1: Find the limit

$$\lim_{n\to\infty}\int_0^\infty n\sin\left(\frac{x}{n}\right)\frac{1}{x(1+x^2)}\,dx.$$

We will use the Lebesgue Dominated Convergence Theorem to find this limit. To apply LDCT, we

(a) Determine the function f that the sequence $\{f_n\}$ converges to. $f := \lim_{n\to\infty} f_n$.

$$\begin{aligned}\lim_{n\to\infty} f_n(x) &= \frac{1}{1+x^2}\lim_{n\to\infty}\frac{\sin\left(\frac{x}{n}\right)}{\frac{x}{n}}\\ &= \frac{1}{1+x^2}\lim_{n\to\infty}\frac{\frac{1}{n}\cos\left(\frac{x}{n}\right)}{\frac{1}{n}}\\ &= \frac{1}{1+x^2}\lim_{n\to\infty}\cos\left(\frac{x}{n}\right)\\ &= \frac{1}{1+x^2}\end{aligned}$$

So let

$$\boxed{f(x) := \frac{1}{1+x^2}}$$

(b) Show there exists some integrable function g that bounds $|f_n|$ for all n, $|f_n| \le g\ \ \forall\ n$.

$$\begin{aligned}|f_n| &= \left|\left(\frac{n}{x}\right)\sin\left(\frac{x}{n}\right)\cdot\frac{1}{1+x^2}\right|\\ &= \frac{1}{1+x^2}\left|\frac{\sin\left(\frac{x}{n}\right)}{\left(\frac{x}{n}\right)}\right|\\ &= \frac{1}{1+x^2}\left|\frac{\sin\left(\frac{x}{n}\right)-\sin(0)}{\frac{x}{n}-0}\right|\\ &= \frac{1}{1+x^2}\left|\cos(c)\right| \qquad c\in\left(0,\frac{x}{n}\right)\\ &\le \frac{1}{1+x^2}\cdot 1\\ &= \frac{1}{1+x^2}\end{aligned}$$

So let

$$\boxed{g(x) := \frac{1}{1+x^2}}$$

Finally, by DCT we have that

$$\lim \int f_n = \int \lim f_n = \int g$$

so that

$$\lim_{n\to\infty} \int_0^\infty n \sin\left(\frac{x}{n}\right) \frac{1}{x(1+x^2)}\,dx = \int_0^\infty \frac{1}{1+x^2}.$$

We have that

$$\begin{aligned}\int_0^\infty \frac{1}{1+x^2} &= \lim_{k\to\infty} \int_0^k \frac{1}{1+x^2} \\ &= \lim_{k\to\infty} \arctan k - 0 \\ &= \frac{\pi}{2}.\end{aligned}$$

Thus,

$$\boxed{\lim_{n\to\infty} \int_0^\infty n \sin\left(\frac{x}{n}\right) \frac{1}{x(1+x^2)}\,dx = \frac{\pi}{2}}$$

2: Let

$$E_n^k = \left(r_n - \frac{1}{2^{n+k}}, r_n - \frac{1}{2^{n+k}}\right).$$

and

$$G = \bigcap_{k=1}^\infty \bigcup_{n=1}^\infty E_n^k.$$

We observe that $\bigcup\limits_{n=1}^\infty E_n^k$ is open as it is the union of open sets and $G = \bigcap\limits_{k=1}^\infty \left(\bigcup\limits_{n=1}^\infty E_n^k\right)$ is G_δ as it is a countable union of open sets.

Since $\mathbb{Q} \subset G$, it suffices to show density by showing that any irrational is arbitrarily close to G. So let $y \in \mathbb{R}\backslash\mathbb{Q}$ and $\varepsilon > 0$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists\, n \in \mathbb{N}$ s.t. $r_n \in B_\varepsilon(x)$. Moreover, $\exists\, k$ s.t.

$$\left(r_n - \frac{1}{2^{n+k}}, r_n - \frac{1}{2^{n+k}}\right) \subset B_\varepsilon(x).$$

