3.23.1) Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on X. What does the connectedness on one imply about the other?

Claim: If $\mathcal{T}' \supset \mathcal{T}$, then

$$(X, \mathcal{T}') \text{ connected } \rightarrow (X, \mathcal{T}) \text{ connected.}$$

($\Rightarrow$) Proof by contrapositive. Suppose $(X, \mathcal{T})$ is not connected and is separeated by some $A, B \in \mathcal{T}$. We have

$$X = A \cup B.$$

But A, B are elements of $\mathcal{T}'$. Implying that X is not connected under $\mathcal{T}$. The converse doesn't necessarily hold. There may be a separation involving a set not in $\mathcal{T}$.

$$(X, \mathcal{T}') = A \cup B, \quad \text{for some } A \notin \mathcal{T}.$$

3.23.2) Let $\{A_n\}$ be a sequence of connected spaces of X s.t. $A_n \cap A_{n+1} \neq \emptyset$. Show that $\bigcup A_n$ connected. Prove by induction.

($\Rightarrow$) By Theorem 23.3, $A_1 \cup A_2$ is connected since the they share some common point in the intersection by hypothesis. Thus it case is true for $n = 1$,

$$\bigcup_{i=1}^{n+1} A_i$$

Suppose that it is true for $n = k - 1$, then

$$\bigcup_{i=1}^{k} A_i.$$

Since $A_k \cap A_{k+1} \neq \emptyset$,

$$\left(\bigcup_{i=1}^{k} A_i \right) \cap A_{k+1} \neq \emptyset,$$

implying

$$\bigcup_{i=1}^{k+1} A_i$$

is connected. Therefore,

$$\bigcup A_i$$

is connected.

3.23.5) A space is *totally disconnected* if its only connected subspaces are one-point sets. Show that if X has the discrete topology, then X is totally disconnected. Does the converse hold?

($\Rightarrow$) Let X have the discrete topology and suppose X is not totally disconnected. Then $\exists$ $Y \subset X$ connected with more than one element.

For some $y \in Y$,

$$Y = \{y\} \cup (Y - \{y\})$$

forms a separation of Y. Each set is open in the discrete topology and are non-empty since Y has more than one element. Thus Y is not connected, contradicting the assumption that Y is connected. Therefore, no such Y exists and X is therefore totally disconnected.

($\Leftarrow$) I believe the converse is true. Prove by contrapositive.

Suppose X does not have the discrete topology. Then for some $a \in X$, the smallest open set contains another element. WLOG, let

$$\{a, b\}$$

be the smallest set containing a. This implies that b is a limit point of a. Therefore,

$$\{a, b\} = \{a\} \cup \{b\}$$

does not constitute a separation of $\{a, b\}$ by Lemma 23.1 because the each set may not contain any limit points of the other. Thus X is not totally disconnected.

3.23.10) $\{X_\alpha\}_{\alpha \in J}$ indexed family of connected spaces. Let X be the product space,

$$X = \prod_{\alpha \in J} X_\alpha.$$

Let $\vec{a} = (a_\alpha)$ be a fixed point of X.

a) $K \subset J$ finite. X_K subspace of X consisting of all points $\vec{x} = (x_\alpha)$ s.t. $x_\alpha = a_\alpha$ for $\alpha \notin K$. Show X_K connected.

   ($\Rightarrow$) Let $K = \{\alpha_{i_1}, ..., \alpha_{i_k}\}$ and $J - K = \{\alpha_{j_1}, ..., \alpha_{j_m, ...}\}$. The slice

$$\begin{aligned} T_{\alpha_{i_1}} &= ... \times X_{\alpha_{i_1}} \times \{a_{\alpha_{j_2}}\} \times ... \times \{a_{\alpha_{jm}}\} \times ... \\ &= X_{\alpha_{i_1}} \times \prod_{i \in K-J} \{a_i\} \end{aligned}$$

   is homeomorphic to $X_{\alpha_{i_1}}$ and is therefore connected.

   Next, the subspace X_K is a finite union of these.

$$\begin{aligned} X_K &= \bigcup_{i \in K} T_i \\ &= ... \times X_{\alpha_{i_1}} \times X_{\alpha_{i_2}} \times ... \times X_{\alpha_{i_k}} \times .... \end{aligned}$$

   By Theorem 23.6, X_K is connected.

b) $Y = \bigcup_{K \subset J} X_K$ share a common point $\vec{a}$ and is therefore connected by Theorem 23.3.

c) We will apply Theorem 23.4 to show that X is connected, i.e. $Y \subset X \subset \bar{Y}$. It suffices to show $X \subset \bar{Y}$.

($\Rightarrow$) Let $\vec{x} = (x_\alpha)_{\alpha \in J} \in X$. Let U be open in the product topology containing $\vec{x}$, such that

$$U = \prod_{\alpha \in J; \alpha \neq \alpha_k} \times U_{\alpha_k}.$$

Everything set except one is the entire X_α. I got confused typing this so am not sure if the notation is correct. U intersects Y at $\vec{a}$ so that $X \subset \bar{Y}$. By theorem, X is connected.

2.24.2) Let $f : S^1 \to \mathbb{R}$ be continuous. Show $\exists\, x \in S^1$ s.t. $f(x) = f(-x)$.

($\Rightarrow$) S^1 is path connected. The punctured plane $\mathbb{R}^2 - \mathbf{0}$ is path connected and is therefore connected. Under the continuous surjective map $g : \mathbb{R}^2 - \mathbf{0} \to S^1$,

$$g(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||},$$

S^1 is path connected and is therefore connected. Then $f(S^1) \subset \mathbb{R}$ is connected.

