Chapter 2 Problems

Panya Sukphranee

A Mathematical Introduction to Robotic Manipulation

1. Let $a, b, c, \in \mathbb{R}^3$ with the standard basis $\{\vec{e_1}, \vec{e_2}, \vec{e_3}\}$. Using Einstein summation notation,

   (a)

$$\begin{aligned}
\vec{a} \cdot (\vec{b} \times \vec{c}) &= (a^i \vec{e_i}) \cdot (\epsilon^l{}_{jk} b^j c^k) \vec{e_l} && (1)\\
&= \delta_{il} a^i \epsilon^l{}_{jk} b^j c^k && (2)\\
&= \sum_i \epsilon^i{}_{jk} a^i b^j c^k && (3)\\
&= \epsilon_{ijk} a^i b^j c^k && (4)\\
&= \epsilon_{kij} a^i b^j c^k && (5)\\
&= (\vec{a} \times \vec{b})_k c^k && (6)\\
&= (\vec{a} \times \vec{b}) \cdot \vec{c} && (7)
\end{aligned}$$

   (b)

$$\begin{aligned}
\vec{a} \times (\vec{b} \times \vec{c}) &= \epsilon^i{}_{jk} a^j (\epsilon^k{}_{lm} b^l c^m) \vec{e_i} && (8)\\
&= (\epsilon^i{}_{jk} \epsilon^k{}_{lm} a^j b^l c^m) \vec{e_i} && (9)\\
&= \sum_i (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})(a^j b^l c^m) \vec{e_i} && (10)\\
&= \sum_j (a^j b^i c^j) \vec{e_i} - (a^j b^j c^i) \vec{e_i} && (11)\\
&= (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{a} \cdot \vec{b}) \vec{c} && (12)
\end{aligned}$$

2. Let $g, h \in SE(3)$, $\bar{g} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix}$ and $\bar{h} = \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix}$

   (a) Closure

$$\begin{aligned}
\bar{g}\bar{h} &= \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix} && (13)\\
&= \begin{bmatrix} R_g R_h & R_g p_h + p_g \\ 0 & 1 \end{bmatrix} && (14)
\end{aligned}$$

Therefore, $gh = (R_g R_h, R_g p_h + p_g) \in SE(3)$

(b) Identity element exists

Let $e = (\mathbb{I}_{3x3}, 0) \in SE(3)$,

$$\bar{g}\bar{e} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbb{I}_{3x3} & 0 \\ 0 & 1 \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \tag{16}$$

$$= \bar{g} \tag{17}$$

and

$$\bar{e}\bar{g} = \begin{bmatrix} \mathbb{I}_{3x3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \tag{19}$$

$$= \bar{g} \tag{20}$$

Thus, $ge = eg = g \Rightarrow e \in SE(3)$ is the identity element.

(c) Inverse Exists Let $g \in SE(3)$, $\bar{g} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix}$.

Consider $\bar{h} = \begin{bmatrix} R_g^\top & -R_g^\top p_g \\ 0 & 1 \end{bmatrix} \in \mathbb{R}_{4x4}$

$$\bar{g}\bar{h} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_g^\top & -R_g^\top p_g \\ 0 & 1 \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} R_g R_g^\top & -R_g R_g^\top p_g + p_g \\ 0 & 1 \end{bmatrix} \tag{22}$$

$$= \begin{bmatrix} \mathbb{I}_{3x3} & 0 \\ 0 & 1 \end{bmatrix} \tag{23}$$

$$= \bar{g} \tag{24}$$

On the other hand,

$$\bar{h}\bar{g} = \begin{bmatrix} R_g^\top & -R_g^\top p_g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} R_g^\top R_g & R_g^\top p_g - R_g^\top p_g \\ 0 & 1 \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} \mathbb{I}_{3x3} & 0 \\ 0 & 1 \end{bmatrix} \tag{27}$$

$$= \bar{g} \tag{28}$$

Since $\bar{h} = \bar{g}^{-1}$, $g^{-1} = (R_g^\top, -R_g^\top p_g)$.

(d) Associativity

Let $f, g, h \in SE(3)$, then $\bar{f} = \begin{bmatrix} R_f & p_f \\ 0 & 1 \end{bmatrix}$ $\bar{g} = \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix}$ $\bar{h} = \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix}$

$$\begin{aligned}
(\bar{f}\bar{g})\bar{h} &= \Big(\begin{bmatrix} R_f & p_f \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \Big) \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix} && (29) \\
&= \Big(\begin{bmatrix} R_f R_g & R_f p_g + p_f \\ 0 & 1 \end{bmatrix} \Big) \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix} && (30) \\
&= \begin{bmatrix} R_f R_g R_h & R_f R_g p_h + R_f p_g + p_f \\ 0 & 1 \end{bmatrix} && (31)
\end{aligned}$$

