Chapter 13

Eigenvalue

Numerical Methods
Fall 2019

Mathematics, 1

- Up until now, heterogeneous systems:

$$[A]\,\{x\} = \{b\}$$

- What about homogeneous systems:

$$[A]\,\{x\} = 0$$

- Trivial solution:

$$\{x\} = 0$$

- Is there another way of formulating the system so that the solution would be meaningful???

Mathematics, 2

- What about a homogeneous system like:

$$
\begin{aligned}
(a_{11}-\lambda)x_1 + \quad a_{12}x_2 + \quad a_{13}x_3 &= 0 \\
a_{21}x_1 + (a_{22}-\lambda)x_2 + \quad a_{23}x_3 &= 0 \\
a_{31}x_1 + \quad a_{32}x_2 + (a_{33}-\lambda)x_3 &= 0
\end{aligned}
$$

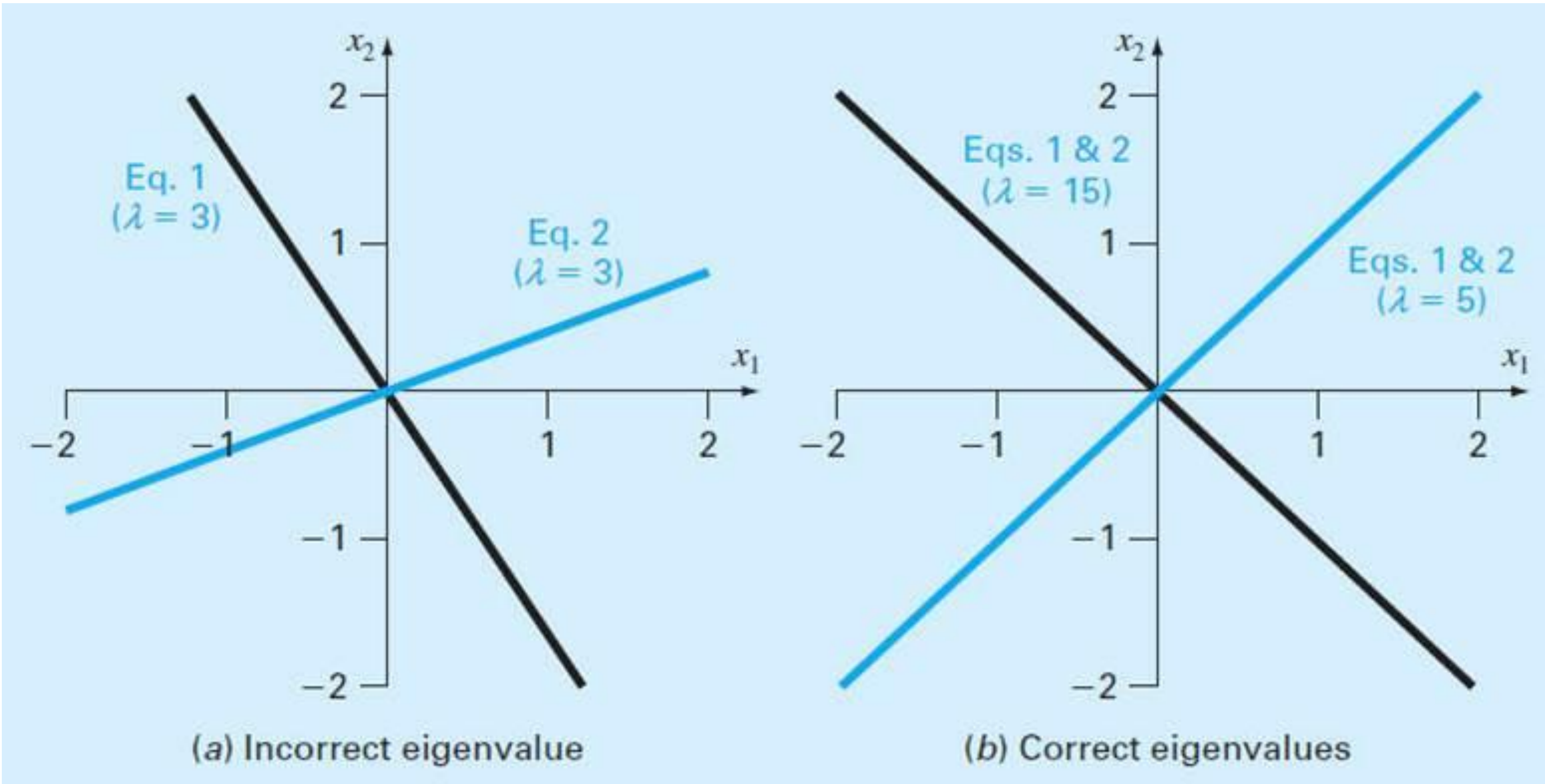
- or in matrix form

$$[[A]-\lambda[I]]\{x\} = 0$$

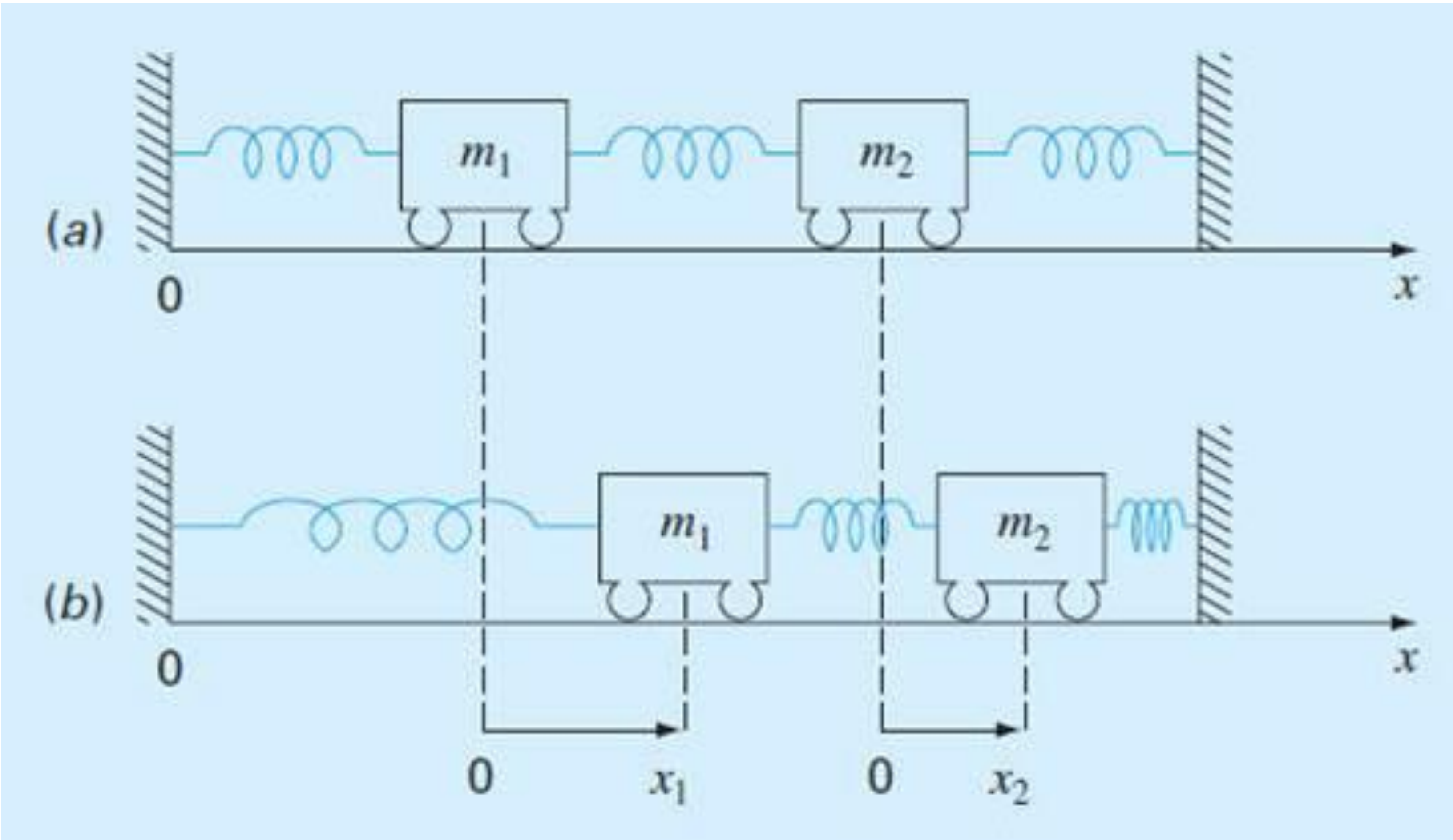
- For this case, there could be a value of λ that makes the equations equal zero. This is called an ***eigenvalue***.

