# MAT 444 Homework Assignments

**Prob. 21** Let $\omega \in A^r(M)$, write

$$\omega = a_{i_1 \ldots i_r}(x^1, \ldots, x^n)\, dx^{i_1} \wedge \ldots \wedge dx^{i_r}$$

where $(x^1, \ldots, x^n)$ are local coordinates of an $n$-dimensional differentiable manifold $M$. Show that

$$d\omega = (d a_{i_1 \ldots i_r}) \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_r}. \quad \text{—} (*)$$

Assume the following property: Suppose $\omega_1 \in A^r(M)$. Then, for any $\omega_2 \in A(M)$,

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2$$

(This is the "product rule" for exterior differentiation.

**Prob. 22** Use the result $(*)$ in Prob. 21 to verify the "product rule" (stated in that problem), when both $\omega_1 \in A^r(M)$ and $\omega_2 \in A^s(M)$ are monomials, that is, when

$$\omega_1 = a\, dx^{i_1} \wedge \ldots \wedge dx^{i_r} \;;\; \omega_2 = b\, dx^{i_1} \wedge \ldots \wedge dx^{i_s},$$

where $a$ and $b$ are both smooth functions on $M$.

**Prob. 23** Use Eq $(*)$ in Prob. 21 to show that, for $f \in A^0(M)$ ($f$ is a function on $M$),

$$d^2 f = 0.$$

**Prob. 24** The Lie Derivative of a tangent vector field $\vec{Y}$ with respect to ~~the~~ the tangent vector field $\vec{X}$ [ or the Lie Bracket of the tangent vector fields $\vec{X}$ and $\vec{Y}$ ] is defined by

$$L_{\vec{X}} \vec{Y} = [\vec{X}, \vec{Y}] \equiv \vec{X}\vec{Y} - \vec{Y}\vec{X} ,$$

where $\vec{X}\vec{Y}$ means the composition of the actions of the tangent vector fields $\vec{Y}$ followed by $\vec{X}$ in succession on a smooth function on $M$.

a) Show that the Lie bracket of two tangent vector fields is also a tangent vector field.

b) If $\vec{X} = X^i \partial_i$ and $\vec{Y} = Y^i \partial_i$, show that

$$[\vec{X}, \vec{Y}]^i = (L_{\vec{X}} \vec{Y})^i = X^j \partial_j Y^i - Y^j \partial_j X^i .$$

**Prob. 25** For a monomial $\omega \in A^r(N)$ [$\dim N = n$], that is $\omega = b_{i_1 \ldots i_r}(y_1, \ldots, y_n)\, dy^{i_1} \wedge \ldots \wedge dy^{i_r}$,

prove that

$$(f^* \circ d)\,\omega = (d \circ f^*)\,\omega .$$

This result actually holds for all $\omega \in A(N)$.