E1: Let $(a,b) = \bigcup_{i=1}^{n}(x_i, y_i) \in \mathbb{R}$ and $f_i : (x_i, y_i) \to \mathbb{R}$ be uniformly continuous functions satisfying the compatibility condition

$$f_i\big|_{(x_i,y_i)\cap(x_j,y_j)} = f_j\big|_{(x_i,y_i)\cap(x_j,y_j)}.$$

Let $f : (a,b) \to \mathbb{R}$ be given by gluing functions together. We'll show that f is uniformly continuous.

($\Rightarrow$) Let $\epsilon > 0$. Then $\forall i \in [i, n]$,

$$\exists\, \delta_i > 0\; |x - y| < \delta_i \Rightarrow |f_i(x) - f_i(x)| < \epsilon.$$

Since $n < \infty$, there exists a smallest element of $\{\delta_i\}$ so that we can let

$$\delta = min\{\delta_i \mid i \in [1, n]\}.$$

Thus, $\forall\ x, y \in (a,b)\ s.t.\ |x - y| < \delta$, we have that

$$|f(x) - f(y)| = |f_i(x) - f_i(y)| < \epsilon,$$

for some interval i containing x, y. Therefore f is uniformly continuous.

E2: Let $(a,b) = \bigcup_{i=1}^{\infty}(x_i, y_i) \in \mathbb{R}$ and $f_i : (x_i, y_i) \to \mathbb{R}$ be uniformly continuous functions satisfying the compatibility condition

$$f_i\big|_{(x_i,y_i)\cap(x_j,y_j)} = f_j\big|_{(x_i,y_i)\cap(x_j,y_j)}.$$

Let $f : (a,b) \to \mathbb{R}$ be given by gluing functions together. We'll show by counterexample that f is not necessarily uniformly continuous.

($\Rightarrow$) Let $f_i(x) = \frac{1}{x}$ *and* $(x_i, y_i) = (\frac{1}{i}, 1)$. We see that $(0,1)$ is covered by the union. $(0,1) = \bigcup_{i=1}^{\infty}(\frac{1}{i}, 1)$.

We show that for any $\epsilon > 0$, each interval requires a different delta and no minimum exists. We can see this by first considering some $m, n \in (\frac{1}{i}, 1)$ for some i. Note that

$$\frac{1}{i} < m, n < 1 \Rightarrow \frac{1}{m}, \frac{1}{n} < 1.$$

Then

$$\begin{aligned}
\left|f(m) - f(n)\right| &= \left|\frac{1}{m} - \frac{1}{n}\right| \\
&= \left|\frac{n-m}{nm}\right| \\
&< \left|n - m\right| i^2 \\
&< \epsilon \quad \text{if } \delta_i = \frac{\epsilon}{i^2}.
\end{aligned}$$

The set $\{\delta_i\}$ has no minimum which implies that the is no δ that guarantees

$$\left|x - y\right| < \delta \longrightarrow \left|f(x) - f(y)\right| < \epsilon.$$

Thus f is not uniformly continuous.

E3: (Munkres 1.2.1) Let $f : A \to B, \quad A_0 \subset A$ and $B_0 \subset B$.

a) Show $A_0 \subset f^{-1}(f(A_0))$ and equality holds if f is injective.

($\Rightarrow$) Let $f(A_0) \subset B$, the inverse is defined to be

$$f^{-1}(f(A_0)) = \{a \in A \mid f(a) \in f(A_0)\} \subset A.$$

Since $\forall\ a \in A_0, f(a) \in f(A_0)$, we have that

$$A_0 \subset f^{-1}(f(A_0)).$$

Now suppose that f is injective. Let $a \in f^{-1}(f(A_0))$. Then $f(a) = b$ for some $b \in f(A_0)$. But $\exists\ a_0 \in A_0 : f(a_0) = b$. By injectivity, $f(a_0) = f(a)$ implies $a = a_0 \in A_0$. Thus $f^{-1}(f(A_0)) \subset A_0$ and

$$A_0 = f^{-1}(f(A_0)).$$

b) Show $f(f^{-1}(B_0)) \subset B_0$ and equality holds if f is surjective.

