$\boxed{\text{Prob. 8}}$

$$[G(\alpha\vec{u}+\beta\vec{v})](\vec{w}) \overset{\text{by definition}}{=} (\alpha\vec{u}+\beta\vec{v}, \vec{w})$$

$$= \alpha(\vec{u},\vec{w})+\beta(\vec{v},\vec{w}) = \alpha[G(u)](\vec{w})+\beta[G(\vec{v})](\vec{w})$$

$$= \left(\alpha G(\vec{u})+\beta G(\vec{v})\right)(\vec{w}),$$

for any $\vec{u},\vec{v},\vec{w}\in\mathbb{V}$; $\alpha,\beta\in\mathbb{F}$

$\therefore\quad G(\alpha\vec{u}+\beta\vec{v}) = \alpha G(\vec{u})+\beta G(\vec{v})$

$\therefore\quad G$ is linear.

$\boxed{\text{Prob. 9}}$ Let $\dim\mathbb{V} = \dim\mathbb{W} = n$

Choose a basis $\{\vec{e}_1,\dots,\vec{e}_n\}$ of $\mathbb{V}$

and a basis $\{\vec{d}_1,\dots,\vec{d}_n\}$ of $\mathbb{W}$.

Suppose $A:\mathbb{V}\to\mathbb{W}$ is an invertible linear map on $\mathbb{V}$. The matrix representation of A with respect to $\{\vec{e}_i\}$ and $\{\vec{d}_j\}$ is given by the matrix (a_i^j) in

$$A\vec{e}_i = a_i^j\vec{d}_j$$

or

$$\begin{pmatrix} A\vec{e}_1 \\ \vdots \\ A\vec{e}_n \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ & & \vdots & \\ a_n^1 & a_n^2 & \cdots & a_n^n \end{pmatrix} \begin{pmatrix} \vec{d}_1 \\ \vdots \\ \vec{d}_n \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \vec{d}_1 \\ \vdots \\ \vec{d}_n \end{pmatrix} \underset{\text{since } A \text{ is invertible}}{=} \begin{pmatrix} A^{-1} \end{pmatrix} \begin{pmatrix} A\vec{e}_1 \\ \vdots \\ A\vec{e}_n \end{pmatrix}$$

where A is the $n\times n$ matrix

$$\begin{pmatrix} a_1^1 & \cdots & a_1^n \\ & \vdots & \\ a_n^1 & \cdots & a_n^n \end{pmatrix}.$$

$\boxed{\text{Prob 9}}$ (cont'd)

Hence

$$\vec{d}_i = (A^{-1})_i^j A(e_j)$$

By linearity of A, we have

$$\vec{d}_i = A\left((A^{-1})_i^j \vec{e}_j\right)$$

So each basis vector in $\mathbb{W}$ has a pre-image under A in $\mathbb{V}$. Let $\vec{w} \in \mathbb{W}$ be an arbitrary vector in $\mathbb{W}$ and write

$$\vec{w} = w^i \vec{d}_i \quad .$$

Then

$$\vec{w} = w^i A\left((A^{-1})_i^j \vec{e}_j\right)$$

by linearity of A

$$= w^i (A^{-1})_i^j A(\vec{e}_j)$$

$$= A\left(w^i (A^{-1})_i^j \vec{e}_j\right)$$

$\therefore$ ~~$\vec{w}$ has a pre~~ every $\vec{w} \in \mathbb{W}$ has a preimage under A in $\mathbb{V}$.

$\therefore$ A is surjective.

$\boxed{\text{Prob 10}}$ $(\vec{u}, \vec{w}) \equiv [G(u)](\vec{w}) = 0$ for all $\vec{w} \in \mathbb{V}$

$\Rightarrow \vec{u} = 0$ because the scalar product is non-degenerate.

$\therefore$ if $G(\vec{u}) \in \mathbb{V}^*$ is the zero map on $\mathbb{V}$, then $\vec{u} = 0$

or $\quad G(\vec{u}) = \vec{0} \Rightarrow \vec{u} = \vec{0}$

$\therefore \quad \text{Ker } G = \vec{0}$.

Let $\vec{u}_1, \vec{u}_2 \in \mathbb{V}$, then

$$G(\vec{u}_1) = G(\vec{u}_2) \xRightarrow{\text{by linearity of } G} G(\vec{u}_1 - \vec{u}_2) = \vec{0}$$

$$\Rightarrow \vec{u}_1 - \vec{u}_2 = \vec{0} \quad (\text{Since Ker } G = \vec{0})$$

$$\Rightarrow \vec{u}_1 = \vec{u}_2$$

$\therefore$ G is injective. (equivalently, G is invertible)

We have already proved in Prob 9 that any invertible linear map between 2 vector spaces of the same dimension is surjective. Since G is both linear (proved in Prob. 8) and ~~inject~~ invertible, it must be surjective also. Hence G is injective and surjective, or, ~~they~~ G is an isomorphism.

$\boxed{\text{Prob. 11}}$ Suppose $G: \mathbb{V} \to \mathbb{V}^*$ is defined by

$$[G(\vec{v})](\vec{w}) = (\vec{v}, \vec{w})$$

where $(\vec{v}, \vec{w})$ is some bilinear form on $\mathbb{V}$.

Assume G is bijective. So it is injective.

Then $\quad \text{Ker } G = \vec{0}$

that is, if $G(\vec{v}) = \vec{0}$ ~~for~~, then $\vec{v} = 0$

that is, $(\vec{v}, \vec{w}) = 0$ for all $\vec{w} \in \mathbb{V} \Rightarrow \vec{v} = 0$

This shows that the bilinear form is non-degenerate.