MAT 4170: Modern Algebra I **Name:** **Panya Sukphranee**

Homework 8

1. a) $f : (\mathbb{Z}, +) \to (\mathbb{R}, +)$, $f(n) = n$.
   Homomorphism. $ker\phi = \{0\}.Im\phi = \mathbb{Z} \subseteq \mathbb{R}$.

   b) $g : (\mathbb{R}, +) \to (\mathbb{Z}, +)$, $f(x) =$ greatest integer $\leq x$.
   Not Homomorphism.

   c) $\phi : \mathbb{Z}_6 \to \mathbb{Z}_2$, $\phi(n) = n\ (mod 2)$
   Homomorphism. $Im\phi = \mathbb{Z}_2, ker\phi = \{0, 2, 4\}$.

   d) $f : \mathbb{Z} \to C_4$, $f(n) = R^{2n}\ \forall\, n \in \mathbb{Z}$.
   IDK

2. Let $\phi : G \to G'$ be a homomorphism.
   ($\Rightarrow$) Suppose ϕ is $1-1$. We know that $e_G \in ker\phi$. If some element $a \in ker\phi$, then $\phi(a) = e_G$. Being $1-1$, this implies that $a = e_G$. Thus, e_G is the only element in the kernel.

   $$ker\phi = \{e_G\}.$$

   ($\Leftarrow$) Suppose $\ker\phi = \{e_G\}$. Let $a, b \in G$ and assume $\phi(a) = (b)$.

   $$e_{G'} = \phi(a)\phi^{-1}(b) = \phi(a)\phi(b^{-1}) = \phi(ab^{-1}).$$

   This implies that $ab^{-1} \in ker\phi$ and thus $ab^{-1} = e_G$ by the hypothesis. Thus, $a = b$. Since $\phi(a) = \phi(b)$ implies $a = b$, we have that ϕ is $1-1$.

3. $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.

   a) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ isn't isomorphic to $\mathbb{Z}_4$. Every element in G has order 2 and Z_4 is cyclic. So no element maps to a generator of $\mathbb{Z}_4$.

G	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

   b) $S_3 \times Z_2$ is isomporphic to A_4 judging by the size of the groups. The other groups have 12 elements.

4. Let $H \leq G$. Show $\forall\, a \in G$,
$$aH = H \iff a \in H.$$
(⇒)
Let $a \in H$. $aH = \{ah | h \in H\}$.
Because of closure, $aH \subseteq H$. We need to show that $H \subseteq aH$. To do so, we'll show that every element in H is also in aH.
Let $x \in H$. We know that the element $a^{-1}x$ is also in H since it's' a group. It follows that
$$a(a^{-1}x) = x \in aH.$$
Since $x \in aH$, $H \subseteq aH$. Finally,
$$aH = H.$$
(⇐) Suppose $aH = H$. Since $e \in H$,
$$ae \in aH = H.$$
But $ae = a$, so $a \in H$.

5. $\dfrac{\mathbb{Z}}{3\mathbb{Z}} = \{H_0, H_1, H_2\}$ defined below. This set forms a group.
$$H_0 = \{2k | k \in \mathbb{Z}\}$$
$$H_1 = \{2k + 1 | k \in \mathbb{Z}\}$$
$$H_2 = \{2k + 2 | k \in \mathbb{Z}\}$$
The identity element is H_0 and H_1, H_2 are inverses of each other.

6. a) Let $H = \{I, R^2\}$. The left cosets are
$$H_1 = \{R, R^3\}, H_2 = \{FR, FR^3\}, H_3 = \{F, FR^2\}.$$
This forms a group and looks isomorphic to Z_4 to me.

b) Let $H = \{I, FR\}$. I'm a bit confused about this problem. The remaining elements in D_4 are
$$\{R, R^2, R^3, F, FR^2, FR^3\}.$$
So the left cosets are
$$H_1 = \{R, F\}$$
$$H_2 = \{R^2, FR\}$$
$$H_3 = \{R^3, FR^2\}$$
$$H_4 = \{FR^3, R^2\}$$
One of the left cosets constructed with the element R^2 gives
$$\{R^2, FR\}.$$
That intersects with H. So this set does not form a group?

c) $H = \{0, 4, 8\}$. The left cosest are
$$H_1 = \{1, 5, 9\}, H_2 = \{2, 6, 10\}, H_3\{3, 7, 11\}.$$
This is a group isomorphic to rotations of a square. The identity is H and $H_1 \mapsto R$.