Chapter 1 - 18) Prove the following assertion: Every measurable function is the limit a.e. of a sequence of continuous functions.

($\Rightarrow$) Given a measureable function f, Theorem 4.2 says there exists a sequence of step functions $\{\phi_k(x)\}$ s.t. $|\phi_k(x)| \leq |\phi_{k+1}(x)|$ and $\lim\limits_{k\to\infty} \phi_k(x) = f(x)$ for all x. We can extend $\phi_k(x) \to \phi'_k(x)$, where $\phi'_k(x)$ joins discontinuities with a sloped line segment to make it continuous.

For example,

$$\chi_{[0,1]} \to \chi'_{[0,1]} = \begin{cases} 1 & x \in [0,1] \\ y_1(x) & x \in [-\varepsilon, 0] \\ y_2(x) & x \in [1, 1+\varepsilon] \\ 0 & elsewhere \end{cases}$$

The simple functions $\phi'_k(x)$ are continuous. We can require that the line segements be such that $m\{x : \phi'_k(x) - \phi_k(x) > 0\} < 2^{-k}$. We then have that $\phi'_k(x)$ are continuous functions converging to $f(x)$.

Chapter 1 - 21) Prove that there is a continuous function that maps a Lebesgue measurable set to a non-measurable set. [Hint: Consider a non-measurable subset of [0,1] and its inverse image in $\mathcal{C}$ by the function F in Exercise 2.]

First, consider the ternary expansion of an element in the Cantor set $x \in \mathcal{C}$,

$$x = \sum_{k=1}^{\infty} a_k 3^{-k} \qquad k = \{0, 2\}.$$

The coefficients of this expansion tell us what "path" to take to get to x. Since each interval has the middle $\frac{1}{3}$ removed, we choose between the 0^{th} and 2^{nd} sub-intervals, hence $\{0, 2\}$.

We now look at the function $F : \mathcal{C} \to [0,1]$ given in the book

$$F(x) = \sum_{k=1}^{\infty} \frac{a_k}{2} \frac{1}{2^k}.$$

The coefficient $\dfrac{a_k}{2}$ turns the $\{0,2\}$ to $\{0,1\}$ and therefore specifies which half-interval path to take at level k, $\frac{1}{2^k}$. Since every number $\beta \in [0,1]$ has a binary expansion, this function is onto. I think it's clear to see that $F(x)$ is an increasing function. $F(x)$ can be made continuous by specifying $F(x)$ to be some constant for all $x \notin \mathcal{C}$.

Finally, the set of equivalence class representatives $\mathcal{N} \subset [0,1]$ from a previous exercise is not measurable. Since $F(x)$ is onto, the pre-image is a subset of the Cantor set,

$$F^{-1}(\mathcal{N}) \subset \mathcal{C}.$$

By subadditivity, $m_*\left(F^{-1}(\mathcal{N})\right) \leq m_*\left(\mathcal{C}\right) = 0$ implies $F^{-1}(\mathcal{N})$ is a measurable set mapped to a non-measurable set by $F(x)$.

Chapter 1 - 22) Let $\chi_{[0,1]}$ be the characteristic function of $[0,1]$. Show that there is no everywhere continuous function f on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]}(x) a.e.$$

$(\Rightarrow)$

Suppose for the sake of contradiction that $\exists\ g(x)$ continuous everywhere s.t.

$$g(x) = \chi_{[0,1]}\ a.e.$$

Then $g(x) \neq \chi_{[0,1]}$ on sets of measure 0. We first construct two sequences $\{x_n\}$ and $\{y_n\}$ converging to 0.

Consider the interval $I_n = \left(0, \dfrac{1}{n}\right)$. Since $m(I_n) > 0$, there exists a $x_n \in I_n$ s.t. $g(x_n) = \chi_{[0,1]}$. Therefore, we can construct a sequence converging to 0 from the right. Similarly, we can construct a sequence $\{x_n\} \to 0$ from the left.

But $g(x_n) \to 0$ and $g(y_n) \to 1$, thus contradicting that $g(x)$ is continuous everywhere.

