A Mathematical Introduction to Robotic Manipulation

Chapter 2 Solutions

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Updated: October 1, 2019

1. Let $a, b, c \in \mathbb{R}^3$ be 3-vectors and let $\cdot$ and $\times$ denote the dot product and cross product in $\mathbb{R}^3$. Verify the following identities:

   (a) $a \cdot (b \times c) = (a \times b) \cdot c$

   (b) $a \times (b \times c) = (a \cdot b)b - (a \cdot b)c$

   Let $a, b, c, \in \mathbb{R}^3$ with the standard basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$. Using Einstein summation notation,

   (a)

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (a^i \vec{e_i}) \cdot (\epsilon^l{}_{jk} b^j c^k) \vec{e_l} \tag{1}$$
$$= \delta_{il} a^i \epsilon^l{}_{jk} b^j c^k \tag{2}$$
$$= \sum_i \epsilon^i{}_{jk} a^i b^j c^k \tag{3}$$
$$= \epsilon_{ijk} a^i b^j c^k \tag{4}$$
$$= \epsilon_{kij} a^i b^j c^k \tag{5}$$
$$= (\vec{a} \times \vec{b})_k c^k \tag{6}$$
$$= (\vec{a} \times \vec{b}) \cdot \vec{c} \tag{7}$$

   (b)

$$\vec{a} \times (\vec{b} \times \vec{c}) = \epsilon^i{}_{jk} a^j (\epsilon^k{}_{lm} b^l c^m) \vec{e_i} \tag{8}$$
$$= (\epsilon^i{}_{jk} \epsilon^k{}_{lm} a^j b^l c^m) \vec{e_i} \tag{9}$$
$$= \sum_i (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(a^j b^l c^m) \vec{e_i} \tag{10}$$
$$= \sum_j (a^j b^i c^j) \vec{e_i} - (a^j b^j c^i) \vec{e_i} \tag{11}$$
$$= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} \tag{12}$$

2. Using the homogeneous representation, show that $SE(3)$ satisfies the axioms of a group, with the group multiplication given by the usual matrix multiplication.

Let $g, h \in SE(3)$, $\bar{g} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix}$ and $\bar{h} = \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix}$

(a) Closure

$$\bar{g}\bar{h} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix} \tag{13}$$

$$= \begin{bmatrix} R_g R_h & R_g p_h + p_g \\ 0 & 1 \end{bmatrix} \tag{14}$$

Therefore, $gh = (R_g R_h, R_g p_h + p_g) \in SE(3)$

(b) Identity element exists
Let $e = (\mathbb{I}_{3x3}, 0) \in SE(3)$,

$$\bar{g}\bar{e} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbb{I}_{3x3} & 0 \\ 0 & 1 \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \tag{16}$$

$$= \bar{g} \tag{17}$$

and

$$\bar{e}\bar{g} = \begin{bmatrix} \mathbb{I}_{3x3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \tag{19}$$

$$= \bar{g} \tag{20}$$

Thus, $ge = eg = g \Rightarrow e \in SE(3)$ is the identity element.

(c) Inverse Exists Let $g \in SE(3)$, $\bar{g} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix}$.

Consider $\bar{h} = \begin{bmatrix} R_g^\top & -R_g^\top p_g \\ 0 & 1 \end{bmatrix} \in \mathbb{R}_{4x4}$

$$\bar{g}\bar{h} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_g^\top & -R_g^\top p_g \\ 0 & 1 \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} R_g R_g^\top & -R_g R_g^\top p_g + p_g \\ 0 & 1 \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} \mathbb{I}_{3x3} & 0 \\ 0 & 1 \end{bmatrix} \tag{23}$$

$$= \bar{g} \tag{24}$$

On the other hand,

$$\bar{h}\bar{g} = \begin{bmatrix} R_g^\top & -R_g^\top p_g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \tag{25}$$
$$= \begin{bmatrix} R_g^\top R_g & R_g^\top p_g - R_g^\top p_g \\ 0 & 1 \end{bmatrix} \tag{26}$$
$$= \begin{bmatrix} \mathbb{I}_{3x3} & 0 \\ 0 & 1 \end{bmatrix} \tag{27}$$
$$= \bar{g} \tag{28}$$

Since $\bar{h} = \bar{g}^{-1}$, $g^{-1} = (R_g^\top, -R_g^\top p_g)$.

