D Log-likelihood, Gradient and Hessian derivations

Introduction. Let us revisit our model with two parameters - inverse temperature, β, and learning rate α - that we aim to estimate. We know that the domain of these model parameters are [0,∞] and [0,1], respectively. However, given an unconstrained optimization approach (i.e., trust region), we need to perform our estimation on unconstrained parameters. Thus, our motivation behind reparameterization of parameters using x_1 and x_2:

$$\beta = e^{x_1} \tag{D.1}$$

$$\alpha = \frac{1}{1+e^{-x_2}} \tag{D.2}$$

This reparameterization allows our parameters - x_1 and x_2 - to have an unconstrained domain of [-∞, ∞] while preserving the appropriate domains of β and α. We now introduce the softmax choice function and Q-learning rule formula for choosing either machine A or machine B on trial t:

$$P(A_t) = \frac{e^{Q_{1_t}}}{e^{Q_{1_t}} + e^{Q_{2_t}}} \tag{D.3}$$

$$Q_{A_t} = Q_{A_{t-1}} + \alpha(r_{t-1} - Q_{A_{t-1}}) \tag{D.4}$$

$$P(B_t) = \frac{e^{Q_{2_t}}}{e^{Q_{1_t}} + e^{Q_{2_t}}} \tag{D.5}$$

$$Q_{B_t} = Q_{B_{t-1}} + \alpha(r_{t-1} - Q_{B_{t-1}}) \tag{D.6}$$

where β and α have been reparameterized and $Q_{1_t} = \beta Q_{A_t}$ and $Q_{2_t} = \beta Q_{B_t}$. Equations (C.3) and (C.4) represent the softmax choice function and Q-learning rule for a participants' choice of machine A, respectively. Likewise, equations (C.5) and (C.6) for machine B. **For the purpose of explaining the remaining derivations, we will perform the derivations for a particular choice (i.e., the probability a participant chooses machine A)**. This means we will focus on equations (C.3) and (C.4). Let's begin by algebraically converting equation (C.4) into one indexed by 1_t (for choosing machine A on trial t):

$$\begin{aligned}
Q_{A_t} &= Q_{A_{t-1}} + \alpha(r_{t-1} - Q_{A_{t-1}}) && \text{(C.4)}\\
\beta Q_{A_t} &= \beta Q_{A_{t-1}} + \alpha(\beta r_{t-1} - \beta Q_{A_{t-1}}) && \text{(multiply by } \beta)\\
Q_{1_t} &= Q_{1_{t-1}} + \alpha(\beta r_{t-1} - Q_{1_{t-1}}) && (Q_{1_t} = \beta Q_{A_t})
\end{aligned}$$

Hence,

$$Q_{1_t} = Q_{1_{t-1}} + \alpha(\beta r_{t-1} - Q_{1_{t-1}}) \tag{D.7}$$

Log-likelihood. Now that we have setup the general model we can begin performing our derivations. We will first start with the derivation of the log-likelihood for choosing machine A on a particular trial t (LL_{A_t}):

$$\begin{aligned}
P(A_t) &= \frac{e^{Q_{1_t}}}{e^{Q_{1_t}} + e^{Q_{2_t}}} && \text{(C.3)}\\
\log(P(A_t)) &= \log\left(\frac{e^{Q_{1_t}}}{e^{Q_{1_t}} + e^{Q_{2_t}}}\right) && (\textit{log} \text{ both sides})\\
&= \log(e^{Q_{1_t}}) - \log(e^{Q_{1_t}} + e^{Q_{2_t}}) && (\textit{log} \text{ quotient rule})\\
&= Q_{1_t} - \log(e^{Q_{1_t}} + e^{Q_{2_t}}) && (\log(e^x) = x)
\end{aligned}$$

Thus,

$$LL_{A_t} = \log(P(A_t)) = Q_{1_t} - \log(e^{Q_{1_t}} + e^{Q_{2_t}}) \tag{D.8}$$

