1a) **Prove the following Lusin Theorem.**

Let f be measurable and finite valued on E with $m(E) < \infty$. Then for all $\varepsilon > 0$, there exists a closed set $F_\varepsilon \subset E$ and $m(E - F_\varepsilon) < \varepsilon$ and $f|_{F_\varepsilon}$ is a continuous function.

($\Rightarrow$) I follow the proof in the lecture notes and explain the process. The set theory and DeMorgan's Law involved had me stumped for a while. We begin be choosing some $\epsilon > 0$. By Theorem 4.3, $\exists$ a sequence of step functions $\{f_n\} \to f$ *a.e.*.

Since step functions are rectangles, we have that f_n is continuous on some $E_n \subset E$, and since $f_n \to f$ *a.e.*, E_n can be so that it's complement in E is

$$m\left(\tilde{E_n}\right) := m\left(E \backslash E_n\right) < \frac{\varepsilon}{3 \cdot 2^n}$$

The $\tilde{E_n}$ are subsets where f_n are not continuous and we want to cap the measure of $\bigcup \tilde{E_n}$ to $\sum m(\tilde{E_n}) < \dfrac{\varepsilon}{3}$.

Now by Egorov's Theorem, we can find a closed set $F \subset E$ s.t. $m(E \backslash F) < \dfrac{\varepsilon}{3}$ and f_n are uniformly continuous on F. Since we want to show f is continuous when restricted to some subset of E, we consider

$$\tilde{F} = F \backslash \left(\bigcup_{n=1}^{\infty} E_n \right).$$

Thus, $f_n \to f$ uniformly convergent and f_n are continuous $\forall\ n$ on $\tilde{F}$. Now,

$$E \backslash \tilde{F}$$

is the region of E where we do not have *both* uniform convergence and continuity. This region is capped by

$$m\left(E \backslash \tilde{F}\right) \leq m\left(E \backslash F\right) + \sum m\left(E \backslash \tilde{E_n}\right) = \frac{2}{3}\varepsilon.$$

We do not set cap to ε yet because, although Egorov's Theorem has that F is closed, it's not clear that the set minus operation resulting in $\tilde{F}$ is also closed. So we may use Theorem 3.4(iii) to approximate $\tilde{F}$. So let $F_\varepsilon \subset \tilde{F}$ be a closed set s.t.

$$m\left(\tilde{F} \backslash F_\varepsilon\right) < \frac{\varepsilon}{3}.$$

Then we have that

$$m(E \backslash F_\varepsilon) < m(E \backslash \tilde{F}) + m(\tilde{F} \backslash F_\varepsilon) < \varepsilon$$

which completes the proof.

1b) **If $f = \chi_{[0,1]}$ and regard f as defined on $[-1,2]$, what would be F_ε? Does it contradict Chapter 1 Ex22?**

($\Rightarrow$) Let $\varepsilon > 0$ and

$$F_\varepsilon = [-1,2] \backslash \left\{ \left(-\frac{\varepsilon}{4}, \frac{\varepsilon}{4} \right) \cup \left(1 - \frac{\varepsilon}{4}, 1 + \frac{\varepsilon}{4} \right) \right\}$$

This doesn't contradict CH1.22 because $f|_\varepsilon$ is not everywhere continuous.

1c) **Suppose that $f : \mathbb{R} \to \mathbb{R}$ in 1a), Then we can extend $f|_{F_\varepsilon}$ to be a continuous function on $\mathbb{R}$. (Hint: Complement of closed sets is a countable union of open disjoint intervals).**

($\Rightarrow$) The set $E \backslash F_\varepsilon$ is open and therefore can be written as countable union of open intervals,

$$E \backslash F_\varepsilon = \bigcup (a_k, b_k).$$

We can connect line segments in each of (a_k, b_k) so that

$$f(x) \to f(a_k) \ \ as \ x \to a_k$$
$$f(x) \to f(b_k) \ \ as \ x \to b_k$$

Ch2.6) **Integrability of f on $\mathbb{R}$ does not necessarily imply the convergence of $f(x)$ to 0 as $x \to \infty$.**

(a) **There exists a positive continuous function f on $\mathbb{R}$ so that f is integrable on $\mathbb{R}$, but yet $\limsup x \to \infty f(x) = \infty$.**

First, let

$$\tilde{f}(x) = \begin{cases} n & x \in [n, n + n^{-3}) \\ 0 & elsewhere \end{cases}$$

Then

$$\tilde{f}(x) = \sum_{n=1}^{\infty} n \chi_{[n, n+n^{-3})}$$

so that

$$\begin{aligned}
\int_{\mathbb{R}} \tilde{f}(x) dx &= \sum_{n=1}^{\infty} n \left\{ m \left([n, n + n^{-3}) \right) \right\} \\
&= \sum_{n=1}^{\infty} \frac{n}{n^3} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{\pi^2}{6}
\end{aligned}$$

We can make $\tilde{f}(x)$ continuous by making the rectangles into trapezoids and call if $f(x)$. Then $f(x) \leq \tilde{f}(x) \ \ \forall\, x$ implies

$$\int f(x)dx \leq \int \tilde{f}(x)dx < \frac{\pi^2}{6}.$$

Which measn $f(x)$ is integrable but $\limsup_{x\to\infty} f(x) = +\infty$.