Example 13.1

Graphical Depiction of Eigenvalues



Physical Background:
Oscillations or *Vibrations* of Mass–Spring Systems



Model With Force Balances $(F = ma)$

$$m_1 \frac{d^2 x_1}{dt^2} = -k x_1 + k(x_2 - x_1)$$
$$m_2 \frac{d^2 x_2}{dt^2} = -k(x_2 - x_1) - k x_2$$

- Collect terms:

$$m_1 \frac{d^2 x_1}{dt^2} - k(-2x_1 + x_2) = 0$$
$$m_2 \frac{d^2 x_2}{dt^2} - k(x_1 - 2x_2) = 0$$

Assume a Sinusoidal Solution

$$x_i = X_i \sin(\omega t) \qquad \text{where} \qquad \omega = \frac{2\pi}{T_p}$$

- Differentiate twice:

$$x_i'' = -X_i \omega^2 \sin(\omega t)$$

- Substitute back into system and collect terms

$$\left(\frac{2k}{m_1} - \omega^2\right)X_1 \qquad - \frac{k}{m_1}X_2 = 0$$

$$\frac{k}{m_2}X_1 + \left(\frac{2k}{m_2} - \omega^2\right)X_2 = 0$$

- Given: $m_1 = m_2 = 40$ kg; $k = 200$ N/m

$$(10 - \omega^2)X_1 - \qquad 5X_2 = 0$$

$$-5X_1 + (10 - \omega^2)X_2 = 0$$

- This is now a homogeneous system where the eigenvalue represents the square of the fundamental frequency.

Solution: The Polynomial Method

$$\begin{bmatrix} 10-\omega^2 & -5 \\ -5 & 10-\omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

- Evaluate the determinant to yield a polynomial

$$\begin{vmatrix} 10-\omega^2 & -5 \\ -5 & 10-\omega^2 \end{vmatrix} = (\omega^2)^2 - 20\omega^2 + 75$$

- The two roots of this "characteristic polynomial" are the system's eigenvalues:

$$\omega^2 = \begin{matrix} 15 \\ 5 \end{matrix} \qquad \text{or} \qquad \omega = \begin{matrix} 3.873\ \text{Hz} \\ 2.36\ \text{Hz} \end{matrix}$$

