10.2.2) $a \in (X,d)$ said to be isolated iff $\exists r > 0 : B_r(a) = \{a\}$.

(a) Show $a \in (X,d)$ not cluster point iff a an isolated point.

($\Rightarrow$) Suppose $a \in (X,d)$ is an isolated point. Then by definition it is not a cluster point. For $\exists\, d := r > 0$ s.t. $B_r(a) = \{a\}$ does not have infinitely many points.

($\Leftarrow$) Suppose $a \in (X,d)$ is not a cluster point. Then $\exists \delta > 0$ s.t. $B_\delta(a)$ has a finite number of points. Let

$$r := \min_{x \in B_\delta(a)} \{\rho\,(x,a)\}$$

Then $B_r(a) = \{a\}$ implying that a is an isolated point.

(b) Let (X,d) be a discrete space and let $a \in (X,d)$. Letting $r = \frac{1}{2}$, we have that

$$B_r(a) = \{a\}$$

Thus implying that a is an isolated point. Since a was an arbitrary point in X, X has no cluster points.

10.2.3) a is a cluster point for some $E \subset X$ iff $\exists$ sequence $x_n \in E - \{a\} : x_n \to a$ as $n \to \infty$.

($\Rightarrow$) Let a be a cluster point and consider the sequence $\delta_n = \dfrac{1}{n}$. We can construct the sequence $\{x_n\}$ by choosing x_n to be a point in the sequence of balls around a, $B_{\delta_n}(a)$. This sequence converges because $\forall\, \delta > 0,\ \exists\ \dfrac{1}{n} < \delta$ implying that there are an infinite amount of points of the sequence in $B_\delta(a)$. Therefore a is a cluster point.

($\Leftarrow$) If $\{x_n\}$ is a sequence converging to a, then $B_\delta(a) - \{a\}$ contains infinitely many points of $\{x_n\}$ by definition of convergence. Thus a is a cluster point.

10.2.4) (a) Let $E \subset X$ be non-empty. Prove a is a cluster point of E iff $\forall r > 0,\ E \cap B_r(a) - \{a\}$ is non-empty.

($\Rightarrow$) If a is a cluster point of E, then $E \cap B_r(a)$ contains infintely many points. It follows that $E \cap B_r(a) - \{a\}$ contains infinitely many points and is therefore non-empty.

($\Leftarrow$) Now suppose that $E \cap B_r(a) - \{a\} \neq \emptyset\ \forall\, r > 0$ but that a is not a cluster point of E. Then $\exists$ finitely many points in $E \cap B_r(a) - \{a\}$. It follows that there is some closest point and therefore if

$$r := \min_{x \neq a} B_1(a)$$

we have that $E \cap B_r(a) = \emptyset$.

(b) If $E \subset \mathbb{R}$ is bounded with infinitely many points then, Bolzano-Weierstrass says that there exists a convergent subsequence.

10.2.9) Let (X,d) be a metric space satisfying the Bolzano-Weierstrass property.

(a) Suppose that E is a closed and bounded subset of X and $x_n \in E$. We will show there is a convergent subsequence. We construct the convergent subsequence similar to way it's done in $\mathbb{R}$.

Let $M := \max_{x,y\in E} d(x,y)$ be the radius of E.
Let $B1 = E$ and pick x_1 to be a point in E.
Now, either $B_{M/2}(x_0)$ or $E - B_{M/2}(x_0)$ has infinitely many points. Let $B2$ be the infinite set and choose $x_2 \in B2$.
Now, either $B_{M/4}(x_1)$ or $E - B_{M/4}(x_1)$ has infinitely many points. Let $B3$ be the infinite set and choose $x_3 \in B3$.
We continue on like this. The $B\#$ we choose has infinitely many points and gets smaller. Thus is converges to some point in E.

(b)

10.3.1) Find the interior, closure, and boundary of each of the following subsets of $\mathbb{R}$.

(a) $E = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
   (1) Interior: None
   (2) Closure: $[0, 1]$
   (3) Boundary: E

(b) $E = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+1}, \frac{1}{n}\right)$
   (1) Interior: E
   (2) Closure: $[0, 1]$
   (3) Boundary: $\bigcup_{n=1}^{\infty} \left\{\dfrac{1}{n+1}, \dfrac{1}{n}\right\}$

(c) $E = \bigcup (-n, n)$
   (1) Interior: $\mathbb{R}$
   (2) Closure: $\mathbb{R}$
   (3) Boundary: $\emptyset$

(d) $E = \mathbb{Q}$
   (1) Interior: $\emptyset$
   (2) Closure: $\mathbb{R}$
   (3) Boundary: $\mathbb{R}\backslash\mathbb{Q}$

10.3.2) Identify which sets are open, closed, or neither. Find $E^O, \overline{E}, \partial E$.