To show the Lebesgue measure is 0, let's hold k and observe that the measure of the union of E_n^k is bounded

$$\begin{aligned} m\left(\bigcup_{n=1}^\infty E_n^k\right) &\leq \sum_{n=1}^\infty \frac{1}{2^{n+k-1}} \\ &= \frac{1}{2^{k-1} \sum\limits_{n=1}^\infty \frac{1}{2^n}} \\ &= \frac{2}{2^{k-1}} < \infty. \end{aligned}$$

Since $G \subset \bigcup_{n=1}^{\infty} E_n^k \ \forall \ k$,

$$\begin{aligned} m(G) &\leq \limsup_{k \to]infty} m\left(\bigcup_{n=1}^{\infty} E_n^k\right) \\ &< \frac{1}{2^k} \end{aligned}$$

Thus,

$$\boxed{m(G) = 0}$$

3: Consider the set

$$E_m^n = \left\{ x : \ \forall \ |y - x| < \frac{1}{m} \Rightarrow |f(x) - f(y)| < \frac{1}{n} \right\}.$$

Fix n and m for a moment and observe that every point in this set is continuous on an $\frac{1}{m}$ neighborhood of itself. These points may be continuous within a larger neighborhood but it doesn't matter.

Let's take some $x_0 \in E_m^n$. Then for each point y in this neighborhood, we can find a neighborhood containing y and contained in the x-neighborhood. i.e. $\exists \ m'$ s.t. $B_{\frac{1}{m'}}(y) \subset B_{\frac{1}{m}}(x_0)$. Therefore, $y \in E_{m'}^n$. It follows that

$$\bigcup_{m=1}^{\infty} E_m^n$$

is the union of all $\frac{1}{m}$ neighborhoods where f is within $\frac{1}{n}$. Since m is arbitrary, $\bigcup_{m=1}^{\infty} E_m^n$ is the union of all continuous points all neighborhoods centered at them. Therefore it is open.

Finally,

$$\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_m^n$$

is a countable intersection of open sets and returns all continous points of $\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} E_m^n$.

(a) This shows that the set of continuity points is a G_δ set.

(b) $\mathbb{Q}$ cannot be the exact set of continuity points for any function because it is F_σ. $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ which is a countable union of closed sets.

4: We will define a sequence of functions $f_n(x)$ s.t. $f_n : [0,1] \to [0,\infty)$, $f_n \to 0 \ \forall \ x \in [0,1]$, $\int_0^1 f_n \, dx \to 0$, and $\sup\limits_n f_n$ not integrable.

We show the first three functions $f_1, f_2,$ *and* f_3 then define f_n following the pattern.

Let $f_1(x)$ be a triangle of height $1+1=2$ with bases at $\dfrac{1}{2}$ and $\dfrac{1}{1}$. Then

$$\int f_1(x)dx = \frac{1}{2}.$$

Let $f_2(x)$ be a triangle of height $2+1=3$ with bases at $\dfrac{1}{3}$ and $\dfrac{1}{2}$. Then

$$\int f_2(x)dx = \frac{1}{4}.$$

Let $f_3(x)$ be a triangle of height $3+1=4$ with bases at $\dfrac{1}{4}$ and $\dfrac{1}{3}$. Then

$$\int f_2(x)dx = \frac{1}{6}.$$

In general, $f_n(x)$ be a triangle of height $n+1$ with bases at $\dfrac{1}{n}$ and $\dfrac{1}{n+1}$,

$$f_n(x) = \begin{cases} triangle & x \in \left(\frac{1}{n+1}, \frac{1}{n}\right) \\ 0 & x \notin \left(\frac{1}{n+1}, \frac{1}{n}\right) \end{cases}$$

(a) We have that $\int f_n = \dfrac{1}{2n}$ so that $\lim\limits_{n\to\infty} \int f_n \to 0$.

(b) $\sup\limits_{n\in\mathbb{N}; x\in[0,1]} f_n(x) = \infty$ is not integrable is clear I think.

(c) The Dominated Convergence Theorem doesn't hold because no such integrable function g s.t. $|f_n| \le g$ *a.e.* exists.

If such a g existed, then $\int g < +\infty$ implies that $g < M$ *a.e.* (for some M). Then $|f_n| < M$ *a.e.* But f_n are all continuous implies that

$$m\left(\{x : |f_n| > M\}\right) \neq 0 \ \ \forall\, n.$$

5: Let E be a measurable set such that $m(E) < \infty$. We say that f_n converges in measure to f on E if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} m(x \in E : |f_n(x) - f(x)| \ge \varepsilon) = 0.$$

(a) Show that if f_n converges to f a.e., then f_n converges in measure to f.

