Homework #4 (Ch10, Ch11, Ch12, & Ch13) Solution

Chapter 9, p270, PROBLEMS

Prob. 9.1

9.1 The flop counts for the tridiagonal algorithm in Fig. 9.6 can be summarized as

	Mult/Div	Add/Subtr	Total
Forward elimination	$3(n-1)$	$2(n-1)$	$5(n-1)$
Back substitution	$2n-1$	$n-1$	$3n-2$
Total	$5n-4$	$3n-3$	$\mathbf{8n-7}$

Thus, as n increases, the effort is much, much less than for a full matrix solved with Gauss elimination which is proportional to n^3.

9.4 (a) The determinant can be evaluated as

$$D = 0\begin{vmatrix} 2 & -1 \\ -2 & 0 \end{vmatrix} - (-3)\begin{vmatrix} 1 & -1 \\ 5 & 0 \end{vmatrix} + 7\begin{vmatrix} 1 & 2 \\ 5 & -2 \end{vmatrix}$$
$$D = 0(-2) + 3(5) + 7(-12) = -69$$

(b) Cramer's rule

$$x_1 = \frac{\begin{vmatrix} 4 & -3 & 7 \\ 0 & 2 & -1 \\ 3 & -2 & 0 \end{vmatrix}}{-69} = 0.5942 \qquad x_2 = \frac{\begin{vmatrix} 0 & 4 & 7 \\ 1 & 0 & -1 \\ 5 & 3 & 0 \end{vmatrix}}{-69} = -0.0145 \qquad x_3 = \frac{\begin{vmatrix} 0 & -3 & 4 \\ 1 & 2 & 0 \\ 5 & -2 & 3 \end{vmatrix}}{-69} = 0.5652$$

(c) Pivoting is necessary, so switch the first and third rows,

$$\begin{aligned} 5x_1 - 2x_2 \qquad &= 3 \\ x_1 + 2x_2 - x_3 &= 0 \\ -3x_2 + 7x_3 &= 4 \end{aligned}$$

Multiply pivot row 1 by $1/5$ and subtract the result from the second row to eliminate the a_{21} term.

$$\begin{aligned} 5x_1 - 2x_2 \qquad &= 3 \\ 2.4x_2 - x_3 &= -0.6 \\ -3x_2 + 7x_3 &= 4 \end{aligned}$$

Pivoting is necessary so switch the second and third row,

$$\begin{aligned} 5x_1 - 2x_2 \qquad &= 3 \\ -3x_2 + 7x_3 &= 4 \\ 2.4x_2 - x_3 &= -0.6 \end{aligned}$$

Multiply pivot row 2 by $2.4/(-3)$ and subtract the result from the third row to eliminate the a_{32} term.

$$\begin{aligned} 5x_1 - 2x_2 \qquad &= 3 \\ -3x_2 + 7x_3 &= 4 \\ 4.6x_3 &= 2.6 \end{aligned}$$

The solution can then be obtained by back substitution

9.7 (a) Pivoting is necessary, so switch the first and third rows,

$$
\begin{aligned}
-8x_1 + x_2 - 2x_3 &= -20 \\
-3x_1 - x_2 + 7x_3 &= -34 \\
2x_1 - 6x_2 - x_3 &= -38
\end{aligned}
$$

Multiply the first equation by $-3/(-8)$ and subtract the result from the second equation to eliminate the a_{21} term from the second equation. Then, multiply the first equation by $2/(-8)$ and subtract the result from the third equation to eliminate the a_{31} term from the third equation.

$$
\begin{aligned}
-8x_1 \quad + x_2 \quad - 2x_3 &= -20 \\
-1.375x_2 + 7.75x_3 &= -26.5 \\
-5.75x_2 \quad - 1.5x_3 &= -43
\end{aligned}
$$

Pivoting is necessary so switch the second and third row,

$$
\begin{aligned}
-8x_1 \quad + x_2 \quad - 2x_3 &= -20 \\
-5.75x_2 \quad - 1.5x_3 &= -43 \\
-1.375x_2 + 7.75x_3 &= -26.5
\end{aligned}
$$

Multiply pivot row 2 by $-1.375/(-5.75)$ and subtract the result from the third row to eliminate the a_{32} term.

