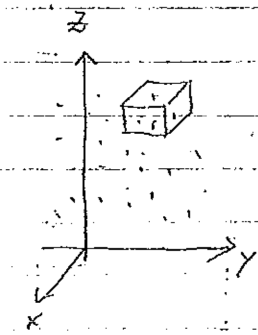
PHY 407 Chap 6 HW Solutions

(6.3) The Maxwell-Boltzmann distribution

$$f(\vec{p},\vec{r}) = C e^{-\left(\frac{\vec{p}^2}{2m} + U(\vec{r})\right)/kT}$$

has to satisfy the constraint

$$\int d^3p \, f(\vec{p},\vec{r}) = n(\vec{r}).$$



In this case $U(\vec{r}) = mgz$

$$\therefore \quad C \int d^3p \, e^{-\frac{\vec{p}^2}{2mkT}} e^{-\frac{mgz}{kT}} = n$$

or

$$n(z) = e^{-\frac{mgz}{kT}} \Big[\underbrace{C \int d^3p \, e^{-\vec{p}^2/(2mkT)}}_{\text{independent of } z \text{ if } T \text{ is not a function of } z} \Big]$$

$\therefore$ setting $z=0$, we see that

$$n(0) = C \int d^3p \, e^{-\vec{p}^2/(2mkT)},$$

so that

$$\boxed{n(z) = n(0) \, e^{-\frac{mgz}{kT}}}$$

Note that this result is valid only if T is independent of the height z.

(6.7) For a relativistic gas (assuming $c=1$)

$$E^2 = p^2 + m^2 \,, \quad p^2 = \vec{p}\cdot\vec{p}$$

$$\therefore \boxed{f(\vec{p}) = C e^{-\sqrt{p^2+m^2}/(kT)}}$$

The pressure P is given by

$$P = \int_{p_x>0} d^3p\,(2p_x)v_x\, f(\vec{p}) \,.$$

Review on special relativity:

Define $\gamma \equiv \dfrac{1}{\sqrt{1-v^2/c^2}}$, then

$$\underbrace{\vec{p} = \gamma m \vec{v} \,, \quad E = \gamma m c^2}$$

$$\Downarrow$$

$$\frac{E^2}{c^2} = \vec{p}\cdot\vec{p} + m^2c^2 \quad \text{or} \quad E^2 = p^2c^2 + m^2c^4 \quad (p^2 \equiv \vec{p}\cdot\vec{p})$$

$$\therefore \quad v_x = \frac{p_x}{\gamma m} = \frac{p_x c^2}{E} = \frac{p_x c^2}{\sqrt{p^2c^2+m^2c^4}} \longrightarrow \frac{p_x}{\sqrt{p^2+m^2}} \quad (c=1)$$

(b) $$\therefore P = \int_{p_x>0} d^3p \cdot \frac{2p_x^2}{\sqrt{p^2+m^2}}\, f(\vec{p}) = \frac{1}{3}\int d^3p\, \frac{p^2}{\sqrt{p^2+m^2}}\, f(\vec{p})$$

integrated over all $\vec{p}$-space

Write
$$\frac{p^2}{\sqrt{p^2+m^2}} = \sqrt{p^2+m^2} - \frac{m^2}{\sqrt{p^2+m^2}}$$

in the ultra-relativistic limit, $p^2 \gg m^2$, so the 2nd term can be neglected

(6.7) cont'd

$$\therefore \quad P \approx \frac{1}{3}\int d^3p \underbrace{\sqrt{p^2+m^2}}_{E} f(\vec{p}) \quad \left(\begin{array}{c}\text{in ultra-}\\ \text{relativistic}\\ \text{limit}\end{array}\right)$$

Since $\frac{U}{V} \equiv \int d^3p \, E \, f(\vec{p})$ (definition of total internal energy)

$$\boxed{PV \sim \frac{U}{3} \quad \text{in the ultra-relativistic limit}}$$

(c) Define the velocity distribution $f(v)$ by

$$f(v)dv = C\exp\left(-\beta\sqrt{p^2+m^2}\right)d^3p$$

where $p = \gamma m v = \frac{mv}{\sqrt{1-v^2}}$.