INTERPRETATION

$$\omega^2 = 5/s^2$$
$$\omega = 2.236/s$$
$$T_p = 2\pi/2.236 = 2.81\text{ s}$$

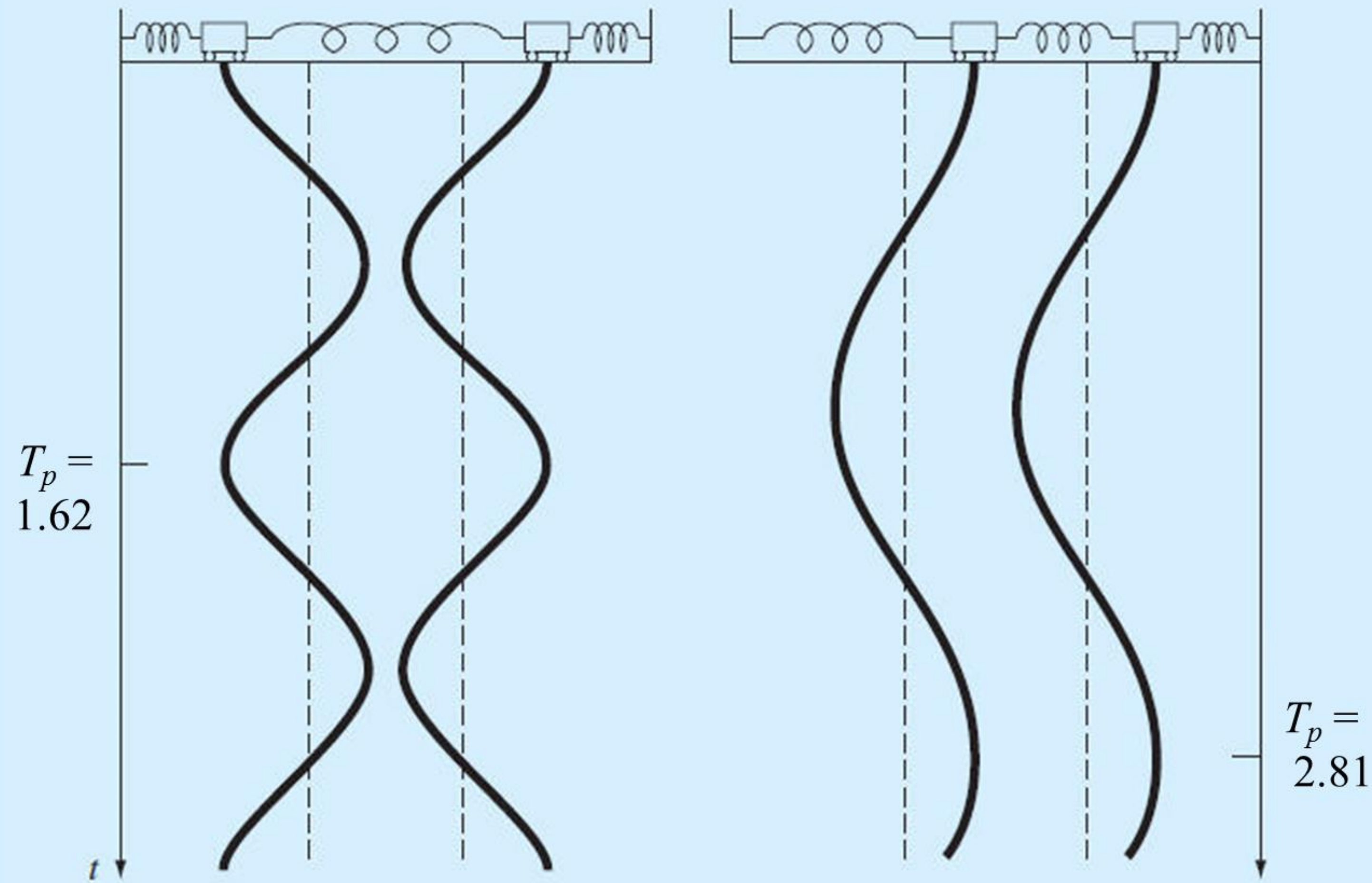
$$\omega^2 = 15/s^2$$
$$\omega = 3.873/s$$
$$T_p = 2\pi/3.373 = 1.62\text{ s}$$