($\Rightarrow$) Let $b \in f(f^{-1}(B_0))$, then $\exists\ a \in f^{-1}(B_0)$ *s.t.* $b = f(a)$. But $f(a) \in B_0$ implying that $b \in B_0$. Thus,

$$f(f^{-1}(B_0)) \subset B_0.$$

Now suppose f is surjective and let $b \in B_0$ be arbitrary.

The hypothesis implies that exists $a \in A$ *s.t.* $b = f(a)$, implying $a \in f^{-1}(B_0)$ and therefore $f(x) \in f(f^{-1}(B_0))$. But $b = f(a)$ so that $B_0 \subset f(f^{-1}(B_0))$. Finally,

$$B = f(f^{-1}(B_0)).$$

E4: (Munkres 1.2.2) Let $f : A \to B$ and let $A_i \subset A$ and $B_i \subset B$ for $i = 0, 1$.

a) $B_0 \subset B_1 \Rightarrow f^{-1}(B_0) \subset f^{-1}(B_1)$.

($\Rightarrow$) Let $a_0 \in f^{-1}(B_0)$. Then $f(a_0) \in B_0 \subset B_1$. Consequently, $a_0 \in f^{-1}(B_1)$ and $f^{-1}(B_0) \subset f^{-1}(B_1)$.

b) $f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1)$.

($\Rightarrow$) Let $a \in f^{-1}(B_0 \cup B_1)$. Then $f(a) \in B_0 \cup B_1$. Suppose WLOG $f(a) \in B_0$. Then, $a \in f^{-1}(B_0) \subset f^{-1}(B_0) \cup f^{-1}(B_1)$. Therefore

$$f^{-1}(B_0 \cup B_1) \subset f^{-1}(B_0) \cup f^{-1}(B_1).$$

Now let $a \in f^{-1}(B_0) \cup f^{-1}(B_1)$. Suppose WLOG $a \in f^{-1}(B_0)$. Then $f(a) \in B_0 \subset B_0 \cup B_1$ implying $a \in f^{-1}(B_0 \cup B_1)$. We have that

$$f^{-1}(B_0) \cup f^{-1}(B_1) \subset f^{-1}(B_0 \cup B_1).$$

Finally,

$$f^{-1}(B_0 \cup B_1) = f^{-1}(B_0) \cup f^{-1}(B_1).$$

c) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$.

($\Rightarrow$)

$$\begin{aligned} a \in f^{-1}(B_0 \cap B_1) &\Rightarrow f(a) \in (B_0 \cap B_1) \\ &\Rightarrow f(a) \in B_0 \textit{ and } f(a) \in B_1 \\ &\Rightarrow a \in f^{-1}(B_0) \textit{ and } a \in f^{-1}(B_1) \\ &\Rightarrow a \in f^{-1}(B_0 \cap B_1). \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned} a \in (f^{-1}(B_0) \cap f^{-1}(B_1)) &\Rightarrow a \in f^{-1}(B_0) \textit{ and } a \in f^{-1}(B_1) \\ &\Rightarrow f(a) \in B_0 \textit{ and } \; f(a) \in B_1 \\ &\Rightarrow f(a) \in B_0 \cap B_1 \\ &\Rightarrow a \in f^{-1}(B_0 \cap B_1). \end{aligned}$$

d) $f^{-1}(B_0 - B_1) = f^{-1}(B_0) - f^{-1}(B_1)$.

($\Rightarrow$)

$$\begin{aligned} a \in f^{-1}(B_0 - B_1) &\Rightarrow f(a) \in B_0 - B_1 \\ &\Rightarrow f(a) \in B_0, \quad f(a) \notin B_1 \\ &\Rightarrow a \in f^{-1}(B_0), \quad a \notin f^{-1}(B_1) \\ &\Rightarrow a \in f^{-1}(B_0) - f^{-1}(B_1). \end{aligned}$$

($\Leftarrow$) Let $a \in f^{-1}(B_0) - f^{-1}(B_1)$,

$$\begin{aligned} a \in f^{-1}(B_0) - f^{-1}(B_1) &\Rightarrow a \in f^{-1}(B_0), \quad a \notin f^{-1}(B_1) \\ &\Rightarrow f(a) \in B_0, \quad f(a) \notin B_1 \\ &\Rightarrow f(a) \in B_0 - B_1 \\ &\Rightarrow a \in f^{-1}(B_0 - B_1). \end{aligned}$$

e) $A_0 \subset A_1 \Rightarrow f(A_0) \subset f(A_1)$.

