaluating each element of G. We can do this by inspecting the Cayley Table. Take 3 for example. We know that $3 \in < 3 >$. Generally, $a \in < a >$. Next we look at the entry of row 3 column 3 to evaluate $3 \cdot 3 = 5$. This means that $5 \in < 3 >$. To evaluate $3^3 = 3 \cdot 3^2 = 3 \cdot 5$, we look at row 3 column 5. This entry being 7 implies $7 \in < 3 >$. We continue doing this until reaching the identity element 1. All of the powers of $a \in G$ results in 1 before exhausting all possible values of G. This means that no element of G generates G. Therefore it is not a cyclic group.

6. (3 points) Let $\mathbb{C}^* = \{a + ib \mid a, b \in \mathbb{R}$ and not both a and b are zero$\}$ be the group of non-zero complex numbers with operation multiplication. For each positive integer n, the set of solutions to the equation $x^n - 1 = 0$ forms a subgroup of $\mathbb{C}^*$ called **the nth roots of unity**.

(a) Find all of the solutions in $\mathbb{C}^*$ to the equation $x^n - 1 = 0$ for $n = 2$, 3, and 4.

($\Rightarrow$) The n^{th} roots of unity are represented by points lying equally spaced apart on a unit circle in the complex plane with $(1, 0)$ being one of them. The points are separated by angle $\frac{2\pi}{n}$ radians.

- $n = 2 \Rightarrow \{1, -1\}$
- $n = 3 \Rightarrow \{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\}$
- $n = 4 \Rightarrow \{1, i, -1, -i\}$

(b) Check that the n solutions in $\mathbb{C}^*$ of the equation $x^n - 1 = 0$ each form a multiplicative subgroup of $\mathbb{C}^*$ for $n = 2$, 3, and 4 by building the table for each group.

	1	-1
1	1	-1
-1	-1	1

Table 1: n=2

	1	$-\frac{1}{2}(1 + i\sqrt{3})$	$\frac{1}{2}(-1 + i\sqrt{3})$
1	1	$-\frac{1}{2}(1 + i\sqrt{3})$	$\frac{1}{2}(-1 + i\sqrt{3})$
$-\frac{1}{2}(1 + i\sqrt{3})$	$-\frac{1}{2}(1 + i\sqrt{3})$	$-\frac{1}{2}(1 + i\sqrt{3})$	1
$\frac{1}{2}(-1 + i\sqrt{3})$	$\frac{1}{2}(-1 + i\sqrt{3})$	1	$\frac{1}{2}(-1 + i\sqrt{3})$

Table 2: n=3

	1	i	-1	-i
1	1	i	-1	-i
i	i	-1	-i	1
-1	-1	-i	1	i
-i	-i	1	i	-1

Table 3: n=4

(c) For $n = 2$, 3, and 4, plot the roots of $x^n = 1$ on the complex plane and draw the picture of what shape you get back when you connect the dots.

See next page. It was too difficult to LaTex.