MAT 444 [redacted] **Problem Solutions** [redacted]

Prob 1[redacted] Consider the transformation of coordinate frames given by

$$e_i = g_i^j \, e'_j \,,$$

where $g^{-1} = g^T$, that is g is an **orthogonal matrix**.

(a)

$$(\delta')^{ij} = g^i_k g^j_l \, \delta^{kl} = \sum_k g^i_k g^j_k = (g^T)^k_i g^j_k = (g^T g)^j_i = \delta^j_i = \delta^{ij} \,.$$

(b)

$$(\delta')_{ij} = (g^{-1})^k_i (g^{-1})^l_j \, \delta_{kl} = \sum_k (g^{-1})^k_i (g^{-1})^k_j = \sum_k (g^{-1})^k_i (g^T)^k_j$$
$$= (g^{-1})^k_i g^j_k = (g^{-1} g)^j_i = \delta^j_i = \delta_{ij} \,.$$

(c)

$$(\delta')^j_i = g^j_k (g^{-1})^l_i \, \delta^k_l = (g^{-1})^k_i g^j_k = (g^{-1} g)^j_i = \delta^j_i \,.$$

(d)

$$(\varepsilon')^i_{jk} = g^i_l (g^{-1})^m_j (g^{-1})^n_k \, \varepsilon^l_{mn} = g^i_l g^j_m g^k_n \, \varepsilon^{lmn} \,.$$

When (ijk) is an **even permutation** of (123), the RHS of the last equality in the above equation equals $\det(g) = 1$ [since $g \in SO(3)$]. In this case $(\varepsilon')^i_{jk} = 1$.

When (ijk) is an **odd permutation** of (123), the RHS equals -1, which implies $(\varepsilon')^i_{jk} = -1$.

Finally, when any two of the three indices (ijk) are equal (for example, $i = j$), the RHS is a 3×3 determinant of a matrix with two identical columns. So the determinant vanishes and $(\varepsilon')^i_{jk} = 0$.

Prob 2.[redacted] In the expression $\mathcal{I}^{ij} = \delta^{ij} x^l x_l - x^i x^j$, we want to verify that both terms transform like a $(2,0)$-tensor. In the first term, $x^l x_l$ is a scalar, while δ^{ij} transforms like a $(2,0)$-tensor [as shown in Problem 1[redacted](a)], so the whole term transforms like a $(2,0)$-tensor. For the second term, we have,[redacted]

$$x'^i x'^j = g^i_k g^j_l \, x^k x^l \,.$$

Hence the second term transforms like a $(2,0)$-tensor also. So $\mathcal{I}^{ij}$ is a $(2,0)$-tensor.

[redacted]

Prob. 3

[redacted] We want to show that

$$\varepsilon^{i}{}_{jk}\,\varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell} \qquad \text{—— } (*)$$

When $j=k$, $\varepsilon^{i}{}_{jk}=0$, hence LHS$=0$; RHS $= \delta_{j\ell}\delta_{jm} - \delta_{jm}\delta_{j\ell} = 0$.

When $\ell = m$, $\varepsilon_{i\ell m}=0$, hence LHS$=0$; RHS $= \delta_{j\ell}\delta_{k\ell} - \delta_{j\ell}\delta_{k\ell} = 0$.

It remains to show (*) when $j \neq k$ and $\ell \neq m$.

Consider the following cases:

j	k	ℓ	m
1	2	1	2
1	2	2	3
1	2	1	3
2	3	1	2
2	3	2	3
2	3	1	3
1	3	1	2
1	3	2	3
1	3	1	3

All other cases involve
i) $j \leftrightarrow k$ ii) $\ell \leftrightarrow m$ iii) $j \leftrightarrow k$ and $\ell \leftrightarrow m$,
where $\leftrightarrow$ means interchanging.

