Q11) Let A be the subset of $[0,1]$ which consists of all numbers which do not have teh digit 4 appearing in their decimal exanpsion. Find $m(A)$.

We find the measure of the complement, A^c, and subtract it from $m([0,1]) = 1$.

Proceed by partitioning the interval $[0,1]$ into 10 segments with the partition

$$\{0, .1, .2, .3, .4, .5, .6, .7, .8, .9, 1\}.$$

We have 1 interval $[.4, .5) \subset A^c$ of length $\frac{1}{10}$ so that

$$m(A^c) = \frac{1}{10}.$$

The remaining 9 intervals each have a 4 in the 10^{-2} position. The subintervals are of length $\frac{1}{10^2}$.

$$[.14, .15), [.24, .25), ..., [.94, .95).$$

So far, the measure of A^c is

$$\frac{1}{10} + 9 \cdot \frac{1}{10^2}.$$

Continuing on, there are $9 \cdot 9 = 9^2$ intervals with a 4 in the 10^{-3} position. So that the sum is now

$$\frac{1}{10} + 9 \cdot \frac{1}{10^2} + 9^2 \cdot \frac{1}{10^3}.$$

We have a geometric series with $a = \frac{1}{10}$ and $r = \frac{9}{10}$

$$\frac{1}{10} + \frac{1}{10}\left(\frac{9}{10}\right) + \frac{1}{10}\left(\frac{9}{10}\right)^2 + \frac{1}{10}\left(\frac{9}{10}\right)^3 + \dots$$

which converges to 1. Therefore $m(A^c) = 1 \Rightarrow m(A) = 0$.

Q24) Does there exists an enumeration $\{r_n\}_{n=1}^{\infty}$ of the rationals such that the complement of

$$\bigcup_{n=1}^{\infty} \left(r_n - \frac{1}{n}, r_n + \frac{1}{n}\right)$$

is non-empty.

There are infinitely many intervals of decreasing radius that we want to assign to each rational.

Start by fixing an interval $A = [0,1]$. I then go through the intervals in decreasing radius order

$$1, \frac{1}{2}, \frac{1}{3}, \dots$$

The first interval of radius 1 (i.e. $n = 1$), will be assigned to some $r_1 \notin [0,1]$ s.t.

$$(r_1 - 1, r_1 + 1) \cap [0,1] = \emptyset$$

The next interval of radius $\frac{1}{2}$, assign $r_2 = \frac{1}{2} \in [0,1]$. This interval is contained in $[0,1]$. Assign the interval of radius $\frac{1}{3}$ to $r_3 = \frac{1}{3} \in [0,1]$, which is also contained in $[0,1]$.

The interval of radius $\frac{1}{4} = \frac{1}{2^2}$ will be assigned to some $r_4 \notin [0,1]$ in a way that doesn't intersect $[0,1]$.

We continue assigning intervals in this manner. If the radius is the perfect square of something, i.e.

$$\frac{1}{n} = \frac{1}{m^2} \quad m \in \mathbb{N},$$

assign it to a rational outside of $[0,1]$. Otherwise, assign the interval to a rational in $[0,1]$.

The size of this convering is finite

$$m(\mathbb{Q}) \leq 1 + \sum_{n=1}^{\infty} \frac{1}{n^2} \longrightarrow 1 + \frac{\pi^2}{3},$$

therefore, $m(\mathbb{Q}^c) = \infty \neq \emptyset$.

Q37) It suffices to show that the measure of Γ is 0 on $[0,1]$. Since $\mathbb{R}$ can be partitioned into countable closed and bounded intervals, the measure holds for all subintervals.

Let $\varepsilon > 0$. Since f is assumed to be uniformly continuous, $\exists\, N \in \mathbb{N}$ s.t.

$$|x - y| < \frac{1}{N} \Rightarrow |f(x) - f(y)| < \varepsilon.$$

So we divide $[0,1]$ into N subintervals of length $\dfrac{1}{N}$. Let

$$A_i := [i, i+1] \times [f(i), f(i+1)].$$

We have that

$$\bigcup_{k=0}^{N-1} A_k$$

covers Γ. So that

$$\begin{aligned} m_*(\Gamma) &\leq \sum_{k=0}^{N-1} |A_k| \\ &= \sum_{k=0}^{N-1} \frac{\epsilon}{N} \\ &= \epsilon \end{aligned}$$

Since ε arbitrary, $m(\Gamma) = 0$.