10.4.1) Identify which are compact and which are not. If not compact, find smallest compact set H such that $E \subset H$.

(a) $\left\{\frac{1}{k} : k \in \mathbb{N}\right\} \cup \{0\}$
Compact

(b) $\{(x, y) \in \mathbb{R}^2 : a \leq x^2 + y^2 \leq b\}$, $0 < a < b$
Compact

(c) $\left\{(x, y) \in \mathbb{R}^2 : y = sin\left(\frac{1}{x}\right) \quad forsomex \in (0, 1]\right\}$
Not Compact. $\left\{y = sin\left(\frac{1}{x}\right) |x \in (0, 1]\right\} \cup \{(0, 0)\}$

(d) $\{(x, y) \in \mathbb{R}^2 : |xy| \leq 1\}$

10.4.2) Let $A, B \subseteq X$ be compact subsets. Prove $A \cup B$ and $A \cap B$ are compact.

(a) $A \cap B \subseteq A$ is closed subset and is therefore compact by 10.45

(b) Let $\mathcal{C} = \{U_\alpha\}$ be an open cover for $A \cup B$. Then $\exists$ finite subsets $\mathcal{C}', \mathcal{C}'' \subset \mathcal{C}$ covering A, B, respectively. Then $\mathcal{C}' \cup \mathcal{C}''$ cover $A \cup B$.

10.4.3) Suppose that $E \subseteq \mathbb{R}$ is compact and non-empty. Prove $\sup E, \inf E \in E$.

Since E compact, E is closed. $\exists$ a sequence $\{x_n\} \to supE$, since every neighborhood of $supE$ intersects E. But E being closed implies $supE \in E$. Similarly, $infE \in E$.

10.4.5) Prove that if V open in separable (X, ρ), then $\exists$ open balls $B_1, B_2, ...$ s.t.

$$V = \bigcup_{j \in \mathbb{N}} B_j.$$

Prove every open set in $\mathbb{R}$ is a countable union of open intervals.

Let $Z \subset X$ be a countable subset of X. For all $v \in V$, $\exists$ $B(v) \subset V$ containing v. But seperability says $\exists$ $\{x_\alpha\} \subset Z$ converging to v. So we choose some $x_j \in \{x_\alpha\} \cap B(v)$.

Now we have that

$$v = \bigcup_{v \in V} B_{\varepsilon_v}(v) = \bigcup_{j \in \mathbb{N}} B_j$$

10.4.6) Suppose (X, ρ) is separable and satisfies Bolzano-Weierstrass Property (BWP), that Y complete, and $E \subseteq X$ bounded. Prove $f : E \to Y$ uniformly continuous on E iff f can be extended to $\overline{E}$.

($\Rightarrow$) We extend f to g by defining $g(x) = f(x)\ \forall\, x \in E$ and $g(x) = lim\{f(x_i)\}\ \ \forall\, x \in \partial E$, where $\{x_i\}$ is any sequence converging to x.

10.4.7) Suppose (X, ρ) satisfies BWP and that $A, B \subseteq X$ compact subsets. Prove that if $A \cap B = \emptyset$ and if

$$dist(A, B) := inf\,\{\rho\,(x, y) : x \in A, y \in B\}$$

then $dist(A, B) > 0$. Show that even in $\mathbb{R}^2$, $\exists$ subsets A, B which are closed and satisfy $A \cap B = \emptyset$, but $dist(A, B) = 0$.

Suppose $dist(A, B) = 0$. $\forall\, x_0, y_0$, $\exists x$

10.4.8) (a) Prove if $H_1, H_2, ...$ is a nested sequence of nonempty compact sets in X, then

$$\bigcap_{k=1}^{\infty} H_k \neq \emptyset$$

($\Rightarrow$) Suppose $H_1 \supset H_2 \supset H_3 \supset \ldots$ and suppose for the sake of contradiction that

$$\bigcap_{k=1}^{\infty} H_k = \emptyset.$$

We have that

$$X = X - \emptyset = X - \bigcap_{k=1}^{\infty} = \bigcup_{k=1}^{\infty} H_i^c$$

Therefore $\mathcal{H} := \{H_i^c\}$ covers H_1. Since the H_i^c are nested, we can pick some H_N^c that covers H_1. But this is contradiction because

$$H_N^c \cap H_N = \emptyset$$

$H_1 \subseteq H_N^c \Rightarrow H_1 \cap H_N = \emptyset$. Contradicting $H_N \subseteq H_1$. Therefore,

$$\bigcap_{k=1}^{\infty} H_k \neq \emptyset.$$

(b) Prove that $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ closed and bounded but not compact in metric space $\mathbb{Q}$.

I dont know

(c) Show that Cantor's Intersection Theorem does not hold in an arbitrary metric space if *compact* is replaced by *closed and bounded.*

I dont know

10.4.9) Prove that the BWP does not hold for $(\mathcal{C}[a, b], \|f\|)$.

($\Rightarrow$) Show by counter-example. Let $f_n(x) := x^n$ on $\mathcal{C}[0, 1]$. The sequence $\{f_n\}$ is bounded as each f_n is bounded by 1 relative to the zero function f_0.

Now,

$$f_n \to 0 \qquad \forall\, x \in [0, 1), n \in \mathbb{N}$$

and

$$f_n = 1 \qquad x = 1, n \in \mathbb{N}$$

Thus the sequence converges to a piecewise function which is not in $\mathcal{C}[0,1]$

$$f_n \to f = \begin{cases} 1 & x \in [0,1) \\ 0 & x = 1 \end{cases}$$

and therefore $\nexists$ convergent subsequence in $\mathcal{C}[0,1]$.

10.4.10) Let (X, ρ) metric space.

(a) Prove $E \subseteq$ compact $\rightarrow$ E sequentially compact

($\Rightarrow$) Let $E \subseteq X$ be compact and $\{x_n\}$ be a sequence in E. Since E compact, it is closed and bounded. Then $\exists$ a convergent subsequnce $\{x_{n-k}\} \longrightarrow x$. By closure, $x \in E$ and therefore $x \in E$. Thus E is sequentially compact.

(b) Prove if X separable and satisfies BWP, then

$$E \subseteq X \text{ sequentially compact} \iff E \text{ compact.}$$

($\Rightarrow$) Suppose E sequentially compact. By problem 10.1.10, sequentially compact implies closed and bounded. By Heine-Borel, that is equivalent to compact (since X seperable and BWP).