Let $h : S^1 \to \mathbb{R}$ be given by $h(s) = f(s) - f(-s)$. h is the continuous by composition of continuous maps.

$h(-s) = f(-s) - f(s) = -(f(s) - f(-s)) = -h(s)$.

If $h(s) = 0$ for all s then done. Otherwise, IVT tells us that $\exists\, s_0$ s.t. $0 = h(s_0) = f(s_0) - f(-s_0)$. Then such s_0 is our desired point.

2.24.3) Let $f : X \to X$ be continuous. Let $X = [0, 1]$, show there exists a fixed point. What if $X = [0, 1)$ or $(0, 1)$?

($\Rightarrow$) $[0, 1]$ is an interval in continuum $\mathbb{R}$ and is therefore connected. $f([0, 1]) \subset [0, 1]$ is also connected being the image under a continuous map.

We have that $f(0) \geq 0$ and $f(1) \leq 1$.

Consider another continuous map $g : f(X) \to \mathbb{R}$

$$g(x) = f(x) - x.$$

We have that

$$g(0) = f(0) - 0 \geq 0$$

and

$$g(1) = f(1) - 1 \leq 0.$$

By IVT, we have that $\exists\, k$ s.t. $g(k) = 0$. Thus,

$$0 = f(k) - k$$

so that $f(k) = k$ is a fixed point as desired.

3.26.3 Finite union of compact subspaces of X is compact.

($\Rightarrow$) Let $\{Y_i\}_I$ be finite collection of compact subspaces of X. Suppose that the collection $\{V_\alpha\}$ covers $\bigcup\limits_{i=1}^{n} Y_i$.

$\Rightarrow$ It follows that $\forall\, i$, $\{V_\alpha\}$ covers Y_i.
$\Rightarrow$ $\{V_{\alpha_i}\}$ finite subcover of Y_i since it's compact.
$\Rightarrow$ $\bigcup\limits_I \{V_{\alpha_i}\}$ finite union of finite subcovers covers $\bigcup\limits_I Y_i$
$\Rightarrow$ Thus union is compact.

3.26.9 $\mathbb{R}$ uncountable. Let A be countable subset of $\mathbb{R}^2$. Show $\mathbb{R}^2 - A$ is path connected.

$$A = \{p_1, p_2, ...\} = \{(x_1, y_1), (x_2, y_2), ...\}.$$

For all $q, r \in \mathbb{R}^2 - A$, if a straight line joining does not intersect A then we're done. Otherwise, if a straight line from r to s intersects some $p_k \in A$, then we can go break the path to go around it since there around uncountable number of points in $\mathbb{R}^2$. I'm not sure if this is a good proof. I followed Example 4.

3.26.4) Let Y be a compact subspace of metric space (X, d). Show Y bounded and closed.

($\Rightarrow$) Let $\bigcup\limits_{y \in Y} B(y, \epsilon)$ cover Y. Then Y compact implies $\bigcup\limits_{y \in W} B(y, \epsilon)$ is a finite subcover of Y, where $W \subset Y$ is finite. $|W| = m < \infty$.

For all $r, s \in Y$,

$$d(r, s) \leq d(r, y_1) + d(r, y_2) + ... + d(r, y_m) < m\epsilon.$$

We can get rid of superflous balls so that $m\epsilon$ bounded above. Therefore, Y is bounded.

Next is an example of a metric space in which not every closed bounded space is compact.

Let $(X, \mathcal{T})$ be infinite with discrete topology and

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

Let $Y \subset X$ be an infinite subset. Y is closed as $X - Y$ is open, being the union of singletons it comprises of. Y is bounded by 1. But it is not compact. For the covering,

$$\bigcup_{y \in Y} \{y\}$$

has no finite subcover that covers Y.

3.26.5) Let A, B be disjoint compact subspaces of X Hausdorff. Show $\exists\, U, V$ disjoint and open containing A, B, respectively.

($\Rightarrow$) Let $b \in B$. By Lemma 26.4, $\exists\, U_b \supset A$ and $V_b \ni b$ disjoint. Then

$$\bigcap_{b \in B} U_b \supset A$$

is disjoint from B. For if $b_0 \in B$, $b_0 \notin U_{b_0}$ for some U_{b_0}. Then $b_0 \notin \bigcap_{b \in B} U_b$.

Using the same argument,

$$\bigcap_{a \in A} V_a \supset B$$

disjoint from A. Thus

$$U = \bigcap_{b \in B} U_b$$

and

$$V = \bigcap_{a \in A} V_a$$

are the sets we are looking for.

3.26.12) Let $p : X \to Y$ be a closed continuous surjective map s.t. $p^{-1}(\{y\})$ is compact for each $y \in Y$. Show

$$Y \text{ compact} \to X \text{ compact}.$$

We'll show this by showing p^{-1} is continuous and using Theorem 26.5 that says the image of compact under continous function is compact.

($\Rightarrow$) Let $U \ni p^{-1}(\{y\})$ in X. We'll show $\exists\, W \subset Y$ st $p^{-1}(W) \subset U$.

First show that $y \in p(U)$.

$$\begin{aligned} p(p^{-1}(\{y\})) &\subset p(U) \\ \{y\} &\subset p(U) && \text{by surjectivity of p.} \\ y &\in p(U). \end{aligned}$$

On the other hand $X - U$ closed $\Rightarrow p(X - U) = Y - p(U)$ closed. Therefore $p(U)$ open in Y.

Next, we have that $\exists W$ st $y \in W \subset p(U)$. So that

$$p^{-1}(W) \subset p^{-1}(p(U)) = U.$$

Thus p^{-1} is continuous. So if Y compact, then $X = p^{-1}(Y)$ is compact by Theorem 26.5.