Prob 15

(5) is the $\overbrace{\text{subspace}}^{\text{5-dimensional}}$ of symmetric, traceless tensors, so a basis set would be

$$\{\vec e_1 \otimes \vec e_1 \,,\ \vec e_2 \otimes \vec e_2 \,,\ \vec e_1 \otimes \vec e_2 \,,\ \vec e_1 \otimes \vec e_3 \,,\ \vec e_2 \otimes \vec e_3\}$$

A tensor in this subspace can be written

$$\tilde S = \tilde S^{11}\, \vec e_1 \otimes \vec e_1 + \tilde S^{22}\, \vec e_2 \otimes \vec e_2 + \tilde S^{12}\, \vec e_1 \otimes \vec e_2$$
$$+ \tilde S^{13}\, \vec e_1 \otimes \vec e_3 + \tilde S^{23}\, \vec e_2 \times \vec e_3$$

where $\tilde S^{ij} = S^{ij} - \dfrac{\delta^{ij}\, \mathcal{T}}{3}$ $\quad$ ($\mathcal{T}$ = trace of $T = \sum_i T^{ii}$)

$$S^{ij} = T^{ij} + T^{ji}$$

with $T = T^{ij}\, \vec e_i \otimes \vec e_j$ a general tensor in $T^2(\mathbb{V})$.

(1) is the 1-dimensional subspace with basis vector

$$\vec e_1 \otimes \vec e_1 + \vec e_2 \otimes \vec e_2 + \vec e_3 \otimes \vec e_3$$

(3) is the $\overbrace{\text{anti-symmetric}}^{\text{3-dimensional}}$ subspace of $T^2(\mathbb{V})$ with basis set

$$\{\vec e_1 \otimes \vec e_2 - \vec e_2 \otimes \vec e_1 \,,\ \vec e_1 \otimes \vec e_3 - \vec e_3 \otimes \vec e_1 \,,\ \vec e_2 \otimes \vec e_3 - \vec e_3 \otimes \vec e_2\}$$

A general tensor here can be written

$$A = A^{12}(\vec e_1 \otimes \vec e_2 - \vec e_2 \otimes \vec e_1) + A^{13}(\vec e_1 \otimes \vec e_3 - \vec e_3 \otimes \vec e_1) + A^{23}(\vec e_2 \otimes \vec e_3 - \vec e_3 \otimes \vec e_2)$$

where $A^{ij} = T^{ij} - T^{ji}$

Prob. 16 The dimension of $\Lambda^3(\mathbb{V})$ with $\dim \mathbb{V} = 5$ is

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!\,2!} = 10$$

A basis is $\{\vec{e}_i \wedge \vec{e}_j \wedge \vec{e}_k\}$, $(1 \le i < j < k \le 5)$

Written out explicitly, it is the following set:

$$\begin{array}{l}
\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3 \\
\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_4 \\
\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_5 \\
\vec{e}_1 \wedge \vec{e}_3 \wedge \vec{e}_4 \\
\vec{e}_1 \wedge \vec{e}_3 \wedge \vec{e}_5 \\
\vec{e}_1 \wedge \vec{e}_4 \wedge \vec{e}_5 \\
\vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_4 \\
\vec{e}_2 \wedge \vec{e}_3 \wedge \vec{e}_5 \\
\vec{e}_2 \wedge \vec{e}_4 \wedge \vec{e}_5 \\
\vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5
\end{array}$$

Prob. 17 A general tensor in $T^3(\mathbb{V})$ $\dim \mathbb{V} = 3$ can be written

$$T = T^{ijk}\, \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \qquad 1 \le i, j, k \le 3$$

A symmetric tensor $\in P^3(\mathbb{V})$ can be written

$$S = S^{ijk}\, \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k$$

$\boxed{\text{Prob 17}}$ cont'd

symmetrizer

where $S^{ijk} = [S_3(T)]^{ijk}$

$$= \frac{1}{3!}\left(T^{ijk} + T^{jki} + T^{kij} + T^{ikj} + T^{kji} + T^{jik}\right)$$

