MAT 444 Midterm 2 Spring 2017

Time allowed: 65 minutes

Do all 5 problems (each worth 20 points); show details for partial credit

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1. Suppose a four-dimensional vector space $\mathbb{V}$ has a basis $\{e_1, e_2, e_3, e_4\}$. In terms of these basis vectors give a basis for each of
   (a) $\Lambda^1(\mathbb{V})$,
   (b) $\Lambda^2(\mathbb{V})$,
   (c) $\Lambda^3(\mathbb{V})$,
   (d) $\Lambda^4(\mathbb{V})$.

2. Suppose $dim(\mathbb{V}) = 5$, $\xi \in \Lambda^2(\mathbb{V})$ and $\eta \in \Lambda^3(\mathbb{V})$. Explain the difference between $\xi \otimes \eta$ and $\xi \wedge \eta$ and define $\xi \wedge \eta$ in terms of $\xi \otimes \eta$. Express $\xi \wedge \eta$ in terms of $\eta \wedge \xi$. Give an example each for ξ and η in terms of a basis $\{e_i\}$ $(1 \le i \le 5)$ of $\mathbb{V}$.

3. Suppose $\{e_1, e_2, e_3\}$ is a basis of a three-dimensional vector space $\mathbb{V}$, and $\{e^{*1}, e^{*2}, e^{*3}\}$ its dual basis in $\mathbb{V}^*$. Compute
$$(e_1 \wedge e_2 \wedge e_3)\left(e^{*3}, e^{*2}, e^{*1}\right) ,$$
interpreted as the action of the wedge product on the ordered triplet. Show your calculation carefully.

4. The matrix representation of a linear map $A : \mathbb{V} \longrightarrow \mathbb{V}$ with respect to a basis $\{e_1, e_2, e_3\}$ of $\mathbb{V}$ is given by
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} .$$
Let $\varphi = e^{*1} \wedge e^{*2} \wedge e^{*3} \in \Lambda^3(\mathbb{V}^*)$. Calculate $A^*\varphi$ in terms of φ, where A^* is the pullback map $A^* : \Lambda^3(\mathbb{V}^*) \longrightarrow \Lambda^3(\mathbb{V}^*)$.

5. In the three-dimensional Euclidean manifold $\mathbb{R}^3$ consider the transformation between the Cartesian coordinates (x, y, z) and the spherical polar coordinates (r, θ, ϕ).
   (a) In the matrix equation
   $$(dx \quad dy \quad dz) = (dr \quad d\theta \quad d\phi)\, g' ,$$
   find the 3×3 matrix g'.
   (b) Use the result in (a) to calculate $dx \wedge dy \wedge dz$ in terms of $dr \wedge d\theta \wedge d\phi$.

MAT 444 MT2 Solutions

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① (a) $\Lambda^1(\mathbb{V}) = \mathbb{V}$.

$\therefore$ a basis is $\boxed{\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}}$

(b) The dimension of $\Lambda^2(\mathbb{V}) = \dfrac{4!}{2!(4-2)!} = 6$.

A basis is $\boxed{\{\vec{e}_1\wedge\vec{e}_2, \vec{e}_1\wedge\vec{e}_3, \vec{e}_1\wedge\vec{e}_4, \vec{e}_2\wedge\vec{e}_3, \vec{e}_2\wedge\vec{e}_4, \vec{e}_3\wedge\vec{e}_4\}}$.

(c) The dimension of $\Lambda^3(\mathbb{V}) = \dfrac{4!}{3!(4-3)!} = \dfrac{4!}{3!} = 4$.

A basis is $\boxed{\{\vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3, \vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_4, \vec{e}_1\wedge\vec{e}_3\wedge\vec{e}_4, \vec{e}_2\wedge\vec{e}_3\wedge\vec{e}_4\}}$.

(d) $\Lambda^4(\mathbb{V})$ is one-dimensional. A basis is

$$\boxed{\{\vec{e}_1\wedge\vec{e}_2\wedge\vec{e}_3\wedge\vec{e}_4\}}.$$

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② $\xi\otimes\eta$ is the tensor product of $\xi\in\Lambda^2(V)$ and $\eta\in\Lambda^3(\mathbb{V})$, when $\Lambda^2(V)$ and $\Lambda^3(\mathbb{V})$ are considered as vector spaces. Note that even as ξ and η are each totally antisymmetric, $\xi\otimes\eta$ is not necessarily totally antisymmetric.

