The matrices $d^{(j)}(\pi)$ satisfy the following property:

$$(d^{(j)}(\pi))_m^{m'} = \delta_{-m}^{m'}\,(-1)^{j-m}\,. \tag{22.41}$$

This fact will not be proved here but it can be seen to be a direct consequence of (22.48) below. The reader should check that it is satisfied by both $d^{(1/2)}(\pi)$ and $d^{(1)}(\pi)$ as given explicitly by (22.13) and (22.28), respectively.

(5) *Symmetry under sign reversal of the magnetic quantum numbers m and m' of the matrix element $(d^{(j)}(\beta))_m^{m'}$.* We have the following symmetry property:

$$(d^{(j)}(\beta))_m^{m'} = (d^{(j)}(-\beta))_{m'}^{m} = (-1)^{m'-m}\,(d^{(j)}(\beta))_{-m}^{-m'}\,. \tag{22.42}$$

Proof. Since all $d^{(j)}$ matrices commute, we have

$$d^{(j)}(\beta) = d^{(j)}(\pi)\,d^{(j)}(\beta)\,d^{(j)}(-\pi)\,. \tag{22.43}$$

Thus

$$\begin{aligned}
(d^{(j)}(\beta))_m^{m'} &= \sum_{l,n}(d^{(j)}(\pi))_l^{m'}\,(d^{(j)}(\beta))_n^l\,(d^{(j)}(-\pi))_m^n \\
&= \sum_{l,n}\delta_{-l}^{m'}\,(-1)^{j-l}\,(d^{(j)}(\beta))_n^l\,(d^{(j)}(\pi))_n^m \\
&= \sum_n(-1)^{j+m'}\,(d^{(j)}(\beta))_n^{-m'}\,\delta_{-n}^m\,(-1)^{j-n} \\
&= (-1)^{j+m'}\,(d^{(j)}(\beta))_{-m}^{-m'}\,(-1)^{j+m} = (-1)^{2j+m+m'}\,(d^{(j)}(\beta))_{-m}^{-m'}\,,
\end{aligned} \tag{22.44}$$

where in the second and third equalities we have used (22.41). Since $(-1)^{2(j+m)} = 1$, we have $(-1)^{2j+m} = (-1)^{-m}$, and thus $(-1)^{2j+m+m'} = (-1)^{m'-m}$. The result (22.42) follows. □

(6) *Relation to the spherical harmonics.* For $j = l$, l being an integer, we have the following relationships:

$$Y_l^m(\theta,\varphi) = (\cancel{-}1)^m\left(\frac{2l+1}{4\pi}\right)^{1/2}\{(D^{(l)}(\varphi,\theta,0))_0^m\}^* \tag{22.45}$$

$$P_l^m(\cos\theta) = (-1)^m\left(\frac{(l+|m|)!}{(l-|m|)!}\right)^{1/2}(d^{(l)}(\theta))_0^m\,, \tag{22.46}$$

$$P_l(\cos\theta) = P_l^0(\cos\theta) = (d^{(l)}(\theta))_0^0\,. \tag{22.47}$$

Finally, we give (without proof) a general formula for $d^{(j)}(\theta)$. For all allowed values of j,

$$\boxed{\begin{aligned}\left(d^{(j)}(\theta)\right)^{m}_{m'} = \sum_k & \frac{(-1)^k \sqrt{(j+m)(j-m)(j+m')(j-m')}}{k!\,(j+m-k)!(j-m'-k)!(k+m'-m)!} \\ & \times (\cos(\theta/2))^{2j+m-m'-2k}\,(\sin(\theta/2))^{2k+m'-m}\end{aligned}}\;, \qquad (22.48)$$

where the sum over k runs through the $2j+1$ values $0, 1, 2, \ldots, 2j$; but includes only terms for which the factorials have meaning. This equation, together with (22.3), give the complete expression for all rotation matrices.

$\boxed{\textbf{Problem 22.3}}$ Work out the spin 3/2 representations of J_1, J_2 and J_3 by computing the matrix elements of $J_\pm = J_1 \pm iJ_2$. Express your answers as 4 by 4 matrices.