Gradient. Given our log-likelihood, we are ready to compute the gradient of LL_{A_t} with respect to each of our parameters - x_1 and x_2 - as symbolized by $\frac{\partial LL_{A_t}}{\partial x_1}$ and $\frac{\partial LL_{A_t}}{\partial x_2}$, respectively :

Starting with $\frac{\partial LL_{A_t}}{\partial x_1}$:

$$\begin{aligned}
LL_{A_t} &= Q_{1_t} - \log(e^{Q_{1_t}} + e^{Q_{2_t}}) && \text{(C.8)}\\
\frac{\partial LL_{A_t}}{\partial x_1} &= \frac{\partial Q_{1_t}}{\partial x_1} - \left[\frac{1}{e^{Q_{1_t}} + e^{Q_{2_t}}}\right]\left[\frac{\partial Q_{1_t}}{\partial x_1}e^{Q_{1_t}} + \frac{\partial Q_{2_t}}{\partial x_1}e^{Q_{2_t}}\right] && \text{(partial derivative w.r.t } x_1)\\
&= \frac{\partial Q_{1_t}}{\partial x_1} - \left[P(A_t)\frac{\partial Q_{1_t}}{\partial x_1} + P(B_t)\frac{\partial Q_{2_t}}{\partial x_1}\right] && \text{(C.3) and (C.5)}
\end{aligned}$$

from which we get

$$\frac{\partial LL_{A_t}}{\partial x_1} = \frac{\partial Q_{1_t}}{\partial x_1} - \left[P(A_t)\frac{\partial Q_{1_t}}{\partial x_1} + P(B_t)\frac{\partial Q_{2_t}}{\partial x_1}\right] \tag{D.9}$$

Second, we need to solve for $\frac{\partial Q_{1_t}}{\partial x_1}$ in equation (C.9). To accomplish this we will refer to equation (C.7) and differentiate with respect to x_1:

Solving $\frac{\partial Q_{1_t}}{\partial x_1}$:

$$\begin{aligned}
Q_{1_t} &= Q_{1_{t-1}} + \alpha(\beta r_{t-1} - Q_{1_{t-1}}) && \text{(C.7)} \\
\frac{\partial Q_{1_t}}{\partial x_1} &= \frac{\partial Q_{1_{t-1}}}{\partial x_1} + \alpha r_{t-1}\frac{\partial \beta}{\partial x_1} - \alpha\frac{\partial Q_{1_{t-1}}}{\partial x_1} && \text{(partial derivative w.r.t } x_1\text{)} \\
&= (1-\alpha)\frac{\partial Q_{1_{t-1}}}{\partial x_1} + \alpha r_{t-1}\frac{\partial \beta}{\partial x_1} && \text{(simplify using common factor)}
\end{aligned}$$

Thus,

$$\frac{\partial Q_{1_t}}{\partial x_1} = (1-\alpha)\frac{\partial Q_{1_{t-1}}}{\partial x_1} + \alpha r_{t-1}\frac{\partial \beta}{\partial x_1} \tag{D.10}$$

The last step needed to evaluate equation (C.9) is to evaluate the term, $\frac{\partial \beta}{\partial x_1}$, from equation (C.10):

Getting $\frac{\partial \beta}{\partial x_1}$:

$$\begin{aligned}
\beta &= e^{x_1} && \text{(C.1)} \\
\frac{\partial \beta}{\partial x_1} &= \frac{\partial e^{x_1}}{\partial x_1} && \text{(partial derivative w.r.t } x_1\text{)} \\
&= e^{x_1} && (\tfrac{\partial}{\partial x}e^x = e^x) \\
&= \beta && \text{(C.1)}
\end{aligned}$$

Consequently,

$$\frac{\partial \beta}{\partial x_1} = \beta \tag{D.11}$$

Finally, we expand equation (C.9) using equations (C.10) and (C.11) to obtain the gradient of LL_{A_t} with respect to x_1.

