CS 468 (Spring 2013) Differential Geometry for Computer Science Project: One Point Isometric Matching with the Heat Kernel

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1 Introduction

Finding structure-preserving maps between two surfaces is one of the most fundamental problems in geometry processing applications, including shape matching, recognition, as well as in computer graphics and animation. The objective is to find an intrinsic isometry $f : M_1 \to M_2$ between two meshes/manifolds M_1 and M_2 such that pairwise geodesics distances between points are preserved. The problem might initially seem daunting since verifying isometry for every pair of points would take a quadratic complexity. In this project, however, we will study the method proposed by Ovsjanikov et al. 2010, "One Point Isometric Matching with the Heat Kernel", which claims that under mild genericity conditions we can recover an isometry defined on entire shapes using only a single correspondance. The paper shows that by analyzing the heat kernel or, intuitively, the amount of heat transferred from one point to the other, one can match two corresponding points with the same heat kernel. Moreover, a map that preserves the heat kernel is an isometry and hence we can bypass the pairwise geodesic distance preservation verification altogether.

We will first explain the mathematical background of the heat kernel and the heat kernel map, as well as its properties. We will then apply the theory into an algorithm for finding approximate isometries. We test our discrete implementation on a few applications and in the end, provide analytic results and observations.

2 Heat Kernel and Heat Kernel Map

2.1 Heat Kernel

Let M be a compact Riemannian manifold without boundary and let $u(x,t) : M \times \mathbb{R}^+ \to \mathbb{R}^+$ be the amount of heat at a point $x \in M$ at time t, for some initial heat distribution $f : M \to \mathbb{R}^+$ at time 0. Then $u(x,t)$ satisfies the heat equation:

$$\frac{\partial u}{\partial t} = -\Delta_M u, \text{ and } \lim_{t\to 0} u(x,t) = f(x),$$

where Δ_M is the Laplace-Beltrami operator of M. The general solutions to this problem at time t can be written as

$$u(x,t) = \left(e^{-t\Delta_M} f\right)(x),$$

where the $e^{-t\Delta_M}$ is the operator exponential. The *heat kernel* on M is then defined to be the unique function $k_t^M(x,y) : \mathbb{R}^+ \times M \times M \to \mathbb{R}^+$, such that

$$u(x,t) = \left(e^{-t\Delta_M} f\right)(x) = \int_M k_t^M(x,y) f(y) dy.$$

As we mentioned above, the heat kernel $k_t^M(x,y)$ measures the amount of heat transferred from point x to point y in time t. Another useful note, which we will not be giving proofs here, is that the heat kernel on any compact manifold has the following spectral expansion:

$$k_t^M(x,y) = \sum_{i=0}^{\infty} e^{-t\lambda_i^M} \phi_i^M(x)\phi_i^M(y),$$

where λ_i^M and ϕ_i^M are the eigenvalues and eigenfunctions of the Laplace-Beltrami operator on M, respectively. Furthermore, the eigenvalues λ_i are all non-negative. We will use this formula as the main calculation of the heat kernel and the heat kernel map in our algorithm.

The heat kernel has many nice properties, but the one that is the most useful to us is the following:

Lemma 1 Let M and N be two compact and connected Riemannian manifolds without boundary. Then a map $T : M \to N$ is an isometry if and only if $k_t^M(x, y) = k_t^M(T(x), T(y))$ for all $x, y \in M, t > 0$.

In other words, the heat kernel is an isometry invariant. This stems from the intrinsic character of the Laplace-Beltrami operator. In fact, the converse is also true, though we shall not give the full statement here. The important point here is that any map that preserves the heat kernel must be distance-preserving as well.

2.2 Heat Kernel Map

We next define the textitHeat Kernel Map. Given a compact manifold M and a fixed point p, the Heat Kernel Map $\Phi_p^M : M \to F$ is defined by

$$\Phi_p^M(x) = k_t^M(p, x), \text{for all point } x \in M,$$

where F is the space of functions from $\mathbb{R}^+$ to $\mathbb{R}^+$. So given a fixed point p, we assign each point $x \in M$ a real-value function of one parameter t given by $k_t^M(x, y)$. The main property that forms as the base of our algorithm is that the Heat Kernel Map is *injective*, i.e. $\Phi_p^M(x) = \Phi_p^M(y)$ if and only if $x = y$. However, this is only true under mild genericity conditions. We first define two terms:

- *Generic Manifolds*: A compact Riemannian manifold M is generic if its Laplace-Beltrami operator does not have repeated eigenvalues. It is known that this condition holds for a generic Riemannian metric on any smooth manifold of dimension at least 2. So if M does not satisfy this condition, we can make an infinitesimal perturbation of the metric to make it generic.
- *Generic Points*: A point p is generic if $\phi_i^M(p) \neq 0$ for every eigenfunction ϕ_i^M of the Laplace-Beltrami operator of M.

