3.29.5) $f : X_1 \to X_2$ homeomorphism of locally compact Hausdorff spaces. Show f extends to a homeomorphism of their one-point compactifications.

Let Y_1, Y_2 be the one-point compactifications of X_1, X_2, respectively. Let p, q be the appended points of X_1, X_2, respectively.

Define

$$g(x) = \begin{cases} f(x) & x \in X_1 \\ q & x = p \end{cases}$$

$g(x)$ is bijective. We need to show that $g(x)$ is continuous. It suffices to consider open sets about p and $q = g(p)$.

Let $U_2 \ni q$ be an open neighborhood of q. The set $C_2 := Y_2 - U_2$ is closed and thus compact in Y_2. Then

$$C_1 := g^{-1}(U_2) = f^{-1}(U_2)$$

is closed in Y_1 and does not contain p. By Lemma 26.4, $\exists$ $U_1 \ni p$ disjoint from and open set containing C_1. It follows that $g(U_1) \ni g(p)$ is contained in U_2.

3.29.8) Show that the one-point compactification of $\mathbb{Z}_+$ is homeomorphic with the subspace

$$\{0\} \cup \left\{ \frac{1}{n} | n \in \mathbb{Z}_+ \right\} \subset \mathbb{R}.$$

We first recognize that $\mathbb{Z}_+$ is locally compact Hausdorff. $\forall$ $x \in \mathbb{Z}_+$, the density of $\mathbb{R}$ guarantees that

$$x \in (x - \epsilon, x + \epsilon) \subset [x - \epsilon, x + \epsilon].$$

By Theorem 29.1, $\exists$ $Y_{\mathbb{Z}_+}$ one-pact compactification of $\mathbb{Z}_+$.

Next, we see that

$$S = \{0\} \cup \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}$$

is the one-pact compactification of

$$S' = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}_+ \right\}.$$

The map $f : Z_+ \to S'$ given by

$$f(n) = \frac{1}{n}$$

is a homeomorphism. By the previous problem, f extends to a homeomorphism between their one-point compactifications. Thus S is homemorphic to Z_+.

7.43.1) (X, d) metric space.

(a) Suppose for some $\epsilon > 0$, every ϵ-ball in X has a compact closure. Show that X is complete.

Let (x_n) be a Cauchy sequence and $\epsilon > 0$. For some $n, m \geq N$, we have

$$d(x_m, x_n) < \frac{\epsilon}{2}.$$

Then $B(x_N, \frac{\epsilon}{2}) \subset B(x_N, \epsilon)$ and contains infinitely many points of (x_n). Then,

$$\overline{B\left(x_N, \frac{\epsilon}{2}\right)} \subset \overline{B(x_N, \epsilon)}.$$

It follows that $\overline{B\left(x_N, \frac{\epsilon}{2}\right)}$ is closed an since $\overline{B(x_N, \epsilon)}$ is compact. Thus (x_n) converges in X and X is therefore complete.

(b) Let $X = \left\{\frac{1}{n} \mid n \in Z_+\right\}$. For each $x \in X$, there exists some $\epsilon > 0$ st $B(x, \epsilon) = \{x\}$. This closure is itself. But X is not complete since it doesn't contain 0.

7.43.2) $(X, d_X), (Y, d_Y)$ metric spaces. Y complete. $A \subset X$, $f : A \to Y$ continuous.

Define $g : \overline{A} \to Y$ by

$$g(x) = \lim_{n\to\infty} f(x_n)$$

for $(x_n) \to x \in \overline{A}$. We can define this because Y is complete. By Theorem 21.3, g is continuous. It is uniformly continuous since $\overline{A}$ is bounded.

To show g is unique, let h be another extension and let $x \in A'$ and $(x_n) \to x$.

Let $y_1 = h(x)$, $y_2 = g(x)$. $\forall\ \epsilon > 0$, let

$$n \geq N \to d(y_1, f(x_n)) < \frac{\epsilon}{2}$$

and

$$n \geq M \to d(y_2, f(x_n)) < \frac{\epsilon}{2}.$$

Then $n \geq max\{N, M\} \to d(y_1, y_2) < \epsilon$. Thus $h = g$.

7.43.3) $\overline{d}(x, y) = min\{d(x, y), 1\}$

(a) Show d is metrically equivalent to the standard bounded metric $\overline{d}$ derived from d.

($\Rightarrow$) First show that $i : (X, d) \to (X, d')$ is uniformly continuous. $i(x) = x$.
Let $\delta = \epsilon$.

$$d(x, y) < \delta \longrightarrow \overline{d}(x, y) < \epsilon.$$

This is trivial by definition for if $\epsilon < 1$, $\overline{d}(x, y) < \epsilon$. Otherwise, $1 \leq \epsilon$, $\overline{d}(x, y) = 1$.

