Homework #2 (Ch4) Solution

Chapter 4, p83, PROBLEMS

1) Problem 4.3

$$(61565)_8 = (6\times 8^4)+(1\times 8^3)+(5\times 8^2)+(6\times 8^1)+(5\times 8^0)$$
$$= 6(4096)+1(512)+5(64)+6(8)+5(1)$$
$$= 24576+512+320+48+5 = 25,461$$
$$(2.71)_8 = (2\times 8^0)+(7\times 8^{-1})+(1\times 8^{-2}) = 2(1)+7(0.125)+1(0.015625) = 2.890625$$

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2) Problem 4.11

Use $\varepsilon_s = 0.5\times 10^{2-2} = 0.5\%$. The true value = $\cos(\pi/4) = 0.707107\ldots$

zero-order:

$$\cos\left(\frac{\pi}{4}\right) \cong 1$$
$$\varepsilon_t = \left|\frac{0.707107-1}{0.707107}\right|100\% = 41.42\%$$

first-order:

$$\cos\left(\frac{\pi}{4}\right) \cong 1-\frac{(\pi/4)^2}{2} = 0.691575$$
$$\varepsilon_t = \left|\frac{0.707107-0.691575}{0.707107}\right|100\% = 2.19\% \qquad \varepsilon_a = \left|\frac{0.691575-1}{0.691575}\right|100\% = 44.6\%$$

second-order:

$$\cos\left(\frac{\pi}{4}\right) \cong 0.691575+\frac{(\pi/4)^4}{24} = 0.707429$$
$$\varepsilon_t = \left|\frac{0.707107-0.707429}{0.707107}\right|100\% = 0.456\% \qquad \varepsilon_a = \left|\frac{0.707429-0.691575}{0.707429}\right|100\% = 2.24\%$$

third-order:

$$\cos\left(\frac{\pi}{4}\right) \cong 0.707429-\frac{(\pi/4)^6}{720} = 0.707103$$
$$\varepsilon_t = \left|\frac{0.707107-0.707103}{0.707107}\right|100\% = 0.0005\% \qquad \varepsilon_a = \left|\frac{0.707103-0.707429}{0.707103}\right|100\% = 0.046\%$$

Because $\varepsilon_a < 0.5\%$, we can terminate the computation.

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3) Problem 4.13

The true value is $f(3) = 554$.

<u>zero order:</u>

$f(3) \cong f(1) = -62$ $\qquad \varepsilon_t = \left|\frac{554-(-62)}{554}\right| 100\% = 111.19\%$

<u>first order:</u>

$f'(1) = 75(1)^2 - 12(1) + 7 = 70$

$f(3) \cong -62 + 70(2) = 78$ $\qquad \varepsilon_t = \left|\frac{554-78}{554}\right| 100\% = 85.92\%$

<u>second order:</u>

$f''(1) = 150(1) - 12 = 138$

$f(3) \cong 78 + \frac{138}{2}(2)^2 = 354$ $\qquad \varepsilon_t = \left|\frac{554-354}{554}\right| 100\% = 36.10\%$

<u>third order:</u>

$f^{(3)}(1) = 150$

$f(3) = 354 + \frac{150}{6}(2)^3 = 554$ $\qquad \varepsilon_t = \left|\frac{554-554}{554}\right| 100\% = 0.0\%$

Because we are working with a third-order polynomial, the error is zero. This is due to the fact that cubics have zero fourth and higher derivatives.

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4) Problem 4.16

The first derivative of the function at $x = 2$ can be evaluated as

$$f'(2) = 75(2)^2 - 12(2) + 7 = 283$$

The points needed to form the finite divided differences can be computed as

$x_{i-1} = 1.75$ $\qquad f(x_{i-1}) = 39.85938$
$x_i = 2.0$ $\qquad f(x_i) = 102$
$x_{i+1} = 2.25$ $\qquad f(x_{i+1}) = 182.1406$

forward:

$$f'(2) = \frac{182.1406 - 102}{0.25} = 320.5625 \qquad |E_t| = |283 - 320.5625| = 37.5625$$

backward:

$$f'(2) = \frac{102 - 39.85938}{0.25} = 248.5625 \qquad |E_t| = |283 - 248.5625| = 34.4375$$

centered:

$$f'(2) = \frac{182.1406 - 39.85938}{0.5} = 284.5625 \quad E_t = 283 - 284.5625 = -1.5625$$

