7.3.1) (a) $\sum_{k=0}^{\infty} = \frac{kx^k}{(2k+1)^2}$.

We use the Ratio Test to find the ROC. The k^{th} term is $|a_k| = \frac{k}{(2k+1)^2}$.

$$\begin{aligned}\lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} &= \lim_{k\to\infty} \left| \frac{k}{k+1} \cdot \frac{(2k+3)^2}{(2k+1)^2} \right| \\ &= 1\end{aligned}$$

We now test the endpoints. For $x = 1$,

$$\sum_{k=0}^{\infty} \frac{k}{(2k+1)^2}$$

diverges by the Comparison Test with $\frac{1}{k}$. For $x = -1$,

$$\sum_{k=0}^{\infty} (-1)^k \frac{k}{(2k+1)^2}$$

converges by the Alternating Series Test. Therefore the IOC is $(-1, 1]$.

(b) $\sum_{k=0}^{\infty} (2 + (-1)^k)^k x^{2k}$

Rewrite the series to

$$\sum_{k=0}^{\infty} \left(\sqrt{2 + (-1)^k}\right)^{2k} x^{2k}$$

Then use the Root Test to find the ROC,

$$\limsup_{2k\to\infty} |a_{2k}|^{1/2k} = \limsup_{2k\to\infty} \begin{cases} 0 & k = 0 \\ 1 & k - odd \\ \sqrt{3} & k - even \end{cases} = \sqrt{3}$$

The ROC is $\frac{1}{\sqrt{3}}$. For the endpoints $x = \pm\frac{1}{\sqrt{3}}$,

$$\sum_{k=0}^{\infty} \left(2 + (-1)^k\right)^k \left(\frac{1}{3}\right)^k = \begin{cases} \left(\frac{1}{3}\right)^k & k - odd \\ 1 & k - even \end{cases}$$

diverges. Therefore, the interval of convergence is $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$

(c) $\sum_{k=0}^{\infty} 3^{k^2} x^{k^2}$

This series sums up every $j = k^2$ term of the series $\sum_j 3^j x^j$. By the Root Test

$$\limsup_{k\to\infty} |a_{k^2}|^{1/k^2} = 3$$

Thus, ROC is $\frac{1}{3}$. The series does not converge at endpoints $x = -\frac{1}{3}, \frac{1}{3}$. We can see that the series for each are $\sum(-1)^k$ and $\sum 1^k$, respectively. Therefore the IOC is $(-\frac{1}{3}, \frac{1}{3})$.

(d) $\sum_{k=0}^{\infty} k^{k^2} x^{k^3}$

We can rewrite the above as

$$\sum_{k=0}^{\infty} \left(k^{1/k}\right)^{k^3} x^{k^3}$$

and see this as the sum of every $j = k^3$ term of $\sum a_j x^j$ (where $a_j = j^{1/j}$). By the Root Test,

$$\limsup_{k\to\infty} \left|k^{k^2}\right|^{1/k^3} = \limsup_{k\to\infty} \left|k^{1/k}\right| = 1$$

and therefore the ROC is 1. The series diverges at both endpoints, $x = \pm 1$. Therefore the IOC is $(-1, 1)$.

7.3.2) Find the interval of convergence of each power series

(a) $\sum_{k=0}^{\infty} \frac{x^k}{2^k}$

The k-th coefficient is $a_k = \dfrac{1}{2^k}$. We use the Ratio Test,

$$\lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|} = \lim_{k\to\infty} 2 = 2$$

to determine that the ROC is 2. For the endpoints $x = \pm 1$, the series diverges. Therfore the IOC is $(-1, 1)$.

(b) $\sum_{k=0}^{\infty} ((-1)^k + 3)^k (x-1)^k$

Applying the Root Test k-th term, we have that

$$\begin{aligned}\lim_{k\to\infty} \left[((-1)^k + 3)^k\right]^{1/k} &= \lim_{k\to\infty} ((-1)^k + 3) \\ &= \begin{cases} 4 & k - even \\ 2 & k - odd \end{cases}\end{aligned}$$

Therefore, the ROC is $min \left\{\dfrac{1}{2}, \dfrac{1}{4}\right\} = \dfrac{1}{4}$. So far, the IOC is $\left(\dfrac{3}{4}, \dfrac{5}{4}\right)$. Now we test the endpoints. Note that the k-th coefficient of the series is

$$a_k = \begin{cases} 4^k & k - even \\ 2^k & k - odd \end{cases}$$

Let $x = \dfrac{3}{4} \Rightarrow x - 1 = -\dfrac{1}{4}$. The **even** terms of this series are

$$4^k \left(-\frac{1}{4}\right) = 1,$$

the series diverges. For the other endpoint $x = \dfrac{5}{4}$, $x - 1 = \dfrac{1}{4}$, the even terms are 1 and therefore make the series diverge. Thus, the IOC is

$$\left(-\frac{3}{4}, \frac{5}{4}\right)$$

(c) $\sum\limits_{k=1}^{\infty} log\left(\dfrac{k+1}{k}\right) x^k$

We find the ROC by the ratio test

$$\lim_{k\to\infty} \left|\frac{k+1}{k} \cdot \frac{k+2}{k+1}\right| = 1$$

imply $R = 1$ and so far the IOC is $(-1, 1)$. Now test the endpoints, let $x = -1$. Note that

$$\lim_{k\to\infty} log\left(\frac{k+1}{k}\right) = 0$$

so that the series

$$\sum_{k=1}^{\infty} log\left(\frac{k+1}{k}\right)(-1)^k$$

converges by the Alternating Series Test. For the other endpoint $x = 1$, the series becomes a telescoping series where

$$\begin{aligned}\sum_{k=1}^{\infty} log\left(\frac{k+1}{k}\right) &= \sum_{k=1}^{\infty} log\,(k+1) - \log k \\ &= \lim_{n\to\infty} log\,(n+1) \\ &= \infty\end{aligned}$$

diverges. Therefore IOC is $[-1, 1)$.