For i) and ii) both the LHS and RHS of (*) change sign.
For iii) both sides are unchanged. So we just need to prove (*) for the cases listed in the table. We have, for example,

$$\varepsilon^{i}{}_{12}\,\varepsilon_{i12} = \varepsilon^{3}{}_{12}\,\varepsilon_{312} = 1.1 = 1 \;;\; \delta_{11}\delta_{22} - \delta_{12}\delta_{21} = \delta_{11}\delta_{22} = 1.1 = 1;$$

$$\varepsilon^{i}{}_{12}\,\varepsilon_{i23} = \varepsilon^{3}{}_{12}\,\varepsilon_{323} = 1.0 = 0 \;;\; \delta_{12}\delta_{23} - \delta_{13}\delta_{22} = 0,$$

$$\varepsilon^{i}{}_{12}\,\varepsilon_{i13} = \varepsilon^{3}{}_{12}\,\varepsilon_{313} = 1.0 = 0 \;;\; \delta_{11}\delta_{23} - \delta_{13}\delta_{21} = 0$$

(*) can be proved similarly for the other 6 quadruplets of values for j, k, ℓ, m shown in the table.

$\boxed{\text{Prob. 4}}$

$$(\vec{A}\times(\vec{B}\times\vec{C}))^i = \varepsilon^i{}_{jk} A^j (\vec{B}\times\vec{C})^k$$

$$= \varepsilon^i{}_{jk} A^j \varepsilon^k{}_{lm} B^l C^m = \varepsilon^i{}_{jk}\varepsilon^k{}_{lm} A^j B^l C^m$$

$$= \varepsilon^k{}_{ij}\, \varepsilon_{klm} A^j B^l C^m$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A^j B^l C^m$$

$$= (\delta_{il} B^l)(\delta_{jm} A^j C^m) - (\delta_{im} C^m)(\delta_{jl} A^j B^l)$$

$$= B^i A_m C^m - C^i A_l B^l = B^i(\vec{A}\cdot\vec{C}) - C^i(\vec{A}\cdot\vec{B})$$

$$\therefore\ \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$\boxed{\text{Prob. 5}}$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A_i (\vec{B}\times\vec{C})^i = A_i \varepsilon^i{}_{jk} B^j C^k$$

$$= \varepsilon^i{}_{jk} A_i B^j C^k = \varepsilon_{ijk} A^i B^j C^k$$

$$= \varepsilon_{jki} B^j C^k A^i = \varepsilon^j{}_{ki} B_j C^k A^i$$

$$= B_j (\varepsilon^j{}_{ki} C^k A^i) = B_j (\vec{C}\times\vec{A})^j$$

$$= \vec{B}\cdot(\vec{C}\times\vec{A}) .$$

Similarly,

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = A^i (\vec{B}\times\vec{C})_i = \varepsilon_{ijk} A^i B^j C^k$$

$$= \varepsilon_{kij} C^k A^i B^j = \varepsilon^k{}_{ij} C_k A^i B^j$$

$$= C_k (\varepsilon^k{}_{ij} A^i B^j) = C_k (\vec{A}\times\vec{B})^k$$

$$= \vec{C}\cdot(\vec{A}\times\vec{B})$$

$\boxed{\text{Prob. 6}}$ A, being linear, maps $\vec{0} \in \mathbb{V}$ to $\vec{0}$ in $\mathbb{W}$. Hence $\vec{0} \in \operatorname{Ker} A$. Suppose $\vec{v}_1, \vec{v}_2 \in \operatorname{Ker} A$, then

$$A(\vec{v}_1 + \vec{v}_2) = A(\vec{v}_1) + A(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0},$$

which implies $\vec{v}_1 + \vec{v}_2 \in \operatorname{Ker} A$. If $\vec{v} \in \operatorname{Ker} A$, then $A(\alpha\vec{v}) = \alpha A(\vec{v}) = \alpha \cdot \vec{0} = \vec{0}$, for all $\alpha \in \mathbb{F}$, implying $\alpha\vec{v} \in \operatorname{Ker} A$. So $\operatorname{Ker} A \subset \mathbb{V}$ is a vector subspace of $\mathbb{V}$.