$$
\begin{aligned}
-8x_1 \quad + x_2 \qquad - 2x_3 &= -20 \\
-5.75x_2 \qquad - 1.5x_3 &= -43 \\
8.108696x_3 &= -16.21739
\end{aligned}
$$

At this point, the determinant can be computed as

$$D = -8 \times -5.75 \times 8.108696 \times (-1)^2 = 373$$

The solution can then be obtained by back substitution

$$
\begin{aligned}
x_3 &= \frac{-16.21739}{8.108696} = -2 \\
x_2 &= \frac{-43 + 1.5(-2)}{-5.75} = 8 \\
x_1 &= \frac{-20 + 2(-2) - 1(8)}{-8} = 4
\end{aligned}
$$

(b) Check:

$$
\begin{aligned}
2(4) - 6(8) - (-2) &= -38 \\
-3(4) - (8) + 7(-2) &= -34 \\
-8(4) + (8) - 2(-2) &= -20
\end{aligned}
$$

9.8 Multiply the first equation by –0.4/0.8 and subtract the result from the second equation to eliminate the x_1 term from the second equation.

$$\begin{bmatrix} 0.8 & -0.4 & \\ & 0.6 & -0.4 \\ & -0.4 & 0.8 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 41 \\ 45.5 \\ 105 \end{Bmatrix}$$

Multiply pivot row 2 by –0.4/0.6 and subtract the result from the third row to eliminate the x_2 term.

$$\begin{bmatrix} 0.8 & -0.4 & \\ & 0.6 & -0.4 \\ & & 0.533333 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 41 \\ 45.5 \\ 135.3333 \end{Bmatrix}$$

The solution can then be obtained by back substitution

$$x_3 = \frac{135.3333}{0.533333} = 253.75$$
$$x_2 = \frac{45.5-(-0.4)253.75}{0.6} = 245$$
$$x_1 = \frac{41-(-0.4)245}{0.8} = 173.75$$

(b) Check:

$$0.8(173.75) - 0.4(245) = 41$$
$$-0.4(173.75) + 0.8(245) - 0.4(253.75) = 25$$
$$-0.4(245) + 0.8(253.75) = 105$$

Chapter 10, p287, PROBLEMS

1) Problem 10.1

The flop counts for LU decomposition can be determined in a similar fashion as was done for Gauss elimination. The major difference is that the elimination is only implemented for the left-hand side coefficients. Thus, for every iteration of the inner loop, there are n multiplications/divisions and $n - 1$ addition/subtractions. The computations can be summarized as

Outer Loop k	Inner Loop i	Addition/Subtraction flops	Multiplication/Division flops
1	2, n	$(n-1)(n-1)$	$(n-1)n$
2	3, n	$(n-2)(n-2)$	$(n-2)(n-1)$
.	.		
.	.		
.	.		
k	$k+1, n$	$(n-k)(n-k)$	$(n-k)(n+1-k)$
.	.		
.	.		
.	.		
$n-1$	n, n	(1)(1)	(1)(2)

Therefore, the total addition/subtraction flops for elimination can be computed as

$$\sum_{k=1}^{n-1}(n-k)(n-k) = \sum_{k=1}^{n-1}\left[n^2 - 2nk + k^2\right]$$

Applying some of the relationships from Eq. (8.14) yields

$$\sum_{k=1}^{n-1}\left[n^2 - 2nk + k^2\right] = \frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6}$$

A similar analysis for the multiplication/division flops yields

$$\sum_{k=1}^{n-1}(n-k)(n+1-k) = \frac{n^3}{3} - \frac{n}{3}$$

$$\left[n^3 + O(n^2)\right] - \left[n^3 + O(n)\right] + \left[\frac{1}{3}n^3 + O(n^2)\right] = \frac{n^3}{3} + O(n^2)$$

Summing these results gives

$$\frac{2n^3}{3} - \frac{n^2}{2} - \frac{n}{6}$$

For forward substitution, the numbers of multiplications and subtractions are the same and equal to

$$\sum_{i=1}^{n-1} i = \frac{(n-1)n}{2} = \frac{n^2}{2} - \frac{n}{2}$$

Back substitution is the same as for Gauss elimination: $n^2/2 - n/2$ subtractions and $n^2/2 + n/2$ multiplications/divisions. The entire number of flops can be summarized as

