1) Prove or Disprove

The identity element is 0. But 1 has no inverse. Not a group.

Table 1

+	0	1
0	0	1
1	1	2

Not a group. The identity is 1 but 0 has no inverse.

Table 2

x	0	1
0	0	0
1	0	1

Is a group. The identity is b.

Table 3

*	a	b
a	b	a
b	a	b

Is a group. The set of even integers is the identity.

Table 4: GGG

+	EVEN	ODD
EVEN	EVEN	ODD
ODD	ODD	EVEN

2) a) $(\mathbb{Q}^*, \cdot)$ **Group**

Associative because multiplication of $\mathbb{R}$ numbers is associative. The identity is 1 and is in the set. $\forall \frac{a}{b} \in \mathbb{Q}*$ (note $a \neq 0$), the inverse is $\frac{b}{a}$.

b) $(\mathbb{I}, +)$, where $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}^*$ **Group**

Addition is associative because $\mathbb{I} \subseteq \mathbb{R}$ and addition of real numbers is associative. By definition of $\mathbb{I}$ above, $0 \in \mathbb{I}$ and is the identity. For every irrational number $g \in \mathbb{I}$, the additive inverse $-g = (-1)g$ is also a irrational number and is therefore in $\mathbb{I}$. So $(\mathbb{I}, +)$ is a group.

c) $(3\mathbb{Z}, +)$ **Group**

The set is a subset of $\mathbb{Z}$ so is associative. The identity, 0, is in $\mathbb{Z}$. The inverse of each element, a, is $-a$ and is in the set.

3) a) $\mathbb{Z}_4$ and the symmetries of a rectangle each have for elements but they are not the same group because each element of the rectangle's symmetry is its own inverse while each of $\mathbb{Z}_4$'s elements is not.

Table 5: Rectangle

	R_0	R_{180}	H	V
R_0	R_0	R_{180}	H	V
R_{180}	R_{180}	R_0	V	H
H	H	V	R_0	R_{180}
V	V	H	R_{180}	R_0

Table 6: $\mathbb{Z}_4$

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

b) $U(12)$ and the rectangle symmetry group are the same. Notice that every element is its inverse. There's a correspondence between elements of the two tables:

$$R_0 \mapsto 1 \ : R_{180} \mapsto 5 \ : H \mapsto 7 \ : V \mapsto 11$$

Table 7: $U(12)$

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

4) $G = \{3^m 6^n | m, n \in \mathbb{Z}\}$. Let

$$a = 3^m 6^n \ : b = 3^o 6^p \ : c = 3^q 6^r$$

Associative Property:

$$(ab)c = a^{(m+o)+q} b^{(n+p)+r} = a^{m+(o+q)} b^{n+(p+r)} = a(bc)$$

Identity Element is $1 = 3^0 6^0$:

$$a \cdot 1 = 3^{m+0} 6^{n+0} = 3^m 6^n = a$$

Inverse: $\forall\, a = 3^m 6^n$, the element $a^{-1} = 3^{-m} 6^{-n}$ is in the set and is its inverse.

5) a) Let $A, B \in GL_2(\mathbb{R})$. Since $det(AB) = det(A)det(B) = 1 \cdot 1$, $AB \in GL_2(\mathbb{R})$. Thus, $GL_2(\mathbb{R})$ is closed under matrix multiplication.

b) Matrix multiplication is associative. The components are equal.

$$\begin{aligned}
[(AB)C]_{mn} &= \sum_k (AB)_{mk} C_{kn} \\
&= \sum_k sum)i A_{mi} B_{ik} C_{kn} \\
&= \sum_i A_{mi} (\sum_k B_{ik}) C_{kn} \\
&= \sum_i A_{mi} (BC)_{in} \\
&= [A(BC)]_{mn}
\end{aligned}$$

c) From linear algebra, a matrix is invertible iff its determinant is non-zero. Moreover,

$$det(A^{-1}) = \frac{1}{det(A)}.$$

Therefore each element in $GL_2(\mathbb{R})$ has an inverse in the set.

d) Its not Abelian. Consider scaling a vectors y-component by some factor and rotating it versus rotating it first and then scaling it. We end up with different vectors.

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

6) a) Let $n \in \mathbb{N}$.

$$(ab^n a^{-1})^n = (aba^{-1})(aba^{-1})...(aba^{-1}) = ab^n a^{-1}$$

The inner terms consist of $(a^{-1}a) = I$, leaving n b operating in each other.

b) Let $n \in \mathbb{Z}$, $n < 0$. Note that $-n > 0$.

$$(aba^{-1})^n = ((aba^{-1})^{-1})^{-n}$$

Since the power of the outer parenthesis, $-n$, is positive, we can use part a) for it.

$$\begin{aligned} ((aba^{-1})^{-1})^{-n} &= (ab^{-1}a^{-1})^{-n} && \text{socks-shoe property} \\ &= a(b^{-1})^{-n}a^{-1} \\ &= ab^n a^{-1}. \end{aligned}$$

Since $b^0 = I$, $(ab^n a^{-1})^n = ab^n a^{-1}$ holds for all integers.