$$\begin{aligned}
\bar{f}(\bar{g}\bar{h}) &= \begin{bmatrix} R_f & p_f \\ 0 & 1 \end{bmatrix} \Big(\begin{bmatrix} R_g & p_g \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_h & p_h \\ 0 & 1 \end{bmatrix} \Big) && (32) \\
&= \begin{bmatrix} R_f & p_f \\ 0 & 1 \end{bmatrix} \Big(\begin{bmatrix} R_g R_h & R_g p_h + p_g \\ 0 & 1 \end{bmatrix} \Big) && (33) \\
&= \begin{bmatrix} R_f R_g R_h & R_f R_g p_h + R_f p_g + p_f \\ 0 & 1 \end{bmatrix} && (34)
\end{aligned}$$

Thus, $(fg)h = f(gh)$.

3. Properties of Rotation Matrices

(a) Find the eigenvalues and eigenvectors of $\widehat{w} = \begin{bmatrix} 0 & -\omega^3 & \omega^2 \\ \omega^3 & 0 & -\omega^1 \\ -\omega^2 & \omega^1 & 0 \end{bmatrix}$, $\|\vec{w}\| = 1$.

$$\begin{aligned}
|\widehat{w} - \lambda \mathbb{I}| &= \begin{vmatrix} -\lambda & -\omega^3 & \omega^2 \\ \omega^3 & -\lambda & -\omega^1 \\ -\omega^2 & \omega^1 & -\lambda \end{vmatrix} && (35) \\
&= -\lambda(\lambda^2 + \omega_1^2) - \omega_3(\lambda\omega_3 - \omega_1\omega_2) - \omega_2(\omega_1\omega_3 + \lambda\omega_2) && (36)
\end{aligned}$$

(being in Euclidean $\mathbb{R}^3$, we can lower the ω indices for simplicity)

$$\begin{aligned}
& && (37) \\
&= -\lambda^3 - \lambda\omega_1^2 - \lambda\omega_3^2 + \omega_1\omega_2\omega_3 - \omega_1\omega_2\omega_3 - \lambda\omega_2^2 && (38) \\
&= -\lambda^3 - \lambda \|\omega\|^2 && (39) \\
&= -\lambda^3 - \lambda && (40) \\
&= 0 && (41)
\end{aligned}$$

implies $\lambda = 0, \pm i$. We find the corresponding eigenvectors.

i. $\lambda = 0$. Solve $\widehat{w}\vec{x} = \vec{0}$. Wlog, $\omega_3 \neq 0$, row reduce the following:

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{\omega_2}{\omega_3} \\ 1 & 0 & -\frac{\omega_1}{\omega_3} \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -\frac{\omega_1}{\omega_3} \\ 0 & 1 & -\frac{\omega_2}{\omega_3} \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{\omega_1}{\omega_3} \\ 0 & 1 & -\frac{\omega_2}{\omega_3} \\ 0 & \omega_1 & -\frac{\omega_1\omega_2}{\omega_3} \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -\frac{\omega_1}{\omega_3} \\ 0 & 1 & -\frac{\omega_2}{\omega_3} \\ 0 & 0 & 0 \end{bmatrix} \to t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

4. Properties of skew-symmetric matrices

(a) It was easier to solve part (b) first and use those results for this problem. Let $\vec{x} \in \mathbb{R}^3$.

$$\begin{aligned} (R\widehat{w}R^\top)\vec{x} &= R\widehat{w}(R^\top\vec{x}) && (42) \\ &= R(\vec{w} \times (R^\top\vec{x})) && (43) \\ &= R\vec{w} \times RR^\top\vec{x} && (44) \\ &= (\widehat{R\vec{w}})\vec{x} && (45) \end{aligned}$$

(b) If $R \in SO(3)$ and $v, w \in \mathbb{R}^3$, then $R(\vec{v}\times\vec{w}) = (R\vec{v})\times(R\vec{w})$. We'll show equality by comparing the $l-th$ component of each side. The result utilizes the relationship between the dot product and multiplication by transpose.

$$\begin{aligned} [R(\vec{v} \times \vec{w})]^l &= \vec{e_l} \cdot R(\vec{v} \times \vec{w}) && (46) \\ &= \vec{e_l}^\top R(\vec{v} \times \vec{w}) && (47) \\ &= (R^\top\vec{e_l})^\top(\vec{v} \times \vec{w}) && (48) \\ &= (R^\top\vec{e_l}) \cdot (\vec{v} \times \vec{w}) && (49) \end{aligned}$$

This is the determinant of matrix with columns $R^\top\vec{e_l}, \vec{v}$, and $\vec{w}$, respectively.

