3.27.2) (X, d) metric space; $A \subset X$.

(a) Show that $d(x, A) = 0 \iff x \in \overline{A}$.

($\Rightarrow$) Suppose $x \in \overline{A}$.
Then $\forall\ \epsilon > 0,\ B(x, \epsilon) \cap A \neq \emptyset$. Thus $\exists\ a \in A$ s.t. $d(x, a) < \epsilon$. Since $d(x, a) > 0$, it follows that

$$d(x, A) = inf\{d(x, a) \mid a \in A\} = 0.$$

($\Leftarrow$) Suppose $0 = d(x, A)$.
Then $0 = inf\{d(x, a) \mid a \in A\}$ so that $\forall\ \epsilon > 0$, $\exists\ a \in A$ s.t.

$$0 < d(x, a) < \epsilon.$$

This implies that every $B(x, \epsilon)$ intersects A. Thus $x \in \overline{A}$.

(b) Show that if A is compact, $d(x, A) = d(x, a)$ for some $a \in A$.

$d(x, A) = inf\{d(x, a) \mid a \in A\}$. Then for all $\epsilon > 0$,

$$\exists\ a_n \text{ s.t. } k \leq d(x, a_n) < k + \epsilon.$$

Consequently, $\exists$ sequence $\{a_n\}$ s.t. $\{f(a_n)\} \to d(x, A)$.
A being compact in a metric space implies it is closed and bounded. So there is a convergent subsequence $\{a_{n_k}\} \to a_0$. By Lemma 21.2, $a \in \overline{A}$. Therefore, a_0 is the element of A s.t.

$$d(x, A) = d(x, a_0).$$

(c) Show $U(A, \epsilon) = \bigcup_{a \in A} B_d(a, \epsilon)$.

$$U(A, \epsilon) = \{x \mid d(x, A) < \epsilon\}.$$

($\Rightarrow$) Let $x \in U(A, \epsilon)$ and define the distance of x to A to be $\delta = d(x, A)$. We have that the following inequality holds

$$\delta < \epsilon.$$

It follows that $\exists$ some point $a_x \in A$ s.t.

$$\delta \leq d(x, a_x) < \epsilon,$$

implying that $x \in B_d(a_x, \epsilon)$. Thus

$$U(A, \epsilon) \subset \bigcup_{a \in A} B_d(a, \epsilon).$$

($\Leftarrow$) Now let $u \in \bigcup_{a \in A} B_d(a, \epsilon)$. Then u is contained is some $B_d(a, \epsilon)$ so that

$$d(u, a) < \epsilon.$$

But $d(u, a) \geq d(u, A)$. Chaining the inequalities together

$$d(u, A) \leq d(u, a) < \epsilon$$

tells us that $u \in U(A, \epsilon)$; i.e. u is in the ϵ-neighborhood of A. Thus $\bigcup_{a \in A} B_d(a, \epsilon) \subset U(A, \epsilon)$. Finally,

$$\bigcup_{a \in A} B_d(a, \epsilon) = U(A, \epsilon).$$

(d) Suppose A compact; $U \supset A$ open (U contains A). Show there exists some $U(A, \epsilon)$ contained in U.

($\Rightarrow$) We know that A is closed implying that $X - A$ is open and $\overline{A} = A$.
For all $u \in U - A$,

$$d(u, A) > 0 \text{ since } u \notin \overline{A}.$$

We know that $d(u, A)$ is continuous so that some $\exists$

$$\delta := min\{d(u, A) \mid u \in U - A\}.$$

We now show that this δ neigborhood of A is contained in U

$$A \subset U(A, \delta) \subset U.$$

Let $v \in U(A, \delta)$. Letting $\delta' = min\{d(v, A), \delta - d(v, A)\}$, we have that

$$B_d(v, \delta') \subset U.$$

(e) Show (d) need not hold if A closed but not compact.
In the discrete topology, every set is open and every set is closed. So $(0, 1)$ is closed in this topology but is not compact. With $U = (0, 1)$, there is no $(U, \epsilon > 0)$ s.t.

$$(0, 1) \subset ((0, 1), \epsilon) \subset (0, 1).$$

3.28.6) Let (X, d) compact metric space, $f : X \to X$. isometry. Show f is bijective and hence homeomorphism.

f is continuous by choosing $\delta = \epsilon$ and using the metric version of $\epsilon - \delta$ continuity.

$$d(f(x), f(y)) = d(x, y) < \delta = \epsilon.$$

We now show f is injective by contrapositive. By definition, f injective if

$$f(a) = f(b) \longrightarrow a = b.$$

So,

$$\begin{aligned}(a \neq b) &\Rightarrow d(a,b) \geq \epsilon \\ &\Rightarrow d(f(a), f(b)) \geq \epsilon \\ &\Rightarrow f(a) \neq f(b).\end{aligned}$$

Now we show f is surjective by contradiction. Let $a \notin f(X)$. Then $\exists\, B(a,\epsilon)$ disjoint from $f(X)$. Create a sequence

$$\begin{aligned}x_1 &= a \\ x_2 &= f(x_1) \\ &\vdots \\ x_{n+1} &= f(x_n).\end{aligned}$$

Now we note that for all $n \neq m$, $d(x_n, x_m) < \epsilon \Rightarrow d(x_1, x_{m-n+1}) < \epsilon$. This contradicts that $B(a,\epsilon) = B(x_1,\epsilon)$ is disjoint from $f(X)$. Thus,

$$d(x_n, x_m) \geq \epsilon \;\forall\; n \neq m.$$

Since X is limit point compact, $x_n \to x$ for some x. But no $B(x,\epsilon)$ can contain infinitely many points because of above. Therefore, we have a contradiction and f is therefore surjective.

