I think Lemma 37.1 does a lot of the heavy lifting in proving Tychonoff's Theorem. It says that for a given collection $\mathcal{A}$ of subsets of X satisfying the finite intersection property, there's is a maximum collection $\mathcal{D}$ it may be extended to that doesn't break the finite interesction property. The importance of this is that we want the maximum constraints for points under consideration when making an argument. I discuss this Lemma, state the next Lemma and then discuss Tychonoff's Theorem.

Lemma 37.1) Let X be a set; let $\mathcal{A}$ be a collection of subsets of X having the finite intersection property. Then there is a collection $\mathcal{D}$ if subsets of X such that $\mathcal{D}$ contains $\mathcal{A}$, and $\mathcal{D}$ has the finite intersection property, and no collection of subsets of X that properly contains $\mathcal{D}$ has this property.

($\Rightarrow$) We have a set X. Let $\mathcal{A}$ be a collection of subsets of X satisfying the finite intersection property.

$$\mathcal{A} = \{A_i \subset X | \cap_i A_i \neq \emptyset.\}$$

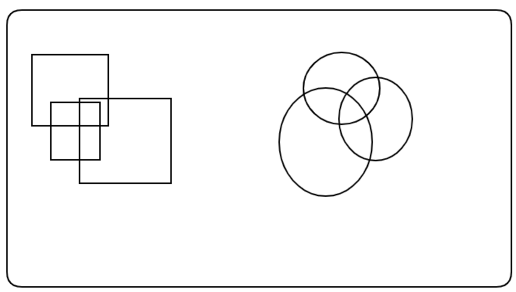

Figure 1: Possibly Multiple $\mathcal{A}$'s

We can have many of these $\mathcal{A}$ as seen in the diagram. With this $\mathcal{A}$ under consideration, we let $\mathbb{A}$ be the set of all extensions of $\mathcal{A}$ that doesn't break the finite intersection property.

$$\mathbb{A} = \{\mathcal{B} \supset \mathcal{A} | \bigcap_{B \in \mathcal{B}} B \neq \emptyset.\}$$

$\mathcal{B}$'s are collections containing $\mathcal{A}$ and every set in each $\mathcal{B}$ share a common point.

At this point, $\mathbb{A}$ is set up and wish to apply Zorn's Lemma to it. This lemma tells us that $\mathbb{A}$ has a maximum element $\mathcal{D}$. Two things need to be done:

1) We need to impose a partial ordering on $\mathbb{A}$.

   The ordering we impose is $\subsetneq$. The choice of ordering being inclusion is clear since we're stricly talking about sets. The specific choice of proper inclusion ($\subsetneq$) is not clear at this point to me however.

Note that inclusion is a partial order since some set may include another. But some sets may have points not contained in the other. Disjoint sets are not in the picture here because every element of $\mathbb{A}$ has $\mathcal{A}$ in common.

2) We need to show that every simply ordered subset of $\mathbb{B} \subset \mathbb{A}$ is has an upper bound.

Let $\mathbb{B} \subset \mathbb{A}$ be a simply ordered (totally orderd) subset. Using indices for the sake of simplicity, we have something that looks like

$$\mathbb{B} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, ...\}.$$

This means that relative to $\subsetneq$, every collection $\mathcal{B}_i$ is comparable. Thus every collection is strictly contained in another. There are no collections that cannot be compared. We then have something like this

$$\mathcal{B}_{i_1} \subsetneq \mathcal{B}_{i_2} \subsetneq \mathcal{B}_{i_3} \subsetneq ...$$

There's no mention of the finiteness of $\mathbb{B}$, so we can't assume the at some $\mathcal{B}_k \in \mathbb{B}$ is an upper bound. But we show that the union of all the sets in $\mathbb{B}$ is an upperbound for $\mathbb{B}$.

$$\bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B}.$$

Although it's clear that the above contains every element of $\mathbb{B}$, we need to explicitly show that the it is actually in $\mathbb{A}$ also,

$$\mathcal{C} := \bigcup_{\mathcal{B} \in \mathbb{B}} \mathcal{B} \in \mathbb{A}.$$