(d) Associativity

Let $f, g, h \in SE(3)$, then $\bar{f} = \begin{bmatrix} R_f & p_f \\ 0 & 1 \end{bmatrix}$ $\bar{g} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix}$ $\bar{h} = \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix}$

$$(\bar{f}\bar{g})\bar{h} = \left(\begin{bmatrix} R_f & p_f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix} \tag{29}$$
$$= \left(\begin{bmatrix} R_f R_g & R_f p_g + p_f \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix} \tag{30}$$
$$= \begin{bmatrix} R_f R_g R_h & R_f R_g p_h + R_f p_g + p_f \\ 0 & 1 \end{bmatrix} \tag{31}$$

$$\bar{f}(\bar{g}\bar{h}) = \begin{bmatrix} R_f & p_f \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix} \right) \tag{32}$$
$$= \begin{bmatrix} R_f & p_f \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} R_g R_h & R_g p_h + p_g \\ 0 & 1 \end{bmatrix} \right) \tag{33}$$
$$= \begin{bmatrix} R_f R_g R_h & R_f R_g p_h + R_f p_g + p_f \\ 0 & 1 \end{bmatrix} \tag{34}$$

Thus, $(fg)h = f(gh)$.

3. *Properties of Rotation Matrices*
   Let $R \in SO(3)$ be a rotation matrix generated by rotating about a unit vector ω by θ radians. That is, R satisfies $R = e^{\hat{\omega}\theta}$.

   (a) Show that the eigenvalues of $\hat{\omega}$ are $0, i$, and i, where $i = \sqrt{-1}$. are the corresponding eigenvectors?

   (b) Show that the eigenvalues of R are $1, e^{i\theta}, and e^{-i\theta}$. is the eigenvector whose eigenvalue is 1?

   (c) Let $R = [r_1\ r_2\ r_3]$ be a rotation matrix. Show that $det R = r_1^\top (r_2 \times r_3)$.

(a) Find the eigenvalues and eigenvectors of $\widehat{w} = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}$, $\|\vec{w}\| = 1$.

$$\begin{aligned}
|\widehat{w} - \lambda \mathbb{I}| &= \begin{vmatrix} -\lambda & -\omega^3 & \omega^2 \\ \omega^3 & -\lambda & -\omega^1 \\ -\omega^2 & \omega^1 & -\lambda \end{vmatrix} && (35) \\
&= -\lambda(\lambda^2 + \omega_1^2) - \omega_3(\lambda\omega_3 - \omega_1\omega_2) - \omega_2(\omega_1\omega_3 + \lambda\omega_2) && (36)
\end{aligned}$$

(being in Euclidean $\mathbb{R}^3$, we can lower the ω indices for simplicity)

$$\begin{aligned}
& && (37) \\
&= -\lambda^3 - \lambda\omega_1^2 - \lambda\omega_3^2 + \omega_1\omega_2\omega_3 - \omega_1\omega_2\omega_3 - \lambda\omega_2^2 && (38) \\
&= -\lambda^3 - \lambda \|\omega\|^2 && (39) \\
&= -\lambda^3 - \lambda && (40) \\
&= 0 && (41)
\end{aligned}$$

implies $\lambda = 0, \pm i$. We find the corresponding eigenvectors.

i. $\lambda = 0$. Solve $\widehat{w}\vec{x} = \vec{0}$. Wlog, $\omega_3 \neq 0$, row reduce the following:

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{\omega_2}{\omega_3} \\ 1 & 0 & -\frac{\omega_1}{\omega_3} \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 0 & -\frac{\omega_1}{\omega_3} \\ 0 & 1 & -\frac{\omega_2}{\omega_3} \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{\omega_1}{\omega_3} \\ 0 & 1 & -\frac{\omega_2}{\omega_3} \\ 0 & \omega_1 & -\frac{\omega_1\omega_2}{\omega_3} \end{bmatrix} \to$$
$$\begin{bmatrix} 1 & 0 & -\frac{\omega_1}{\omega_3} \\ 0 & 1 & -\frac{\omega_2}{\omega_3} \\ 0 & 0 & 0 \end{bmatrix} \to t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

4. *Properties of skew-symmetric matrices* Show that the following properties of skew-symmetric matrices are true:

(a) If $R \in SO(3)$ and $w \in \mathbb{R}^3$, then $R\hat{w}R^\top = \widehat{Rw}$
It was easier to solve part (b) first and use those results for this problem. Let $\vec{x} \in \mathbb{R}^3$.