$$(10-\omega^2)X_1 - \quad 5\,X_2 = 0$$
$$-5\,X_1 + (10-\omega^2)\,X_2 = 0$$

$$(10-5)\,X_1 - \quad 5\,X_2 = 0$$
$$-5\,X_1 + (10-5)\,X_2 = 0$$
$$5\,X_1 - 5\,X_2 = 0$$
$$-5\,X_1 + 5\,X_2 = 0$$
$$X_1 = X_2$$
$$X = \begin{pmatrix} -0.7071 \\ -0.7071 \end{pmatrix}$$

$$(10-15)\,X_1 - \quad 5\,X_2 = 0$$
$$-5\,X_1 + (10-15)\,X_2 = 0$$
$$-5\,X_1 - 5\,X_2 = 0$$
$$-5\,X_1 - 5\,X_2 = 0$$
$$X_1 = -X_2$$
$$X = \begin{pmatrix} -0.7071 \\ 0.7071 \end{pmatrix}$$

Principle Modes of Vibration, 1



(*a*) First mode

(*b*) Second mode

The Power Method

- Iterative method to compute the largest eigenvalue ***and*** its associated eigenvector.

$$\big[[A]-\lambda[I]\big]\{x\}=0$$
$$[A]\{x\}=\lambda\{x\}$$

- Simple Algorithm:

```
function [eval, evect] = powereig(A,es,maxit)
n=length(A);
evect=ones(n,1);eval=1;iter=0;ea=100; %initialize
while(1)
  evalold=eval;                %save old eigenvalue value
  evect=A*evect;               %determine eigenvector as
[A]*{x}
  eval=max(abs(evect));        %determine new eigenvalue
  evect=evect./eval;           %normalize eigenvector to
eigenvalue
  iter=iter+1;
  if eval~=0, ea = abs((eval-evalold)/eval)*100; end
  if ea<=es | iter >= maxit,break,end
end
```

Example: The Power Method, 1

- First iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ 20 \end{Bmatrix} = 20 \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

- Second iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 40 \\ -20 \\ 40 \end{Bmatrix} = 40 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

$$|\varepsilon_a| = \left| \frac{40 - 20}{40} \right| \times 100\% = 50\%$$

Example: The Power Method, 2

- Third iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 60 \\ -80 \\ 60 \end{Bmatrix} = -80 \begin{Bmatrix} -0.75 \\ 1 \\ -0.75 \end{Bmatrix}$$

$$|\varepsilon_a| = \left| \frac{-80 - 40}{-80} \right| \times 100\% = 150\%$$

- Fourth iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} -0.75 \\ 1 \\ -0.75 \end{Bmatrix} = \begin{Bmatrix} -50 \\ 75 \\ -50 \end{Bmatrix} = 70 \begin{Bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{Bmatrix}$$

$$|\varepsilon_a| = \left| \frac{70 - (-80)}{70} \right| \times 100\% = 214\%$$

Example: The Power Method, 3

- Fifth iteration:

$$\begin{bmatrix} 40 & -20 & 0 \\ -20 & 40 & -20 \\ 0 & -20 & 40 \end{bmatrix} \begin{Bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{Bmatrix} = \begin{Bmatrix} -48.51714 \\ 68.51714 \\ -48.51714 \end{Bmatrix}$$

$$= 68.51714 \begin{Bmatrix} -0.71429 \\ 1 \\ -0.71429 \end{Bmatrix}$$

$$|\varepsilon_a| = \left| \frac{68.51714 - 70}{70} \right| \times 100\% = 2.08\%$$

- The process can be continued to determine the largest eigenvalue (= 68.284) with the associated eigenvector [–0.7071 1 –0.7071]

- Note that the smallest eigenvalue and its associate d eigenvector can be determined by applying the p ower method to the inverse of A

Determining Eigenvalues & Eigenvectors with MATLAB

```
>> A = [10 -5;-5 10]

A =
    10    -5
    -5    10

>> [v,lambda] = eig(A)

v =
   -0.7071   -0.7071
   -0.7071    0.7071
lambda =
     5     0
     0    15
```