B1: Let A be a nonempty subset of $\mathbb{R}$ that is bounded above and let $s = supA$.

a) Show that if A is open the $s \notin A$.

Suppose for the sake of contradiction that A is open and $s \in A$. Then $\exists\ \epsilon > 0$ s.t. $V_\epsilon(s) \subseteq A$. In $\mathbb{R}$, that means the interval $(s-\epsilon, s+\epsilon) \subseteq A$. But s being the supremum of A says that

$$s + \epsilon \leq s.$$

This inequality cannot hold since $\epsilon > 0$. Thus, $s \notin A$.

b) Show that if A is closed then $s \in A$.

We want to show that $s = supA$ is a limit point of A. With A being closed, A contains it's limit points, particularly s. s being the supremum of A implies that $\forall \epsilon > 0$, $\exists a \in A$ s.t.

$$s - \epsilon < a \leq s.$$

Recalling the definition of a limit point, s is a limit point if

$$A \setminus \{s\} \cap V_\epsilon(s) \neq \emptyset.$$

or

$$A \setminus \{s\} \cap (s-\epsilon, s+\epsilon) \neq \emptyset.$$

This implies we need to show that there exist a number strictly less than s. But $\mathbb{R}$ is dense so that

$$\frac{a - s + \epsilon}{2}$$

is in the intersection. Thus, s is a limit point of A. Since A is closed, it contains its limit points and $s \in A$.

B2: Prove that the set $A = \left\{\frac{1}{n} \mid n \in \mathbb{N} \text{ and } n \geq 2\right\} \cup \{0\}$ is compact.

Let $\mathcal{U}$ be an open cover for A. For all $x \in A$, x is in some $U_x \in \mathcal{U}$. Since such U_x is open, some ϵ-neighborhood of x is contained in it,

$$x \in V_\epsilon(x) \subseteq U_x.$$

In particular, some ϵ-neighborhood of 0 is contained in some $U_0 \in \mathcal{U}$,

$$(-\epsilon, \epsilon) \subseteq U_0 \in \mathcal{U}.$$

By the Archimidean Propert, $\exists\ N \in \mathbb{N}$ s.t. $n \geq N \to \frac{1}{n} < \epsilon$. This means that the open set U_0 covers all points of A with $n \geq N$. Since there are a finite amount of points in A with $n < N$, $\mathcal{U}$ has a finite subcover

$$\{U_0, U_2, ..., U_{N-1}\}$$

that contains A. Since $\mathcal{U}$ is arbitrary, A is compact.

B3: Prove the following about compact sets

a) Show that an arbitrary intersection of compact sets is compact.

Let $\mathcal{W}$ be a collection of compact sets. The intersection

$$Y = \bigcap_{G \in \mathcal{W}} G$$

is contained in all $G \in \mathcal{W}$ and is closed. Therefore, Y is compact.

b) Show that a finite union of compact sets is compact.

Let $\{A_i\}$ be a collection of compact sets. Let $\bigcup_{i\in I} A_i$ be a finite union those sets. Note that the indexing set I is finite. Let $\mathcal{W}$ be an open cover for $\bigcup_{i\in I} A_i$. It follows that $\mathcal{W}$ is an open cover for each A_i. Since each A_i is compact, there's a finite subcover each of them,

$$\mathcal{W}'_i \subseteq \mathcal{W}.$$

The finite union of them, $\bigcup_{i\in I} \mathcal{W}'_i$, is a finite subset of $\mathcal{W}$ covering $\bigcup_{i\in I} A_i$. Therefore, $\mathcal{W}$ has a finite subcover. Since $\mathcal{W}$ is arbitrary,

$$\bigcup_{i\in I} A_i$$

is compact.

c) Show that any closed subset of a compact set is compact.
Let $B \subseteq A$ be closed and $\mathcal{U}$ be an open cover for B. B being closed implies B^c is open and the union $\mathcal{U} \cup B^c$ covers A. Since A is compact, some finite subcover $\mathcal{W} \subseteq \mathcal{U} \cup B^c$ covers A. Being a subset of A, $\mathcal{W}$ also covers B. But the B^c part doesn't intersect B so we can throw it out. Thus $\mathcal{W} \setminus B^c$ is a finite subcover of $\mathcal{U}$ that covers B. Since $\mathcal{U}$ is arbitrary and has a finite subcover, B is compact.