	Mult/Div	Add/Subtr	Total
Forward elimination	$\frac{n^3}{3}-\frac{n}{3}$	$\frac{n^3}{3}-\frac{n^2}{2}+\frac{n}{6}$	$\frac{2n^3}{3}-\frac{n^2}{2}-\frac{n}{6}$
Forward substitution	$\frac{n^2}{2}-\frac{n}{2}$	$\frac{n^2}{2}-\frac{n}{2}$	n^2-n
Back substitution	$\frac{n^2}{2}+\frac{n}{2}$	$\frac{n^2}{2}-\frac{n}{2}$	n^2
Total	$\frac{n^3}{3}+n^2-\frac{n}{3}$	$\frac{n^3}{3}+\frac{n^2}{2}-\frac{5n}{6}$	$\frac{2n^3}{3}+\frac{3n^2}{2}-\frac{7n}{6}$

The total number of flops is identical to that obtained with standard Gauss elimination.

■

2) Problem 10.3

10.3 (a) The coefficient a_{21} is eliminated by multiplying row 1 by $f_{21} = -0.3$ and subtracting the result from row 2. a_{31} is eliminated by multiplying row 1 by $f_{31} = 0.1$ and subtracting the result from row 3. The factors f_{21} and f_{31} can be stored in a_{21} and a_{31}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & 0.8 & 5.1 \end{bmatrix}$$

a_{32} is eliminated by multiplying row 2 by $f_{32} = -0.14815$ and subtracting the result from row 3. The factor f_{32} can be stored in a_{32}.

$$\begin{bmatrix} 10 & 2 & -1 \\ -0.3 & -5.4 & 1.7 \\ 0.1 & -0.14815 & 5.3519 \end{bmatrix}$$

Therefore, the LU decomposition is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.14815 & 1 \end{bmatrix} \qquad [U] = \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.3519 \end{bmatrix}$$

These two matrices can be multiplied to yield the original system. For example, using MATLAB to perform the multiplication gives

```
>> L =[1 0 0;-0.3 1 0;0.1 -0.14815 1]
>> U =[10 2 -1;0 -5.4 1.7;0 0 5.3519]
>> A=L*U

A =
   10.0000     2.0000    -1.0000
   -3.0000    -6.0000     2.0000
    1.0000     1.0000     5.0000
```

3) Problem 10.5

10.5 The system can be written in matrix form as

$$[A]=\begin{bmatrix} 2 & -6 & -1 \\ -3 & -1 & 7 \\ -8 & 1 & -2 \end{bmatrix} \qquad \{b\}=\begin{Bmatrix} -38 \\ -34 \\ -40 \end{Bmatrix} \qquad [\mathrm{P}]=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Partial pivot:

$$[A]=\begin{bmatrix} -8 & 1 & -2 \\ -3 & -1 & 7 \\ 2 & -6 & -1 \end{bmatrix} \qquad [\mathrm{P}]=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Compute factors:

$$f_{21}=-3/-8=0.375 \qquad f_{31}=2/(-8)=-0.25$$

Forward eliminate and store factors in zeros:

$$[A]=\begin{bmatrix} -8 & 1 & -2 \\ 0.375 & -1.375 & 7.75 \\ -0.25 & -5.75 & -1.5 \end{bmatrix}$$

Pivot again

$$[A]=\begin{bmatrix} -8 & 1 & -2 \\ -0.25 & -5.75 & -1.5 \\ 0.375 & -1.375 & 7.75 \end{bmatrix} \qquad [\mathrm{P}]=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Compute factors:

$$f_{32}=-1.375/(-5.75)=0.23913$$

Forward eliminate and store factor in zero:

$$[LU] = \begin{bmatrix} -8 & 1 & -2 \\ -0.25 & -5.75 & -1.5 \\ 0.375 & 0.23913 & 8.1087 \end{bmatrix}$$

Therefore, the LU decomposition is

$$[L] = \begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.375 & 0.23913 & 1 \end{bmatrix} \qquad [U] = \begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 8.1087 \end{bmatrix}$$