We have $$\frac{dp}{dv} = m\left(\frac{1}{\sqrt{1-v^2}} + \frac{v^2}{(1-v^2)^{3/2}}\right)$$

$$= \frac{m}{\sqrt{1-v^2}}\left(1+\frac{v^2}{1-v^2}\right) = \frac{m}{(1-v^2)^{3/2}}$$

Thus $$d^3p = 4\pi p^2 dp = \frac{4\pi m^2 v^2}{(1-v^2)} \cdot \frac{m\,dv}{(1-v^2)^{3/2}}$$

or $d^3p = \frac{4\pi m^3 v^2 dv}{(1-v^2)^{5/2}}$. We then have

$$\boxed{f(v) = \frac{4\pi C m^3 v^2}{(1-v^2)^{5/2}} \exp\left(-\frac{\sqrt{p^2+m^2}}{kT}\right), \; p = \gamma m v}$$

(6.7) cont'd

(a) To find the most probable velocity we calculate $\frac{df}{dv}$, set the derivative equal to zero, and then solve for v:

$$\frac{1}{4\pi C m^3}\frac{df}{dv} = \frac{d}{dv}\left[\frac{v^2}{(1-v^2)^{5/2}}\, e^{-\sqrt{p^2+m^2}/(kT)}\right]$$

$$= \frac{2v}{(1-v^2)^{5/2}}\exp\left(-\frac{\sqrt{p^2+m^2}}{kT}\right) + \frac{5}{2}\,\frac{v^2(2v)}{(1-v^2)^{7/2}}\exp\left(-\frac{\sqrt{p^2+m^2}}{kT}\right)$$

$$+ \frac{v^2}{(1-v^2)^{5/2}}\,\frac{d}{dv}\left(e^{-\frac{\sqrt{p^2+m^2}}{kT}}\right).$$

First calculate

$$\frac{d}{dv}\left(e^{-\frac{\sqrt{p^2+m^2}}{kT}}\right) = -\frac{1}{kT\cdot 2}\,\frac{2p\frac{dp}{dv}}{\sqrt{p^2+m^2}}\cdot e^{-\sqrt{p^2+m^2}/(kT)}$$

Using the above result for dp/dv, we have

$$\frac{d}{dv}\left(\exp\left\{-\frac{\sqrt{p^2+m^2}}{kT}\right\}\right) = -\left(\frac{1}{kT}\right)\frac{p\,\frac{m}{(1-v^2)^{3/2}}}{p\left(1+\frac{m^2}{p^2}\right)^{1/2}}\, e^{-\sqrt{p^2+m^2}/(kT)}$$

Now $1+\left(\frac{m}{p}\right)^2 = 1 + \frac{1}{\gamma^2 v^2} = 1 + \frac{1-v^2}{v^2} = \frac{1}{v^2}$,

(6.7) cont'd

$$\therefore \frac{d}{dv}\left(\exp\left\{-\frac{\sqrt{(p^2+m^2)}}{kT}\right\}\right)$$

$$= -\frac{mv}{kT(1-v^2)^{3/2}}\exp\left(-\frac{\sqrt{(p^2+m^2)}}{kT}\right).$$

Thus

$$\frac{1}{4\pi C m^3}\frac{df}{dv} = \exp\left(-\frac{\sqrt{(p^2+m^2)}}{kT}\right) \times$$

$$\left\{\frac{(2v)}{(1-v^2)^{5/2}} + \frac{5}{2}\frac{(2v)\,v^2}{(1-v^2)^{7/2}} - \frac{m(2v)v^2}{2kT(1-v^2)^{8/2}}\right\}.$$

Setting the derivative equal to zero, we obtain the following equation for v (the most probable velocity):

$$\boxed{(1-v^2)^{3/2} + \frac{5}{2}v^2(1-v^2)^{1/2} - \frac{mv^2}{2kT} = 0\ .}$$

We will find approximate solutions for the

non-relativistic limit : $\frac{kT}{mc^2} \ll 1$ and the

ultra-relativistic limit : $\frac{kT}{mc^2} \gg 1$.

In the nonrelativistic limit the condition

$\frac{kT}{mc^2} \ll 1$ is equivalent to $v^2 \ll 1$

$\therefore$ the above equation implies $1 - \frac{mv^2}{2kT} \approx 0$

or, on restoring the c

$$\boxed{\frac{v}{c} \sim \sqrt{\frac{2kT}{mc^2}}\quad \text{, non-relativistic limit}}$$

This agrees with the result from classical mechanics.

