1: (a) A function $f(\alpha)$ is concave if any line joining any two points on the graph is below the graph. That is to say, $\forall\, a, b \in \mathbb{R}$,

$$f(a(1-t)+bt) \geq f(a)(1-t) + f(b)t \quad 0<t<1.$$

We know from Calculus that $ln(\alpha)$ is a concave function and therefore satisfies this relationship for all $a, b \in \mathbb{R}^+$.

$$\boxed{ln(a(1-t)+bt) \geq ln(a)(1-t) + ln(b)t \quad 0<t<1}.$$

(b) As t is parameterized from 0 to 1, the relationship between t and $1-t$ is

$$t + (1-t) = 1.$$

We can reparameterize t to exploit some properties of $ln(x)$. So let $t = \dfrac{1}{p}$ with $\infty < p < 1$. Then $1 - t = \dfrac{1}{q}$ with $1 < q < \infty$. Rewriting the concavity relationship of $ln(x)$ gives us

$$ln\left(\frac{1}{q}a + \frac{1}{p}b\right) \geq \frac{1}{q}ln(a) + \frac{1}{p}ln(b).$$

Since the result above is true $\forall a, b \in R^+$, we can utilize function composition by letting $a = |f(x)|^q$ and $b = |g(x)|^p$. So we have

$$\begin{aligned} ln\left(\frac{1}{q}\,|f(x)|^q + \frac{1}{p}\,|g(x)|^p\right) &\geq \frac{1}{q}ln\left(|f(x)|^q\right) + \frac{1}{p}ln\left(|g(x)|^p\right) \\ &= \ln\left(|f(x)|\,|g(x)|\right) \end{aligned}$$

implying

$$\boxed{|f(x)|\,|g(x)| \leq \frac{1}{q}\,|f(x)|^q + \frac{1}{p}\,|g(x)|^p}$$

(c) **Holder Inequality**. Suppose $\dfrac{1}{p} + \dfrac{1}{q} = 1$ so that we can use the inequality from part b. Let

$$\left(\int |f|^p\right)^{1/p} = A$$
$$\left(\int |g|^q\right)^{1/q} = B$$

Dividing both sides by A, B respectively gets

$$\left(\int \left|\frac{f}{A}\right|^p\right)^{1/p} = 1$$
$$\left(\int \left|\frac{f}{A}\right|^p\right)^{1/q} = 1$$

Now we use the boxed inequality on $|f|$ and $|g|$,

$$\left|\frac{f}{A}\right|\left|\frac{g}{B}\right| \leq \frac{1}{p}\left|\frac{f}{A}\right|^p + \frac{1}{q}\left|\frac{g}{B}\right|^q$$

Apply the integral,

$$\begin{aligned}
\int \left|\frac{f}{A}\right|\left|\frac{g}{B}\right| &\leq \frac{1}{p}\int\left|\frac{f}{A}\right|^p + \frac{1}{q}\int\left|\frac{g}{B}\right|^q \\
\frac{1}{AB}\int |fg| &\leq \frac{1}{p}+\frac{1}{q} \\
\frac{1}{AB}\int |fg| &\leq 1 \\
\int |fg| &\leq AB
\end{aligned}$$

$$\boxed{\int |fg| \leq \left(\int |f|^p\right)^{1/p}\left(\int |g|^q\right)^{1/q}.}$$

2: For $1 \leq p < \infty$, let

$$L^p\left(\mathbb{R}^d\right) = \left\{f : \int |f|^p < \infty\right\}$$

(a) **Minkowski Inequality.** Show

$$\left(\int |f+g|^p\right)^{1/p} \leq \left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}.$$

For easier viewing, write the above as

$$\boxed{\|f+g\|_p \leq \|f\|_p + \|g\|_p}.$$

We start off with the hint,

$$\int |f+g|^p \leq \int |f|\,|f+g|^{p-1} + \int |g|\,|f+g|^{p-1}$$

Use the inequality from 1c on each term on the right-hand side and pulilng out the common term, we get

$$\int |f+g|^p \leq \left\{\left(\int |f|^p\right)^{1/p} + \left\{\int |g|^p\right\}^{1/p}\right\}\left(\int |f+g|^{\frac{p-1}{q}}\right)^{1/q}$$

