$\boxed{\text{Problem 27.2}}$ We will first calculate

$$\langle j_1 j_2 (j_{12}) j_3 ; j \mid j_2 j_3 (j_{23}) j_1 ; j \rangle$$

by direct expansion in terms of the $|j_1 m_1\rangle$, $|j_2 m_2\rangle$, $|j_3 m_3\rangle$ states. From the solution of Prob. 26.1, we had

$$|j_1 j_2 (j_{12}) j_3 ; j m\rangle$$

$$= \sum_{\substack{m_{12}, m_3 \\ m_1, m_2}} |j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle \times \langle j_1 j_2 ; m_1 m_2 \mid j_{12} m_{12}\rangle \langle j_{12} j_3 ; m_{12} m_3 \mid j m\rangle$$

Similarly, we have

$$|j_2 j_3 (j_{23}) j_1 ; j m\rangle \equiv |j_{23} j_1 ; j m\rangle$$

$$= \sum_{m_{23}, m_1'} |j_{23} j_1 ; m_{23} m_1'\rangle \langle j_{23} j_1 ; m_{23} m_1' \mid j m\rangle$$

$$= \sum_{m_{23}, m_1'} |j_{23} m_{23}\rangle |j_1 m_1'\rangle \langle j_{23} j_1 ; m_{23} m_1' \mid j m\rangle$$

$$= \sum_{\substack{m_{23}, m_1' \\ m_2', m_3'}} |j_2 j_3 ; m_2' m_3'\rangle \langle j_2 j_3 \, m_2' m_3' \mid j_{23} m_{23}\rangle |j_1 m_1'\rangle \times \langle j_{23} j_1 ; m_{23} m_1' \mid j m\rangle$$

$$= \sum_{\substack{m_{23}, m_1' \\ m_2', m_3'}} |j_2 m_2'\rangle |j_3 m_3'\rangle |j_1 m_1'\rangle \times \langle j_2 j_3 ; m_2' m_3' \mid j_{23} m_{23}\rangle \langle j_{23} j_1 ; m_{23} m_1' \mid j m\rangle$$

Recall from Prob. 26.1, that

$$\langle j_1 j_2 (j_{12}) j_3 ; j \mid j_2 j_3 (j_{23}) j_1 ; j \rangle$$

$$= \langle j_1 j_2 (j_{12}) j_3 ; j m \mid j_2 j_3 (j_{23}) j_1 ; j m \rangle$$

is independent of m.

$\boxed{\text{Prob. 27.2}}$ cont'd

$$\therefore \langle j_1 j_2 (j_{12}) j_3 ; j \,|\, j_2 j_3 (j_{23}) j_1 ; \rangle$$

$$= \sum_{\substack{m_{12}, m_3 \\ m_1, m_2 \\ m_{23}, m_1' \\ m_2', m_3'}} \langle j_{12} m_{12} | j_1 j_2 ; m_1 m_2 \rangle \langle j m | j_{12} j_3 ; m_{12} m_3 \rangle \times \langle j_2 j_3 ; m_2' m_3' | j_{23} m_{23} \rangle \langle j_{23} j_1 ; m_{23} m_1' | j' m \rangle \times \underbrace{\langle j_1 m_1 | \langle j_2 m_2 | \langle j_3 m_3 | \cdot | j_2 m_2' \rangle | j_3 m_3' \rangle | j_1 m_1' \rangle}_{= \delta_{m_1 m_1'} \delta_{m_2 m_2'} \delta_{m_3 m_3'}}$$

$$= \sum_{m_{12} m_{23} m_1 m_2 m_3} \langle j_1 j_2 ; m_1 m_2 | j_{12} m_{12} \rangle \langle j_{12} j_3 ; m_{12} m_3 | j m \rangle \times \langle j_2 j_3 ; m_2 m_3 | j_{23} m_{23} \rangle \langle j_{23} j_1 ; m_{23} m_1 | j m \rangle$$