$$\begin{aligned}
(R\widehat{w}R^\top)\vec{x} &= R\widehat{w}(R^\top\vec{x}) && (42) \\
&= R(\vec{w} \times (R^\top\vec{x})) && (43) \\
&= R\vec{w} \times RR^\top\vec{x} && (44) \\
&= (\widehat{R\vec{w}})\vec{x} && (45)
\end{aligned}$$

(b) If $R \in SO(3)$ and $v, w \in \mathbb{R}^3$, then $R(\vec{v} \times \vec{w}) = (R\vec{v}) \times (R\vec{w})$.
We'll show equality by comparing the l^{th} component of each side. The result utilizes the relationship between the dot product and multiplication by transpose.

$$\begin{aligned}
[R(\vec{v} \times \vec{w})]^l &= \vec{e_l} \cdot R(\vec{v} \times \vec{w}) && (46)\\
&= \vec{e_l}^\top R(\vec{v} \times \vec{w}) && (47)\\
&= (R^\top \vec{e_l})^\top (\vec{v} \times \vec{w}) && (48)\\
&= (R^\top \vec{e_l}) \cdot (\vec{v} \times \vec{w}) && (49)
\end{aligned}$$

This is the determinant of matrix with columns $R^\top \vec{e_l}, \vec{v}$, and $\vec{w}$, respectively.

$$= det([R^\top \vec{e_l}, \vec{v}, \vec{w}]) \qquad (50)$$

Since $R^\top R = \mathbb{I}$,

$$\begin{aligned}
&= det(R^\top R[R^\top \vec{e_l}, \vec{v}, \vec{w}]) && (51)\\
&= det(R^\top [\vec{e_l}, R\vec{v}, R\vec{w}]) && (52)\\
&= det(R^\top) det([\vec{e_l}, R\vec{v}, R\vec{w}]) && (53)\\
&= det([\vec{e_l}, R\vec{v}, R\vec{w}]) && (54)\\
&= \vec{e_l} \cdot (R\vec{v} \times R\vec{w}) && (55)\\
&= (R\vec{v} \times R\vec{w})^l && (56)
\end{aligned}$$

Thus, $R(\vec{v} \times \vec{w}) = (R\vec{v}) \times (R\vec{w})$

(c) Show that $so(3)$ is a vector space. Determine its dimension and give a basis for $so(3)$.
Clearly, $0_{3\times 3} \in so(3)$.
Let $\alpha, \beta \in so(3), c \in \mathbb{R}$.

$$(c\alpha + \beta)^\top = c\alpha^\top + \beta^\top = -c\alpha - \beta = -(c\alpha + \beta).$$

$dim(so(3)) = 3$, spanned by the basis $\{ \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \}$

5. Cayley Parameters

Cayley Parameterization, like the exponential map, is a mapping from $so(3)$ to $SO(3)$. In this problem we show that $R_a = (\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a})$ is indeed an element of $SO(3)$, given $\widehat{a} \in so(3)$. The derivation of this mapping can be found going in the opposite direction. i.e. by letting $R_a \in SO(3)$ and showing $\widehat{a} \in so(3)$; this involves using the diagonals of the parallelogram formed by some vector $\vec{v}$ and its transformation $R_a\vec{v}$.

(a) Show $R_a = (\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) \in SO(3)$.
Since, $\widehat{a}$ is anti-symmetric, $\widehat{a}^\top = -\widehat{a}$. Therefore, $(\mathbb{I} \pm \widehat{a})^\top = (\mathbb{I} \mp \widehat{a})$. Recall from Linear Algebra that

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)^\top = B^\top A^\top$$
$$(A^{-1})^\top = (A^\top)^{-1}$$

Note that the transformations $(\mathbb{I} - \widehat{a})^{-1}$ and $(\mathbb{I} + \widehat{a})$ commute. To show this, we write $(\mathbb{I} + \widehat{a})$ in terms of $(\mathbb{I} - \widehat{a})$.