Clearly, $\vec{0} \in \mathbb{W}$ is in $\operatorname{Im} A$, since $A(\vec{0}) = \vec{0}$. If $\vec{w}_1, \vec{w}_2 \in \operatorname{Im} A$, then there exist $\vec{v}_1, \vec{v}_2 \in \mathbb{V}$ such that

$$A(\vec{v}_1) = \vec{w}_1, \quad A(\vec{v}_2) = \vec{w}_2.$$

So $A(\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2) = \alpha_1 A(\vec{v}_1) + \alpha_2 A(\vec{v}_2) = \alpha_1\vec{w}_1 + \alpha_2\vec{w}_2$.

Hence $\alpha_1\vec{w}_1 + \alpha_2\vec{w}_2 \in \operatorname{Im} A$. So $\operatorname{Im} A$ is a vector subspace of $\mathbb{W}$.

Let $\{\vec{v}_1, \ldots, \vec{v}_r\}$ be a basis of $\operatorname{Ker} A$ and $\{\vec{w}_1{}', \ldots, \vec{w}_s{}'\}$ be a basis of $\operatorname{Im} A$. For each $1 \le i \le s$, let $\vec{v}_i{}' \in \mathbb{V}$ such that $A(\vec{v}_i{}') = \vec{w}_i{}'$. Consider the set of vectors

$$\{\vec{v}_1, \ldots, \vec{v}_r, \vec{v}_1{}', \ldots, \vec{v}_s{}'\}.$$

We will show that it forms a basis of $\mathbb{V}$.

Choose an arbitrary $\vec{v} \in \mathbb{V}$. Then $A(\vec{v}) \in \operatorname{Im} A$. So we can write ~~(using Einstein summation convention)~~

$$A(\vec{v}) = \sum_{i=1}^{s} c^i \vec{w}_i{}' = \sum_{i=1}^{s} c^i A(\vec{v}_i{}') \quad ; \quad c^i \in \mathbb{F}.$$

From the linearity of A we have

$$A\left(\vec{v} - \sum_{i=1}^{s} c^i \vec{v}_i{}'\right) = 0.$$

Thus $\vec{v} - \sum_{i=1}^{s} c^i \vec{v}_i{}' \in \operatorname{Ker} A$, so we can write

$$\vec{v} - \sum_{i=1}^{s} c^i \vec{v}_i{}' = \sum_{i=1}^{r} d^i \vec{v}_i \quad ; \quad d^i \in \mathbb{F},$$

$\boxed{\text{Prob. 6}}$ cont'd.

$$\text{or} \qquad \vec{v} = \sum_{i=1}^{s} c^i \vec{v}_i{}' + \sum_{i=1}^{r} d^i \vec{v}_i ,$$

that is, $\vec{v}$ is a linear combination of $\{\vec{v}_1, \dots, \vec{v}_r, \vec{v}_1{}', \dots, \vec{v}_s{}'\}$. We still have to show that this set is linearly independent.

Suppose
$$\sum_{i=1}^{r} a^i \vec{v}_i + \sum_{j=1}^{s} b^j \vec{v}_j{}' = \vec{0} , \quad a^i, b^j \in \mathbb{F} , \ 1 \le i \le r , \ 1 \le j \le s .$$

Then, since $A(\vec{0}) = \vec{0}$, we have

$$\vec{0} = A\left(\sum_{i=1}^{r} a^i \vec{v}_i + \sum_{j=1}^{s} b^j \vec{v}_j{}' \right) = \sum_{i=1}^{r} a^i \overbrace{A(\vec{v}_i)}^{= \vec{0} \text{ (since } \vec{v}_i \in \operatorname{Ker} A)} + \sum_{j=1}^{s} b^j A(\vec{v}_j{}')$$

