10.1.1) Let $a, b \in X$ and suppose $\rho(a,b) < \varepsilon \; \forall \, \varepsilon > 0$. We will show $a = b$ by contradiction. So suppose for the sake on contradiction that the hypothesis holds but $a \neq b$. The $\rho(a,b) > 0$. Letting $\varepsilon = \rho(a,b)$, we have that $\rho(a,b) < \rho(a,b)$ which is a contradiction.

10.1.2) Prove $\{x_k\}$ bounded in X iff $\sup\limits_{k \in \mathbb{N}} \rho(x_k, a) < \infty \; \forall \, a \in X$.

($\Rightarrow$) Suppose that $\{x_k\}$ is bounded and let $b \in X$ and $M > 0$ by s.t. $\rho(x_k, b) \leq M \; \forall \, k$. Now pick some and hold some point $a \in X$. We have that

$$
\begin{aligned}
\sup_{k \in \mathbb{N}} \rho(x_k) \, a &\leq \sup_{k \in \mathbb{N}} \{\rho(x_k, b) + \rho(b, a)\} \\
&= \sup_{k \in \mathbb{N}} \rho(x_k, b) + \rho(b, a) \\
&= M + \rho(b, a) \\
&< \infty.
\end{aligned}
$$

($\Leftarrow$) Suppose that $M := \sup\limits_{k \in \mathbb{N}} \rho(x_k, a) < \infty \; \forall \, a \in X$. Let $a = x_1$. Then $\rho(x_1, x_k) \leq M$.

10.1.4) Let $a \in X$. Prove that if $x_n = a \; \forall \, n \in \mathbb{N}$, then x_n converges. What does it converge to?

(a) Let $\varepsilon > 0$, $\rho(x_n, a) = 0 < \varepsilon \; \forall \, n \in \mathbb{N}$. By definition $\{x_n\} \to a$.

(b) Suppose $X = \mathbb{R}$ with the discrete metric. Prove that $x_n \to a$ as $n \to \infty$ iff $x_n = a$ for large n.

($\Rightarrow$) Suppose $\{x_n\} \to a$. Let $\varepsilon = \frac{1}{2}$, then for some $N \in \mathbb{N}$, if $n \geq N$, then $\rho(x_n, a) = 0$. This implies $x_n = a$ for large n.

($\Leftarrow$) Suppose that $x_n = a$ for large n. i.e. $\exists \; N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow \rho(x_n, a) = 0$. For that same N, $\rho(x_n, a) < \varepsilon \; \forall \, \varepsilon > 0$.

10.1.5) (a) Let x_n, y_n be sequences in X which converge to the same point. Prove that $\rho(x_n, y_n) \to 0$ as $n \to 0$.

Let $\varepsilon > 0$. Then $\exists N, M \in \mathbb{N}$ s.t. $\rho(x_n, a) < \frac{\varepsilon}{2}$ and $\rho(y_n, a) < \frac{\varepsilon}{2}$. If $n \geq max\{N, M\}$, then

$$
\rho(x_n, y_n) \leq \rho(x_n, a) + \rho(y_n, a) < \varepsilon
$$

(b) Show that the converse is false.

10.1.6) Let $\{x_n\}$ be a Cauchy sequence in X. Prove $\{x_n\}$ converges if at least one subsequence converges.

($\Rightarrow$) Suppose $\{x_{n_i}\}$ is a convergent subsequence of $\{x_n\}$ that converges to $x \in X$. Let $\varepsilon > 0$. Then $\exists N'$ s.t. $\rho(x_n, x) < \frac{\varepsilon}{2}$.

Cauchy implies that $\exists N$ s.t. $\rho(x_n, x_{n_i}) < \frac{\varepsilon}{2}$. Therefore, $\rho(x_n, x) < \varepsilon$ for $n > max\{N, N'\}$.

($\Leftarrow$) Suppose $\{x_n\} \to x$. $\{x_{2n}\}$ is a subsequence of $\{x_n\}$ which converges.