$$\begin{aligned}
(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) &= -(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) \\
&= -(\mathbb{I} - \widehat{a})^{-1}(-2\mathbb{I} + \mathbb{I} - \widehat{a}) \\
&= -(\mathbb{I} - \widehat{a})^{-1}(-2\mathbb{I} + (\mathbb{I} - \widehat{a})) \\
&= (\mathbb{I} - \widehat{a})^{-1}(2\mathbb{I} - (\mathbb{I} - \widehat{a})) \\
&= 2(\mathbb{I} - \widehat{a})^{-1} - (\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} - \widehat{a}) \\
&= 2(\mathbb{I} - \widehat{a})^{-1} - \mathbb{I} \\
&= 2(\mathbb{I} - \widehat{a})^{-1} - (\mathbb{I} - \widehat{a})(\mathbb{I} - \widehat{a})^{-1} \\
&= [2\mathbb{I} - (\mathbb{I} - \widehat{a})](\mathbb{I} - \widehat{a})^{-1} \\
&= (\mathbb{I} + \widehat{a})(\mathbb{I} - \widehat{a})^{-1}
\end{aligned}$$

Now,

$$\begin{aligned}
R_a^\top R_a &= [(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a})]^\top (\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) && (57) \\
&= (\mathbb{I} + \widehat{a})^\top [(\mathbb{I} - \widehat{a})^{-1}]^\top (\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) && (58) \\
&= (\mathbb{I} - \widehat{a})(\mathbb{I} + \widehat{a})^{-1}(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) && (59) \\
&= (\mathbb{I} - \widehat{a})(\mathbb{I} + \widehat{a})^{-1}(\mathbb{I} + \widehat{a})(\mathbb{I} - \widehat{a})^{-1} && (60) \\
&= \mathbb{I} && (61)
\end{aligned}$$

Similarly, $R_a R_a^\top = \mathbb{I}$. **Thus, R_a is orthogonal**. Now, we will show

$$det R_a = 1$$

$$\begin{aligned}
|(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a})| &= |(\mathbb{I} - \widehat{a})^{-1}||(\mathbb{I} + \widehat{a})| && (62) \\
&= |(\mathbb{I} - \widehat{a})^{-1}||(\mathbb{I} + \widehat{a})^\top| && (63) \\
&= |(\mathbb{I} - \widehat{a})^{-1}||(\mathbb{I} - \widehat{a})| && (64) \\
&= 1 && (65)
\end{aligned}$$

Since, $|R_a| = 1$, $R_a \in SO(3)$. $\square$

6. Unit Quaternions

   Let $Q = (q_0, \vec{q})$ and $P = (p_0, \vec{p})$ be quaternions, where $q_0, p_0 \in \mathbb{R}$ are the scalar parts of Q and P and $\vec{q}, \vec{p}$ are the vector parts.

   (a) Show H_1, the set of unit quaternions satisfies the axioms a group.
   Let $p, q, r \in H_1$.

      i. Closure
      $\|pq\| = \sqrt{(pq)(pq)^*} = \sqrt{pqq^*p^*} = \sqrt{p\,\|q\|^2\,p^*} = \sqrt{\|q\|^2\,\|p\|^2} = \|p\|\,\|q\| = 1.$
      $pq \in H_1$.

ii. Associativity come back to this problem after figuring out some formatting issues
$p = (p_0, \vec{p}), q = (q_0, \vec{q}), q = (r_0, \vec{r})$

$$\begin{aligned}
p(qr) =&(p_0, \vec{p})[(q_0, \vec{q})(r_0, \vec{r})] \\
=&(p_0, \vec{p})[(q_0 r_0 - \vec{q} \cdot \vec{r}, \vec{q} \times \vec{r} + q_0 \vec{r} + r_0 \vec{q})] \\
=&(p_0[q_0 r_0 - \vec{q} \cdot \vec{r}] - \vec{p} \cdot [\vec{q} \times \vec{r} + q_0 \vec{r} + r_0 \vec{q}], \\
&\vec{p} \times [\vec{q} \times \vec{r} + q_0 \vec{r} + r_0 \vec{q}] + p_0[\vec{q} \times \vec{r} + q_0 \vec{r} + r_0 \vec{q}] + [q_0 r_0 - \vec{q} \cdot \vec{r}]\vec{p}) \\
=&(p_0[q_0 r_0 - \vec{q} \cdot \vec{r}] - \vec{p} \cdot [\vec{q} \times \vec{r} + q_0 \vec{r} + r_0 \vec{q}], \\
&[\vec{p} \times \vec{q} + p_0 \vec{q} + q_0 \vec{p}] \times \vec{r} + [p_0 q_0 \vec{r} - (\vec{q} \cdot \vec{r})\vec{p}] + r_0[\vec{p} \times \vec{q} + p_0 \vec{q} + q_0 \vec{p}])
\end{aligned}$$

