Note: Any problem with a * will not be graded for points but should provide an additional challenge.

C1: Let (a_n) be a sequence.

(a) Let (a_{n_j}) be a subsequence of (a_n). Show that if (a_n) converges to a then (a_{n_j}) converges to a.

($\Rightarrow$) Suppose $(a_n) \to a$. Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N} \ni \{n \geq N \to |a_n - a| < \epsilon\}$. That is to say that the N^{th} term and beyond of the sequence are all within ϵ distance of a. We will show that the N^{th} term and beyond of the sub-sequence is also within ϵ distance of a. Since (a_{n_j}) is a sub-sequence of (a_n), the order is preserved and the j^{th} term of the subsequence, a_{n_j} is either a_j or to the right of it. Thus, $n_n \geq n \geq N$ implies that the N^{th} term and beyond of the subsequence are the terms of (a_n) that are within ϵ distance of a. It follows that $(a_{n_j}) \to a$.

(b) Let (a_n) be a convergent sequence of positive numbers. Define (b_n) by $b_n = (-1)^n a_n$. Show that b_n converges if and only if (a_n) converges to 0.

($\Rightarrow$) Suppose $(a_n) \to 0$. Let $\epsilon > 0$ be given and N chosen s.t.

$$n \geq N \to |a_n - 0| = |a_n| < \epsilon.$$

We can see that the same N works for the sequence (b_n),

$$|b_n - 0| = |b_n| = |(-1)^n a_n| = |a_n| < \epsilon.$$

Thus, $(a_n) \to 0$ implies (b_n) converges.

($\Leftarrow$) We'll show by contrapositive that (b_n) converges implies $(a_n) \to 0$. Since (a_n) converges, we'll assume that $(a_n) \to L \neq 0$. With (a_n) being a sequence of positive terms, $a_n > 0$. But it's safe to say that $a_n \geq 0$ so that we can use the Order Limit Theorem to conclude $L \geq 0$. But the hypothesis being that $L \neq 0$ implies $L > 0$. But the odd terms of (b_n) converge to $-L$ while the even terms converge to L. It follows that (b_n) does not converge. Therefore, L must equal 0.

C2: We will show the second part of the Nested Interval Theorem. Let $I_n = [a_n, b_n]$ be a collection of nested intervals ($I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$). Show that if the sequence $(b_n - a_n)$ goes to zero then $\bigcap_{n=1}^{\infty} I_n$ has exactly one point.

(⇒) By the Nested Interval Theorem,

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Let $c \in \bigcap_{n=1}^{\infty} I_n$. We have that for all n,

$$a_n \leq c \leq b_n.$$

The sequences (a_n) and (b_n) are both monotone and bounded and therefore both converge to a and b, respectively. By the Order Limit Theorem,

$$a \leq c \leq b.$$

We will show that $a = c = b$. Let $\epsilon > 0$ be given and choose P s.t. $|b_n - b| < \epsilon$ whenever $n \geq P$. Now choose Q s.t. $|a_n - b_n| < 0$ whenever $n \geq Q$. Let

$$N = max\{P, Q\}.$$

We have that for $n \geq N$,

$$\begin{aligned}
|a_n - b| &= |a_n - b_n + b_n - b| \\
&\leq |a_n - b_n| + |b_n - b| \\
&< 0 + \epsilon \\
&= \epsilon.
\end{aligned}$$

Thus $(a_n) \to b$. Since the limit is unique, $a = b$. The sequences (a_n) ad (b_n) converge to the same point. We also have that

$$a \leq c \leq a,$$

implying that the intersection contains only one point.

C3: Let $(a_n), (b_n),$ and (c_n) be sequences, where $a_n \leq b_n \leq c_n$ for ever n. Show that if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

Note. This result is often known as the Squeeze Theorem.

(⇒) By the Algebraic Limit Theorem, the sequence $(c_n - a_n)$ converges to 0. Subtracting a_n from the inequality gives us

$$0 \leq b_n - a_n \leq c_n - a_n,$$

which gives us

$$0 \leq |b_n - a_n| \leq |c_n - a_n|.$$

Thus $(b_n) \to L$. Let $\epsilon > 0$ be given and choose P s.t. $|c_n - a_n| < 0$ whenever $n \geq P$. Now choose Q s.t. $|a_n - L| < \epsilon$ whenever $n \geq Q$. Let

$$N = max\{P, Q\}.$$

For $n \geq N$, we have

$$\begin{aligned} |b_n - L| &= |b_n - a_n + a_n - L| \\ &\leq |b_n - a_n| + |a_n - L| \\ &\leq |c_n - a_n| + |a_n - L| \\ &< 0 + \epsilon \\ &= \epsilon. \end{aligned}$$

C4: Define the sequence (a_n) recursively by $a_n = \sqrt{a_{n-1} + 6}$ with $a_1 = 10$.

(a) Show that (a_n) converges.

($\Rightarrow$) We will show by Induction that with $a_1 = 10$, the terms of this sequence are decreasing. With the first term given, $n = 2$ is the base case.

$$a_2 = \sqrt{a_1 + 6} = \sqrt{10 + 6} = 4 \leq 10 = a_1.$$

Suppose that it is true for some $n = k$,

$$a_k \leq a_{k-1}.$$

Since the square root function is increasing, this implies $\sqrt{a_k} \leq \sqrt{a_{k-1}}$.
By Induction, the terms of this sequence, with $a_1 = 10$, is decreasing. Then

$$a_{k+1} = \sqrt{a_k + 6} \leq \sqrt{a_{k-1} + 6} = a_k.$$

The terms of this sequence are non-negative and is therefore bounded below by 0 and above by 10. By the Monotone Convergence Theorem, this sequence converges.

(b) Show that $\lim_{n\to\infty} a_n = 3$.

$$\begin{aligned} a_n &= \sqrt{a_{n-1} + 6} \\ a_n^2 &= a_{n-1} + 6 \\ \lim_{n\to\infty} a_n^2 &= \lim_{n\to\infty} \{a_{n-1} + 6\} \\ \lim_{n\to\infty} a_n^2 &= \lim_{n\to\infty} a_{n-1} + \lim_{n\to\infty} 6 && \text{Algebraic Order Limits} \\ L^2 &= L + 6 \\ L^2 - L - 6 &= 0 \\ (L-3)(L+2) &= 0 \end{aligned}$$

The limit is $L = 3$ because it can't be -2 since $a_n \geq 0$ for all n implies $L \geq 0$.