Since $\mathcal{A}$ is clearly in $\mathbb{C}$, we only need to show $\mathcal{C}$ satisfies the finite intersection property. Consider the finite subset of $\mathcal{C}$,

$$\{C_1, C_2, ..., C_n\}.$$

$C_i \in \mathcal{B}_i \ \forall \ i$. Finite and total ordering implies every $C_i \in \mathcal{B}_k \ \forall$ some k. $\mathcal{B}_k$ having finite intersection property implies

$$\bigcap C_i \neq \emptyset.$$

By Zorn's Lemma, $\mathbb{A}$ contains maximum element $\mathcal{D}$ w.r.t. $\subsetneq$.

Lemma 37.2) Let X be a set; let $\mathcal{D}$ be a collection of subsets of X that is maximal with respect to the finite intersection property. Then:

(a) Any finite intersection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.

(b) If A is a subset of X that interesects every elemetn of $\mathcal{D}$, then A is an element of $\mathcal{D}$.

Tychonoff Theorem) An arbitrary product of compact spaces is compact in the product topology.

($\Rightarrow$)

Let

$$X = \prod_{\alpha \in J} X_\alpha, \; X_\alpha \text{ compact.}$$

To show that X is compact, we use Theorem 26.9 that says X compact is equivalent to the condition:

For every collection $\mathcal{C}$ of closed sets in X with the finite intersection property, the intersection of all those sets are non-empty.

To being, we let $\mathcal{C}$ be a collections of subsets of X with the finite intersection property. By Lemma 37.1, $\mathcal{C}$ can be extended to some maximum $\mathcal{D}$ and not break the finite intersection property.

Note: Extending $\mathcal{C}$ to the maximal element $\mathcal{D}$ allows us to have the maximum constraints on points under consideration. Per the discussion at the beginning of 37*, it allows us to work with the most general set of points.*

With the above being said, showing $\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset$ is non-empty provides a more general argument than showing our initial choice $\bigcap_{C \in \mathcal{C}} \overline{C} \neq \emptyset$ is non-empty. So it suffices to show that

$$\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset.$$

Although Lemma 37.1 doesn't care about whether the sets are opened, closed, or both, Theorem 26.9 specifically states we need closed sets. Extending $D \in \mathcal{D}$ to their closures $\overline{D}$ doesn't break the finite intersection property.

We need to show that some $\mathbf{x} \in \bigcap_{D \in \mathcal{D}} \overline{D}$. We choose $\mathbf{x} = (x_\alpha)$ s.t.

$$x_\alpha \in \bigcap_{D \in \mathcal{D}} \overline{\pi_\alpha(D)}.$$

According to Munkres, this is possible because of the compactness of X_α. *It's not clear to me how compactness of X_α factors in.*

Moving forward, let $U_\alpha \ni x_\alpha$ neighborhood. Some U_α intersects $\pi_\alpha(D) \;\; \forall\, D \in \mathcal{D}$. There's some $y_\alpha \in U_\alpha \cap \pi_\alpha(D)$. Then

$$\mathbf{y} \in \pi_\alpha^{-1}(U_\alpha) \cap D.$$

were $\mathbf{y} = (y_\alpha)_{\alpha \in J}$. This tells us that every sub-basis element $\pi_\alpha^{-1}(U_\alpha) \ni \mathbf{x}$ intersects every $D \in \mathcal{D}$. By Lemma 37.2b, this sub-basis element is also in $\mathcal{D}$,

$$\pi_\alpha^{-1}(U_\alpha) \in \mathcal{D}.$$

Recalling that basis elements can be derived from finite intersections of sub-basis elements, Lemma 37.2a tells us that every basis element containing $B_\mathbf{x}$ are also part of $\mathcal{D}$. $B_\mathbf{x}$ intersects every element $D \in \mathcal{D}$ by the finite intersection property. Thus $\mathbf{x} \in \overline{D}$ for all $\in \mathcal{D}$. Therefore,

$$\bigcap_{D \in \mathcal{D}} \overline{D} \neq \emptyset.$$

By Theorem 26.9,

$$X = \prod_{\alpha \in J} X_\alpha$$

is compact.