$$\begin{aligned}
(pq)r =&[(p_0, \vec{p})(q_0, \vec{q})](r_0, \vec{r}) \\
=&[(p_0 q_0 - \vec{p} \cdot \vec{q}, \vec{p} \times \vec{q} + p_0 \vec{q} + q_0 \vec{p})](r_0, \vec{r}) \\
=&([p_0 q_0 - \vec{p} \cdot \vec{q}]r_0 - [\vec{p} \times \vec{q} + p_0 \vec{q} + q_0 \vec{p}] \cdot \vec{r}, \\
&[\vec{p} \times \vec{q} + p_0 \vec{q} + q_0 \vec{p}] \times \vec{r} + [p_0 q_0 - \vec{p} \cdot \vec{q}]\vec{r} + r_0[\vec{p} \times \vec{q} + p_0 \vec{q} + q_0 \vec{p}])
\end{aligned}$$

iii. Identity
Let $\mathbf{1} = [1, \vec{0}]$.

$$\begin{aligned}
\mathbf{1}Q &= [1, \vec{0}][q, \vec{q}] \\
&= [1q - \vec{0} \cdot \vec{q}, \vec{0} \times \vec{q} + 1\vec{q} + q\vec{0}] \\
&= [q, \vec{q}]
\end{aligned}$$

iv. Inverse
$\forall Q \in H_1, Q^* \in H_1$ since $\|Q^*\| = \|Q\| = 1$. $Q^*Q = QQ^* = \mathbf{1}$.

(b) Let $\vec{x}$ be a point and let X be a quaternion whose scalar part is zero and whose vector part is equal to $\vec{x}$ (such a quaternion is called a pure quaternion). Show that if Q is a unit quaternion, the product QXQ^* is a pure quaternion and the vector part of QXQ^* satisfies

$$(q_0^2 - \vec{q} \cdot \vec{q})\vec{x} + 2(q_0(\vec{q} \times \vec{x}) + (\vec{x} \cdot \vec{q})\vec{q})$$

Verify that the vector part describes the point to which $\vec{x}$ is rotated under the rotation associated with Q.
Let $\vec{x} \in \mathbb{R}^3$. For $Q \in H_1$,

$$\begin{aligned}
QXQ^* &= (q_0, \vec{q})[(0, \vec{x})(q_0, -\vec{q})] \\
&= (q_0, \vec{q})[(\vec{x} \cdot \vec{q}, \vec{q} \times \vec{x} + q_0 \vec{x})] \\
&= (q_0(\vec{x} \cdot \vec{q}) - \vec{q} \cdot (\vec{q} \times \vec{x} + q_0 \vec{x}), \vec{q} \times (\vec{q} \times \vec{x} + q_0 \vec{x}) + q_0(\vec{q} \times \vec{x} + q_0 \vec{x}) + (\vec{q} \cdot \vec{x})\vec{q}) \\
&= (0, 2(\vec{q} \cdot \vec{x})\vec{q} + 2q_0(\vec{q} \times \vec{x}) + (q_0^2 - \vec{q} \cdot \vec{q})\vec{x}) \text{ bac-cab identity} \\
&= (0, 2((\vec{q} \cdot \vec{x})\vec{q} + q_0(\vec{q} \times \vec{x})) + (q_0^2 - \vec{q} \cdot \vec{q})\vec{x})
\end{aligned}$$

The vector part being

$$(q_0^2 - \vec{q} \cdot \vec{q})\vec{x} + 2((\vec{q} \cdot \vec{x})\vec{q} + q_0(\vec{q} \times \vec{x})).$$

Since $Q \in H_1, \exists \alpha \in (-\pi, \pi) : Q = [\cos\alpha, \hat{n}\sin\alpha]$, where $\hat{n} = \frac{\vec{q}}{\|\vec{q}\|}$ and $\cos\alpha = q_0$. We can rewrite the above equation in terms as

$$\boxed{\cos(2\alpha)\vec{x} + (1 - \cos(2\alpha))(\hat{n} \cdot \vec{x})\hat{n} + \sin(2\alpha)(\hat{n} \times \vec{x})}$$

On the other hand, it can be shown using vector calculus that a vector $\vec{x} \in \mathbb{R}^3$ rotated by θ about a unit vector $\hat{n}$ results in the vector

$$\boxed{\cos(\theta)\vec{x} + (1 - \cos(\theta))(\hat{n} \cdot \vec{x})\hat{n} + \sin(\theta)(\hat{n} \times \vec{x})}$$

(c) Show that the set of unit quaternions is a two-to-one covering of $SO(3)$. That is, for each $R \in SO(3)$, there exist two distinct unit quaternions which can be used to represent this rotation.