Both the forward and backward differences should have errors approximately equal to

$$|E_t| \approx \frac{f''(x_i)}{2} h$$

The second derivative can be evaluated as

$$f''(2) = 150(2) - 12 = 288$$

Therefore,

$$|E_t| \approx \frac{288}{2} 0.25 = 36$$

which is similar in magnitude to the computed errors. For the central difference,

$$E_t \approx -\frac{f^{(3)}(x_i)}{6} h^2$$

The third derivative of the function is 150 and

$$E_t \approx -\frac{150}{6}(0.25)^2 = -1.5625$$

which is exact. This occurs because the underlying function is a cubic equation that has zero fourth and higher derivatives.

True value:

$$f''(x) = 150x - 12$$
$$f''(2) = 150(2) - 12 = 288$$

<u>$h = 0.25$:</u>

$$f''(2) = \frac{f(2.25) - 2f(2) + f(1.75)}{0.25^2} = \frac{182.1406 - 2(102) + 39.85938}{0.25^2} = 288$$

<u>$h = 0.125$:</u>

$$f''(2) = \frac{f(2.125) - 2f(2) + f(1.875)}{0.125^2} = \frac{139.6738 - 2(102) + 68.82617}{0.125^2} = 288$$

Both results are exact because the errors are a function of 4th and higher derivatives which are zero for a 3rd-order polynomial.

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5) Problem 4.17

The second derivative of the function at $x = 2$ can be evaluated as

$$f''(2) = 150(2) - 12 = 288$$

For $h = 0.2$,

$$f''(2) = \frac{164.56 - 2(102) + 50.96}{(0.2)^2} = 288$$

For $h = 0.1$,

$$f''(2) = \frac{131.765 - 2(102) + 75.115}{(0.1)^2} = 288$$

Both are exact because the errors are a function of fourth and higher derivatives which are zero for a 3rd-order polynomial.

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6) Problem 4.24

First we can evaluate the exact values using the standard formula with double-precision arithmetic as

$$\begin{matrix} x_1 \\ x_2 \end{matrix} = \frac{5{,}000.002 \pm \sqrt{(5{,}000.002)^2 - 4(1)10}}{2(1)} = \begin{matrix} 5{,}000 \\ 0.002 \end{matrix}$$

We can then determine the square root term with 5-digit arithmetic and chopping

$$\sqrt{(5{,}000.0)^2 - 4(1)10} = \sqrt{2{,}500{,}000 - 4(1)10} = \sqrt{24{,}999{,}960} \xrightarrow{\text{chopping}} \sqrt{24{,}999{,}000}$$
$$= 4{,}999.996 \xrightarrow{\text{chopping}} 4{,}999.9$$

<u>Standard quadratic formula:</u>

$$x_1 = \frac{5{,}000.0 + 4{,}999.9}{2} = \frac{9{,}999.9}{2} = 4{,}999.95 \xrightarrow{\text{chopping}} 4{,}999.9$$
$$x_2 = \frac{5{,}000.0 - 4{,}999.9}{2} = \frac{0.1}{2} = 0.05$$

Thus, although the first root is reasonably close to the true value ($\varepsilon_t = 0.002\%$), the second is considerably off ($\varepsilon_t = 2400\%$) due primarily to subtractive cancellation.

<u>Equation (3.13):</u>

$$x_1 = \frac{-2(10)}{-5{,}000.0 + 4{,}999.9} = \frac{-20}{-0.1} = 200$$
$$x_2 = \frac{-2(10)}{-5{,}000.0 - 4{,}999.9} = \frac{-20}{-9{,}999.9} = 0.002$$

For this case, the second root is well approximated, whereas the first is considerably off ($\varepsilon_t = 96\%$). Again, the culprit is the subtraction of two nearly equal numbers.

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