MAT 444 HW Solutions

Prob. 21 Assume that

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2 \quad (\text{product rule})$$

where $\omega_1 \in A^r(M)$ and ω_2 is any differential form in $A(M)$.

Let $\omega = a_{i_1 \dots i_r}(x^1, \dots, x^n)\, dx^{i_1} \wedge \dots \wedge dx^{i_r}$

be an r-form on M. The functions $a_{i_1} \dots a_{i_r}(x^1, \dots, x^n)$ can be considered as 0-forms on M.

$$\therefore \quad d\omega = (d a_{i_1 \dots i_r}) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r} + (-1)^0 a_{i_1 \dots i_r} \wedge d(dx^{i_1} \wedge \dots \wedge dx^{i_r})$$

Use the product rule again:

$$d(dx^{i_1} \wedge \dots \wedge dx^{i_r}) = \cancel{d(dx^{i_1}) + (-1)^1 dx^{i_1} \wedge d(}$$

$$= \underbrace{d^2 x^{i_1}}_{=0} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_r} + (-1)^1 dx^{i_1} \wedge d(dx^{i_2} \wedge \dots \wedge dx^{i_r})$$

$$= 0 - dx^{i_1} \wedge \left(\underbrace{d^2 x^{i_2}}_{=0} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_r} + (-1)^1 dx^{i_2} \wedge \underbrace{d(dx^{i_3} \wedge \dots \wedge dx^{i_r})}_{=0} \right)$$

$$= 0$$

$$\therefore \quad \boxed{d\omega = d a_{i_1 \dots i_r} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}}$$

or

$$\boxed{d\omega = \frac{\partial a_{i_1 \dots i_r}}{\partial x^k}\, dx^k \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}}$$

Prob. 22 Suppose $\omega_1 = a\, dx^{i_1} \wedge \dots \wedge dx^{i_r} \in A^r(M)$
$\omega_2 = b\, dx^{j_1} \wedge \dots \wedge dx^{j_s} \in A^s(M)$.

Then
$$d(\omega_1 \wedge \omega_2) = d(ab) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s}$$
$$= (b\,da + a\,db) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s}$$
$$= (da \wedge dx^{i_1} \wedge \dots \wedge dx^{i_r}) \wedge (b\, dx^{j_1} \wedge \dots \wedge dx^{j_s})$$
$$+ (-1)^r (a\, dx^{i_1} \wedge \dots \wedge dx^{i_r}) \wedge (db \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s})$$
$$= d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2 ,$$

where, in the RHS of the third equality, the term db has been moved to the right by r "slots", and thus a factor of $(-1)^r$ is acquired according to the anticommutative rule of exterior products.

Prob. 23 Write $df = \dfrac{\partial f}{\partial x^i} dx^i$

$$\therefore\ d^2 f = d(df) = d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i = \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i$$
$$= \frac{1}{2} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i + \frac{1}{2} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i$$
$$= \frac{1}{2} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i + \frac{1}{2} \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j \quad (i \leftrightarrow j \text{ in the 2nd term})$$
$$= \frac{1}{2} \underbrace{\left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j} \right)}_{0} dx^j \wedge dx^i = 0$$

since the 2nd derivatives are equal to each other within the parentheses.

Prob. 24 Let $\vec{X}, \vec{Y}$ be two smooth tangent vector fields given by

$$\vec{X} = X^i \partial_i \ , \quad \vec{Y} = Y^i \partial_i$$

Using the definition of a tangent vector field $\vec{X}$ is a map from the set of smooth functions on M to the whole set of functions on M given by

$$(\vec{X} f)(x) = \vec{X}_x f \quad \text{where } f \in C^\infty(M),$$

a smooth tangent vector field is one where $\vec{X} f$ is also a smooth function on M, i.e. $\vec{X} f \in C^\infty(M)$.