We now perform a similar derivation procedure for the gradient of LL_{A_t} with respect to the other parameter, x_2:

Starting with $\frac{\partial LL_{A_t}}{\partial x_2}$:

$$\begin{aligned}
LL_{A_t} &= Q_{1_t} - \log(e^{Q_{1_t}} + e^{Q_{2_t}}) && \text{(C.8)} \\
\frac{\partial LL_{A_t}}{\partial x_2} &= \frac{\partial Q_{1_t}}{\partial x_2} - \left[\frac{1}{e^{Q_{1_t}} + e^{Q_{2_t}}}\right]\left[\frac{\partial Q_{1_t}}{\partial x_2}e^{Q_{1_t}} + \frac{\partial Q_{2_t}}{\partial x_2}e^{Q_{2_t}}\right] && \text{(partial derivative w.r.t } x_2\text{)} \\
&= \frac{\partial Q_{1_t}}{\partial x_2} - \left[P(A_t)\frac{\partial Q_{1_t}}{\partial x_2} + P(B_t)\frac{\partial Q_{2_t}}{\partial x_2}\right] && \text{(C.3) and (C.5)}
\end{aligned}$$

we get

$$\frac{\partial LL_{A_t}}{\partial x_2} = \frac{\partial Q_{1_t}}{\partial x_2} - \left[P(A_t)\frac{\partial Q_{1_t}}{\partial x_2} + P(B_t)\frac{\partial Q_{2_t}}{\partial x_2}\right] \tag{D.12}$$

To solve for $\frac{\partial Q_{1_t}}{\partial x_2}$ in equation (C.12), we will refer to equation (C.7) and differentiate with respect to x_2:

Solving $\frac{\partial Q_{1_t}}{\partial x_2}$:

$$
\begin{aligned}
Q_{1_t} &= Q_{1_{t-1}} + \alpha(\beta r_{t-1} - Q_{1_{t-1}}) && \text{(C.7)}\\
\frac{\partial Q_{1_t}}{\partial x_2} &= \frac{\partial Q_{1_{t-1}}}{\partial x_2} + \alpha\Big[-\frac{\partial Q_{1_{t-1}}}{\partial x_2}\Big] + (\beta r_{t-1} - Q_{1_{t-1}})\Big[\frac{\partial \alpha}{\partial x_2}\Big] && \text{(partial derivative w.r.t } x_2\text{)}\\
&= (1-\alpha)\frac{\partial Q_{1_{t-1}}}{\partial x_2} + (\beta r_{t-1} - Q_{1_{t-1}})\frac{\partial \alpha}{\partial x_2} && \text{(simplify using common factor)}
\end{aligned}
$$

we obtain

$$
\frac{\partial Q_{1_t}}{\partial x_2} = (1-\alpha)\frac{\partial Q_{1_{t-1}}}{\partial x_2} + (\beta r_{t-1} - Q_{1_{t-1}})\frac{\partial \alpha}{\partial x_2} \tag{D.13}
$$

The last step needed to evaluate equation (C.12) is to evaluate the term, $\frac{\partial \alpha}{\partial x_2}$, from equation (C.13):

Finally, $\frac{\partial \alpha}{\partial x_2}$:

$$
\begin{aligned}
\alpha &= \frac{1}{1+e^{-x_2}} && \text{(C.2)}\\
&= (1+e^{-x_2})^{-1} && \text{(rewritten form)}\\
\frac{\partial \alpha}{\partial x_2} &= -(1+e^{-x_2})^{-2}(-e^{-x_2}) && \text{(partial derivative w.r.t } x_2\text{)}\\
&= (1+e^{-x_2})^{-2}(e^{-x_2}) && \text{(divide by -1)}\\
&= (\alpha)^2(e^{-x_2}) && \text{(since } \alpha = (1+e^{-x_2})^{-1}\text{)}\\
&= (\alpha)^2\Big[\frac{1-\alpha}{\alpha}\Big] && \text{(solve for } e^{-x_2} \text{ in (C.2))}\\
&= \alpha(1-\alpha) && \text{(simplify)}
\end{aligned}
$$

Consequently,

$$
\frac{\partial \alpha}{\partial x_2} = \alpha(1-\alpha) \tag{D.14}
$$

Finally, we expand equation (C.12) using equations (C.13) and (C.14) to obtain the gradient of LL_{A_t} with respect to x_2.

