1: Recall the Weierstrass approximation theorem which states that for all $\varepsilon > 0$ and all continuous functions on $[0, 1]$, we can find a polynomial P such that

$$\sup_{x\in[0,1]} |f(x) - P(x)| < \varepsilon$$

i.e. every continuous function is a uniform limit of a sequence of polynomials

(a) Show polynomials dense in $L^p([0, 1])$.

Let $f \in L^p([0, 1])$. Then $|f|^p < \infty$ *a.e.* implying$|f| < \infty$ *a.e.* and $f \in L^1([0, 1])$. By Theorem 2.4, continuous functions on compact support are dense in $L^1(\mathbb{R}^d)$. As we are in $[0, 1]$, there exists a sequence of continuous functions $\{f_n\} \to f$:

$$|f_n - f| < \frac{\varepsilon}{2} \;\; \forall\, \varepsilon > 0.$$

By the Weierstrass Approximation Theorem, $\forall\, l f_n$, $\exists$ polynomial $P_n(x)$ s.t.

$$|f_n - P_n| < \frac{\varepsilon}{2} \;\; \forall\, \varepsilon > 0.$$

By the Triangle Inequality,

$$|P_n - f| \leq |P_n - f_n| + |f_n - f| < \varepsilon.$$

Then

$$\begin{aligned}\left(\int_{[0,1]} |P_n - f|^p\right)^{1/p} &< \left(\int_{[0,1]} \varepsilon^p\right)^{1/p} \\ &= (m([0, 1]) \cdot \varepsilon^p)^{1/p} \\ &= \varepsilon.\end{aligned}$$

(b) Show that $L^p([0, 1])$ is separable.

I think the idea of separability is analogous to having a spanning set as in linear algebra. But since we are in a space of functions that are infinite, we can settle with a set that doesn't exactly characterize every function but is at least able to approximate every function, i.e. dense. We also need countability to be able to describe such approximations.
We can let

$$\mathcal{T} = \left\{ \sum_{k=0}^{n} a_k x^k \mid a_k \in \mathbb{Q}, n \in \mathbb{N} \right\}.$$

$\mathcal{T}$ is countable since the coefficients of each term is in $\mathbb{Q}$. This set is dense wrt polynomials as the rational coefficients are dense. Let $a_k \in \mathbb{Q}$, $b_k \in \mathbb{R}$ and observe that we can choose coefficients a_k to make the sum arbitrarily small:

$$\begin{aligned}\left|\sum_{k=0}^{n} a_k x^k - \sum_{k=0}^{n} b_k x^k\right| &= \left|\sum_{k=0}^{n}(a_k - b_k)x^k\right| \\ &\leq \sum_{k=0}^{n}\left|(a_k - b_k)x^k\right| \\ &= \varepsilon.\end{aligned}$$

So, $\forall\ f \in L^p([0,1])$ and $\varepsilon > 0$, the above and part (a) tells us that we can find a polynomial f_n and an element in $q(x) \in \mathcal{T}$ s.t.

$$|f - q| \leq |f - f_n| + |f_n - q| < \varepsilon.$$

Thus, $\mathcal{T}$ is a countable dense subset of $L^p([0,1])$ and is therefore separable.

(c) Show that $L^p(\mathbb{R})$ is separable.

If $f \in L^p(\mathbb{R})$, f is integrable, measurable, and finite-valued on some E with $m(E) < \infty$. By Lusin's Theorem, $\forall\ \varepsilon > 0$, $\exists\ A_\varepsilon \subset E$ closed s.t.

$$m(E \backslash A_\varepsilon) < \varepsilon \text{ and } f|_{A_\varepsilon} \text{ is continuous.}$$

Since A_ε compact, we can extend the results of part (b) from $[0,1]$ to A_ε. Therefore, on A_ε, $\exists\ q \in \mathcal{T}$ s.t.

$$\|q - f\|_p < \varepsilon.$$

This implies that $\mathcal{T}$ is dense in $L^p(\mathbb{R})$ and therefore $L^p(\mathbb{R})$ is separable.