Note that there exist non-generic manifolds. The simplest case is the sphere, where for every point p, $k_t^M(p, x) = k_t^M(p, y)$ for all t, whenever $d(p, x) = d(p, y)$. Leaving that aside, we present the following theorem:

Theorem 2 Let M be a generic compact manifold with no boundary and p a generic point on M. Then the Heat Kernel Map is injective, i.e. $\Phi_p^M(x) = \Phi_p^M(y)$ if and only if $x = y$.

Collorary 2.1 If M and N be two generic connected compact manifolds without boundary, and p a generic point on M. Then, if $f, g : M \to N$, are two isometries such that $f(p) = g(p)$, then $f(x) = g(x)$ for all $x \in M$.

This means that if we know that a point $p \in M$ must map to a point $q \in N$ under some unknown isometry $f : M \to N$, then we can compute the Heat Kernel Map Φ_p^M of points on M and the Heat Kernel Map Φ_q^M of points on N, and every for every point $x \in M$, its corresponding point $f(x)$ is the only point such that $\Phi_p^M(x) = \Phi_q^N(f(x))$.

However, even if we can find the corresponding point for every point x, we still need to verify that this map is an isometry. Fortunately, we have the following theorem:

Theorem 3 Let M and N be two generic connected compact manifolds and p a generic point on M. Then, any map f such that $f(p)$ is generic and $k_t^M(p, x) = k_t^N(f(p), f(x))$, for every time $t > 0$ and every point $x \in M$, f is an isometry.

This theorem is important since it ensures that preservation of the Heat Kernel Map is sufficient to force pairwise isometric consistency. Hence finding a correspondence for each point x using the Heat Kermel Map of a single point gives us the desired isometry.

3 Algorithm

The above properties of the heat kernel and the Heat Kernel Map suggests the following algorithm in finding approximate isometries in practice. We first find a set of landmark corresponding points and then extending them to every point on the manifold. Given two corresponding manifolds M and N, our algorithm goes as follows:

(i) Feature Detection
We first find a set of feature points $P \subseteq M$ and $Q \subseteq N$. We use a local descriptor Heat Kernel signature for a large time t to identify feature points. If $k_t(p,p) > k_t(x,x)$ for all x in the two-ring neighborhood of p, then p is a feature point on M.

Figure 1: Using Heat Kernel Signature, we can identify the feature points on each mesh. Here the feature points, which are mostly the protuding parts, are depicted by purple color.
(left) Armadillo (right) Ant

(ii) Single Feature Matching
Given a set of feature points $P \subseteq M$ and $Q \subseteq N$, we will find a single correspondance to be used for our Heat Kernel Map. We pick a random point $p \in P$ (or by user selection) and iterate over all possible $q_i \in Q$ to find the best match. To evaluate the potential of each (p, q_i), we compute the Heat Kernel Map Φ_p^M and $\Phi_{q_i}^N$ by discretizing the function $k_t^M(p,x)$ into the set of time samples t_j for every point $x \in M$. In other words, for every point $x \in M$, we compute a vector of length $|\{t_j\}|$ whose component j is $k_{t_j}^M(p,x)$. Similarly, we compute the $k_{t_j}^N(q,y)$ for each $y \in N$. Then for each pair (p, q_i) we evalute the error or the distance between them by

$$E(p, q_i) = \sum_{x \in M} \min_{y \in N} \|\Phi_p^M(x) - \Phi_{q_i}^N(y)\|^2$$

where $\|\Phi_p^M(x) - \Phi_{q_i}^N(y)\| = \sup_j t_j \left| k_{t_j}^M(p,x) - k_{t_j}^N(q_i,y) \right|$. The additional t_j at the front of the product keeps the distances within limit for t near 0. In theory, if $E(p,q) = 0$, then there exists an isometry map betwee p and q. In practice, however, due to numerical unstability and discretization of meshes, we use the pairs (p, q_i) with small $E(p, q_i)$.

