CHAPTER 16

16.1 The angular frequency can be computed as $\omega_0 = 2\pi/24 = 0.261799$. The various summations required for the normal equations can be set up as

t	y	$\cos(\omega_0 t)$	$\sin(\omega_0 t)$	$\sin(\omega_0 t)\cos(\omega_0 t)$	$\cos^2(\omega_0 t)$	$\sin^2(\omega_0 t)$	$y\cos(\omega_0 t)$	$y\sin(\omega_0 t)$
0	7.6	1.00000	0.00000	0.00000	1.00000	0.00000	7.60000	0.00000
2	7.2	0.86603	0.50000	0.43301	0.75000	0.25000	6.23538	3.60000
4	7	0.50000	0.86603	0.43301	0.25000	0.75000	3.50000	6.06218
5	6.5	0.25882	0.96593	0.25000	0.06699	0.93301	1.68232	6.27852
7	7.5	-0.25882	0.96593	-0.25000	0.06699	0.93301	-1.94114	7.24444
9	7.2	-0.70711	0.70711	-0.50000	0.50000	0.50000	-5.09117	5.09117
12	8.9	-1.00000	0.00000	0.00000	1.00000	0.00000	-8.90000	0.00000
15	9.1	-0.70711	-0.70711	0.50000	0.50000	0.50000	-6.43467	-6.43467
20	8.9	0.50000	-0.86603	-0.43301	0.25000	0.75000	4.45000	-7.70763
22	7.9	0.86603	-0.50000	-0.43301	0.75000	0.25000	6.84160	-3.95000
24	7	1.00000	0.00000	0.00000	1.00000	0.00000	7.00000	0.00000
sum→	**84.8**	**2.31784**	**1.93185**	**0.00000**	**6.13397**	**4.86603**	**14.94232**	**10.18401**

The normal equations can be assembled as

$$\begin{bmatrix} 11 & 2.317837 & 1.931852 \\ 2.317837 & 6.13397 & 0 \\ 1.931852 & 0 & 4.86603 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} 84.8 \\ 14.9423 \\ 10.184 \end{Bmatrix}$$

This system can be solved for $A_0 = 8.0270$, $A_1 = -0.59717$, and $B_1 = -1.09392$. Therefore, the best-fit sinusoid is

$$pH = 8.0270 - 0.59717 \cos(\omega_0 t) - 1.09392 \sin(\omega_0 t)$$

The result can also be expressed in the alternate form of Eq. 16.2 by computing the amplitude,

$$C_1 = \sqrt{(-0.59717)^2 + (-1.09392)^2} = 1.2463$$

and the phase shift

$$\theta = \arctan\left(-\frac{-1.09392}{-0.59717}\right) + \pi = 2.0705 * \times \frac{24 \text{ hr}}{2\pi} = 7.9087 \text{ hr}$$

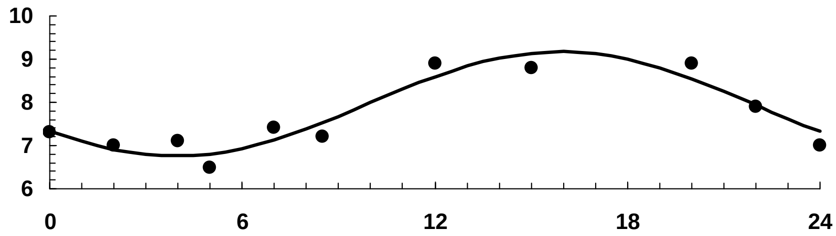
Therefore, the fit can also be expressed as

$$pH = 8.0270 + 1.2463 \cos\left(\omega_0 (t + 7.9087)\right)$$

Consequently, the mean is 8.0270 and the amplitude is 1.2463. To determine the time of the maximum, inspection of Fig. 16.3 indicates that a positive phase shift represents the time prior to midnight that the peak occurs. Therefore the time of the maximum can be computed as

$$t_{\max} = 24 - 7.0987 = 16.0913 \text{ hrs}$$

This is equal to about 16:05:29 or 4:05:29 PM. The data and the model can be plotted as