$\xi\wedge\eta$ is the exterior (wedge) product of ξ and η. It is totally antisymmetric, and is defined in terms of $\xi\otimes\eta$ by

$$\xi\wedge\eta = A_5(\xi\otimes\eta)$$

where A_5 is the antisymmetrizer (operator) given by

$$A_5 = \frac{1}{5!}\sum_{\sigma\in S(5)}(\operatorname{sgn}\sigma)\cdot\sigma$$ ($S(5)$ is the permutation group of 5 objects)

(2) cont'd. An example of each ξ and η are given by

$$\xi = 2\,\vec{e}_1 \wedge \vec{e}_5 + 3\,\vec{e}_2 \wedge \vec{e}_4 - \vec{e}_4 \wedge \vec{e}_5 \quad ;$$

$$\eta = \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_5 - 2\,\vec{e}_3 \wedge \vec{e}_4 \wedge \vec{e}_5 \quad .$$

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(3) We use the <u>evaluation formula</u> for exterior products:

$$(\vec{e}_{i_1} \wedge \dots \wedge \vec{e}_{i_r})(\vec{v}^{*1}, \vec{v}^{*2}, \dots, \vec{v}^{*r})$$

$$= \frac{1}{r!} \begin{vmatrix} \langle \vec{e}_{i_1}, \vec{v}^{*1} \rangle & \cdots\cdots & \langle \vec{e}_{i_1}, \vec{v}^{*r} \rangle \\ \langle \vec{e}_{i_2}, \vec{v}^{*1} \rangle & \cdots\cdots & \langle \vec{e}_{i_2}, \vec{v}^{*r} \rangle \\ & \vdots & \\ \langle \vec{e}_{i_r}, \vec{v}^{*1} \rangle & \cdots\cdots & \langle \vec{e}_{i_r}, \vec{v}^{*r} \rangle \end{vmatrix} .$$

Hence $(\vec{e}_1 \wedge \vec{e}_2 \wedge \vec{e}_3)(\vec{e}^{*3}, \vec{e}^{*2}, \vec{e}^{*1})$

$$= \frac{1}{3!} \begin{vmatrix} \langle \vec{e}_1, \vec{e}^{*3} \rangle & \langle \vec{e}_1, \vec{e}^{*2} \rangle & \langle \vec{e}_1, \vec{e}^{*1} \rangle \\ \langle \vec{e}_2, \vec{e}^{*3} \rangle & \langle \vec{e}_2, \vec{e}^{*2} \rangle & \langle \vec{e}_2, \vec{e}^{*1} \rangle \\ \langle \vec{e}_3, \vec{e}^{*3} \rangle & \langle \vec{e}_3, \vec{e}^{*2} \rangle & \langle \vec{e}_3, \vec{e}^{*1} \rangle \end{vmatrix}$$

$$= \frac{1}{3!} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \frac{1}{6}(-1) = \boxed{-\frac{1}{6}} = \frac{1}{3!}\,\delta^{321}_{123}$$

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④ Write $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

This induces the pullback map $A^* : \Lambda^3(V^*) \to \Lambda^3(V^*)$. We have, for <u>any</u> $\varphi \in \Lambda^3(V^*)$,

$$A^* \varphi = (\det A)\, \varphi$$

$$\det A = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 1(2) + 2(-4) + 3(-1) = -9$$

$$\therefore \quad A^*(\vec{e}^{\,*1} \wedge \vec{e}^{\,*2} \wedge \vec{e}^{\,*3}) = (-9)(\vec{e}^{\,*1} \wedge \vec{e}^{\,*2} \wedge \vec{e}^{\,*3})$$

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⑤ Use $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ to get

(a)
$$\begin{aligned} dx &= \sin\theta\cos\phi\, dr + r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi \\ dy &= \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi \\ dz &= \cos\theta\, dr - r\sin\theta\, d\theta \end{aligned}$$

So we can write (in matrix form)

$$(dx \;\; dy \;\; dz) = (dr \;\; d\theta \;\; d\phi) \underbrace{\begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix}}_{g'}$$

(b) $dx \wedge dy \wedge dz = (\det g')\, dr \wedge d\theta \wedge d\phi$

$\det g' = r^2 \sin\theta$ (by straightforward calculation)

$$\therefore \boxed{dx \wedge dy \wedge dz = r^2 \sin\theta\, dr \wedge d\theta \wedge d\phi}$$