B4: Let A be a subset of $\mathbb{R}$.

a) Let $\mathcal{U}$ be the collection of all open subsets that are subsets of A. Prove that

$$A^o = \bigcup_{G \in \mathcal{U}} G.$$

($\Rightarrow$) We know that $A^o \subseteq A$. We want to show that A^o is open by showing that, $\forall x \in A^o$ and some $\epsilon > 0$,

$$V_\epsilon(x) \subseteq A^o.$$

First take some interior point $x \in A^o$. By definition, there is some positive number $\epsilon > 0$ s.t. x's ϵ-neighborhood is contained in A.

$$x \in V_\epsilon(x) \subseteq A.$$

Let $y \in V_\epsilon(x)$ be some arbitrary point in this x's ϵ-neighborhood. By letting

$$\epsilon' = \epsilon - d(x, y),$$

$V_{\epsilon'}(y) \subseteq A$. Thus, $y \in A^o$. Since every point in $V_\epsilon(x)$ is an interior point, $V_\epsilon(x) \subseteq A^o$. Thus, A^o is open and

$$A^o \subseteq \bigcup_{G \in \mathcal{U}} G.$$

($\Leftarrow$)Now we show that interior of A^o contains the union above. For all $x \in \bigcup_{G \in \mathcal{U}} G$, x is in some $G \subseteq A$. Since G is open, some ϵ-neighborhood of x is contained in $G \subseteq A$,

$$V_\epsilon(x) \subseteq G \subseteq A, \ \epsilon > 0.$$

Thus x is an interior point. So we have that

$$\bigcup_{G \in \mathcal{U}} G \subseteq A^o.$$

Because the union and the interior are contained in each other,

$$A^o = \bigcup_{G \in \mathcal{U}} G.$$

b) ($\Rightarrow$) The set $\overline{A} = A \cup L_A$ contains A and A's limit points. To show that the set is closed, we will show that the set contains the limit points of L_A also.
Let $y \in L_{L_A}$, i.e. y is some limit point of L_A. Then $\forall \epsilon > 0$, $\exists$ some limit point $z \in L_A$ contained in $V_\epsilon(y)$. Let $\epsilon' = \epsilon - d(z, y)$. Then $V_{\epsilon'}(z) \subseteq V_\epsilon(y)$. But each $V_{\epsilon'}(z)$ contains an element of A since $z \in L_A$. Thus, every ϵ-neighborhood of y contains an element of A. Therefore the limit points of L_A are limit points of A and are consequently contained in $\overline{A}$.
We just showed that $\overline{A}$ is closed. Therefore $\overline{A} \in \mathcal{C}$. It follows that

$$\bigcap_{G \in \mathcal{U}} G \subseteq \overline{A}.$$

($\Leftarrow$) Since $A \subseteq G$ for all $G \in \mathcal{U}$,

$$A \in \bigcap_{G \in \mathcal{U}} G.$$

But the arbitrary intersection of closed sets is closed. So that L_A is also contained in the set. Thus

$$\overline{A} \subseteq \bigcap_{G \in \mathcal{U}} G.$$

Therefore,

$$\overline{A} = \bigcap_{G \in \mathcal{U}} G.$$

c) Let $\mathcal{U}$ be an collection of all open subsets of A. By De Morgan's Laws,

$$(A^o)^c = (\bigcup_{G\in\mathcal{U}} G)^c = \bigcap_{G\in\mathcal{U}} G^c$$

But $G \subseteq A^o \subseteq A \Rightarrow A^c \subseteq G^c$. With G^c being closed, use the result of $4b$ to get

$$\bigcap_{G\in\mathcal{U}} G^c = \overline{A^c}.$$

Thus,

$$(A^o)^c = \overline{A^c}.$$