[using the phase convention that all C-G coefficients are real]

We now let

$j_1 = 1$, $j_2 = \frac{1}{2}$, $j_3 = \frac{1}{2}$, $j = m = 1$, $j_{12} = \frac{1}{2}$, $j_{23} = 0$. Then

$$\langle j_1 j_2 (j_{12}) j_3 ; j \,|\, j_2 j_3 (j_{23}) j_1 ; j \rangle$$

$$= \langle 1, \tfrac{1}{2} (\tfrac{1}{2}) \tfrac{1}{2} ; 1 \,|\, \tfrac{1}{2}, \tfrac{1}{2} (0) 1 ; 1 \rangle$$

$$= \sum_{m_{12} m_{23} m_1 m_2 m_3} \langle \overset{j_1}{1}, \overset{j_2}{\tfrac{1}{2}} ; m_1 m_2 | \overset{j_{12}}{\tfrac{1}{2}}, m_{12} \rangle \langle \overset{j_{12}}{\tfrac{1}{2}}, \overset{j_3}{\tfrac{1}{2}} ; m_{12} m_3 | \overset{j}{1}, \overset{m}{1} \rangle \times \langle \overset{j_2}{\tfrac{1}{2}}, \overset{j_3}{\tfrac{1}{2}} ; m_2 m_3 | \overset{j_{23}}{0}, \underset{=0}{m_{23}} \rangle \langle \overset{j_{23}}{0}, \overset{j_1}{1} ; m_{23} m_1 | \overset{j}{1}, \overset{m}{1} \rangle$$

must be equal to 0 since $j_{23} = 0$

in this sum m_{23} can only take the value zero.

Prob. 27.2 cont'd

$$= \sum_{\substack{m_{12}\, \cancel{m_{23}}\, m_1 \\ m_2\, m_3}} \langle 1, \tfrac{1}{2}; m_1 m_2 | \tfrac{1}{2}, m_{12} \rangle \langle \tfrac{1}{2}, \tfrac{1}{2}; m_{12} m_3 | 1, 1 \rangle \langle \tfrac{1}{2}, \tfrac{1}{2}; m_2 m_3 | 0, 0 \rangle \langle 0, 1; 0\, m_1 | 1, 1 \rangle$$

$m_1 = 1$ (since in this CG coeff. $0 + m_1 = 1$)

$$= \sum_{m_{12}\, m_2\, m_3} \langle 1, \tfrac{1}{2}; 1, m_2 | \tfrac{1}{2}, m_{12} \rangle \langle \tfrac{1}{2}, \tfrac{1}{2}; m_{12} m_3 | 1, 1 \rangle \langle \tfrac{1}{2}, \tfrac{1}{2}; m_2 m_3 | 0, 0 \rangle \underbrace{\langle 0, 1; 0\, 1 | 1, 1 \rangle}_{= 1 \text{ (since } |1,1\rangle = |0,1;0,1\rangle)}$$

$$= \sum_{m_{12}\, m_2\, m_3} \langle 1, \tfrac{1}{2}; 1, m_2 | \tfrac{1}{2}, m_{12} \rangle \underbrace{\langle \tfrac{1}{2}, \tfrac{1}{2}; m_{12} m_3 | 1, 1 \rangle}_{\substack{\text{for this C.G coeff.} \neq 0,\ m_{12}+m_3 = 1, \\ \text{so } m_{12} = m_3 = 1/2}} \langle \tfrac{1}{2}, \tfrac{1}{2}; m_2, m_3 | 0\, 0 \rangle$$

$$= \sum_{m_2} \langle 1, \tfrac{1}{2}; 1, m_2 | \tfrac{1}{2}, \tfrac{1}{2} \rangle \langle \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2} | 1, 1 \rangle \underbrace{\langle \tfrac{1}{2}, \tfrac{1}{2}; m_2, \tfrac{1}{2} | 0\, 0 \rangle}_{\substack{\text{non-vanishing of this} \\ \text{CG coeff. requires } m_2 = -\frac{1}{2}}}$$