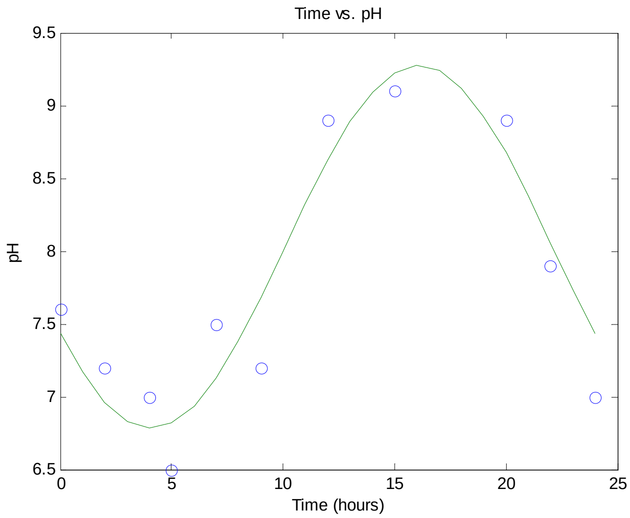


The same calculation can be implemented with MATLAB. The coefficients can also be determined with the following script:

```
clear,clc,clf,format compact
w0 = 2*pi/24;
t = [0 2 4 5 7 9 12 15 20 22 24]';
pH = [7.6 7.2 7 6.5 7.5 7.2 8.9 9.1 8.9 7.9 7]';
Z = [ones(size(t)) cos(w0*t) sin(w0*t)];
a = (Z'*Z)\(Z'*pH);
Sr = sum((pH - Z*a).^2);
syx = sqrt(Sr/(length(t)-length(a)));
tp = [0:24]; pHp = a(1)+a(2)*cos(w0*tp)+a(3)*sin(w0*tp);
plot(t,pH,'o',tp,pHp)
title('Time vs. pH')
xlabel('Time (hours)'), ylabel('pH')
mean=a(1)
theta = atan2(-a(3),a(2))*24/(2*pi)
Amplitude = sqrt(a(2)^2+a(3)^2)
time_Max_pH = 24-theta
```

Output:

```
mean =
    8.0270
theta =
    7.9087
Amplitude =
    1.2463
time_Max_pH =
   16.0913
```



16.2 The angular frequency can be computed as $\omega_0 = 2\pi/360 = 0.017453$. Because the data are equispaced, the coefficients can be determined with Eqs. 16.14-16.16. The various summations required to set up the model can be determined as

t	Radiation	$\cos(\omega_0 t)$	$\sin(\omega_0 t)$	$y\cos(\omega_0 t)$	$y\sin(\omega_0 t)$
15	144	0.96593	0.25882	139.093	37.270
45	188	0.70711	0.70711	132.936	132.936
75	245	0.25882	0.96593	63.411	236.652
105	311	-0.25882	0.96593	-80.493	300.403
135	351	-0.70711	0.70711	-248.194	248.194
165	359	-0.96593	0.25882	-346.767	92.916
195	308	-0.96593	-0.25882	-297.505	-79.716
225	287	-0.70711	-0.70711	-202.940	-202.940
255	260	-0.25882	-0.96593	-67.293	-251.141
285	211	0.25882	-0.96593	54.611	-203.810
315	159	0.70711	-0.70711	112.430	-112.430
345	131	0.96593	-0.25882	126.536	-33.905
sum→	**2954**			**-614.175**	**164.429**

The coefficients can be determined as

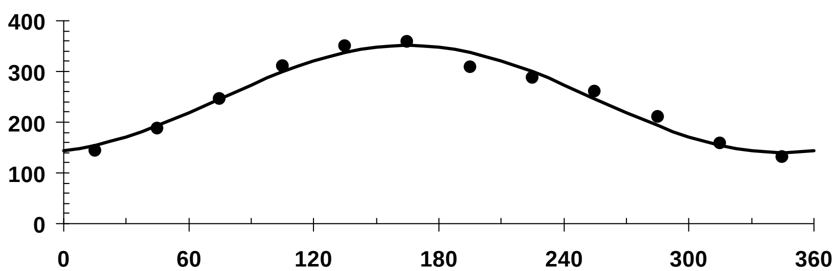
$$A_0 = \frac{\Sigma y}{N} = \frac{2954}{12} = 246.1667 \qquad A_1 = \frac{2}{N}\Sigma y\cos(\omega_0 t) = \frac{2}{12}(-614.175) = -102.363$$

