APRIL 26 M444
MIDTERM I 28th FRIDAY

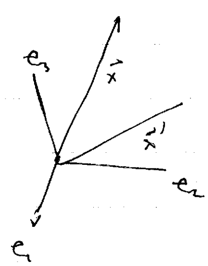
Look into
↳ Representation Theory of a Group

Suppose $T^2(V)$, $\dim V = 2$, $\{\vec{e}_1, \vec{e}_2\}$,
$\exists$ 4 basis elements $e_1 \otimes e_1$, $e_2 \otimes e_2$, $e_1 \otimes e_2$, $e_2 \otimes e_1$

($\Rightarrow$) ANY TENSOR CAN BE WRITTEN AS $T^{ij} e_i \otimes e_j$

Symmetric : $e_1 \otimes e_1$, $e_2 \otimes e_2$, $e_1 \otimes e_2 + e_2 \otimes e_1$ $\quad P^2(V)$ ← forms irreducible

Anti-symm : $e_1 \otimes e_2 - e_2 \otimes e_1$ $\quad \Lambda^2(V)$



(x^1, x^2, x^3)
(x'^1, x'^2, x'^3)
$R\vec{x} = \vec{x}'$

$$(\Rightarrow) \; (x'^1, x'^2, x'^3) = (x^1, x^2, x^3) \begin{pmatrix} R_1^1 & R_1^2 & R_1^3 \\ R_2^1 & R_2^2 & R_2^2 \\ R_3^1 & R_3^2 & R_3^3 \end{pmatrix}$$

$\det R = 1 \Rightarrow RT' = \mathbb{1}$

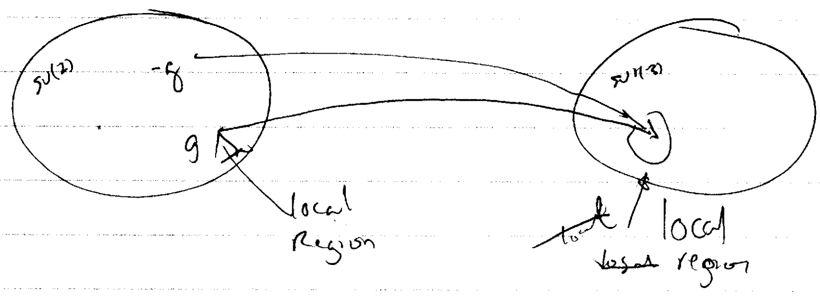
{set of all 3D rot. matrices} $\subset$ {set of all 3×3 ; $RR^T = \mathbb{1}$}
// called Rotation group.
// set of transf on 3D-vectors
If HIGHER DIM, Represent by linear ops

For dim = 2, use $SU(2)$ special unitary $\det = 1$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times2}$, $U^{-1} = U^{\dagger} = (U^T)^*$

$ad - bc = 1$, ex) $\begin{pmatrix} a^* & b \\ b & c^* \end{pmatrix}$ $\quad |a|^2 + |b|^2 = 1$

Note need to specify 2 - $\mathbb{C}$ #'s, 4 $\mathbb{R}$ #s can ~~assoc~~ make association

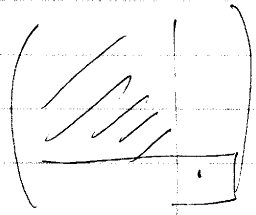
$SO(3) \quad SU(2)$ not ~~1A~~ 1-1 (2-1)



for $R^2(4)$

~~$4 = 3 \otimes 1$~~

$4 \cdot 3 \oplus 1$



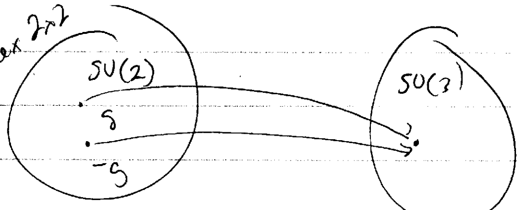
Def Group homomorphism, ...