$$= \sum_{j=1}^{s} b^j A(\vec{v}_j{}') = \sum_{j=1}^{s} b^j \vec{w}_j{}' .$$

Since $\{\vec{w}_i{}'\}$, being a basis of $\operatorname{Im} A$, is linearly independent,

$$b^i = 0 , \quad 1 \le i \le s .$$

Thus
$$\sum_{i=1}^{r} a^i \vec{v}_i = 0$$

But $\{\vec{v}_i\}$ is linearly independent, since it is a basis of $\operatorname{Ker} A$. Therefore

$$a^i = 0 , \quad 1 \le i \le r .$$

So we have shown that the linearly independent set
$$\{\vec{v}_1, \dots, \vec{v}_r, \vec{v}_1{}', \dots, \vec{v}_s{}'\}$$
is a basis of $\mathbb{V}$. Finally we conclude

$$\dim(\mathbb{V}) = \dim(\operatorname{Ker} A) + \dim(\operatorname{Im} A) .$$

$\boxed{\text{Prob. 7}}$ $\dfrac{\mathbb{V}}{\mathbb{W}}$ is made into a vector space by the following definitions of vector addition and scalar multiplication:

$$\{\vec{v}_1\} + \{\vec{v}_2\} = \{\vec{v}_1 + \vec{v}_2\} \quad , \vec{v}_1, \vec{v}_2 \in \mathbb{V}$$
$$\alpha\{\vec{v}\} = \{\alpha\vec{v}\} \quad , \alpha \in \mathbb{F}$$

To show these definitions make sense we have to show that they are independent of the <u>representative</u> of the equivalence class $\{\vec{v}\}$.

Suppose $\vec{v}_1{}'$ is another representative of $\{\vec{v}_1\}$ (besides $\vec{v}_1$) and $\vec{v}_2{}'$ is another representative of $\{\vec{v}_2\}$ (besides $\vec{v}_2$).

let $\vec{v}_1{}' = \vec{v}_1 + \vec{w}_1$, $\vec{v}_2{}' = \vec{v}_2 + \vec{w}_2$, $\vec{w}_1, \vec{w}_2 \in \mathbb{W}$

Then $\{\vec{v}_1{}'\} + \{\vec{v}_2{}'\} = \{\vec{v}_1{}' + \vec{v}_2{}'\}$ by def. of vector addition in $\mathbb{V}/\mathbb{W}$

$$= \{\vec{v}_1 + \vec{w}_1 + \vec{v}_2 + \vec{w}_2\} = \{\vec{v}_1 + \vec{v}_2 + (\vec{w}_1 + \vec{w}_2)\}$$

$$= \{\vec{v}_1 + \vec{v}_2\} \qquad (\text{since } \vec{w}_1 + \vec{w}_2 \in \mathbb{W}, \ \mathbb{W} \text{ being a subspace})$$

$$= \{\vec{v}_1\} + \{\vec{v}_2\} .$$

Also, for $\alpha \in \mathbb{F}$, $\vec{v}\,'$ another representative of $\{\vec{v}\}$ (besides $\vec{v}$), let $\vec{v}\,' = \vec{v} + \vec{w}$, $\vec{w} \in \mathbb{W}$

$$\alpha\{\vec{v}\,'\} = \alpha\{\vec{v} + \vec{w}\} = \alpha\{\vec{v}\} = \{\alpha\vec{v}\}$$

by def. of scalar multiplication in $\mathbb{V}/\mathbb{W}$

$$= \{\alpha(\vec{v}\,' - \vec{w})\} = \{\alpha\vec{v}\,' - \alpha\vec{w}\}$$

$$= \{\alpha\vec{v}\,'\}$$

So we have shown that both vector addition and scalar multiplication in $\mathbb{V}/\mathbb{W}$ are independent of the representatives.

$\boxed{\text{Prob. 7}}$ cont'd.

