P14.1 One can 'complete squares' and write the potential $V(x)$ as follows: this term leads to a constant force $-\alpha$.

$$V(x) = \frac{1}{2} m\omega^2 x^2 + \alpha x = \frac{1}{2} m\omega^2 \left(x^2 + \underbrace{\frac{2\alpha}{m\omega^2}}_{\beta} x\right)$$

$$= \frac{1}{2} m\omega^2 \left(x^2 + \beta x + \frac{\beta^2}{4} - \frac{\beta^2}{4}\right) = \frac{1}{2} m\omega^2 \left\{ \left(x + \frac{\beta}{2}\right)^2 - \frac{\beta^2}{4} \right\}$$

$$= \frac{1}{2} m\omega^2 \left(x + \frac{\beta}{2}\right)^2 - \frac{1}{2} m\omega^2 \cdot \frac{1}{4} \cdot \frac{4\alpha^2}{m^2\omega^4}$$

$$= \frac{1}{2} m\omega^2 x'^2 - \frac{\alpha^2}{2m\omega^2} \qquad \left(x' \equiv x + \frac{\beta}{2}\right)$$

$\therefore$ The Schrödinger eq. can be written as

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x'^2 - \frac{\alpha^2}{2m\omega^2}\right)\psi(x) = E\psi(x)$$

Since $\frac{d^2}{dx^2} = \frac{d^2}{dx'^2}$, we have

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx'^2} + \frac{1}{2} m\omega^2 x'^2\right)\psi(x') = \overbrace{\left(E + \frac{\alpha^2}{2m\omega^2}\right)}^{E'}\psi(x')$$

But this is the Sch. equation for the simple harmonic oscillator problem without the constant force.

$$\therefore \quad E' = \left(n + \frac{1}{2}\right)\hbar\omega = E + \frac{\alpha^2}{2m\omega^2}$$

$$\therefore \quad \boxed{E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{\alpha^2}{2m\omega^2}}$$

The new energies with a constant force added to the harmonic force are just shifted from those without the Constant force by a constant amount.

$\boxed{P14.2}$ Fermion operator <u>anticommutation</u> <u>relations</u> are:

$$\{a, a^\dagger\} = 1 \quad , \quad \{a, a\} = \{a^\dagger, a^\dagger\} = 0$$

The number operator is still $N \equiv a^\dagger a$. We have

$$Na = a^\dagger a . a = (1 - aa^\dagger)a = a - aa^\dagger a = a(1-N) \quad -(1)$$

$$Na^\dagger = a^\dagger a a^\dagger = a^\dagger(1 - a^\dagger a) = a^\dagger - a^\dagger a^\dagger a = a^\dagger(1-N) \quad -(2)$$

Let $|\nu\rangle$ be an eigenvalue of N, and $|\nu\rangle$ the corresponding non-zero eigenvector. Then

$$N|\nu\rangle = \nu|\nu\rangle \quad , \quad \langle\nu|\nu\rangle > 0$$

$\therefore \langle\nu|a^\dagger a|\nu\rangle$ = norm of $a|\nu\rangle = \langle\nu|N|\nu\rangle = \nu\langle\nu|\nu\rangle$

$\therefore$ if $\nu = 0$, $a|\nu\rangle$ is the zero-vector, i.e.

$$\boxed{a|0\rangle = 0} \leftarrow \text{the zero-vector.}$$

The above equation also implies $\nu \geq 0$. Now,

norm of $a|1\rangle = \langle 1|a^\dagger a|1\rangle = \langle 1|N|1\rangle = \langle 1|1\rangle = 1$

~~and $Na|1\rangle = a(1-N)|1\rangle = a|1\rangle - aN|1\rangle$~~

~~$= a|1\rangle - a|1\rangle = 0 \leftarrow$ the zero vector.~~

$$N a|\nu\rangle \overset{\text{by (1)}}{=} a(1-N)|\nu\rangle = a|\nu\rangle - aN|\nu\rangle = a|\nu\rangle - \nu a|\nu\rangle$$
$$= (1-\nu)a|\nu\rangle$$

$\therefore a|\nu\rangle$ is an eigenvector of N with eigenvalue $(1-\nu)$. Since all eigenvalues are ≥ 0, we have $1-\nu \geq 0$. This implies that $\boxed{\nu \text{ can only be } 0 \text{ or } 1}$. ~~For $\nu = 1$, we have~~

~~$Na|1\rangle = (1-1)a|\nu\rangle = 0.$~~

Also, $a|1\rangle$ is an eigenvector of N with eigenvalue 0.