$$= \underbrace{\langle 1, \tfrac{1}{2}; 1, -\tfrac{1}{2} | \tfrac{1}{2}, \tfrac{1}{2} \rangle}_{} \underbrace{\langle \tfrac{1}{2}, \tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2} | 1, 1 \rangle}_{= 1\ [\text{by } (27.11)]} \underbrace{\langle \tfrac{1}{2}, \tfrac{1}{2}; -\tfrac{1}{2}, \tfrac{1}{2} | 0\, 0 \rangle}_{= -\frac{1}{\sqrt{2}}\ [\text{by } (27.11)]}$$

this is of the form
$\langle \ell, \tfrac{1}{2}; m-m', m' | j, m \rangle$ where
$\ell = 1$, $m = \tfrac{1}{2}$, $m' = -\tfrac{1}{2}$, $j = \tfrac{1}{2} = \ell - \tfrac{1}{2}$.
By (27.38), it is equal to

$$\sqrt{\frac{\ell + m + 1/2}{2\ell + 1}} = \sqrt{\frac{1 + 1/2 + 1/2}{3}} = \sqrt{\frac{2}{3}}$$

$$\therefore \langle 1, \tfrac{1}{2} (\tfrac{1}{2}) \tfrac{1}{2}; 1 | \tfrac{1}{2}, \tfrac{1}{2} (0) 1; 1 \rangle = \boxed{-\sqrt{\frac{1}{3}}}.$$

Prob. 27.2 cont'd We will now calculate the above matrix element using a 6-j symbol by means of the formula

$$\langle j_1 j_2 (j_{12}) j_3 ; j \mid j_2 j_3 (j_{23}) j_1 ; j \rangle$$
$$= (-1)^{j_1+j_2+j_3+j} \sqrt{(2j_{12}+1)(2j_{23}+1)} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix},$$

where the 6-j symbol is given by

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ \ell_1 & \ell_2 & \ell_3 \end{Bmatrix} = (-1)^{j_1+j_2+\ell_1+\ell_2} \Delta(j_1 j_2 j_3)\Delta(\ell_1 \ell_2 j_3)\Delta(\ell_1 j_2 \ell_3)\Delta(j_1 \ell_2 \ell_3)$$
$$\times \sum_k \frac{(-1)^k (j_1+j_2+\ell_1+\ell_2+1-k)!}{\left\{ k!(j_1+j_2-j_3-k)!(\ell_1+\ell_2-j_3-k)!(j_1+\ell_2-\ell_3-k)!(\ell_1+j_2-\ell_3-k)! \times (-j_1-\ell_1+j_3+\ell_3+k)!(-j_2-\ell_2+j_3+\ell_3+k)! \right\}},$$

and $$\Delta(abc) \equiv \sqrt{\frac{(a+b-c)!(a-b+c)!(b+c-a)!}{(a+b+c+1)!}}.$$

In the above summation over k, only those integer values of k are included which do not produce factorials of negative numbers. We have

$$\begin{array}{c} j_1\ j_2\ (j_{12})\ j_3\ ;\ j \qquad j_2\ j_3\ (j_{23})\ j_1\ ;\ j \\ \langle 1, \tfrac{1}{2} (\tfrac{1}{2}) \tfrac{1}{2} ; 1 \mid \tfrac{1}{2}, \tfrac{1}{2} (0) 1 ; 1 \rangle \end{array}$$
$$= (-1)^{1+\frac{1}{2}+\frac{1}{2}+1} \sqrt{(2\cdot\tfrac{1}{2}+1)(2\cdot 0+1)} \begin{Bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \end{Bmatrix}$$
$$= (-\sqrt{2}) \begin{Bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \end{Bmatrix}$$

