2.13.1) Let X be a topological space. $A \subset X$. Suppose $\forall\, x \in A$, there is an open U containing x s.t. $U \subset A$. Show that A is open in X.

Note: Open in X is understood as open in $(X, \mathcal{T})$.

$(\Rightarrow)$ Consider the set $\bigcup_{x\in A} U_x$, where $U_x \;:\; x \in U_x \subset A$. This set is open in X as it is a union of open sets of X. We can see that $A \subset \bigcup_{x\in A} U_x$ by construction. Now, $\forall\, x \in \bigcup_{x\in A} U_x$, $x \in U_x \subset A$. Thus

$$A = \bigcup_{x\in A} U_x$$

is open in X.

2.13.3) Let X be a set; let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ either is countable or is all of X. Then $\mathcal{T}_c$ is a topology on X. Is the colection

$$\mathcal{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

$(\Rightarrow)$ We show that $\mathcal{T}_c$ satisfies the definition of a topology.

(a) $\boxed{\emptyset \in \mathcal{T}_c}$ since

$$X - \emptyset = X$$

is all of X.

(b) Because the empty set is countably.

$$X - X = \emptyset$$

implies $\boxed{X \in \mathcal{T}_c}$.

(c) Consider a collection $\{U_\alpha\} \subset \mathcal{T}_c$ and its union $\bigcup U_\alpha$. We have that

$$X - \bigcup U_\alpha = \bigcap (X - U_\alpha) \subset X - U_\alpha,$$

for all α. But $X - U_\alpha$ is countable implies that the subset $X - \bigcup U_\alpha$ is countable. Thus,

$$\boxed{\bigcup U_\alpha \in \mathcal{T}_c.}$$

(d) Consider a finite collection $\{U_\alpha\} \subset \mathcal{T}_c$ and its intersection $\bigcap\limits_{\alpha \in I} U_\alpha$. We have that

$$X - \bigcap_{\alpha \in I} U_\alpha = \bigcup_{\alpha \in I} X - U_\alpha$$

is a countable set because a countable union of countable sets is countable by Theorem 7.5. Thus

$$\boxed{\bigcap_{\alpha \in I} U_\alpha \in \mathcal{T}_c.}$$

2.13.4a) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X. We will show that $\bigcap \mathcal{T}_\alpha$ is a topology on X.

Since $\mathcal{T}_\alpha$ are all topologies on X,

$$\emptyset, X \in \mathcal{T}_\alpha, \ \forall\, \alpha$$

implying

$$\boxed{\emptyset, X \in \bigcap \mathcal{T}_\alpha.}$$

Now suppose $\{U_i\}$ is an arbitrary collection of sets in $\bigcap \mathcal{T}_\alpha$. Then

$$\{U_i\} \subset \bigcap \mathcal{T}_\alpha \Rightarrow \{U_i\} \subset \mathcal{T}_\alpha$$

for all α. With all $\mathcal{T}_\alpha$ being topologies, it follows that $\bigcup U_i \in \mathcal{T}_\alpha$ for all α. Which implies

$$\boxed{\bigcup U_i \in \bigcap \mathcal{T}_\alpha.}$$

If $\{U_i\}_{i \in I}$ is a finite collection of sets in $\bigcap \mathcal{T}_\alpha$, then $\{U_i\}_{i \in I} \in \mathcal{T}_\alpha$ for all α. Again, with all $\mathcal{T}_\alpha$ being topologies,

$$\bigcap_{i \in I} U_i \in T_\alpha$$

for all α implies

$$\boxed{\bigcap_{i \in I} U_i \in \bigcap T_\alpha.}$$

Therefore, $\bigcap T_\alpha$ is a topology on X.

Question: Is $\bigcup \mathcal{T}_\alpha$ a topology on X? No, counterexample: Consider two non-comparable topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X. Since neither are contained in the other, there exists $U_1 \in \mathcal{T}_1$ and $U_2 \in \mathcal{T}_2$ s.t. $U_1 \notin \mathcal{T}_2$ and $U_2 \notin \mathcal{T}_1$. Then $U_1 \cup U_2$ is not defined in either $\mathcal{T}_1$ nor $\mathcal{T}_2$. Thus, $U_1 \cup U_2 \notin \mathcal{T}_1 \cup \mathcal{T}_2$.

2.13.4b) Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on X.

(a) $\exists$ unique largest topology contained in all $\mathcal{T}_\alpha$
$\mathcal{T}_S = \{\emptyset, X\}$. $\mathcal{T}_S \subset \mathcal{T}_\alpha\ \forall\alpha$. Any proper subset of $\mathcal{T}_S$ is not a topology.

(b) $\exists$ unique smallest topology containing all $\mathcal{T}_\alpha$
$\mathcal{T}_L = \mathcal{P}(X)$, power set of X. This set contains all possible $\mathcal{T}_\alpha$. Note that some T_α's it contains may not be comparable.

2.13.8a) Apply Lemma 13.2 to show that the countable collection

$$\mathcal{B} = \{(a,b) \mid a < b,\ \ a \textit{ and } b \textit{ rational}\}$$

is a basis that generates the standard topology on $\mathbb{R}$.