Let $b \in f(A_0)$, then $\exists\ a \in A_0 \subset A_1$ s.t. $f(a) = b$. But $b = f(a) \in f(A_1)$. Therefore $f(A_0) \subset f(A_1)$.

f) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.

($\Rightarrow$) Let $b \in f(A_0 \cup A_1)$.

Then $\exists\ a \in A_0 \cup A_1$ s.t. $f(a) \in b$. Suppose WLOG $a \in A_0$, then $b = f(a) \in f(A_0) \subset f(A_0) \cup f(A_1)$.

($\Leftarrow$) Let $b \in f(A_0) \cup f(A_1)$. Suppose WLOG $b \in f(A_0)$.

Then $\exists\ a \in A_0$ s.t. $b = f(a)$. But $A_0 \subset A_0 \cup A_1 \Rightarrow f(a) \in f(A_0 \cup A_1)$.

g) Let $b \in f(A_0 \cap A_1)$, $\exists a \in A_0 \cap A_1$ s.t. $b = f(a)$.

$$\begin{aligned} &\Rightarrow a \in A_0 \textit{ and } a \in A_1 \\ &\Rightarrow b \in f(A_0) \textit{ and } b \in f(A_1) \\ &\Rightarrow b \in f(A_0) \cap f(A_1) \end{aligned}$$

Therefore,

$$f(A_0 \cap A_1) \subset f(A_0) \cap f(A_1).$$

If f is injective, we know from a previous problem that

$$A_0 \cap A_1 = f^{-1}(f(A_0 \cap A_1)).$$

Let $b \in f(A_0) \cap f(A_1)$. Then $\exists\ a \in f^{-1}(f(A_0) \cap f(A_1))$ *s.t.* $b = f(a)$.

It follows that $a \in (A_0 \cap A_1)$ and therefore $f(a) \in f(A_0 \cap A_1)$. This shows us that

$$f(A_0) \cap f(A_1) \subset f(A_0 \cap A_1).$$

Finally,

$$f(A_0 \cap A_1) = f(A_0) \cap f(A_1).$$

(Munkres 1.3.1) Define $(x_0, y_0) \sim (x_1, y_1)$ if $y_0 - x_0^2 = y_1 - x_1^2$.

(a) **Reflexivity**
Since $y_0 - x_0^2 = y_0 - x_0^2$, $(x_0, y_0) \sim (x_0, y_0)$.

(b) **Symmetry**
If $(x_0, y_0) \sim (x_1, y_1)$, then $y_0 - x_0^2 = y_1 - x_1^2$.
Symmetry of equality implies $y_1 - x_1^2 = y_0 - x_0^2 \Rightarrow (x_1, y_1) \sim (x_0, y_0)$.

(c) **Transitivity**
Suppose $(x_0, y_0) \sim (x_1, y_1)$ and $(x_1, y_1) \sim (x_2, y_2)$. Then

$$y_0 - x_0^2 = y_1 - x_1^2 = y_2 - x_2^2 \Rightarrow y_0 - x_0^2 = y_2 - x_2^2.$$

Therefore, $(x_0, y_0) \sim (x_2, y_2)$.

This set of equivalence classes are parabolic curves.

(Munkres 1.3.2) Let C be a relation on A. Define the restriction of C to $A_0 \subset A$ to be the relation $C \cap (A_0 \times A_0)$.

(a) **Reflexive**
Let $a_0 \in A_0 \subset A$ then $(a_0, b_0) \in A_0 \times A_0$. But $a_0 \in C \Rightarrow (a_0, a_0) \subset A$. But $(a_0, a_0) \in C$ by reflextivity of C. Therefore

$$a_0 \in Ccap(A_0 \times A_0).$$

(b) **Symmetric**
Let $(a_0, b_0) \in C \cap (A_0 \times A_0)$. Then $(a_0, b_0) \in (A_0 \times A_0)$ implies $(b_0, a_0) \in (A_0 \times A_0)$. Now $(b_0, a_0) \in C$ since C is a relation. Thus

$$(b_0, a_0) \in C \cap (A_0 \times A_0).$$

(c) **Transitive**
Suppose $(a_0, b_0), (b_0 c0) \in C \cap (A_0 \times A_0)$. We can see that $a_0, b_0, c_0 \in A_0$ implying $(a_0, c_0) \in A_0 \times A_0$. Transitivity of C implies $(a_0, c_0) \in C$. Therefore,

$$(a_0, c_0) \in C \cap (A_0 \times A_0).$$

(Munkres 1.3.13) **Theorem**. If an ordered set A has the Least Upper Bound property (LUB), then it has the Greatest Lower Bound property.

