$\boxed{\text{Prob. 26}}$ The <u>metric tensor</u> $g_{ij}(x)$ on a $D$-dimensional ~~differentiable~~ Riemannian manifold with local coordinates $(x^1, \ldots, x^D)$ is defined locally by

$$ds^2 = g_{ij}\, dx^i \otimes dx^j$$

where $ds^2$ is the square of an infinitesimal arc length on the manifold, which is <u>invariant</u> under a local differentiable change of coordinates (diffeomorphisms)

$$(x^1, \ldots, x^D) \rightarrow (x'^1, \ldots, x'^D)$$

(a) Express $g'_{ij}(x')$ in terms of the $g_{ij}(x)$, that is, show how the covariant [rank (0,2)] tensor field $g(x)$ transforms under diffeomorphisms.

<u>Hint</u>: set $g_{ij}(x)\, dx^i \otimes dx^j = g'_{kl}(x')\, dx'^k \otimes dx'^l$

Let $J_{ij} \equiv \dfrac{\partial x^i}{\partial x'^j}$ be the Jacobean matrix of the transformation

$$J = \det(J_{ij}), \quad g = \det(g_{ij}(x)), \quad g' = \det(g'_{ij}(x'))$$

(b) Show that the volume elements in the different coordinate systems are related by

$$dx^1 \wedge \ldots \wedge dx^D = (dx'^1 \wedge \ldots \wedge dx'^D) . J$$

(c) Show that $g' = g J^2$

(d) Show that the invariant volume element is given by $d^D x \sqrt{g}$ (on writing $d^D x = dx^1 \wedge \ldots \wedge dx^D$), that is, show that

$$d^D x \sqrt{g} = d^D x' \sqrt{g'}$$

$\boxed{\text{Prob. 27}}$ Prove the following statements in 3-d vector calculus by using the tensorial index method:

$$\nabla \cdot (\nabla \times \vec{A}) = 0 ,$$
$$\nabla \times (\nabla f) = 0 ,$$

where $\vec{A}(x)$ is a vector field, and $f(x)$ is a scalar field in 3-d Euclidean space!

[ These statements have been proved using the exterior derivative $d$ in class. They follow directly from the general fact that $d^2 = 0$ ].

$\boxed{\text{Prob. 28}}$ Prob. 7.1 of Chap. 7 in supplementary notes.

$\boxed{\text{Prob. 29}}$ Prob. 7.2 of Chap. 7 in Supplementary notes.

$\boxed{\text{Prob. 30}}$ Prob. 7.3 of Chap. 7 in supplementary notes.