$$B_1 = \frac{2}{N}\Sigma y\sin(\omega_0 t) = \frac{2}{12}(164.429) = 27.4048$$

Therefore, the best-fit sinusoid is

$$R = 246.1667 - 102.363\cos(0.017453t) + 27.4048\sin(0.017453t)$$

The data and the model can be plotted as



The value for mid-August can be computed as

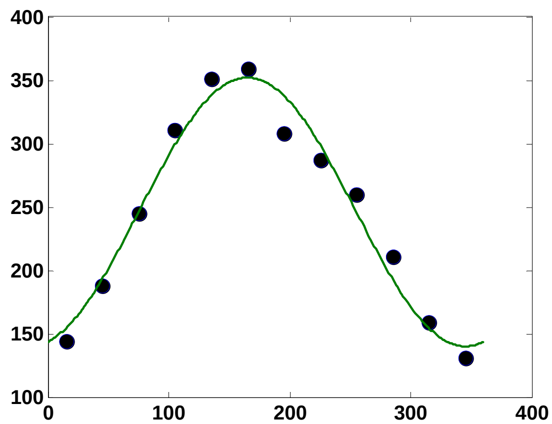
$$R = 246.1667 - 102.363\cos(0.017453(225)) + 27.4048\sin(0.017453(225)) = 299.1698$$

The same calculation can be implemented with MATLAB. The coefficients can be determined with the following script:

```
clear,clc,clf
w0=2*pi/360;
t=[15 45 75 105 135 165 195 225 255 285 315 345]';
R=[144 188 245 311 351 359 308 287 260 211 159 131]';
Z=[ones(size(t)) cos(w0*t) sin(w0*t)];
a=(Z'*Z)\(Z'*R)
theta = atan2(-a(3),a(2))*360/(2*pi)
Amplitude = sqrt(a(2)^2+a(3)^2)
```

```
RadAug = a(1)+a(2)*cos(w0*225)+a(3)*sin(w0*225)
tp=[0:360];
Rp=a(1)+a(2)*cos(w0*tp)+a(3)*sin(w0*tp);
plot(t,R,'o',tp,Rp)

a =
  246.1667
 -102.3625
   27.4048
theta =
 -165.0121
Amplitude =
  105.9675
RadAug =
  299.1698
```



16.3 In the following equations, $\omega_0 = 2\pi/T$

$$\frac{\int_0^T \sin(\omega_0 t)\,dt}{T} = \frac{-\omega_0\left[\cos(\omega_0 t)\right]_0^T}{T} = \frac{-\omega_0}{T}(\cos 2\pi - \cos 0) = 0$$

$$\frac{\int_0^T \cos(\omega_0 t)\,dt}{T} = \frac{\omega_0\left[\sin(\omega_0 t)\right]_0^T}{T} = \frac{\omega_0}{T}(\sin 2\pi - \sin 0) = 0$$

$$\frac{\int_0^T \sin^2(\omega_0 t)\,dt}{T} = \frac{\left[\frac{t}{2} - \frac{\sin(2\omega_0 t)}{4\omega_0}\right]_0^T}{T} = \frac{\frac{T}{2} - \frac{\sin 4\pi}{4\omega_0} - 0 + 0}{T} = \frac{1}{2}$$

$$\frac{\int_0^T \cos^2(\omega_0 t)\,dt}{T} = \frac{\left[\frac{t}{2} + \frac{\sin(2\omega_0 t)}{4\omega_0}\right]_0^T}{T} = \frac{\frac{T}{2} + \frac{\sin 4\pi}{4\omega_0} - 0 - 0}{T} = \frac{1}{2}$$

$$\frac{\int_0^T \cos(\omega_0 t)\sin(\omega_0 t)\,dt}{T} = \left[\frac{\sin^2(\omega_0 t)}{2T\omega_0}\right]_0^T = \frac{\sin^2 2\pi}{2\omega_0 T} - 0 = 0$$

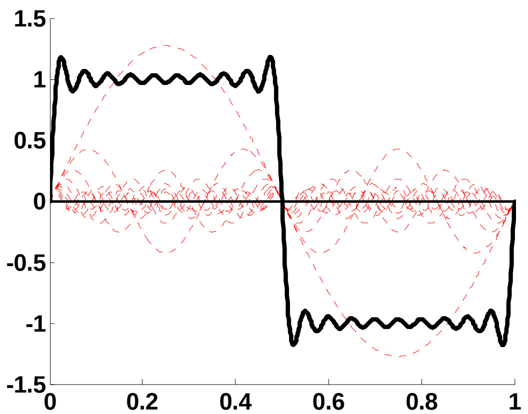
16.4 The function is:

```
function [t,f] = FourierSquare(A0,T,n)
t=[0:T/256:4*T];
nn=length(t);
f=zeros(n,nn);
```

```
s=zeros(nn);
for ii = 1:n
  for j = 1:nn
    f(ii,j)=f(ii,j)+4*A0/(2*ii-1)/pi*sin(2*pi*(2*ii-1)*t(j));
    s(j)=s(j)+f(ii,j);
  end
end
hold on
for ii=1:n
  plot(t,f(ii,:),'r:')
end
plot(t,s,'k-','linewidth',2)
hold off
end
```

