Lecture 1
Mathematical Modeling, Numerical Methods, and Problem Solving

Numerical Methods
Fall 2019

A Simple Mathematical Model

- A mathematical model can be broadly defined as a formulation or equation that expresses the essential features of a physical system or process in mathematical terms.

- Mathematical models can be represented by a functional relationship between dependent variables, independent variables, parameters, and forcing functions.

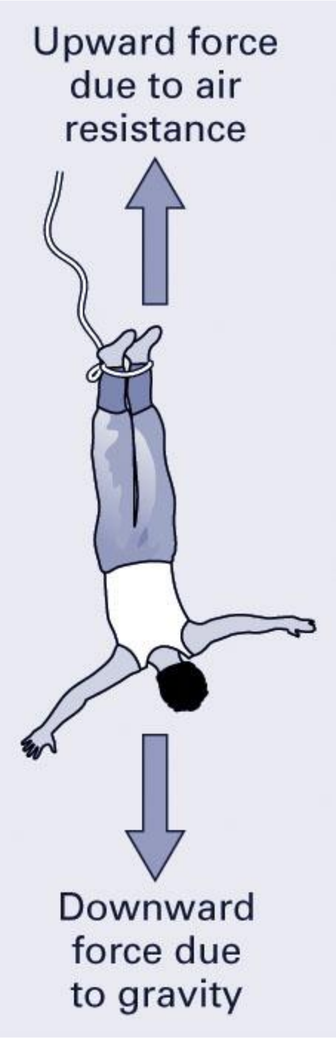
Mathematical Model

$$\begin{matrix}\text{Dependent} \\ \text{variable}\end{matrix} = f\left(\begin{matrix}\text{independent} \\ \text{variables}\end{matrix}, \text{parameters}, \begin{matrix}\text{forcing} \\ \text{functions}\end{matrix}\right)$$

- *Dependent variable* – a characteristic that usually reflects the behavior or state of the system
- *Independent variables* – dimensions, such as time and space, along which the system's behavior is being determined
- *Parameters* – constants reflective of the system's properties or composition
- *Forcing functions* – external influences acting upon the system

Mathematical Model

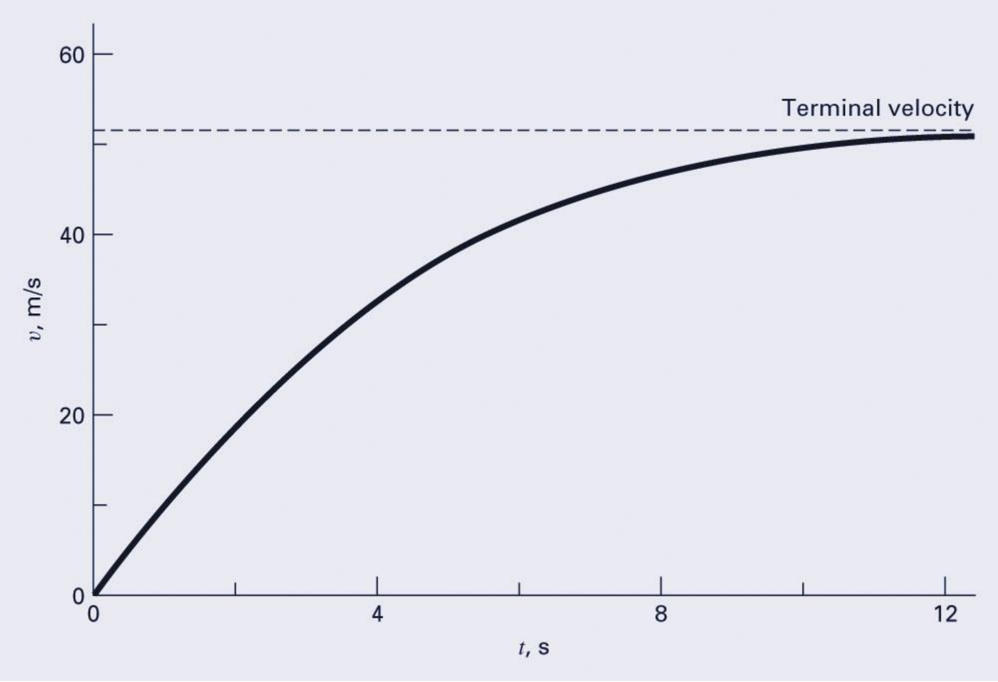
- Assuming a bungee jumper is in mid-flight, an analytical model for the jumper's velocity, accounting for drag,
  - $v(t) = \sqrt{\frac{gm}{c_d}} \tanh\left(\sqrt{\frac{gc_d}{m}}\, t\right)$



