MAT 444 HW Solutions

Prob. 26 (a) $g_{ij}(x)\,dx^i \otimes dx^j = g_{ij}(x)\,\dfrac{\partial x^i}{\partial x'^k}\,dx'^k \otimes \dfrac{\partial x^j}{\partial x'^l}\,dx'^l$

$$= g_{ij}(x)\,\frac{\partial x^i}{\partial x'^k}\cdot\frac{\partial x^j}{\partial x'^l}\,dx'^k \otimes dx'^l$$

$$= g'_{kl}(x')\,dx'^k \otimes dx'^l$$

all x's expressed in terms of x''s.

$$\therefore \boxed{g'_{kl}(x') = g_{ij}(x(x'))\,\frac{\partial x^i}{\partial x'^k}\cdot\frac{\partial x^j}{\partial x'^l}}$$

(b) $dx^i = \dfrac{\partial x^i}{\partial x'^j}\,dx'^j$, $J_{ij} \equiv \dfrac{\partial x^i}{\partial x'^j}$ $J \equiv \det(J_{ij})$

$$\therefore dx^1\wedge\dots\wedge dx^D = \det\left(\frac{\partial x^i}{\partial x'^j}\right) dx'^1\wedge\dots\wedge dx'^D$$

$$\therefore \boxed{dx^1\wedge\dots\wedge dx^D = J\,dx'^1\wedge\dots\wedge dx'^D}$$

we write this is $d^Dx = J\,d^Dx'$

(c) The result in (a) can be written as

$$g'_{kl} = \sum_{ij} g_{ij}\,J_{ik}\,J_{jl} = \sum_{ij} (J^T)_{ki}\,g_{ij}\,J_{jl}$$

The RHS is matrix multiplication. $\therefore$ We have the matrix equation $g' = \underset{\sim}{J}^T\cdot\underset{\sim}{g}\,\underset{\sim}{J}$ (where $\underset{\sim}{g}$ denote the matrix (g_{ij}) etc.)

$\therefore$ Taking determinants on both sides, and realizing that $\det(J^T) = \det(J)$, we have

$$\boxed{g' = g J^2}$$

(d) $d^Dx'\sqrt{g'} = \dfrac{d^Dx}{J}\cdot\sqrt{gJ^2} = d^Dx\sqrt{g}$.

$\therefore$ $\boxed{d^Dx\sqrt{g} \text{ is the invariant volume element.}}$

$\boxed{\text{Prob. 27}}$ Use rectangular coordinates:

$$(\vec{A} \times \vec{B})^i = \varepsilon^i{}_{jk} A^j B^k$$

$\therefore (\nabla \times \vec{A})^i = \varepsilon^{ij}{}_k \, \partial_j A^k$ (Einstein summation convention used)

$\therefore \nabla \cdot (\nabla \times \vec{A}) = \partial_i (\nabla \times \vec{A})^i = \partial_i (\varepsilon^{ij}{}_k \, \partial_j A^k)$

$$= \varepsilon^{ij}{}_k (\partial_i \partial_j A^k)$$

in this sum only terms with distinct i, j, k contribute. When $i \neq j$, $\varepsilon^{ij}{}_k = -\varepsilon^{ji}{}_k$, but $\partial_i \partial_j A^k = \partial_j \partial_i A^k$

$\therefore$ The RHS of the 3rd equality in the above equation vanishes, and so

$\nabla \cdot (\nabla \times \vec{A}) = 0$ (divergence of a curl vanishes)

$$(\nabla f)_i = \partial_i f \left(= \frac{\partial f}{\partial x^i} \right)$$

$\therefore [\nabla \times (\nabla f)]^i = \varepsilon^{ijk} \partial_j (\nabla f)_k = \varepsilon^{ijk} \partial_j \partial_k f$

$= 0$ (for the same reason as in the evaluation of the divergence of a curl)

$\therefore \nabla \times \nabla f = 0$ (curl of a gradient vanishes)