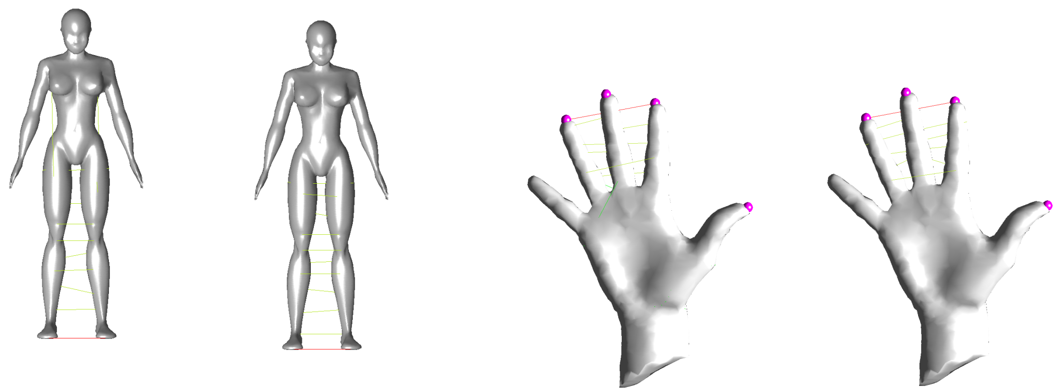

Figure 2: Comparison between the intrinsic symmetric detection using simple Heat Kernel Map (left models) and the augmented with Heat Kernel Signature (right models)

(iii) Correspondence Propagation
Now that we have acquire a correspondance (p, q), and assuming the genericity described above, Theorem 3 claims that there will be a unique point $y \in N$ such that $\|\Phi_p^M(x) - \Phi_q^N(y)\| = 0$. In practice, we can find for each point $x \in M$ the best candidate

$$f(x) = \text{argmin}_{y \in N} \|\Phi_p^M(x) - \Phi_q^N(y)\|.$$

So using only one correspondance, we can find for every point $x \in M$ its nearest neighbor from N in the space of Heat Kernel Maps, and hence its isometric map, given the genericity conditions above.

(iv) Local Descriptors & Correspondance Augmentation
In practice, not all shapes are isometric nor generic. There are a few heuristics that we can implement to improve the quality of our map. First, we can introduce another $|\{t_j\}|$ coordinates for each point by adding $k_{t_j}^M(x, x)$ for all time $\{t_j\}$. Another heuristic is to augment the Heat Kernel Map by another landmark correspondance (p_2, q_2). So each point $x \in M$ will now consists of $2|\{t_j\}|$ components, $k_{t_j}^M(p, x)$ and $k_{t_j}^M(p_2, x)$. It should also be noted that introducing these extra coordinates does not affect commensurability of all the quantities and hence we can still apply the formulas in the previous two steps.

As mentioned earlier, we can compute the heat kernel by $k_t^M(x, y) = \sum_{i=0}^{\infty} e^{-t\lambda_i^M} \phi_i^M(x)\phi_i^M(y)$, which is easy and fast to implement. Here we use the cotangent weight scheme to compute the discrete Laplace-Beltrami operator on triangular meshes due to its simplicity and comparable performance. Unfortunately, all of the following results were implemented in MATLAB and with high number of vertices it runs slowly with for loops and out of memory if using matrix operations. If I had planned ahead, the algorithm could be much faster using other data structures. (Although I think a better explanation would be there are mistakes in my implementation.)

4 Results

We test this algorithm in two following schemes. The data are from the Surface Correspondence Benchmark from Princeton University gathered by Kim et al. 2011 [KLF11].

4.1 Symmetric Detection

We can use the algorithm to find intrinsic symmetries in the shapes. In fact, this is simply a special case for intrinsic isometric shape mapping in such a way that instead of mapping two different shapes, we want to find an isometric map within the certain shape, i.e. two identical shapes. Figure 2 shows the result of implementing our algorithm with a straight-faced human and a hand. The red line represents the landmark match that we use to generate the Heat Kernel Map and hence the isometry. The green lines are the the propagated correspondences described in step iii in our algorithm. Here we use 10 random time samples from 0.001 to 2.

On the left, we implement the basic approach of our algorithm and compare them to the right where we augment the Heat Kernel Signature in hope to improve the match. As you can see, without the HKS, the isometry does a good job but still has some stray matches. In the human figure, the thigh sometimes is matched with the breast and in the hand figure, some of the finger's sides are matched with the bottom of the palm. All of these are either removed or reduced greatly with the introduction of the HKS as you can see on the right figures.

It should be noted that at the time of implementation, I have troubles finding the right scale for the time samples. If I use the time that are too large, it will blows up the computation and most of the match will gather around certain points. I suspect this is due to the fact that we multiply t_j in front of the Heat Kernel Map distances formula. Although it scales well with small t, with large t combined with numerical error in the computation, this extra t can amplify the error and reduces the accuracy in our algorithm.