- Dependent variable – *velocity v*
- Independent variables – *time t*
- Parameters – *mass m, drag coefficient c_d*
- Forcing function – *gravitational acceleration g*

Mathematical Model Results

- Using a computer (or a calculator), the model can be used to generate a graphical representation of the system. For example, the graph below represents the velocity of a 68.1 kilogram jumper, assuming a drag coefficient of 0.25 kilograms per mile



Numerical Modeling

- Example – the bungee jumper velocity equation from before is the analytical solution to the differential equation

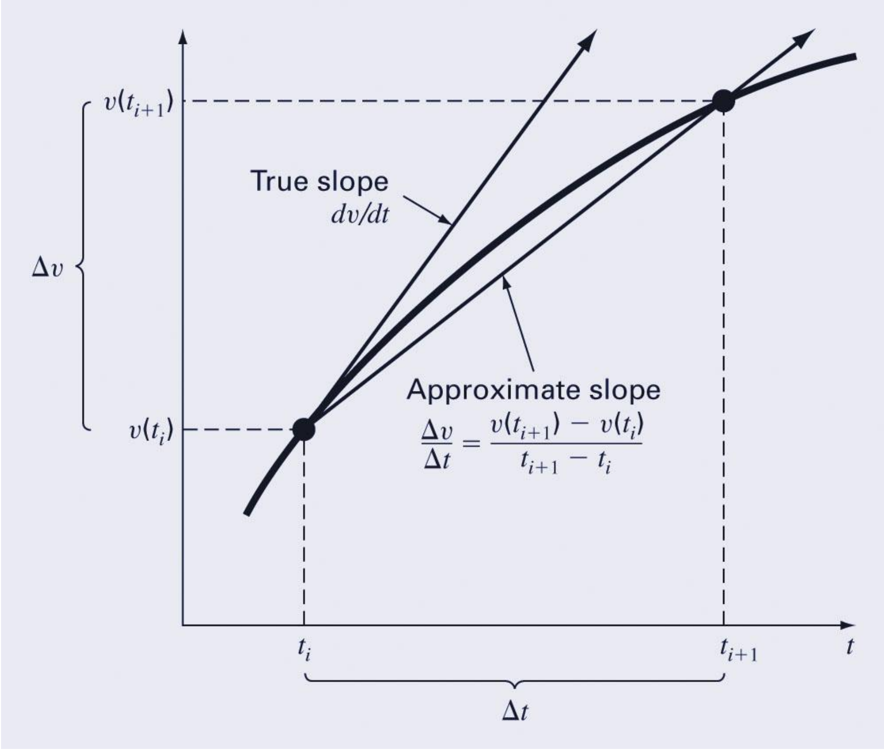
$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

where the change in velocity is determined by the gravitational forces acting on the jumper minus the drag force.

Numerical Methods

- To solve the problem using a numerical method, note that the time rate of change of velocity can be approximated as:

$$\frac{dv}{dt} \approx \frac{\Delta v}{\Delta t} = \frac{v(t_{i+1}) - v(ti)}{t_{i+1} - ti}$$