The function can be run (with $n = 10$) to generate the resulting plot

```
>> [t,f] = FourierSquare(1,0.25,10);
```



16.5 $a_0 = 0$

$$a_k = \frac{2}{T}\int_{-T/2}^{T/2} -2t\cos(k\omega_0 t)\, dt$$

$$= -\frac{4}{T}\left[\frac{1}{(k\omega_0)^2}\cos(k\omega_0 t) + \frac{t}{k\omega_0}\sin(k\omega_0 t)\right]_{-T/2}^{T/2}$$

$$b_k = \frac{2}{T}\int_{-T/2}^{T/2} -2t\sin(k\omega_0 t)\, dt$$

$$= -\frac{4}{T}\left[\frac{1}{(k\omega_0)^2}\sin(k\omega_0 t) - \frac{t}{k\omega_0}\cos(k\omega_0 t)\right]_{-T/2}^{T/2}$$

On the basis of these, all a's = 0.

For k = odd: $\qquad b_k = \dfrac{2}{k\pi}$

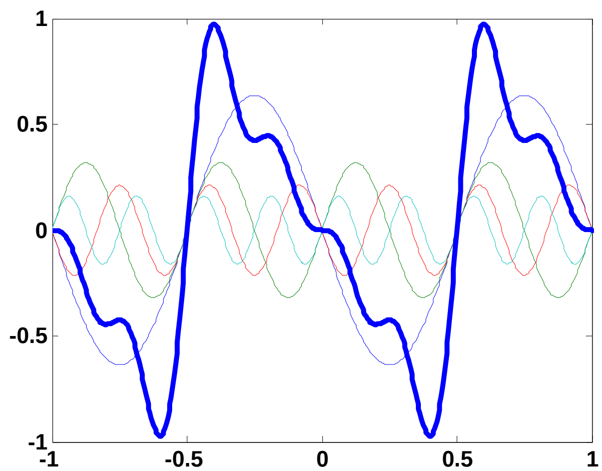
For k = even: $\quad b_k = -\dfrac{2}{k\pi}$

Therefore, the series is

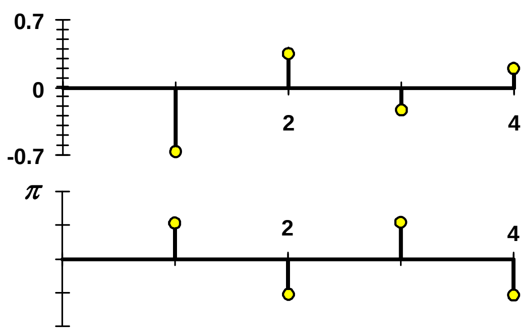
$$f(t) = -\frac{2}{\pi}\sin(\omega_0 t) + \frac{1}{\pi}\sin(2\omega_0 t) - \frac{2}{3\pi}\sin(3\omega_0 t) + \frac{1}{2\pi}\sin(4\omega_0 t) + \cdots$$

The following script plots the first 4 terms:

```
clear,clc,clf
T=1;w0=2*pi/T;
tp=[-1:1/256:1];
term1=-2/pi*sin(w0*tp);
term2=1/pi*sin(2*w0*tp);
term3=-2/(3*pi)*sin(3*w0*tp);
term4=1/(2*pi)*sin(4*w0*tp);
summ=term1+term2+term3+term4;
plot(tp,term1,tp,term2,tp,term3,tp,term4)
hold on
plot(tp,summ,'linewidth',2)
hold off
```



Here are the amplitude and phase spectra:



16.6 $a_0 = 0.5$

$$a_k = \frac{2}{2}\left[\int_{-1}^{0} -t\cos(k\pi t)\ dt + \int_{0}^{1} t\cos(k\pi t)\ dt\right]$$

$$= 1\left\{\left[-\frac{\cos(k\pi t)}{(k\pi)^2} - \frac{t\sin(k\pi t)}{k\pi}\right]_{-1}^{0} + \left[\frac{\cos(k\pi t)}{(k\pi)^2} + \frac{t\sin(k\pi t)}{k\pi}\right]_{0}^{1}\right\}$$

