Q32a) Let $\mathcal{N}$ denote the non-measurable subset of $I = [0,1]$ constructed at the end of Section 3. Prove that if E is measurable, then $m(E) = 0$.

$(\Rightarrow)$

We can begin by discussing the set $\mathcal{N}$. An equivalence class is first defined on $[0,1]$ by

$$x \sim y \quad if\ x - y \in \mathbb{Q}.$$

The equivalence classes form a partition of $[0,1]$. We can see that the subset of rationals

$$\mathbb{Q} \cap [0,1]$$

forms one class. The remaining classes are formed by irrational numbers with their rational translations

$$[\alpha] = \{\alpha + p \mid \alpha \in \mathbb{I}, p \in \mathbb{Q}\}.$$

The set $\mathcal{N} = \{x_\alpha\}$ is constructed by taking one representative from each equivalence class. It is shown in the text that rational translations do not overlap

$$\mathcal{N}_k = \mathcal{N} + r_k$$

are all disjoint. Here, $\{r_k\}_{n=1}^{\infty}$ is an enumeration of $\mathbb{Q} \cap [0,1]$. Therefore any translations of E_k of $E \subset \mathcal{N}$ also do not overlap.

Now suppose that $m(E)$ exists. We show that $m(E) = 0$. Following the book's proof, consider all of the rational translation so that

$$E \subset \bigcup_{k=1}^{\infty} E_k \subset [-1,2].$$

Since the translations are disjoint, their measures sum so that

$$0 \leq \sum_{k=1}^{\infty} m(E_k) \leq 3.$$

But $m(E_k) = m(E)$ as the translations do not affect measure. Therefore,

$$0 \leq \sum_{k=1}^{\infty} m(E) \leq 3.$$

This implies $m(E) = 0$ because if $m(E) > 0$, then the sum is infinite.

Q33) Let $\mathcal{N}$ denote the non-measurable set constructed in the text (and used in the previous problem). Show that the set $\mathcal{N}^c = I - \mathcal{N}$ satisfies $m_*(\mathcal{N}^c) = 1$ and conclude that if $E_1 = \mathcal{N}$, $E_2 = \mathcal{N}^c$, then

$$m_*(E_1) + m_*(E_2) \neq m_*(E_1 \cup E_2)$$

(⇒)

We have that $\mathcal{N}^c \subset I$ implies that $m_* (\mathcal{N}^c) \leq 1$.

Now assume for the sake of contradiction that $m_* (\mathcal{N}^c) < 1$. Then there exists some $\varepsilon > 0$ and $\mathcal{U}$ s.t.

$$\mathcal{N}^c \subset \mathcal{U} \subset I$$

and

$$m_* (\mathcal{U}) < 1 - \varepsilon.$$

The set $\mathcal{U} - \mathcal{N}^c \subset \mathcal{N}$ has a measure 0 by the previous exercise as it is a subset of $\mathcal{N}$. But $\mathcal{N}$ and $\mathcal{N}^c$ are disjoint, implying

$$\begin{aligned} I &= \mathcal{N} \cup \mathcal{N}^c \\ m(I) &= m(\mathcal{N}) + m(\mathcal{N}^c) \\ 1 &\leq m(\mathcal{U}) + 0 \\ 1 &\leq 1 - \varepsilon \end{aligned}$$

which is a contradiction since $\varepsilon > 0$. Thus $m_* (\mathcal{N}^c) = 1$.

The second part of this question is a bit confusing. $\mathcal{N}$ is not measurable but if it was then the proof in the book shows $m(\mathcal{N}) > 0$ so that

$$1 < m_* (E_1) + m_* (E_2) \neq m_* (E_1 \cup E_2) = 1.$$

36) (a) We follow Rudin's proof. Credit to Aswin Rangasamy Venkatesan, we worked on it together but he did more of the heavy lifting.

Let

$$S = \{(p, q) | p, q \in \mathbb{Q}\}$$

be the set of intervals with rational endpoints. As S is constructed from rational endpoints and are of rational length, S is countable.

We perform the following algorithm:

(1) Take a open interval, i.e. $I_k \in S$

(2) Construct a Fat Cantor Set from s and call this A_k

(3) In the process of constructing the above FCS, an interval is removed at the first level. We construct a FCS from this interval and call is B_k.

The two Fat Cantor sets constructed are clearly disjoint, $A_k \cap B_k = \emptyset$.

We now begin by choosing $I_1 = (0, 1) \in S$ and follow the above algorithm. Let $C_1 = A_1 \cup B_1$. The set $I_1 - C_1$ now contains some $s_2 \in S$. We perform the algorithm above on s_2 and let $C_2 = A_2 \cup B_2$. Continue on with this process.

Note that the C_k are disjoint from each other, i.e. $(A_k \cup B_k) \cap (A_{k-1} \cup B_{k-1})$ $\forall\, k$.

It follows that

$$A = \bigcup_{k=1}^{\infty} A_i \text{ and } B = \bigcup_{k=1}^{\infty} B_i$$

is CTDP.

Now, to complete the problem, any open $I \subset [0,1]$ contains an $s \in S$ so that $I \cap A \neq \emptyset$ and $I \cap B \neq \emptyset$ and

$$m(I \cap A) > 0 \text{ and } m(I \cap B) > 0.$$

Q1) Show that the Cantor Set $\mathcal{C}$ is constructed in the text is totally disconnected and perfect.

(a) Let $x, y \in \mathcal{C}$ be distinct points so that $|x - y| > \left(\frac{1}{3}\right)^k$ for some k. It follows that at some level k, x and y are in disjoint intervals,

$$x \in \sigma_1 \text{ and } y \in \sigma_2, \text{ where } \sigma_i \in \Sigma^k.$$

Thus $\mathcal{C}$ is totally disconnected.

Now, let $x \in \mathcal{C}$ and $\varepsilon > 0$. $\exists\ k$ s.t.

$$B_{(1/3)^{k-1}}(x)$$

contains some $y \in \mathcal{C}$. Thus x is not an isolated point.