3.28.7) Let (X,d) metric space.

(a) f contraction, X compact; show f has unique fixed point.
($\Rightarrow$) f contraction

$$d(f(x), f(y)) \leq \alpha d(x,y)$$

implies f is continuous and X metric implies X Hausdorff.

By hypothesis of X compact, we have that $f(X)$ is closed and also compact. It follows that $f^n(X)$ is closed and compact $\forall\, n$.

Define $A_n = f^n(X)$. Since images under f are nested

$$f^{n+1}(X) \subset f^n(X),$$

so that $A = \bigcap\limits_{n=1}^{\infty} f^n(X)$ is non-empty.

The diameter $d\{f^n(X)\} \leq \alpha^n d\{X\}$ implies points of A are arbitrarily close. Therefore A contains one point.

$$A = \{a\}.$$

How can I show a is a fixed point?

(b) f shrinking map and X compact, show f has unique fixed point.

f is continuous since

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y) < d(x, y).$$

Let $x \in A$ and define x_n s.t. $x = f^{n+1}(x_n)$. x^n is $n+1$ hops away from x under f.

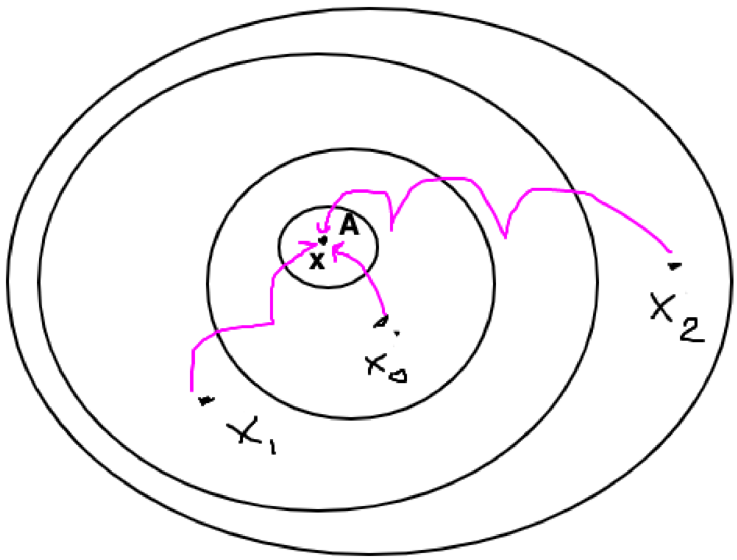

Figure 1: Caption

I'm stuck.

(c) Show $f(x)$ is a shrinking map but not contraction on $X = [0, 1]$.

$$f(x) = x - \frac{x^2}{2}.$$

(⇒) f is a shrinking map:

$$\begin{aligned}
d(f(a), f(b)) &= \left|\left(a - \frac{a^2}{2}\right) - \left(b - \frac{b^2}{2}\right)\right| && (1)\\
&= \left|(a-b) - \frac{1}{2}(a^2 - b^2)\right| && (2)\\
&= \left|(a-b)\left[1 - \frac{1}{2}(a+b)\right]\right| && (3)\\
&= |a-b|\left|1 - \frac{1}{2}(a+b)\right| && (4)\\
&< |a-b| && (5)\\
&= d(a,b). && (6)
\end{aligned}$$

But f is not because of line (4). Since $a \neq b$, $a > b$ WLOG and $\frac{1}{2}(a+b)$ never 0 or 1. Thus,

$$\left|1 - \frac{1}{2}(a+b)\right| \in (0,1).$$

The above term can be made arbitrarily close to 1 implying no such α exists to make f a contraction.

3.29.1) Show $\mathbb{Q} \subset \mathbb{R}$ not locally compact.

Consider the set $[a,b] \cap \mathbb{Q}$. This set is closed in $\mathbb{Q}$ as it is the intersection of $\mathbb{Q}$ and a closed set in $\mathbb{R}$. Because rationals and irrationals are dense, we can construct a sequence in $[a,b]$ that converges to an irrational number. But such sequence doesn't converge in $[a,b] \cap \mathbb{Q}$ implying it is not sequentially compact. Therefore, sets of the form

$$[a,b] \cap \mathbb{Q}$$

are not compact.

Now, let $x \in \mathbb{Q}$ and suppose $\exists$ compact subset $C \subset \mathbb{Q}$ containing some neighborhood $(c,d) \cap \mathbb{Q}$ of x. i.e.,

$$(c,d) \cap \mathbb{Q} \subset C.$$

Density of irrationals implies that some $[a,b] \subset (c,d)$ contains x. Consequently, C contains the closed set $[a,b] \cap \mathbb{Q}$ implying that

$$[a,b] \cap \mathbb{Q}$$

is compact, which is a contradiction. Thus, $\mathbb{Q}$ is not locally compact at any point and therefore $\mathbb{Q}$ not locally compact.

3.29.2) $\{X_\alpha\}$ indexed family of non-empty spaces.

(a) Suppose $\prod X_\alpha$ is locally compact.

First show each X_α is locally compact. Let $x_\alpha \in X_\alpha$. Then $\pi^{-1}(x) \in \prod X_\alpha$ and by hypothesis

$$C \supset U_x \ni \pi^{-1}(x_\alpha)$$

is some compact subspace containing some neighborhood of x. Under the projection map, which is continuous, we have

$$\pi_\alpha(C) \supset \pi_\alpha(U_x) \ni x_\alpha.$$

Thus X_α is locally compact $\forall\ \alpha$.