$\therefore a|1\rangle \propto |0\rangle$ ← assumed to be normalized. But since norm of $a|1\rangle$ is 1 we have $\boxed{a|1\rangle = |0\rangle}$

$\boxed{P14.2}$ norm of $a^\dagger|0\rangle$ is given by

$$\langle 0|aa^\dagger|0\rangle = \langle 0|1-a^\dagger a|0\rangle = \langle 0|0\rangle - 0.\langle 0|0\rangle = \langle 0|0\rangle = 1$$

($|0\rangle$ is a non-zero normalized state)

At the same time

$$Na^\dagger|\nu\rangle \underset{\text{from (2)}}{=} a^\dagger(1-N)|\nu\rangle = a^\dagger|\nu\rangle - \nu a^\dagger|\nu\rangle = (1-\nu)a^\dagger|\nu\rangle$$

$\therefore$ $a^\dagger|\nu\rangle$ is an eigenvector of N with eigenvalue $(1-\nu)$.

$\therefore$ $a^\dagger|0\rangle$ is an eigenvector of N with eigenvalue 1

$\therefore$ $a^\dagger|0\rangle \propto |1\rangle$

But the norm of $a^\dagger|0\rangle$ = norm of $|1\rangle = 1$

$$\therefore \boxed{a^\dagger|0\rangle = |1\rangle}$$

Finally,

$$\text{norm of } a^\dagger|1\rangle = \langle 1|aa^\dagger|1\rangle \underset{\text{from } \{a,a^\dagger\} = a^\dagger a + aa^\dagger = 1}{=} \langle 1|1-a^\dagger a|1\rangle = 0$$

$$\therefore \boxed{a^\dagger|1\rangle = 0}$$ the zero-vector

Since the possible eigenvalues of the number operator N are either 0 or 1, a particular quantum state of a system of identical fermions cannot accommodate more than one particle. This is the <u>Pauli Exclusion Principle</u>.

Note: The fact that $a^\dagger|1\rangle = 0$ is also consistent with $\{a^\dagger, a^\dagger\} = 2a^\dagger a^\dagger = 0 \Rightarrow a^\dagger a^\dagger = 0$. Since then $a^\dagger(\underbrace{a^\dagger|0\rangle}_{|1\rangle}) = a^\dagger|1\rangle = 0$.

$\boxed{\text{P14.3}}$ (a) For $z \in \mathbb{C}$, define a <u>coherent</u> <u>state</u> $|z\rangle$ by

$$|z\rangle \equiv e^{(za^\dagger - z^* a)}|0\rangle \quad , \text{ where } [a, a^\dagger] = 1$$

Use the ~~[illegible]~~ <u>Baker-Cambell-Hausdorff</u> <u>Theorem</u> (P12.1) for

$$A = za^\dagger \ , \quad B = -z^* a$$

Then, the operator C satisfying $e^A e^B = e^C$ is given by

$$C = za^\dagger - z^* a + \frac{1}{2}\underbrace{[za^\dagger, -z^* a]}_{\substack{\shortparallel \\ |z|^2 [a^\dagger, -a] \,=\, |z|^2}}$$

$$\therefore \quad \underbrace{e^{za^\dagger - z^* a + |z|^2/2}}_{\substack{\shortparallel \\ e^{|z|^2/2}\, e^{za^\dagger - z^* a}}} = e^{za^\dagger} e^{-z^* a}$$

or $$e^{za^\dagger - z^* a} = e^{-|z|^2/2}\, e^{za^\dagger} e^{-z^* a}$$

$$\therefore \quad |z\rangle = e^{-|z|^2/2}\, e^{za^\dagger} \underbrace{e^{-z^* a}|0\rangle}_{\left(1 - z^* a + (-1)^2 \frac{(z^*)^2 a^2}{2} + \cdots\right)|0\rangle \,=\, |0\rangle}$$