An antisymmetric tensor $\in \Lambda^3(\mathbb{V})$ can be written

$$A = A^{ijk}\,\vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k$$

antisymmetrizer

where $A^{ijk} = [A_3(T)]^{ijk}$

$$= \frac{1}{3!}\left(T^{ijk} + T^{jki} + T^{kij} - T^{ikj} - T^{kji} - T^{jik}\right)$$

$\Lambda^3(\mathbb{V})$ is <u>one-dimensional</u>, since the only independent component is A^{123}

A basis of $\Lambda^3(\mathbb{V})$ consists of the exterior vector

$$\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3$$

$P^3(\mathbb{V})$ is <u>ten-dimensional</u>. The 10 independent components are

$S^{111}, S^{222}, S^{333}, S^{112}, S^{113}, S^{221}, S^{223}, S^{331}, S^{332}, S^{123}$.

A basis of $P^3(\mathbb{V})$ is the set

$\{\vec{e}_1 \otimes \vec{e}_1 \otimes \vec{e}_1,\ \vec{e}_2 \otimes \vec{e}_2 \otimes \vec{e}_2,\ \vec{e}_3 \otimes \vec{e}_3 \otimes \vec{e}_3,\ \vec{e}_1 \otimes \vec{e}_1 \otimes \vec{e}_2,$
$\vec{e}_1 \otimes \vec{e}_1 \otimes \vec{e}_3,\ \vec{e}_2 \otimes \vec{e}_2 \otimes \vec{e}_1,\ \vec{e}_2 \otimes \vec{e}_2 \otimes \vec{e}_3,\ \vec{e}_3 \otimes \vec{e}_3 \otimes \vec{e}_1,$
$\vec{e}_3 \otimes \vec{e}_3 \otimes \vec{e}_2,\ \vec{e}_1 \otimes \vec{e}_2 \otimes \vec{e}_3\}$.

$\boxed{\text{Prob. 17}}$ So $P^3(\mathbb{V}) \oplus \Lambda^3(\mathbb{V})$ is 11-dimensional.

$T^3(\mathbb{V})$, however, is 27-dimensional.

$\boxed{\text{Prob. 18}}$ Let $\varphi_1, \varphi_2 \in \Lambda^*(\mathbb{W}^*)$. We need to show that

$$A^*(\alpha_1\varphi_1 + \alpha_2\varphi_2) = \alpha_1 A^*(\varphi_1) + \alpha_2 A^*(\varphi_2) \quad \text{—— } (*)$$

where $\alpha_1, \alpha_2 \in \mathbb{F}$.

First, note that $\alpha_1\varphi_1 + \alpha_2\varphi_2 \in \Lambda^r(\mathbb{W}^*)$. We thus have

$$A^*(\alpha_1\varphi_1 + \alpha_2\varphi_2)(\vec{v}_1, \dots, \vec{v}_r) \quad \text{for any } \vec{v}_1, \dots, \vec{v}_r \in \mathbb{V}$$

$$= (\alpha_1\varphi_1 + \alpha_2\varphi_2)(A\vec{v}_1, \dots, A\vec{v}_r)$$

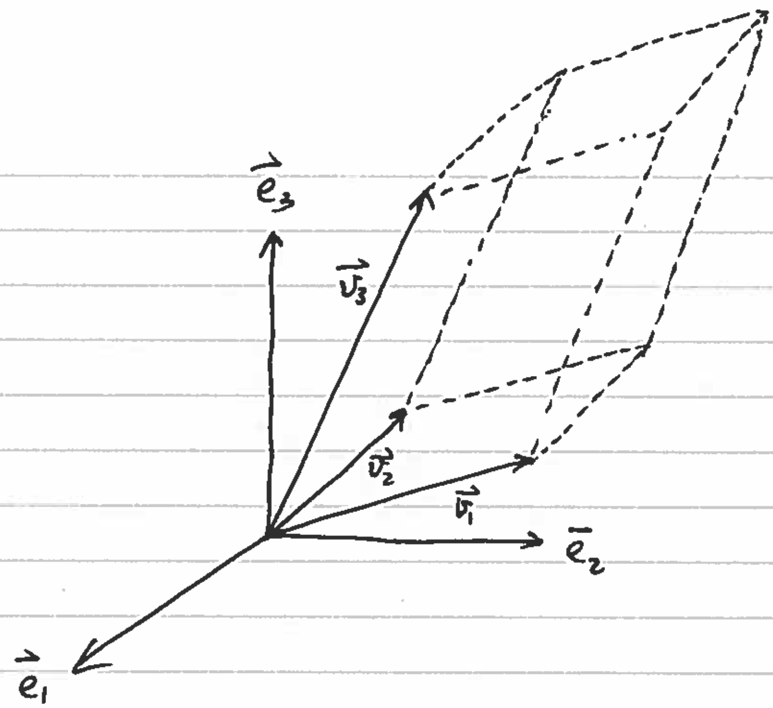
$$= (\alpha_1\varphi_1)(A\vec{v}_1, \dots, A\vec{v}_r) + (\alpha_2\varphi_2)(A\vec{v}_1, \dots, A\vec{v}_r)$$