Prob. 27.2 cont'd

$$\begin{Bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 \end{Bmatrix} = (-1)^{1+\frac{1}{2}+\frac{1}{2}+1} \Delta\left(1,\tfrac{1}{2},\tfrac{1}{2}\right)\Delta\left(\tfrac{1}{2},1,\tfrac{1}{2}\right)\Delta\left(\tfrac{1}{2},\tfrac{1}{2},0\right)\Delta(1,1,0)$$

$$\times \sum_k \frac{(-1)^k \left(1+\frac{1}{2}+\frac{1}{2}+1+1-k\right)!}{\left\{k!\left(1+\frac{1}{2}-\frac{1}{2}-k\right)!\left(\frac{1}{2}+1-\frac{1}{2}-k\right)!(1+1-0-k)!\left(\frac{1}{2}+\frac{1}{2}-0-k\right)! \times \left(-1-\frac{1}{2}+\frac{1}{2}+0+k\right)!\left(-\frac{1}{2}-1+\frac{1}{2}+0+k\right)!\right\}}$$

The factor involving the sum over k

$$= \sum_k \frac{(-1)^k (4-k)!}{k!(1-k)!(1-k)!(2-k)!(1-k)!(k-1)!(k-1)!}$$

In order for no negative factorials to occur, k ~~in the sum~~ can only take the value $k=1$. Thus

$$\sum_k \cdots\cdots = \frac{(-1)^1(4-1)!}{1!0!0!1!0!0!0!} = (-1)\,3! = -6$$

Also
$$\Delta\left(1,\tfrac{1}{2},\tfrac{1}{2}\right) = \sqrt{\frac{\left(1+\frac{1}{2}-\frac{1}{2}\right)!\left(1-\frac{1}{2}+\frac{1}{2}\right)!\left(\frac{1}{2}+\frac{1}{2}-1\right)!}{\left(1+\frac{1}{2}+\frac{1}{2}+1\right)!}} = \sqrt{\frac{1!\,1!\,0!}{3!}} = \sqrt{\frac{1}{6}}$$

$$\Delta\left(\tfrac{1}{2},1,\tfrac{1}{2}\right) = \sqrt{\frac{\left(\frac{1}{2}+1-\frac{1}{2}\right)!\left(\frac{1}{2}-1+\frac{1}{2}\right)!\left(1+\frac{1}{2}-\frac{1}{2}\right)!}{\left(\frac{1}{2}+1+\frac{1}{2}+1\right)!}} = \sqrt{\frac{1!\,0!\,1!}{3!}} = \sqrt{\frac{1}{6}}$$

$$\Delta\left(\tfrac{1}{2},\tfrac{1}{2},0\right) = \sqrt{\frac{\left(\frac{1}{2}+\frac{1}{2}-0\right)!\left(\frac{1}{2}-\frac{1}{2}+0\right)!\left(\frac{1}{2}+0-\frac{1}{2}\right)!}{\left(\frac{1}{2}+\frac{1}{2}+0+1\right)!}} = \sqrt{\frac{1!\,0!\,0!}{2!}} = \sqrt{\frac{1}{2}}$$

$$\Delta(1,1,0) = \sqrt{\frac{(1+1-0)!\,(1-1+0)!\,(1+0-1)!}{(1+1+0+1)!}} = \sqrt{\frac{2!}{3!}} = \sqrt{\frac{1}{3}}$$

Gathering the above results we have

$$\left\langle 1,\tfrac{1}{2}\left(\tfrac{1}{2}\right)\tfrac{1}{2};1 \,\middle|\, \tfrac{1}{2},\tfrac{1}{2}(0)1;1\right\rangle = \left(-\sqrt{2}\right)(-1)^3\sqrt{\frac{1}{6}}\sqrt{\frac{1}{6}}\sqrt{\frac{1}{2}}\sqrt{\frac{1}{3}}\cdot(-6)$$

$$= \boxed{-\sqrt{\frac{1}{3}}}\,, \text{ as before.}$$