1 Let A, B, C, and D be sets. Let $f : C \to D$, $g : B \to C$, and $h : A \to B$ and $x \in A$.

$$\begin{aligned}(f \circ (g \circ h))x &= f \circ g(h(x)) \\ &= f(g(h(x))) \\ &= (f \circ g)h(x) \\ &= ((f \circ g) \circ h)x\end{aligned}$$

The composition of functions $f \circ (g \circ h)$ and $(f \circ g) \circ h$. take x to the same element in D for any $x \in A$. Thus, $f \circ (g \circ h) = (f \circ g) \circ h$.

2 There are $3! = 6$ permutations of $A = \{1, 2, 3\}$. We can construct a permutation f by mapping 1 to one of three possible elements. Then map 2 to one of two possible elements and 3 is mapped to the remaining element.

Define I, R, and F as

$$id : 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3$$

$$R : 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$$

and

$$F : 1 \mapsto 2, 2 \mapsto 1, 3 \mapsto 3.$$

		I	R	F	R^2	FR	RF
123	I	I	R	F	R^2	FR	RF
231	R	R	R^2	RF	I	F	FR
213	F	F	FR	I	RF	R	R^2
312	R^2	R^2	I	FR	R	RF	F
321	FR	FR	RF	R^2	F	I	R
132	RF	RF	F	R	FR	R^2	I

The table has the same pattern as the table of triangle symmetries

3 (i) non-square rectangle

We can rotate by 0°, 180°, flip about a horizontal, and flip about a vertical axis to map the rectangle onto itself.

Symmetries: $\{R_{0°}, R_{180°}, H, V\}$

	R_0	R_{180}	H	V
R_0	R_0	R_{180}	H	V
R_{180}	R_{180}	R_0	V	H
H	H	V	R_0	R_{180}
V	V	H	R_{180}	R_0

(ii) The square has 8 symmetries. R^n is a counterclockwise rotation by 90° n-times. D is a flip about a diagonal going through the upper left and bottom right corners.

	I	R	R^2	R^3	D	RD	R^2D	R^3D
I	I	R	R^2	R^3	D	RD	R^2D	R^3D
R	R	R^2	R^3	I	RD	R^2D	R^3D	D
R^2	R^2	R^3	I	R	R^2D	R^3D	D	RD
R^3	R^3	I	R	R^2	R^3D	D	RD	R^2D
D	D	R^3D	R^2D	RD	I	R^3	R^2	R
RD	RD	D	R^3D	R^2D	R	I	R^3	R^2
R^2D	R^2D	RD	D	R^3D	R^2	R	I	R^3
R^3D	R^3D	R^2D	RD	D	R^3	R^2	R	I

4 $F^2(RF)^3F^{-1}(R^2)^{-1}F^3$

$$\begin{aligned}
F^2(RF)^3F^{-1}(R^2)^{-1}F^3 &= I(RF)^2(RF)F^{-1}(R^2)^{-1}F && \text{expand terms, rule 1, 4}\\
&= (RF)^2R(R^2)^{-1}F && \text{rule 1, 4}\\
&= (RF)^2R(R^2)^{-1}F && \text{rule 1, 4}\\
&= (RF)(RF)R(R^2)^{-1}F && \text{expand terms}\\
&= R(FR)(FR)(R^2)^{-1}F && \text{Rule 4 (x2)}\\
&= R(FR)(FR)(R^2)^{-1}F && \text{Rule 4 (x2)}\\
&= R(R^2F)(FR)(R^2)^{-1}F && \text{Rule 2}\\
&= IIR(R^2)^{-1}F && \text{Rule 1}\\
&= R(R^2)^{-1}F && \text{Rule 3}\\
&= RRF && \text{Rule 1,4 imply } (R^2)^{-1} = R\\
&= FR && \text{Rule 2}
\end{aligned}$$

5 (a)

$$\begin{aligned}
(RF)(RF) &= R(F(RF)) && \text{Rule 4}\\
&= R((FR)F) && \text{Rule 4}\\
&= R((R^2F)F) && \text{Rule 2}\\
&= R(R^2I) && \text{Rule 3}\\
&= R^3 && \text{Rule 4}\\
&= I && \text{Rule 1}
\end{aligned}$$

(b)

$$\begin{aligned}
F &= IF && \text{Rule 3}\\
&= R^3F && \text{Rule 1}\\
&= R(R^2F) && \text{expand, Rule 4}\\
&= RFR
\end{aligned}$$

(6) The six symmetries of a triangle expressed in matrix form:

$$
\begin{aligned}
I &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
R &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \\
F &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\
R^2 &= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \\
FR &= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \\
RF &= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}
\end{aligned}
$$

All of them are invertible since the determinant for each is non-zero. They should be invertible since each operation can be undone.