(d) Compare the number of additions and multiplications needed to perform the following operations:

i. Compose two rotation matrices.

ii. Compose two quaternions.

iii. Apply a rotation matrix to a vector.

iv. Apply a quaternion to a vector [as in part (b)].

Count a subtraction as an addition, and a division as a multiplication.

(e) Show that a rigid body rotating at unit velocity about a unit vector in $w \in \mathbb{R}^3$ can be represented by the quaternion differential equations where $\cdot$ represents quaternion multiplication.

7. A rigid body moving in $\mathbb{R}^2$ has three degrees of freedom (two com- ponents of translation and one of rotation), a rigid body moving in $\mathbb{R}^3$ has six degrees of freedom (three each of translation and rota- tion). Show that a rigid body moving in $\mathbb{R}^n$ will have $\frac{1}{2}(n + n^2)$ degrees of freedom. How many are translational and how many are rotational?

$SE(n) = \mathbb{R}^n \oplus SO(n)$. $SO(n)$ is generated by $so(n)$ is of dimension $\frac{1}{2}(n^2 - n)$. Therefore, $\frac{1}{2}(n^2 - n) + n = \boxed{\frac{1}{2}(n^2 + n)}$.

8. *Properties of the matrix exponential*
Let Λ be a matrix in $\mathbb{R}^{n \times n}$. The exponential of Λ is defined as

$$e^{\Lambda} = I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + ...$$

(a) Choose a matrix norm and show that the above series converges.

(b) Let $g \in R^{n \times n}$ be an invertible matrix. Show the following equality:

$$g e^{\Lambda} g^{-1} = e^{g\Lambda g^{-1}}$$

$$\begin{aligned}
ge^{\Lambda}g^{-1} &= g(I + \Lambda + \frac{\Lambda^2}{2!} + \frac{\Lambda^3}{3!} + ...)g^{-1} \\
&= I + g\Lambda g^{-1} + g\frac{\Lambda^2}{2!}g^{-1} + g\frac{\Lambda^3}{3!}g^{-1} + ... \\
&= I + g\Lambda g^{-1} + \frac{(g\Lambda g^{-1})^2}{2!} + \frac{(g\Lambda g^{-1})^3}{3!} + ... \\
&= e^{g\Lambda g^{-1}}
\end{aligned}$$

(c) Verify that

$$\frac{d}{dt}e^{\Lambda\theta} = (\Lambda\dot{\theta})e^{\Lambda\theta} = e^{\Lambda\theta}(\Lambda\dot{\theta}).$$

$$\begin{aligned}
\frac{d}{dt}e^{\Lambda\theta} &= \frac{d}{dt}(I + \sum_{k=1}^{\infty}\frac{(\Lambda\theta)^k}{k!}) \\
&= \sum_{k=1}^{\infty}\frac{d}{dt}\frac{\Lambda^k\theta^k}{k!} \\
&= \sum_{k=1}^{\infty}\frac{\Lambda^k}{k!}\frac{d}{dt}\theta^k \\
&= \sum_{k=1}^{\infty}\frac{\Lambda^k}{k!}k\theta^{k-1}\dot{\theta} \\
&= \Lambda\dot{\theta}\sum_{k=1}^{\infty}\frac{\Lambda^{k-1}}{k-1!}\theta^{k-1} \\
&= \Lambda\dot{\theta}e^{\Lambda\theta}
\end{aligned}$$

9. *Projection maps and proof of Proposition 2.9*
This problem completes the proof of Proposition 2.9 using the properties of projection maps on linear spaces. Assume $w \in so(3)$ and $\|\vec{w}\| = 1$.

