M-144

APRIL 14

$\mathbb{V} \times \mathbb{W}$, elements are ordered pairs.
Not v.s. until define linear structure.

h

$\Rightarrow$ $\mathbb{V} \otimes \mathbb{W}$ is a v.s.

has to be constructed so that

no dim

dim $\mathbb{V}$ × dim $\mathbb{W}$

fig 1

$$\begin{tikzcd} \mathbb{V}\times\mathbb{W} \arrow[r, "h"] \arrow[dr, "f"'] & \mathbb{V}\otimes\mathbb{W} \arrow[d, "\phi"] \\ & \mathbb{Z} \end{tikzcd}$$

$f = \phi \circ h$, ϕ unique

~~i.e. ∀ bilinear map~~

(*) We went through fact that
$\mathbb{V} \otimes \mathbb{W} = \mathcal{L}(\mathbb{V}^*, \mathbb{W}^*; \mathbb{F})$

no relation

$(\vec{v} \otimes \vec{w})(\vec{v}^*, \vec{w}^*) =$
$\langle \vec{v}, \vec{v}^* \rangle \cdot \langle \vec{w}, \vec{w}^* \rangle$

monomial

$a_1(\vec{v}_1 \otimes \vec{w}_1) + a_2(\vec{v}_2 \otimes \vec{w}_2) + \ldots + a_n(\vec{v}_n \otimes \vec{w}_n)$

– refer to fig 1, given f, how to define ϕ?

$\phi(\vec{v} \otimes \vec{w}) = f(\vec{v}, \vec{w})$

Let dim $\mathbb{V} = n$
dim $\mathbb{W} = m$

$\Rightarrow \{\vec{a}_1, \ldots, \vec{a}_n\}, \{\vec{b}_1, \ldots, \vec{b}_m\}$
bases

$\vec{v} = v^i \vec{a}_i$, $\vec{w} = w^j \vec{b}_j$

$\vec{v} \otimes \vec{w} = (v^i \vec{a}_i) \otimes (w^j \vec{b}_j)$

$= v^i w^j (\vec{a}_i \otimes \vec{b}_j)$

want tens prod to obey certain rules, e.g. assoc. mult.

Suppose we have three tens. of rank r, s, t, resp.

$\phi \quad \psi \quad \mathfrak{z}$

can $\phi \otimes (\psi \otimes \mathfrak{z})$,
$(\phi \otimes \psi) \otimes \mathfrak{z}$

we can prove they are equal

$\Rightarrow \phi \otimes \psi \otimes \mathfrak{z}$

$\frac{n!}{r!(n-r)!}\ \frac{n!}{s!(n-s)!}$

ex) supp $r=s=t=1$,

$[\phi\otimes(\psi\otimes\zeta)](\vec{v}^*, \vec{w}^*, \vec{z}^*)$ should be starred.

$= \phi(\vec{v}^*)\cdot(\psi\otimes\zeta)(\vec{w}^*, \vec{z}^*)$
$= \phi(\vec{v}^*)\cdot\psi(\vec{w}^*)\cdot\zeta(\vec{z}^*)$
$= (\text{"} \quad\quad \text{"})\cdot\zeta(\vec{z}^*)$

$= \ldots [(\phi\otimes\psi)\otimes\zeta](\vec{v}^*, \vec{w}^*, \vec{z}^*)$

IN PHYS USUALLY CONSIDER PROD OF:

CONSIDER

$\mathbb{V}$, $\mathbb{V}^*$

$\nexists$ canonical rel between pts on diff tan space @ some pt. on manifold

$\mathbb{V}^r_s = \underbrace{\mathbb{V}\otimes\ldots\otimes\mathbb{V}}_{r}\otimes\underbrace{\mathbb{V}^*\otimes\ldots\otimes\mathbb{V}^*}_{s}$

dim n^{r+s}.

basis for $\mathbb{V} = \{\vec{e}_i\}$
$\mathbb{V}^* = \cancel{\{\vec{e}^*\}}\ \cancel{\{\vec{e}_i\}}$
$= \{\vec{e}^{*i}\}$

basis for $\underbrace{\mathbb{V}\otimes\ldots\otimes\mathbb{V}}_{r}$

$\vec{e}_{i_1}\otimes\ldots\otimes\vec{e}_{i_r}$
$\otimes\,\vec{e}^{*j_1}\otimes\ldots\otimes\vec{e}^{*j_s}$

$\dim \mathbb{V}^r_s = n^{r+s}$

$T\in\mathbb{V}^r_s$

$T = T^{i_1\ldots i_r}_{j_1\ldots j_s}$

$(r+s)$- sums

$T^{i_1\ldots i_r}_{j_1\ldots j_s} = T(\vec{e}^{*i_1},\ldots,\vec{e}^{*i_r}, \vec{e}_{j_1},\ldots,\vec{e}_{j_s})$