$$= \alpha_1 (A^*\varphi_1)(\vec{v}_1, \dots, \vec{v}_r) + \alpha_2 (A^*\varphi_2)(\vec{v}_1, \dots, \vec{v}_r)$$

$$= (\alpha_1 A^*\varphi_1 + \alpha_2 A^*\varphi_2)(\vec{v}_1, \dots, \vec{v}_r).$$

$\therefore$ $(*)$ ~~is~~ follows.

Prob. 19



$\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3$ is an <u>oriented</u> frame in $\mathbb{R}^3$. (The order 123 gives the orientation). The volume of the parallelopiped formed by the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is given, from 3-d vector analysis, by

$$|\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3)| .$$

If we express $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in terms of components with respect to $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ by

$$\vec{v}_1 = v_1^1 \vec{e}_1 + v_1^2 \vec{e}_2 + v_1^3 \vec{e}_3 , \quad \vec{v}_2 = v_2^1 \vec{e}_1 + v_2^2 \vec{e}_2 + v_2^3 \vec{e}_3$$

$$\vec{v}_3 = v_3^1 \vec{e}_1 + v_3^2 \vec{e}_2 + v_3^3 \vec{e}_3 ,$$

we see that

$$\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) = \begin{vmatrix} v_1^1 & v_1^2 & v_1^3 \\ v_2^1 & v_2^2 & v_2^3 \\ v_3^1 & v_3^2 & v_3^3 \end{vmatrix}$$

$\boxed{\text{Prob 19}}$ cont'd

This is precisely the pairing

$$\langle \vec{v}_1 \wedge \vec{v}_2 \wedge \vec{v}_3 ,\ \vec{e}^{*1} \wedge \vec{e}^{*2} \wedge \vec{e}^{*3} \rangle \qquad -(*)$$

where $\{\vec{e}^{*i}\}$ is the dual basis to $\{\vec{e}_i\}$.

$\vec{e}^{*1} \wedge \vec{e}^{*2} \wedge \vec{e}^{*3}$ is called the volume form. The pairing given by $(*)$ is actually called the oriented volume.

$\boxed{\text{Prob. 20}}$ $$\det(A) = \sum_{\sigma \in S(n)} (\operatorname{sgn} \sigma)\, a_1^{\sigma(1)} a_2^{\sigma(2)} \cdots a_n^{\sigma(n)}$$

$$= \sum_{\sigma} (\operatorname{sgn} \sigma)\, a^1_{\sigma^{-1}(1)} a^2_{\sigma^{-1}(2)} \cdots a^n_{\sigma^{-1}(n)}$$

$$= \sum_{\sigma^{-1}} (\operatorname{sgn} \sigma^{-1})\, a^1_{\sigma^{-1}(1)} a^2_{\sigma^{-1}(2)} \cdots a^n_{\sigma^{-1}(n)}$$

$$= \sum_{\sigma \in S(n)} (\operatorname{sgn} \sigma)\, a^1_{\sigma(1)} a^2_{\sigma(2)} \cdots a^n_{\sigma(n)} .$$

In the 3rd equality, we have used the fact that $\operatorname{sgn} \sigma = \operatorname{sgn} \sigma^{-1}$. In the last equality we have used $\sum_{\sigma} = \sum_{\sigma^{-1}}$.