MAY 3 M444

$\Lambda^r(\mathbb{V})$, $\dim \mathbb{V} = n$

$\dim \Lambda^r(\mathbb{V}) = \binom{n}{r}$, $\{\vec{e}_1, \ldots, \vec{e}_n\}$ for $\mathbb{V}$

Basis $\{e_{i_1} \wedge \ldots \wedge e_{i_r}\}$ $\quad 1 \leq i_1 < i_2 < \ldots < i_r \leq n$

ex) $T^3(\mathbb{V})$ $\quad \dim \mathbb{V} = 4$

$\Lambda^3(\mathbb{V})$ $\quad \dim \{\Lambda^3(\mathbb{V})\} = \binom{4}{3} = 4$

$\Rightarrow$ Basis $\{e_1 \wedge e_2 \wedge e_3,\ e_1 \wedge e_2 \wedge e_4,\ e_1 \wedge e_3 \wedge e_4,\ e_2 \wedge e_3 \wedge e_4\}$

$A = A^{123} e_1 \wedge e_2 \wedge e_3 + A^{124} e_1 \wedge e_2 \wedge e_4 + \ldots$

if $\Lambda^n(\mathbb{V})$, $\dim \mathbb{V} = n$, basis $e_1 \wedge \ldots \wedge e_n$

$(\vec{e}_{i_1} \wedge \ldots \wedge \vec{e}_{i_r})(v^{*1}, \ldots, v^{*r})$, $\quad v^{*i} \in \mathbb{V}^*$

$$= \frac{1}{r!} \sum_{\sigma \in S_r} (\operatorname{sgn} \sigma)\, \sigma \underbrace{(e_{i_1} \otimes \ldots \otimes e_{i_r})(v^{*1}, \ldots, v^{*r})}$$

$(e_{i_1} \otimes \ldots \otimes e_{i_r})(v^{*\sigma(1)}, \ldots, v^{*\sigma(r)})$

$\Rightarrow \langle e_{i_1}, v^{*\sigma(1)} \rangle \langle e_{i_2}, v^{*\sigma(2)} \rangle \ldots \langle e_{i_r}, v^{*\sigma(r)} \rangle$

$$= \frac{1}{r!} \sum_{\sigma \in S_r} (\operatorname{sgn} \sigma) \langle e_{i_1}, v^{*\sigma(1)} \rangle \langle e_{i_2}, v^{*\sigma(2)} \rangle \ldots \langle e_{i_r}, v^{*\sigma(r)} \rangle$$

$= \frac{1}{r!} \Big| \langle e_{i_1}, v^{*\sigma(1)} \rangle \quad \ldots \ldots \quad \langle e_{i_1}, v^{*\sigma}$

$$= \frac{1}{r!}\begin{vmatrix} \langle e_{i_1}, v^{*1}\rangle & \cdots & \langle e_{i_1}, v^{*r}\rangle \\ \langle e_{i_2}, v^{*1}\rangle & \cdots & \langle e_{i_2}, v^{*r}\rangle \\ \vdots & & \vdots \\ \langle e_{i_r}, v^{*1}\rangle & \cdots & \langle e_{i_r}, v^{*r}\rangle \end{vmatrix}$$

determinant invariant under similarity ~~&~~ transform
(i.e. is an invariant quantity)
$\det A \neq 0 \Leftrightarrow$ invertible recall

NOW consider
$(\vec{e}_{i_1} \wedge \cdots \wedge \vec{e}_{i_r})(e^{*j_1}, \ldots, e^{*j_r})$,

we have $\frac{1}{r!}\begin{vmatrix} \langle e_{i_1}, e^{*j_1}\rangle & \cdots & \langle e_{i_1}, e^{*j_r}\rangle \\ \vdots & & \\ \end{vmatrix}$

$\delta_{i_1}^{j_1}$ $\qquad$ $\delta_{i_1}^{j_r}$

$\frac{1}{r!}\delta_{i_1 \cdots i_r}^{j_1 \cdots j_r}$ Generalized Kronecker

$$= \begin{cases} 1 & \text{if } (i_1, \ldots, i_r) \text{ all distinct } \& \ (j_1, \ldots, j_r) \ \underline{\text{even}} \text{ perm of } (i_1, \ldots, i_r) \\ -1 & \text{if } (i_1, \ldots, i_r) \quad " \quad " \quad \underline{\text{odd}} \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon^{j_1 j_2 j_3} = \delta_{1\,2\,3}^{j_1 j_2 j_3}$$

ex) $(e_1 \wedge e_2 \wedge e_3)(e^{*3}, e^{*2}, e^{*1}) = \frac{1}{3!}\begin{vmatrix} \langle e_1, e^{3}\rangle & \langle e_1, e^{2}\rangle & \langle e_1, e^{1}\rangle \\ \langle e_2, e^{3}\rangle & \langle e_2, e^{2}\rangle & \langle e_2, e^{1}\rangle \\ \langle e_3, e^{3}\rangle & \langle e_3, e^{2}\rangle & \langle e_3, e^{1}\rangle \end{vmatrix} = \frac{1}{3!}$

$\vec{v}_1 \wedge \dots \wedge \vec{v}_r \in \Lambda^r(\mathbb{V})$ & $v^{*1} \wedge \dots \wedge v^{*r} \in \Lambda^r(\mathbb{V}^*)$

are dual spaces of each other.

$\langle \vec{v}_1 \wedge \dots \wedge \vec{v}_r , \vec{v}^{*1} \wedge \dots \wedge \vec{v}^{*r} \rangle = \det\left(\langle v_\alpha , v^{*\beta} \rangle\right)$

Pullback Map

general lin map f

$f: \mathbb{V} \to \mathbb{W}$, choose $r \leq \min\{\mathbb{V}, \mathbb{W}\}$

consider $\mathbb{V}^*, \mathbb{W}^*$, $\Lambda^r(\mathbb{W}^*), \Lambda^r(\mathbb{V}^*)$

obtain $f^* : \Lambda^r(\mathbb{W}^*) \longrightarrow \Lambda^r(\mathbb{V}^*)$

$\phi \in \Lambda^r(\mathbb{W}^*)$, $(f^*\phi)(v_1, \dots, v_r) \overset{\text{by def}}{=} \phi(f(\vec{v}_1), \dots, f(\vec{v}_r))$

WHY IT makes sense? $f(v_i) \in \mathbb{W}$

f linear $\Rightarrow f^*$ linear