Hessian. Now that we have completed our gradient computation, we are ready to construct our Hessian matrix. Let's compute the second-order partial derivatives of LL_{A_t} with respect to each of our parameters - x_1 and x_2 - as symbolized by $\frac{\partial^2 LL_{A_t}}{\partial x_1^2}$, $\frac{\partial^2 LL_{A_t}}{\partial x_1 \partial x_2}$, $\frac{\partial^2 LL_{A_t}}{\partial x_2 \partial x_1}$ and $\frac{\partial^2 LL_{A_t}}{\partial x_2^2}$. Let's first look at our computed gradients where we replace $P(A_t)$ and $P(B_t)$ with $e^{LL_{A_t}}$ and 1-$e^{LL_{A_t}}$, respectively, from equations (C.9) and (C.12) given that we focus on particpants' decision of choosing machine A:

$$
\frac{\partial LL_{A_t}}{\partial x_1} = \frac{\partial Q_{1_t}}{\partial x_1} - \Big[e^{LL_{A_t}}\frac{\partial Q_{1_t}}{\partial x_1} + (1-e^{LL_{A_t}})\frac{\partial Q_{2_t}}{\partial x_1}\Big] \tag{D.15}
$$

$$
\frac{\partial LL_{A_t}}{\partial x_2} = \frac{\partial Q_{1_t}}{\partial x_2} - \Big[e^{LL_{A_t}}\frac{\partial Q_{1_t}}{\partial x_2} + (1-e^{LL_{A_t}})\frac{\partial Q_{2_t}}{\partial x_2}\Big] \tag{D.16}
$$

Note that we can replace the probabilities with $e^{LL_{A_t}}$ because $LL_{A_t} = \log(P(A_t))$ so if we take the exponential from both sides we get $P(A_t) = e^{LL_{A_t}}$ and since $P(B_t) =$ 1-$P(A_t)$ we get $P(B_t) =$ 1-$e^{LL_{A_t}}$.

Using equations (C.15) and (C.16), we can compute the elements of the Hessian. Let's begin with the second-order direct partial derivative of LL_{A_t} w.r.t x_1:

Solve for $\frac{\partial^2 LL_{A_t}}{\partial x_1^2}$:

$$\frac{\partial LL_{A_t}}{\partial x_1} = \frac{\partial Q_{1_t}}{\partial x_1} - \left[e^{LL_{A_t}}\frac{\partial Q_{1_t}}{\partial x_1} + (1-e^{LL_{A_t}})\frac{\partial Q_{2_t}}{\partial x_1}\right] \tag{C.15}$$

$$\begin{aligned}
\frac{\partial^2 LL_{A_t}}{\partial x_1^2} &= \frac{\partial^2 Q_{1_t}}{\partial x_1^2} - \left[e^{LL_{A_t}}\frac{\partial^2 Q_{1_t}}{\partial x_1^2} + \frac{\partial Q_{1_t}}{\partial x_1}\left(\frac{\partial LL_{A_t}}{\partial x_1}e^{LL_{A_t}}\right)\right] \\
&\quad - \left[(1-e^{LL_{A_t}})\frac{\partial^2 Q_{2_t}}{\partial x_1^2} + \frac{\partial Q_{2_t}}{\partial x_1}\left(-\frac{\partial LL_{A_t}}{\partial x_1}e^{LL_{A_t}}\right)\right] && \text{(direct partial derivative w.r.t } x_1\text{)} \\
&= \frac{\partial^2 Q_{1_t}}{\partial x_1^2} - \left[e^{LL_{A_t}}\frac{\partial^2 Q_{1_t}}{\partial x_1^2} + (1-e^{LL_{A_t}})\frac{\partial^2 Q_{2_t}}{\partial x_1^2}\right] \\
&\quad - \frac{\partial LL_{A_t}}{\partial x_1}e^{LL_{A_t}}\left(\frac{\partial Q_{1_t}}{\partial x_1} - \frac{\partial Q_{2_t}}{\partial x_1}\right) && \text{(expand and simplify)}
\end{aligned}$$

