Chapter 16

Fourier Analysis

Numerical Methods
Fall 2019

Chapter Objectives, 1

- Understanding sinusoids and how they can be used for curve fitting.
- Knowing how to use least-squares regression to fit a sinusoid to data.
- Knowing how to fit a Fourier series to a periodic function.
- Understanding the relationship between sinusoids and complex exponentials based on Euler's formula.
- Recognizing the benefits of analyzing mathematical function or signals in the frequency domain (i.e., as a function of frequency).
- Understanding how the Fourier integral and transform extend Fourier analysis to aperiodic functions.

Chapter Objectives, 2

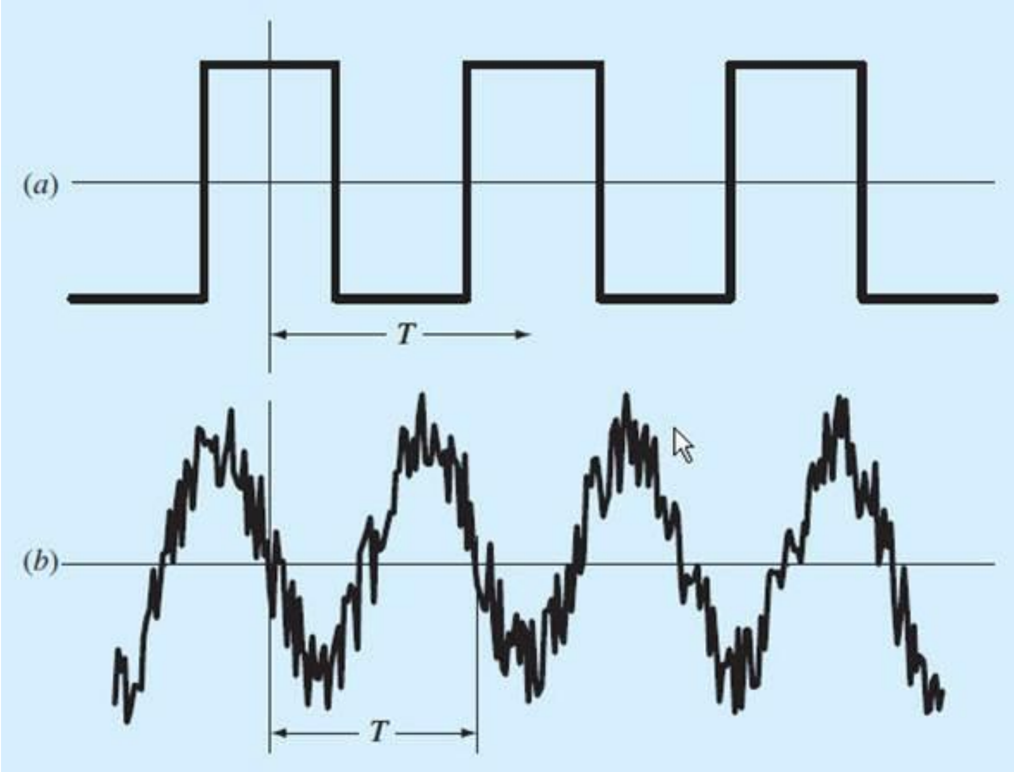
- Understanding how the discrete Fourier transform (DFT) extends Fourier analysis to discrete signals.
- Recognizing how discrete sampling affects the ability of the DFT to distinguish frequencies. In particular, know how to compute and interpret the Nyquist frequency.
- Recognizing how the fast Fourier transform (FFT) provides a highly efficient means to compute the DFT for cases where the data record length is a power of 2.
- Knowing how to use the MATLAB function `fft` to compute a DFT and understand how to interpret the results.
- Knowing how to compute and interpret a power spectrum.

