7.45.1) If X_n metrizable with d_n, then

$$D(\mathbf{x}, \mathbf{y}) = sup \left\{ \frac{\overline{d_i}(x_i, y_i)}{i} \right\}$$

is a metric for $X = \prod X_n$. Show X totally bounded if X_n totally bounded under d_n.

We'll consider $0 < \epsilon < 1$ because $\epsilon \geq 1$ is a trivial. For $B(x, 1)$ covers all of X.

Consider $B_D(\mathbf{x}, \epsilon)$ and take N to be s.t. $\frac{1}{N} < \epsilon$. Observe that for all

$$\alpha = \{1, 2, ..., N - 1\},$$

$\frac{1}{\alpha} \geq \epsilon$.
We have that for all α,

$$\frac{\overline{d_\alpha}(x_\alpha, y_\alpha)}{\alpha} < \epsilon \leq \frac{1}{\alpha}.$$

It follows that

$$\overline{d_\alpha}(x_\alpha, y_\alpha) < 1$$

and therefore

$$d_\alpha(x_\alpha, y_\alpha) < 1.$$

Since each X_n is totally bounded, there are finite $B_{d_\alpha}(x_\alpha, \epsilon)$ covering X_α. The remaining X_n, $(n \neq \alpha)$, are covered by one ϵ-ball. Let N_α be the number of ϵ-balls covering X_α.

We then have that there are a finite maximum of $\prod_\alpha N_\alpha$ open sets covering X.

7.46.1) Show that the set $B_C(f, \epsilon)$ form a basis for a topology on Y^X.

(a) We can construct a $B_C(f, \epsilon)$ containing $f \in Y^X$.

(b) Consider some $h \in B_C(f_1, \epsilon_1) \cap B_C(f_2, \epsilon_2)$.

Observe that $\exists$ some positive

$$\delta_i := \epsilon_i - sup\{d(h(x), f_i(x)) \mid x \in C\}$$

so that

$$B_C(h, \delta_i) \subset B_C(f_i, \epsilon_i).$$

Letting $\delta = min_i\{d_i\}$, we have that

$$B_C(h, \delta) \subset B_C(f_1, \epsilon_1) \cap B_C(f_2, \epsilon_2).$$

Thus, $B_C(f,\epsilon)$ satisfies being a basis for a topology on Y^X.

7.46.2) X space; (Y,d) metric space. For Y^X,

$$(uniform) \supset (compact\ convergence) \supset (pointwise\ convergence).$$

If X compact, first two coincide. If X discrete, second two coincide.

The proof will be in steps.

(a) Compact Convergence $\subset$ Uniform.

Let $B_C(f,\epsilon)$ be a basis element in the Compact Convergence Topology containing $f \in Y^X$.

$$B_C(f,\epsilon) = sup\{d(f(x),g(x)) \mid x \in C\} < \epsilon.$$

Let $\delta = min\{\epsilon, 1\}$. Then

$$\overline{\rho}(f,g) = sup\{\overline{d}(f(x),g(x)) \mid x \in C\} < \delta$$

implies

$$\overline{\rho}(f,g) = sup\{d(f(x),g(x)) \mid x \in C\} < \delta < \epsilon.$$

Thus

$$B_{\overline{\rho}}(f,\delta) \subset B_D(f,\epsilon).$$

(b) Pointwise Convergence $\subset$ Compact Convergence.

Let

$$S = \bigcap_{k=0}^{n} S(x_k, U_k)$$

be a finite intersection of sub-basis elements for $x_k \in C$. S is therefore a basis element of Pointwise Convergence Topology containing f.

$$f \in S.$$

We have that $f(x_k) \in U_k$. Let δ_k be s.t. $B_C(f,\delta_k) \subset U_k$ and

$$\delta = min\{d_k\}.$$

Then

$$B_C(f,\delta) \subset S.$$

(c) X compact $\rightarrow$ Compact Convergence $\supset$ Uniform.

Let $\delta < \epsilon < 1$ so that $\overline{d} = d$. It follows that on all compact C,

$$B_C(f,\delta) \subset B_\rho(f,\epsilon).$$

(d) X discrete $\to f \in \{f\} \subset B_C(f, \epsilon)$.
So Pointwise Convergence $\supset$ Compact Convergence.

7.46.3) Show that the set $\mathcal{B}(\mathbb{R}, \mathbb{R})$ is closed in the uniform topology but not in the topology of compact convergence.

($\Rightarrow$) Let f be a limit point of $\mathbb{R}^{\mathbb{R}}$ under the uniform topology. Then $\exists$ sequence $(f_n) \to f$, where each f_n is bounded. Choose N s.t.

$$\overline{\rho}(f_N, f) < \frac{1}{2}$$

with $diam(f_N) = M$. It follows that

$$d(f_N(x), f(x)) < \frac{1}{2}$$

since $\overline{d} = d$ for $\epsilon < 1$.

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f(y)) \\ &< \frac{1}{2} + M + \frac{1}{2} \\ &= M + 1. \end{aligned}$$

Thus $f \in \mathcal{B}(\mathbb{R}, \mathbb{R})$ implying $\mathcal{B}(\mathbb{R}, \mathbb{R})$ is closed.