The zero element in $\mathbb{V}/\mathbb{W}$ is $\mathbb{W}$ itself, since for $\vec{w} \in \mathbb{W}$;

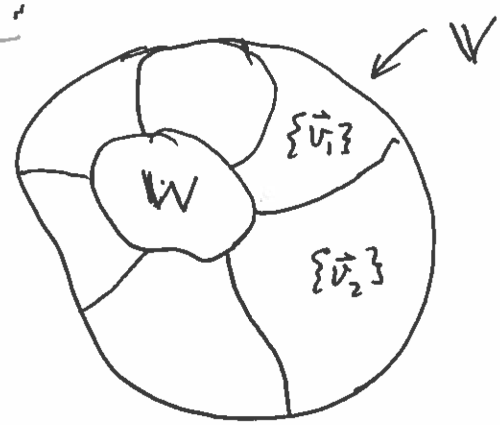
$$\{\vec{v}\} + \{\vec{w}\} = \{\vec{v} + \vec{w}\} = \{\vec{v}\}$$

Next we show that <u>$\pi : \mathbb{V} \to \mathbb{V}/\mathbb{W}$ is a linear map.</u>

$$\pi(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) = \{\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2\}$$

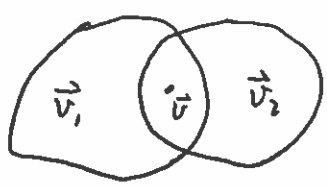
$$= \alpha_1 \{\vec{v}_1\} + \alpha_2 \{\vec{v}_2\} = \alpha_1 \pi(\vec{v}_1) + \alpha_2 \pi(\vec{v}_2).$$

We know that <u>all the equivalence classes in $\mathbb{V}/\mathbb{W}$ are disjoint.</u>



For if this is not the case, suppose $\{\vec{v}_1\}, \{\vec{v}_2\}$ are two distinct equivalence classes, and there exists a $\vec{v}$ which is a representative of both $\{\vec{v}_1\}$ and $\{\vec{v}_2\}$. Then we have

$\vec{v}_1$ and $\vec{v}_2$ are representatives of different equivalence classes



$$\vec{v} - \vec{v}_1 = \vec{w}_1 \in \mathbb{W} \quad \text{and} \quad \vec{v} - \vec{v}_2 = \vec{w}_2 \in \mathbb{W}, \text{ which imply}$$

$$\vec{v}_2 - \vec{v}_1 = \vec{w}_1 - \vec{w}_2 \in \mathbb{W},$$

which implies $\vec{v}_1$ and $\vec{v}_2$ are both representatives of the same equivalence class. Contradiction. So all the equivalence classes are disjoint, and $\mathbb{W}$ is an equivalence class.

$\boxed{\text{Prob. 7}}$ cont'd. Pick an equivalence class $\{\vec{v}\}$ that is not $\mathbb{W}$. Then

$$\Pi(\vec{v}) = \{\vec{v}\} \neq \mathbb{W}, \text{ or } \Pi(\vec{v}) \neq \vec{0} \in \mathbb{V}/\mathbb{W}$$

$$\text{but } \Pi(\vec{w}) = \{\vec{w}\} = \mathbb{W} = \vec{0} \in \mathbb{V}/\mathbb{W}.$$

Hence $\quad \text{Ker}\,\Pi = \mathbb{W}$.

So we can apply the result of Prob 7 to the linear map

$$\Pi : \mathbb{V} \to \mathbb{V}/\mathbb{W} \quad \text{and}$$

get

$$\dim(\mathbb{V}) = \dim(\text{Ker}\,\Pi) + \dim(\text{Im}\,\Pi)$$

$$= \dim \mathbb{W} + \dim(\text{Im}\,\Pi)$$

But the map Π is clearly onto (surjective)

$$\therefore \quad \text{Im}\,\Pi = \mathbb{V}/\mathbb{W}$$

$$\therefore \quad \dim(\mathbb{V}) = \dim \mathbb{W} + \dim(\mathbb{V}/\mathbb{W})$$

or $\quad \dim(\mathbb{V}/\mathbb{W}) = \dim \mathbb{V} - \dim \mathbb{W}$