7.3.3) Suppose that $\sum\limits_{k=0}^{\infty} a_k x^k$ has a radius of convergence $R \in (0, \infty)$.

(a) Find the ROC of $\sum\limits_{k=0}^{\infty} a_k x^{2k}$

For clarity, rewrite this series as $\sum\limits_{j=0}^{\infty} b_j x^j$ where $\begin{cases} a_k & if\ j = 2k \\ 0 & if\ j \neq 2k \end{cases}$ The ROC of this series is

$$\limsup_{j\to\infty} |b_j|^{1/j} = \begin{cases} \limsup\limits_{k\to\infty} |a_k|^{\frac{1}{2k}} & if\ k - odd \\ 0 & if\ k - even \end{cases}$$
$$= \begin{cases} \left(\limsup\limits_{k\to\infty} |a_k|^{\frac{1}{k}}\right)^{1/2} & if\ k - odd \\ 0 & if\ k - even \end{cases}$$
$$= \begin{cases} \sqrt{\frac{1}{R}} & if\ k - odd \\ 0 & if\ k - even \end{cases}$$

so that the ROC is $min\left\{\sqrt{R}, \infty\right\} = \sqrt{R}.$

(b) Find the ROC of $\sum\limits_{k=0}^{\infty} a_k^2 x^k$

The ROC is

$$\lim_{k\to\infty} \frac{|a_k|^2}{|a_{k+1}|^2} = \lim_{k\to\infty} \left(\frac{|a_k|}{|a_{k+1}|}\right)^2$$
$$= \left(\lim_{k\to\infty} \frac{|a_k|^2}{|a_{k+1}|}\right)^2$$
$$= R^2$$

7.3.4) Suppose $|a_k| \leq |b_k|$ for large k and $\sum\limits_{k=0}^{\infty} b_k x^k$ converges on $I := I_b$. We show that $\sum\limits_{k=0}^{\infty} a_k x^k$ converges on I_b.

$$\frac{1}{R_a} = \limsup_{k\to\infty} |a_k|^{1/k} = \limsup_{k\to\infty} |b_k|^{1/k} = \frac{1}{R_b}.$$

Therefore, $R_a \leq R_b$ and $I := I_b \subset I_a$ so that $\lim\limits_{k\to\infty} |a_k|^{1/k}$ converges on I. If I is not open then the result would not hold. Show by counterexample, let $a_k = \frac{(-1)^k}{k}$ and $b_k = \frac{1}{k}$. Then, $\sum b_k x^k$ converges at $x = -1$ and therefore $[-1, 1)$. But $\sum a_k x^k$ does not.

7.3.5) Suppose that $\{a_k\}_{k=0}^{\infty}$ is a bounded sequence of $\mathbb{R}$ numbers. Prove that

$$f(x) := \sum_{k=0}^{\infty} a_k x^k$$

has a positive radius of convergence.

The sequence being bounded tells us that for some $M > 0$, $0 \leq |a_k| \leq M$. Therefore,

$$0 \leq \limsup_{k\to\infty} |a_k|^{1/k} \leq \limsup_{k\to\infty} M^{1/k} = \lim_{k\to\infty} M^{1/k} = 1$$

So that $0 \leq \frac{1}{R} \leq 1$. Therefore,

$$R \geq 1.$$

7.3.6) A series $\sum_{=0}^{\infty} a_k$ Abel summable iff

$$\lim_{r\to 1^-} \sum_{k=0}^{\infty} a_k r^k = L$$

(a) Prove that if $\sum_{k=0}^{\infty} a_k \longrightarrow L$ then $\sum_{k=0}^{\infty} a_k$ Abel summable.

The hypotheses $\sum a_k = L$ implies $\{a_k\}$ is bounded and by problem 7.3.5, the function

$$f(x) := \sum_{k=0}^{\infty} a_k x^k$$

has a ROC ≥ 1 and converges at the point $x = 1$. By Abels Theorem, f is continous on $[0, 1]$ so that

$$\lim_{x\to 1^-} f(x) = f(1) = L.$$

Thus $f(x)$ is Abel summable.

(b) Find the Abel sum of $\sum_{k=0}^{\infty} (-1)^k$.

The terms $a_k = (-1)^k$. We will find the limit

$$L = \lim_{r\to 1^-} \sum_{k=0}^{\infty} (-1)^k r^k$$

Since $r < 1$, the sum is a Geometric Series

$$\sum_{k=0}^{\infty} (-1)^k r^k = \frac{1}{1+k}$$

Then

$$L = \lim_{r\to 1^-} \frac{1}{1+k} = \frac{1}{2}.$$