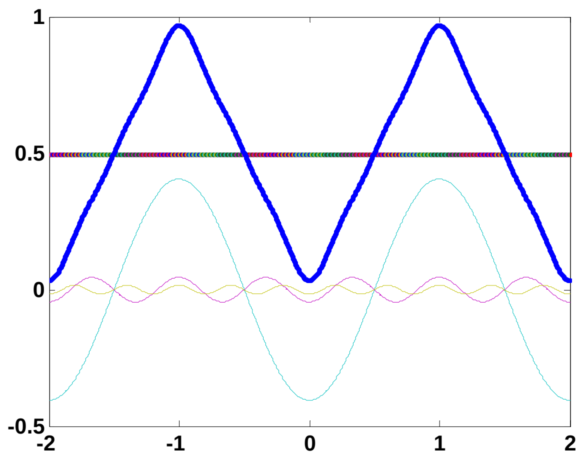
$$= \frac{2}{(k\pi)^2}(\cos k\pi - 1)$$

$b_k = 0$

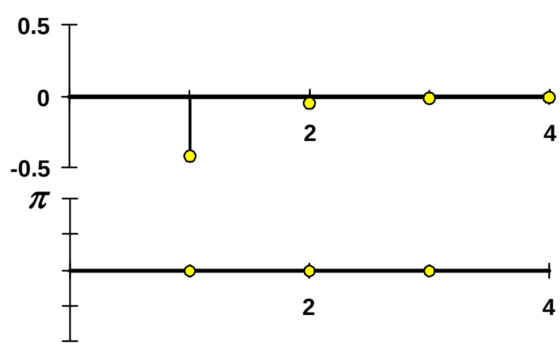
Substituting these coefficients into Eq. (16.17) gives

$$f(t) = \frac{1}{2} - \frac{4}{\pi^2}\cos(\pi t) - \frac{4}{9\pi^2}\cos(3\pi t) - \frac{4}{25\pi^2}\cos(5\pi t) + \cdots$$

This function for the first 4 terms is displayed below:



Here are the amplitude and phase spectra:



16.7 The Maclaurin series expansions are

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Euler's formula with positive exponent is

$$e^{ix} = \cos x + i \sin x$$

Substitute series into the formula

$$1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

Understanding that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, …:

$$1 + ix - \frac{x^2}{2} - \frac{x^3}{3!}i + \frac{x^4}{4!} + \frac{x^5}{5!}i - \frac{x^6}{6!} - \frac{x^7}{7!}i + \cdots = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots$$

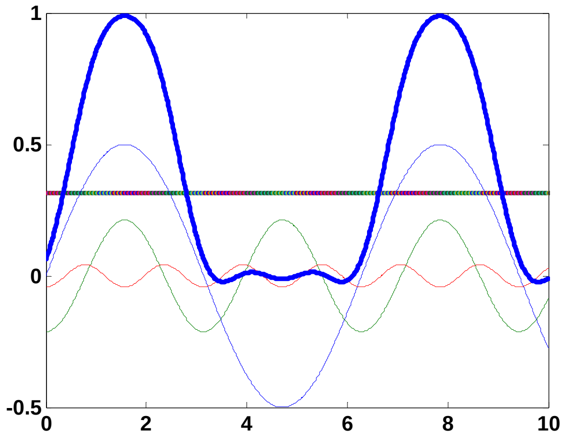
Collecting imaginary and real parts on the left-hand side yields

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + \left(ix - \frac{x^3}{3!}i + \frac{x^5}{5!}i - \frac{x^7}{7!}i + \cdots\right) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots$$

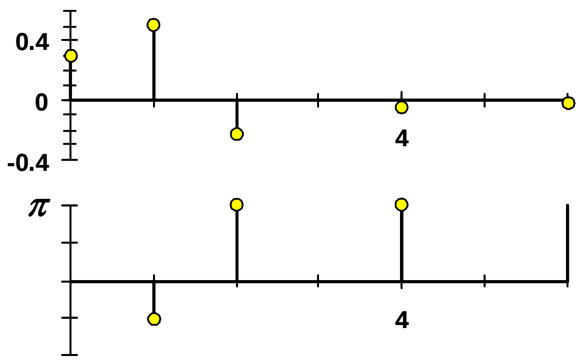
Thus, we can see that the left and right-hand sides are equivalent.