(6.7) cont'd

In the relativistic limit, $v^2 \lesssim 1$, so the above equation reduces to

$$\frac{5}{2}(1-v^2)^{1/2} \simeq \frac{m}{2kT}$$

$$\Rightarrow \quad v^2 \sim \left\{1 - \left(\frac{m}{5kT}\right)^2\right\}$$

or

$$\boxed{\frac{v}{c} \sim 1 - \frac{1}{2}\left(\frac{mc^2}{5kT}\right)^2 \text{, ultra-relativistic limit}}$$

(d) Relativistic results become noticeable when $\frac{kT}{mc^2}$ is appreciable, say, about 10%.

For H_2 gas, where $mc^2 \sim 2 \times 10^9$ eV, we need

$kT \sim 0.1 \times 2 \times 10^9$ eV. Since $k \times (300°K) \sim \frac{1}{40}$ eV

we have $$T \sim \frac{300 \times 0.1 \times 2 \times 10^9}{1/40} \, °K$$

or $T \sim 2 \times 10^{12}$ °K

(6.8) The Doppler formula for a moving source is

$$f = f_0 \left(1 + \frac{v_x}{c}\right)$$

which implies $v_x = c\left(\frac{f - f_0}{f_0}\right)$.

(a) The Maxwell-Boltzmann distribution function (for the frequency) is then given by

$$P(f) = C e^{-\frac{m v_x^2}{2kT}} = C \exp\left(-\frac{mc^2}{2kT}\frac{(f-f_0)^2}{f_0^2}\right)$$

Normalization requires that

$$\int_0^\infty P(f)\,df = 1 .$$

For $kT \ll mc^2$, the lower limit of integration can be extended to $-\infty$. Thus we have

$$C \int_{-\infty}^{\infty} df \, \exp\left(-\frac{mc^2}{2kT f_0^2}(f-f_0)^2\right)$$

$$= C \sqrt{\frac{2\pi k T f_0^2}{mc^2}} = 1 \Rightarrow C = \sqrt{\frac{mc^2}{2\pi k T f_0^2}}$$

Thus

$$\boxed{P(f) = \sqrt{\left(\frac{mc^2}{2\pi kT f_0^2}\right)} \exp\left(-\frac{mc^2}{2kT f_0^2}(f-f_0)^2\right)}$$

This gives the so-called <u>Doppler line-shape</u>. The Doppler line width $\Delta f = 2\sqrt{\overline{(f-f_0)^2}}$

(6.8) cont'd

(b) $$\overline{(f-f_0)^2} = C\int_0^{\infty} df\,(f-f_0)^2 e^{-\lambda(f-f_0)^2}, \quad \lambda \equiv \frac{mc^2}{2kTf_0^2}$$

$$\simeq \sqrt{\frac{\lambda}{\pi}}\int_{-\infty}^{\infty} df\,(f-f_0)^2 e^{-\lambda(f-f_0)^2}$$

$$= \sqrt{\frac{\lambda}{\pi}}\cdot\frac{\sqrt{\pi}}{2\lambda^{3/2}} = \frac{1}{2\lambda} = \frac{kT}{mc^2}f_0^2$$

Thus

$$\boxed{\Delta f \simeq 2f_0\sqrt{\frac{kT}{mc^2}}}$$

(c) Hence $$\frac{(\Delta f)_{H_2}}{(\Delta f)_{O_2}} \simeq \sqrt{\frac{m_{O_2}}{m_{H_2}}} = \sqrt{\frac{32}{2}} = 4.$$

(6.12) The escape velocity at the surface of the earth is

$$v_e = \sqrt{\frac{2GM}{R}}$$ where G = gravitational constant, M = mass of earth, R = radius of earth

$$\sim \sqrt{2gR}$$

$$\sim 10^4 \text{ m/s}.$$

This is to be compared with the most probable speed at STP of $v_0 = \sqrt{\frac{2kT}{m}} \sim 2.2\times 10^3$ m/s.