Forward substitution. First pre-multiply right-hand side vector $\{b\}$ by $[P]$ to give

$$[P]\{b\} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} -38 \\ -34 \\ -40 \end{Bmatrix} = \begin{Bmatrix} -40 \\ -38 \\ -34 \end{Bmatrix}$$

Therefore,

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.25 & 1 & 0 \\ 0.375 & 0.23913 & 1 \end{bmatrix} \{d\} = \begin{Bmatrix} -40 \\ -38 \\ -34 \end{Bmatrix}$$

which can be solved for

$$d_1 = -40$$
$$d_2 = -38 - 0.25(-40) = -48$$
$$d_3 = -34 - 0.375(-40) - 0.23913(-48) = -7.52174$$

Back substitution:

$$\begin{bmatrix} -8 & 1 & -2 \\ 0 & -5.75 & -1.5 \\ 0 & 0 & 8.1087 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} -40 \\ -48 \\ -7.52174 \end{Bmatrix}$$

$$x_3 = \frac{-7.52174}{8.1087} = -0.92761$$
$$x_2 = \frac{-48 + 1.5(-0.92761)}{-5.75} = 8.589812$$
$$x_1 = \frac{-40 + 2(-0.92761) - 1(8.589812)}{8} = 6.30563$$

■

4) Problem 10.8 (a), (c)

10.8 (a) For the first row ($i = 1$), Eq. (10.15) is employed to compute

$$u_{11} = \sqrt{a_{11}} = \sqrt{8} = 2.828427$$

Then, Eq. (10.16) can be used to determine

$$u_{12} = \frac{a_{12}}{u_{11}} = \frac{20}{2.828427} = 7.071068$$

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{15}{2.8288427} = 5.303301$$

For the second row ($i = 2$),

$$u_{22} = \sqrt{a_{22} - u_{12}^2} = \sqrt{80 - (7.071068)^2} = 5.477226$$

$$u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}} = \frac{50 - 7.071068(5.303301)}{5.477226} = 2.282177$$

For the third row ($i = 3$),

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2} = \sqrt{60 - 5.303301^2 - 2.282177^2} = 5.163978$$

Thus, the Cholesky decomposition yields

$$[U] = \begin{bmatrix} 2.828427 & 7.071068 & 5.303301 \\ & 5.477226 & 2.282177 \\ & & 5.163978 \end{bmatrix}$$

The validity of this decomposition can be verified by substituting it and its transpose into Eq. (10.14) to see if their product yields the original matrix $[A]$.

```
>> U =[2.828427 7.071068 5.303301; 0 5.477226 2.282177;0 0
5.163978];
>> A=U'*U

A =
    8.0000   20.0000   15.0000
   20.0000   80.0000   50.0000
   15.0000   50.0000   60.0000
```

(b)

```
>> A = [8 20 15;20 80 50;15 50 60];
>> U = chol(A)

U =
    2.8284    7.0711    5.3033
         0    5.4772    2.2822
         0         0    5.1640
```

(c) The solution can be obtained by hand or by MATLAB. Using MATLAB:

```
>> b = [50;250;100];
>> d=U'\b
```

■

Chapter 11, p300, PROBLEMS

1) Problem 11.1

The matrix to be evaluated is

$$\begin{bmatrix} 10 & 2 & -1 \\ -3 & -6 & 2 \\ 1 & 1 & 5 \end{bmatrix}$$

First, compute the LU decomposition. Multiply the first row by $f_{21} = -3/10 = -0.3$ and subtract the result from the second row to eliminate the a_{21} term. Then, multiply the first row by $f_{31} = 1/10 = 0.1$ and subtrac the result from the third row to eliminate the a_{31} term. The result is

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0.8 & 5.1 \end{bmatrix}$$

Multiply the second row by $f_{32} = 0.8/(-5.4) = -0.148148$ and subtract the result from the third row to eliminate the a_{32} term.

