$\boxed{\text{Prob. 12}}$ The spin state vector of an electron, a spin 1/2 particle, is usually given as a two-component vector with respect to an orthonormal basis $\{\vec{e}_1, \vec{e}_2\}$ in spin space, where $\vec{e}_1$ and $\vec{e}_2$ are the normalized spin-up and spin-down vectors, respectively. A two-electron system state vector is then given in general by the tensor product:

$$S = S^{ij}\,\vec{e}_i \otimes \vec{e}_j \quad ; \quad i, j = 1, 2,$$

which is a vector in a 4-dimensional vector space with components S^{ij}, so it can be viewed as a contravariant rank-(2,0) tensor also. In physics, the spin wave vectors are usually just written as

$$\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle\}, \quad — (*)$$

$|\uparrow\downarrow\rangle$, for example, actually means $|\uparrow\rangle \otimes |\downarrow\rangle$, with $|\uparrow\rangle$ and $|\downarrow\rangle$ designating spin-up and spin-down single-electron state vectors, respectively. Construct an orthonormal basis in the tensor-product space with basis vectors that are either symmetric or anti-symmetric with respect to interchanges of electrons. Write a 4x4 transformation matrix between the basis (*) and the new basis with definite symmetries.

Prob 13 The total state vector of an electron is given by the tensor product of an orbital part and a spin part. Consider a p-electron (orbital angular momentum quantum number $l = 1$). Use the Dirac notation $|1\rangle, |0\rangle, |-1\rangle$ for the basis orbital state vectors (where the numbers indicate the z-component of the orbital angular momentum), and $|\uparrow\rangle, |\downarrow\rangle$ for the spin-up and spin down spin state vectors. What is the dimensionality of the tensor product space. Construct a set of basis states in the tensor product space for this electron.

Prob. 14 Consider a vector space $\mathbb{V}$, with $\dim \mathbb{V} = 2$. Show that $P^2(\mathbb{V})$ and $\Lambda^2(\mathbb{V})$ are irreducible subspaces of $SU(2)$ (special unitary group of dimension 2, or the group of all unitary 2×2 matrices with determinant $+1$). Show that $\dim(P^2(\mathbb{V})) = 3$ and $\dim(\Lambda^2(\mathbb{V})) = 1$.