Q5) Suppose E is a given set, and $\mathcal{O}_n$ is the open set:

$$\mathcal{O}_n = \left\{ x : d(x,E) < \frac{1}{n} \right\}$$

Show:

(a) If E is compact, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.

If E is compact, then $m(E) < \infty$. So every $\mathcal{O}_n \supset E$ has measure

$$m(\mathcal{O}_n) = m(E) + \frac{k}{n} \qquad \text{for some k}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} m(\mathcal{O}_n) &= \lim_{n\to\infty} \left\{ m(E) + \frac{k}{n} \right\} \\ &= m(E). \end{aligned}$$

(b) However, the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

(1) Let $E = \mathbb{Z}$, a closed and unbounded set. Then

$$\mathcal{O}_n = \bigcup_{z\in\mathbb{Z}} \left(z - \frac{1}{n}, z + \frac{1}{n} \right)$$

which implies that

$$m(\mathcal{O}_n) = \sum_{k=1}^{\infty} \frac{2}{n} = \frac{2}{n} \sum_{k=1}^{\infty} 1$$

The measure of $\mathcal{O}_n$ is infinite.

Q26) Suppose $A \subset E \subset B$, where A and B are measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is measurable.

We first have that

$$\begin{aligned} m(B) &= m(A) + m(B - A) \\ m(B) - m(A) &= m(B - A) \\ 0 &= m(B - A). \end{aligned}$$

Now $E \subset B$ implies $E - A \subset B - A$. By Monotonicity,

$$m(E - A) \leq m(B - A) = 0$$

This shows that

$$m(E) = m(E - A) + m(A)$$

exists and therefore E is measurable.$\square$

Q8) Suppose L is a linear transformation of $\mathbb{R}^d$. Show that if E is a measurable subset of $\mathbb{R}^d$, then so is $L(E)$, be proceeding as follows:

(a) Note that if E is compact, so is $L(E)$. Hence if E is an F_σ set, so is $L(E)$.

($\Rightarrow$) Assuming we are using l_2 norm, the linear transformation L is continuous and from Topology, the continuous image of a compact set is compact. Now, for all coverings of $L(E)$, there's some finite (i.e. countable) subcovering of $L(E)$,

$$L(E) \subset \bigcup_{\alpha=1}^{N} V_\alpha$$

It follows that,

$$L(E) = \bigcup_{\alpha=1}^{N} \left(L(E) \cap V_\alpha\right).$$

Showing that $L(E)$ is a countable union of closed sets and is therefore in F_σ.$\square$

(b) Because L automatically satisfies the inequality

$$|L(x) - L(x')| \leq M\,|x - x'|$$

for some M, we can see that L maps any cube of side length l into a cube of side length $c_d M l$, with $c_d = 2\sqrt{d}$. Now if $m(E) = 0$, there is a collection of cubes $\{Q_j\}$ such that $E \subset \bigcup_j Q_j$, and $\sum_j m(Q_j) < \varepsilon$. Thus $m_*\left(L(E)\right) \leq c'\varepsilon$, and hence $m(L(E)) = 0$. Finally, use Corollary 3.5.

One can show that $m(L(E)) = |detL|\, m(E)$; see Problem 4 in the next chapter.

($\Rightarrow$) Assuming we are using the l_2 norm, L continuous implies L is uniformly continuous on the compact set E. Thus, $\exists$ M s.t.,

$$|L(x) - L(x')| \leq M\,|x - x'|.$$

Since $m(E) = 0 < \infty$, $\exists$ a collection $\{Q_j\}$ s.t.

$$E \subset \bigcup_{j=1}^{N} Q_j$$

and

$$m(E) \le \sum_{j=1}^{N} m(Q_j) = \sum_{j=1}^{N} l^d = Nl^d < \varepsilon.$$

Now,

$$L(E) \subset L(\bigcup_{j=1}^{N} Q_j) = \bigcup_{j=1}^{N} L(Q_j)$$

so that

$$m_* \left(L(E)\right) \le \sum_{j=1}^{N} m_* \left(L(Q_j)\right) \le \sum_{j=1}^{N} \left(2\sqrt{d}M\right)^d Nl^d =: c'\varepsilon.$$

Thus,

$$m(E) = 0.\square$$

Q16) **The Borel-Cantelli lemma** Suppose $(E_k)_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty$$