($\Rightarrow$) Suppose $f_n \longrightarrow f$ *a.e.* on E, with $m(E) < +\infty$. Consider the set B of points where $f_n \longrightarrow f$,

$$B := \{x : f_n \longrightarrow f\}$$

and note that $m\left(E \backslash B\right) = 0$ by hypothesis. Now let $\varepsilon > 0$ and by Egoroff, $\exists\, A_\varepsilon \subset B$ s.t.

$$m\left(B \backslash A_\varepsilon\right) < \varepsilon \text{ and } f_n \longrightarrow f, \text{ uniformly.}$$

Restating the above argument, we have for $\varepsilon > 0$, $\exists\, A_\varepsilon \subset B$ and $N_\varepsilon \in \mathbb{N}$ s.t.

$$\boxed{m(B \backslash A_\varepsilon) \text{ and } |f_n - f| < \varepsilon}.$$

Then the set of x where $|f_n - f| \geq \varepsilon$ is contained outside of A_ε,

$$\{x : |f_n - f| \geq \varepsilon\} \subset \left(E \backslash B\right) \cup \left(B \backslash A_\varepsilon\right).$$

Therefore,

$$\begin{aligned} m\left(\{x : |f_n - f| \geq \varepsilon\}\right) &\leq m\left(E \backslash B\right) + m\left(B \backslash A_\varepsilon\right) \\ &= 0 + \varepsilon \\ &= \varepsilon. \end{aligned}$$

Therefore

$$\boxed{\lim_{n\to\infty} m\left(x : |f_n - f| \geq \varepsilon\right) = 0.}$$

(b) We will show by counter-example that the converse is not true using the **Typewriter Sequence**

$$f_n(x) = \mathbb{1}_{\left[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}\right]}, \quad 2^k \leq n < 2^{k+1}.$$

We make some observations about this function:

(1) The parameter k is determined by the index n. Which in turn determines how [0,1] is partitioned.

(2) This function outputs a rectangle of height 1 and width $\dfrac{1}{2^k}$.

(3) The endpoints of the rectangle remain in $[0, 1]$ since $2^k \leq n < 2^{k+1}$ imply

$$0 \leq \frac{n-2^k}{2^k} < 1 \textit{ and } \frac{n-2^k+1}{2^k} < 1 + \frac{1}{2^k}.$$

The left inequality tells us that the *left endpoint* is strictly less than 1. And since the width of the rectangle is $\frac{1}{2^k}$, the right inequality tells us the *right endpoint* ≤ 1.

(4) As the sequence evolves, the rectangles endpoints move and/or width decrease in size. The "and/or" comes from the fact that the width only decreases when the index n pushes k higher. The parameter k remains constant for a finite number of n's.

This sequence converges in measure as

$$m\{x : |f_n(x) - f(x)| \geq \varepsilon\} \leq \frac{1}{2^k}$$

implies

$$\lim_{n\to\infty} m\{x : |f_n(x) - f(x)| \geq \varepsilon\} = 0$$

....but $f_n \not\longrightarrow f$ since the $f_n \longrightarrow f$ nowhere.

(c) Suppose that $f_n \to f$ in measure on E and $\exists\, M > 0$ s.t. $|f_n| \leq M$ *a.e.*.

(1) We first make a claim that f is bounded by M,

$$|f| \leq M \text{ } a.e.$$

Intuitively, since each f_n is bounded and the the set where f_n and f deviate has measure 0, f must be bounded by the same number.

