M444 5/10/17

Thm Suppose $f: V \to W$ is linear.
Then $\forall$ $\Phi \in \Lambda^r(W^*)$ & $\Psi \in \Lambda^s(W^*)$

$f^*: \Lambda^t(W^*) \longrightarrow \Lambda^t(V^*)$
$f^*\Phi$, $f^*\Psi$
$f^*(\Phi \wedge \Psi) = f^*(\Phi) \wedge f^*(\Psi)$

Thm: The vects $v_1, \dots, v_r \in V$ are L.D. iff
$v_1 \wedge \dots \wedge v_r = 0$

Pf: ($\Rightarrow$) Supp $v_1, \dots, v_r$ L.D., WLOG
$v_r = \alpha_1 v_1 + \dots + \alpha_{r-1} v_{r-1}$,

$v_1 \wedge \dots \wedge v_{r-1} \wedge (\alpha_1 v_1 + \dots + \alpha_{r-1} v_{r-1})$
$= \alpha_1 (v_1 \wedge \dots \wedge v_{r-1} \wedge v_1) + \alpha_2 (v_1 \wedge \dots \wedge v_{r-1} \wedge v_2) + \dots$
$= 0 + \dots + 0$
$= 0$

($\Leftarrow$) ~~Suppose~~ WTP $v_1 \wedge \dots \wedge v_r = 0 \Rightarrow \{v_1, \dots, v_r\}$ are L.D.
Proof By contrapositive
Suppose $\{v_1, \dots, v_r\}$ are L.I. $r \leq n = \dim V$
~~$v_1 \wedge \dots \wedge v_r = \alpha \, v^{\alpha_1} \dots v^{\alpha_r} e_{\alpha_1} \wedge \dots \wedge e_{\alpha_r}$~~
extend set $\{v_1, \dots, v_r; v_{r+1}, \dots, v_n\}$
$v_1 \wedge v_2 \wedge \dots \wedge v_r \wedge v_{r+1} \wedge \dots \wedge v_n \neq 0$
$\Rightarrow v_1 \wedge \dots \wedge v_r \neq 0$

Cartans Lemma

Suppose $\{v_1,\dots,v_r\}$ & $\{w_1,\dots,w_r\}$ are two sets of vectors in $\mathbb{V}$ s.t.

$$\sum_{\alpha=1}^{r} v_\alpha \wedge w_\alpha = 0.$$

If $v_1,\dots,v_r$ are L.I., then $w_\alpha = \sum_{\beta=1}^{r} a_{\alpha\beta} \vec{v}_\beta$ $\quad 1\le\alpha\le r$

w/ ~~α~~ $a_{\alpha\beta} = a_{\beta\alpha}$ (symm)

<u>PROOF:</u> Since $v_1,\dots,v_r$ are L.I. $\quad r\le n = \dim \mathbb{V}$

Then extend to $\{v_1,\dots,v_r, v_{r+1},\dots,v_n\}$ of $\mathbb{V}$.

~~Then~~ $$w_\alpha = \sum_{\beta=1}^{r} a_{\alpha\beta}\vec{v}_\beta + \sum_{i=r+1}^{n} a_{\alpha i}\vec{v}_i$$

$$\Rightarrow \sum_{\alpha=1}^{r} v_\alpha \wedge \left(\sum_{\beta=1}^{r} a_{\alpha\beta}\vec{v}_\beta + \sum_{i=r+1}^{n} a_{\alpha i}\vec{v}_i\right)$$

$$= \sum_{\alpha,\beta=1}^{r} a_{\alpha\beta}\, v_\alpha \wedge v_\beta + \sum_{\alpha=1}^{r}\sum_{i=r+1}^{n} a_{\alpha i}\, v_\alpha \wedge v_i$$

$\Vert_{\alpha=\beta} \Rightarrow 0$

L.I.

$$= \sum_{1\le\alpha<\beta\le r} (a_{\alpha\beta} - a_{\beta\alpha})\, \underbrace{v_\alpha \wedge v_\beta} + \sum_{\alpha=1}^{r}\sum_{i=r+1}^{n} a_{\alpha i}\, \underbrace{\vec{v}_\alpha \wedge \vec{v}_i}$$

Forms basis of $\Lambda^2(\mathbb{V})$

$\{v_i \wedge v_j : 1\le i<j\le n\}$

$\Rightarrow a_{\alpha i} = 0 \quad \forall \quad 1\le\alpha\le r,\ r+1\le i\le n$ &

~~α~~ $a_{\alpha\beta} = a_{\beta\alpha}$

Thm Suppose $\{v_1, \ldots, v_r\} \in \mathbb{V}$ are lin. indep.
and $\omega \in \Lambda^r(\mathbb{V})$.

Then $\omega = \vec{v}_1 \wedge \psi_1 + \ldots + \vec{v}_r \wedge \psi_r$

where $\psi_1, \ldots, \psi_r \in \Lambda^{p-1}(\mathbb{V})$, iff

$v_1 \wedge \ldots \wedge v_r \wedge \omega = 0$

PROOF: Suppose $\omega = \vec{v}_1 \wedge \psi_1 + \ldots + \vec{v}_r \wedge \psi_r$

$$\vec{v}_1 \wedge \ldots \wedge \vec{v}_r \wedge \omega = \vec{v}_1 \wedge \ldots \wedge \vec{v}_r \wedge (\vec{v}_1 \wedge \psi_1 + \ldots + \vec{v}_r \wedge \psi_r)$$

$$= \hat{\vec{v}}_1 \wedge \ldots \wedge \vec{v}_r \wedge \vec{v}_1 \wedge \psi_1 + \ldots$$

$$= 0$$

Sufficiently suppose $\vec{v}_1 \wedge \ldots \wedge \vec{v}_r \wedge \omega = 0$ $(*)$

true if $p+r > n$. so suppose $p+r \leq n$

Extend to $\{v_1, \ldots, v_r, v_{r+1}, \ldots, v_n\}$ of $\mathbb{V}$

Then $\omega \in \Lambda^p(\mathbb{V})$ can be expressed as

$$\omega = \vec{v}_1 \wedge \psi_1 + \ldots + \vec{v}_r \wedge \psi_r + \underbrace{\sum_{r+1 \leq \alpha_1 < \ldots < \alpha_p \leq n} \xi^{\alpha_1 \ldots \alpha_p} v_{\alpha_1} \wedge \ldots \wedge v_{\alpha_p}}$$

$\psi_i \in \Lambda^{p-1}(\mathbb{V})$

– implies $= 0$ $\Rightarrow$ coeff $\xi = 0$