MAT 444

[redacted] Problem Solutions for [redacted] Prob. 28, 29, 30

Prob [redacted] 28

To prove (7.19), that is,

$$\nabla(fg) = f\,\nabla g + g\,\nabla f\ ,$$

we note that

$$d(fg) = f\,dg + g\,df\ .$$

Since

$$df = (\nabla f)\cdot d\boldsymbol{r}$$

for any function f, we have

$$\nabla(fg)\cdot d\boldsymbol{r} = (f\nabla g)\cdot d\boldsymbol{r} + (g\nabla f)\cdot d\boldsymbol{r}\ .$$

Eq.(7.19) then follows.

To prove (7.20), that is,

$$\nabla\times(f\boldsymbol{X}) = (\nabla f)\times\boldsymbol{X} + f\,\nabla\times\boldsymbol{X}\ ,$$

we let

$$\boldsymbol{X} = X^1\epsilon_1 + X^2\epsilon_2 + X^3\epsilon_3\ ,\quad \omega = X^1dx + X^2dy + X^3dz = X_i\,dx^i\ ,$$

and note that, by the rules of exterior differentiation,

$$d(f\omega) = (df)\wedge\omega + f\,d\omega\ .$$

Since ω is a 1-form, $f\omega = fX^1dx + fX^2dy + fX^3dz$ is a 1-form also; and so

$$\begin{aligned} d(f\omega) &= \partial_j(fX_i)\,dx^j\wedge dx^i \\ &= (\nabla\times f\boldsymbol{X})^1\,dy\wedge dz + (\nabla\times f\boldsymbol{X})^2\,dz\wedge dx + (\nabla\times f\boldsymbol{X})^3\,dx\wedge dy\ . \end{aligned}$$

On the other hand,

$$\begin{aligned} (df)\wedge\omega &= (\partial_j f)X_i\,dx^j\wedge dx^i \\ &= ((\nabla f)\times\boldsymbol{X})^1\ dy\wedge dz + ((\nabla f)\times\boldsymbol{X})^2\ dz\wedge dx + ((\nabla f)\times\boldsymbol{X})^3\ dx\wedge dy\ ; \end{aligned}$$

and

$$\begin{aligned} f d\omega &= f(\partial_j X_i)\,dx^j\wedge dx^i \\ &= f\ \left[(\nabla\times\boldsymbol{X})^1\,dy\wedge dz + (\nabla\times\boldsymbol{X})^2\,dz\wedge dx + (\nabla\times\boldsymbol{X})^3\,dx\wedge dy\right]\ . \end{aligned}$$

The last four equations imply (7.20).

To prove (7.21), that is,

$$\nabla\cdot(f\boldsymbol{X}) = (\nabla f)\cdot\boldsymbol{X} + f\,\nabla\cdot\boldsymbol{X}\ ,$$

We set the vector field $\boldsymbol{X}$ and the 2-form ψ as follows:

$$\boldsymbol{X} = X^1\epsilon_1 + X^2\epsilon_2 + X^3\epsilon_3 \,, \quad \psi = X^1\, dy \wedge dz + X^2\, dz \wedge dx + X^3\, dx \wedge dy \,.$$

Then

$$d(f\psi) = df \wedge \psi + f\, d\psi \,.$$

But we know that

$$f d\psi = (f \nabla \cdot \boldsymbol{X})\, dx \wedge dy \wedge dz \,,$$

and

$$d(f\psi) = \nabla \cdot (f\boldsymbol{X})\, dx \wedge dy \wedge dz \,.$$

Also

$$\begin{aligned} df \wedge \psi &= (\partial_i f) dx^i \wedge [X^1\, dy \wedge dz + X^2\, dz \wedge dx + X^3\, dx \wedge dy] \\ &= \left[(\partial_x f)X^1 + (\partial_y f)X^2 + (\partial_z f)X^3\right]\, dx \wedge dy \wedge dz \\ &= (\nabla f \cdot \boldsymbol{X})\, dx \wedge dy \wedge dz \,. \end{aligned}$$

Hence (7.21) follows.