16.8 Here is a script to generate the plot of the first 4 terms:

```
clear,clc,clf
t=0:.01:10;
term1=1/pi;
term2=0.5*sin(t);
term3=-2/(3*pi)*cos(2*t);
term4=-2/(15*pi)*cos(4*t);
summ=term1+term2+term3+term4;
plot(t,term1,t,term2,t,term3,t,term4)
hold on
plot(t,summ,'linewidth',2)
hold off
```



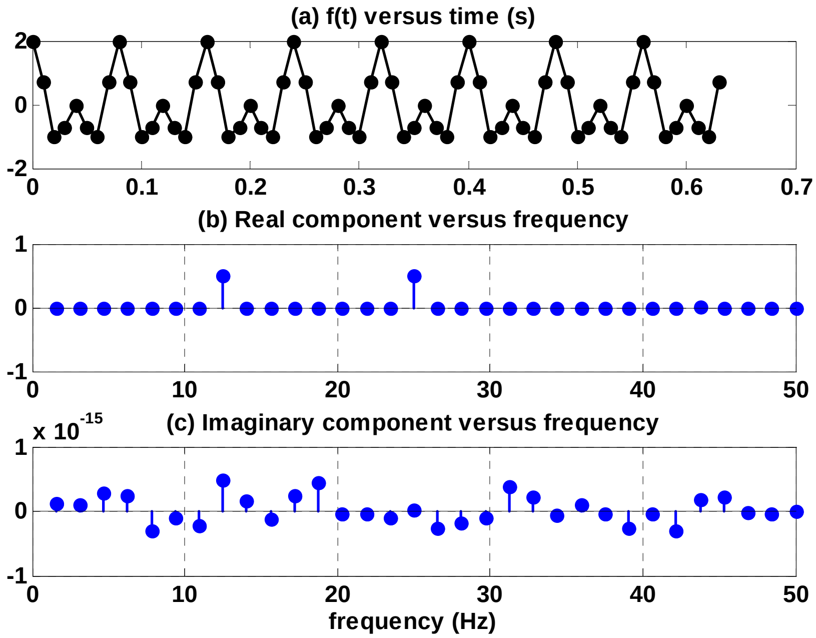
(b) The amplitude and phase line spectra can be developed as:

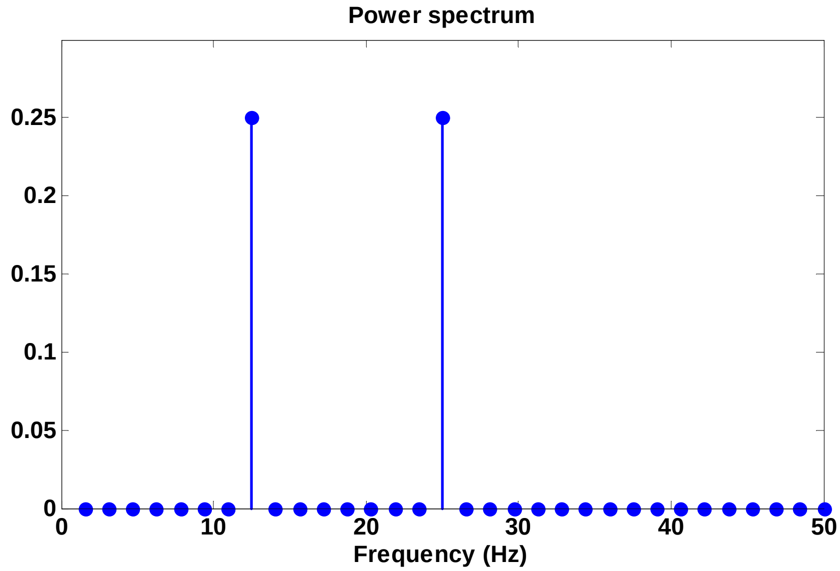


16.9

```
clear,clc,clf
n=64; dt=0.01; fs=1/dt;T=n/fs;
tspan=(0:n-1)/fs;
y=cos(2*pi*12.5*tspan)+cos(2*pi*25*tspan);
subplot(3,1,1);
plot(tspan,y,'-ok','linewidth',2,'MarkerFaceColor','black');
title('(a) f(t) versus time (s)');
Y=fft(y)/n;
nyquist=fs/2;fmin=1/T;
f = linspace(fmin,nyquist,n/2);
Y(1)=[];YP=Y(1:n/2);
subplot(3,1,2)
stem(f,real(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(b) Real component versus frequency')
subplot(3,1,3)
stem(f,imag(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(c) Imaginary component versus frequency')
xlabel('frequency (Hz)')
pause
% compute and display the power spectrum
clf
Pyy = abs(Y(1:n/2)).^2;
stem(f,Pyy,'linewidth',2,'MarkerFaceColor','blue')
title('Power spectrum')
```