(a) The fraction of the gas that can escape, i.e., have $p > m v_e$, is given by

(6.12) cont'd

$$f = \frac{1}{n}\int_{mv_e}^{\infty} 4\pi p^2 dp \cdot \frac{n}{(2\pi mkT)^{3/2}} e^{-p^2/2m}$$

$$= \frac{4\pi}{(2\pi mkT)^{3/2}} \int_{mv_e}^{\infty} dp\, p^2 e^{-p^2/(2mkT)}$$

Let $x^2 \equiv \frac{p^2}{2mkT}$, then

$$f = \frac{4\pi}{(2\pi mkT)^{3/2}} \sqrt{2mkT} \int_{\sqrt{\frac{m}{2kT}}v_e}^{\infty} dx\,(2mkT)\, x^2 e^{-x^2}$$

$$= \frac{4}{\sqrt{\pi}} \int_{\frac{v_e}{v_0} \equiv y}^{\infty} dx\, x^2 e^{-x^2} \qquad v_0 = \sqrt{\frac{2kT}{m}}$$

(from prob (6.10)) $\simeq \dfrac{4}{\sqrt{\pi}} \dfrac{e^{-y^2}}{2y}\left(y^2 + \dfrac{1}{2}\right)$, $\quad y \sim \dfrac{10^4}{2.2\times 10^3} \sim 4.5$

$$\simeq \frac{2y}{\sqrt{\pi}} e^{-y^2}.$$

For $y \sim 4.5$, we have $f \sim 8\times 10^{-9}$

(b) The time it takes for an atom to escape is the time it takes for the atom to get from sea-level to the top of the atmosphere. Based on random walk, the result of problem (5.10) shows that this time is

$$t \approx \frac{L^2 \tau}{\lambda \cdot \lambda} = \frac{L^2}{\lambda v_\theta}$$

where L = height of atmosphere $\simeq 100$ km $\simeq 10^5$ m, take $\lambda \sim$ mean free path $\sim 3\times 10^{-6}$ m.

$$\therefore\ t \sim \frac{(10^5)^2}{3\times 10^{-6}\times 10^4} \text{ sec} \simeq 3\times 10^{11} \text{ sec} \sim 10^4 \text{ yrs}$$

(6.13) The number of atoms with momentum magnitude between p and $p+dp$ is

$$V\, 4\pi p^2 f(p)\, dp\,,$$

where V is the volume and $f(p)$ is the Maxwell-Boltzmann distribution. Hence

$$\Delta N = 4\pi V \int_{p_0}^{\infty} dp\, p^2 f(p)\,,$$

$$\Delta E = 4\pi V \int_{p_0}^{\infty} dp\, p^2 \cdot \frac{p^2}{2m} f(p)\,.$$

Using $f(p) = \dfrac{n}{(2\pi m kT)^{3/2}} e^{-p^2/(2mkT)}$, $N = nV$, and $E = \dfrac{3NkT}{2}$

we have

(a) $$\frac{\Delta N}{N} = \frac{4\pi V}{nV} \cdot \frac{n}{(2\pi m kT)^{3/2}} \int_{p_0}^{\infty} dp\, p^2 e^{-p^2/(2mkT)}$$

$$= \frac{4}{\sqrt{\pi}} \int_{y \equiv \frac{p_0}{\sqrt{2mkT}}}^{\infty} dx\, x^2 e^{-x^2} \qquad \left(x \equiv \frac{p}{\sqrt{2mkT}} \right)$$

or, using (6.10) $$\frac{\Delta N}{N} \simeq \frac{2}{\sqrt{\pi}}\, y\, e^{-y^2} \qquad \left(\text{for } y^2 = \frac{p_0^2}{2mkT} \gg 1 \right)$$

(6.13) cont'd

Defining $\epsilon_0 \equiv \frac{p_0^2}{2m}$, we have

$$\boxed{\frac{\Delta N}{N} \simeq \frac{2}{\sqrt{\pi}}\left(\frac{\epsilon_0}{kT}\right)^{1/2} e^{-\epsilon_0/(kT)}}$$

