Prob. 12 The symmetric states in $P^2(V)$ are spanned by the <u>triplet</u>

$$|\uparrow\uparrow\rangle\ ,\ |\downarrow\downarrow\rangle \quad \text{and} \quad \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$$

The antisymmetric states in $\Lambda^2(V)$ are spanned by the <u>singlet</u>

$$\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

On writing $\vec{e}_1 = |\uparrow\rangle$ and $\vec{e}_2 = |\downarrow\rangle$, the triplet states can be written as the tensor products

$$\vec{e}_1 \otimes \vec{e}_1\ ,\ \vec{e}_2 \otimes \vec{e}_2 \quad \text{and} \quad \frac{1}{\sqrt{2}}\left(\vec{e}_1 \otimes e_2 + \vec{e}_2 \otimes \vec{e}_1\right)$$

The singlet state can be written

$$\frac{1}{\sqrt{2}}\left(\vec{e}_1 \otimes \vec{e}_2 - \vec{e}_2 \otimes \vec{e}_1\right)$$

The 4×4 transformation matrix between the 2 bases is given by

$$\begin{pmatrix} |\uparrow\uparrow\rangle \\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) \\ |\downarrow\downarrow\rangle \\ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} |\uparrow\uparrow\rangle \\ |\uparrow\downarrow\rangle \\ |\downarrow\uparrow\rangle \\ |\downarrow\downarrow\rangle \end{pmatrix}$$

Assuming that the single-electron spin basis states are orthonormal: $(\vec{e}_i, \vec{e}_j) = \delta_{ij}$, the $1/\sqrt{2}$ factors will ensure that the triplet and singlet states are all <u>orthormal with</u> respect to each other.

Prob. 13

In quantum mechanics the orbit states of an electron are denoted $|l,m\rangle$, where $m = l, l-1, \ldots, -l$, a total of $2l+1$ states for each l. So, for a p-electron $(l=1)$, we have

$$|1,1\rangle \equiv |1\rangle$$
$$|1,0\rangle \equiv |0\rangle$$
$$|1,-1\rangle \equiv |-1\rangle$$

The spin states are $|s, m_s\rangle = |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle$.

We write

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle \equiv |\uparrow\rangle \quad (\text{spin up})$$
$$|\tfrac{1}{2}, -\tfrac{1}{2}\rangle \equiv |\downarrow\rangle \quad (\text{spin down})$$

A basis for the tensor product space (tensored from the orbit space and the spin space) is then

$$|1\rangle \otimes |\tfrac{1}{2}\rangle \equiv |1, \uparrow\rangle,$$
$$|1\rangle \otimes |-\tfrac{1}{2}\rangle \equiv |1, \downarrow\rangle,$$
$$|0\rangle \otimes |\tfrac{1}{2}\rangle \equiv |0, \uparrow\rangle,$$
$$|0\rangle \otimes |-\tfrac{1}{2}\rangle \equiv |0, \downarrow\rangle,$$
$$|-1\rangle \otimes |\tfrac{1}{2}\rangle \equiv |-1, \uparrow\rangle$$
$$|-1\rangle \otimes |-\tfrac{1}{2}\rangle \equiv |-1, \downarrow\rangle.$$

So the tensor product space is a 6-dimensional vector space. In this way one can make tensor product spaces for <u>multi-electron atoms</u>.

$\boxed{\text{Prob 14}}$ Any element in $SU(2)$ can be written

$$g = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad a, b \in \mathbb{C} \ (\cancel{\text{compact}} \text{ complex numbers})$$

with ~~provided~~ $|a|^2 + |b|^2 = 1$. A matrix of this form is unitary since $gg^\dagger = 1$, where $g^\dagger = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix}$ (complex-conjugated transpose)