($\Leftarrow$) Let $\delta = min\{\epsilon, 1\}$. We have two cases:

If $\delta := \epsilon \leq 1$, then

$$\overline{d}(x, y) < \epsilon \longrightarrow d(x, y) < \epsilon.$$

If $\delta := 1 < \epsilon$, then

$$\overline{d}(x, y) < 1 \longrightarrow d(x, y) < 1 < \epsilon.$$

Thus, i and i^{-1} are uniformly continuous and thus d, d' are metrically equivalent.

(b) Show if d and $\overline{d}$ are metrically equivalent, then X is complete in d iff it is complete in $\overline{d}$.

($\Rightarrow$) Suppose (X, d) complete. Let $(x_n) \to x$ be a convergent Cauchy sequence in (X, d).

By continuity of $i : (X, d) \to (X, d')$, Theorem 21.3 tell us that

$$x_n \longrightarrow x \iff i(x_n) \longrightarrow i(x)$$

thus $x_n \longrightarrow x$ in (X, d'). The converse holds by symmetry.

7.43.9) (X, d) metric space. $\tilde{X}$ set of all Cauchy sequences

$$\mathbf{x} = (x_1, x_2, ...)$$

of points of X. Define $x \sim y$ if

$$d(x_n, y_n) \longrightarrow 0.$$

Let $Y = \{[x] |$ equivalence class of sequences$\}$. Define D on Y by

$$D([\mathbf{x}], [\mathbf{y}]) = \lim_{n\to\infty} d(x_n, y_n).$$

(a) Show $\sim$ is an equivalence relation and D well-defined.

(1) $\sim$ is relflexive since $d(x_n, x_n) = 0 \; \forall \; n$.

(2) $\sim$ symmetric by symmetry of metric d.

(3) Given $\epsilon > 0$, N, M s.t.

$$d(x_n, y_n) < \frac{\epsilon}{2} \ \forall \ n \geq N,$$

and

$$d(y_m, z_m) < \frac{\epsilon}{2} \ \forall \ m \geq M.$$

It follows that $d(x_n, z_n) \longrightarrow 0$. Thus $\sim$ is transitive.

(4) Let $a \in [x]$ and $b \in [y]$.

$$\begin{aligned}
D([\mathbf{a}], [\mathbf{b}]) &= \lim_{n\to\infty} d(a_n, b_n) \\
&= \lim_{n\to\infty} d(x_n, a_n) + \lim_{n\to\infty} d(a_n, b_n) + \lim_{n\to\infty} d(b_n, y_n) \\
&= \left[\lim_{n\to\infty} d(x_n, a_n) + \lim_{n\to\infty} d(a_n, b_n)\right] + \lim_{n\to\infty} d(b_n, y_n) \\
&\geq \lim_{n\to\infty} d(x_n, b_n) + \lim_{n\to\infty} d(b_n, y_n) \\
&\geq \lim_{n\to\infty} d(x_n, y_n) \\
&= D([\mathbf{x}], [\mathbf{y}])
\end{aligned}$$

The same argument will show that $D([x], [y]) \leq D([a], [b])$. Thus

$$D([\mathbf{x}], [\mathbf{y}]) = D([\mathbf{a}], [\mathbf{b}]).$$

(b) $h : X \to Y$ by $h(x) = [(x, x, ...)]$. Show h isometric imbedding.
We first observe that h is injective. For if

$$h(x) = [(x, x, x, ...)] = [(y, y, y, ...)] = h(y),$$

then $d(x, y) \longrightarrow 0$ implying $x = y$. h is clearly surjective onto its image $h(X)$. Thus we have a bijection. We need to show that h is a homeomorhism.

$$D([x], [y]) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

By the chain of equalities above,

$$d(x, y) < \delta := \epsilon$$

implies

$$D([x], [y]) < \epsilon,$$

giving us h uniformly continuous. It follows that h^{-1} is also uniformly continuous. Thus h is a homeomorphism and an isometric imbedding.

(c) Show that h is dense. Let $\mathbf{x} = (x_1, x_2, ...) \in \tilde{X}$. Want to show that the sequence $h(x_n) \longrightarrow [\mathbf{x}]$ in Y.

We will show that $\forall\, \epsilon > 0,\ \ (n \geq N)$ s.t.

$$D(h(x_n), [\mathbf{x}]) < \epsilon.$$

Expanding upon the definition of this metric, we want to show that

$$\begin{aligned} D(h(x_n), [\mathbf{x}]) &= \lim_{k \to \infty} d(x_n, x_k) \\ &< \epsilon. \end{aligned}$$

Since (x_n) is Cauchy, such $N \leq n$ exists and thus

$$h(x_n) \longrightarrow [\mathbf{x}].$$

(d) cant think, i dont know

(e) cant think, i dont know