Q9) Consider the construction of the Fat Cantor set,

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in^n} I_\sigma$$

At level k, we take out 2^{k-1} intervals of length $\left(\frac{1}{4}\right)^k$. The sets we remove are countable and denote them $\mathcal{O}_i$. We take the union of all of them

$$\mathcal{O} = \bigcup \mathcal{O}_i.$$

The measure of $\mathcal{O}$ is

$$\begin{aligned}\sum_{k=1}^{\infty} 2^{k-1} \left(\frac{1}{4}\right)^k &= \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \\ &= \frac{1}{2} \left\{ \frac{1}{1-\frac{1}{2}} - 1 \right\} \\ &= \frac{1}{2}\end{aligned}$$

This implies that the measure of the Fat Cantor Set

$$m(\mathcal{C}) = \frac{1}{2}$$

But $\partial \mathcal{C} = \mathcal{C}$ and $\partial \mathcal{C} = \partial \mathcal{O}$. Therefore,

$$m\left(\mathcal{O}\right) = \frac{1}{2}.$$

Q10) Consider the Cantor set in the previous problem. At level k, we remove 2^{k-1} intervals

$$\{R_{k1}, R_{k2}, \ldots\}$$

Now define continuous piecewise functions F_k, where $F_k(x) = 0$ is x is at the center of the removed interval. $F_k(x) = 1$ if x is in not in any of the removed intervals. And $0 \leq F_k(x) \leq 1$ for all $x \in [0, 1]$.

Define $f_n = \prod_{k=1}^{n} F_1 \ldots F_n$.

(a) I believe it is clear that $\forall\, x \in [0, 1]$, $0 \leq f_n(x) \leq 1$. Since $0 \leq F_{n+1} \leq 1$,

$$f_{n+1} = F_{n+1} \cdot f_n(x) \leq 1.$$

$$f_n(x) \geq f_{n+1}(x)$$

Since $f_n(x)$ is bounded and monotonic, $f_n(x)$ converges pointwise to $f(x)$.

(b) Let $x \in \mathcal{C}$ so that $f(x) = 1$. We can construct a sequence $\{x_n\} \longrightarrow x$ s.t. $f(x_n) = 0$. Let $\varepsilon > 0$. Then there is a interval small enough contained in an ε neighborhood of x. i.e. $\exists\ N, \sigma \in \Sigma^N$ such that

$$x \in I_\sigma \subset B_\varepsilon(x)$$

Take x_i to be the center of a removed interval in the path taken. We then have

$$f(x_n) = 0 \ \ \forall \ \ n$$

Thus $f(x)$ is not continuous on $\mathcal{C}$.

(c) The function is discontinuous on $\mathcal{C}$ which has a non-zero measure of $\frac{1}{2}$. This implies that f is not Reimann integrable.

Q19) (a) Show that if either A or B is open, then $A + B$ is open.

Suppose WLOG that A is open. Then since $A + \{b\}$ is open, we have that

$$A + B = \bigcup_{b \in B} A + \{b\}$$

is open.

(b) If A and B are closed, then A^c and B^c are open, so that

$$A^c + B^c = (A + B)^c$$

is in G_σ. Consequently $A + B \in F_\sigma$ and is therefore measureable.

(c) Let $A = \{-n : n \in \mathbb{N}\}$ and $B = \left\{n + \frac{1}{n} : n \in \mathbb{N}\right\}$. Then A, B both closed but

$$A + B = \{1/n : n \in \mathbb{N}\}$$

is not closed.