MAY 1st MAT444

Exterior Algebra

$\wedge$ (Exterior Product)

$\Lambda^r(V)$ rank r anti-symm tensors

Suppose $\dim V = n$

$T \in \Lambda^r(V)$ is an exterior r-vector

$\Lambda^0(V) = \mathbb{F}$, $\Lambda^1(V) = V$

<u>Exterior Product</u> of two tensors of different ranks

$\xi \in \Lambda^k(V)$ & $\eta \in \Lambda^l(V)$

not neccessarily antisymm

$\xi \wedge \eta$ is defined by $A_{k+l}(\xi \otimes \eta) = \xi \wedge \eta$

The result is then an exterior $(k+l)$-vector.

Suppose some $T \in T^r(V)$, $A_r(T) = \frac{1}{r!} \sum_{\sigma \in S_r} (\operatorname{sgn} \sigma)(\sigma T)$

wedge prod ! commute

<u>Thm</u>: The ext prod. satisfies the following properties.

let $\xi, \xi_1, \xi_2 \in \Lambda^k(V)$,

$\eta, \eta_2, \eta_3 \in \Lambda^l(V)$. TFA TRUE

① $(\xi_1 + \xi_2) \wedge \eta = \xi_1 \wedge \eta + \xi_2 \wedge \eta$ (distro)

$\xi \wedge (\eta_1 + \eta_2) = \xi \wedge \eta_1 + \xi \wedge \eta_2$

② $\xi \wedge \eta = (-1)^{kl} \eta \wedge \xi$ (anti-commutative law)

③ $(\xi \wedge \eta) \wedge \zeta = \xi \wedge (\eta \wedge \zeta) = \xi \wedge \eta \wedge \zeta$

PROOF OF 2

$$\xi \wedge \eta = \frac{1}{(k+l)!} \sum_{\tau \in S_{k+l}} (\operatorname{sgn} \tau)\, \tau(\xi \otimes \eta)$$

note $\tau(\xi \wedge \eta) = (\operatorname{sgn} \tau)(\xi \wedge \eta)$

consider $\tau = \begin{pmatrix} 1 \dots & k & \vdots & k+1 \dots\dots & k+l \\ l+1 \dots & k+l & \vdots & 1 \dots\dots & l \end{pmatrix}$

ex) $k=3$, $l=2$ $\quad \tau = \begin{pmatrix} 1 & 2 & 3 & \vdots & 4 & 5 \\ 3 & 4 & 5 & \vdots & 1 & 2 \end{pmatrix}$

$\operatorname{sgn} \tau = (-1)^{kl} = +1$

$$(\xi \wedge \eta)(v^{*1}, \dots, v^{*(k+l)}) =$$
$$\frac{(-1)^{kl}}{(k+l)!}\, \tau \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) [\sigma(\xi \otimes \eta)](v^{*1}, \dots, v^{*(k+l)})$$

Recall: $(\sigma T)(v^{*1}, \dots, v^{*r}) = T(v^{*\sigma(1)}, \dots, v^{*\sigma(r)})$

$$= \frac{(-1)^{kl}}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)(\xi \otimes \eta)(v^{*\sigma(1)}, \dots, v^{*\sigma(k)}, v^{*\sigma(k+1)}, \dots, v^{*\sigma(k+l)})$$

$$= \frac{(-1)^{k}}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)(\xi \otimes \eta)(v^{*\sigma \cdot \tau(1)}, \dots, v^{*\sigma \cdot \tau(k)}, v^{*\sigma \cdot \tau(k+1)}, \dots, v^{*\sigma \circ \tau(k+l)})$$

$$= \frac{(-1)^{kl}}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \cdot \xi(v^{*\sigma(l+1)}, \dots, v^{*\sigma(k+l)})\, \eta(v^{*\sigma(1)}, \dots, v^{*\sigma(l)})$$

$$= (-1)^{kl} (\eta \wedge \xi)(v^{*1}, \dots, v^{*l}, v^{*l+1}, \dots, v^{*k+l})$$

$\xi \in \Lambda^1(V)$, $\eta \in \Lambda^1(V)$ $\Rightarrow$ $\xi \wedge \eta = (-1)^{1\cdot 1} \eta \wedge \xi = -\eta \wedge \xi$

$\xi \wedge \xi = -\xi \wedge \xi = 0$ $\Rightarrow$ same rank 1 tensor ~~same~~ wedge together

$\psi_1 \wedge \psi_2 \wedge \xi \wedge \ldots \wedge \xi \wedge \ldots = 0$

Suppose $\{\vec{e}_1, \ldots, \vec{e}_n\}$ basis in V.

Then base in $T^r(V)$ $\quad e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r}$.

If choose $\underbrace{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r}}$ $= A_r(e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_r})$

non-zero only if
$e_{i_1}, \ldots, e_{i_r}$ all distinct

un change

If $\xi \in \Lambda^r(V)$ $\qquad r \leq (n = \dim V)$

$A_r(\xi) =$

$$= \xi^{i_1 \ldots i_r} A_r(e_{i_1} \otimes \ldots \otimes e_{i_r})$$

$$= \xi^{i_1 \ldots i_r} e_{i_1} \wedge \ldots \wedge e_{i_r}$$

A basis for $\Lambda^r(V)$ is $\{e_{i_1} \wedge \ldots \wedge e_{i_r}\}$ where
$1 \leq i_1 < \ldots < i_r \leq n$