(a) $E = \{(x, y) : x^2 + 4y^2 \leq 1\}$
   (1) Closed
   (2) $E^o = \{(x, y) : x^2 + 4y^2 < 1\}$
   (3) $\overline{E} = \{(x, y) : x^2 + 4y^2 \leq 1\}$

(4) $\partial E = \{(x, y) : x^2 + 4y^2 = 1\}$

(b) $E = \{(x, y) : x^2 - 2x + y^2 = 0\} \cup \{(x, 0) : x \in [2, 3]\}$

(1) Closed

(2) $E^o = \emptyset$

(3) $\overline{E} = E$

(4) $\partial E = E$

(c) $E = \{(x, y) : y \geq x^2, 0 \leq y < 1\}$

(1) Closed

(2) $E^o = \emptyset$

(3) $\overline{E} = E \cup \{(-1, 1), (1, 1)\}$

(4) $\partial E = E$

(d) $E = \{(x, y) : x^2 - y^2 < 1, -1 < y < 1\}$

(1) Closed

(2) $E^o = E$

(3) $\overline{E} = \{(x, y) : x^2 - y^2 \leq 1, -1 \leq y \leq 1\}$

(4) $\partial E = ((x, y) : x^2 - y^2 \leq 1) \cup \left\{(x, -1) : -\sqrt{2} \leq x \leq \sqrt{2}\right\} \cup \left\{(x, 1) : -\sqrt{2} \leq x \leq \sqrt{2}\right\}$

10.3.3) Let $a \in X$, $s < r$,

$$V = \{x \in X : s < \rho(x, a) < r\}$$

and

$$E = \{x \in X : s \leq \rho(x, a) \leq r\}.$$

Prove V is open and E is closed.

($\Rightarrow$) $\forall\, v \in V$, let $r := min\{\rho(v, a) - s, r - \rho(v, a)\}$; i.e. let r be the distance to the closest boundary. Then

$$B_r(v) \subseteq V$$

Since, every point in V is contained in some open set contained in V, V is open.

The set $E = \{x \in X : \rho(x, a) \leq r\} \cap \{x \in X : \rho(x, a) \geq s\}$ is closed as it is the finite union of closed sets.

10.3.4) Suppose $A \subseteq B \subseteq X$. Prove $\overline{A} \subseteq \overline{B}$ and $A^o \subset B^o$.

Since $A^o \subseteq B$, we have by Theorem 10.34 that $\boxed{A^o \subseteq B^o}$. Since $A \subseteq \overline{B}$ and $\overline{B}$ is closed, Theorem 10.34 also tells us that $\boxed{\overline{A} \subseteq \overline{B}}$.

10.3.5) Let $E \neq \emptyset$ and closed in X, $x \notin E$. Prove that $\inf\limits_{x \in E} \rho(x, a) > 0$.

($\Rightarrow$) We have that $x \in E^c$, which is open in X. Therefore, $\exists \varepsilon > 0$ s.t. $B_r(x) \cap E = \emptyset$ since the ball is contained in E^c. Then $\rho(x, a) > \varepsilon\ \forall\, a \in E$. It follows that $\inf\limits_{a \in E} \rho(x, a) \geq \varepsilon > 0$.

10.3.7) Show the following

(a) $\exists\, A, B \subset \mathbb{R}\, :\, (A \cup B)^o \neq A^o \cup B^o$.
Let $A = [0, 1]$ and $B = [1, 2]$. $(A \cup B)^o = (1, 2)$ but $A^o \cup B^o = (0, 1) \cup (1, 2)$.

(b) $\exists\, A, B \subset \mathbb{R}\, :\, \overline{A \cap B} \neq \overline{A} \cap \overline{B}$.
Let $A = (0, 1)$ and $B = (1, 2)$. Then $\overline{A \cap B} = \emptyset$ but $\overline{A} \cap \overline{B} = \{1\}$.

(c) $\exists\, A, B \subset \mathbb{R}\, :\, \partial(A \cup B) \neq \partial A \cup \partial B$ and

$$\partial(A \cap B) \neq (A \cap \partial B) \cup (B \cap \partial A) \cup (\partial A \cap \partial B).$$

Let $A = (0, 1)$ and $B = [1, 2]$
(1) $\{0, 2\} = \partial(A \cup B) \neq \partial A \cup \partial B = \{0, 1, 2\}$
(2) $\emptyset = \partial(A \cap B) \neq \{1\} = \partial A \cap \partial B$.

