2.16.2) If $\mathcal{T}$ and $\mathcal{T}'$ are on X and $\mathcal{T}'$ is strictly finer that $\mathcal{T}$, what can you say about the corresponding subspace on the subset Y of X?

We have that $\mathcal{T} \subsetneq \mathcal{T}'$. Let $Y \subset X$. The corresponding subspace topologies are

$$\mathcal{T}_Y = \{Y \cap U \mid U \in \mathcal{T}\}$$
$$\mathcal{T}'_Y = \{Y \cap U' \mid U' \in \mathcal{T}'\}$$

We'll show that $\mathcal{T}_Y$ is at least contained in $\mathcal{T}'_Y$. But containment in the other direction depends on Y.

Let $\mathcal{O} \in \mathcal{T}_Y$. We have that $\mathcal{O} = Y \cap U$ for some $U \in \mathcal{T}$.
But $\mathcal{T} \subset \mathcal{T}'$ implies $\mathcal{O} = Y \cap U \in \mathcal{T}'_Y \Rightarrow$

$$\boxed{\mathcal{T}_Y \subset \mathcal{T}'_Y.}$$

Now consider,

$$X = \{a, b, c, d, e\},$$

and the following two topoologies on X. We write them without braces for simplicity.

$$\mathcal{T} = \{\emptyset, abcde, abc, cde, c\}$$
$$\mathcal{T}' = \{\emptyset, abcde, abc, cde, c, bc, bcde\}.$$

The two examples show choices of Y that the strictly finer condition doesn't necessarily carry over to subspace topologies.

ex1) $Y = \{bcde\}$

$$\mathcal{T}_Y = \{\emptyset, bcde, bc, cde, c\}$$
$$\mathcal{T}'_Y = \{\emptyset, bcde, bc, cde, c\}$$

ex2) $Y = \{ab\}$

$$\mathcal{T}_Y = \{\emptyset, ab\}$$
$$\mathcal{T}'_Y = \{\emptyset, ab, b\}$$

2.16.3) Consider $Y = [-1, 1]$ as a subspace of $\mathbb{R}$.

(a) $A = \{x \mid \frac{1}{2} < |x| < 1\}$. **Open in $[-1, 1]$ and $\mathbb{R}$.**

$$\begin{aligned} A &= \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \\ &= \left\{\left(-1, -\frac{1}{2}\right) \cap [-1, 1]\right\} \cup \left\{\left(\frac{1}{2}, 1\right) \cap [-1, 1]\right\} \end{aligned}$$

(b) $B = \{x \mid \frac{1}{2} < |x| \leq 1\}$. **Open in** $[-1, 1]$.

$$\begin{aligned} B &= \left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right] \\ &= \left\{\left(-2, -\frac{1}{2}\right) \cap [-1, 1]\right\} \cup \left\{\left(\frac{1}{2}, 2\right) \cap [-1, 1]\right\} \end{aligned}$$

(c) $C = \{x \mid \frac{1}{2} \leq |x| < 1\}$. **Not open in either.**

(d) $D = \{x \mid \frac{1}{2} \leq |x| \leq 1\}$. **Not open in either.**

(e) $E = \{x \mid 0 < |x| < 1,\ \frac{1}{x} \notin \mathbb{Z}_+\}$. **Open in** $[-1, 1]$ **and** $\mathbb{R}$.

$$\begin{aligned} E &= \bigcup_{k=1}^{\infty} \left(\frac{1}{k+1}, \frac{1}{k}\right) \\ &= \bigcup_{k=1}^{\infty} \left(\frac{1}{k+1}, \frac{1}{k}\right) \cap [-1, 1] \end{aligned}$$

2.16.4) Show $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are open maps.

($\Rightarrow$) Let $\mathcal{O} \subset_o X \times Y$ ($\subset_o$ denotes open set). Then $\mathcal{O}$ is the union of some basis elements

$$\mathcal{O} = \bigcup_{\alpha} (U_\alpha \times V_\alpha), \qquad U_\alpha \subset_o X,\ V_\alpha \subset_o Y.$$

$\pi_1(\mathcal{O}) = \bigcup_{\alpha} U_\alpha$ is a union of open sets of X and is therefore open. Similarly,

$\pi_2(\mathcal{O}) = \bigcup_{\alpha} V_\alpha$ is a union of open sets of Y and is therefore open.

2.17.3) Let $A \subset_c X$ and $B \subset_c Y$. Show that $A \times B \subset_c X \times Y$. ($\subset_c$ closed set)

($\Rightarrow$)

$A \times B = (A \times Y) \cap (X \times B) \Rightarrow$

$$\begin{aligned} (X \times Y) - (A \times B) &= (X \times Y) - \{(A \times Y) \cap (X \times B)\} \\ &= \{(X \times Y) - (A \times Y)\} \cup \{(X \times Y) - (X \times B)\} \\ &= \{(X - A) \times Y\} \cup \{X \times (Y - B)\}. \end{aligned}$$

Since $(X - A)$ and $(Y - B)$ are open in their respective spaces, $A \times B$ is open in $X \times Y$.

2.17.6 Let A, B, A_α be subsets of a space X.

(a) Show $A \subset B \to \bar{A} \subset \bar{B}$.