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

Therefore, the LU decomposition is

$$[L]\{U\} = \begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix} \begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix}$$

The first column of the matrix inverse can be determined by performing the forward-substitution solution procedure with a unit vector (with 1 in the first row) as the right-hand-side vector. Thus, the lower-triangular system, can be set up as,

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

and solved with forward substitution for $\{d\}^T = [1\ 0.3\ -0.055556]$. This vector can then be used as the right-hand side of the upper triangular system,

$$\begin{bmatrix} 10 & 2 & -1 \\ 0 & -5.4 & 1.7 \\ 0 & 0 & 5.351852 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0.3 \\ -0.055556 \end{Bmatrix}$$

which can be solved by back substitution for the first column of the matrix inverse,

$$[A]^{-1} = \begin{bmatrix} 0.110727 & 0 & 0 \\ -0.058824 & 0 & 0 \\ -0.010381 & 0 & 0 \end{bmatrix}$$

To determine the second column, Eq. (9.8) is formulated as

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.3 & 1 & 0 \\ 0.1 & -0.148148 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

This can be solved with forward substitution for $\{d\}^T = [0\ 1\ 0.148148]$, and the results are used with $[U]$ to determine $\{x\}$ by back substitution to generate the second column of the matrix inverse,

$$[A]^{-1} = \begin{bmatrix} 0.110727 & 0.038062 & 0 \\ -0.058824 & -0.176471 & 0 \\ -0.010381 & 0.027682 & 0 \end{bmatrix}$$

Finally, the same procedures can be implemented with $\{b\}^T = [0\ 0\ 1]$ to solve for $\{d\}^T = \lfloor 0 \quad 0 \quad 1 \rfloor$, and the results are used with $[U]$ to determine $\{x\}$ by back substitution to generate the third column of the matrix inverse,

$$[A]^{-1} = \begin{bmatrix} 0.110727 & 0.038062 & 0.00692 \\ -0.058824 & -0.176471 & 0.058824 \\ -0.010381 & 0.027682 & 0.186851 \end{bmatrix}$$

This result can be checked by multiplying it times the original matrix to give the identity matrix. The following MATLAB session can be used to implement this check,

```
>> A = [10 2 -1;-3 -6 2;1 1 5];
>> AI = [0.110727 0.038062 0.00692;
-0.058824 -0.176471 0.058824;
-0.010381 0.027682 0.186851];
>> A*AI

ans =
    1.0000   -0.0000   -0.0000
    0.0000    1.0000   -0.0000
   -0.0000    0.0000    1.0000
```

■

Problem 11.3

The following solution is generated with MATLAB.

(a)
```
>> A = [15 -3 -1;-3 18 -6;-4 -1 12];
>> format long
>> AI = inv(A)

AI =
   0.07253886010363   0.01278065630397   0.01243523316062
   0.02072538860104   0.06079447322971   0.03212435233161
   0.02590673575130   0.00932642487047   0.09015544041451
```

(b)
```
>> b = [4000 1500 2400]';
>> format short
>> c = AI*b

c =
  339.1710
  251.1917
  333.9896
```

(c) The impact of a load to reactor 3 on the concentration of reactor 1 is specified by the element a_{13}^{-1} = 0.0124352. Therefore, the increase in the mass input to reactor 3 needed to induce a 10 g/m^3 rise in the concentration of reactor 1 can be computed as

$$\Delta b_3 = \frac{10}{0.0124352} = 804.1667 \frac{\text{g}}{\text{d}}$$

(d) The decrease in the concentration of the third reactor will be

$$\Delta c_3 = 0.0259067(500) + 0.009326(250) = 12.9534 + 2.3316 = 15.285 \frac{\text{g}}{\text{m}^3}$$

■

Problem 11.6

The matrix can be scaled by dividing each row by the element with the largest absolute value

```
>> A = [8/(-10) 2/(-10) 1;1 1/(-9) 3/(-9);1 -1/15 6/15]

A =
   -0.8000   -0.2000    1.0000
    1.0000   -0.1111   -0.3333
    1.0000   -0.0667    0.4000
```

MATLAB can then be used to determine each of the norms,

```
>> norm(A,'fro')
ans =
    1.9920

>> norm(A,1)
ans =
    2.8000

>> norm(A,inf)
ans =
     2
```

■

Problem 11.7

Prob. 11.2:

```
>> A = [-8 1 -2;2 -6 -1;-3 -1 7];
>> norm(A,'fro')

ans =
    13

>> norm(A,inf)

ans =
    11
```

Prob. 11.3:

```
>> A = [15 -3 -1;-3 18 -6;-4 -1 12]
>> norm(A,'fro')

ans =
   27.6586

>> norm(A,inf)

ans =
    27
```

■