To prove (7.22), that is,

$$\nabla \cdot (\boldsymbol{X} \times \boldsymbol{Y}) = \boldsymbol{Y} \cdot (\nabla \times \boldsymbol{X}) - \boldsymbol{X} \cdot (\nabla \times \boldsymbol{Y}) \,,$$

we let

$$\boldsymbol{X} = X^1\epsilon_1 + X^2\epsilon_2 + X^3\epsilon_3 \,, \quad \boldsymbol{Y} = Y^1\epsilon_1 + Y^2\epsilon_2 + Y^3\epsilon_3 \,;$$

and define the 1-forms

$$\theta = X^1 dx + X^2 dy + X^3 dz = X_i dx^i \,, \quad \phi = Y^1 dx + Y^2 dy + Y^3 dz = Y_j dx^j \,,$$

and the 2-form

$$\psi = \theta \wedge \phi \,.$$

So, since θ is a 1-form,

$$d\psi = d\theta \wedge \phi - \theta \wedge d\phi \,.$$

We have

$$\begin{aligned} \psi &= X_i Y_j\, dx^i \wedge dx^j \\ &= \left(X^1Y^2 - X^2Y^1\right)\, dx \wedge dy + \left(X^3Y^1 - X^1Y^3\right)\, dz \wedge dx + \left(X^2Y^3 - X^3Y^2\right)\, dy \wedge dz \\ &= (\boldsymbol{X} \times \boldsymbol{Y})^1\, dy \wedge dz + (\boldsymbol{X} \times \boldsymbol{Y})^2\, dz \wedge dx + (\boldsymbol{X} \times \boldsymbol{Y})^3\, dx \wedge dy \,. \end{aligned}$$

So

$$d\psi = \nabla \cdot (\boldsymbol{X} \times \boldsymbol{Y})\, dx \wedge dy \wedge dz \,.$$

On the other hand,

$$\begin{aligned} d\theta &= (\partial_j X_i)\, dx^j \wedge dx^i = (\nabla \times \boldsymbol{X})^1\, dy \wedge dz + (\nabla \times \boldsymbol{X})^2\, dz \wedge dx + (\nabla \times \boldsymbol{X})^3\, dx \wedge dy \,, \\ d\phi &= (\partial_j Y_i)\, dx^j \wedge dx^i = (\nabla \times \boldsymbol{Y})^1\, dy \wedge dz + (\nabla \times \boldsymbol{Y})^2\, dz \wedge dx + (\nabla \times \boldsymbol{Y})^3\, dx \wedge dy \,; \end{aligned}$$

and so

$$\begin{aligned} d\theta \wedge \phi &= \left[Y^1(\nabla \times \boldsymbol{X})^1 + Y^2(\nabla \times \boldsymbol{X})^2 + Y^3(\nabla \times \boldsymbol{X})^3\right]\, dx \wedge dy \wedge dz \\ &= \left[\boldsymbol{Y} \cdot (\nabla \times \boldsymbol{X})\right] dx \wedge dy \wedge dz\ , \end{aligned}$$

and

$$\begin{aligned} \theta \wedge d\phi &= \left[X^1(\nabla \times \boldsymbol{Y})^1 + X^2(\nabla \times \boldsymbol{Y})^2 + X^3(\nabla \times \boldsymbol{Y})^3\right]\, dx \wedge dy \wedge dz \\ &= \left[\boldsymbol{X} \cdot (\nabla \times \boldsymbol{Y})\right] dx \wedge dy \wedge dz\ . \end{aligned}$$

Eq.(7.22) follows.

