APRIL 19 MAT444

$T \in V_2^3$ $\qquad \{\vec{e}_1, \ldots, \vec{e}_n\}$ basis for $\mathbb{V}$

$T^{ijk}_{lm}\ \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k \otimes \vec{e}^{*l} \otimes \vec{e}^{*m}$ // recall order matters

contraction:

contraction operator $C_{jl}(T) = T^{ijk}_{lm} \langle \vec{e}_j, \vec{e}^{*l} \rangle \vec{e}_i \otimes \vec{e}_k \otimes \vec{e}^{*m}$

$= T^{ijk}_{lm}\, \delta^l_j\, \vec{e}_i \otimes \vec{e}_k \otimes \vec{e}^{*m}$

$= T^{ijk}_{jm}\, \vec{e}_i \otimes \vec{e}_k \otimes \vec{e}^{*m}$ // contraction possible only when mixed

$= T^{ik}_{m}$

metric tensor induces ~~creates~~ ~~an~~ a canonical isom between $\mathbb{V}$ & $\mathbb{V}^*$

Suppose $T = \vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_3 \otimes \vec{\omega}^{*1} \otimes \vec{\omega}^{*2}$

$C_{25} = \underbrace{\langle \vec{v}_2, \vec{\omega}^{*2} \rangle}_{\in \mathbb{F}} \vec{v}_1 \otimes \vec{v}_3 \otimes \vec{\omega}^{*1}$ // position 2 & 5

$\otimes$ Tensor Alg.

$\underbrace{\mathbb{V} \otimes \ldots \otimes \mathbb{V}}$ // r-times

$T^3(\mathbb{V})$ $\qquad T^3(\mathbb{V}) \oplus T^2(\mathbb{V})$ some op $\oplus$

$T^2(\mathbb{V})$,

$\otimes$ lets us tensor diff ranked tens.

e.g.) $\phi \otimes \psi \otimes \zeta$

e.g.) $\phi \otimes (\psi_1 + \psi_2)$ // ordinary addition, ψ_i's same rank

$= \phi \otimes \psi_1 + \phi \otimes \psi_2$

Define formal sum

finite $\sum_{\oplus r} T^r(\mathbb{V})$, ex) $T^0(\mathbb{V}) \oplus T^2(\mathbb{V}) \oplus T^4(\mathbb{V})$ $\quad r=\{0,2,4\}$

e.g.) in ord alg

$$\begin{array}{rl} & 2+x^2+x^4 \\ + & 3x^2+4x^3+6x^5 \\ \hline \oplus: & 2+4x^2+4x^3+x^4+6x^5 \end{array}$$

i.e. only add together equal rank

$$(2x^2+x^7)(2+4x^2+4x^3+x^4+6x^5)$$

$$\phi_1 \otimes (\phi_2 \oplus \phi_3 \oplus \phi_4 \oplus \phi_5)$$

$$= (\phi_1 \otimes \phi_2) \oplus (\phi_1 \otimes \phi_2) \oplus \ldots$$

$T^3(\mathbb{V})$

general tensor in above written

$T = T^{ijk}\, \vec{e}_i \otimes \vec{e}_j \otimes \vec{e}_k$

if $\dim \mathbb{V} = 2$, $\exists$ 8 components

in gen, no particular relationship

$\mathcal{S}(3) = \{\sigma \mid \sigma \text{ is perm. of } (1,2,3)\}$

$\sigma_1 = \text{id}$, $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ Meiven

$\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $\quad \sigma_5 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $\quad \sigma_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$$\sigma_2\sigma_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \sigma_6$$

$$\sigma_4\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ \cancel{3} & 3 & 1 \\ 1 & 3 & 2 \end{pmatrix} = \sigma_5$$

$\cancel{\vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_3}$

$x \in T^3(\mathbb{V})$

(σx)

$x(\vec{v}^{*1}, \vec{v}^{*2}, \vec{v}^{*3})$

$$(\sigma x)(\vec{v}^{*1}, \vec{v}^{*2}, \vec{v}^{*3}) \underset{\text{def}}{=} x(\vec{v}^{*\sigma(1)}, \vec{v}^{*\sigma(2)}, \vec{v}^{*\sigma(3)})$$

$x \in T^r(\mathbb{V})$, $x = \vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_3$

$\sigma x \overset{?}{=} \vec{v}_3 \otimes \vec{v}_1 \otimes \vec{v}_2$ — no, the def permutes args, so σ^{-1} on the tensor vects

$\sigma x = \vec{v}_{\sigma^{-1}(1)} \otimes \vec{v}_{\sigma^{-1}(2)} \otimes \vec{v}_{\sigma^{-1}(3)}$

$$(\vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_3)(\vec{v}^{*1}, \vec{v}^{*2}, \vec{v}^{*3})$$
$$= \langle \vec{v}_1, \vec{v}^{*1} \rangle \langle \vec{v}_2, \vec{v}^{*2} \rangle \langle \vec{v}_3, \vec{v}^{*3} \rangle$$

ex) $(\sigma x)(\vec{v}^{*1}, \vec{v}^{*2}, \vec{v}^{*3})$
$$= x(\vec{v}^{*3}, \vec{v}^{*1}, \vec{v}^{*2}) = (\vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_3)(\vec{v}^{*3}, \vec{v}^{*1}, \vec{v}^{*2})$$
$$= \langle \vec{v}_1, \vec{v}^{*3} \rangle \langle \vec{v}_2, \vec{v}^{*1} \rangle \langle \vec{v}_3, \vec{v}^{*2} \rangle$$

~~But wrong of perm~~ $\cancel{\sigma(\vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_3) = \langle \vec{v}_2, \vec{v}^{*1} \rangle \langle \vec{v}_3, \vec{v}^{*2} \rangle \langle \vec{v}_1, \vec{v}^{*3} \rangle}$

$\sigma(\vec{v}_1 \otimes \vec{v}_2 \otimes \vec{v}_3)$ is wrong, gives wrong pairing

if $x = \vec{v}_1 \otimes \dots \otimes \vec{v}_r$ then $\sigma x = \vec{v}_{\sigma^{-1}(1)} \otimes \dots \otimes \vec{v}_{\sigma^{-1}(r)}$