($\Rightarrow$) Let $x \in \mathbb{R}$ and $U \subset \mathbb{R}$ be an open containing x under the standard topology. Then $U = (\alpha, \beta)$ for some $\alpha, \beta \in \mathbb{R}$. We have that

$$\alpha < x < \beta.$$

But it's known from that $\mathbb{R}$ is dense and therefore $\exists\ a, b \in \mathbb{Q}$ s.t.

$$\alpha < a < x < b < \beta.$$

That is to say that $\exists$ some $W = (a,b) \in$ s.t.

$$x \in W \subset U.$$

By Lemma 13.2, $\mathcal{B}$ is a basis for the standard topology of $\mathbb{R}$.

E1: Show that the set of open subsets of $\mathbb{R}^2$ in the standard topology is uncountable.

Consider the point $\mathbf{0} := (0,0) \in \mathbb{R}^2$ and let $\mathcal{S}$ be a subset of open discs of radius $r < 1$ containing $\mathbf{0}$ in $\mathbb{R}^2$. Denote such discs of radius r centered at x by $B_x(r)$.

$$\mathcal{S} = \{B_x(\epsilon) \mid 0 < \epsilon < 1\}.$$

We can characterize each disc by it's radius. But since the set $(0,1)$ is uncountable, $\mathcal{S}$ is uncountable. $\mathcal{S}$ being a subset of open sets in $\mathbb{R}^2$ implies that the set of open sets in $\mathbb{R}^2$ is not countable.

E2: Does the standard topology on $\mathbb{R}^2$ admit a countable basis $\mathcal{B}$?

($\Rightarrow$) We've seen that

$$\mathcal{B} = \{(a,b) \mid a < b,\ \ a \textit{ and } b \textit{ rational}\}$$

is a countable basis on $\mathbb{R}$. Theorem 15.1, 7.6 tells us that $\mathcal{B} \times \mathcal{B}$ is a countable basis for $\mathbb{R} \times \mathbb{R}$.

$$\mathcal{B} \times \mathcal{B} = \{(a,b) \times (c,d) \mid (a,b), (c,d) \in \mathcal{B}\}.$$

We want to show that $\forall\, x \in R^2$ and $B_x(r) \ni x$, $\exists$ a rectangle in $\mathcal{B}\times\mathcal{B}$ containing x contained in the disc.

($\Rightarrow$) Consider $x \in B_x(r) \subset R^2$. Let $\delta' = \frac{r}{\sqrt{2}}$. Let $\delta \in \mathbb{Q}$ between $(0, \delta')$. Then

$$(x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 + \delta)$$

contains x and is contained in the disc. By Lemma 13.3, $\mathbb{R} \times \mathbb{R} \supset \mathbb{R}^2$.

Now show inclusion the other way. Consider $x \in (a, b) \times (c, d) \subset \mathbb{R} \times \mathbb{R}$. Let

$$r = min\{|x_1 - a|, |x_1 - b|, |x_2 - c|, |x_2 - d|\}.$$

Then $B_x(r)$ is a disc centered at x contained in the above basis element. By Lemma 13.3, $\mathbb{R} \times \mathbb{R} \subset \mathbb{R}^2$. Thus, $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

E3: Show that the two topologies on $X_1 \times X_2 \times X_3$ resulting from the two natural identifications $(X_1 \times X_2) \times X_3$, $X_1 \times (X_2 \times X_3)$ are the same.

- $(X_1 \times X_2) \times X_3 \leftrightarrow \mathcal{T}_1$
- $X_1 \times (X_2 \times X_3) \leftrightarrow \mathcal{T}_2$

($\Rightarrow$) Let $\mathbf{x} = (x_1, x_2, x_3) \in X_1 \times X_2 \times X_3$. Under the identification, there exists a basis element of $(X_1 \times X_2) \times X_3$ containing $\mathbf{x}$. That is to say $\exists$ open sets $\mathcal{O}_{12}$, U_3 in $X_1 \times X_2$ and X_3, respectively, such that:

$$(x_1, x_2) \in \mathcal{O}_{12},\ x_3 \in U_3.$$

$\mathcal{O}_{12}$ being open in $X_1 \times X_2$ implies $\exists\, U_1, U_2$ open in X_1, X_2, respectively s.t.:

- $x_1 \in U_1 \subset X_1$, U_1 open in X_1
- $x_2 \in U_2 \subset X_2$, U_2 open in X_2
- $U_1 \times U_2 \subset \mathcal{O}_{12}$

U_2, U_3 open implies $U_2 \times U_3$ is a basis element of $X_2 \times X_3$ containing (x_2, x_3). Then $U_2 \times U_3$ is open in $X_2 \times X_3$. It follows that $U_1 \times (U_2 \times U_3)$ is a basis element of $X_1 \times (X_2 \times X_3)$ containing $\mathbf{x} = (x_1, x_2, x_3)$ contained in $\mathcal{O}_{12},\ x_3 \in U_3$. By Lemma 13.3, $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, $\mathcal{T}_2 \subset \mathcal{T}_1$. Similar argument starting with the second identification shows us that $\mathcal{T}_1 \subset \mathcal{T}_2$. Thus $\mathcal{T}_1 = \mathcal{T}_2$.