Prob [redacted] 29

This is actually Prob. 27

Proof of (7.17):

$$(\nabla \times (\nabla f))^i = \varepsilon^{ijk}\, \partial_j (\nabla f)_k = \varepsilon^{ijk}\, \partial_j \partial_k f\ .$$

Hence

$$\begin{aligned} (\nabla \times (\nabla f))^1 &= \varepsilon^{123}\partial_2\partial_3 f + \varepsilon^{132}\partial_3\partial_2 f = \partial_2\partial_3 f - \partial_3\partial_2 f = 0\ , \\ (\nabla \times (\nabla f))^2 &= \varepsilon^{231}\partial_3\partial_1 f + \varepsilon^{213}\partial_1\partial_3 f = \partial_3\partial_1 f - \partial_1\partial_3 f = 0\ , \\ (\nabla \times (\nabla f))^3 &= \varepsilon^{312}\partial_1\partial_2 f + \varepsilon^{321}\partial_2\partial_1 f = \partial_1\partial_2 f - \partial_2\partial_1 f = 0\ , \end{aligned}$$

Proof of (7.18):

$$\begin{aligned} \nabla \cdot (\nabla \times \boldsymbol{X}) &= \partial_i (\nabla \times \boldsymbol{X})^i = \partial_i (\varepsilon^{ij}{}_k \partial_j X^k) = \varepsilon^{ij}{}_k\, \partial_i \partial_j X^k \\ &= (\varepsilon^{23}{}_1\, \partial_2\partial_3 X^1 + \varepsilon^{32}{}_1\, \partial_3\partial_2 X^1) + (\varepsilon^{31}{}_2\, \partial_3\partial_1 X^2 + \varepsilon^{13}{}_2\, \partial_1\partial_3 X^2) \\ &\quad + (\varepsilon^{12}{}_3\, \partial_1\partial_2 X^3 + \varepsilon^{21}{}_3\, \partial_2\partial_1 X^3) = 0 + 0 + 0 = 0\ . \end{aligned}$$

Proof of (7.19):

$$\begin{aligned} \nabla(fg) &= \sum_i \partial_i (fg)\, \boldsymbol{\epsilon}_i = \sum_i (f\partial_i g + g\partial_i f)\, \boldsymbol{\epsilon}_i = f \sum_i (\partial_i g)\, \boldsymbol{\epsilon}_i + g \sum_i (\partial_i f)\, \boldsymbol{\epsilon}_i \\ &= f\nabla g + g\nabla f\ . \end{aligned}$$

Proof of (7.20):

$$\begin{aligned} (\nabla \times (f\boldsymbol{X}))^i &= \varepsilon^{ij}{}_k \partial_j (f X^k) = \varepsilon^{ij}{}_k \left((\partial_j f) X^k + f \partial_j X^k\right) \\ &= \varepsilon^{ij}{}_k (\nabla f)_j X^k + f \varepsilon^{ij}{}_k \partial_j X^k = (\nabla f \times \boldsymbol{X})^i + f (\nabla \times \boldsymbol{X})^i\ . \end{aligned}$$

Proof of (7.21):

$$\nabla \cdot (f\boldsymbol{X}) = \partial_i (f X^i) = (\partial_i f) X^i + f \partial_i X^i = (\nabla f) \cdot \boldsymbol{X} + f \nabla \cdot \boldsymbol{X}\ .$$

Proof of (7.22):

$$\begin{aligned} \nabla \cdot (\boldsymbol{X} \times \boldsymbol{Y}) &= \partial_i (\boldsymbol{X} \times \boldsymbol{Y})^i = \varepsilon^i{}_{jk} \partial_i (X^j Y^k) \\ &= \varepsilon^i{}_{jk} (\partial_i X^j) Y^k + \varepsilon^i{}_{jk} X^j \partial_i Y^k = (\varepsilon_k{}^i{}_j \partial_i X^j) Y^k - \varepsilon_j{}^i{}_k (\partial_i Y^k) X^j \\ &= (\nabla \times \boldsymbol{X})_k Y^k - (\nabla \times \boldsymbol{Y})_j X^j = (\nabla \times \boldsymbol{X}) \cdot \boldsymbol{Y} - (\nabla \times \boldsymbol{Y}) \cdot \boldsymbol{X}\ . \end{aligned}$$

