MAT 444 MT1 Solutions

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① An orthogonal transformation between $\{\vec{e}_i\}$ and $\{\vec{e}\,'_j\}$ given by $\vec{e}_i = a_i^{\ j}\,\vec{e}\,'_j$ is one that satisfies

$$a a^T = 1 \quad (\text{where } a^T \text{ is the transpose of } a)$$

that is,

$$a_i^{\ j}(a^T)_j^{\ k} = \delta_i^k$$

i.e.,

$$\sum_j a_i^{\ j} a_k^{\ j} = \delta_{ik}$$

In $T^{ij} = \delta^{ij}x^l x_l - x^i x^j$

$x^l x_l$ is invariant (since it is a scalar):

$$x'^l x'_l = a^l_{\ i}(a^{-1})^j_{\ l}\, x^i x_j$$

$$= a^l_{\ i}(a^T)^j_{\ l}\, x^i x_j = \delta_i^j x^i x_j = x^i x_i$$

δ^{ij} transforms as a (2,0)-tensor since

$$\delta'^{ij} = a^i_{\ l} a^j_{\ k}\,\delta^{lk} = \sum_l a^i_{\ l} a^j_{\ l} = \sum_l (a^T)^l_{\ j} a^i_{\ l}$$

$$= (a^T a)^i_{\ j} = \delta^i_j = \delta^{ij}$$

$x^i x^j$ manifestly transforms as a (2,0)-tensor.

since $x'^i x'^j = a^i_{\ l} a^j_{\ k} x^l x^k$

∴ T^{ij} transforms as a (2,0) tensor

(1) cont'd

$$T^{ij} = \delta^{ij} x^l x_l - x^i x^j$$

is a symmetric tensor. Therefore, there are 6 independent components in T:

$$T = \begin{pmatrix} T^{11} & T^{12} & T^{13} \\ & T^{22} & T^{23} \\ & & T^{33} \end{pmatrix}.$$

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(2) A metric tensor g_{ij} on a vector space $\mathbb{V}$ is a rank $(0,2)$-tensor g_{ij} which defines a scalar product in $\mathbb{V}$: if $\vec{x}, \vec{y} \in \mathbb{V}$, then $(\vec{x}, \vec{y}) = g_{ij} x^i y^j$

g_{ij} has to be 1) symmetric
~~and~~ 2) non-degenerate
and 3) bi-linear.

The canonical isomorphism $G: \mathbb{V} \to \mathbb{V}^*$ $(\vec{v} \mapsto \vec{v}^*)$ is given by:

For all $\vec{w} \in \mathbb{V}$

$$\langle v^*, \vec{w} \rangle = (\vec{v}, \vec{w}).$$

The non-degeneracy of $(\, , \,)$ guarantees that for any $\vec{v} \in \mathbb{V}$, $G(\vec{v}) = v^* \in \mathbb{V}^*$ is unique.

(3) From two vector spaces $\mathbb{V}$ and $\mathbb{W}$, we first form the Cartesian product $\mathbb{V} \times \mathbb{W}$ (the set of ordered pairs $(\vec{v}, \vec{w})$, with $\vec{v} \in \mathbb{V}$ and $\vec{w} \in \mathbb{W}$). One then constructs a vector space $\mathbb{V} \otimes \mathbb{W}$ (tensor product of $\mathbb{V}$ and $\mathbb{W}$) and a bilinear map $h: \mathbb{V} \times \mathbb{W} \to \mathbb{V} \otimes \mathbb{W}$ satisfying the following: For any bilinear map $f: \mathbb{V} \times \mathbb{W} \to \mathbb{Z}$ (another vector space) there exists a unique linear map $g: \mathbb{V} \otimes \mathbb{W} \to \mathbb{Z}$ such that

$$f = g \circ h : \mathbb{V} \times \mathbb{W} \to \mathbb{Z}$$

The above is represented by the following <u>commuting diagram</u>:

$$\begin{tikzcd}
\mathbb{V} \times \mathbb{W} \arrow[r, "h"] \arrow[dr, "f"'] & \mathbb{V} \otimes \mathbb{W} \arrow[d, "g"] \\
& \mathbb{Z}
\end{tikzcd}$$

This is <u>not</u> a vector space

this <u>is</u> a vector space

(3) cont'd.

$$\mathbb{V} \otimes \mathbb{W} = \mathcal{L}(\mathbb{V}^*, \mathbb{W}^*; \mathbb{F})$$

where $\mathcal{L}(\mathbb{V}^*, \mathbb{W}^*; \mathbb{F})$ is the vector space of bilinear functions on $\mathbb{V}^* \times \mathbb{W}^*$, that is, if $f \in \mathcal{L}(\mathbb{V}^*, \mathbb{W}^*; \mathbb{F})$, $f: \mathbb{V}^* \times \mathbb{W}^* \to \mathbb{F}$ maps an ordered pair $(\vec{v}^*, \vec{w}^*) \in \mathbb{V}^* \times \mathbb{W}^*$ to a scalar in $\mathbb{F}$ in a bilinear fashion.

Suppose $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is a basis in $\mathbb{V}$
and $\{\vec{d}_1, \ldots\ldots, \vec{d}_m\}$ is a basis in $\mathbb{W}$,
then a basis in $\mathbb{V} \otimes \mathbb{W}$ is

$$\{\vec{e}_i \otimes \vec{d}_j\} \quad ; \quad 1 \le i \le n \;,\; 1 \le j \le m,$$

so that dim $\mathbb{V} \otimes \mathbb{W}$ is nm.

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(4) With $\vec{e}_i = a_i^j \, \vec{e}'_j$

$$T'^{ij}_{k} = a^i_\ell \, a^j_m \, (a^{-1})^n_k \, T^{\ell m}_n$$

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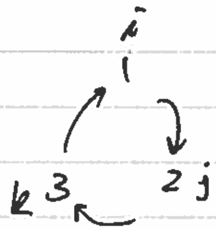
(5)



i	j	k
1	2	3
2	3	1
3	1	2
1	3	2
3	2	1
2	1	3

If $T = T^{ijk}\, \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_j$

$$S^{ijk} = [S_3(T)]^{ijk}$$
$$= \frac{1}{3!}\left(T^{ijk} + T^{jki} + T^{kij} + T^{ikj} + T^{kji} + T^{jik} \right)$$

$$A^{ijk} = [A_3(T)]^{ijk}$$
$$= \frac{1}{3!}\left(T^{ijk} + T^{jki} + T^{kij} - T^{ikj} - T^{kji} - T^{jik} \right)$$

$\Lambda^3(\mathbb{V})$ is <u>one-dimensional</u> since the only independent ~~element~~ component is A^{123}.

$P^3(\mathbb{V})$ is <u>ten-</u> <u>dimensional</u>. The independent components are

$$S^{111}, S^{222}, S^{333}, S^{112}, S^{113}, S^{221}, S^{223}, S^{331}, S^{332}, S^{123}.$$