Indeed $$\begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}\begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & -ab + ab \\ -a^*b^* + a^*b^* & |b|^2 + |a|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\{\vec{e}_1, \vec{e}_2\}$ be a basis of $\mathbb{V}$. Then (as shown in Prob 12), $\mathbb{P}^2(\mathbb{V})$, the subspace of $\mathbb{V} \otimes \mathbb{V}$ consisting of symmetric states, is spanned by the <u>triplet</u>

$$\vec{e}_1 \otimes \vec{e}_1 \quad , \quad \vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1 \quad , \quad \vec{e}_2 \otimes \vec{e}_2$$

and $\Lambda^2(\mathbb{V})$, the one-dimensional subspace of $\mathbb{V} \otimes \mathbb{V}$ consisting of antisymmetric states, is spanned by the <u>singlet</u>

$$\vec{e}_1 \otimes \vec{e}_2 - \vec{e}_2 \otimes \vec{e}_1$$

For arbitrary $g \in SU(2)$, we have

$$g(\vec{e}_1 \otimes \vec{e}_2 - \vec{e}_2 \otimes \vec{e}_1) = g(\vec{e}_1) \otimes g(\vec{e}_2) - g(\vec{e}_2) \otimes g(\vec{e}_1)$$

The action of $g \in SU(2)$ on $\mathbb{V}$ is given by

$$g(\vec{e}_i) = g_i^{\ j}\, \vec{e}_j$$

$\boxed{\text{Prob. 14}}$ (cont'd)

$$\therefore \quad g(\vec{e}_1 \otimes \vec{e}_2) - g(\vec{e}_2 \otimes \vec{e}_1)$$

$$= (g_1^{\,1}\vec{e}_1 + g_1^{\,2}\vec{e}_2) \otimes (g_2^{\,1}\vec{e}_1 + g_2^{\,2}\vec{e}_2)$$

$$- (g_2^{\,1}\vec{e}_1 + g_2^{\,2}\vec{e}_2) \otimes (g_1^{\,1}\vec{e}_1 + g_1^{\,2}\vec{e}_2)$$

$$= \cancel{g_1^{\,1}g_2^{\,1}(\vec{e}_1 \otimes \vec{e}_1)} + g_1^{\,2}g_2^{\,1}(\vec{e}_2 \otimes \vec{e}_1) + g_1^{\,1}g_2^{\,2}(\vec{e}_1 \otimes \vec{e}_2) + \cancel{g_1^{\,2}g_2^{\,2}(\vec{e}_2 \otimes \vec{e}_2)}$$

$$\cancel{-g_2^{\,1}g_1^{\,1}(\vec{e}_1 \otimes \vec{e}_1)} - g_2^{\,2}g_1^{\,1}(\vec{e}_2 \otimes \vec{e}_1) - g_2^{\,1}g_1^{\,2}(\vec{e}_1 \otimes \vec{e}_2) \cancel{- g_2^{\,2}g_1^{\,2}(\vec{e}_2 \otimes \vec{e}_2)}$$

$$= (g_1^{\,1}g_2^{\,2} - g_1^{\,2}g_2^{\,1})(\vec{e}_1 \otimes \vec{e}_2) - (g_1^{\,1}g_2^{\,2} - g_1^{\,2}g_2^{\,1})(\vec{e}_2 \otimes \vec{e}_1)$$

$$= (g_1^{\,1}g_2^{\,2} - g_1^{\,2}g_2^{\,1})(\vec{e}_1 \otimes \vec{e}_2 - \vec{e}_2 \otimes \vec{e}_1)$$

$$= (|a|^2 + |b|^2)(\vec{e}_1 \otimes \vec{e}_2 - \vec{e}_2 \otimes \vec{e}_1) = \vec{e}_1 \otimes \vec{e}_2 - \vec{e}_2 \otimes \vec{e}_1$$