10.3.8) Let $Y \subseteq X$ be a subspace.

(a) Show V open in Y iff open $U \subset X$ s.t. $V = U \cap Y$. $(\Rightarrow)$ Suppose V is open in Y. Then $\forall\, v \in V$, $\exists B_Y(v) = B_X(v) \cap Y$ contained in V (and containing v). We have the following chain of equalities

$$V = \bigcup_{v \in V} B_Y(v) = \bigcup_{v \in V} (B_X(v) \cap Y) = \left(\bigcup_{v \in V} B_X(v)\right) \cap Y.$$

$(\Leftarrow)$ Suppose $\exists$ open $U \subset X$ s.t. $V = U \cap Y$. Then every $v \in V$ is contained in U. Therefore, $\exists\, B_X(v) \subseteq U$ containing v. Then
$B_Y(v) := B_X(v) \cap Y$ is open in Y containing v. Thus $B_Y(v) \subseteq V$.

(b) Prove $E \subseteq Y$ closed iff $\exists$ closed set $A \subseteq X$ s.t. $E = A \cap Y$.

$(\Rightarrow)$ E closed in $Y \Rightarrow Y \backslash E$ open in Y. By part a), $\exists\ \subset X$ open s.t. $Y \backslash E = \tilde{A} \cap Y$.
Let $A := X - \tilde{A}$, which is an open set. We have

$$\begin{aligned} A \cap Y &= (X - \tilde{A}) \cap Y \\ &= (X \cap Y) - (\tilde{A} cap Y) \\ &= Y - (Y \backslash E) \\ &= Y - Y \backslash E \\ &= E \end{aligned}$$

$(\Leftarrow)$ Suppose $\exists$ a closed set $A \subseteq X$ s.t. $E = A \cap Y$. Show $E \subseteq Y$ is closed in Y.
The set $X \backslash A$ is open in X and therefore it's intersection with Y

$$(X \backslash A) \cap Y$$

is open in Y. This set is $Y \backslash E$

$$(X \backslash A) \cap Y = (X \cap Y) \setminus (A \cap Y) = Y \backslash E.$$

which implies E is closed in Y.

10.3.10) Let $V \subseteq X$.

(a) Prove V is open in X iff there is a collection of open balls $\{B_\alpha : \alpha \in A\}$ s.t.

$$V = \bigcup_{\alpha \in A} B_\alpha$$

($\Rightarrow$) Suppose V open in X. Consider

$$\bigcup_{v \in V} B_{\varepsilon_v}(v), \qquad B_{\varepsilon_v}(v) \subset V$$

It follows that

$$V \subset \bigcup_{v \in V} B_{\varepsilon_v}(v)$$

Inclusion in the other direction is trivial and therefore

$$V = \bigcup_{v \in V} B_{\varepsilon_v}(v)$$

(b) What happens to this result if *open* is replaced by *closed*?
Not sure I understand this question.

10.3.11) Let $E \subset X$ be closed.

(a) Prove $\partial E \subseteq E$
Let $x \in \partial E$. Then $B_\varepsilon(x) \cap E \neq \emptyset$, implying that it is a cluster point. But E closed implies $x \in E$ since E contains its cluster points. Thus $\partial E \subseteq E$.

(b) Prove $\partial E = E$ iff $E^o = \emptyset$
($\Rightarrow$) Suppose $\partial E = E$ and let $x \in E$. Then $B_r(x) \cap E^c \neq \emptyset \; \forall \, r > 0$. i.e. Every open ball containing x intersects E^c. This means that ball around x is contained in E. Thus E^o is empty, $E = \emptyset$.
($\Leftarrow$) Suppose $E^o = \emptyset$ and let $x \in E$. Consider some ball around x, $B_r(x)$. Since $B_r(x)$ is open and no open set is contained in E by hypothesis, we have that

$$B_r(x) \cap E \neq \emptyset.$$

Thus $x \in \partial E \Rightarrow E \subseteq \partial E$. Inclusion in the other direction is trivial and therefore

$$E = \partial E.$$

(c) Show that b) is false if E is not closed.
Suppose E is not closed. Then some cluster point z of E s.t. $z \notin E$. But

$$B_\varepsilon(z) \cap E \neq \emptyset$$

and

$$B_\varepsilon(z) \cap E^c \neq \emptyset$$

implies $z \in \partial E$. Thus $\partial E \not\subset E$.