(since $a|0\rangle = 0$)

$$\therefore \quad |z\rangle = e^{-|z|^2/2}\, e^{za^\dagger}|0\rangle$$

$$= e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{(za^\dagger)^n}{n!}|0\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \underbrace{\frac{(a^\dagger)^n |0\rangle}{\sqrt{n!}}}_{\substack{\shortparallel \ [\text{by } (14.27)] \\ |n\rangle}}$$

$$\therefore \quad \boxed{|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{n!}|n\rangle}$$

So $|z\rangle$ is expanded as a linear combination of the complete $\underbrace{\,}_{\text{orthonormal}}$ set $\{|n\rangle\}$.

$\boxed{\text{P14.3}}$ cont'd

(b) Use the result in (a):

$$|z\rangle = e^{-|z|^2/2} \sum_n \frac{z^n}{\sqrt{n!}} |n\rangle$$

$$\therefore \langle z| = e^{-|z|^2/2} \sum_m \frac{(z^*)^m}{\sqrt{m!}} \langle m|$$

$$\therefore \langle z|z\rangle = e^{-|z|^2} \sum_{n,m} \frac{z^n (z^*)^m}{\sqrt{n!\, m!}} \underbrace{\langle m|n\rangle}_{\delta_{mn}}$$

$$= e^{-|z|^2} \sum_n \frac{z^n (z^*)^n}{\sqrt{n!\, n!}} = e^{-|z|^2} \sum_{n=0}^{\infty} \frac{(z z^*)^n}{n!}$$

$$= e^{-|z|^2} \underbrace{\sum_{n=0}^{\infty} \frac{(|z|^2)^n}{n!}}_{= e^{|z|^2}} = 1 .$$

(c)
$$\langle z|w\rangle = e^{-|z|^2/2} e^{-|w|^2/2} \sum_{n,m} \frac{(z^*)^n}{\sqrt{n!}} \frac{w^m}{\sqrt{m!}} \underbrace{\langle n|m\rangle}_{\delta_{nm}}$$

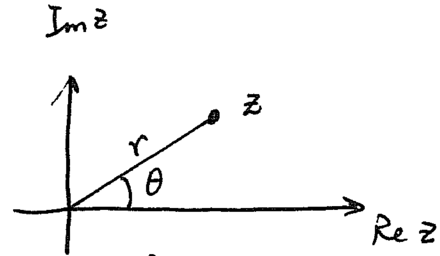
$$= e^{-|z|^2/2} e^{-|w|^2/2} \underbrace{\sum_{n=0}^{\infty} \frac{(z^*)^n w^n}{n!}}_{= e^{z^* w}}$$

$$= e^{-|z|^2/2 - |w|^2/2 + z^* w}$$

$$\therefore |\langle z|w\rangle|^2 = e^{-|z|^2/2 - |w|^2/2 + z^* w} \, e^{-|z|^2/2 - |w|^2/2 + z w^*}$$

$$= e^{-(|z|^2 + |w|^2 - z^* w - z w^*)} = e^{-|z-w|^2}$$

P14.3 Cont'd



$d^2z = r\,dr\,d\theta$

(d) $\displaystyle\int \frac{d^2z}{\pi} |z\rangle\langle z|$

$$= \int \frac{d^2z}{\pi} \Bigg(\underbrace{e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle}_{|z\rangle} \Bigg) \Bigg(\underbrace{e^{-|z|^2/2} \sum_{m=0}^{\infty} \frac{(z^*)^m}{\sqrt{m!}} \langle m|}_{\langle z|} \Bigg)$$

$$= \int_0^\infty dr\, r \int_0^{2\pi} d\theta \cdot \frac{1}{\pi} e^{-|z|^2} \sum_{n,m} \frac{z^n (z^*)^m}{\sqrt{n!\,m!}} |n\rangle\langle m| \; ,$$