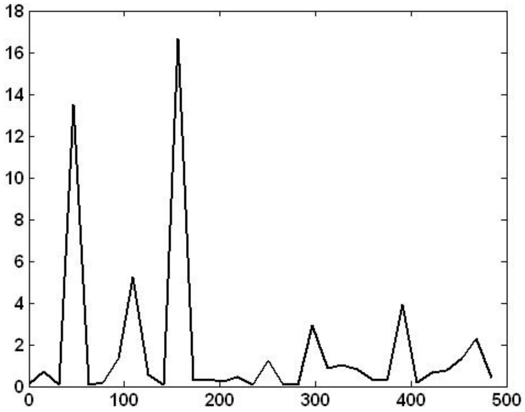
```
xlabel('Frequency (Hz)');ylim([0 0.3])
```



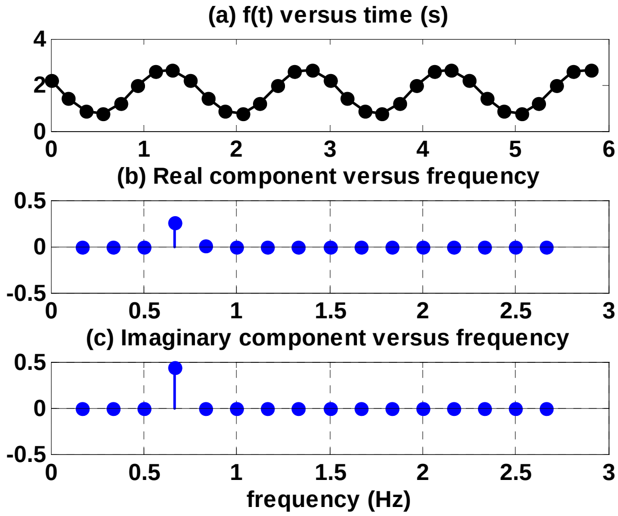


16.10 The following MATLAB session develops the fft along with a plot of the power spectral density versus frequency.

```
>> t=0:63;
>> y=cos(10*2*pi*t/63)+sin(3*2*pi*t/63)+randn(size(t));
>> Y=fft(y,64);
>> Pyy=Y.*conj(Y)/64;
>> f=1000*(0:31)/64;
>> plot(f,Pyy(1:32))
```

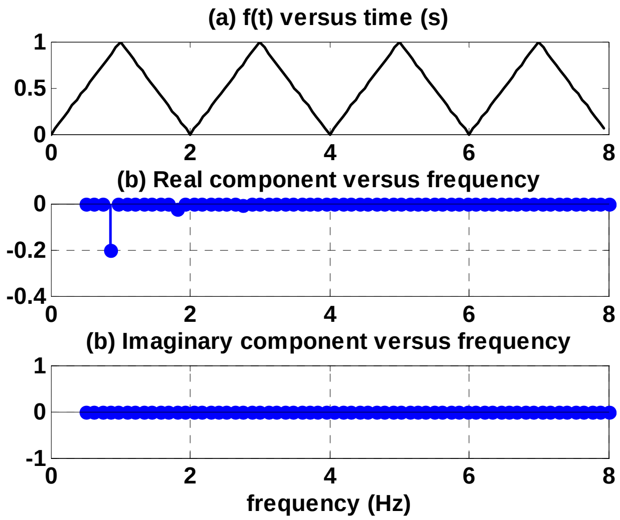


16.11
```
clear,clc,clf
T=6;n=32;dt=T/n;fs=1/dt;
tspan=linspace(0,6-dt,32);
w0=2*pi/1.5;
y=1.7+cos(w0*tspan+pi/3);
subplot(3,1,1);
plot(tspan,y,'-ok','linewidth',2,'MarkerFaceColor','black');
title('(a) f(t) versus time (s)');
Y=fft(y)/n;
nyquist=fs/2;fmin=1/T;
f = linspace(fmin,nyquist,n/2);
Y(1)=[];YP=Y(1:n/2);
subplot(3,1,2)
stem(f,real(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(b) Real component versus frequency')
subplot(3,1,3)
stem(f,imag(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(c) Imaginary component versus frequency')
xlabel('frequency (Hz)')
```