For the sake of contradiction, suppose $|f| > M$ and

$$m(\{x : |f| > M\}) > \varepsilon \text{ some } \varepsilon.$$

Then by the density of $\mathbb{R}$, $\exists\, \varepsilon' > 0$ s.t.

$$|f| - \varepsilon' > M \iff |f| > M + \varepsilon'.$$

This implies that $|f - f_n| + |f_n| > M + \varepsilon'$ so that

$$|f - f_n| > \varepsilon' \;\; \forall\, n \in \mathbb{N}$$

and

$$m\left(\bigcap_{n=1}^{\infty} \{x : |f_n - f| > \varepsilon'\}\right) > \varepsilon.$$

WLOG, let $\varepsilon := \min\{\varepsilon, \varepsilon'\}$ so

$$m\left(\bigcap_{n=1}^{\infty} \{x : |f_n - f| > \varepsilon\}\right) > \varepsilon.$$

But $\forall\; n \in N$,

$$\bigcap_{n=1}^{\infty} \{x : |f_n - f| > \varepsilon\} \subset \{x : |f_n - f| > \varepsilon\}.$$

which implies

$$\begin{aligned} m\left(\bigcap_{n=1}^{\infty} \{x : |f_n - f| > \varepsilon\}\right) &\leq \limsup m\{x : |f_n - f| > \varepsilon\} \\ &= \lim m(\{x : |f_n - f| > \varepsilon\}) \\ &= 0. \end{aligned}$$

This is a contradiction so that

$$\boxed{|f| \leq M \ a.e.}$$

(2) Now we show $\int_E |f_n - f| \to 0$. Let $\varepsilon > 0$ and let

$$E_\varepsilon^{n'} = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$$
$$E_\varepsilon^{n} = \{x : |f_n(x) - f(x)| < \varepsilon\}.$$

We split the integral over the two sets and use the triangle inequality to get our desired result.

$$\begin{aligned}\int_E |f_n - f| &= \int_{E_\varepsilon} |f_n - f| + \int_{E_\varepsilon{}'} |f_n - f| \\ &< \varepsilon m(E_\varepsilon) + m(E_\varepsilon') \left(|f_n| + |f|\right) \\ &< \varepsilon m(E) + 2M\varepsilon. \quad \square\end{aligned}$$

6: Let $f \in L^p(\mathbb{R}^d)$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

($\Rightarrow$)

Suppose $\|g\|_q = 1$.

$$\left|\int fg\right| \leq \int |fg| \leq \|f\|_p \|g\|_q = \|f\|_p$$

implies

$$\boxed{\sup\left\{\left|\int fg\right| : \|g\|_q = 1\right\} \leq \|f\|_p}$$

($\Leftarrow$)

Let

$$g := \frac{|f|^{p-1} \cdot sgn(f)}{\|f\|_p^{p-1}}$$

and note that $\frac{1}{p} + \frac{1}{q} = 1$ imply $\boxed{q = \dfrac{p}{p-1}}$

(a) Check that $\|g\|_q = 1$.

$$\begin{aligned}\|g\|_q &= \left(\int \left|\frac{|f|^{p-1}\, sgn(f)}{\|f\|_p^{p-1}}\right|\right)^{1/q} \\ &= \left(\int \frac{|f|^{p-1}}{\|f\|_p^{p-1}}\right)^{1/q} \\ &= \left[\left(\frac{1}{\|f\|_p^p}\right) \cdot \left(\int |f|^p\right)\right]^{1/q}\end{aligned}$$

But

$$\|f\|_p^p = \int |f|^p$$

So that the quantity inside the bracket equals 1 and therefore

$$\boxed{\|g\|_q = 1^{1/q} = 1}.$$

(b) On the other hand, $\left|\int fg\right| = \|f\|_p$ as shown below

$$\begin{aligned}\left|\int fg\right| &= \left|\int f \cdot \frac{|f|^{p-1}\, sgn(f)}{\|f\|_p^{p-1}}\right| \\ &= \left|\int \frac{|f|^p}{\|f\|_p^{p-1}}\right| \\ &= \|f\|_p^{1-p} \cdot \int |f|^p \\ &= \|f\|_p^{1-p} \cdot \|f\|_p^p \\ &= \|f\|_p.\end{aligned}$$

This shows us that $\|f\|_p \in \left\{\left|\int fg\right| : \|g\|_q = 1\right\}$ so that

$$\|f\|_p \le \sup\left\{\left|\int fg\right| : \|g\|_q = 1\right\}$$

Finally, both inequalities result in

$$\boxed{\|f\|_p = \sup\left\{\left|\int fg\right| : \|g\|_q = 1\right\}}$$