$|z| = r$, write $z = re^{i\theta}$, $z^* = re^{-i\theta}$

$$= \int_0^\infty dr\, r \int_0^{2\pi} d\theta\, \frac{1}{\pi} e^{-r^2} \sum_{n,m} r^n e^{in\theta} r^m e^{-im\theta} \cdot |n\rangle\langle m|$$

$$= \frac{1}{\pi} \sum_{n,m} \frac{|n\rangle\langle m|}{\sqrt{n!\,m!}} \int_0^\infty dr\, r^{n+m+1} e^{-r^2} \underbrace{\int_0^{2\pi} d\theta\, e^{i(n-m)\theta}}_{= 2\pi\delta_{nm}}$$ (from P7.3)

$$= 2 \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int_0^\infty dr\, r\, r^{2n} e^{-r^2}$$

To do the integral, let $\xi \equiv r^2$ $\therefore d\xi = 2r\,dr$. So

$$\int \frac{d^2z}{\pi} |z\rangle\langle z| = \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int_0^\infty d\xi\, \xi^n e^{-\xi}$$

$\displaystyle\int_0^\infty d\xi\, \xi^n e^{-\xi} = \Gamma(n+1) = n!$ (gamma function)

$$= \sum_{n=0}^{\infty} |n\rangle\langle n| = 1 \,.$$ (since $\{|n\rangle\}$ is a complete set)

$\boxed{\text{P14.3}}$ cont'd.

(e) $$a|z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\underbrace{a|n\rangle}_{=\sqrt{n}\,|n-1\rangle}$$ [from (14.25)] if $n \neq 0$

$$\therefore a|z\rangle = e^{-|z|^2/2}\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle \qquad (a|0\rangle = 0)$$

$$= e^{-|z|^2/2}\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{(n-1)!}}|n-1\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^{n+1}}{\sqrt{n!}}|n\rangle$$

$$= z\,.\,e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle = z|z\rangle.$$

(f) Since $a|z\rangle = z|z\rangle$

$$\langle z|a^\dagger = z^*\langle z|$$

$$\therefore \langle z|a^\dagger a|z\rangle = zz^*\underbrace{\langle z|z\rangle}_{=1 \text{ (from (b))}} = |z|^2$$

Since $a^\dagger a$ is the number operator, this result states that the mean number of bosons in the state $|z\rangle$ is $|z|^2$.

(g) From the result in (a), namely,

$$|z\rangle = e^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle ,$$

it is clear that

$$\langle n|z\rangle = \cancel{e^{-|z|^2/2}\sum_{n=0}^{\infty}} \; e^{-|z|^2/2}\frac{z^n}{\sqrt{n!}}$$

$$\therefore |\langle n|z\rangle|^2 = e^{-|z|^2}\frac{z^n(z^*)^n}{n!} = e^{-|z|^2}\frac{|z|^{2n}}{n!}$$

$\boxed{P14.3}$ (g) cont'd.

$$P(n) = \frac{e^{-|z|^2}|z|^{2n}}{n!}$$ is a <u>Poisson distribution</u>. It gives the probability of finding n bosons in the state $|z\rangle$, with the mean number of bosons being $|z|^2$.

(h) $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$, $\Delta q = \sqrt{\langle q^2\rangle - \langle q\rangle^2}$,

where $\langle A\rangle = \langle z|A|z\rangle$, for an observable A

$q = \sqrt{\frac{\hbar}{m\omega}}\cdot\frac{1}{\sqrt{2}}(a+a^\dagger)$, $p = \sqrt{m\hbar\omega}\left(\frac{-i}{\sqrt{2}}\right)(a-a^\dagger)$. So

$$\langle q\rangle = \langle z|q|z\rangle = \sqrt{\frac{\hbar}{m\omega}}\cdot\frac{1}{\sqrt{2}}\langle z|a+a^\dagger|z\rangle = \sqrt{\frac{\hbar}{m\omega}}\cdot\frac{1}{\sqrt{2}}\left(\underbrace{\langle z|a|z\rangle}_{z}+\underbrace{\langle z|a^\dagger|z\rangle}_{z^*}\right)$$