$$= det([R^\top\vec{e_l}, \vec{v}, \vec{w}]) \quad (50)$$

Since $R^\top R = \mathbb{I}$,

$$\begin{aligned} &= det(R^\top R[R^\top\vec{e_l}, \vec{v}, \vec{w}]) && (51) \\ &= det(R^\top[\vec{e_l}, R\vec{v}, R\vec{w}]) && (52) \\ &= det(R^\top)det([\vec{e_l}, R\vec{v}, R\vec{w}]) && (53) \\ &= det([\vec{e_l}, R\vec{v}, R\vec{w}]) && (54) \\ &= \vec{e_l} \cdot (R\vec{v} \times R\vec{w}) && (55) \\ &= (R\vec{v} \times R\vec{w})^l && (56) \end{aligned}$$

Thus, $R(\vec{v} \times \vec{w}) = (R\vec{v}) \times (R\vec{w})$

5. Cayley Parameters

Cayley Parameterization, like the exponential map, is a mapping from $so(3)$ to $SO(3)$. In this problem we show that $R_a = (\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a})$ is indeed an element of $SO(3)$, given $\widehat{a} \in so(3)$. The derivation of this mapping can be found going in the opposite direction. i.e. by letting $R_a \in SO(3)$ and showing $\widehat{a} \in so(3)$; this involves using the diagonals of the parallelogram formed by some vector $\vec{v}$ and its transformation $R_a\vec{v}$.

(a) Show $R_a = (\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) \in SO(3)$.
Since, $\widehat{a}$ is anti-symmetric, $\widehat{a}^\top = -\widehat{a}$. Therefore, $(\mathbb{I} \pm \widehat{a})^\top = (\mathbb{I} \mp \widehat{a})$. Recall from Linear Algebra that

$$\begin{aligned}(AB)^{-1} &= B^{-1}A^{-1}\\ (AB)^\top &= B^\top A^\top\\ (A^{-1})^\top &= (A^\top)^{-1}\end{aligned}$$

Note that the transformations $(\mathbb{I} - \widehat{a})^{-1}$ and $(\mathbb{I} + \widehat{a})$ commute. To show this, we write $(\mathbb{I} + \widehat{a})$ in terms of $(\mathbb{I} - \widehat{a})$.

$$\begin{aligned}(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) &= -(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a})\\ &= -(\mathbb{I} - \widehat{a})^{-1}(-2\mathbb{I} + \mathbb{I} - \widehat{a})\\ &= -(\mathbb{I} - \widehat{a})^{-1}(-2\mathbb{I} + (\mathbb{I} - \widehat{a}))\\ &= (\mathbb{I} - \widehat{a})^{-1}(2\mathbb{I} - (\mathbb{I} - \widehat{a}))\\ &= 2(\mathbb{I} - \widehat{a})^{-1} - (\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} - \widehat{a})\\ &= 2(\mathbb{I} - \widehat{a})^{-1} - \mathbb{I}\\ &= 2(\mathbb{I} - \widehat{a})^{-1} - (\mathbb{I} - \widehat{a})(\mathbb{I} - \widehat{a})^{-1}\\ &= [2\mathbb{I} - (\mathbb{I} - \widehat{a})](\mathbb{I} - \widehat{a})^{-1}\\ &= (\mathbb{I} + \widehat{a})(\mathbb{I} - \widehat{a})^{-1}\end{aligned}$$

Now,

$$\begin{aligned}R_a^\top R_a &= [(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a})]^\top(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) && (57)\\ &= (\mathbb{I} + \widehat{a})^\top[(\mathbb{I} - \widehat{a})^{-1}]^\top(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) && (58)\\ &= (\mathbb{I} - \widehat{a})(\mathbb{I} + \widehat{a})^{-1}(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a}) && (59)\\ &= (\mathbb{I} - \widehat{a})(\mathbb{I} + \widehat{a})^{-1}(\mathbb{I} + \widehat{a})(\mathbb{I} - \widehat{a})^{-1} && (60)\\ &= \mathbb{I} && (61)\end{aligned}$$

Similarly, $R_aR_a^\top = \mathbb{I}$. **Thus, R_a is orthogonal**. Now, we will show

$$det R_a = 1$$

$$\begin{aligned}|(\mathbb{I} - \widehat{a})^{-1}(\mathbb{I} + \widehat{a})| &= |(\mathbb{I} - \widehat{a})^{-1}||(\mathbb{I} + \widehat{a})| && (62)\\ &= |(\mathbb{I} - \widehat{a})^{-1}||(\mathbb{I} + \widehat{a})^\top| && (63)\\ &= |(\mathbb{I} - \widehat{a})^{-1}||(\mathbb{I} - \widehat{a})| && (64)\\ &= 1 && (65)\end{aligned}$$

Since, $|R_a| = 1$, $R_a \in SO(3)$. $\square$