the bottom left corner of the lattice, where $m = -j_1 - j_2$ and $n(m) = 1$ again. The above results can be summarized as follows.

$$\begin{aligned} n(m) &= 0 && \text{if} \quad |m| > j_1 + j_2 \,, \\ n(m) &= j_1 + j_2 + 1 - |m| && \text{if} \quad j_1 + j_2 \geq |m| \geq |\, j_1 - j_2 \,| \,, \\ n(m) &= 2j_{min} + 1 && \text{if} \quad |\, j_1 - j_2 \,| \geq |m| \geq 0 \,, \end{aligned} \tag{26.18}$$

where j_{min} = the smaller of j_1 and j_2. It then follows from (26.17) that

$$a_j = \begin{cases} 1 \,, & \text{for} \quad |\, j_1 - j_2 \,| \leq j \leq j_1 + j_2 \,, \\ 0 \,, & \text{otherwise} \,. \end{cases} \tag{26.19}$$

Finally we have the following important result: The direct product representations for $SO(3)$ can be reduced as

$$\boxed{D^{(j_1)} \otimes D^{(j_2)} = \textstyle\sum_{\oplus j} D^{(j)}} \quad , \tag{26.20}$$

where $j_1 + j_2 \geq j \geq |\, j_1 - j_2 \,|$ in steps of one. Equation (26.8) is a special case of this general result.

$\boxed{\text{Problem 26.1}}$ In the study of spectroscopy one must often go from one angular momentum coupling scheme to another. The simplest example is that of three commuting angular momenta:

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{J}_3 \,.$$

This addition can be done in several ways. For example, one can add $\boldsymbol{J}_1$ to $\boldsymbol{J}_2$ first to give $\boldsymbol{J}_{12}$, and then add $\boldsymbol{J}_3$ to give $\boldsymbol{J}$. Let the state formed by this coupling scheme be denoted by $|\, j_1 j_2 (j_{12}) j_3;\, jm \,\rangle$, where $j_i(j_i+1)$ is the eigenvalue of $\boldsymbol{J}_i^2$, $j(j+1)$ is the eigenvalue of $\boldsymbol{J}^2$, and m is the eigenvalue of J_z. Using the angular momentum commutation rules show that the matrix elements

$$\langle\, j_1 j_2 (j_{12}) j_3;\, jm \,|\, j_1 j_3 (j_{13}) j_2;\, jm \,\rangle$$

do not depend on m. (Hint: Use the commutation rule $[\, J_+,\, J_- \,] = 2J_z$.)

Give an explicit formula for

$|\, j_1 j_2 (j_{12}) j_3 ; j m \rangle$

in terms of Clebsch-Gordan coefficients and the tensor product states

$|\, j_1 m_1 \rangle |\, j_2 m_2 \rangle |\, j_3 m_3 \rangle$.

The quantity in braces is called a 6-j **symbol**, and is given in general by

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} = (-1)^{j_1+j_2+l_1+l_2}\, \Delta(j_1 j_2 j_3)\Delta(l_1 l_2 j_3)\Delta(l_1 j_2 l_3)\Delta(j_1 l_2 l_3)$$

$+1 \qquad k!$

$$\times \sum_k [\,(-1)^k (j_1+j_2+l_1+l_2-k)!/\{(j_1+j_2-j_3-k)!(l_1+l_2-j_3-k)!$$
$$(j_1+l_2-l_3-k)!(l_1+j_2-l_3-k)!(-j_1-l_1+j_3+l_3+k)!(-j_2-l_2+j_3+l_3+k)!\}\,]\,,$$

where

$j_{12} = \frac{1}{2},\ j_{23} = 0$

$$\Delta(abc) \equiv \sqrt{\frac{(a+b-c)!(a-b+c)!(b+c-a)!}{(a+b+c+1)!}}\,.$$

Verify the above results for the special case $j_1 = 1$, $j_2 = 1/2$ and $j_3 = 1/2$, and for $j = m = 1$, by expanding all eigenfunctions in terms of the appropriate $|\, j_1, m_1\,\rangle$, $|\, j_2, m_2\,\rangle$ and $|\, j_3, m_3\,\rangle$ states, and by making use of the CG coefficients derived in this chapter.

Problem 27.3 Suppose a system with angular momentum J_1 is coupled to another with angular momentum J_2. Let $T^{(k)}$ be an irreducible tensor operator that acts only on the first system. Prove the following relationship between reduced matrix elements:

$$\langle\, \tau', j_1' j_2';\, j' \,\|\, T^{(k)} \,\|\, \tau, j_1, j_2;\, j\,\rangle$$
$$= (-1)^{j_1'+j_2'+j'+k}\, \delta_{j_2' j_2} \sqrt{(2j'+1)(2j+1)} \begin{Bmatrix} j' & k & j \\ j_1 & j_2' & j_1' \end{Bmatrix} \langle\, \tau',\, j_1' \,\|\, T^{(k)} \,\|\, \tau,\, j_1\,\rangle\,.$$

j