Chapter 1 - 28) Let E be a subset of $\mathbb{R}$ with $m_*\left(E\right) > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$m_*\left(E \cap I\right) \geq \alpha m_*\left(I\right).$$

Loosely speaking, this estimate shows that E contains almost a whole interval. [Hint: Choose an open set $\mathcal{O}$ that contains E, and such that $m_*\left(E\right) \geq \alpha m_*\left(\mathcal{O}\right)$. Write $\mathcal{O}$ as the countable union of disjoint open intervals, and show that one of these intervals must satisfy the desired property.]

$(\Rightarrow)$ Let $\varepsilon > 0$ and $\mathcal{O} \supset E$ be an open covering of E s.t.

$$m_*\left(\mathcal{O}\right) \leq m_*\left(E\right) + \varepsilon.$$

We perform some algebra to write a lower bound on the measure of $m_*\left(E\right)$ in terms of $m_*\left(\mathcal{O}\right)$. Let $c \in \mathbb{R}$ s.t. $\varepsilon = c m_*\left(E\right)$.

$$\begin{aligned} m_* (E) &\geq m_* (\mathcal{O}) - \varepsilon \\ &= m_* (\mathcal{O}) - c m_* (E) \\ &\dots \\ (1+c) m_* (E) &\geq m_* (\mathcal{O}) \\ m_* (E) &\geq \frac{1}{1+c} \, m_* (\mathcal{O}) \end{aligned}$$

So that with $\alpha := \dfrac{1}{1+c}$,

$$\boxed{m_* (E) \geq \alpha m_* (\mathcal{O})}.$$

We can write $\mathcal{O} = \bigcup U_i$ as a union of open sets so that

$$E = \bigcup E \cap U_i$$

and

$$\sum m_* (E \cap U_i) \geq \alpha \sum m_* (U_i)$$

Note that since α and ε are related, the covering $\mathcal{O}$ depends on α. We claim that one of the U_i satisfies

$$m_* (E \cap U_i) \geq \alpha m_* (U_i) .$$

Suppose for the sake of contradiction that every U_i,

$$m_* (E \cap U_i) < \alpha m_* (U_i) .$$

Then the following leads to a contradiction,

$$m_* (E) = m_* \left(\bigcup E \cap U_i \right) = \sum m_* (E \cap U_i) < \alpha \sum m_* (U_i) = \alpha m_* (\mathcal{O})$$

$$m_* (E) < \alpha m_* (\mathcal{O}) .$$

Thus, $\exists \, U_i$ s.t.

$$m_* (E \cap U_i) \geq \alpha m_* (U_i) .$$

Chapter 1 - 29) Suppose E is a measurable subset of $\mathbb{R}$ with $m(E) > 0$. Prove that the **difference set** of E, which is defined by

$$\Delta := \{ z \in \mathbb{R} : z = x - y \text{ for some } x, y \in E \}$$

contains an open interval centered at the origin. If E contains an interval, then the conclusion is straightforward. In general, one may rely on Exercise 28.

[Hint: Indeed, by Exercise 28, there exists an open interval I so that $m(E \cap I) \geq (9/10)m(I)$. If we denote $E \cap I$ by E_0, and suppose that the difference set of E_0 does not contain an open interval around the origin, then for arbitrarily small a the sets E_0 and $E_0 + a$ are disjoint. From the fact that $(E_0 \cup (E_0 + a)) \subset (I \cup (I + a))$ we get a contradiction, since the left-hand side has measure $2m(E_0)$, while the right-hand side has measure only slightly larger than $m(I)$.]

($\Rightarrow$) Suppose for the sake of contradiction that $\nexists$ open I centered at 0, i.e. $I \ni 0$. But we can see that $0 \in \Delta$, so there is a a sufficiently small (i.e. close to 0) s.t.

$$(E + a) \cap E = \emptyset$$

The translation by a does not result in overlap.

Now, by the previous exercise, we have that $\exists\ I$ s.t.

$$m_* (E \cap I) \geq \left(\frac{9}{10}\right) m_* (I) .$$

We'll denote this $E_0 := E \cap I$ and note that $E_0 \subset I$. By subadditivity, monotonicity, etc., we have that

$$m(E_0) + m(E_0 + a) \leq m(I) + |a|$$

implying

$$2m(E_0) \leq m(I) + |a|$$

which is a contradiction. So that Δ contains some open interval I.