Suppose $A \subset B$ and let $a \in \bar{A}$. Then every neighborhood U_a of a intersets A. If follows that $U_a \cap B \neq \emptyset$. Therefore $a \in \bar{B} \Rightarrow$

$$\boxed{\overline{A} \subset \overline{B}}.$$

(b) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

($\Rightarrow$) Let $x \in \overline{A \cup B}$. Then for all neighborhood U_x of x,

$$U_x \cap (A \cup B) \neq \emptyset.$$

This tells us that $U_x \cap A \neq \emptyset$ or $U_x \cap B \neq \emptyset$. Therefore, $x \in \overline{A}$ or $x \in \overline{B} \Rightarrow x \in \overline{A} \cup \overline{B}$. So that

$$\overline{A \cup B} \subset \overline{A} \cup \overline{B}.$$

($\Leftarrow$) Let $x \in \overline{A} \cup \overline{B}$ Then $\forall$ neighborhood U_x of x,

$$U_x \cap A \neq \emptyset \text{ or } U_x \cap B \neq \emptyset.$$

$A, B \subset A \cup B \Rightarrow U_x \cap (A \cup B) \neq \emptyset$. Thus,

$$\overline{A} \cup \overline{B} \subset \overline{A \cup B}.$$

Therefore,

$$\boxed{\overline{A \cup B} = \overline{A} \cup \overline{B}}.$$

(c) Show $\overline{\bigcup A_\alpha} \supset \bigcup \overline{A_\alpha}$. Give an example where equality fails.

Let $x \in \bigcup \overline{A_\alpha}$ and denote U_x an open set containing x.
We have $x \in \overline{A_\alpha}$ for some $\alpha \Rightarrow\ \forall\, U_x, U_x \cap A_\alpha \neq \emptyset$

$$\begin{aligned} &\Rightarrow U_x \cap (\bigcup A_\alpha) \neq \emptyset \\ &\Rightarrow x \in \overline{\bigcup A_\alpha} \\ &\Rightarrow \bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}. \end{aligned}$$

ex) Let $A_\alpha = \left(\frac{1}{\alpha}, 1\right) \Rightarrow \overline{A_\alpha} = [\frac{1}{\alpha}, 1]$.

The union $\bigcup A_\alpha = (0,1) \Rightarrow \overline{\bigcup A_\alpha} \ni 0$.

But $0 \notin \overline{A_\alpha}\ \forall\, \alpha \Rightarrow 0 \notin \bigcup \overline{A_\alpha}$.

Thus equality fails here.

2.17.7) Criticize the proof of Problem 2.17.7.

The problem is with the assumption that every neighborhood of x intersects the union implies that, that neighborhood always intersects the same set. i.e.

$$U_x \cap \bigcup A_\alpha \neq \emptyset \Rightarrow U_x \cap A_\alpha \neq \emptyset, \quad \textit{some } \alpha.$$

α may depend on the neighborhood, $\alpha = \alpha(U_x)$.

2.17.9) Let $A \subset X, B \subset Y$. Show that in space $X \times Y$,

$$\overline{A \times B} = \overline{A} \times \overline{B}.$$

($\Rightarrow$) Suppose $(x, y) \in \overline{A \times B}$. We know that for all open sets containing (x, y), $\mathcal{O}_{(x,y)}$,

$$\mathcal{O}_{(x,y)} \cap (A \times B) \neq \emptyset.$$

Let $x \in U_x, y \in V_y$, both open sets in their respective spaces. We have that the product contains (x, y) and is open in $X \times Y$

$$(x, y) \in (U_x \times V_y) \subset_o X \times Y.$$

Hypothesis implies

$$\begin{aligned} &\Rightarrow (U_x \times V_y) \cap (A \times B) \neq \emptyset \\ &\Rightarrow (U_x \cap A) \times (V_y \cap B) \neq \emptyset \\ &\Rightarrow (U_x \cap A) \neq \emptyset \textit{ and } (V_y \cap B) \neq \emptyset. \end{aligned}$$

Since U_x and V_y were arbitrary open sets, this implies $x \in \overline{A}$ and $y \in \overline{B}$. Therefore, $(x, y) \in \overline{A} \times \overline{B}$ and

$$\boxed{\overline{A \times B} \subset \overline{A} \times \overline{B}}$$

($\Leftarrow$) Suppose $(x, y) \in \overline{A} \times \overline{B}$. Then $x \in \overline{A}$ and $y \in \overline{B}$.

Now consider some open $\mathcal{O} \subset X \times Y$ containing (x, y). Then $\exists\ U_x \ni x, V_y \ni y$ (open in X, Y respectively) s.t.

$$(x, y) \in (U_x \times V_y) \subset \mathcal{O}.$$

It follows from the hypotheses that

$$U_x \cap A \neq \emptyset \textit{ and } V_y \cap B \neq \emptyset.$$

$$\begin{aligned} &\Rightarrow (U_x \cap A) \times (V_y \cap B) \neq \emptyset \\ &\Rightarrow (U_x \times V_y) \cap (A \times B) \neq \emptyset \\ &\Rightarrow \mathcal{O} \cap (A \times B) \neq \emptyset. \end{aligned}$$

Since $\mathcal{O}$ is was an arbitrary neighborhood of (x, y), we have that

$$\boxed{\overline{A} \times \overline{B} \subset \overline{A \times B}}.$$

Finally,

$$\boxed{\overline{A \times B} = \overline{A} \times \overline{B}}.$$