①

MAY 5 M444

$A: \mathbb{V} \to \mathbb{W}$
$A^*: \Lambda^r(\mathbb{W}^*) \to \Lambda^r(\mathbb{V}^*)$

introduce $\varphi \in \Lambda^r(\mathbb{W}^*)$
$(A^*\varphi)(v_1, \dots, v_r) = \varphi(Av_1, \dots, Av_r)$

$A: \mathbb{V} \to \mathbb{V}$, $A^*: \Lambda^r(\mathbb{V}^*) \to \Lambda^r(\mathbb{V}^*)$
$A^*: \Lambda^n(\mathbb{V}^*) \to \Lambda^n(\mathbb{V}^*)$

$\dim(\Lambda^n(\mathbb{V}^*)) = 1$
any ~~$\varphi \in \Lambda^n$~~ $\varphi \in \Lambda^n(\mathbb{V}^*) = \alpha\, e^{*1} \wedge \dots \wedge e^{*n}$

NOW $A^*\varphi = (\text{som \#})\,\varphi \qquad (\varphi \in \Lambda^n(\mathbb{V}^*))$
a transformation on 1-dim space $\Rightarrow$ some scalar mult of the 1-dim vect. the ~~sect~~ # is dependent on a. i.e. (som #) = det A

$A^*\varphi = (\det A)\,\varphi$
Suppose $A\vec{v} = \alpha\vec{v}$, calc det using above

$(\Rightarrow)\ (A^*\varphi)(v_1, \dots, v_n) = \varphi(Av_1, \dots, Av_n) = \varphi(\alpha v_1, \dots, \alpha v_n)$

$= \underbrace{\alpha^n \varphi(v_1, \dots, v_n)}_{\in \Lambda^n(\mathbb{V}^*)} \Rightarrow A^*\varphi = \alpha^n \varphi$
$\Rightarrow \boxed{\det A = \alpha^n}$

$A(e_1) = \alpha$, $A(e_2) = \alpha, \dots$, $A(e_n) = \alpha \Rightarrow \begin{pmatrix} \alpha & & 0 \\ & \alpha & \\ 0 & & \alpha \ddots \end{pmatrix}$

$\begin{pmatrix} \alpha & & 0 \\ & \alpha & \\ 0 & & \ddots \end{pmatrix} = \alpha \mathbb{I}_n$, If $A = \alpha \mathbb{I}_n$, then $\det A = \alpha^n$

all important property of determinant

$AB = C$, $\det(AB) = (\det A)(\det B)$ $\forall$

Then $\forall$ $\varphi \in \Lambda^n(\mathbb{V}^*)$ & $v_1, \dots, v_n \in \mathbb{V}$

$$(C^*\varphi)(v_1, \dots, v_n) = \varphi(Cv_1, \dots, Cv_n) = \varphi(ABv_1, \dots, ABv_n)$$
$$= \varphi(A(Bv_1), \dots, A(Bv_n)) = (A^*\varphi)(Bv_1, \dots, Bv_n)$$
$$= (B^*(A^*\varphi))(v_1, \dots, v_n) \Rightarrow C^* = B^*A^*$$

$\Rightarrow$ $C^*\varphi = (\det C)\varphi$ & $(B^*A^*)\varphi = \cancel{\det} B^*(A^*\varphi)$
$= (\det AB)\varphi$ $= (\det B)(A^*\varphi)$
$= (\det B)(\det A)\varphi$

under Δ-coords , $B^{-1}AB$ is new rep.

SHOW invariance :

$$\det(B^{-1}AB) = \det(B^{-1})\det A \det B$$
$$= (\det B)^{-1} \det A \det B = \det A$$

Let $\varphi \in \Lambda^n(\mathbb{V}^*)$, $A^*\varphi = (\det A)\cdot\varphi$ (*)

$$(A^*\varphi)(e_1, \dots, e_n) = \varphi(Ae_1, \dots, Ae_n) = (\det A)\varphi(e_1, \dots, e_n)$$

Now $Ae_1 = a_1^{i_1}\vec{e}_{i_1}$, ... , $A(\vec{e}_n) = a_n^{i_n}\vec{e}_{i_n}$

$$\Rightarrow \varphi(a_1^{i_1}e_{i_1}, \dots, a_n^{i_n}e_{i_n}) = a_1^{i_1} \dots a_n^{i_n} \varphi(e_{i_1}, \dots, e_{i_n})$$

But $\varphi \in \Lambda^n(\mathbb{V}^*)$

$$\sum_{\sigma \in S_n} a_1^{\sigma(1)} \dots a_n^{\sigma(n)} \varphi(e_{\sigma(1)}, \dots, e_{\sigma(n)})$$

$$= \sum_{\sigma \in S_n} a_1^{\sigma(1)} \cdots a_n^{\sigma(n)} (\sigma\phi)(e_1, \dots, e_n)$$

ϕ antisym $\Rightarrow \sigma\phi = (\operatorname{sgn} \sigma)\, \phi$

$$\underbrace{= \sum_{\sigma \in S_n} a_1^{\sigma(1)} \cdots a_n^{\sigma(n)} (\operatorname{sgn} \sigma)}_{= (*)} \phi(e_1, \dots, e_n)$$