Prob [redacted] 30

We will show how to use both methods to establish the identity. First consider the tensorial index method. Start with

$$(\boldsymbol{X} \times \boldsymbol{Y})^i = \varepsilon^i{}_{jk}\, X^j Y^k \,.$$

Then

$$\begin{aligned}
(\nabla \times (\boldsymbol{X} \times \boldsymbol{Y}))^i &= \varepsilon^{ij}{}_k \partial_j (\boldsymbol{X} \times \boldsymbol{Y})^k = \varepsilon^{ij}{}_k\, \partial_j (\varepsilon^k{}_{lm} X^l Y^m) \\
&= \varepsilon^{ij}{}_k \varepsilon^k{}_{lm}\, \partial_j (X^l Y^m) = \varepsilon_k{}^{ij} \varepsilon^k{}_{lm}\, \partial_j (X^l Y^m) \\
&= (\delta^i_l \delta^j_m - \delta^i_m \delta^j_l)\, \partial_j (X^l Y^m) = \partial_j (X^i Y^j) - \partial_j (X^j Y^i) \\
&= X^i \partial_j Y^j + (Y^j \partial_j) X^i - (X^j \partial_j) Y^i - Y^i (\partial_j X^j) \\
&= (\nabla \cdot \boldsymbol{Y}) X^i + (\boldsymbol{Y} \cdot \nabla) X^i - (\boldsymbol{X} \cdot \nabla) Y^i - (\nabla \cdot \boldsymbol{X}) Y^i \,.
\end{aligned}$$

This proves (7.26).

To use differential forms to prove (7.26) we define the 1-forms θ and ϕ and the 2-form ψ as in the last problem, that is,

$$\theta = X_i dx^i \,, \quad \phi = Y_j dx^j \,, \quad \psi = \theta \wedge \phi = X_i Y_j\, dx^i \wedge dx^j \,.$$

Focus on the left-hand-side of (7.26) first. To find the curl of a vector field we know that we have to exteriorly differentiate a 1-form. But $\boldsymbol{X} \times \boldsymbol{Y}$ corresponds to the 2-form ψ. So we have to use the Hodge star operator $*$ to make ψ into the 1-form $*\psi$ before exteriorly differentiating. The result $d(*\psi)$ will be a 2-form. The right-hand-side of (7.26), however, corresponds to a 1-form. This suggests that, to prove (7.26), we have to compute $(*d*)\psi$. We start with

$$*\psi = *(X_i Y_j\, dx^i \wedge dx^j) = X_i Y_j * (dx^i \wedge dx^j) = \varepsilon^{ij}{}_k\, X_i Y_j dx^k \,.$$

Then

$$d(*\psi) = \varepsilon^{ij}{}_k\, d(X_i Y_j\, dx^k) = \varepsilon^{ij}{}_k\, \partial_l (X_i Y_j)\, dx^l \wedge dx^k \,.$$

So

$$\begin{aligned}
(*d*)\psi &= \varepsilon^{ij}{}_k \varepsilon^{lk}{}_m\, \partial_l (X_i Y_j) dx^m = \varepsilon_k{}^{ij} \varepsilon^k{}_m{}^l\, \partial_l (X_i Y_j) dx^m \\
&= (\delta^i_m \delta^{jl} - \delta^{il} \delta^j_m)\, \partial_l (X_i Y_j) dx^m = \partial_j (X_i Y^j) dx^i - \partial_i (X^i Y_j) dx^j \\
&= X_i (\partial_j Y^j) dx^i + (Y^j \partial_j) X_i dx^i - (X^i \partial_i) Y_j dx^j - (\partial_i X^i) Y_j dx^j \\
&= \{(\nabla \cdot \boldsymbol{Y}) X_i + (\boldsymbol{Y} \cdot \nabla) X_i - (\boldsymbol{X} \cdot \nabla) Y_i - (\nabla \cdot \boldsymbol{X}) Y_i\}\, dx^i \,.
\end{aligned}$$

This also proves (7.26).