10.1.7) Prove that $\mathbb{R}$ with the discrete metric is complete.

Suppose $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon < 1$. Then $\exists N \in \mathbb{N}$ s.t. $n, m > N$ implies $\rho(x_n, x_m) < \varepsilon$. Since we are using the discrete metric, $\rho(x_n, x_m) = 0$ implies $x_n = x_m$. Therefore $\{x_n\} \to x_k$, for some element x_k in the sequence which is an element of X.

10.1.8) (a) Prove $C[a,b]$ in example 10.6 is complete. The norm of f is defined to be $\|f\| = \sup\limits_{x\in[a,b]} |f(x)|$ and metric $\rho(f,g) = \|f-g\|$.

Suppose that $\{f_k\}$ is a Cauchy sequence of functions s.t. $f_k \in C[a,b]$. Then $\exists\, N \in \mathbb{N}$ s.t. $m, n \geq N$ implies $\rho(()\, f_m, f_n) < \varepsilon$. So

$$\sup_{x\in[a,b]} |f_m(x) - f_n(x)| < \varepsilon \;\forall\, m, n \geq N.$$

Being a Cauchy sequence implies f_k is uniformly convergent by Lemma 7.11 which implies that $\{f_k\} \to f$. By Thm 7.9, f is continuous on $[a,b]$. Thus $\{f_k\}$ converges in $C[a,b]$ and $C[a,b]$ is therefore complete.

(b) Define $f_1 := \int_a^b |f(x)|\, dx$ and define $dist(f,g) := \|f-g\|_1$. We show that $dist$ is a metric. $dist(f,g) \geq 0$ is clear and $f = g \iff dist(f,g) = 0$. $dist$ is symmetric as the order of arguments doesn't affect the integral. For the triangle inequality, let $h \in C[a,b]$ also. We have that

$$|f-g| \leq |f-h| + |h-g|\,.$$

Since both quantities are positive, integrating does not affect the inequality, therefore

$$\begin{aligned}\int_a^b |f-g| &\leq \int_a^b |f-h| + |h-g| \\ &= \int_a^b |f-h| + \int_a^b |h-g|\end{aligned}$$

which says that

$$\rho(f,g) \leq \rho(f,h) + \rho(h,g)\,.$$

10.1.9) (a) Consider $x \in B_r(a)$. The distance between x and the "edge" of the ball is less than $r - \rho(x,a)$. So define

$$\varepsilon < r - \rho(x,a)\,.$$

Then $\forall\, y \in B_\varepsilon(x)$, we have the following inequality

$$\begin{aligned}\rho(y,a) &\leq \rho(y,x) + \rho(x,a) \\ &< r - \rho(x,a) + \rho(x,a) \\ &= r\end{aligned}$$

Thus $y \in B_r(a)$.

(b) Let $a \neq b$ be distinct in X. Prove $\exists r > 0 : B_r(a) \cap B_r(b) = \emptyset$. Let

$$r := \frac{1}{2}\rho\,(a, b)$$

and suppose for the sake of contradiction that $y \in B_r(a) \cap B_r(b)$. Then we have that

$$\begin{aligned} 2r &> \rho\,(a, y) + (y, b) \\ &\leq \rho\,(a, b) \\ &= 2r \end{aligned}$$

which implies that $r > r$, which is a contradiction.

(c) Consider two balls $B_r(a)$ and $B_s(b)$ and a point $x \in B_r(a) \cap B_s(b)$.
The distance between x and the closest edge of each balls are $r - \rho\,(a, x)$ and $s - \rho\,(x, b)$, respectively. Let

$$c < min\,\{r - \rho\,(a, x)\,, s - \rho\,(x, b)\}$$

It follows that $B_c(x) \subset B_r(a) \cap B_s(b)$. The distance between x and the furthest edge is $r + \rho\,(a, x)$ and $s + \rho\,(x, b)$. So we define

$$d > max\,\{r + \rho\,(a, x)\,, s + \rho\,(x, b)\}$$

It follows that

$$B_d(x) \supset B_r(a) \cup B_s(b).$$