resulting in

$$\frac{\partial^2 LL_{A_t}}{\partial x_1^2} = \frac{\partial^2 Q_{1_t}}{\partial x_1^2} - \left[e^{LL_{A_t}}\frac{\partial^2 Q_{1_t}}{\partial x_1^2} + (1-e^{LL_{A_t}})\frac{\partial^2 Q_{2_t}}{\partial x_1^2}\right] - \frac{\partial LL_{A_t}}{\partial x_1}e^{LL_{A_t}}\left(\frac{\partial Q_{1_t}}{\partial x_1} - \frac{\partial Q_{2_t}}{\partial x_1}\right) \tag{D.17}$$

Now, we need to solve for $\frac{\partial^2 Q_{1_t}}{\partial x_1^2}$ in equation (C.17). To accomplish this we will refer to equation (C.10) and differentiate with respect to x_1:

Solve for $\frac{\partial^2 Q_{1_t}}{\partial x_1^2}$:

$$\begin{aligned}
\frac{\partial Q_{1_t}}{\partial x_1} &= (1-\alpha)\frac{\partial Q_{1_{t-1}}}{\partial x_1} + \alpha r_{t-1}\frac{\partial \beta}{\partial x_1} && \text{(C.10)} \\
\frac{\partial^2 Q_{1_t}}{\partial x_1^2} &= (1-\alpha)\frac{\partial^2 Q_{1_{t-1}}}{\partial x_1^2} + \alpha r_{t-1}\beta && \text{(direct partial derivative w.r.t } x_1\text{)}
\end{aligned}$$

resulting in

$$\frac{\partial^2 Q_{1_t}}{\partial x_1^2} = (1-\alpha)\frac{\partial^2 Q_{1_{t-1}}}{\partial x_1^2} + \alpha r_{t-1}\beta \tag{D.18}$$

Finally, we expand equation (C.17) using equation (C.22) to obtain the second-order direct partial derivative of LL_{A_t} with respect to x_1.

We will now derive the remaining Hessian components - $\frac{\partial^2 LL_{A_t}}{\partial x_1 \partial x_2}$, $\frac{\partial^2 LL_{A_t}}{\partial x_2 \partial x_1}$ and $\frac{\partial^2 LL_{A_t}}{\partial x_2^2}$.

Solve for $\frac{\partial^2 LL_{A_t}}{\partial x_1 \partial x_2}$:

$$\begin{aligned}
\frac{\partial^2 LL_{A_t}}{\partial x_1 \partial x_2} &= \frac{\partial^2 Q_{1_t}}{\partial x_1 \partial x_2} - \left[e^{LL_{A_t}}\frac{\partial^2 Q_{1_t}}{\partial x_1 \partial x_2} + \frac{\partial Q_{1_t}}{\partial x_1}\left(\frac{\partial LL_{A_t}}{\partial x_2}e^{LL_{A_t}}\right)\right] \\
&\quad - \left[(1-e^{LL_{A_t}})\frac{\partial^2 Q_{2_t}}{\partial x_1 \partial x_2} + \frac{\partial Q_{2_t}}{\partial x_1}\left(-\frac{\partial LL_{A_t}}{\partial x_2}e^{LL_{A_t}}\right)\right] && \text{(cross partial derivative of (C.16) of } x_2 \text{ w.r.t } x_1\text{)}
\end{aligned}$$