Now rewrite with the norm notaion as

$$\|(f+g)^p\|_1 \leq \left(\|f\|_p + \|g\|_p\right)\|(f+g)^{p-1}\|_q$$

The term $\|(f+g)^{p-1}\|_q$ can be rewritten as $(\|f+g\|_p)^{p/q}$ as shown below:

$$\begin{aligned}\|(f+g)^{p-1}\|_q &= \left(\int |f+g|^{(p-1)q}\right)^{1/q} \\ &= \left(\int |f+g|^{p}\right)^{1/q} \\ &= \left(\int |f+g|^{p}\right)^{\frac{1}{p}\cdot\frac{q}{p}} \\ &= \|f+g\|_p^{\frac{q}{p}}\end{aligned}$$

So we have that

$$\|(f+g)^p\|_1 \leq (\|f\|_p + \|g\|_p)\,\|f+g\|_p^{\frac{q}{p}}$$

The term on the left of the inequality, $\|(f+g)^p\|_1$ can be rewritten as $\|f+g\|_p^p$ since

$$\|(f+g)^p\|_1 = \left[\left(\int |f+g|^p\right)^{\frac{1}{p}}\right]^p = \|f+g\|_p^p.$$

Then

$$\begin{aligned}\|f+g\|_p^p &\leq (\|f\|_p + \|g\|_p)\,\|f+g\|_p^{\frac{q}{p}} \\ \|f+g\|_p^{p-\frac{q}{p}} &\leq \|f\|_p + \|g\|_p\end{aligned}$$

Since $p - \frac{q}{p} = 1$,

$$\boxed{\|f+g\|_p \leq \|f\|_p + \|g\|_p}$$

(b) Show $\|\cdot\|_p$ is a norm.

(1) **Positive-Definite** $\|f\|_p \geq 0$ is clear so we will show that $\|f\|_p = 0 \iff f = 0$ a.e.

$$\begin{aligned}\|f\|_p = 0 &\Rightarrow \int |f|^p = 0 \\ &\Rightarrow |f|^p = 0 \ a.e. \text{ result after BCT} \\ &\Rightarrow f = 0 \ a.e.\end{aligned}$$

In the other direction, we have

$$\begin{aligned}f = 0 \ a.e. &\Rightarrow |f|^p = 0 \ a.e. \\ &\Rightarrow \int |f|^p = 0 \text{ Lemma 1.2}\end{aligned}$$

Therefore,

$$\boxed{\|f\|_p = 0 \iff f = 0 \ a.e.}$$

(2) **Linearity**

$$\begin{aligned}\|\lambda f\|_p &= \left(\int |\lambda f|^p\right)^{1/p} \\ &= \left(|\lambda|^p \int |f|^p\right)^{1/p} \\ &= \lambda \left(\int |f|^p\right)^{1/p}\end{aligned}$$

(3) **Triangle Inequality** Minkowski Inequality in previous problem

(c) Show that $L^p(\mathbb{R}^d)$ is a metric space with respect to the metric

$$d(f,g) = \|f-g\|_p.$$

Let $\{f_n\}$ be a Cauchy sequence in $\|\cdot\|_p$ and $\{f_{n_k}\}$ be a subsequence s.t.

$$\|f_{n_{k+1}} - f_{n_k}\|_p < 2^{-k}.$$

We define $f(x)$ to be a telescoping series of this subsequence

$$f(x) = f_{n_1} + \sum_{k=1}^{\infty} \left(f_{n_{k+1}} - f_{n_k}\right).$$

Note that the N^{th} partial sum is the $N+1$ term by telescoping

$$f_N(x) = f_{N+1}(x)$$

and that $f_n \nearrow f$. By the triangle inequality, we have

$$\begin{aligned}\|f\|_p &\le \|f_{n_1}\|_p + \sum_{k=1}^{\infty} \|f_{n_{k+1}} - f_{n_k}\|_p \\ &\le \|f_{n_1}\|_p + 2^{-k} \\ &< \infty\end{aligned}$$

which implies $f \in L^p(\mathbb{R}^d)$. That is to say, the subsequence converges to some $f \in L^p(\mathbb{R}^d)$. We now use the triangle inequality to show that the Cauchy sequence converges to f. So let $\varepsilon > 0$,

$$\|f_n - f\|_p \le \|f_n - f_{n_k}\|_p + \|f_{n_k} - f\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$