Problem 1 An <u>orthogonal transformation</u> $A : \mathbb{V} \to \mathbb{V}$ is one satisfying $AA^T = \mathbb{1}$ (where A^T is the transpose of A). Verify the following tensorial properties under orthogonal transformations for the Kronecker deltas and the Levi-Civita tensor:

(a) δ^{ij} transforms as a (2,0)-type tensor;

(b) δ_{ij} transforms as a (0,2)-type tensor;

(c) $\delta_i^{\,j}$ transforms as a (1,1)-type tensor;

(d) $\varepsilon^i{}_{jk}$ transforms as a (1,2)-type tensor.

Problem 2 Show that the <u>inertia</u> <u>tensor</u> dencity

$$\mathcal{I}^{ij} \equiv \delta^{ij} x^l x_l - x^i x^j \quad ; \quad i,j = 1,2,3$$

in classical mechanics, where the x^i are Cartesian coordinates of a mass element, transforms as a (2,0)-type tensor under orthogonal transformations of the coordinates.

Problem 3 Verify the following contraction property of the Levi-Civita tensor:

$$\varepsilon^k{}_{ij}\, \varepsilon_{klm} = \delta_{il}\, \delta_{jm} - \delta_{im}\, \delta_{jl} \,.$$

Problem 4 Given $\vec{A} \times \vec{B} = (\varepsilon^i{}_{jk} A^j B^k)\, \vec{e}_i$,

Use the contraction property of the Levi-Civita tensor in Prob. 3 to prove that

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \,.$$

[This is called the "bac" minus "cab" identity.]

Prob. 5 Use the properties of the Levi-Civita tensor to show that

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B}).$$

Prob. 6 Let $\vec{A}: \mathbb{V} \to \mathbb{W}$ be a linear operator, where $\mathbb{V}$ and $\mathbb{W}$ are vector spaces over the same field. Prove the following relationship between the dimensions of the various vector spaces:

$$\dim(\mathbb{V}) = \dim(\mathrm{Ker}\, A) + \dim(\mathrm{Im}\, A),$$

where Ker A and Im A are the kernel and image of the operator A, respectively.

(Hint: First show that Ker A and Im A are vector subspaces of $\mathbb{V}$ and $\mathbb{W}$, respectively).

Prob. 7 Let $\mathbb{W} \subset \mathbb{V}$ be a vector subspace of the vector space $\mathbb{V}$. Define the <u>quotient space</u> $\mathbb{V}/\mathbb{W}$ to be the space of <u>equivalence classes</u> $\{\vec{v}\}$, with the equivalence relationship $\sim$ defined by

$$\vec{v}_1 \sim \vec{v}_2 \quad \text{if} \quad \vec{v}_1 - \vec{v}_2 \in \mathbb{W}.$$

Consider the <u>projection map</u> $\pi: \mathbb{V} \to \mathbb{V}/\mathbb{W}$ defined by

$$\pi(\vec{v}) = \{\vec{v}\}.$$

Use the result in Prob. 6 to show that

$$\dim\left(\frac{\mathbb{V}}{\mathbb{W}}\right) = \dim(\mathbb{V}) - \dim(\mathbb{W})$$

(Hint: $\mathrm{Ker}\,\pi = \mathbb{W}$).