resulting in

$$\frac{\partial^2 LL_{A_t}}{\partial x_1 \partial x_2} = \frac{\partial^2 Q_{1_t}}{\partial x_1 \partial x_2} - \left[e^{LL_{A_t}}\frac{\partial^2 Q_{1_t}}{\partial x_1 \partial x_2} + (1-e^{LL_{A_t}})\left(\frac{\partial^2 Q_{2_t}}{\partial x_1 \partial x_2}\right)\right] - \frac{\partial LL_{A_t}}{\partial x_2}e^{LL_{A_t}}\left(\frac{\partial Q_{1_t}}{\partial x_1} - \frac{\partial Q_{2_t}}{\partial x_1}\right) \tag{D.19}$$

Now, we need to solve for $\frac{\partial^2 Q_{1_t}}{\partial x_1 \partial x_2}$ in equation (C.19). To accomplish this we will refer to equation (C.10) and differentiate with respect to x_1:

Solve for $\frac{\partial^2 Q_{1_t}}{\partial x_1 \partial x_2}$:

$$
\begin{aligned}
\frac{\partial Q_{1_t}}{\partial x_1} &= (1-\alpha)\frac{\partial Q_{1_{t-1}}}{\partial x_1} + \alpha r_{t-1}\frac{\partial \beta}{\partial x_1} && \text{(C.10)}\\
\frac{\partial^2 Q_1}{\partial x_1 \partial x_2} &= (1-\alpha)\frac{\partial^2 Q_1}{\partial x_1 \partial x_2} - \frac{\partial \alpha}{\partial x_2}\left(\frac{\partial Q_{1_{t-1}}}{\partial x_1}\right) + \frac{\partial \alpha}{\partial x_2} r_{t-1}\beta && \text{(cross partial derivative of } x_2 \text{ w.r.t } x_1)\\
&= (1-\alpha)\frac{\partial^2 Q_1}{\partial x_1 \partial x_2} - (\alpha(1-\alpha))\frac{\partial Q_{1_{t-1}}}{\partial x_1} + (\alpha(1-\alpha))r_{t-1}\beta && (\tfrac{\partial \alpha}{\partial x_2} = \alpha(1\text{-}\alpha))
\end{aligned}
$$

resulting in

$$
\frac{\partial^2 Q_{1_t}}{\partial x_1 \partial x_2} = (1-\alpha)\frac{\partial^2 Q_{1_t}}{\partial x_1 \partial x_2} - (\alpha(1-\alpha))\left[\frac{\partial Q_{1_{t-1}}}{\partial x_1} - r_{t-1}\beta\right] \tag{D.20}
$$

Finally, we expand equation (C.19) using equation (C.20) to obtain the second-order cross partial derivative of LL_{A_t} of x_2 with respect to x_1.

Solve for $\frac{\partial^2 LL_{A_t}}{\partial x_2^2}$:

$$
\begin{aligned}
\frac{\partial LL_{A_t}}{\partial x_2} &= \frac{\partial Q_{1_t}}{\partial x_2} - \left[e^{LL_{A_t}}\frac{\partial Q_{1_t}}{\partial x_2} + (1-e^{LL_{A_t}})\frac{\partial Q_{2_t}}{\partial x_2}\right] && \text{(C.16)}\\
\frac{\partial^2 LL_{A_t}}{\partial x_2^2} &= \frac{\partial^2 Q_{1_t}}{\partial x_2^2} - \left[e^{LL_{A_t}}\frac{\partial^2 Q_{1_t}}{\partial x_2^2} + \frac{\partial Q_{1_t}}{\partial x_2}\left(\frac{\partial LL_{A_t}}{\partial x_2}e^{LL_{A_t}}\right)\right]\\
&\quad - \left[(1-e^{LL_{A_t}})\frac{\partial^2 Q_{2_t}}{\partial x_2^2} + \frac{\partial Q_{2_t}}{\partial x_2}\left(-\frac{\partial LL_{A_t}}{\partial x_2}e^{LL_{A_t}}\right)\right] && \text{(direct partial derivative w.r.t } x_2)\\
&= \frac{\partial^2 Q_{1_t}}{\partial x_2^2} - \left[e^{LL_{A_t}}\frac{\partial^2 Q_{1_t}}{\partial x_2^2} + (1-e^{LL_{A_t}})\frac{\partial^2 Q_{2_t}}{\partial x_2^2}\right]\\
&\quad - \frac{\partial LL_{A_t}}{\partial x_2}e^{LL_{A_t}}\left(\frac{\partial Q_{1_t}}{\partial x_2} - \frac{\partial Q_{2_t}}{\partial x_2}\right) && \text{(expand and simplify)}
\end{aligned}
$$