(b) **However, if we assume that f is uniformly continuous on $\mathbb{R}$ and integrable, then as $\limsup_{|x|\to\infty} f(x) = 0$.**

The difference with uniformly continuous vs just continuous is that there's some limit to the steepness of the curves. So we cannot make the regions under the curve to get smaller fast enough to make the area converge.

So assume the hypothesis is and

$$\limsup_{|x|\to\infty} f(x) = c > 0.$$

Then show this that this leads to a contradiction.

Let $\varepsilon > 0$. By uniform continuity and the fact that $f(x) = c$ for infinitely many x, we can find infinitely many points that map to a neighborhood of c. We let $0 < d < \varepsilon$ and choose a sequence $\{x_i\}$ s.t. $|f(x_i) - c| < d$. So each x_i is in some $\delta > 0$ neighborhood of some $f^{-1}(c)$.

Now, $\varepsilon - d > 0$ is the size of the gap if x_i is chosen s.t. $f(x_i) = d$ (i.e. least value possible and furthest distance from c). Then $\exists$ a δ' neighborhood of x_i s.t.

$$|x - x_i| < \delta' \Rightarrow |f(x) - f(x_i)| < \varepsilon - d.$$

Then the rectangle of area $d \cdot 2\delta'$ is under the curve. Thus

$$\infty = \sum 2d\delta' \geq \int f dx$$

which is a contradiction.

[Hint: for (a), construct a continuous version of the function equal to n on the segment $[n, n + 1/n^3), n \geq 1]$.

Ch2.9) **Tchebychev inequality**. Suppose $f \geq 0$, and f is integrable. If $\alpha > 0$ and $E_\alpha = \{x : f(x) > \alpha\}$, prove that

$$m(E_\alpha) \leq \frac{1}{\alpha}\int_{\mathbb{R}} f.$$

$$\infty > \int_{\mathbb{R}} f dx \geq \int_{E_\alpha} f dx > \alpha m(E_\alpha)$$

implies the result we want.

Ch2.10) **Suppose $f \geq 0$, and let $E_{2^k} = \left\{x : f(x) > 2^k\right\}$ and $F_k = \left\{x : 2^k < f(x) \leq 2^{k+1}\right\}$. If f is finite almost everywhere, then**

$$\bigcup_{k=-\infty}^{\infty} F_k = \{f(x) > 0\},$$

and the sets F_k are disjoint.

Prove that f is integrable if and only if

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty, \textbf{ if and only if } \sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty.$$

($\Rightarrow$) If f is integrable then

$$\begin{aligned}
\infty &> \int f dx \\
&= \sum_{k=-\infty}^{\infty} \int f \chi_{F_k} dx \\
&> \sum_{k=-\infty}^{\infty} \int 2^k \chi_{F_k} dx \\
&= \sum_{k=-\infty}^{\infty} 2^k m(F_k)
\end{aligned}$$

so that

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty.$$

Now, we show

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) = 2 \sum_{k=-\infty}^{\infty} 2^k m(F_k).$$

$$\begin{aligned}\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) &= \sum_{k=-\infty}^{\infty}\sum_{j\geq k} 2^k m(F_j)\\ &= \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{j} 2^k m(F_j)\\ &= \sum_{j=-\infty}^{\infty} m(F_j)\sum_{k=-\infty}^{j} 2^k\\ &= \sum_{j=-\infty}^{\infty} m(F_j)2^{j+1}\\ &= 2\sum_{j=-\infty}^{\infty} m(F_j)2^{j}\\ &= 2\sum_{k=-\infty}^{\infty} m(F_k)2^{k}\end{aligned}$$

Therefore,

$$\sum_{k=-\infty}^{\infty} 2^k m(E_{2^k}) < \infty \iff \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$$

Since

$$\int f dx < \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty,$$

we have our desired result.

Use this result to verify the following assertions. Let

$$f(x) = \begin{cases} |x|^{-a} & if\ |x| \leq 1 \\ 0 & otherwise \end{cases}$$

and

$$g(x) = \begin{cases} |x|^{-b} & if\ |x| > 1 \\ 0 & otherwise \end{cases}$$

Then f is integrable on $\mathbb{R}^d$ iff $a < d$; also g integrable on $\mathbb{R}^d$ iff $b > d$.

