A1: Let A and B be bounded subsets of $\mathbb{R}$. Define

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\}.$$

($\Rightarrow$)
Since A and B are bounded sets,

$$\begin{aligned} s &= \sup A, t = \sup B, \\ u &= \inf A, v = \inf B \end{aligned}$$

exists. The set A + B is bounded since

$$u + v \leq a + b \leq s + t, \ \forall a \in A, b \in B.$$

Therefore sup $(A + B)$ exists. Let $\epsilon > 0$,

$$s + t - \epsilon = (s - \frac{\epsilon}{2}) + (t - \frac{\epsilon}{2}).$$

Then $\exists\, a_0 \in A$ and $b_0 \in B$ s.t. $s - \dfrac{\epsilon}{2} < a_0$ and $t - \dfrac{\epsilon}{2} < b_0$. Hence, $s + t - \epsilon < a_0 + b_0$. Therefore, sup (A+B) = sup A + sup B.

A2: Let A be a nonempty subset of $\mathbb{R}$ that is bounded above. Define B to be the set of upper bounds of A. Prove that B is bounded below and that

$$\sup A = \inf B.$$

($\Rightarrow$)

Since A is bounded above, $\alpha = \sup$ A exists and is, by definition of B, an element of B. The inequality,

$$a \leq \alpha \leq b, \ \forall a \in A, b \in B$$

implies B is bounded below and $\beta = \inf$ B exists. In particular, $\beta \leq \alpha$.

Suppose $\beta < \alpha$. Define $\epsilon := \alpha - \beta > 0$. Then $\exists a_0 \in A$ s.t.

$$\begin{aligned} a_0 &> \text{sup A} - \epsilon \\ &= \text{sup A - (sup A - inf B)} \\ &= \text{inf B} \\ &= \beta. \end{aligned}$$

Contradiction. Therefore sup A = inf B.

A3: Let x and y be real numbers with $x > y > 0$ and let n be a natural number. Prove the following.

(a) $x^n > y^n$.

For n = 2,

$$y^2 < xy < x^2.$$

Suppose

$$y^k < x^k,\ k \in \mathbb{N}$$

then

$$y^{k+1} < xy^k < x^{k+1}.$$

Thus,

$$0 < y < x \Rightarrow y^n < x^n.$$

(b) $x^n - y^n < nx^{n-1}(x-y)$

$$\begin{aligned} x^n - y^n &= (x-y)\sum_{k=0}^{n-1} x^{n-1-k}y^k \\ &= (x-y)\sum_{k=0}^{n-1} x^{n-k}\left(\frac{y}{x}\right)^k \\ &< (x-y)\sum_{k=0}^{n-1} x^{n-1},\quad \left(\frac{y}{x}\right) < 1 \\ &= (x-y)(n-1)x^{n-1} \\ \\ &< nx^{n-1}(x-y). \end{aligned}$$

A4: Assume that $a \in \mathbb{R}$ and $a > 0$ and that $n \in \mathbb{N}$.

$(\Rightarrow)$

WLOG assume $a > 1$. If $0 < a < 1$, then $\frac{1}{a} > 1$, so that the positive n-th root of a is the reciprocal of the positive n-th root of $\frac{1}{a}$. Define

$$A = \{x \in \mathbb{R} \mid x^n < a\}$$

The set is non-empty since $1 \in A$. The binomial expansion of $(a+1)^n$

$$(a+1)^n = \sum_{k=0}^{n} \binom{n}{k} a^k > a$$

implies A is bounded above and therefore $s = \sup$ A exists. $1 \in A$ implies $s \geq 1 > 0$. Let $b \in \mathbb{R}$ s.t. $b^n > a$. Assume b is positive? Let $k \in \mathbb{N}$ s.t. $0 < (b - \frac{1}{k}) < b$. By A3.b, we have

$$\begin{aligned} b^n - (b - \frac{1}{k})^n &< nb^{n-1}(b - b + \frac{1}{k}) \\ &= \frac{n}{k} b^{n-1}. \end{aligned}$$

So,

$$b^n - \frac{n}{k} b^{n-1} < (b - \frac{1}{k})^n.$$

For $k > \dfrac{nb^{n-1}}{b^n - a}$,

$$a < b^n - \frac{n}{k} b^{n-1} < (b - \frac{1}{k})^n$$

implies that $b - \frac{1}{k} \notin A$. Moreover, $b - \frac{1}{k}$ is an upper bound on A and b is not the LUB. Therefore $s = \sup$ A cannot be a number s.t. $s^n > a$. Therefore s is s.t. $s^n \leq a$. We now show that s is s.t. $s^n \geq a$.

Let $c \in \mathbb{R}$ s.t. $c^n < a, k \in \mathbb{N}$. Choose $k \in \mathbb{N}$ s.t.

$$k > \frac{n(c+1)^n}{a - c^n}$$

Then

$$\left(c + \frac{1}{k}\right)^n < a.$$

This implies that $s = \sup$ A cannot be s.t. $s^n < a$ because that would contradict it being an upper bound. Therefore s is s.t. $s^n \geq a$. It follows that s is s.t. $s^n = a$. s must be unique because if s' is a different positive root,

$$s' < s \text{ or } s < s'$$

implies that

$$(s')^n < a \text{ or } (s')^n > a.$$