Now consider the function

$$\begin{aligned} f(x) &= sinh(x) \\ &= \frac{e^x - e^{-x}}{2} \end{aligned}$$

and the family of functions

$$f_n(x) = \frac{e^x - e^{-x}}{e^{\frac{x}{n}} + e^{-\frac{x}{n}}}.$$

The function $f_n(x)$ is a modified version of

$$tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

which is bounded. So that each $f_n(x) \in \mathcal{B}(\mathbb{R}, \mathbb{R})$.

Observe that on any compact interval C,

$$f_n|_C \to f|_C$$

since the denominator of f_n approaches 2.

By Theorem 46.2, $f_n \to f$. But f is not bounded as it behaves like $e^{\pm x}$ as $x \to \infty$ and therefore $\mathcal{B}(\mathbb{R}, \mathbb{R})$ is not closed.

7.46.4) $f_n(x) = \dfrac{x}{n}$. Which topologies does it converge in?

(a) Uniform Topology NO
$f_n(x)$ is not bounded $\forall\, n$. So no neighborhood of $\mathbb{R}^{\mathbb{R}}$ contain infinitely many f_n.

(b) Compact Convergence Topology YES
Be restricting x-values to compact sets, i.e. closed intervals, the sequence

$$f_n|_C \to 0.$$

By Theorem 46.2, $f_n \to f$.

(c) Pointwise Convergence Topology YES
$\forall x \in X,\ f_n(x) \to 0.$

For which topologies does

$$f_n(x) = \frac{1}{n^3\left[x - \dfrac{1}{n}\right]^2 + 1}$$

converge?

(a) Uniform Topology
Maybe

(b) Compact Convergence Topology
Yes. For $x \in [a, b]$,

$$\begin{aligned}\frac{1}{n^3\left[x - \dfrac{1}{n}\right]^2 + 1} &\longrightarrow \frac{1}{n^3\,[x]^2 + 1}\\ &\longrightarrow \frac{1}{\infty}\\ &\longrightarrow 0.\end{aligned}$$

(c) Pointwise Convergence Topology
Yes. Becuase Compact Convergence Topology is finer than this one.

7.46.5) Consider $f_n : (-1, 1) \to \mathbb{R}$, defined by

$$f_n(x) = \sum_{k=1}^{n} kx^k.$$

(a) Show (f_n) converges under compact convergence topology.
($\Rightarrow$) For all $x \in C \subset (-1, 1)$,

$$\begin{aligned} f_n(x) &= \sum_{k=1}^{n} kx^k \\ &\leq \sum_{k=1}^{n} nx^k. \end{aligned}$$

These are partial sums of the geometric series so we know they converge on out interval. So $f_n \to f$.

(b) Show (f_n) doesn't converge in the uniform topology.
($\Rightarrow$) The supremum of x^k is 1. So for all $\epsilon < 1$, we can find an x s.t. $x^k > \epsilon$. Thus, no ϵ ball contains f_n.

7.51.1) Show that if $h, h' : X \to Y$ are homotopic and $k, k' : Y \to Z$ are homotopic, then $k \cdot h$ and $k' \cdot h'$ are homotopic.

($\Rightarrow$) Let

$$F(x,t) = \begin{cases} F(x,0) = h(x) \\ F(x,1) = h'(x) \end{cases}$$

and

$$G(x,t) = \begin{cases} G(y,0) = k(y) \\ G(y,1) = k'(y) \end{cases}$$

$H(x,t) = G(F(x,t),t) : X \times I \to Z$ is a homotopy between $k \circ h$. We can see that,

$$\begin{aligned} H(x,0) &= G(F(x,0),0) \\ &= G(h(x),0) \\ &= k(h(x)) \\ &= (k \circ h)(x) \end{aligned}$$

and

$$\begin{aligned} H(x,1) &= G(F(x,1),1) \\ &= G(h'(x),1) \\ &= k'(h'(x)) \\ &= (k' \circ h')(x) \end{aligned}$$

7.51.2) $[X,Y]$ denote the set of homotopy classes of maps of X into Y.

(a) Let $I = [0,1]$. Show that $\forall\, X$, the set $[X,I]$ has a single element.

For any continuous map $f : X \to I$, we can deform it to 0,

$$F(x,t) = tf(x).$$

So that all continuous maps are homotopic to $g(x) = 0$ and therefore there is only one equivalence class in the set

$$[X,I].$$

(b) Show that if Y is path connected, the set $[I,Y]$ has a single element.

Consider any two paths $h,g : I \to Y$. We can deform one to the other with a line straight line Homotopy:

$$H(x,t) = t \cdot h(x) + (1-t)g(x).$$

Therefore there is only one equivalence class.

7.51.3) X contractible if $i_X : X \to X$ is nullhomotopic.

(a) Show I and $\mathbb{R}$ homotopic. Let

$$F(x,t) = t \cdot i_X(x).$$

(b) Show contractible space is path connected.
For $x, y \in X$, define the homotopy

$$\begin{aligned} F(x,t) &= (1-t) \cdot i_X(x) + t \cdot 0 \\ G(y,t) &= t \cdot i_X(y) + (1-t) \cdot 0 \end{aligned}$$

Then the composition is a path from x to y going through 0.

(c) -

(d) -