$$\therefore \langle q\rangle^2 = \frac{\hbar}{2m\omega}(z+z^*)^2$$

$$\langle p\rangle = \langle z|p|z\rangle = \sqrt{m\hbar\omega}\left(\frac{-i}{\sqrt{2}}\right)\langle z|a-a^\dagger|z\rangle = -\frac{i}{\sqrt{2}}\sqrt{m\hbar\omega}\,(z-z^*)$$

$$\therefore \langle p\rangle^2 = -\frac{m\hbar\omega}{2}(z-z^*)^2$$

$$\langle q^2\rangle = \langle z|q^2|z\rangle = \frac{\hbar}{2m\omega}\langle z|(a+a^\dagger)(a+a^\dagger)|z\rangle$$
$$= \frac{\hbar}{2m\omega}\langle z|a^2+(a^\dagger)^2+a^\dagger a+aa^\dagger|z\rangle.$$

We have

$$\langle z|a^2|z\rangle = \langle z|a.a|z\rangle = z\langle z|a|z\rangle = z^2\langle z|z\rangle = z^2$$

$$\langle z|a^\dagger a^\dagger|z\rangle = z^*\langle z|a^\dagger|z\rangle = (z^*)^2\langle z|z\rangle = (z^*)^2$$

$$\langle z|a^\dagger a|z\rangle = |z|^2 \quad [\text{from (f)}]$$

$$\langle z|aa^\dagger|z\rangle \underset{\substack{\uparrow\\ \text{since } aa^\dagger - a^\dagger a = 1}}{=} \langle z|1+a^\dagger a|z\rangle = 1+|z|^2$$

$\boxed{\text{P 14.3}}$ (h) cont'd

$$\therefore \langle q^2 \rangle = \frac{\hbar}{2m\omega}\left(z^2 + (z^*)^2 + 2|z|^2 + 1\right) = \frac{\hbar}{2m\omega}\left\{ z^2 + (z^*)^2 + 2zz^* + 1 \right\}$$

$$= \frac{\hbar}{2m\omega}\left\{ (z+z^*)^2 + 1 \right\}$$

$$\therefore \Delta q = \sqrt{\langle q^2\rangle - \langle q\rangle^2} = \sqrt{\left\{\frac{\hbar}{2m\omega}\left((z+z^*)^2 + 1 - (z+z^*)^2\right)\right\}} = \sqrt{\frac{\hbar}{2m\omega}}$$

Similarly,

$$\langle p^2 \rangle = -\frac{m\hbar\omega}{2}\langle z|(a-a^\dagger)(a-a^\dagger)|z\rangle$$

$$= -\frac{m\hbar\omega}{2}\langle z|a^2 + (a^\dagger)^2 - a^\dagger a - aa^\dagger|z\rangle$$

$$= -\frac{m\hbar\omega}{2}\left(z^2 + (z^*)^2 - 2zz^* - 1\right) = -\frac{m\hbar\omega}{2}\left((z-z^*)^2 - 1\right)$$

$$= \frac{m\hbar\omega}{2}\left(1 - (z-z^*)^2\right)$$

$$\therefore \Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\left\{\frac{m\hbar\omega}{2}\left(1 - (z-z^*)^2 + (z-z^*)^2\right)\right\}} = \sqrt{\frac{m\hbar\omega}{2}}$$

$$\therefore (\Delta q)(\Delta p) = \sqrt{\frac{\hbar}{2m\omega}}\sqrt{\frac{m\hbar\omega}{2}}$$

i.e. $(\Delta q)(\Delta p) = \frac{\hbar}{2}$

This is the situation when the equality holds in the general uncertainty principle

$$(\Delta q)(\Delta p) \geq \frac{\hbar}{2} .$$

So $|z\rangle$ represents a <u>minimum</u> <u>uncertainty</u> <u>state</u>.