ARIL 17 M444

$\dim \mathbb{V} = n$

$$V^r_s = \underbrace{\mathbb{V}\otimes \ldots \otimes \mathbb{V}}_{r\text{-terms}} \otimes \underbrace{\mathbb{V}^* \otimes \ldots \otimes \mathbb{V}^*}_{s\text{-terms}}$$

$$= \mathcal{L}(\underbrace{\mathbb{V}^*, \ldots, \mathbb{V}^*}_{r}, \underbrace{\mathbb{V}, \ldots, \mathbb{V}}_{s}; \mathbb{F})$$

$$\dim V^r_s = n^{r+s}$$

$$\{e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{*j_1} \otimes \ldots \otimes e^{*j_s}\}$$

$$T \in V^r_s = T^{i_1 \ldots i_r}_{j_1 \ldots j_s}\, e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{*j_1} \otimes \ldots \otimes e^{*j_s}$$

ex) $T^{ij} e_i \otimes e_j$ displayed as $\begin{pmatrix} T^{11} & T^{12} & T^{13} \\ T^{21} & T^{22} & T^{23} \\ T^{31} & T^{32} & T^{33} \end{pmatrix}$

If we Δ bases, say $\{\vec{e}_i\} \to \{\vec{e}_i'\}$, components are diff.

$$\vec{e}_i = a^j_i \vec{e}_j'$$

Δ of $\vec{e}_i$'s $\Rightarrow$ $\vec{e}^{*i}$'s change also — HOW Related?

WANT $\vec{e}^{\,\prime *i}(\vec{e}_j') = \delta^i_j$

WHAT ABOUT ~~HOW ARE~~ $\vec{e}^{*i} = b^i_j (\vec{e}')^{*j}$

($\Rightarrow$) WRITE action as $\langle \vec{e}^{*i}, \vec{e}_j \rangle = \vec{e}^{*i}(\vec{e}_j)$

$$\underbrace{\langle \vec{e}^{*i}, \vec{e}_j \rangle}_{\delta^i_j} = \langle b^i_k (\vec{e}')^{*k}, a^l_j \vec{e}_l' \rangle = b^i_k a^l_j \underbrace{\langle \vec{e}^{\,\prime *k}, \vec{e}_l' \rangle}_{\delta^k_l}$$

$$= b^i_l a^l_j = (ab)^i_j$$

$\mathbb{I}$

($\Rightarrow$) $ab = \mathbb{I} \quad \therefore \quad b = a^{-1}$

$$T = T'^{\,i_1 \dots i_r}_{\;j_1 \dots j_s} \; \vec{e}'_{i_1} \otimes \dots \otimes \vec{e}'_{i_r} \otimes \vec{e}'^{*j_1} \otimes \dots \otimes \vec{e}'^{*j_s}$$

$$= T^{k_1 \dots k_r}_{l_1 \dots l_s} \; \vec{e}_{k_1} \otimes \dots \otimes \vec{e}_{k_r} \otimes \vec{e}^{*l_1} \otimes \dots \otimes \vec{e}^{*l_s}$$

$$= T^{k_1 \dots k_r}_{l_1 \dots l_s} \; (a^{i_1}_{k_1} \vec{e}'_{i_1}) \otimes \dots \otimes (a^{i_r}_{k_r} \vec{e}'_{i_r}) \otimes (a^{-1}{}^{l_1}_{j_1} \vec{e}'^{*j_1})$$

$$= T^{k_1 \dots k_r}_{l_1 \dots l_s} \; (a^{i_1}_{k_1}) \dots (a^{i_r}_{k_r}) (a^{-1})^{l_1}_{j_1} \dots (a^{-1})^{l_s}_{j_s} \; e'_{i_1} \otimes \dots$$

$$T'^{\,i_1 \dots i_r}_{\;j_1 \dots j_s} = a^{i_1}_{k_1} \dots a^{i_r}_{k_r} (a^{-1})^{l_1}_{j_1} \dots (a^{-1})^{l_s}_{j_s} \; T^{k_1 \dots k_r}_{l_1 \dots l_s}$$

Scalars invar. under coord transf.

$$T'^{\,i}_{\;j} = a^i_k (a^{-1})^l_j \, T^k_l$$

Linear trans transf as $(1,1)$-tensor $T'^{\,i}_{\;j} = (a^{-1} T a)^i_j$

$$x \in V^{r_1}_{s_1}, \; y \in V^{r_2}_{s_2} \;\Rightarrow\; x \otimes y \in V^{r_1 + r_2}_{s_1 + s_2}$$

In order to def $x \otimes y$, need to show its action

$$(x \otimes y)(\vec{v}^{*1}, \dots, \vec{v}^{*r_1}, \dots \vec{v}^{*(r_1 + r_2)}, \vec{v}_1, \dots, \vec{v}_{s_1}, \dots, \vec{v}_{s_1 + s_2})$$

$$\stackrel{\text{def}}{=} x(\vec{v}^{*1}, \dots, \vec{v}^{*r_1}, \vec{v}_1, \dots, \vec{v}_s) \cdot y(\vec{v}^{*(r_1+1)}, \dots, \vec{v}^{*(r_1+r_2)}, \vec{v}_{s_1+1}, \dots \vec{v}_{s_1+s_2})$$

SHOW Trace invariant under similarity transf

T^{ijk}_{mnop} $(3,4)$-tensor, can contract to $(r-1, s-1)$

T^{ijk}_{imnp}