(a) Given a vector $\vec{w} \in \mathbb{R}^3$, let N_w denote the subspace spanned by w and $N_w^{\perp}$ denote the orthogonal complement. Show that

$$image(\hat{w}) = N_w^{\perp} \text{ and } kernel(\hat{w}) = N_w.$$

i. Let $\vec{x} \in \mathbb{R}^3$.
$\langle \vec{w}, \hat{w}\vec{x} \rangle = \vec{w} \cdot (\vec{w} \times \vec{x}) = \vec{0} \Rightarrow \hat{w}\vec{x} \in N_w^{\perp}$. Thus, $image(\hat{w}) \subseteq N_w^{\perp}$.
$\forall \vec{y} \in N_w^{\perp}$, let $\vec{x} = -\vec{w} \times \vec{y}$. Since $\hat{w}\vec{x} = \vec{y}, \vec{y} \in image(\hat{w})$. Thus, $N_w^{\perp} \subseteq image(\hat{w})$. Consequently, $image(\hat{w}) = N_w^{\perp}$.

ii. $\forall \vec{x} \in N_w$, $\hat{w}\vec{x} = \vec{w} \times (t\vec{w}) = \vec{0}$, for some $t \in \mathbb{R}$.
Thus $N_w \subseteq kernel(\hat{w})$.
Let $\vec{x} \in kernel(\hat{w})$. Since $\mathbb{R}^3 = N_w \oplus N_w^{\perp}$, $\vec{x} = \vec{x}_w + \vec{x}_w^{\perp}$.

$$\vec{0} = \hat{w}\vec{x} = \vec{w} \times (\vec{x}_w + \vec{w}_w^{\perp}) = \vec{0} + \vec{w} \times \vec{x}_w^{\perp}$$

implies $\vec{x}_w^\perp = \vec{0}$. Thus, $\vec{x} = \vec{x}_w \in N_w$ so that $kernel(\hat{w}) \subseteq N_w$.
$kernel(\hat{w}) = N_w$.

(b) Let $V \subseteq \mathbb{R}^n$ be a linear subspace. A projection map is a linear mapping $P_V : \mathbb{R}^n \to V$ which satisfies $image(P_V) = V$ and $P_V(x) = x \, \forall x \in V$. Show that

$$P_{N_w} = ww^\top \text{ and } P_{N_w^\perp} = (I - ww^\top).$$

are both projection maps. (w understood to be vector $\vec{w}$)

i. Let $x \in \mathbb{R}^3$.
$P_{N_w}(x) = ww^\top x = w\langle w, x\rangle \in N_w$ implies $image(P_{N_w}) \subseteq N_w$.
Let $\vec{y} \in N_w$. Then $\vec{y} = \|\vec{y}\| \, \vec{w}$.
$\forall \vec{y} \in N_w$, since

$$P_{N_w}(\vec{y}) = ww^\top \vec{y} = \vec{w}\langle \vec{w}, \|\vec{y}\| \, \vec{w}\rangle = \|\vec{y}\| \, \vec{w} = \vec{y},$$

$\vec{y}$ is the pre-image of $\vec{y}$ under P_{N_w}. Thus $N_w \subseteq image(P_{N_w})$ implies $image(P_{N_w}) = N_w$ and P_{N_w} is a projection map.

ii. REDO Let $\vec{y} \in N_W^\perp$.

$$\langle \vec{w}, (I - \vec{w}\vec{w}^\top)\vec{y}\rangle = \langle \vec{w}, \vec{y}\rangle - \langle \vec{w}, \vec{w}\vec{w}^\top \vec{y}\rangle = \vec{0} - \vec{0} = \vec{0}$$

Thus $(I - ww^\top)\vec{y} \in N_w^\top \Rightarrow image(P_{N_w^\perp}) \subseteq N_w^\perp$.

(c) Calculate the null space of $I - e^{\hat{w}\theta}$ for $\hat{w} \in so(3)$ and $\theta \in (0, 2\pi)$ and show that $(I - e^{\hat{w}\theta}) : N_w^\perp \to N_w^\perp$ is bijective.

i. $ker(I - e^{\hat{w}\theta}) = N_w$

Using Rodrigues' Forumula

$$\begin{aligned}
I - e^{\hat{w}\theta} &= I - (I + (sin\, \|\vec{w}\|\, \theta)\frac{\hat{w}}{\|\vec{w}\|} + (1 - cos\, \|\vec{w}\|\, \theta)\frac{\hat{w}^2}{\|\vec{w}\|^2}) \\
&= -(sin\, \|\vec{w}\|\, \theta)\frac{\hat{w}}{\|\vec{w}\|} - (1 - cos\, \|\vec{w}\|\, \theta)\frac{\hat{w}^2}{\|\vec{w}\|^2} \\
&= \alpha\hat{w} - \beta\hat{w}^2
\end{aligned}$$

where $\alpha = \frac{-sin\|\vec{w}\|\theta}{\|\vec{w}\|}$, $\beta = \frac{1-cos\|\vec{w}\|\theta}{\|\vec{w}\|^2}$.