resulting in

$$
\frac{\partial^2 LL_{A_t}}{\partial x_2^2} = \frac{\partial^2 Q_{1_t}}{\partial x_2^2} - \left[e^{LL_{A_t}}\frac{\partial^2 Q_{1_t}}{\partial x_2^2} + (1-e^{LL_{A_t}})\frac{\partial^2 Q_{2_t}}{\partial x_2^2}\right] - \frac{\partial LL_{A_t}}{\partial x_2}e^{LL_{A_t}}\left(\frac{\partial Q_{1_t}}{\partial x_2} - \frac{\partial Q_{2_t}}{\partial x_2}\right) \tag{D.21}
$$

Now, we need to solve for $\frac{\partial^2 Q_{1_t}}{\partial x_2^2}$ in equation (C.21). To accomplish this we will refer to equation (C.13) and differentiate with respect to x_2:

Solve for $\frac{\partial^2 Q_{1_t}}{\partial x_2^2}$:

$$
\begin{aligned}
\frac{\partial Q_{1_t}}{\partial x_2} &= (1-\alpha)\frac{\partial Q_{1_{t-1}}}{\partial x_2} + (\beta r_{t-1} - Q_{1_{t-1}})\frac{\partial \alpha}{\partial x_2} && \text{(C.13)}\\
\frac{\partial^2 Q_{1_t}}{\partial x_2^2} &= (1-\alpha)\frac{\partial^2 Q_{1_{t-1}}}{\partial x_2^2} + \left(-\frac{\partial \alpha}{\partial x_2}\right)\frac{\partial Q_{1_{t-1}}}{\partial x_2}\\
&\quad + (\beta r_{t-1} - Q_{1_{t-1}})\frac{\partial^2 \alpha}{\partial x_2^2} + \left(-\frac{\partial Q_{1_{t-1}}}{\partial x_2}\right)\frac{\partial \alpha}{\partial x_2} && \text{(direct partial derivative w.r.t } x_2)\\
&= (1-\alpha)\frac{\partial^2 Q_{1_{t-1}}}{\partial x_2^2} - 2(\alpha(1-\alpha))\frac{\partial Q_{1_{t-1}}}{\partial x_2}\\
&\quad + (1-2\alpha)\beta r_{t-1} - Q_{1_{t-1}} && \text{(expand and simplify)}
\end{aligned}
$$

resulting in

$$
\frac{\partial^2 Q_{1_t}}{\partial x_2^2} = (1-\alpha)\frac{\partial^2 Q_{1_{t-1}}}{\partial x_2^2} - 2(\alpha(1-\alpha))\frac{\partial Q_{1_{t-1}}}{\partial x_2} + (1-2\alpha)\beta r_{t-1} - Q_{1_{t-1}} \tag{D.22}
$$

Finally, we expand equation (C.21) using equation (C.22) to obtain the second-order direct partial derivative of LL_{A_t} with respect to x_2.

Solve for $\frac{\partial^2 LL_{A_t}}{\partial x_2 \partial x_1}$:

$$
\begin{aligned}
\frac{\partial^2 LL_{A_t}}{\partial x_2 \partial x_1} = \frac{\partial^2 Q_{1_t}}{\partial x_2 \partial x_1} - \left[e^{LL_{A_t}} \frac{\partial^2 Q_{1_t}}{\partial x_2 \partial x_1} + \frac{\partial Q_{1_t}}{\partial x_2} \left(\frac{\partial LL_{A_t}}{\partial x_1} e^{LL_{A_t}} \right) \right] \\
- \left[(1 - e^{LL_{A_t}}) \frac{\partial^2 Q_{2_t}}{\partial x_2 \partial x_1} + \frac{\partial Q_{2_t}}{\partial x_2} \left(- \frac{\partial LL_{A_t}}{\partial x_1} e^{LL_{A_t}} \right) \right] \quad \text{(cross partial derivative of (C.15) of } x_1 \text{ w.r.t } x_2\text{)}
\end{aligned}
$$

resulting in

$$
\frac{\partial^2 LL_{A_t}}{\partial x_2 \partial x_1} = \frac{\partial^2 Q_{1_t}}{\partial x_2 \partial x_1} - \left[e^{LL_{A_t}} \frac{\partial^2 Q_{1_t}}{\partial x_2 \partial x_1} + (1 - e^{LL_{A_t}}) \left(\frac{\partial^2 Q_{2_t}}{\partial x_2 \partial x_1} \right) \right] - \frac{\partial LL_{A_t}}{\partial x_1} e^{LL_{A_t}} \left(\frac{\partial Q_{1_t}}{\partial x_2} - \frac{\partial Q_{2_t}}{\partial x_2} \right) \tag{D.23}
$$

To solve for $\frac{\partial^2 Q_{1_t}}{\partial x_2 \partial x_1}$ in equation (C.23), we will refer to equation (C.13) and differentiate with respect to x_2:

Solve for $\frac{\partial^2 Q_{1_t}}{\partial x_2 \partial x_1}$:

$$
\begin{aligned}
\frac{\partial Q_{1_t}}{\partial x_2} &= (1-\alpha) \frac{\partial Q_{1_{t-1}}}{\partial x_2} + (\beta r_{t-1} - Q_{1_{t-1}}) \frac{\partial \alpha}{\partial x_2} && \text{(C.13)} \\
\frac{\partial^2 Q_{1_t}}{\partial x_2 \partial x_1} &= (1-\alpha) \frac{\partial^2 Q_{1_{t-1}}}{\partial x_2 \partial x_1} + (\beta r_{t-1} - Q_{1_{t-1}}) \frac{\partial^2 \alpha}{\partial x_2 \partial x_1} \\
&\quad + \left(\frac{\partial \beta}{\partial x_1} r_{t-1} - \frac{\partial Q_{1_{t-1}}}{\partial x_1}\right) \frac{\partial \alpha}{\partial x_2} && \text{(cross partial derivative of } x_1 \text{ w.r.t } x_2\text{)} \\
&= (1-\alpha) \frac{\partial^2 Q_{1_{t-1}}}{\partial x_2 \partial x_1} + (\alpha(1-\alpha)) \left[\beta r_{t-1} - \frac{\partial Q_{1_{t-1}}}{\partial x_1} \right] && \text{(expand and simplify)}
\end{aligned}
$$

resulting in

$$
\frac{\partial^2 Q_{1_t}}{\partial x_2 \partial x_1} = (1-\alpha) \frac{\partial^2 Q_{1_{t-1}}}{\partial x_2 \partial x_1} + (\alpha(1-\alpha)) \left[\beta r_{t-1} - \frac{\partial Q_{1_{t-1}}}{\partial x_1} \right] \tag{D.24}
$$

Finally, we expand equation (C.23) using equation (C.24) to obtain the second-order cross partial derivative of LL_{A_t} of x_1 with respect to x_2.