$\boxed{\text{P.31.3}}$ The Schrödinger equation for the linear harmonic oscillator is

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dq^2} + \underbrace{\frac{1}{2}m\omega^2 q^2}_{V_0}\right)\psi_n(q) = E_n\psi_n(q) \qquad \text{--- (1)}$$

With a constant force added, the perturbation potential is

$$V = \alpha q \qquad (\alpha = \text{constant})$$

We can write the total potential energy as

$$V_0 + V = \frac{1}{2}m\omega^2 q^2 + \alpha q = \frac{1}{2}m\omega^2\left(q^2 + \underbrace{\frac{2\alpha}{m\omega^2}}_{\equiv\beta} q\right)$$

$$= \frac{1}{2}m\omega^2\left(q^2 + \beta q + \frac{\beta^2}{4} - \frac{\beta^2}{4}\right) \quad \text{(completing squares)}$$

$$= \frac{1}{2}m\omega^2\left\{\left(q + \frac{\beta}{2}\right)^2 - \frac{\beta^2}{4}\right\}$$

$$= \frac{1}{2}m\omega^2\left(q + \frac{\beta}{2}\right)^2 - \frac{1}{2}m\omega^2\cdot\frac{1}{4}\cdot\frac{4\alpha^2}{(m\omega^2)^2}$$

$$= \frac{1}{2}m\omega^2\underbrace{\left(q + \frac{\beta}{2}\right)^2}_{\equiv q'^2} - \frac{\alpha^2}{2m\omega^2}$$

$$= \frac{1}{2}m\omega^2 q'^2 - \frac{\alpha^2}{2m\omega^2}, \quad \text{where } q' \equiv q + \frac{\beta}{2}$$

Then Schrödinger equation becomes

new energy eigenvalues

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dq^2} + \frac{1}{2}m\omega^2 q'^2 - \frac{\alpha^2}{2m\omega^2}\right)\psi_n(q) = E_n'\psi_n(q)$$

i.e.

$$\left(\frac{-\hbar^2}{2m}\frac{d^2}{dq^2} + \frac{1}{2}m\omega^2 q'^2\right)\psi_n(q) = \left(E_n' + \frac{\alpha^2}{2m\omega^2}\right)\psi_n(q) \qquad \text{--- (2)}$$

In (2), make the coordinate transformation

$$q \to q + \frac{\beta}{2} = q', \quad \psi_n(q) \to \psi_n'(q')$$

P31.3 cont'd

Then (2) ⇒

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dq'^2}+\frac{1}{2}m\omega^2 q'^2\right)\psi_n'(q') = \left(E_n'+\frac{\alpha^2}{2m\omega^2}\right)\psi_n'(q') \quad \text{— (3)}$$

But (3) is the same equation as (1), with

$$E_n' + \frac{\alpha^2}{2m\omega^2} = \left(n+\frac{1}{2}\right)\hbar\omega$$

$$\therefore \boxed{E_n' = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{\alpha^2}{2m\omega^2}} \quad \text{— (4)}$$

These are the <u>exact</u> new energy eigenvalues

Now let's do perturbation (this turns out to be the harder way!). From (31.13) and (31.37) in text, we have, up to second-order:

$$\boxed{E_n = E_n^{(0)} + \langle n|\overset{\alpha q}{V}|n\rangle + \sum_{m\neq n}\frac{|\langle m|V|n\rangle|^2}{E_n^{(0)}-E_m^{(0)}} + \cdots\cdots} \quad \text{— (5)}$$

where $|n\rangle$ and $E_n^{(0)}$ are the unperturbed eigenstates and energy eigenvalues of the unperturbed Hamiltonian H_0, i.e.

$$H_0|n\rangle = \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle, \quad E_n^{(0)} = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

$$\psi_n^{(0)}(q) = \langle q|n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\, e^{-Q^2/2} H_n(Q) \quad \left[\text{see eqs. (14.37) and (14.57) in text}\right]$$

$Q \equiv \sqrt{\frac{m\omega}{\hbar}}\, q$ ← dimensionless coordinate, $H_n(Q)$ is the n-th Hermite polynomial

$$H_n(Q) = (-1)^n e^{Q^2}\frac{d^n}{dQ^n}e^{-Q^2}; \; n = 0,1,2,3,\ldots\ldots \quad \left[\text{see eq. (14.54) in text}\right]$$

Now the unperturbed Hamiltonian ~~has~~ is symmetric under reflection:

$$H_0(q) = H_0(-q)$$

$\therefore \psi_n^{(0)}(q)$ has definite <u>parity</u> (either even or odd under reflection)

$$\therefore \langle n|V|n\rangle = \alpha\int_{-\infty}^{\infty} dq\, q\, \psi_n^{(0)*}(q)\,\psi_n^{(0)}(q) = 0$$

P31.3 cont'd

$\therefore$ $\boxed{\text{There is no first-order correction to } E_n^{(0)}}$.

