7.2.1) (a) Prove that $\sum_{k=1}^{\infty} sin\left(\frac{x}{k^2}\right)$ converges uniformly on any bounded interval in $\mathbb{R}$.

Let $f_k(x) = sin\left(\frac{x}{k^2}\right)$ and $E \subset \mathbb{R}$ non-empty and bounded. Since E bounded, we let $d := \max_{x \in E} \{|x|\}$. We then have that

$$\left|sin\left(\frac{x}{k^2}\right)\right| \leq \frac{|x|}{k^2} \leq \frac{d}{k^2}.$$

With $M_k := \frac{d}{k^2}$, the Wierstrauss M-Test tells us that

$$\sum_{k=1}^{\infty} sin\left(\frac{x}{k^2}\right)$$

converges absolutely and uniformly.

(b) Prove that $\sum_{k=0}^{\infty} e^{-kx}$ converges uniformly on any closed subinterval of $(0, \infty)$.

Let $\varepsilon > 0$, $f_k(x) = e^{-kx}$, $[a, b] \subset (0, \infty)$. Since e^{-kx} is decreasing and positive,

$$|f_k(x)| = \left|e^{-kx}\right| \leq e^{-ka}$$

We know

$$\sum_{k=0}^{\infty} e^{-ka}$$

converges since it is a Geomtric Series. By the W-M Test,

$$\sum_{k=0}^{\infty} f_k(x)$$

converges uniformly.

7.2.2) Prove that the geomtric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

converges uniformly on any closed interval $[a, b] \subset (-1, 1)$.

Let $f_k(x) = x^k$, $d = max\{|a|, |b|\} < 1$, and $M_k := d^k$. Since $d < 1$,

$$\sum_{k=1}^{\infty} M_k < \infty$$

since it is the Geometric Series. We also have that

$$|f_k(x)| \leq M_k \; \forall \, k \in \mathbb{N}, x \in [a, b]$$

By the W-M Test,

$$\sum_{\infty}^{k=1} x^k$$

converges absolutely and uniformly on all $[a,b] \subset (-1,1)$.

7.2.4) Suppose that

$$f(x) = \sum_{k=1}^{\infty} \frac{cos(kx)}{k^2}.$$

Prove that

$$\int_{\frac{pi}{2}}^{0} f(x)dx = \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^3}.$$

Let $f_k(x) = \dfrac{cos(kx)}{k^2}$, $E = [0,2]$, and $M_k := \dfrac{1}{k^2}$.
We have that

$$|f_k(x)| = \left|\frac{cos(kx)}{k^2}\right| \leq \frac{1}{k^2} = M_k.$$

By the W-M test, we have that

$$\sum_{k=1}^{\infty} \frac{cos(kx)}{k^2}$$

converges uniformly and absolutely on E. Now $\forall\, k$, f_k is integrable and

$$\int_0^{\frac{\pi}{2}} f_k(x) = \frac{sin(\frac{\pi}{2}k)}{k^3}.$$

By Thereom 7.14,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} f(x) &= \int_0^{\frac{\pi}{2}} \sum_{k=1}^{\infty} f_k(x) \\
&= \sum_{k=1}^{\infty} \int_0^{\frac{\pi}{2}} f_k(x) \\
&= \sum_{k=1}^{\infty} \frac{sin(\frac{\pi}{2}k)}{k^3} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}
\end{aligned}$$

The last inequality comes from the fact that even terms of k make the summand 0.

7.2.5) Show that

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k} sin\left(\frac{x}{k+1}\right)$$

converges pointwise in $\mathbb{R}$ and uniformly on each bounded interval in $\mathbb{R}$ to a differentiable function f which satisfies

$$|f(x)| \leq |x| \;\; and \;\; |f'(x)| \leq 1$$

for all $x \in \mathbb{R}$.

Let $E = (a, b)$ be a bounded interval in $\mathbb{R}$,

$$f_k(x) = \frac{1}{k} sin\left(\frac{x}{k+1}\right),$$

and $d := sup\{a, b\}$.
Each $f_k(x)$ is bounded, for

$$\begin{aligned} |f_k(x)| &= \left|\frac{1}{k}\left(\frac{x}{k+1}\right)\right| \\ &= \frac{|x|}{k(k+1)} \\ &\leq \frac{d}{k(k+1)} =: M_k \end{aligned}$$

We know that each M_k converges by Comparison with $\frac{1}{k^2}$. By the WM Test, the series

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{x}{k+1}\right)$$

converges uniformly on E and consequently pointwise on $\mathbb{R}$.

We now show that f, f_k satisfy conditions to use Theorem 7.14ii:

(a)
$$f_k'(x) = \frac{1}{k(k+1)} cos\left(\frac{x}{k+1}\right)$$
is continuous on E and therefore f_k is differentiable on E.

(b) f converges pointwise on $\mathbb{R}$ by previous part.

(c) $\sum_{k=1}^{\infty} f_k'$ converges uniformly:

$$|f_k'| = \left|\frac{1}{k(k+1)} cos\left(\frac{x}{k+1}\right)\right| \leq \frac{1}{k(k+1)} =: M_k$$

By WM Test, $\sum\limits_{k=1}^{\infty} f_k'$ converges uniformly.

By Theorem 7.14ii, f is a diferentiable function on E. f satisfies the following conditions:

(a) -

$$\begin{aligned}
|f(x)| &\leq \sum_{k=1}^{\infty} \frac{|x|}{k(k+1)} \\
&= |x| \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \\
&= |x| \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\
&= |x|.
\end{aligned}$$

(b) -

$$\begin{aligned}
|f'(x)| &= \left| \sum_{k=1}^{\infty} \frac{1}{k(k+1)} cos \left(\frac{x}{k+1} \right) \right| \\
&\leq \sum_{k=1}^{\infty} \left| \frac{1}{k(k+1)} \right| \\
&= 1.
\end{aligned}$$

We use the fact of convergence of the series to pull the summation out, recognize telescoping series, and use the fact that cos is ≤ 1.

7.2.6) Prove that

$$\left| \sum_{k=1}^{\infty} \left(1 - cos \left(\frac{1}{k} \right) \right) \right| \leq 2.$$