Similarly,

$$\frac{\Delta E}{E} = \frac{4\pi V}{\frac{3}{2}nV\,kT}\cdot\frac{1}{2m}\cdot\frac{n}{(2\pi mkT)^{3/2}}\int_{p_0}^{\infty} dp\, p^4 e^{-p^2/(2mkT)}$$

$$= \frac{8}{3\sqrt{\pi}}\,\frac{1}{(2mkT)^{5/2}}(2mkT)^2(2mkT)^{1/2}$$

$$\times \int_{y=\frac{p_0}{\sqrt{2mkT}}}^{\infty} dx\, x^4 e^{-x^2} \qquad \left(x \equiv \frac{p}{\sqrt{2mkT}}\right)$$

again, assuming $\frac{\epsilon_0}{kT} = y^2 = \frac{p_0^2}{2mkT} \gg 1$

$$\simeq \frac{8}{3\sqrt{\pi}}\,\frac{e^{-y^2}}{2y}\cdot y^4 = \frac{4}{3\sqrt{\pi}}\,y^3 e^{-y^2}$$

or

$$\boxed{\frac{\Delta E}{E} \simeq \frac{4}{3\sqrt{\pi}}\left(\frac{\epsilon_0}{kT}\right)^{3/2} e^{-\epsilon_0/(kT)}}$$

(b) From $E = \frac{3}{2}NkT$, we have $kT = \frac{2E}{3N}$

$$\therefore\ k\,dT = \frac{2}{3}\left(\frac{dE}{N} - \frac{E}{N^2}dN\right)$$

$$\therefore\ \frac{dT}{T} = \frac{k\,dT}{kT} = \frac{N}{E}\left(\frac{dE}{N} - \frac{E}{N^2}dN\right) = \frac{dE}{E} - \frac{dN}{N}$$

Since both $\Delta N/N$ and $\Delta E/E$ are $\ll 1$, we have

6.13 cont'd

$$\frac{\Delta T}{T} = \frac{\Delta E}{E} - \frac{\Delta N}{N} \approx \frac{4}{3\sqrt{\pi}} z^{3/2} e^{-z} - \frac{2}{\sqrt{\pi}} z^{1/2} e^{-z}$$

or $$\frac{\Delta T}{T} \simeq \underbrace{\left(\frac{2e^{-z}\sqrt{z}}{\sqrt{\pi}}\right)}_{\frac{\Delta N}{N}}\left(\frac{2}{3}z - 1\right)$$ where $z \equiv \frac{\epsilon_0}{kT} \gg 1$

from $$\frac{\Delta N}{N} \equiv \delta = \frac{2}{\sqrt{\pi}} z^{\frac{1}{2}} e^{-z} \qquad (\delta \ll 1)$$

we have $$\ln\left(\frac{1}{\delta}\right) = z - \frac{1}{2}\ln z + \ln\left(\frac{\sqrt{\pi}}{2}\right)$$

$$\therefore \quad z = \ln\left(\frac{1}{\delta}\right) + \frac{1}{2}\ln z - \ln\left(\frac{\sqrt{\pi}}{2}\right)$$

iterating, we have

$$z \simeq \ln\left(\frac{1}{\delta}\right) + \frac{1}{2}\ln\left(\ln\left(\frac{1}{\delta}\right) + \frac{1}{2}\ln z - \ln\left(\frac{\sqrt{\pi}}{2}\right)\right) - \ln\left(\frac{\sqrt{\pi}}{2}\right)$$

$$\approx -\ln\delta - \ln\left(\frac{\sqrt{\pi}}{2}\right) = -\ln\left(\frac{\sqrt{\pi}}{2}\delta\right) = \ln\left(\frac{2}{\sqrt{\pi}}\left(\frac{1}{\delta}\right)\right)$$

$$\approx \ln\left[\frac{2}{\sqrt{\pi}}\left(\frac{N}{\Delta N}\right)\right] \approx \ln\left(\frac{N}{\Delta N}\right)$$

$$\therefore \quad \boxed{\frac{\Delta T}{T} \approx \left(\frac{\Delta N}{N}\right)\left\{\frac{2}{3}\ln\left(\frac{N}{\Delta N}\right) - 1\right\}}$$

For $z \equiv \frac{\epsilon_0}{kT} \sim 10$, $\frac{\Delta N}{N} \sim \sqrt{10}\, e^{-10} \sim 1.4 \times 10^{-4}$

$$\frac{2}{3}\ln\left(\frac{N}{\Delta N}\right) \sim 8.8 \times \frac{2}{3} \sim 5.9$$

$$\therefore \frac{\Delta T}{T} \sim 1.4 \times 10^{-4} \times 4.9 \quad \text{or} \quad \frac{\Delta T}{T} \sim 6.8 \times 10^{-4}$$