($\Rightarrow$) Suppose A has the LUB property (i.e. for every non-empty that is bounded above, the supremum exists) and let $A_0 \subset A$ be a non-empty subset that is bounded below. We will show that there exists a Greatest Lower Bound for this subset. Define

$$L_{A_0} = \{l \in A \mid l \leq a_0 \ \forall \ a_0 \in A_0\}$$

to be the set of lower bounds of A_0. A_0 being bounded below implies that $L_{A_0} \neq \emptyset$. L_{A_0} is bounded above by every $a \in A_0$. Since A has the LUB property,

$$\exists\ m := sup(L_{A_0}).$$

Thus, $\forall\ l \in L_{A_0}$ (i.e. for every lower bound for A_0) $l \leq m$ implies that

$$m = inf(A_0).$$

is the Greatest Lower Bound of A_0. Thus $LUB \Rightarrow GLB$.

Munkres 1.3.14) Let C be a relation on A. Define new relation D on A by $(b, a) \in D$ if $(a, b) \in C$.

(a) Show C is symmetric iff $C = D$.
($\Rightarrow$) Suppose $C = D$ and $(a, b) \in C$,

$$(a, b) \in C \Rightarrow (b, a) \in D \Rightarrow (b, a) \in C \quad \textit{since C=D.}$$

Thus C is symmetric.
($\Leftarrow$) Suppose C is symmetric, let $(a, b) \in C$.

$$(a, b) \in C \Rightarrow (b, a) \in C \Rightarrow (a, b) \in D.$$

Thus $C \subset D$. Now suppose $(b, a) \in D$. The construction of D implies that $(a, b) \in C$. C symmetric implies $(b, a) \in C$. Thus $D \subset C$. Since D and C are contained in each other,

$$C = D.$$

(b) Show if C is an order relation then D is also an order relation. We'll use $<'$ to denote the order relation of D.

(1) First show that D is not reflexive. We do so by showing that if two elements are related, then they are distinct.
We know by construction of D that if $(b, a) \in D$, then $(a, b) \in C$. Since C is an order relation, $a \neq b$. Thus,

$$(b, a) \in D \Rightarrow b \neq a.$$

(2) We show that D satisfies Comparability, i.e. $\forall\ a \neq b,\ a <' b \ or \ b <' a$.
($\Rightarrow$) $\forall\ a, b \in A$ s.t. $a \neq b$, we can suppose WLOG that $a < b \Rightarrow (a, b) \in C$. It follows that $(b, a) \in D \Rightarrow b <' a$.

(3) Transitivity
($\Rightarrow$) Suppose $(c, b), (b, a) \in D \Rightarrow (b, c), (a, b) \in C$. Transitiviy of C tells us that $(a, c) \in C \Rightarrow (c, a) \in D$.

(c) Prove the converse of 1.3.13. If A has the Greatest Lower Bound (GLB) property, then it has the Least Upper Bound (LUB) property.

($\Rightarrow$) Suppose A has the Greatest Lower Bound Property. We want to show that every non-empty $A_0 \subset A$ that is bounded above has a Least Upper Bound (supremum). Let $A_0 \subset A$ be bounded above. Define the set of all upper bounds of A_0 to be

$$U_{A_0} = \{u \in A \mid a_0 \leq u \ \forall \ a_0 \in A_0\}.$$

We want to show that the above set has a lowest element. Well, A_0 being bounded above implies U_{A_0} isn't empty. By construction of U_{A_0}, A_0 is a non-empty set containing all lower bounds of U_{A_0}. The supremum of A_0 exists by hypothesis, $m := sup(A_0)$. Since,

$$\forall \ u \in U_{A_0}, u \geq m,$$

$m = inf(U_{A_0})$ is the LUB of A_0.