Euler's Method

- Substituting the finite difference into the differential equation gives

$$\frac{dv}{dt} = g - \frac{c_d}{m} v^2$$

$$\frac{v(t_{i+1}) - v(t_i)}{t_{i+1} - t_i} = g - \frac{c_d}{m} v^2$$

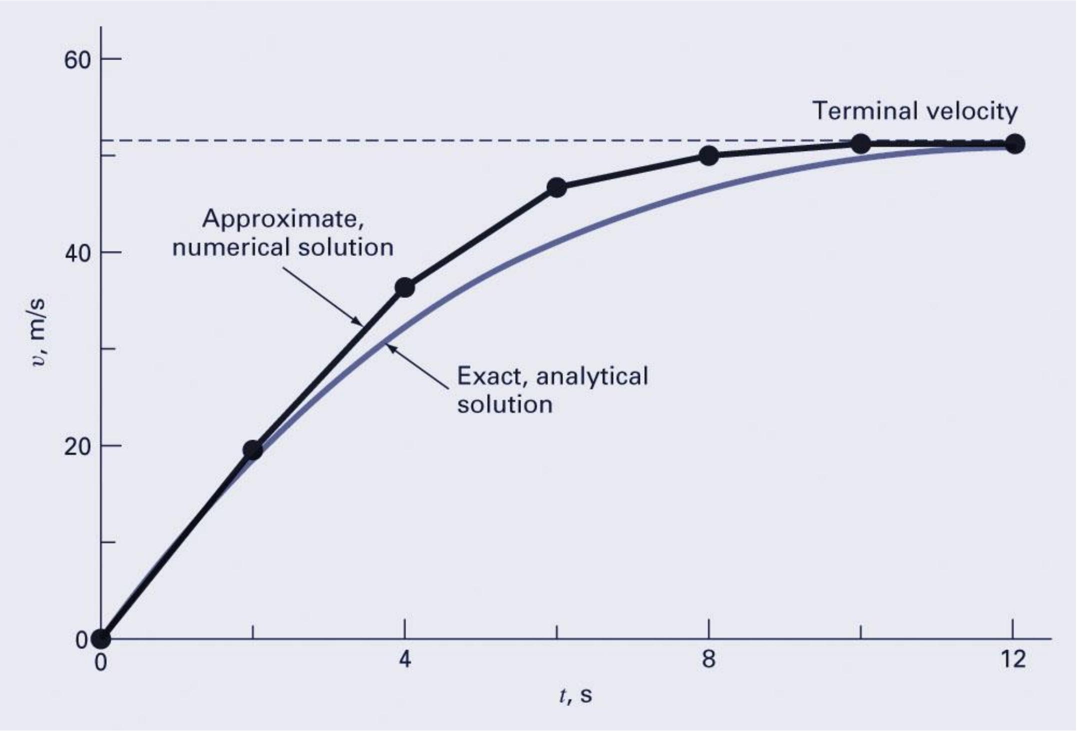
- Solve for

$$v(t_{i+1}) = v(t_i) + \left(g - \frac{c_d}{m} v(t_i)^2 \right) (t_{i+1} - t_i)$$

$$\text{new} = \text{old} + \text{slope} \times \text{step}$$

Numerical Results

- Applying Euler's method in 2 second intervals yields:



- How do we improve the solution?
  - Smaller steps

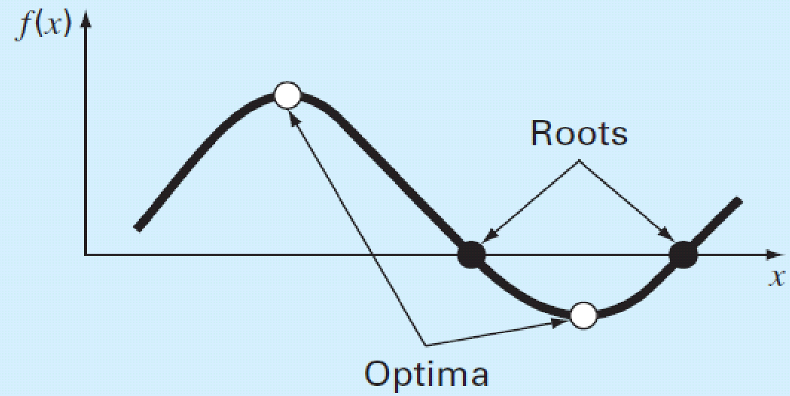
Summary of Numerical Methods

- The book is divided into five categories of numerical methods:

(a) ***Part 2*****: Roots and optimization**

Roots: Solve for x so that $f(x) = 0$

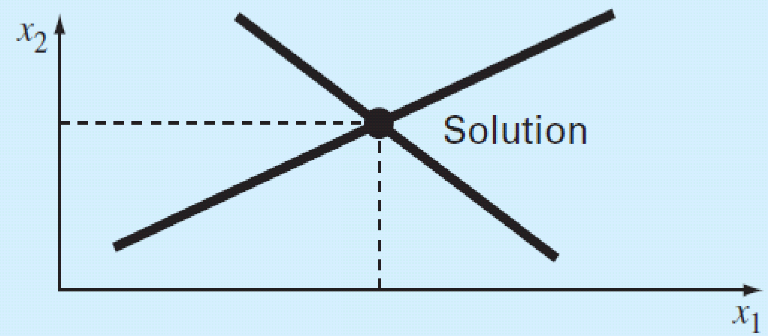
Optimization: Solve for x so that $f'(x) = 0$



(b) ***Part 3*****: Linear algebraic equations**

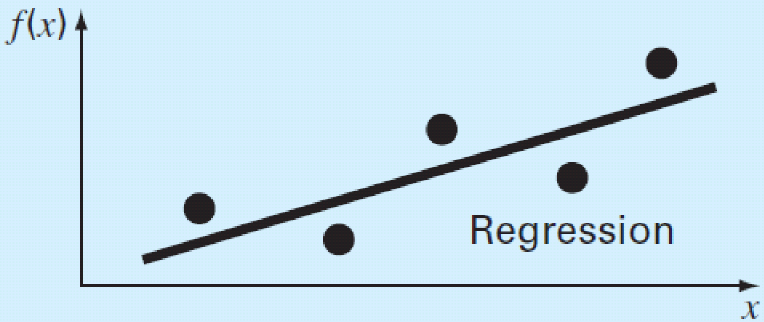
Given the a's and the b's, solve for the x's

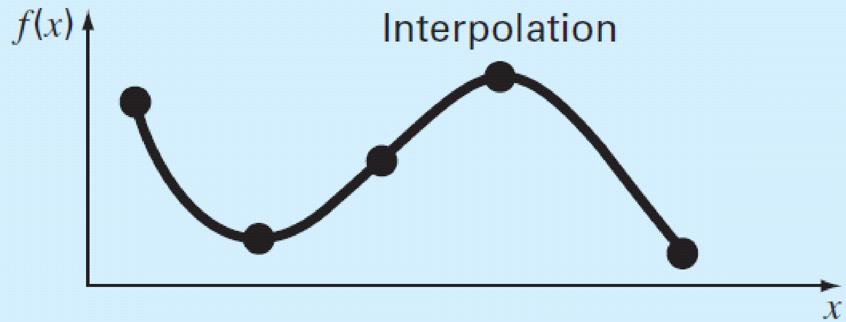
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$



Summary of Numerical Methods

(c) *Part 4*: Curve fitting

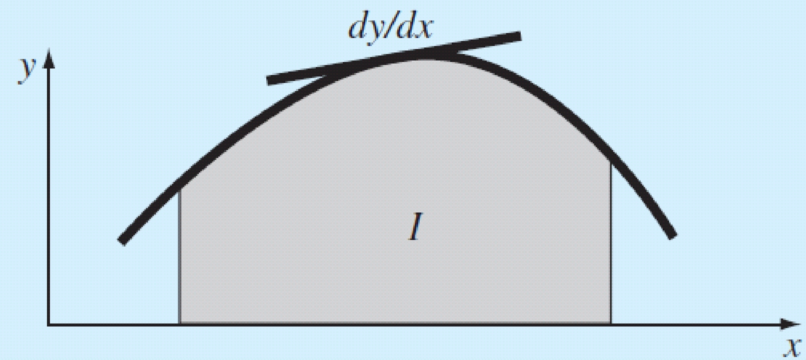




(d) *Part 5*: Integration and differentiation

Integration: Find the area under the curve

Differentiation: Find the slope of the curve



(e) *Part 6*: Differential equations

Given

$$\frac{dy}{dt} \approx \frac{\Delta y}{\Delta t} = f(t, y)$$

solve for y as a function of t

$$y_{i+1} = y_i + f(t_i, y_i)\Delta t$$