D1: a) Let $f(x) = |2x+1|$. By the Triangle Inequality, we have that

$$\begin{aligned}\big||2x+1| - |2y+1|\big| &\le |2x+1-(2y+1)| \\ &= |2x-2y| \\ &= 2\,|x-y|\,.\end{aligned}$$

Since we have that $|f(x)-f(y)| \le 2|x-y|$, $f(x)$ is Lipschitz.

b) Let $f(x)$ be a Lipschitz function and $\epsilon > 0$. Since $f(x)$ is Lipschitz, $\exists\, C > 0$ s.t.

$$|f(x)-f(y)| \le C|x-y|.$$

Let $\delta := \frac{\epsilon}{C}$ and suppose $|x-y| < \delta$. Then

$$|f(x)-f(y)| \le C|x-y| < C\left(\frac{\epsilon}{C}\right) = \epsilon.$$

Since δ does not depend on the points x, y in the domain, $f(x)$ is uniformly continuous.

D2: Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$. Define $g(x) = f(x) - x$. By the Algebraic Continuity Theorem, $g(x)$ is also continuous. We will show that there exist some fixed point c of f. If $g(0) = 0$, then 0 is a fixed point of f. If $g(1) = 0$, then 1 is a fixed point of f. Otherwise, we're left with the case that $g(0) > 0$ and $g(1) < 0$. *To see why, we can imagine a* $[0,1] \times [0,1]$ *box with a line going through* $(0,0)$ *and* $(1,1)$. *Then graph of* f *has its left endpoint somewhere above the origin and its right endpoint somewhere below* $(1,1)$. By the Intermediate Value Theorem, $\exists\, c \in (a,b)$ s.t. $g(c) = 0$, or

$$0 = f(c) - c,$$

implying that $f(c) = c$.

D3: Let $f : \mathbb{R} \to \mathbb{R}$ be a contraction. i.e. $\exists\, c \in (0,1)$ s.t.

$$|f(x)-f(y)| \le c|x-y| \;\forall\, x, y \in \mathbb{R}.$$

a) Define $a_n = f(a_{n-1})$ with $a_1 = 0$. We will prove the sequence (a_n) is Cauchy. First, we show by induction that

$$|a_k - a_2| \le \sum_{i=1}^{k-2} c^i |a_2|,\ (k > 2)$$

and consequently is bounded by some constant

$$|a_k - a_2| < \frac{|a_2|}{1-c}.$$

($\Rightarrow$) Note that it is true for $k = 3$,

$$|a_3 - a_2| \le c|a_2|$$

since

$$|a_3 - a_2| = |f(a_2) - f(a_1)| \leq c|a_2 - a_1| = c|a_2|.$$

Now, suppose that it is true for some $k = j$, then

$$\begin{aligned} |a_{j+1} - a_2| &\leq |a_{j+1} - a_j| + |a_j - a_2| \\ &\leq c^{j-1}|a_2| + \sum_{i=1}^{j-2} c^i |a_2| \\ &= \sum_{i=1}^{j-1} c^i |a_2|. \end{aligned}$$

We get from $|a_{j+1} - a_j|$ *to* $c^{j-1}|a_2|$ *by repeatedly applying* $|a_{j+1} - a_j| \leq c|a_j - a_{j-1}|$. Therefore, it is true for $k = j + 1$ and the inequality is proven. Going back to the original problem, let $\epsilon > 0$.

$$\begin{aligned} |a_m - a_n| &\leq c^{n-2}|a_{m-n+2} - a_2| \\ &< c^{n-2}\frac{|a_2|}{1-c} \\ &< \epsilon. \end{aligned}$$

This is true if

$$c^{n-2} < \epsilon\frac{1-c}{|a_2|}.$$

Since $0 < c < 1$, The Archimedean Property implies that $\exists\ N$ s.t. $n \geq N$ makes this inequality true. Thus the (a_n) is Cauchy and $(a_n) \to a$ for some $a \in \mathbb{R}$.

b) Prove $f(a) = a$.
Problem 1 tells us that f is continuous. Therefore,

$$f(a) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(a_{n-1}) = \lim_{n\to\infty} a_n = a.$$

c) Prove the fixed point is unique.
($\Rightarrow$) Suppose $\exists\, b \in \mathbb{R}$ s.t. $f(b) = b$. Then

$$|b - a| = |f(b) - f(a)| \leq c|b - a|.$$

Since $0 < c$, $|b - a| = 0$ implying $b = a$. Therefore, the fixed point is unique.

D4: a) Prove

$$\lim_{x\to\infty} \frac{x}{x+1} = 1.$$

($\Rightarrow$) Let $\epsilon > 0$ and $N = \frac{1+\epsilon}{\epsilon}$. If $x > N$,

$$x > \frac{1+\epsilon}{\epsilon} = \frac{1}{\epsilon} + 1,$$

so that
$$\left|\frac{1}{x-1}\right| = \frac{1}{x-1} < \epsilon,$$
since $x - 1 > 0$. We have that
$$\begin{aligned}\left|\frac{x}{x+1} - 1\right| &= \left|\frac{1}{x-1}\right| \\ &= \frac{1}{x-1} \\ &< \epsilon.\end{aligned}$$

Thus,
$$\lim_{x\to\infty} \frac{x}{x+1} = 1.$$

b) Let $a_n = f(n)$ and $\epsilon > 0$. Suppose $f(x) \to L$ as $x \to \infty$.
($\Rightarrow$) There exists some N s.t. $x > N \to |f(x) - L| < \epsilon$. Let $M = \lceil N \rceil$. Then $\forall\, n \geq M$,
$$|a_n - L| = |f(n) - L| < \epsilon.$$
Thus $(a_n) \to L$.

c) Let $a_n = sin(\frac{\pi}{2} + 2\pi n)$ and $b_n = sin(\frac{3\pi}{2} + 2\pi n)$. The limits are $1, -1$, respectively. Therefore, limit don't exist.