4.2 Isometric Matching

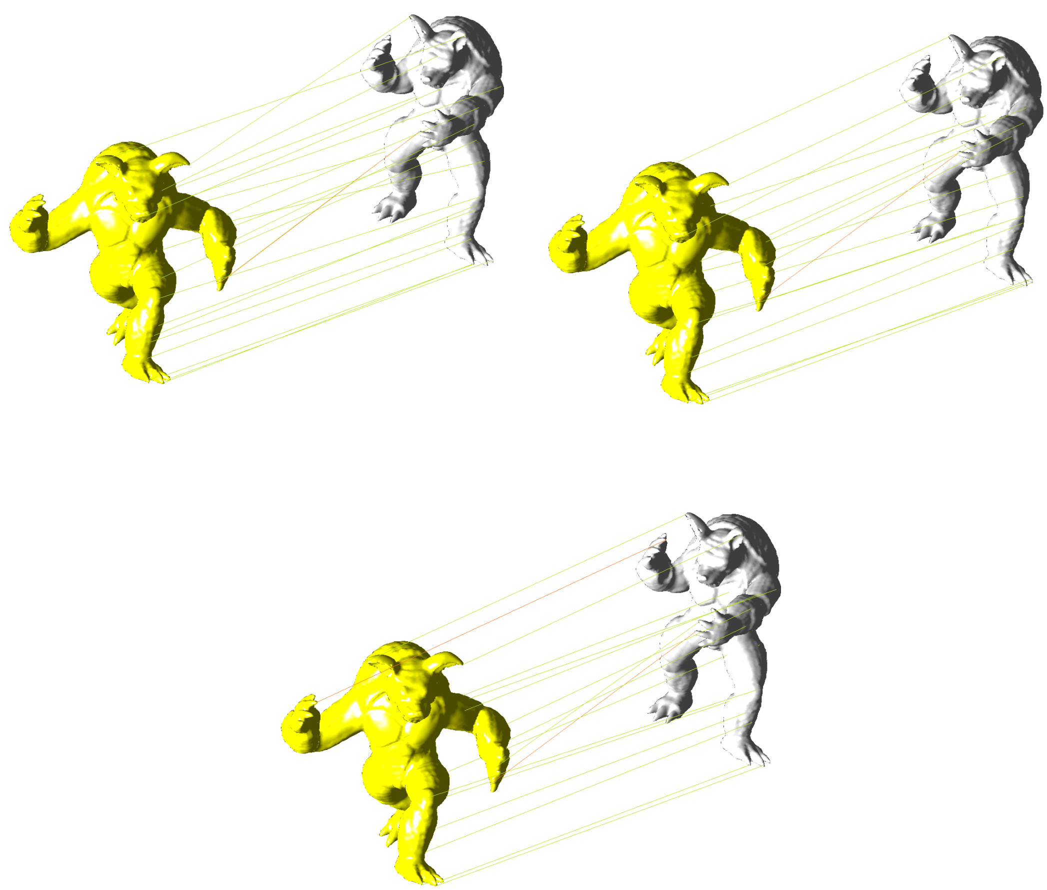

Figure 3: (top left) Single landmark (top right) Single landmark with HKS (bottom) Two landmarks

Figure 3 shows the result of applying the Heat Kernel Map algorithm to the armadillo model with moderate geodesic distortion. As you can see in all cases, the result is a high quality dense map on the entire shape. On the left, however, we use only one landmark correspondence and although it looks to match most of the shape properly, there are still some stray matching (such as the ears). This map can be fixed by computing the Heat Kernel Signature as mentioned before (Figure 3 middle) or adding a second landmark correspondance (Figure 3 right). Figure 4 also shows the mapping using two landmark correspondence between two models with a medium geodesic distortion.

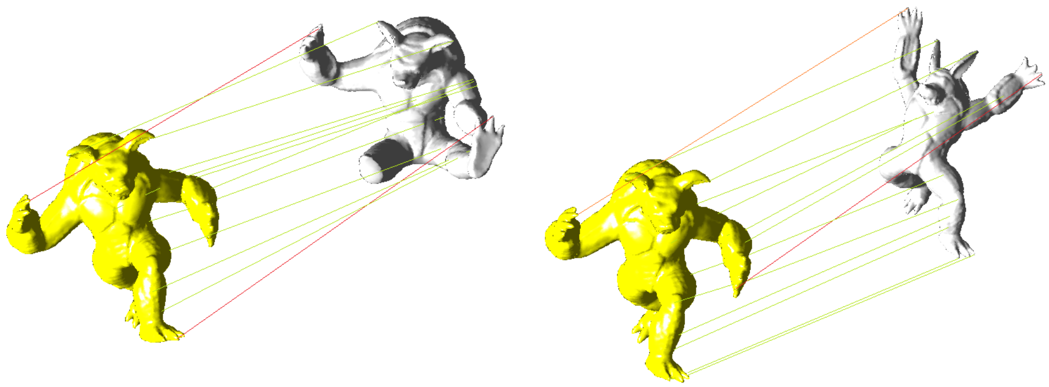

Figure 4: Isometric mapping with medium geodesic distortion

5 Conclusion and Future work

Reference

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