MAT 4170: Modern Algebra I **Name:** **Panya Sukphranee**

Homework 9

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1. Let $Z(G) = \{g \in G \mid gx = xg \text{ for all } g \in G\}$, where G is a group. We will show that $Z(G)$ is normal subgroup of G by applying the Normal Subgroup Test,

$$H \trianglelefteq G \iff xHx^{-1} \subseteq H \text{ for all } x \in G.$$

In terms of $Z(G)$, we have

$$Z(G) \trianglelefteq G \iff xZ(G)x^{-1} \subseteq Z(G) \text{ for all } x \in G.$$

Let $x \in G$ and consider some element $y \in xZ(G)x^{-1}$. Then $y = xzx^{-1}$, for some $z \in Z(G)$. We will show that this element is also in $Z(G)$, i.e. xzx^{-1} commutes with everything else in G. To do so, let's introduce another element $g \in G$ and show that,

$$(xzx^{-1})g = g(xzx^{-1}).$$

($\Rightarrow$)

$$\begin{aligned}
(xzx^{-1})g &= (xx^{-1}z)g \\
&= zg \\
&= gz \\
&= g(xx^{-1})z \\
&= gx(x^{-1}z) \\
&= gx(zx^{-1}) \\
&= g(xzx^{-1}).
\end{aligned}$$

Since $xzx^{-1} \in Z(G)$, $xZ(G)x^{-1} \subseteq Z(G)$. $Z(G)$ passes the Normal Subgroup test and is therefore Normal in G.

2. Let G be a finite group of prime order. If $|G| = 1$, then there's only one element in there. e, which generates it. So let's consider the non-trivial case and suppose that $a \in G$ is not the identity element. We know that $< a >$ forms a subgroup in G. Lagrange's Theorem says that $| < a > |$ divides $|G|$. But $|G|$ being prime implies that $| < a > | = |G|$. It follows that $G =< a >$ is cyclic.

3. Let G be a finite group of order n.

   a) Let $a \in G$. We know that $|a| < \infty$. Since $< a >$ is a subgroup in G, it's order, $| < a > |$ divides G by Lagrange's Theorem. But $| < a > | = |a|$. Therefore, the order of each element divides the order of the group.

b) Let $a \in G$ be an element of order n_a. From part (a), n_a divides n,

$$n = kn_a.$$

Then

$$a^n = a^{kn_a} = (a^{n_a})^k = e^k = e.$$

4. Let $\phi : G \to H$ be a group homomorphism. We will show by the Normal Subgroup Test that $Ker\phi$ is a normal subgroup. Let $g \in G$ and $h \in Ker\phi$.

$$\begin{aligned}\phi(ghg^{-1}) &= \phi(g)e_H\phi(g^{-1}) \\ &= \phi(g)\phi^{-1}(g) \\ &= e_H.\end{aligned}$$

Thus, $ghg^{-1} \in Ker\phi$. Since g and h are arbitrary, we have that

$$gKer(\phi)g^{-1} \subseteq Ker(\phi).$$

5. Let $H = \{1, 15\}$ and $K = \{1, 9\}$. I believe these two sets are isomorphic. We can define $\phi : H \to G$ by,

$$\phi(1) = 1, \phi(15) = 9.$$

This mapping is bijective and preserves multiplication. The cosets formed by each are,

$$\frac{G}{H} = \{\{1, 15\}, \{3, 13\}, \{5, 11\}, \{7, 9\}\}$$

and

$$\frac{G}{K} = \{\{1, 9\}, \{3, 11\}, \{5, 13\}, \{7, 15\}\}.$$

The quotient group $\frac{G}{H}$ is cyclic genterated by $\{3, 13\} and is therefore cyclic$,

$$\{3, 13\} \to \{7, 9\} \to \{5, 11\} \to \{1, 15\}.$$

On the other hand, $\frac{G}{K}$ has all elements of order 2 and cannot be cyclic. Thus, $\frac{G}{K}$ not isomorphic to $\frac{G}{H}$.

6. Let $G = Z_4 \times Z_4$, $H = \{(0, 0), (2, 0), (0, 2), (2, 2)\}$, and $K = \{(0, 0), (1, 3), (2, 0), (3, 2)\}$. The quotient groups are

$$\begin{aligned}\frac{G}{H} = \{&\{(0, 0), (2, 0), (0, 2), (2, 2)\}, \\ &\{(0, 1), (2, 1), (0, 3), (2, 3)\}, \\ &\{(1, 1), (3, 1), (1, 3), (2, 3)\}, \\ &\{(1, 2), (3, 2), (1, 0), (3, 0)\}\}\end{aligned}$$

and

$$\begin{aligned}\frac{G}{H} = \{&\{(0, 0), (1, 3), (2, 0), (3, 2)\}, \\ &\{(0, 1), (1, 0), (2, 1), (3, 3)\}, \\ &\{(0, 2), (1, 1), (2, 2), (3, 0)\}, \\ &\{(0, 3), (1, 2), (2, 3), (3, 1)\}\}.\end{aligned}$$

I can't think and give up on the problem right now. I have to return to this one in the future.