TRIVIAL HOMOMORPHISM $f: G \to G' : f(g) = e' \ \forall g \in G$

complex 2x2

det $\mathbb{1}$

$A^{-1} = A^{\dagger}$



Group Representation

Def: A representation D of a group G is a group homomorphism btwn G & $O(V)$, $D: G \to O(V)$

$O(V)$ group of lin. ops on vect space V. A representation is said to be faithful if homomorphism also isomorphism.

~~G~~ ~~we have a~~

~~$\dim \mathbb{V} = 3$~~ ~~$\mathbb{R}$ vects~~

~~$SO(3) = \{R \mid 3\times 3 \ \mathbb{R}$ matrix~~

$\dim \mathbb{V} = 3$ Real vects

$O(3) = \{ R \mid R \text{ is } 3\times 3 \text{ matrix} : RR^T = \mathbb{1} \}$

Form $\mathbb{V} \otimes \mathbb{V} = T^2(\mathbb{V})$, $\quad \mathbb{V} : \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

$T \in T^2(\mathbb{V}) \Rightarrow T = T^{ij}\, \vec{e}_i \otimes \vec{e}_j$

Rotate $\rightarrow RT = T^{ij} R(\vec{e}_i \otimes \vec{e}_j)$.

say Rotation Δ's $\quad \vec{e}_i \rightarrow \vec{e}_i{}' \quad \Rightarrow \quad \vec{e}_i = R^j_i \vec{e}_j{}'$

$T' = T'^{ij}\, e_i \otimes e_k \Rightarrow T'^{ij} = \cancel{D}\, R^i_k R^j_l T^{kl}$

so $\quad T'^{11} = R^1_l R^1_k T^{lk}$

$\quad = R^1_1 R^1_1 T^{11} + \dots$

$(1\times 9) = (1\times 9)\begin{pmatrix} 9 \times 9 \\ \ \end{pmatrix}$ $\quad$ FORM 9-dim rep of an element of $O(3)$

Top acting on 9-dim space.

Result $\quad 9 = \underbrace{5 \oplus 1}_{\text{sym}} \oplus \underbrace{3}_{\text{anti-symm}}$

Start with general tensor T^{ij} $\quad {}^{\cdot\cdot} = 1,2,3$

indep

$S^{ij} = \cancel{A}\ T^{ij} + T^{ji}$ $\quad \begin{pmatrix} S^{11} & S^{12} & S^{13} \\ S^{21} & S^{22} & S^{23} \\ S^{31} & S^{32} & S^{33} \end{pmatrix} = \begin{pmatrix} 2T^{11} & T^{12}+T^{21} & T^{13}+T^{31} \\ & 2T^{22} & T^{23}+T^{32} \\ & & T^{33} \end{pmatrix}$

$$S'^{ij} = T'^{ij} + T'^{ji}$$
$$= R^i_l R^j_m T^{lm} + R^j_m R^i_l T^{ml}$$
$$= R^i_l R^j_m \left(T^{lm} + T^{ml}\right)$$
$$= R^i_l R^j_m \left(S^{lm}\right)$$

Sub Space of symm tens invariant under rotation

Now $A^{ij} = T^{ij} - T^{ji}$

$$\begin{pmatrix} 0 & T^{12}-T^{21} & T^{13}-T^{31} \\ & 0 & T^{23}-T^{32} \\ & & 0 \end{pmatrix}$$ // 3 indep comp

Traceless symm tens

$$\tilde{S}^{ij} = S^{ij} - \frac{\delta^{ij} T}{D}$$ ← tr. of S^{ij}

D = dim V

$$\tilde{S}^{11} + \tilde{S}^{22} + \tilde{S}^{33} = 0$$

$$\begin{pmatrix} \tilde{S}^{11} & \tilde{S}^{12} & \tilde{S}^{13} \\ & \tilde{S}^{22} & \tilde{S}^{23} \\ & & \tilde{S}^{33} \end{pmatrix}$$

$$9 = \underbrace{\underbrace{5}_{\text{traceless}} \oplus \underbrace{1}_{\text{trace}}}_{\text{sym}} \oplus 3$$