MAT 4170: Modern Algebra I **Name:** **Panya Sukphranee**

Homework 7

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1. a) The group G is isomorphic to $\mathbb{Z}_4$. There are 2 isomorphisms,

$$\{A \mapsto 2, B \mapsto 3, C \mapsto 1, D \mapsto 0\}$$

and

$$\{A \mapsto 2, B \mapsto 1, C \mapsto 3, D \mapsto 0\}$$

   b) With $4! = 24$ permutations of elements of G, $24 - 2 = 22$ of those are not isomorphisms. We see that there are 2 isomorphisms because the element D can only be mapped to the identity and that A can only be mapped to 2 because they are both of order 2. There are 2 positions left to for use to freely choose.

2. We use the result of problem 4 that order of elements are preserved.

   a) The group $U(5)$ is isomorphic to $U(10)$ because the mapping ϕ defined as follows is an isomorphism. Define ϕ by

$$\{1 \mapsto 1, 2 \mapsto 3, 3 \mapsto 7, 4 \mapsto 9\}.$$

   The group $U(5)$ is cyclic and is generated by 2 and 3. On the other hand, $U(12)$ is not cyclic since every element is of order 2. Since isomorphisms preserve every elements order, there is no isomorphism between $U(5)$ and $U(12)$. Thus $U(5)$ is not isomorphic to $U(12)$.

   b) Every element of $U(24)$ is of order 2. But the element $9 \in U(20)$ is of order 4. With isomorphisms preserving order of elements, no such isomorphism exists between them and are therefore not isomorphic.

3. Let $G = \{a + b\sqrt{2} | a, b \in \mathbb{Q}\}$ and $H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} | a, b \in \mathbb{Q} \right\}$.
Define $\phi : G \to H$ by

$$\phi(a + b\sqrt{2}) = \begin{bmatrix} a & 2b \\ b & a \end{bmatrix}.$$

The mapping is onto since both elements of G and H are specified by two rational numbers. We can take a amd b from the matrix and identify an element in G that maps to it, under ϕ. Next we'll show the mapping is 1-1. Suppose

$$\begin{bmatrix} a_1 & 2b_1 \\ b_1 & a_1 \end{bmatrix} = \begin{bmatrix} a_2 & 2b_2 \\ b_2 & a_2 \end{bmatrix}.$$

Rewriting that in terms of some elements of G, we have that

$$\phi(a_1 + b_1\sqrt{2}) = \phi(a_2 + b_2\sqrt{2}).$$

On the other hand, component-wise comparison tells us that $a_1 = a_2$ and $b_1 = b_2$. Thus,

$$1 + b_1\sqrt{2} = a_2 + b_2\sqrt{2}.$$

Therefore, ϕ is 1-1. Finally, we show that $\phi : G \to H$ is operation preserving. Let $g_1 = a_1 + b_1\sqrt{2}$ and $g_2 = a_2 + b_2\sqrt{2}$.

$$\begin{aligned}\phi(g_1 + g_2) &= \begin{bmatrix} a_1 + a_2 & 2b_1 + 2b_2 \\ b_1 + b_2 & a_1 + a_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1 & 2b_1 \\ b_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & 2b_2 \\ b_2 & a_2 \end{bmatrix} \\ &= \phi(g_1) + \phi(g_2).\end{aligned}$$

Thus, $(G, \cdot)$ is isomorphic to $(H, +)$.

4. Suppose $(G, \cdot)$ and $(H, *)$ are groups and $\phi : G \to H$ is an isomorphism. If $|a| = \infty$, we have that for all $n \in \mathbb{Z}$,

$$a^n \neq e_G.$$

Since isomorphisms preserve multiplication, $\phi(a^n) = \phi^n(a)$, implies

$$\phi^n(a) \neq e_H \;\; \forall n \in \mathbb{Z}.$$

Thus, $|\phi(a)| = \infty$. Because ϕ is an isomorphism, so is its inverse and the same arguemnt can be made in the other direction. Therefore,

$$|a| = \infty \iff |\phi(a)| = \infty.$$

We now consider the case that $|a| = n$ is finite and let $|\phi(a)| = m$, which is also finite. $\phi(a^n) = \phi^n(a) = e_H$ implies m divides n so that $m \leq n$. On the other hand,

$$\phi^{-1}[\phi^m(a)] = \phi^{-1}\phi(a^m) = a^m$$

implies n divides m so that $n \leq m$. It follows that $n = m$. Thus,

$$|a| = |\phi(a)|.$$

5. Suppose G is a cyclic group of infinite order generated by some $a \in G$. Define $\phi : G \to \mathbb{Z}$ by

$$\phi(a^k) = k.$$

This map is onto because every $k \in \mathbb{Z}$ is identified with $a^k \in G$. Suppose that

$$\phi(a^l) = \phi(a^k),$$

then $l = k$. Because $|a| = \infty$, Theorem 4.1 tells us that $l = k \iff a^k = a^l$. Thus ϕ is 1-1. Finally, we show that ϕ preserves multiplication.

$$\phi(a^k a^j) = \phi(a^{k+j}) = k + j = \phi(a^k)\phi(a^j).$$

6. Let $\phi_a : G \to G$ be defined by $\phi_a(x) = axa^{-1}$ $\forall x \in G$. We begin by showing that ϕ_a is onto. Let $y \in G$. By closure of G, the element $axa^{-1} \in G$. Since

$$\phi_a(a^{-1}ya) = a(a^{-1}ya)a^{-1} = y,$$

we have that ϕ_a is onto. Now suppose that $\phi_a(x) = axa^{-1} = aya^{-1} = \phi_a(y)$. Applying the Cancellation Property, we have that $x = y$. Thus ϕ_a is 1-1. Finally, we show that ϕ_a is operation preserving. Let $x, y \in G$.

$$\phi_a(xy) = a(xy)a^{-1}.$$

On the other hand,

$$\begin{aligned}\phi_a(x)\phi_a(y) &= (axa^{-1})(aya^{-1}) \\ &= axa^{-1}aya^{-1} \\ &= axya^{-1}.\end{aligned}$$

Thus, ϕ_a is an automorphism.