$N_w \subseteq ker(I - e^{\hat{w}\theta})$ is clear from inspection. Let $\vec{v} \in ker(I - e^{\hat{w}\theta})$. Then

$$\begin{aligned}
\vec{0} &= (I - e^{\hat{w}\theta})\vec{v} \\
&= \alpha\hat{w}\vec{v} - \beta\hat{w}^2\vec{v} \\
&= \alpha(\vec{w} \times \vec{v}) - \beta(\vec{w} \times \vec{w} \times \vec{v}) \\
&= \alpha(\vec{w} \times \vec{v}) - \beta(\vec{w}(\vec{w} \cdot \vec{w}) - \|\vec{w}\|^2 \vec{v})
\end{aligned}$$

Since $(\vec{w} \times \vec{v})$ is orthogonal to $\vec{v}$ and $\vec{w}$, it's component along those directions are both 0.

$$\begin{aligned} \alpha(\vec{w} \times \vec{v}) &= \beta(\vec{w}(\vec{w} \cdot \vec{w}) - \|\vec{w}\|^2 \vec{v}) \\ &= \vec{0} \end{aligned}$$

Thus, $\vec{v} \in ker(\hat{w}) = N_w \Rightarrow ker(I - e^{\hat{w}\theta}) \subseteq N_w \Rightarrow ker(I - e^{\hat{w}\theta}) = N_w \quad \square$.

ii. $(I - e^{\hat{w}\theta}) : N_w^{\perp} \to N_w^{\perp}$ is bijective.

Since $(I - e^{\hat{w}\theta})$ is a linear map and $N_w^{\perp}$ is finite-dimensional, it suffices to show that $(I - e^{\hat{w}\theta})$ is one-to-one.

$$ker(I - e^{\hat{w}\theta}) \cap N_w^{\perp} = \{\vec{0}\}$$

Thus $(I - e^{\hat{w}\theta})$ is 1-1 $\iff$ $(I - e^{\hat{w}\theta})$ onto. $\square$

(d) Let $A = (I - e^{\hat{w}\theta})\hat{w} + \vec{w}\vec{w}^{\top}\theta$, where $\theta \in (0, 2\pi)$. Show that $A : \mathbb{R}^3 \to \mathbb{R}^3$ is invertible.

Suppose $\vec{x} \in ker(A)$. Since $\mathbb{R}^3 = N_w \oplus N_w^{\perp}$, $\vec{x} = \vec{x}_w + \vec{x}_{\perp}$.

$$\begin{aligned} \vec{0} &= (I - e^{\hat{w}\theta})\hat{w}\vec{x} + \vec{w}\vec{w}^{\top}\theta\vec{x} \\ &= (I - e^{\hat{w}\theta})\hat{w}\vec{x} + \theta\vec{x}_w \end{aligned}$$

The terms above being in orthogonal subspaces, $N_w^{\perp}$ and N_w, respectively, implies that $(I - e^{\hat{w}\theta})\hat{w}\vec{x} = \vec{0}$ and $\vec{x}_w = \vec{0}$. Then,

$$\begin{aligned} \vec{x} &= \vec{0} + \vec{x}_{\perp} \\ &= \vec{x}_{\perp} \end{aligned}$$

so that

$$\begin{aligned} (I - e^{\hat{w}\theta})\hat{w}\vec{x}_{\perp} &= \vec{0} \\ \iff \hat{w}\vec{x}_{\perp} &= \vec{0} \end{aligned}$$

since $(I - e^{\hat{w}\theta})$ is bijective. It follows that $\vec{x}_{\perp} = \vec{0}$.
Thus $\vec{x} = \vec{0} \Rightarrow ker(A) = \{\vec{0}\} \Rightarrow A$ is invertible. $\square$

10.

11. *Planar rigid body transformations*
A transformation $g = (p, R) \in SE(2)$ consists of a translation $p \in \mathbb{R}^2$ and a 2×2 rotation matrix R. We represent this in homogeneous coordinates as a 3×3 matrix: