CHAPTER 21

21.1

	x	$f(x)$
x_{i-2}	0.261799388	0.965925826
x_{i-1}	0.523598776	0.866025404
x_i	0.785398163	0.707106781
x_{i+1}	1.047197551	0.5
x_{i+2}	1.308996939	0.258819045

true = $-\sin(\pi/4) = -0.70710678$

The results are summarized as

	first-order	second-order
Forward	–0.79108963	–0.72601275
	–11.877%	–2.674%
Backward	–0.60702442	–0.71974088
	14.154%	–1.787%
Centered	–0.69905703	–0.70699696
	1.138%	0.016%

21.2

	x	$f(x)$
x_{i-2}	1.8	6.049647464
x_{i-1}	1.9	6.685894442
x_i	2	7.389056099
x_{i+1}	2.1	8.166169913
x_{i+2}	2.2	9.025013499

Both the first and second derivatives have the same value,

$$\text{truth} = e^2 = 7.389056099$$

The results are summarized as

	first-order	second-order
First derivative	7.401377351	7.389031439
	-0.166750%	0.000334%
Second derivative	7.395215699	7.389047882
	-0.083361%	0.000111%

21.3 First, we will use forward expansions. The Taylor series expansion about $a = x_i$ and $x = x_{i+2}$ ($2\Delta x$ steps forward) can be written as:

$$f(x_{i+2}) = f(x_i) + f'(x_i)2\Delta x + \frac{1}{2}f''(x_i)(2\Delta x)^2 + \frac{1}{6}f'''(x_i)(2\Delta x)^3 + \frac{1}{24}f^{(4)}(x_i)(2\Delta x)^4 + \frac{1}{120}f^{(5)}(x_i)(2\Delta x)^5 + \cdots$$

$$f(x_{i+2}) = f(x_i) + 2f'(x_i)\Delta x + 2f''(x_i)\Delta x^2 + \frac{8}{6}f'''(x_i)\Delta x^3 + \frac{16}{24}f^{(4)}(x_i)\Delta x^4 + \frac{32}{120}f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(1)}$$

Taylor series expansion about $a = x_i$ and $x = x_{i+1}$ (Δx steps forward):

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{1}{2} f''(x_i)\Delta x^2 + \frac{1}{6} f'''(x_i)\Delta x^3 + \frac{1}{24} f^{(4)}(x_i)\Delta x^4 + \frac{1}{120} f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(2)}$$

Multiply Eq. 2 by 2 and subtract the result from Eq. 1 to yield

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)\Delta x^2 + \frac{6}{6} f'''(x_i)\Delta x^3 + \frac{14}{24} f^{(4)}(x_i)\Delta x^4 + \frac{30}{120} f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(3)}$$

Next, we will use backward expansions. The Taylor series expansion about $a = x_i$ and $x = x_{i-2}$ ($2\Delta x$ steps backward) can be written as:

$$f(x_{i-2}) = f(x_i) + f'(x_i)(-2\Delta x) + \frac{1}{2} f''(x_i)(-2\Delta x)^2 + \frac{1}{6} f'''(x_i)(-2\Delta x)^3 + \frac{1}{24} f^{(4)}(x_i)(-2\Delta x)^4 + \frac{1}{120} f^{(5)}(x_i)(-2\Delta x)^5 + \cdots$$

$$f(x_{i-2}) = f(x_i) - 2f'(x_i)\Delta x + 2f''(x_i)\Delta x^2 - \frac{8}{6} f'''(x_i)\Delta x^3 + \frac{16}{24} f^{(4)}(x_i)\Delta x^4 - \frac{32}{120} f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(4)}$$

Taylor series expansion about $a = x_i$ and $x = x_{i-1}$ (Δx steps backward):

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{1}{2} f''(x_i)\Delta x^2 - \frac{1}{6} f'''(x_i)\Delta x^3 + \frac{1}{24} f^{(4)}(x_i)\Delta x^4 - \frac{1}{120} f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(5)}$$

Multiply Eq. 5 by 2 and subtract the result from Eq. 4 to yield

$$2f(x_{i-1}) - f(x_{i-2}) = f(x_i) - f''(x_i)\Delta x^2 + \frac{6}{6} f'''(x_i)\Delta x^3 - \frac{14}{24} f^{(4)}(x_i)\Delta x^4 + \frac{30}{120} f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(6)}$$

Add Eqs (3) and (6)

$$f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2}) = 2f'''(x_i)\Delta x^3 + \frac{60}{120} f^{(5)}(x_i)\Delta x^5 + \cdots \quad \textbf{(7)}$$

Equation 7 can be solved for

$$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2\Delta x^3} - \frac{\frac{60}{120} f^{(5)}(x_i)\Delta x^5}{2\Delta x^3} + \cdots$$

$$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2\Delta x^3} - \frac{1}{4}f^{(5)}(x_i)\Delta x^2 + \cdots$$

21.4 The true value is $-\sin(\pi/4) = -0.70710678$.

$$D(\pi/3) = \frac{-0.25882 - 0.965926}{2(1.047198)} = -0.58477$$

$$D(\pi/6) = \frac{0.258819 - 0.965926}{2(0.523599)} = -0.67524$$

$$D = \frac{4}{3}(-0.67524) - \frac{1}{3}(-0.58477) = -0.70539$$

21.5 The true value is $1/x = 1/5 = 0.2$.

$$D(2) = \frac{1.94591 - 1.098612}{2(2)} = 0.211824$$

$$D(1) = \frac{1.791759 - 1.386294}{2(1)} = 0.202733$$

$$D = \frac{4}{3}(0.202733) - \frac{1}{3}(0.211824) = 0.199702$$

21.6 The true value

$$f'(0) = 8(0)^3 - 18(0)^2 - 12 = -12$$

Equation (21.21) can be used to compute the derivative as

$x_0 = -0.5$ $\quad$ $f(x_0) = -1.125$
$x_1 = 1$ $\quad$ $f(x_1) = -24$
$x_2 = 2$ $\quad$ $f(x_2) = -48$

$$f'(0) = -1.125\frac{2(0) - 1 - 2}{(-0.5 - 1)(-0.5 - 2)} + (-24)\frac{2(0) - (-0.5) - 2}{(1 - (-0.5))(1 - 2)} + (-48)\frac{2(0) - (-0.5) - 1}{(2 - (-0.5))(2 - 1)}$$
$$= 0.9 - 24 + 9.6 = -13.5$$

Centered difference:

$$f'(0) = \frac{-24 - 12}{1 - (-1)} = -18$$

21.7 At $x = x_i$, Eq. (21.21) is

$$f'(x) = f(x_{i-1})\frac{2x_i - x_i - x_{i+1}}{(x_{i-1} - x_i)(x_{i-1} - x_{i+1})} + f(x_i)\frac{2x_i - x_{i-1} - x_{i+1}}{(x_i - x_{i-1})(x_i - x_{i+1})} + f(x_{i+1})\frac{2x_i - x_{i-1} - x_i}{(x_{i+1} - x_{i-1})(x_{i+1} - x_i)}$$

For equispaced points that are h distance apart, this equation becomes

$$f'(x) = f(x_{i-1})\frac{-h}{-h(-2h)} + f(x_i)\frac{2x_i - (x_i - h) - (x_i + h)}{h(-h)} + f(x_{i+1})\frac{h}{2h(h)}$$

$$= \frac{-f(x_{i-1})}{2h} + 0 + \frac{f(x_{i+1})}{2h} = \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

21.8

```
function [d,ea,iter]=rombdiff(func,x,es,maxit,varargin)
% romberg: Romberg integration quadrature
%   q = romberg(func,a,b,es,maxit,varargin):
%                   Romberg integration.
% input:
%   func = name of function to be integrated
%   a, b = integration limits
%   es = desired relative error (default = 0.000001%)
%   maxit = maximum allowable iterations (default = 50)
%   p1,p2,... = additional parameters used by func
% output:
%   q = integral estimate
%   ea = approximate relative error (%)
%   iter = number of iterations

if nargin<2,error('at least 2 input arguments required'),end
if nargin<3|isempty(es), es=0.000001;end
if nargin<4|isempty(maxit), maxit=50;end
n = 1;
DY(1,1) = dydxnew(func,x,n,varargin{:});
iter = 0;
ea=100;
while iter<maxit
  iter = iter+1;
  n = 2^iter;
  DY(iter+1,1) = dydxnew(func,x,n,varargin{:});
  for k = 2:iter+1
    j = 2+iter-k;
    DY(j,k) = (4^(k-1)*DY(j+1,k-1)-DY(j,k-1))/(4^(k-1)-1);
  end
  if DY(1,iter+1)~=0,ea = abs((DY(1,iter+1)-DY(2,iter))/DY(1,iter+1))*100;end
  if ea<=es, break; end
end
d = DY(1,iter+1);

function d = dydxnew(func,x,n,varargin)
a = x - x/n;
b = x + x/n;
d=(func(b,varargin{:})-func(a,varargin{:}))/(b-a);
```

Test the program by evaluating the derivative of $f(x) = e^{-0.5x}$ at $x = 1$. The exact result is $f'(1) = -0.5e^{-0.5(1)} = -0.30326533$.

```
>> format long
>> f=@(x) exp(-0.5*x);
>> [d,ea,iter]=rombdiff(f,1)

d =
  -0.30326532985552
ea =
    7.603832401115816e-008
iter =
     3
```

21.9
```
function dy=diffuneq(x,y)
% diffuneq: differentiation of unequally-spaced data
%   dy=diffuneq(func,x,es,maxit,varargin):
% input:
%   x = vector of independent variable
%   y = vector of dependent variable
% output:
%   dy = vector of derivative estimates

if nargin<2,error('at least 2 input arguments required'),end
n=length(x);
for i = 1:n
  dy(i) = dyuneq(x, y, n, x(i));
end
end

function dydx=dyuneq(x, y, n, xx)
if xx <= x(2)
  dydx = DyDx(xx, x(1), x(2), x(3), y(1), y(2), y(3));
elseif xx >= x(n - 1)
  dydx = DyDx(xx, x(n - 2), x(n - 1), x(n), y(n - 2), y(n - 1), y(n));
else
  for ii = 2:n - 2
    if xx >= x(ii) & xx <= x(ii + 1)
      if xx - x(ii) < x(ii+1) - xx
        %if the unknown is closer to the lower end of the range,
        %x(ii) will be chosen as the middle point
        dydx = DyDx(xx, x(ii - 1), x(ii), x(ii + 1), y(ii - 1), ...
                y(ii), y(ii + 1));
      elseif xx - x(ii-1) == x(ii+1) - xx & x(ii)-x(ii-1)<x(ii+2)-x(ii+1)
        %if the unknown is at the midpoint of the range
        % and if the interval below the range
        %is less than the interval above the range,
        %x(ii) will be chosen as the middle point
        dydx = DyDx(xx, x(ii - 1), x(ii), x(ii + 1), y(ii - 1), ...
               y(ii), y(ii + 1));
      else
        %otherwise, x(ii+1) is chosen as the middle point
        dydx = DyDx(xx, x(ii), x(ii + 1), x(ii + 2), y(ii), y(ii + 1), ...
             y(ii + 2));
      end
      break
    end
  end
end
end

function dy=DyDx(x, x0, x1, x2, y0, y1, y2)
dy = y0 * (2 * x - x1 - x2) / (x0 - x1) / (x0 - x2) ...
      + y1 * (2 * x - x0 - x2) / (x1 - x0) / (x1 - x2) ...
       + y2 * (2 * x - x0 - x1) / (x2 - x0) / (x2 - x1);
end
```

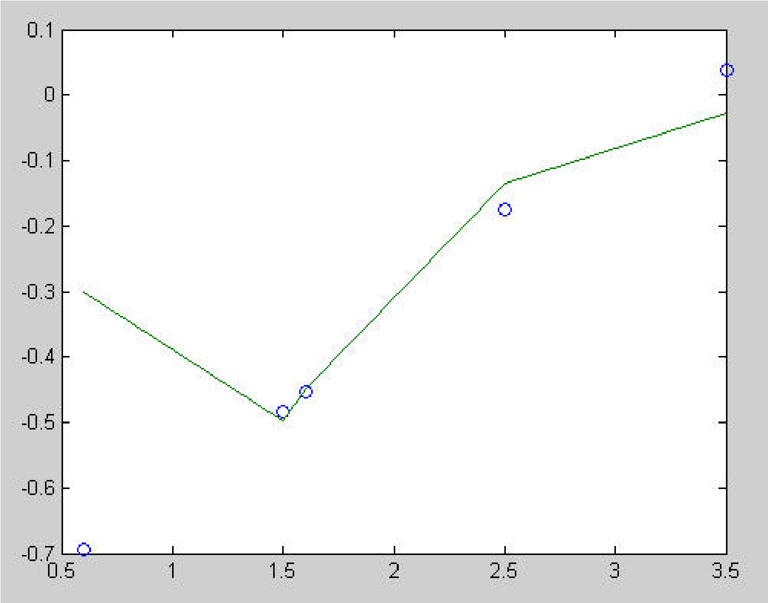
The program can be applied to the data with the result:

```
>> x=[0.6 1.5 1.6 2.5 3.5];
>> fx=[0.9036 0.3734 0.3261 0.08422 0.01596];
>> dydx=diffuneq(x,fx)

dydx =
```

```
   -0.6936    -0.4846    -0.4526    -0.1738     0.0373
```

The results can be compared with the true derivatives which can be calculated with the analytical solution, $f'(x) = 5e^{-2x} - 10xe^{-2x}$. The results can be displayed graphically below where the computed values are represented as points and the true values as the curve.

```
>> dytrue=5*exp(-2*x)-10*x.*exp(-2*x)
dytrue =
   -0.3012    -0.4979    -0.4484    -0.1348    -0.0274
```



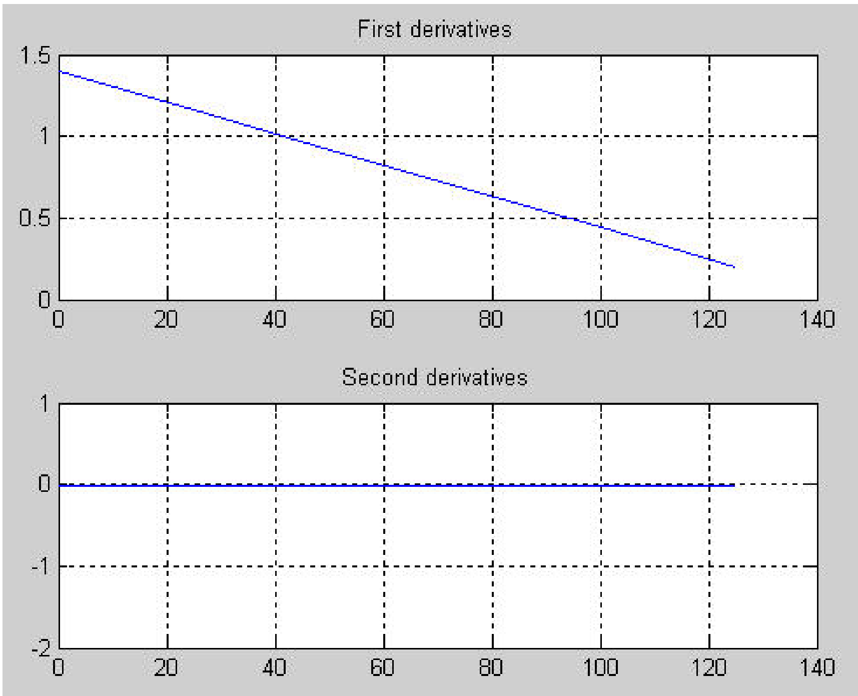
As can be seen, the results leave something to be desired, particularly at the left end of the interval. The poor performance is in part due to the highly irregular spacing of the first three points.

21.10

```
function [dydx, d2ydx2] = diffeq(x,y)
n = length(x);
if length(y)~=n, error('x and y must be same length'), end
if any(diff(diff(x))~=0), error('unequal spacing'), end
if length(x)<4, error('at least 4 values required'), end
dx=x(2)-x(1);
for i=1:n
  if i==1
    dydx(i)=(-y(i+2)+4*y(i+1)-3*y(i))/dx/2;
    d2ydx2(i)=(-y(i+3)+4*y(i+2)-5*y(i+1)+2*y(i))/dx^2;
  elseif i==n
    dydx(i)=(3*y(i)-4*y(i-1)+y(i-2))/dx/2;
    d2ydx2(i)=(2*y(i)-5*y(i-1)+4*y(i-2)-y(i-3))/dx^2;
  else
    dydx(i)=(y(i+1)-y(i-1))/dx/2;
    d2ydx2(i)=(y(i+1)-2*y(i)+y(i-1))/dx^2;
  end
end
subplot(2,1,1);plot(x,dydx);grid;title('First derivatives')
subplot(2,1,2);plot(x,d2ydx2);grid;title('Second derivatives')
```

The M-file can be run for the data from Prob. 21.11:

```
>> t=[0 25 50 75 100 125];
>> y=[0 32 58 78 92 100];
>> [dydx, d2ydx2] = diffeq(t,y)

dydx =
    1.4000    1.1600    0.9200    0.6800    0.4400    0.2000
d2ydx2 =
   -0.0096   -0.0096   -0.0096   -0.0096   -0.0096   -0.0096
```



21.11 The first forward difference formula of $O(h^2)$ from Fig. 21.3 can be used to estimate the velocity for the first point at $t = 0$,

$$f'(0) = \frac{-58 + 4(32) - 3(0)}{2(25)} = 1.4\frac{\text{km}}{\text{s}}$$

The acceleration can be estimated with the second forward difference formula of $O(h^2)$ from Fig. 21.3

$$f''(0) = \frac{-78 + 4(58) - 5(32) + 2(0)}{(25)^2} = -0.0096\frac{\text{km}}{\text{s}^2}$$

For the interior points, centered difference formulas of $O(h^2)$ from Fig. 21.5 can be used to estimate the velocities and accelerations. For example, at the second point at $t = 25$,

$$f'(25) = \frac{58 - 0}{2(25)} = 1.16\frac{\text{km}}{\text{s}}$$

$$f''(25) = \frac{58 - 2(32) + 0}{(25)^2} = -0.0096\frac{\text{km}}{\text{s}^2}$$

For the final point, backward difference formulas of $O(h^2)$ from Fig. 21.4 can be used to estimate the velocities and accelerations. The results for all values are summarized in the following table.

t	*y*	*v*	*a*
0	0	1.40	-0.0096
25	32	1.16	-0.0096
50	58	0.92	-0.0096
75	78	0.68	-0.0096
100	92	0.44	-0.0096
125	100	0.20	-0.0096

21.12 Although the first three points are equally spaced, the remaining values are unequally spaced. Therefore, a good approach is to use Eq. 21.21 to perform the differentiation for all points. The results are summarized below:

t	*x*	*v*	*a*
0	153	70.19231	-47.09922
0.52	185	52.88462	-19.46883
1.04	208	49.94473	-10.82145
1.75	249	37.25169	-26.58748
2.37	261	16.05181	-24.50300
3.25	271	6.59273	-10.80561
3.83	273	0.30382	-10.88031

21.13 (a)

$$v = \frac{dx}{dt} = x'(t_i) = \frac{x(t_{i+1}) - x(t_{i-1})}{2h} = \frac{7.3 - 5.1}{4} = 0.55\,\frac{\text{m}}{\text{s}}$$

$$a = \frac{d^2x}{dt^2} = x''(t_i) = \frac{x(t_{i+1}) - 2x(t_i) + x(t_{i-1})}{h^2} = \frac{7.3 - 2(6.3) + 5.1}{2^2} = -0.05\,\frac{\text{m}}{\text{s}^2}$$

(b)

$$v = \frac{-x(t_{i+2}) + 4x(t_{i+1}) - 3x(t_i)}{2h} = \frac{-8 + 4(7.3) - 3(6.3)}{4} = 0.575\,\frac{\text{m}}{\text{s}}$$

$$a = \frac{-x(t_{i+3}) + 4x(t_{i+2}) - 5x(t_{i+1}) + 2x(t_i)}{h^2} = \frac{-8.4 + 4(8) - 5(7.3) + 2(6.3)}{2^2} = -0.075\,\frac{\text{m}}{\text{s}^2}$$

(c)

$$v = \frac{3x(t_i) - 4x(t_{i-1}) + x(t_{i-2})}{2h} = \frac{3(6.3) - 4(5.1) + 3.4}{4} = 0.475\,\frac{\text{m}}{\text{s}}$$

$$a = \frac{2x(t_i) - 5x(t_{i-1}) + 4x(t_{i-2}) - x(t_{i-3})}{h^2} = \frac{2(6.3) - 5(5.1) + 4(3.4) - 1.8}{2^2} = -0.275\,\frac{\text{m}}{\text{s}^2}$$

21.14

$$\dot{\theta} = \frac{d\theta}{dt} = \frac{\theta(t_{i+1}) - \theta(t_{i-1})}{2h} = \frac{0.67 - 0.70}{4} = -0.0075 \text{ rad/s}$$

$$\dot{r} = \frac{dr}{dt} = \frac{r(t_{i+1}) - r(t_{i-1})}{2h} = \frac{6030 - 5560}{4} = 117.5 \text{ m/s}$$

$$\ddot{\theta} = \frac{d^2\theta}{dt^2} = \frac{\theta(t_{i+1}) - 2\theta(t_i) + \theta(t_{i-1})}{h^2} = \frac{0.67 - 2(0.68) + 0.70}{(2)^2} = 0.0025 \text{ rad/s}^2$$

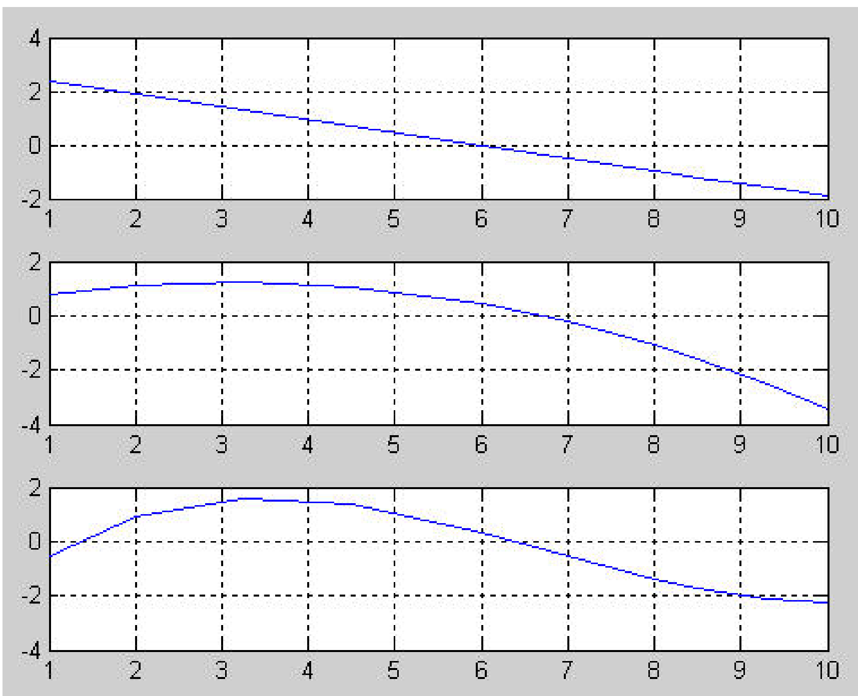
$$\ddot{r} = \frac{d^2r}{dt^2} = \frac{r(t_{i+1}) - 2r(t_i) + r(t_{i-1})}{h^2} = \frac{6030 - 2(5800) + 5560}{(2)^2} = -2.5 \ \text{m/s}^2$$

$$\vec{v} = 117.5\,\vec{e}_r - 43.5\,\vec{e}_\theta$$

$$\vec{a} = -2.82625\,\vec{e}_r + 12.7375\,\vec{e}_\theta$$

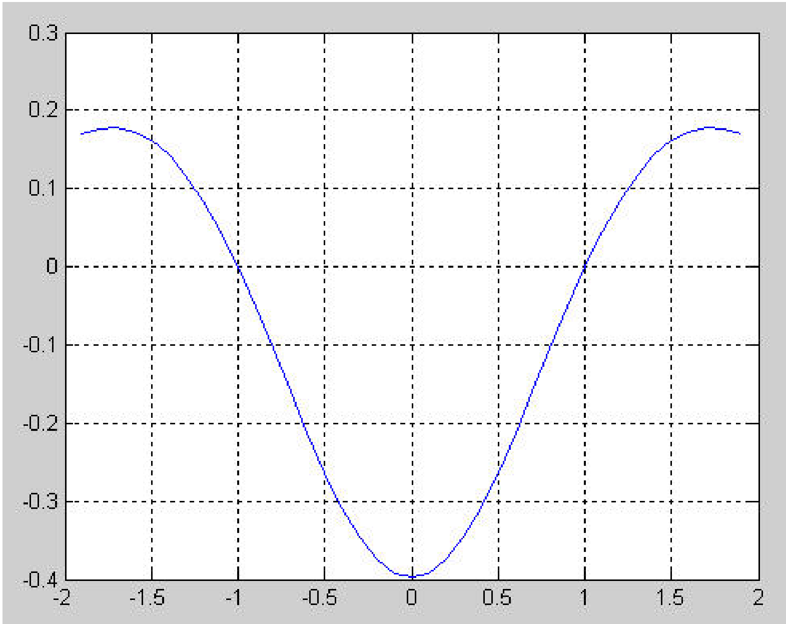
21.15 The following script is developed to solve the problem and generate the plot:

```
t=[1 2 3.25 4.5 6 7 8 8.5 9.3 10];
v=[10 12 11 14 17 16 12 14 14 10];
for i = 2:4
  p=polyfit(t,v,i);
  dvdt=polyder(p);
  for j = 1:length(t)
    vp=polyval(dvdt,t);
  end
  subplot(3,1,i-1),plot(t,vp),grid
end
```



21.16

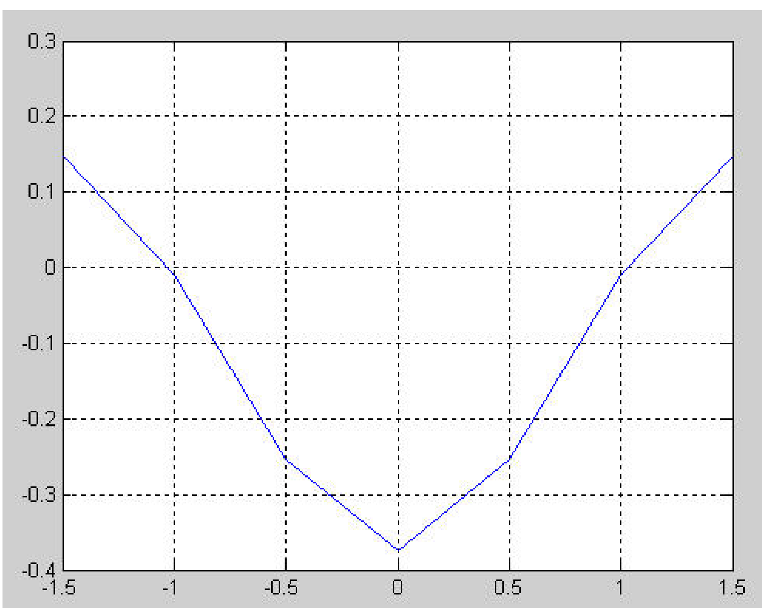
```
>> x=-2:.1:2;
>> f=@(x) 1/sqrt(2*pi)*exp(-(x.^2)/2);
>> y=f(x);
>> d=diff(y)./diff(x);
>> x=-1.95:.1:1.95;
>> d2=diff(d)./diff(x);
>> x=-1.9:.1:1.9;
>> plot(x,d2);grid
```



Thus, inflection points ($d^2y/dx^2 = 0$) occur at –1 and 1.

21.17

```
>> x=[-2 -1.5 -1 -0.5 0 0.5 1 1.5 2];
>> y=[0.05399 0.12952 0.24197 0.35207 0.39894 0.35207 0.24197 0.12952 0.05399];
>> d=diff(y)./diff(x);
>> x=-1.75:.5:1.75;
>> d2=diff(d)./diff(x);
>> x=-1.5:.5:1.5;
>> plot(x,d2);grid
```



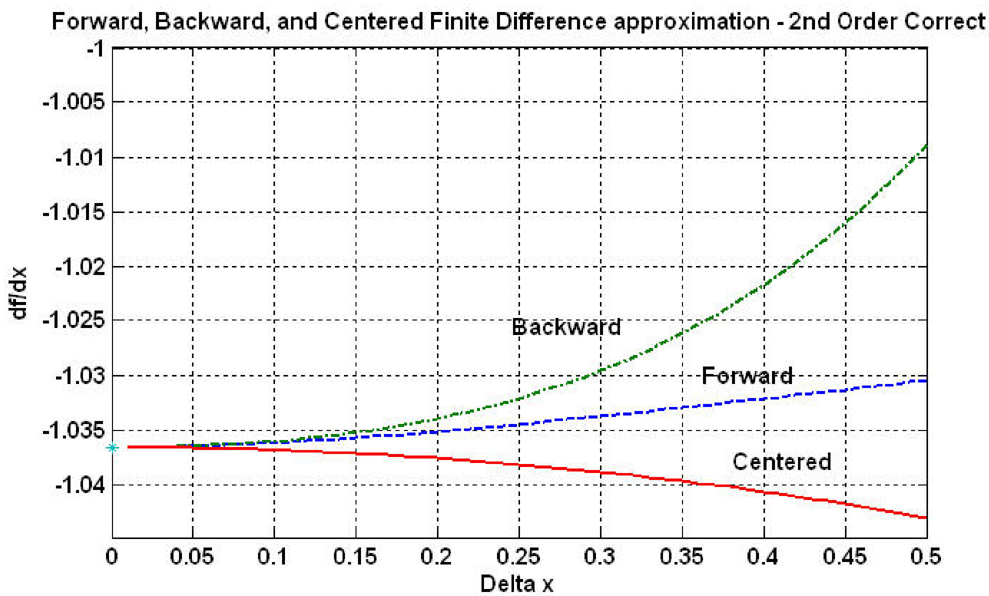
Thus, inflection points ($d^2y/dx^2 = 0$) occur at –1 and 1.

21.18
```
%Centered Finite Difference First & Second Derivatives of Order O(dx^2)
%Using diff(y)
dx=1.;
y=[1.4 2.1 3.3 4.8 6.8 6.6 8.6 7.5 8.9 10.9 10];
dyf=diff(y);
% First Derivative Centered FD using diff
n=length(y);
for i=1:n-2
  dydxc(i)=(dyf(i+1)+dyf(i))/(2*dx);
end
%Second Derivative Centered FD using diff
dy2dx2c=diff(dyf)/(dx*dx);
fprintf('first derivative \n');  fprintf('%f\n', dydxc)
fprintf('second derivative \n'); fprintf('%f\n', dy2dx2c)

first derivative
0.950000
1.350000
1.750000
0.900000
0.900000
0.450000
0.150000
1.700000
0.550000
second derivative
0.500000
0.300000
0.500000
-2.200000
2.200000
-3.100000
2.500000
0.600000
-2.900000
```

21.19 Parts (a) through (d) can be solved with the following script:

```
%  Finite Difference Approximation of slope
%  For f(x)=exp(-2*x)-x
%      f'(x)=-2*exp(-2*x)-1
%  Centered diff. df/dx=(f(i+1)-f(i-1))/2dx         + O(dx^2)
%  Fwd. diff.     df/dx=(-f(i+2)+4f(i+1)-3f(i))/2dx + O(dx^2)
%  Bkwd. diff.    df/dx=(3f(i)-4f(i-1)+f(i-2))/2dx  + O(dx^2)

x=2;
fx=exp(-2*x)-x;
dfdx2=-2*exp(-2*x)-1;
%approximation
dx=0.5:-0.01:.01;
for i=1:length(dx)
  %x-values at i+-dx and +-2dx
  xp(i)=x+dx(i);
  x2p(i)=x+2*dx(i);
  xn(i)=x-dx(i);
  x2n(i)=x-2*dx(i);
  %f(x)-values at i+-dx and +-2dx
  fp(i)=exp(-2*xp(i))-xp(i);
  f2p(i)=exp(-2*x2p(i))-x2p(i);
  fn(i)=exp(-2*xn(i))-xn(i);
```

```
  f2n(i)=exp(-2*x2n(i))-x2n(i);
  %Finite Diff. Approximations
  Cdfdx(i)=(fp(i)-fn(i))/(2*dx(i));
  Fdfdx(i)=(-f2p(i)+4*fp(i)-3*fx)/(2*dx(i));
  Bdfdx(i)=(3*fx-4*fn(i)+f2n(i))/(2*dx(i));
end
dx0=0;
plot(dx,Fdfdx,'--',dx,Bdfdx,'-.',dx,Cdfdx,'-',dx0,dfdx2,'*')
grid
title('Forward, Backward, and Centered Finite Difference approximation - 2nd
Order Correct')
xlabel('Delta x')
ylabel('df/dx')
gtext('Centered'); gtext('Forward');  gtext('Backward')
```



21.20 The flow rate is equal to the derivative of volume with respect to time. Equation (21.21) can be used to compute the derivative as

$x_0 = 1$ $\quad f(x_0) = 1$
$x_1 = 5$ $\quad f(x_1) = 8$
$x_2 = 8$ $\quad f(x_2) = 16.4$

$$f'(7) = 1\frac{2(7)-5-8}{(1-5)(1-8)} + 8\frac{2(7)-1-8}{(5-1)(5-8)} + 16.4\frac{2(7)-1-5}{(8-1)(8-5)} = 0.035714 - 3.33333 + 6.247619 = 2.95$$

Therefore, the flow is equal to 2.95 cm^3/s.

21.21 The velocity at the surface can be computed with Eq. (21.21) as

$x_0 = 0$ $\quad f(x_0) = 0$
$x_1 = 0.002$ $\quad f(x_1) = 0.287$
$x_2 = 0.006$ $\quad f(x_2) = 0.899$

$$f'(0) = 0\frac{2(0)-0.002-0.006}{(0-0.002)(0-0.006)} + 0.287\frac{2(0)-0-0.006}{(0.002-0)(0.002-0.006)} + 0.899\frac{2(0)-0-0.002}{(0.006-0)(0.006-0.002)}$$
$$= 0 + 215.25 - 74.9167 = 140.3333$$

Therefore, the shear stress can be computed as

$$\tau = 1.8\times10^{-5}\,\frac{\text{N}\cdot\text{s}}{\text{m}^2}140.3333\frac{1}{\text{s}} = 0.00253\frac{\text{N}}{\text{m}^2}$$

21.22 Equation (21.21) can be used to compute the derivative as

$x_0 = 0$ $\quad$ $f(x_0) = 0.06$
$x_1 = 1$ $\quad$ $f(x_1) = 0.32$
$x_2 = 3$ $\quad$ $f(x_2) = 0.6$

$$f'(0) = 0.06\frac{2(0)-1-3}{(0-1)(0-3)} + 0.32\frac{2(0)-0-3}{(1-0)(1-3)} + 0.6\frac{2(0)-0-1}{(3-0)(3-1)} = -0.08 + 0.48 - 0.1 = 0.3$$

The mass flux can be computed as

$$\text{Mass flux} = -1.52\times10^{-6}\,\frac{\text{cm}^2}{\text{s}}0.3\times10^{-6}\,\frac{\text{g}}{\text{cm}^4} = -4.56\times10^{-13}\,\frac{\text{g}}{\text{cm}^2\text{s}}$$

where the negative sign connotes transport from the sediments into the lake. The amount of mass transported into the lake can be computed as

$$\text{Mass transport} = 4.56\times10^{-13}\,\frac{\text{g}}{\text{cm}^2\text{s}}(3.6\times10^6\ \text{m}^2)\frac{365\text{ d}}{\text{yr}}\frac{86{,}400\text{ s}}{\text{d}}\frac{10{,}000\text{ cm}^2}{\text{m}^2}\frac{\text{kg}}{1000\text{ g}} = 517.695\text{ kg}$$

21.23 For the equally-spaced points, we can use the second-order formulas from Figs. 21.3 through 21.5. For example, for the first point (t = 0), we can use

$$f'(0) = \frac{-0.77 + 4(0.7) - 3(0.4)}{2(10)} = 0.0415\frac{\text{barrels}}{\text{min}}$$

However, the points around t = 30 are unequally spaced so we must use Eq. (21.21) to compute the derivative as

$x_0 = 20$ $\quad$ $f(x_0) = 0.77$
$x_1 = 30$ $\quad$ $f(x_1) = 0.88$
$x_2 = 45$ $\quad$ $f(x_2) = 1.05$

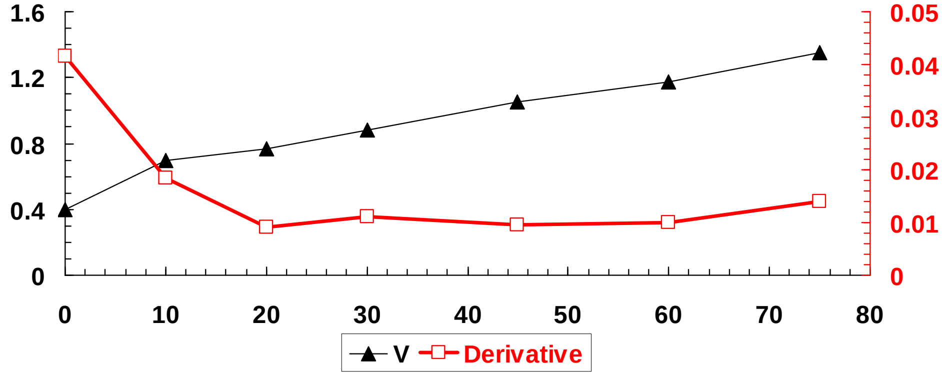
$$f'(30) = 0.77\frac{2(30)-30-45}{(20-30)(20-45)} + 0.88\frac{2(30)-20-45}{(30-20)(30-45)} + 1.05\frac{2(30)-20-30}{(45-20)(45-30)} = 0.011133$$

All the results can be summarized as

t	*V*	Derivative	Method
0	0.4	0.0415	Forward
10	0.7	0.0185	Centered
20	0.77	0.009	Centered
30	0.88	0.011133	Unequal

45	1.05	0.009667	Centered
60	1.17	0.01	Centered
75	1.35	0.014	Backward

The derivatives can be plotted versus time as



21.24 Because the values are equally spaced, we can use the second-order forward difference from Fig. 21.3 to compute the derivative as

$x_0 = 0$ $\quad f(x_0) = 20$
$x_1 = 0.08$ $\quad f(x_1) = 17$
$x_2 = 0.16$ $\quad f(x_2) = 15$

$$f'(0) = \frac{-15 + 4(17) - 3(20)}{0.16} = -43.75 \frac{°\text{C}}{\text{m}}$$

The coefficient of thermal conductivity can then be estimated as k = –60 W/m^2/(–43.75 ^{o}C/m) = 1.37143 W/(^{o}C·m).

21.25 First, we can estimate the areas by numerically differentiating the volume data. Because the values are equally spaced, we can use the second-order difference formulas from Fig. 21.3 to compute the derivatives at each depth. For example, at the first depth, we can use the forward difference to compute

$$A_s(0) = -\frac{dV}{dz}(0) = -\frac{-1{,}963{,}500 + 4(5{,}105{,}100) - 3(9{,}817{,}500)}{8} = 1{,}374{,}450 \text{ m}^2$$

For the interior points, second-order centered differences can be used. For example, at the second point at (z = 4 m),

$$A_s(4) = -\frac{dV}{dz}(4) = -\frac{1{,}963{,}500 - 9{,}817{,}500}{8} = 981{,}750 \text{ m}^2$$

The other interior points can be determined in a similar fashion

$$A_s(8) = -\frac{dV}{dz}(8) = -\frac{392{,}700 - 5{,}105{,}100}{8} = 589{,}050 \text{ m}^2$$

$$A_s(12) = -\frac{dV}{dz}(12) = -\frac{0 - 1{,}963{,}500}{8} = 245{,}437.5 \text{ m}^2$$

For the last point, the second-order backward formula yields

$$A_s(16) = -\frac{dV}{dz}(16) = -\frac{3(0) - 4(392,700) + 1,963,500}{8} = -49,087.5 \text{ m}^2$$

Since this is clearly a physically unrealistic result, we will assume that the bottom area is 0. The results are summarized in the following table along with the other quantities needed to determine the average concentration.

z, m	*V*, m^3	*c*, g/m^3	A_s, m^2	$c \times A_s$
0	9817500	10.2	1374450.0	14019390
4	5105100	8.5	981750.0	8344875
8	1963500	7.4	589050.0	4358970
12	392700	5.2	245437.5	1276275
16	0	4.1	0	0

The necessary integrals can then be evaluated with the multi-segment Simpson's 1/3 rule,

$$\int_0^z A_s(z)\, dz = (16-0)\frac{1,374,450 + 4(981,750 + 245,437.5) + 2(589,050) + 0}{12} = 9,948,400 \text{ m}^3$$

$$\int_0^z c(z)A_s(z)\, dz = (16-0)\frac{14,019,390 + 4(8,344,875 + 1,276,275) + 2(4,358,970) + 0}{12} = 81{,}629{,}240 \text{ g}$$

The average concentration can then be computed as

$$\bar{c} = \frac{\int_0^Z c(z)A_s(z)\, dz}{\int_0^Z A_s(z)\, dz} = \frac{81{,}629{,}240}{9{,}948{,}400} = 8.205263\frac{\text{g}}{\text{m}^3}$$

21.26 For the equispaced data, we can use the second-order finite divided difference formulas from Figs. 21.3 through 21.5. For example, for the first point, we can use

$$\frac{di}{dt} = \frac{-0.32 + 4(0.16) - 3(0)}{0.2 - 0} = 1.6$$

For the intermediate equispaced points, we can use centered differences. For example, for the second point

$$\frac{di}{dt} = \frac{0.32 - 0}{0.2 - 0} = 1.6$$

For the last point, we can use a backward difference

$$\frac{di}{dt} = \frac{3(2) - 4(0.84) + 0.56}{0.7 - 0.3} = 8$$

One of the points (t = 0.3) has unequally spaced neighbors. For this point, we can use Eq. (23.9) to compute the derivative as

$x_0 = 0.2$ $f(x_0) = 0.32$
$x_1 = 0.3$ $f(x_1) = 0.56$
$x_2 = 0.5$ $f(x_2) = 0.84$

$$f'(0) = 0.32\frac{2(0.3)-0.3-0.5}{(0.2-0.3)(0.2-0.5)} + 0.56\frac{2(0.3)-0.2-0.5}{(0.3-0.2)(0.3-0.5)} + 0.84\frac{2(0.3)-0.2-0.3}{(0.5-0.2)(0.5-0.3)} = 2.066667$$

We can then multiply these derivative estimates by the inductance. All the results are summarized below:

t	*i*	*di/dt*	V_L
0	0	1.6	6.4
0.1	0.16	1.6	6.4
0.2	0.32	2	8
0.3	0.56	2.066667	8.266667
0.5	0.84	3.6	14.4
0.7	2	8	32

21.27 We can solve Faraday's law for inductance as

$$L = \frac{\int_0^t V_L \, dt}{i}$$

We can evaluate the integral using a combination of the trapezoidal and Simpson's rules,

$$I = (20-0)\frac{0+4(18)+29}{6} + (80-20)\frac{29+3(44+49)+46}{8} + (120-80)\frac{46+35}{2}$$
$$+(180-120)\frac{35+26}{2} + (280-180)\frac{26+15}{2} + (400-280)\frac{15+7}{2}$$
$$= 336.6667 + 2655 + 1620 + 1830 + 2050 + 1320 = 9811.667$$

The inductance can then be computed as

$$L = \frac{9811.667}{2} = 4905.833\frac{\text{volts ms}}{\text{A}}\frac{\text{s}}{1000\text{ ms}} = 4.905833\text{ H}$$

21.28 Because the data is equispaced, we can use the second-order finite divided difference formulas from Figs. 21.3 through 21.5. For the first point, we can use

$$\frac{dT}{dt} = \frac{-30+4(44.5)-3(80)}{10-0} = -9.2$$

For the intermediate points, we can use centered differences. For example, for the second point

$$\frac{dT}{dt} = \frac{30-80}{10-0} = -5$$

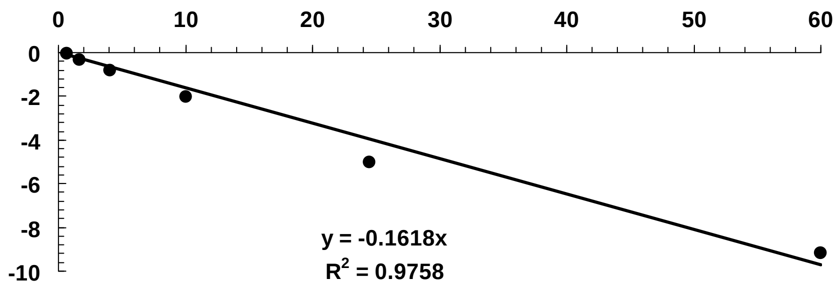
We can analyze the other interior points in a similar fashion. For the last point, we can use a backward difference

$$\frac{dT}{dt} = \frac{3(20.7)-4(21.7)+24.1}{25-15} = -0.06$$

All the values can be tabulated as

t	T	$T - T_a$	dT/dt
0	80	60	-9.2
5	44.5	24.5	-5
10	30	10	-2.04
15	24.1	4.1	-0.83
20	21.7	1.7	-0.34
25	20.7	0.7	-0.06

If Newton's law of cooling holds, we can plot dT/dt versus $T - T_a$ and the points can be fit with a linear regression with zero intercept to estimate the cooling rate. As in the following plot, the result is k = 0.1618/min.



21.29 The following script solves the problem. Note that the derivative is calculated with a centered difference,

$$\frac{dV}{dT} = \frac{V_{450K} - V_{350K}}{100K}$$

The following script evaluates the derivative with centered finite differences and the integral with the `trapz` function,

```
V=[220 250 282.5;4.1 4.7 5.23;2.2 2.5 2.7;1.35 1.49 1.55;1.1 1.2 1.24;.9 .99
1.03;.68 .75 .78;.61 .675 .7;.54 .6 .62];
P=[0.1 5 10 20 25 30 40 45 50]';
T=[350 400 450]';
n=length(V);
dVdt=(V(1:n,3)-V(1:n,1))/(T(3)-T(1));
integrand=V(1:n,2)-T(2)*dVdt;
H=trapz(P,integrand)
```

When the script is run the result is H = 21.4410. The following table displays all the results

P,atm	T=350K	T=400K	T=450K	dVdT	(V - T (dV/dT)p)	Trap
0.1	220	250	282.5	0.625	0	
5	4.1	4.7	5.23	0.0113	0.18	0.441
10	2.2	2.5	2.7	0.005	0.5	1.7
20	1.35	1.49	1.55	0.002	0.69	5.95
25	1.1	1.2	1.24	0.0014	0.64	3.325
30	0.9	0.99	1.03	0.0013	0.47	2.775
40	0.68	0.75	0.78	0.001	0.35	4.1
45	0.61	0.675	0.7	0.0009	0.315	1.6625
50	0.54	0.6	0.62	0.0008	0.28	1.4875
					Total Integral =	**21.441**

21.30 (a) First, the distance can be converted to meters. Then, Eq. (21.21) can be used to compute the derivative at the surface as

$x_0 = 0$ $f(x_0) = 900$
$x_1 = 0.01$ $f(x_1) = 480$
$x_2 = 0.03$ $f(x_2) = 270$

$$f'(0) = 900\frac{2(0)-0.01-0.03}{(0-0.01)(0-0.03)} + 480\frac{2(0)-0-0.03}{(0.01-0)(0.01-0.03)} + 270\frac{2(0)-0-0.01}{(0.03-0)(0.03-0.01)} = -52,500\ \frac{\text{K}}{\text{m}}$$

The heat flux can be computed as

$$\text{Heat flux} = -0.028\frac{\text{J}}{\text{s}\cdot\text{m}\cdot\text{K}}\left(-52,500\frac{\text{K}}{\text{m}}\right) = 1,470\frac{\text{W}}{\text{m}^2}$$

(b) The heat transfer can be computed by multiplying the flux by the area

$$\text{Heat transfer} = 1,470\frac{\text{W}}{\text{m}^2}(200\text{ cm}\times 50\text{ cm})\frac{\text{m}^2}{10{,}000\text{ cm}^2} = 1,470\text{ W}$$

21.31 (a) The pressure drop can be determined by integrating the pressure gradient

$$p = \int_{x_1}^{x_2} -\frac{8\mu Q}{\pi r(x)^4}\,dx$$

After converting the units to meters, a table can be set up holding the data and the integrand. The trapezoidal and Simpson's rules can then be used to integrate this data as shown in the last column of the table.

***x*, m**	***r*, m**	**integrand**	**integral**	**method**
0	0.002	-7957.75		
0.02	0.00135	-38333.2		
0.04	0.00134	-39490.3	-1338.5392	Simp 1/3
0.05	0.0016	-19428.1		
0.06	0.00158	-20430.6		
0.07	0.00142	-31315.3	-713.9319	Simp 3/8
0.1	0.002	-7957.75	-589.0959	Trap
		Sum→	**-2641.5670**	

Therefore, the pressure drop is computed as –2,641.567 N/m^2.

(b) The average radius can also be computed by integration as

$$\bar{r} = \frac{\int_{x_1}^{x_2} r(x)\,dx}{x_2 - x_1}$$

The numerical evaluations can again be determined by a combination of the trapezoidal and Simpson's rules.

***x*, m**	***r*, m**	**integral**	**method**
0	0.00200		
0.02	0.00135		
0.04	0.00134	5.83E-05	Simp 1/3

0.05	0.00160		
0.06	0.00158		
0.07	0.00142	4.61E-05	Simp 3/8
0.1	0.00200	5.13E-05	Trap
	Sum→	**0.0001557**	

Therefore, the average radius is 0.0001557/0.1 = 0.001557 m. This value can be used to compute a pressure drop of

$$\Delta p = \frac{dp}{dx}\Delta x = -\frac{8\mu Q}{\pi r^4}\Delta x = -\frac{8(0.005)0.00001}{\pi(0.001557)^4}0.1 = -2166.95\frac{\text{N}}{\text{m}^2}$$

Thus, there is less pressure drop if the radius is at the constant mean value.

(c) The average Reynolds number can be computed by first determining the average velocity as

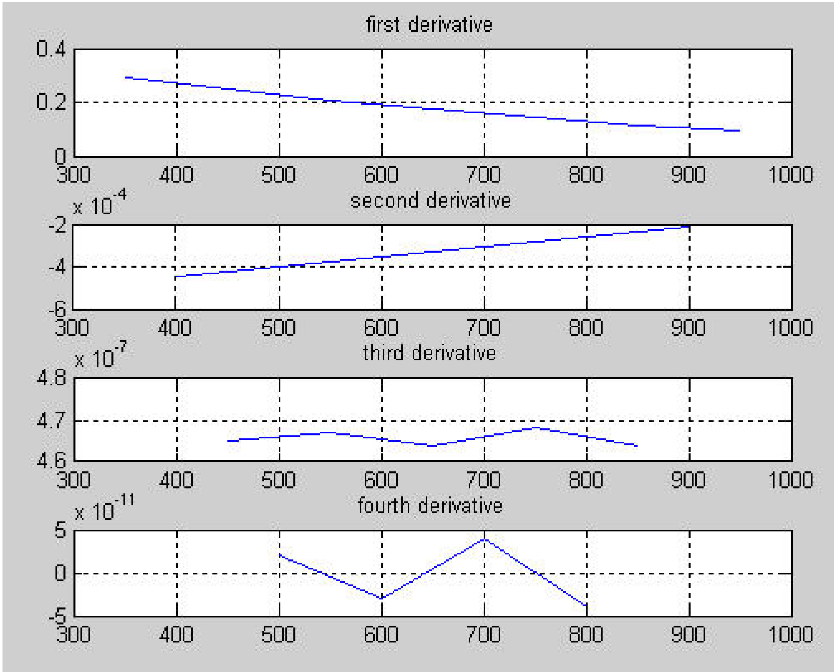
$$v = \frac{Q}{A_c} = \frac{0.00001}{\pi(0.001557)^2} = \frac{0.00001}{7.6152\times 10^{-6}} = 1.31317\frac{\text{m}}{\text{s}}$$

Then the Reynolds number can be computed as

$$\text{Re} = \frac{1\times 10^3(1.31317)0.0031138}{0.005} = 817.796$$

21.32 Using an approach based on Example 21.4, the following script determines first through fourth derivative estimates by taking differences of differences. A plot of the results is also generated.

```
T=[300 400 500 600 700 800 900 1000];
Cp=[82.888 112.136 136.933 157.744 175.036 189.273 200.923 210.450];
dCp=diff(Cp)./diff(T);
n=length(T);
Tm=(T(1:n-1)+T(2:n))./2;
dCp2=diff(dCp)./diff(Tm);
n2=length(Tm);
Tm2=(Tm(1:n2-1)+Tm(2:n2))./2;
dCp3=diff(dCp2)./diff(Tm2);
n3=length(Tm2);
Tm3=(Tm2(1:n3-1)+Tm2(2:n3))./2;
dCp4=diff(dCp3)./diff(Tm3);
n4=length(Tm3);
Tm4=(Tm3(1:n4-1)+Tm3(2:n4))./2;
subplot(4,1,1),plot(Tm,dCp)
grid,xlim([min(T) max(T)]),title('first derivative')
subplot(4,1,2),plot(Tm2,dCp2)
grid,xlim([min(T) max(T)]),title('second derivative')
subplot(4,1,3),plot(Tm3,dCp3)
grid,xlim([min(T) max(T)]),title('third derivative')
subplot(4,1,4),plot(Tm4,dCp4)
grid,xlim([min(T) max(T)]),title('fourth derivative')
```



The fact that (1) the second derivative appears linear, (2) the third derivative is close to being constant, and (3) the fourth derivative oscillates in a random fashion close to zero suggests that the data was generated with a fourth-order polynomial (which in fact was the case).

21.33 Equation (21.21) can be used to estimate the derivatives at each temperature. This can be done using the M-file developed in Prob. 21.9 with the result

```
>> T=[750 800 900 1000];
>> h=[29629 32179 37405 42769];
>> cp=diffuneq(T,h)

cp =
   50.5800   51.4200   52.9500   54.3300
```

These results, which have units of kJ/(kmol K), can be converted to the desired units as in

$$c_p\left(\frac{\text{J}}{\text{kg K}}\right) = c_p\left(\frac{\text{kJ}}{\text{kmol K}}\right) \times \frac{\text{kmol}}{10^3 \text{ mol}} \times \frac{10^3\text{J}}{\text{kJ}} \times \frac{\text{mol}}{(12.011 + 2(15.9994)) \text{ g}} \times \frac{10^3\text{g}}{\text{kg}}$$

```
>> cp=cp*1e3/(12.011+2*15.9994)

cp =
  1.0e+003 *
    1.1493    1.1684    1.2031    1.2345
```

21.34 The following script determines dc/dt using an M-file that implements Eq. (21.21). Linear regression is then used to fit the log transformed equation. The best-fit parameters are then used to compute the model coefficients n and k with the following script:

```
t=[0 5 15 30 45];
c=[0.750 0.594 0.420 0.291 0.223];
dcdt=diffuneq(t,c)
p=polyfit(log10(c),log10(-dcdt),1)
```

```
n=p(1)
k=10^p(2)

>> prob1934

dcdt =
   -0.0358   -0.0266   -0.0139   -0.0066   -0.0025
p =
    2.1419   -1.1121
n =
    2.1419
k =
    0.0773
```

21.35 Finite-divided differences can be used to numerically estimate the derivatives. For the first time (t = 0), a forward difference of $O(h^2)$ can be employed to give

$$\frac{do}{dt}(0) = \frac{-3(10) + 4(7.11) - 4.59}{0.25} = -24.6 \frac{\text{mg}}{\text{L} \cdot \text{d}}$$

Centered differences of $O(h^2)$ can be used for the interior data. For example, for the second interval,

$$\frac{do}{dt}(0.125) = \frac{4.59 - 10}{0.25} = -21.64 \frac{\text{mg}}{\text{L} \cdot \text{d}}$$

A backward difference of $O(h^2)$ can be employed for the last time (t = 0.75) to give

$$\frac{do}{dt}(0.75) = \frac{3(0.03) - 4(0.33) + 1.15}{0.25} = -0.32 \frac{\text{mg}}{\text{L} \cdot \text{d}}$$

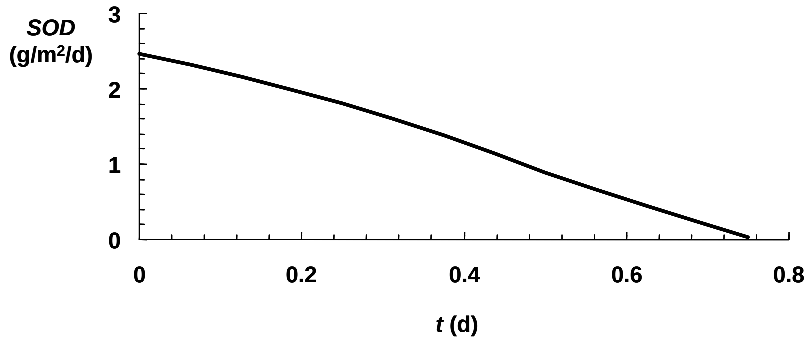
The SODs can then be computed with the formula from the problem statement. For example, for the first point,

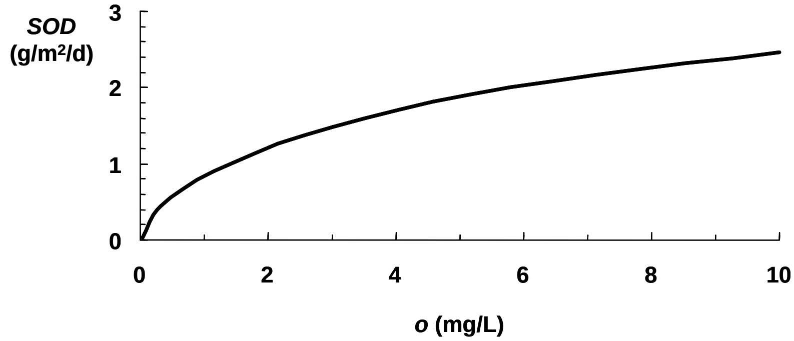
$$SOD = -H \frac{do}{dt} = -0.1(-24.6) = 2.46 \frac{\text{g}}{\text{m}^2\text{d}}$$

All the results can be tabulated along with the SOD:

***t* (d)**	***o* (mg/L)**	***do/dt***	**SOD**
0	10.00	-24.60	2.460
0.125	7.11	-21.64	2.164
0.25	4.59	-18.16	1.816
0.375	2.57	-13.76	1.376
0.5	1.15	-8.96	0.896
0.625	0.33	-4.48	0.448
0.75	0.03	-0.32	0.032

The plots can be developed as





21.36 (a) Displacement can be determined by integrating the slope equation (P21.35),

$$y(x) = \frac{w_0}{120EIL}\int_0^x -5x^4 + 6L^2x^2 - L^4 \; dx$$

Because this is a polynomial, the exact solution can be evaluated analytically as

$$y = \frac{w_0}{120EIL}\left[-x^5 + 2L^2x^3 - L^4x\right] = -0.0000115741x^5 + 0.000208333x^3 - 0.0009375x$$

Numerical solutions can be obtained with the trapezoidal. For example, the displacement at $x = 0.125$ can be computed as

$$\theta(0) = \frac{250000}{120(2\times10^{11})0.0003(3)}\left[-5(0)^4 + 2(3)^2(0)^2 - (3)^4\right] = -9.375\times10^{-4}$$

$$\theta(0.125) = \frac{250000}{120(2\times10^{11})0.0003(3)}\left[-5(0.125)^4 + 2(3)^2(0.125)^2 - (3)^4\right] = -9.27749\times10^{-4}$$

$$y(0.125) = (0.125-0)\frac{-9.375\times10^{-4} - 9.27749\times10^{-4}}{2} = -0.000116578 \text{ m}$$

At $x = 0.25$:

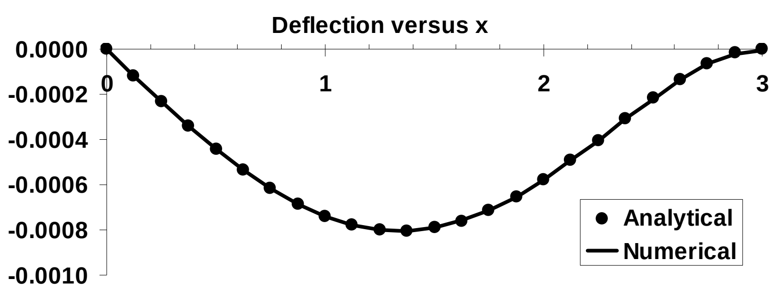
$$\theta(0.25) = \frac{250000}{120(2\times10^{11})0.0003(3)}\left[-5(0.25)^4 + 2(3)^2(0.25)^2 - (3)^4\right] = -8.98664\times10^{-4}$$

$$y(0.25) = -0.000116578 + (0.25-0.125)\frac{9.27749\times10^{-4} - 8.98664\times10^{-4}}{2} = -0.000116578 - 0.000114151 = -0.000230729$$

The remainder of the calculation along with the analytical solution is summarized in the following table and graph:

x	θ	y-analytical	Trap Rule	$y = \Sigma$ Trap Rule
0	-0.0009375	0.0000000	0.0000000	0.0000000
0.125	-0.0009277	-0.0001168	-0.0001166	-0.0001166
0.25	-0.0008987	-0.0002311	-0.0001142	-0.0002307
0.375	-0.0008508	-0.0003407	-0.0001093	-0.0003401
0.5	-0.0007849	-0.0004431	-0.0001022	-0.0004423
0.625	-0.0007022	-0.0005362	-0.0000929	-0.0005352
0.75	-0.0006042	-0.0006180	-0.0000817	-0.0006169
0.875	-0.0004929	-0.0006867	-0.0000686	-0.0006855
1	-0.0003704	-0.0007407	-0.0000540	-0.0007394
1.125	-0.0002392	-0.0007789	-0.0000381	-0.0007775
1.25	-0.0001022	-0.0008003	-0.0000213	-0.0007988
1.375	0.0000373	-0.0008044	-0.0000041	-0.0008029
1.5	0.0001758	-0.0007910	0.0000133	-0.0007896
1.625	0.0003094	-0.0007606	0.0000303	-0.0007593
1.75	0.0004338	-0.0007141	0.0000464	-0.0007128
1.875	0.0005445	-0.0006527	0.0000611	-0.0006517
2	0.0006366	-0.0005787	0.0000738	-0.0005779
2.125	0.0007047	-0.0004946	0.0000838	-0.0004940
2.25	0.0007434	-0.0004037	0.0000905	-0.0004035
2.375	0.0007466	-0.0003102	0.0000931	-0.0003104
2.5	0.0007082	-0.0002188	0.0000909	-0.0002195
2.625	0.0006214	-0.0001352	0.0000831	-0.0001364
2.75	0.0004794	-0.0000658	0.0000688	-0.0000676
2.875	0.0002748	-0.0000179	0.0000471	-0.0000204
3	0.0000000	0.0000000	0.0000172	-0.0000033

The following plot indicates good agreement between the analytical and numerical results. Note that we could improve the results by using a finer step size (i.e., a smaller value of Δx), a more refined numerical integration formula (Simpson's 1/3 rule) or a higher accuracy method (e.g., Romberg integration or the MATLAB `quad` function).



(b) The moment can also be evaluated analytically:

$$M(x) = EI\frac{d\theta}{dx} = \frac{w_0}{120L}(-20x^3 + 12L^2 x)$$

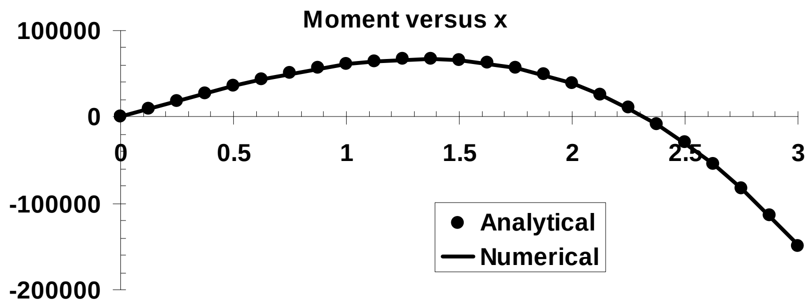
Numerical solutions can be obtained with finite differences. For example, the moment at $x = 0$ can be computed with a forward finite difference estimate for the first derivative (Table 21.3).

$$M(0) = EI\frac{d\theta}{dx} = 2\times10^{11}(0.0003)\frac{-3(-9.375\times10^{-4}) + 4(-9.277\times10^{-4}) - (-8.987\times10^{-4})}{2(0.125)} = 40.69$$

Centered differences (Table 21.5) can then be used for the interior points and a backward difference (Table 21.4) for x = 3. For example, for x = 0.125,

$$M(0.125) = 2\times10^{11}(0.0003)\frac{-8.987\times10^{-4}-(-9.375\times10^{-4})}{2(0.125)} = 9320.75$$

The remainder of the calculation along with the analytical solution is summarized in the following graph and the table at the end of this problem solution.



(c) The shear can also be evaluated analytically:

$$V(x) = EI\frac{d^2\theta}{dx^2} = \frac{w_0}{120L}(-60x^2+12L^2)$$

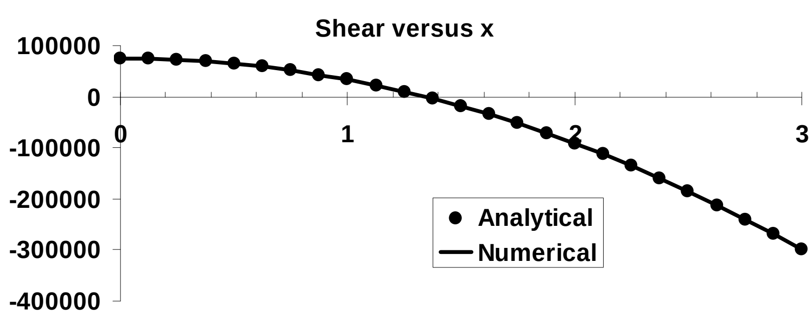
Numerical solutions can be obtained with finite differences. For example, the shear at x = 0 can be computed with a forward finite difference estimate for the second derivative (Table 21.3)

$$V(0) = 2\times10^{11}(0.0003)\frac{-(-8.508\times10^{-4})+4(-8.987\times10^{-4})-5(-9.277\times10^{-4})+2(-9.375\times10^{-4})}{(0.125)^2} = 76193.58$$

Centered differences (Table 21.5) can then be used for the interior points and a backward difference (Table 21.4) for x = 3. For example, for x = 0.125,

$$V(0) = 2\times10^{11}(0.0003)\frac{-8.987\times10^{-4}-2(-9.277\times10^{-4})-9.375\times10^{-4}}{(0.125)^2} = 74240.45$$

The remainder of the calculation along with the analytical solution is summarized in the following graph and the table below.



x	M-anal	M-num	V-anal	V-num
0	0.0	40.7	75000.0	76193.6
0.125	9347.9	9320.7	74349.0	74240.5
0.25	18533.0	18478.7	72395.8	72287.3
0.375	27392.6	27311.2	69140.6	69032.1
0.5	35763.9	35655.4	64583.3	64474.8
0.625	43484.2	43348.5	58724.0	58615.5
0.75	50390.6	50227.9	51562.5	51454.0
0.875	56320.5	56130.6	43099.0	42990.5
1	61111.1	60894.1	33333.3	33224.8
1.125	64599.6	64355.5	22265.6	22157.1
1.25	66623.3	66352.0	9895.8	9787.3
1.375	67019.3	66720.9	-3776.0	-3884.5
1.5	65625.0	65299.5	-18750.0	-18858.5
1.625	62277.6	61924.9	-35026.0	-35134.5
1.75	56814.2	56434.5	-52604.2	-52712.7
1.875	49072.3	48665.4	-71484.4	-71592.9
2	38888.9	38454.9	-91666.7	-91775.2
2.125	26101.3	25640.2	-113151.0	-113259.5
2.25	10546.9	10058.6	-135937.5	-136046.0
2.375	-7937.3	-8452.7	-160026.0	-160134.5
2.5	-29513.9	-30056.4	-185416.7	-185525.2
2.625	-54345.7	-54915.4	-212109.4	-212217.9
2.75	-82595.5	-83192.3	-240104.2	-240212.7
2.875	-114426.0	-115049.9	-269401.0	-269509.5
3	-150000.0	-148738.6	-300000.0	-298806.4

The above results indicate good agreement between the analytical and numerical results. We could improve the results by using a finer step size (i.e., a smaller value of Δx).

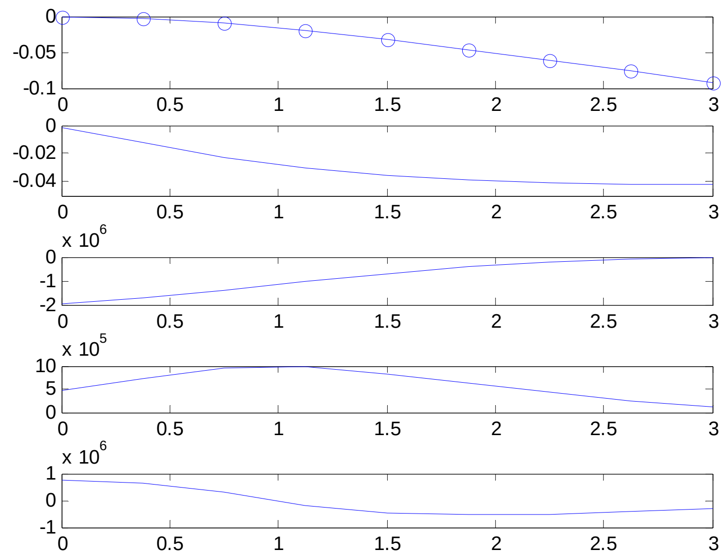
21.37 **Script:**

```
clear,clc,clf
x=[0 0.375 0.75 1.125 1.5 1.875 2.25 2.625 3];
y=[0 -0.2571 -0.9484 -1.9689 -3.2262 -4.6414 -6.1503 -7.7051 -9.275];
y=y/100;
E=200e9;I=0.0003;
[dydx,d2ydx2,d3ydx3,d4ydx4]=DiffSpline(x,y);
M=E*I*d2ydx2;V=E*I*d3ydx3;wx=E*I*d4ydx4;
z=[x' y' dydx M V wx]';
fprintf('     x          y         dy/dx      M(x)       V(x)      d4y/dx4\n');
fprintf('%10.4f %10.4f %10.4f %10.1f %10.1f %10.1f\n',z);
```

Supporting functions:

```
function [dydx,d2ydx2,d3ydx3,d4ydx4]=DiffSpline(x,y)
clf
if any(diff(diff(x))~=0), error('unequal spacing'), end
if any(diff(x)<=0), error('not in ascending order'), end
subplot(5,1,1)
plot(x,y,'o',x,y)
dydx=DiffNum(x,y);
subplot(5,1,2)
plot(x,dydx)
d2ydx2=DiffNum(x,dydx);
subplot(5,1,3)
plot(x,d2ydx2)
d3ydx3=DiffNum(x,d2ydx2);
subplot(5,1,4)
plot(x,d3ydx3)
d4ydx4=DiffNum(x,d3ydx3);
subplot(5,1,5)
```

```
plot(x,d4ydx4)

function dydx=DiffNum(x,y)
n=length(x);
dydx=zeros(n);dx=x(2)-x(1);
dydx(1)=(-3*y(1)+4*y(2)-y(3))/(2*dx);
for i = 2:n-1
  dydx(i)=(y(i+1)-y(i-1))/(2*dx);
end
dydx(n)=(3*y(n)-4*y(n-1)+y(n-2))/(2*dx);
dydx=dydx(:,1);
```



21.38 (a)

$$\frac{\partial f}{\partial x} = 3y + 3 - 3x^2 = 3(1) + 3 - 3(1)^2 = 3$$

$$\frac{\partial f}{\partial y} = 3x - 9y^2 = 3(1) - 9(1)^2 = -6$$

$$\frac{\partial f}{\partial x \partial y} = 3$$

(b)

$$\frac{\partial f}{\partial x} = \frac{f(x+\Delta x, y) - f(x-\Delta x, y)}{2\Delta x} = \frac{f(1.0001,1) - f(0.9999,1)}{2\Delta x}$$

$$= \frac{[3(1.0001)1 + 3(1.0001) - (1.0001)^3 - 3(1)^3] - [3(0.9999)1 + 3(0.9999) - (0.9999)^3 - 3(1)^3]}{0.0002}$$

$$= \frac{2.00029997 - 1.99969997}{0.0002} = 2.99999999$$

$$\frac{\partial f}{\partial y} = \frac{f(x, y+\Delta y) - f(x, y-\Delta y)}{2\Delta y} = \frac{f(1,1.0001) - f(1,0.9999)}{2\Delta y}$$

$$= \frac{[3(1)1.0001 + 3(1) - (1)^3 - 3(1.0001)^3] - [3(1)0.9999 + 3(1) - (1)^3 - 3(0.9999)^3]}{0.0002}$$

$$= \frac{1.99939991 - 2.00059991}{0.0002} = -6.00000003$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y-\Delta y) - f(x-\Delta x, y+\Delta y) + f(x-\Delta x, y-\Delta y)}{4\Delta x \Delta y}$$

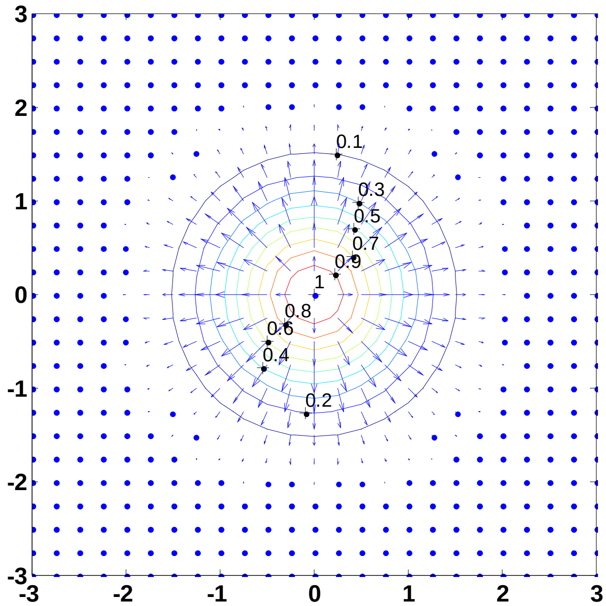
$$= \frac{f(1.0001, 1.0001) - f(1.0001, 0.9999) - f(0.9999, 1.0001) + f(0.9999, 0.9999)}{4\Delta x \Delta y}$$

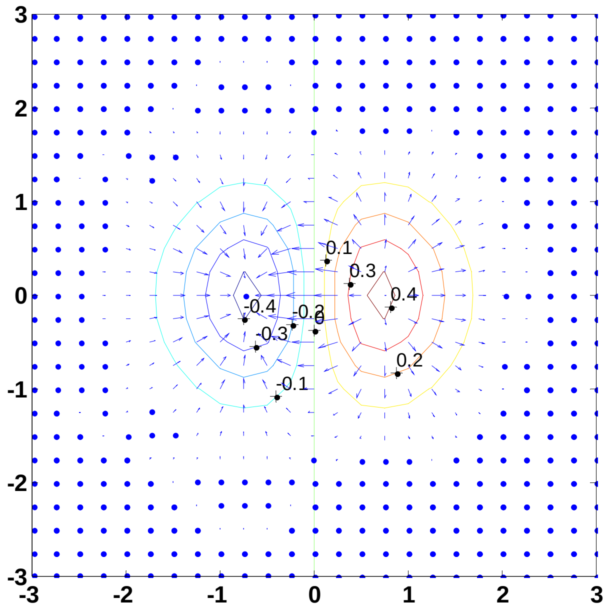
$$= \frac{1.99969991 - 2.00089985 - 1.99909985 + 2.00029991}{4(0.0001)(0.0001)} = 2.999999982$$

21.39 **Script:**

```
clear,clc,clf
f=@(x,y) exp(-(x.^2+y.^2));
[x,y]=meshgrid(-3:0.25:3,-3:0.25:3);
z=f(x,y);
[fx,fy]=gradient(z,0.25);
cs=contour(x,y,z);clabel(cs)
hold on
quiver(x,y,-fx,-fy);axis square
hold off
pause
f=@(x,y) x.*exp(-(x.^2+y.^2));
[x,y]=meshgrid(-3:0.25:3,-3:0.25:3);
z=f(x,y);
[fx,fy]=gradient(z,0.25);
cs=contour(x,y,z);clabel(cs);hold on
hold on
quiver(x,y,-fx,-fy);axis square
hold off
```

Output:

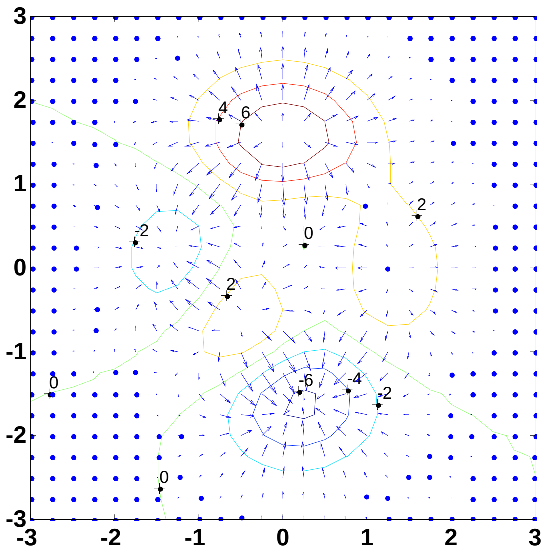




21.40 Script:

```
clear,clc,clf
[x,y]=meshgrid(-3:0.25:3,-3:0.25:3);
z=peaks(x,y);
[fx,fy]=gradient(z,0.25);
cs=contour(x,y,z);clabel(cs)
hold on
quiver(x,y,-fx,-fy);axis square
hold off
```

Result:



3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3
4
6
-2
0
2
2
-6
-4
-2
0
0