a) If $\vec{X}$ and $\vec{Y}$ are smooth, then, for any $f \in C^\infty(M)$

$$\vec{X}(f) \text{ and } \vec{Y}(f) \in C^\infty(M)$$

Thus $\vec{X}(\vec{Y}(f)) - \vec{Y}(\vec{X}(f)) = (\vec{X}\vec{Y})(f) - (\vec{Y}\vec{X})(f)$

$$= (\vec{X}\vec{Y} - \vec{Y}\vec{X})(f) \in C^\infty(M)$$

Thus $[\vec{X}, \vec{Y}] \equiv \vec{X}\vec{Y} - \vec{Y}\vec{X}$ is also a smooth tangent vector field.

b) $[\vec{X}, \vec{Y}]f = (\vec{X}\vec{Y} - \vec{Y}\vec{X})f = X^j \partial_j (Y^i \partial_i f) - Y^j \partial_j (X^i \partial_i f)$

$$= X^j (Y^i \partial_j \partial_i f + \partial_j Y^i \partial_i f)$$
$$- Y^j (X^i \partial_j \partial_i f + \partial_j X^i \partial_i f)$$

$$= X^j \partial_j Y^i \partial_i f - Y^j \partial_j X^i \partial_i f$$

$$= (X^j \partial_j Y^i - Y^j \partial_j X^i) \partial_i f$$

$$\therefore [\vec{X}, \vec{Y}]^i = X^j \partial_j Y^i - Y^j \partial_j X^i .$$

$\boxed{\text{Prob. 25}}$ Let $\omega = b_{i_1 \dots i_r}(y_1, \dots, y_r)\, dy^{i_1} \wedge \dots \wedge dy^{i_r}$

$f(x^1, \dots, x^n) = (y^1, \dots, y^n)$

$$d\omega = \frac{\partial b_{i_1 \dots i_r}}{\partial y^j} dy^j \wedge dy^{i_1} \wedge \dots \wedge dy^{i_r}$$

$$\therefore (f^* \circ d)(\omega) = \frac{\partial b_{i_1 \dots i_r}}{\partial y^j} f^*(dy^j) \wedge f^*(dy^{i_1}) \wedge \dots \wedge f^*(dy^{i_r})$$

$$= \frac{\partial b_{i_1 \dots i_r}}{\partial y^j} \left(\frac{\partial y^j}{\partial x^i} dx^i \right) \wedge \left(\frac{\partial y^{i_1}}{\partial x^{k_1}} dx^{k_1} \right) \wedge \dots \wedge \left(\frac{\partial y^{i_r}}{\partial x^{k_r}} dx^{k_r} \right)$$

$$= \left(\frac{\partial b_{i_1 \dots i_r}}{\partial y^j} \frac{\partial y^j}{\partial x^i} \right) \left(\frac{\partial y^{i_1}}{\partial x^{k_1}} \right) \dots \left(\frac{\partial y^{i_r}}{\partial x^{k_r}} \right) dx^i \wedge dx^{k_1} \wedge \dots \wedge dx^{k_r}$$

$$= \frac{\partial b_{i_1 \dots i_r}}{\partial x^i} \frac{\partial y^{i_1}}{\partial x^{k_1}} \dots \frac{\partial y^{i_r}}{\partial x^{k_r}} dx^i \wedge dx^{k_1} \wedge \dots \wedge dx^{k_r}$$

$$(d \circ f^*)(\omega) = d\left[b_{i_1 \dots i_r} f^*(dy^{i_1}) \wedge \dots \wedge f^*(dy^{i_r}) \right]$$

$$= \frac{\partial b_{i_1 \dots i_r}}{\partial x^i} dx^i \wedge f^*(dy^{i_1}) \wedge \dots \wedge f^*(dy^{i_r})$$

$$= \frac{\partial b_{i_1 \dots i_r}}{\partial x^i} \frac{\partial y^{i_1}}{\partial x^{k_1}} \dots \frac{\partial y^{i_r}}{\partial x^{k_r}} dx^i \wedge dx^{k_1} \wedge \dots \wedge dx^{k_r}$$

$$\therefore (f^* \circ d)(\omega) = (d \circ f^*)(\omega)$$