16.12
```
clear,clc,clf
Tp=2;
n=128;
dt=0.0625;fs=1/dt;
tspan=linspace(0,4*Tp-4*Tp/n,n);
y=abs(2*(tspan/Tp-floor(tspan/Tp+1/2)));
```

```
subplot(3,1,1);
plot(tspan,y,'-k','linewidth',2);
title('(a) f(t) versus time (s)');
Y=fft(y)/n;
nyquist=fs/2;fmin=1/Tp;
f = linspace(fmin,nyquist,n/2);
Y(1)=[];YP=Y(1:n/2);
subplot(3,1,2)
stem(f,real(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(b) Real component versus frequency')
subplot(3,1,3)
stem(f,imag(YP),'linewidth',2,'MarkerFaceColor','blue')
grid;title('(b) Imaginary component versus frequency')
xlabel('frequency (Hz)')
```



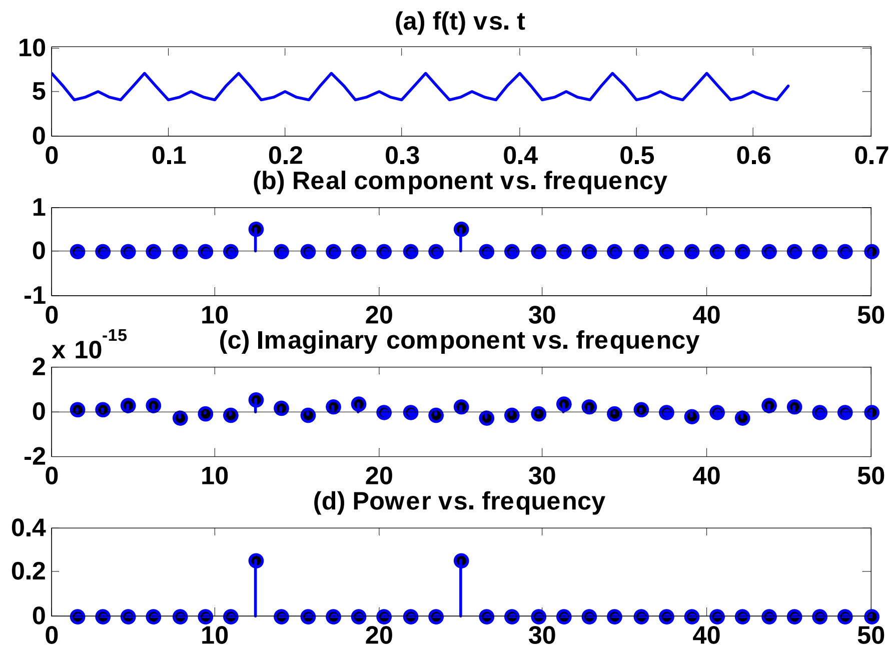
16.13 Here is the function:

```
function Y=fftmakerNew(t,y)
subplot(4,1,1)
plot(t,y)
n=length(t);dt=(max(t)-min(t))/(n-1);
T = t(n)- t(1) + dt;
fs=1/dt;nyquist=fs/2;fmin=1/T;fmax=.5*fs;df=fs/n;
Y=fft(y)/n;
Y';Y(1)=[];
nf=n/2;
f=[fmin:df:fmax];
YP=Y(1:length(f));
subplot(4,1,2)
stem(f,real(YP))
subplot(4,1,3)
stem(f,imag(YP))
P=abs(YP.^2);
subplot(4,1,4)
stem(f,P)
```

Here is a script that uses the function to solve Prob. 16.9:

```
clear,clc,clf,format compact,format short g
n=64; dt=0.01; fs=1/dt;
tspan=(0:n-1)/fs;
y=5+cos(2*pi*12.5*tspan)+cos(2*pi*25*tspan);
Y=fftmakerNew(tspan,y);
```

When this script is run, the result is:


(a) f(t) vs. t
10
5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
(b) Real component vs. frequency
1
0
-1
0
10
20
30
40
50
(c) Imaginary component vs. frequency
x 10^-15
2
0
-2
0
10
20
30
40
50
(d) Power vs. frequency
0.4
0.2
0
0
10
20
30
40
50