2: Let f be a measurable finite-valued function on $[0,1]$, and suppose that $|f(x) - f(y)|$ is integrable on $[0,1] \times [0,1]$. Show that $f(x)$ is integrable on $[0,1]$.

We would like to apply Theorem 3.1 to $|f(x) - f(y)|$. To do so, we need $|f(x) - f(y)|$ to be integrable on $\mathbb{R} \times \mathbb{R}$. Therefore, let

$$g(x,y) = \chi_{[0,1]}(x)\chi_{[0,1]}(y)\left|f(x) - f(y)\right|.$$

Then $g(x,y)$ integrable on $\mathbb{R}\times\mathbb{R}$. We want to show that $\int_0^1 |f(x)|\,dx < \infty$.

$$\begin{aligned}
\int_0^1 |f(x)|\,dx &= \int_{\mathbb{R}} \chi_{[0,1]}(x)\,|f(x)|\,dx \\
&\leq \int_{\mathbb{R}} \chi_{[0,1]}(x)\,|f(x)-f(y)|\,dx + \int_{\mathbb{R}} \chi_{[0,1]}(x)\,|f(y)|\,dx \\
&= \int_{\mathbb{R}} \chi_{[0,1]}(x)\,g^y(x)\,dx + \int_{\mathbb{R}} \chi_{[0,1]}(x)\,|f(y)|\,dx \\
&= \int_{\mathbb{R}} \chi_{[0,1]}(x)\,g^y(x)\,dx + |f(y)| \\
&< \infty
\end{aligned}$$

; □

3: Suppose f is integrable on $\mathbb{R}^d$. For each $\alpha > 0$, let $E_\alpha = \{x : |f(x)| > \alpha\}$. Prove that

$$\int_{\mathbb{R}^d} |f(x)|\,dx = \int_0^\infty .$$

Let's take a look at the set E from the hint, which is a set of tuples satisfying the condition $|f(x)| \geq t$,

$$E = \{(x,t) : |f(x)| \geq t\}.$$

Then the characteristic function of this set, $\chi_E(x,t)$, evaluates to 1 if $(x,t) \in E$ and 0 otherwise

$$\chi_E(x,t) = \begin{cases} 1 & (x,t) \in E \\ 0 & (x,t) \notin E \end{cases}$$

We can re-write this as

$$\chi_E(x,t) = \begin{cases} 1 & \text{\textit{if}}\ |f(x)| \geq t \\ 0 & \text{\textit{if}}\ |f(x)| < t \end{cases}$$

We can view this function as a slice of t by fixing t

$$\chi_E^t(x) = \begin{cases} 1 & (x,t)\ \text{\textit{if}}\ x \in E_t \\ 0 & (x,t)\ \text{\textit{if}}\ x \notin E_t \end{cases}$$

This implies that t-slice is equivalent to the characteristic function of E_t.

$$\boxed{\chi_E^t(x) = \chi_{E_t}(x)}.$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^d} |f|\,dx &= \int_0^\infty \left(\int_{\mathbb{R}^d} \chi_E(x,t)dx \right) dt \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} \chi_E^t(x)dx \right) dt \\
&= \int_0^\infty \left(\int_{\mathbb{R}^d} \chi_{E_t}(x)dx \right) dt \\
&= \int_0^\infty m\left(\chi_{E_t}(x)\right) dt.
\end{aligned}$$

4: Prove that if $f \in L^1(\mathbb{R}^d)$ and

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \xi} dx,$$

then $\hat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

Using the hint, we write

$$\hat{f}(\xi) = \frac{1}{2} \int_{\mathbb{R}^d} \left[f(x) - f(x-\xi')\right] e^{-2\pi i x \xi} dx, \qquad \xi' = \frac{1}{2}\frac{\xi}{|\xi|^2}$$

Why is there a 1/2 factor when the functions are being subtracted?

