1) ($\Rightarrow$) Suppose G is Abelian.

$$\begin{aligned}
(ab)^2 &= (ab)(ab) & \\
&= a(b(ab)) & \text{Associativity} \\
&= a((ba)b) & \text{Associativity} \\
&= a((ab)b) & \text{Abelian} \\
&= a(a(bb)) & \text{Associativity} \\
&= (aa)(bb) & \text{Associativity} \\
&= a^2b^2 &
\end{aligned}$$

($\Leftarrow$) Suppose $(ab)^2 = a^2b^2$. Then

$$\begin{aligned}
(ab)(ab) &= a^2b^2 \\
(ab)(ab) &= aabb \\
a(b(ab)) &= aabb \\
abab &= aabb \\
a^{-1}(abab)b^{-1} &= a^{-1}aabbb^{-1} \\
ba &= ab
\end{aligned}$$

2) Let $a, b \in G$, where G is Abelian.

a) We can see that for $n = 1$, $ab = ab$ is true. Problem number 1 shows us it's also true for $n = 2$. Suppose that for some $n = k$, $k \in \mathbb{N}$,

$$(ab)^k = a^kb^k.$$

Then

$$\begin{aligned}
(ab)^{k+1} &= (ab)^k(ab) & \\
&= a^kb^k(ab) & \\
&= a^kb^k(ba) & \text{Abelian} \\
&= a^kb^{k+1}a & \text{Associativity} \\
&= a^kab^{k+1} & \text{Abelian} \\
&= a^{k+1}b^{k+1} & \text{Associativity}
\end{aligned}$$

Therefore, $(ab)^n = a^nb^n$ for all $n \in \mathbb{N}$.

b) Let n be a negative integer, then $-n$ is a positive integer.

$$\begin{aligned}
(ab)^n &= ((ab)^{-1})^{-n} & \\
&= (b^{-1}a^{-1})^{-n} & \text{Socks-Shoe} \\
&= (b^{-1})^{-n}(b^{-1})^{-n} & \text{Part 2a} \\
&= b^na^n & \\
&= a^nb^n & \text{Abelian}
\end{aligned}$$

c) $(RF)^2 = RFRF = I$ but

$$R^2F^2 = R^2$$

3) Let $|a| = n$ and suppose for the sake of contradiction that $a^k = e$ but n does not divide k. By the Division Algorithm, $\exists$ unique integeres q and r s.t.

$$k = nq + r,\ 0 \leq r < n.$$

Since we assumed that n does not divide k, r cannot be 0, so that

$$0 < r < n.$$

We then have

$$e = a^k = a^{nq+r} = (a^n)^q a^r = e^q a^r = a^r \neq e.$$

Therefore, n divides k.

4) To show the order of $a \in G$ and the order of its inverse a^{-1} are the same, we'll consider the finite and infinite case. Starting with the infinite case, let $|a| = \infty$ and suppose $|a^{-1}| = m < \infty$. In other words, assume the order of a^{-1} is finite. Then

$$(a^{-1})^m = (a^{-1}a^{-1}\ldots a^{-1}) = e.$$

Applying a m-times to both sides results in

$$e = a^m.$$

This implies that a is of some finite order which contradicts that a has infinite order. Therefore, for any infinite order element, it's inverse is also of infinite order.
Now let $|a| = n < \infty$. Then

$$a^n = e.$$

Applying a^{-1} n-times gives us

$$e = (a^{-1})^n.$$

This implies that a^{-1} is of finite order, say m, that divides n. So $n = km$, where $k \geq 1$ since m and n are positive. If $k > 1$, then $m = \dfrac{n}{k} < n$. But

$$(a^{-1})^m = e \to e = a^m.$$

This contradicts the fact that $|a| = n \neq m$. Therefore, $k = 1$ so that $m = n$. Thus, the order of the inverse of an element is the same as the order of the element.

5) Let $S = \mathbb{R} - \{-1\}$ and $a \star b = a + b + ab$.

a) Let $a, b \in S$ and suppose $a \star b \notin S$. We have

$$\begin{aligned} a + b(1+a) &= -1 \\ 1 + a + b(1+a) &= 0 \\ 1(1+a) + b(1+a) &= 0 \\ (1+b)(1+a) &= 0 \end{aligned}$$

implies that either $a \notin S$ or $b \notin S$. This is a contradiction so that $a \star b \in S$.

b) The identity is 0.

$$a \star 0 = a + 0 + a * 0 = a.$$

c) The inverse of $a \in S$ is

$$a^{-1} = -\frac{a}{1+a}.$$

d) $(a \star b) \star c = a \star (b \star c)$

$$\begin{aligned}
a \star (b \star c) &= a \star (b + c + bc) \\
&= a + (b + c + bc) + a(b + c + bc) \\
&= a + b + c + bc + ab + ac + abc \\
(a \star b) \star c &= (a + b + ab) \star c \\
&= (a + b + ab) + c + (a + b + ab)c \\
&= a + b + ab + c + ac + bc + abc
\end{aligned}$$

6) $\mathbb{R}^* \times \mathbb{Z}$ with operation $(a, n) \star (b, m) = (ab, m + n)$.

a) Associativity

$$\begin{aligned}
((a, m) \star (b, n)) \star (c, o) &= (ab, m + n) \star (c, o) \\
&= (ab, m + n + o) \\
&= (a, m) \star (bc, n + o) \\
&= (a, m) \star ((b, n) \star (c, o))
\end{aligned}$$

b) Identity e = (1,0) For all $(a, m) \in S$,

$$(a, m) \star (1, 0) = (a * 1, m + 0) = (a, m).$$

c) The inverse of $(a, m) \in S$ is $(\frac{1}{a}, -m) \in S$.