$\therefore$ the singlet state is invariant under $SU(2)$

Now consider the triplet states in $\mathbb{P}^2(\mathbb{V})$:

$$g(\vec{e}_1 \otimes \vec{e}_1) = g(\vec{e}_1) \otimes g(\vec{e}_1)$$

$$= (g_1^{\,1}\vec{e}_1 + g_1^{\,2}\vec{e}_2) \otimes (g_1^{\,1}\vec{e}_1 + g_1^{\,2}\vec{e}_2)$$

$$= (g_1^{\,1})^2\,\vec{e}_1 \otimes \vec{e}_1 + g_1^{\,2}g_1^{\,1}\,\vec{e}_2 \otimes \vec{e}_1 + g_1^{\,1}g_1^{\,2}\,\vec{e}_1 \otimes \vec{e}_2 + (g_1^{\,2})^2\,\vec{e}_2 \otimes \vec{e}_2$$

$$= a^2(\vec{e}_1 \otimes \vec{e}_1) + ab(\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1) + b^2(\vec{e}_2 \otimes \vec{e}_2)$$

$$\in \mathbb{P}^2(\mathbb{V}).$$

$\boxed{\text{Prob 14}}$ (cont'd)

$$g(\vec{e}_2 \otimes \vec{e}_2) = g(\vec{e}_2) \otimes g(\vec{e}_2)$$
$$= (g_2^1 \vec{e}_1 + g_2^2 \vec{e}_2) \otimes (g_2^1 \vec{e}_1 + g_2^2 \vec{e}_2)$$
$$= (g_2^1)^2 \vec{e}_1 \otimes \vec{e}_1 + g_2^2 g_2^1 \vec{e}_2 \otimes \vec{e}_1 + g_2^1 g_2^2 \vec{e}_1 \otimes \vec{e}_2 + (g_2^2)^2 \vec{e}_2 \otimes \vec{e}_2$$
$$= (b^*)^2 \vec{e}_1 \times \vec{e}_1 - a^* b^* (\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1) + (a^*)^2 \vec{e}_2 \otimes \vec{e}_2$$

$$\in P^2(V)$$

Finally

$$g(\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1) = g(\vec{e}_1) \otimes g(\vec{e}_2) + g(\vec{e}_2) \otimes g(\vec{e}_1)$$
$$= (g_1^1 \vec{e}_1 + g_1^2 \vec{e}_2) \otimes (g_2^1 \vec{e}_1 + g_2^2 \vec{e}_2)$$
$$+ (g_2^1 \vec{e}_1 + g_2^2 \vec{e}_2) \otimes (g_1^1 \vec{e}_1 + g_1^2 \vec{e}_2)$$

$$= g_1^1 g_2^1 \vec{e}_1 \otimes \vec{e}_1 + g_1^2 g_2^1 (\vec{e}_2 \otimes \vec{e}_1) + g_1^1 g_2^2 (\vec{e}_1 \otimes \vec{e}_2) + g_1^2 g_2^2 (\vec{e}_2 \otimes \vec{e}_2)$$
$$+ g_2^1 g_1^1 \vec{e}_1 \otimes \vec{e}_1 + g_2^2 g_1^1 (\vec{e}_2 \otimes \vec{e}_1) + g_2^1 g_1^2 (\vec{e}_1 \otimes \vec{e}_2) + g_2^2 g_1^2 (\vec{e}_2 \otimes \vec{e}_2)$$

$$= 2 g_1^1 g_2^1 (\vec{e}_1 \otimes \vec{e}_1) + 2 g_1^2 g_2^2 (\vec{e}_2 \otimes \vec{e}_2)$$
$$+ (\vec{e}_1 \otimes \vec{e}_2)(g_1^1 g_2^2 + g_2^1 g_1^2)$$
$$+ (\vec{e}_2 \otimes \vec{e}_1)(g_1^2 g_2^1 + g_1^1 g_2^2)$$

$$= -2ab^* (\vec{e}_1 \otimes \vec{e}_1) + 2a^* b (\vec{e}_2 \otimes \vec{e}_2) + (|a|^2 - |b|^2)(\vec{e}_1 \otimes \vec{e}_2 + \vec{e}_2 \otimes \vec{e}_1)$$
$$\in P^2(V)$$

$\therefore$ $P^2(V)$ is <u>invariant under $SU(2)$</u>