We take the absolute value of the integrand as an upper bound

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{2} \int_{\mathbb{R}^d} \left[f(x) - f(x-\xi')\right] e^{-2\pi i x \xi} dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^d} \left|f(x) - f(x-\xi')\right| dx \\
&\leq \|\hat{f}(x) - \hat{f}(x-\xi')\|.
\end{aligned}$$

By Proposition 2.5, $f \in L^1(\mathbb{R}^d) \Rightarrow \|f_{\xi'} - f\|_1 \to 0$ as $\xi' \to 0$. By the definition

$$\xi' = \frac{1}{2}\frac{\xi}{|\xi|^2},$$

$|\xi| \to \infty \Rightarrow \xi' \to 0$. Therefore,

$$|\xi| \to \infty \Rightarrow \|f - f_{\xi'}\| \to 0$$

implying

$$\boxed{|\xi| \to \infty \Rightarrow \hat{f}(\xi) \to 0.}$$

5: If f is integrable on $[0, 2\pi]$, then $\int_0^{2\pi} f(x)e^{-inx}dx \longrightarrow 0$ as $|n| \to \infty$. Show as a consequence that if E is a measurable subset of $[0, 2\pi]$, then

$$\int_E cos^2\left(nx + u_n\right) dx \to \frac{m(E)}{2},\ as\ n \to \infty$$

for any sequence $\{u_n\}$.

($\Rightarrow$) We first use trigonometric identities to rewrite the integrand,

$$\begin{aligned} cos^2(nx + u_n) &= \frac{1}{2}\left[1 + cos(2(nx + u_n))\right] \\ &= \frac{1}{2} + \frac{1}{2}cos(2nx + 2u_n). \end{aligned}$$

Then the integral can be re-written as

$$\begin{aligned} \int_E cos^2(nx + u_n) &= \frac{1}{2}\int_E dx + \frac{1}{2}\int_E cos(2nx + 2u_n)\, dx \\ &= \frac{m(E)}{2} + \frac{1}{2}\int_E cos(2nx + 2u_n)\, dx \\ &= \frac{m(E)}{2} + \frac{1}{2}\int_0^{2\pi} \chi_E(x) cos(2nx + 2u_n)\, dx. \end{aligned}$$

We now want to show that the second term goes to 0 as $|n| \to \infty$. The hypothesis implies that both real and imaginary parts go to 0 as $|n| \to \infty$ for all integrable functions $f(x)$ on $[0, 2\pi]$, i.e.

$$\int_0^{2\pi} f(x)cos(nx)dx \to 0 \tag{1}$$

$$\int_0^{2\pi} f(x)sin(nx)dx \to 0. \tag{2}$$

Using the sum of angles identity for cos and triangle inequality, we have

$$\begin{aligned} \left|\int_0^{2\pi} \chi_E(x)cos(2nx + 2u_n)dx\right| &= \left|\int_E cos(2nx + 2u_n)dx\right| \\ &= \left|\int_E cos(2nx)cos(2u_n) - sin(2nx)sin(2u_n)dx\right| \\ &= \left|cos(2u_n)\int_0^{2\pi} \chi_E(x)cos(2nx)dx - sin(2u_n)\int_0^{2\pi} \chi_E(x)sin(2nx)dx\right| \\ &\le |cos(2u_n)|\left|\int_0^{2\pi} \chi_E(x)cos(2nx)dx\right| + |sin(2u_n)|\left|\int_0^{2\pi} \chi_E(x)sin(2nx)dx\right| \\ &\le 1\cdot\frac{\varepsilon}{2} + 1\cdot\frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

The third line is where we use the (1) and (2), where $\chi_E(x)$ is integrable on $[0, 2\pi]$. $\exists\, N$ s.t. the integrals $\left|\int_0^{2\pi} \chi_E(x)cos(2nx)dx\right| < \frac{\varepsilon}{2}$ and $\left|\int_0^{2\pi} \chi_E(x)sin(2nx)dx\right| < \frac{\varepsilon}{2}$.

Since we have shown that $|n| \to \infty$ implies the second term goes to 0, we have our result.

$$\boxed{\int_E cos^2(nx + u_n) \to \frac{m(E)}{2},\ as\ \ |n| \to \infty.}$$