So we have to calculate the 2nd order term in (5).
The matrix elements in this correction term are given by

$$\langle m|V|n\rangle = \alpha C_m C_n \int_{-\infty}^{\infty} dq\, q\, e^{-Q^2} H_m(Q) H_n(Q)$$

when $C_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}}$, $Q \equiv \sqrt{\frac{m\omega}{\hbar}}\, q$

$$\therefore \langle m|V|n\rangle = \alpha C_m C_n \left(\frac{\hbar}{m\omega}\right) \int_{-\infty}^{\infty} dQ . Q\, H_m(Q) H_n(Q)\, e^{-Q^2}$$

(Note all the normalization constants and wave functions are real)

From Eq. (14.71) of text, we have the recursion relation

$$Q H_m(Q) = \frac{H_{m+1}(Q)}{2} + m H_{m-1}(Q)$$

$$\therefore \langle m|V|n\rangle = \alpha C_m C_n \left(\frac{\hbar}{m\omega}\right) \int_{-\infty}^{\infty} dQ\, e^{-Q^2} H_n(Q) \left(\frac{H_{m+1}(Q)}{2} + m H_{m-1}(Q)\right)$$

note that this is mass of the oscillator, do not confuse with the numerical index m.

$$\langle m|V|n\rangle = \alpha C_m C_n \left(\frac{\hbar}{m\omega}\right)\left[\frac{1}{2}\int_{-\infty}^{\infty} dQ\, e^{-Q^2} H_n H_{m+1} + m \int_{-\infty}^{\infty} dQ\, e^{-Q^2} H_n H_{m-1}\right] \quad -(6)$$

From the orthonormality condition of the $\psi_n^{(0)}$'s, we have

$$\delta_{ij} = \int_{-\infty}^{\infty} dq . \psi_i^{(0)}(q)\, \psi_j^{(0)}(q) = C_i C_j \int dq\, e^{-Q^2} H_i(Q) H_j(Q) \qquad \left(q = \sqrt{\frac{\hbar}{m\omega}}\, Q\right)$$

$$= C_i C_j \sqrt{\frac{\hbar}{m\omega}} \int dQ\, e^{-Q^2} H_i(Q)\, H_j(Q)$$

$$\therefore \int_{-\infty}^{\infty} dQ\, e^{-Q^2} H_i(Q)\, H_j(Q) = \sqrt{\frac{m\omega}{\hbar}}\, \frac{\delta_{ij}}{C_i C_j}$$

$\boxed{\text{P31.3}}$ cont'd

$\therefore$ Eq. (6) becomes

$$\langle m|V|n\rangle = \alpha C_m C_n \left(\frac{\hbar}{m\omega}\right)\left[\frac{1}{2}\sqrt{\frac{m\omega}{\hbar}}\,\frac{\delta_{n,m+1}}{C_n C_{m+1}} + m\sqrt{\frac{m\omega}{\hbar}}\,\frac{\delta_{n,m-1}}{C_n C_{m-1}}\right]$$

$$= \frac{\alpha}{2}\sqrt{\frac{\hbar}{m\omega}}\left(\frac{C_m}{C_{m+1}}\right)\delta_{n,m+1} + \alpha m\sqrt{\frac{\hbar}{m\omega}}\left(\frac{C_m}{C_{m-1}}\right)\delta_{n,m-1}$$

$$= \alpha\sqrt{\frac{\hbar}{m\omega}}\left\{\frac{1}{2}\left(\frac{C_m}{C_{m+1}}\right)\delta_{n,m+1} + m\left(\frac{C_m}{C_{m-1}}\right)\delta_{n,m-1}\right\}$$

$$\therefore |\langle m|V|n\rangle|^2 = \alpha^2\frac{\hbar}{m\omega}\left\{\frac{1}{4}\left(\frac{C_m}{C_{m+1}}\right)^2\delta_{n,m+1} + m^2\left(\frac{C_m}{C_{m-1}}\right)^2\delta_{n,m-1}\right\}$$

The cross terms in the square vanishes since

$$\delta_{n,m-1}\cdot\delta_{n,m+1} = 0 \quad \text{for any pair } (n,m)$$

and ~~the~~ $\delta^2_{n,m} = \delta_{n,m}$.

Thus

$$\sum_{m\neq n}\frac{|\langle m|V|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \frac{\alpha^2\hbar}{m\omega}\left[\frac{\frac{1}{4}\left(\frac{C_{n-1}}{C_n}\right)^2}{E_n^{(0)} - E_{n-1}^{(0)}} + \frac{(n+1)^2\left(\frac{C_{n+1}}{C_n}\right)^2}{E_n^{(0)} - E_{n+1}^{(0)}}\right]$$

$$E_n^{(0)} - E_{n-1}^{(0)} = \hbar\omega \;; \quad E_n^{(0)} - E_{n+1}^{(0)} = -\hbar\omega$$

and ~~from~~ by using $C_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n\, n!}}$, we have

$$\frac{C_{n-1}}{C_n} = \frac{2^{n/2}\sqrt{n!}}{2^{(n-1)/2}\sqrt{(n-1)!}} = \sqrt{2n} \;; \quad \frac{C_{n+1}}{C_n} = \frac{2^{n/2}\sqrt{n!}}{2^{(n+1)/2}\sqrt{(n+1)!}} = \frac{1}{\sqrt{2(n+1)}}$$

$$\therefore \sum_{m\neq n}\frac{|\langle m|V|n\rangle|^2}{E_n^{(0)} - E_m^{(0)}} = \frac{\alpha^2\hbar}{m\omega}\left[\frac{2n}{4\hbar\omega} - \frac{(n+1)^2}{\hbar\omega}\cdot\frac{1}{2(n+1)}\right]$$

$$= \frac{\alpha^2\hbar}{2.m\omega.\hbar\omega}\left[n - (n+1)\right] = \frac{-\alpha^2}{2m\omega^2}$$

$$\therefore \boxed{E_n^{(2)} = \left(n+\frac{1}{2}\right)\hbar\omega - \frac{\alpha^2}{2m\omega^2}}$$

in agreement with our exact result earlier.