Let

$$\begin{aligned} E &= \left\{x \in \mathbb{R}^d : x \in E_k, \text{ for infinitely many } k\right\} \\ &= \limsup_{k\to\infty}(E_k) \end{aligned}$$

(a) Show that E is measurable

($\Rightarrow$) We begin by writing E as

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k\ge n} E_k.$$

We can see that this makes sense by expanding out the first few terms

$$\begin{aligned} \bigcap_{n=1}^{\infty} \bigcup_{k\ge n} E_k = (E_1 \cup E_2 \cup E_3 \cup E_4 \cup \ldots) \cap \\ (E_2 \cup E_3 \cup E_4 \cup \ldots) \cap \\ (E_3 \cup E_4 \cup \ldots) \cap \\ (E_4 \cup \ldots) \cap \end{aligned}$$

Each n^{th} level of intersection removes elements of E_n not in every successive set.

Let us now define

$$\tilde{E_j} = \bigcap_{n=1}^{j} \bigcup_{k \geq n} E_k,$$

which is a measurable set as it is the countable intersection of countable union of measurable sets. It follows that

$$\tilde{E_j} \supset \tilde{E}_{j+1}.$$

By Corollary 3.3, E is measurable with

$$m(E) = \lim_{N \to \infty} m(\tilde{E_N})$$

$\square$.

(b) Prove m(E) = 0

($\Rightarrow$) By monotonicity of above and the fact given $\sum_{k=1}^{\infty} m(E_k) < \infty$,

$$m(E) \to 0 \text{ and } m\left(\tilde{E_j}\right) \geq m\left(\tilde{E}_{j+1}\right)$$

implies

$$m\left(\tilde{E_j}\right) \to 0.$$

By part (a),

$$m(E) = 0.\square$$

$$\left[Hint : Write\ E = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k\right]$$

Q13) The following deals with G_σ and F_σ sets.

(a) Show that a closed set is a G_σ and an open set is F_σ.

[Hint: If F closed, consider $\mathcal{O}_n = \{x : d(x, F) < 1/n\}$.]

($\Rightarrow$) Let $F \in F_\sigma$ and $\mathcal{O}_n \left\{x : (x, F) < \frac{1}{n}\right\}$. We will show that

$$F = \bigcap \mathcal{O}_n.$$

The containment $F \subset \bigcap \mathcal{O}_n$ is clear. Suppose $y \in \bigcap \mathcal{O}_n$,

$$d(y, F) < \frac{1}{n}$$

implies $y =\in F$. Thus, $F = \bigcap \mathcal{O}_n$ is the countable intersection of open sets and is therefore in G_σ.

(b) Give an example of an F_σ which is not a G_σ.

[Hint: Let F be a denumerable set that is dense.]
($\Rightarrow$) Consider the rationals,

$$\mathbb{Q} = \bigcup_{p \in \mathbb{Q}} \{p\} \in F_\sigma.$$

Suppose that $\mathbb{Q} = \bigcap U_n$ is the countable intersection of open sets. WLOG, we can let $U_n = \mathcal{O}_n$ defined above, i.e.

$$U_n = \bigcup_{p \in \mathbb{Q}} B_{1/n}(p)$$

Generality is not loss because open balls are contained in every open set. But U_n being dense implies that $U_n = \mathbb{R} \ \ \forall \ \ n$. In particular, U_n contains irrationals and therefore we do not equality.

$$\mathbb{Q} \notin G_\sigma$$

(c) Give an example of a Borel set which is not a G_σ nor an F_σ.

($\Rightarrow$)
From part (a) and (b),

$$\mathbb{Q} \text{ and } \mathbb{I} := \mathbb{R} - \mathbb{Q}$$

are in F_σ and G_σ, respectively, but not in the other. They are also complements of each other and disjoint. So we can take "half" of each and define,

$$S = \mathbb{Q}^- \cup \mathbb{I}^+.$$

S is closed and is neither F_σ nor .

Baire Category Theorem. If $\mathcal{O}_n$ are open dense sets in $\mathbb{R}^d$, then

$$\bigcap_{n=1}^{\infty} \mathcal{O}_n$$

is also dense in $\mathbb{R}^d$.

Let $F = \mathbb{Q} = \{r_1, r_2, \ldots\}$ and suppose that $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$ for some open sets U_n. Let $\mathcal{O}_n = U_n$.