(a) $2^{-\frac{k}{a}} > |x| \geq 2^{-\dfrac{k+1}{a}}$ tells us that the modulus of $x \in \mathbb{R}^d$ is between these two values. Therefore, the measure of F_k between the volume of the balls of those radii. We multiply the measure of a unit ball v_d by inequalities to the d^{th} power.

$$v_d 2^{-\frac{(k+1)d}{a}} \leq m(F_k) < v_d 2^{-\frac{kd}{a}} \quad (*)$$

We show $a < d$ implies $\sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$. Suppose $a < d$.

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} 2^k m(F_k) &= \sum_{k=0}^{\infty} 2^{-k} m(F_k) + \sum_{k=1}^{\infty} 2^k m(F_k) \\
&\leq \sum_{k=0}^{\infty} 2^{-k} v_d + \sum_{k=1}^{\infty} 2^k m(F_k) \\
&= 2v_d + \sum_{k=1}^{\infty} 2^k m(F_k) \\
&< 2v_d + v_d \sum_{k=1}^{\infty} 2^{k(1-d/a)} \\
&< \infty.
\end{aligned}$$

On the other hand,

$$v_d 2^{-d/a} \sum_{k=1}^{\infty} 2^{k(1-d/a)} < \sum_{k=-\infty}^{\infty} 2^k m(F_k) < \infty$$

implies $1 - \frac{d}{a} < 0 \Rightarrow a < d$.

(b) Consider $F_k = \left\{ x : 2^k < |x|^{-b} < 2^{k+1} \right\}$

$$2^k < |x|^{-b} < 2^{k+1}$$

$\Rightarrow$

$$2^{-k/b} > |x| > 2^{-(k+1)/b}.$$

$x \in \mathbb{R}^d$ satisfying the inequality above exists between two balls of satisfying measures

$$v_d \left(2^{\frac{-kd}{b}} \right) > m(F_k) > v_d \left(2^{-\frac{d(k+1)}{b}} \right)$$

where v_d is the measure of a unit ball. Then

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} 2^k m(F_k) &< v_d \sum_{k=-\infty}^{\infty} 2^{k\left(1-\frac{d}{b}\right)} \\
&< v_d \sum_{k=-\infty}^{0} 2^{k\left(1-\frac{d}{b}\right)} \\
&< v_d \sum_{k=1}^{\infty} 2^{k\left(\frac{d}{b}-1\right)}
\end{aligned}$$

Thus $d < b$ implies $\int g < +\infty$.

Using the other inequality, we have

$$\sum_{k=-\infty}^{\infty} 2^k m(F_k) < v_d \sum_{k=-\infty}^{\infty} 2^{k\left(1-\frac{d}{b}\right)-\frac{d}{b}}$$

so integrability implies $d < b$.

Ch2.11) Prove that if f is integrable on $\mathbb{R}^d$, real-valued, and $\int_E f(x)dx \geq 0$ for every measurable E, then $f(x) \geq 0$ *a.e.* x. As a result, if $\int_E f(x)dx = 0$ for every measurable E, then $f(x) = 0$ *a.e.*

Consider the set $F = \{x : f(x) < 0\}$. Since f is integrable, F is measurable and we will show $m(F) = 0$. Since f is negative on F, for all $n \geq 1$,

$$nf\chi_F \leq f.$$

Then,

$$\begin{aligned}
\int nf\chi_F dx &\leq \int f dx \\
n\int_F f dx &\leq \int f dx \\
\int_F f dx &\leq \frac{1}{n}\int f dx \\
\int_F f dx &\leq 0
\end{aligned}$$

But hypothesis implies

$$0 \leq \int_F f dx \leq 0$$

so that $\int_F f dx = 0$. But $f < 0 \;\; \forall\, x \in F$ implies $m(F) = 0$.

Now, if $\int_E f(x)dx = 0$ for all measurable E. Let $E = \{x : f(x) > 0\}$.

$$\int_E f dx = 0$$

implies $m(E) = 0$ since $f > 0$.

Ch2.15) Consider the function defined over $\mathbb{R}$ by

$$f(x) = \begin{cases} x^{-1/2} & if\ 0 < x < 1 \\ 0 & otherwise \end{cases}$$

For a fixed enumeration $\{r_n\}_{n=1}^{\infty}$ of rationals $\mathbb{Q}$, let

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n).$$

Prove that F is integrable, hence the series defining F converges for almost every $x \in \mathbb{R}$. However, observe that this series is unbounded on every interval, and in fact, any funciton $\tilde{F}$ that agrees with F *a.e.* is unbounded in any interval.