Periodic Functions

- A periodic function is one for which

$$f(t) = f(t + T)$$

where T = the period



Sinusoids

$$f(t) = A_0 + C_1 \cos(\omega_0 t + \theta)$$

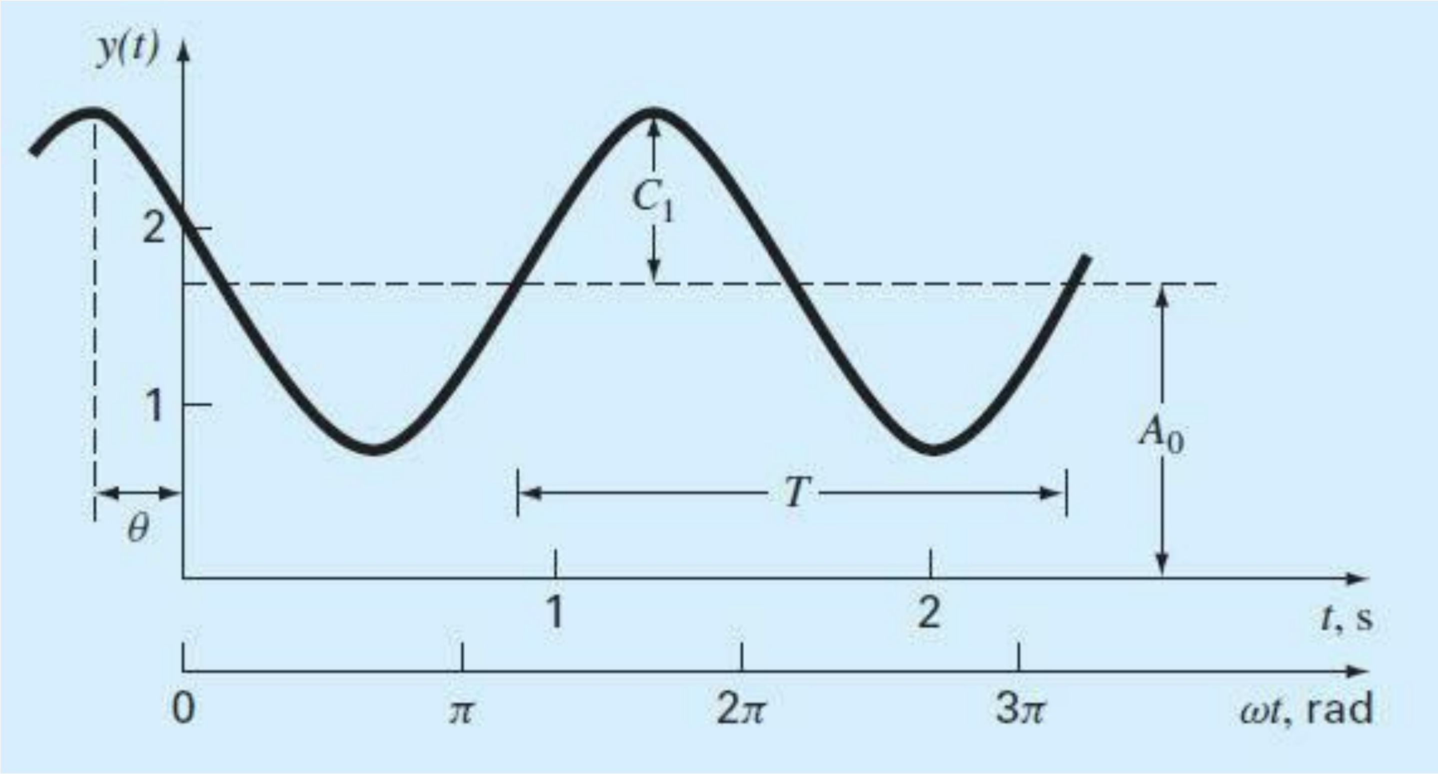
Phase shift

Mean value

Amplitude

Angular frequency

$$\omega_0 = 2\pi f = \frac{2\pi}{T}$$



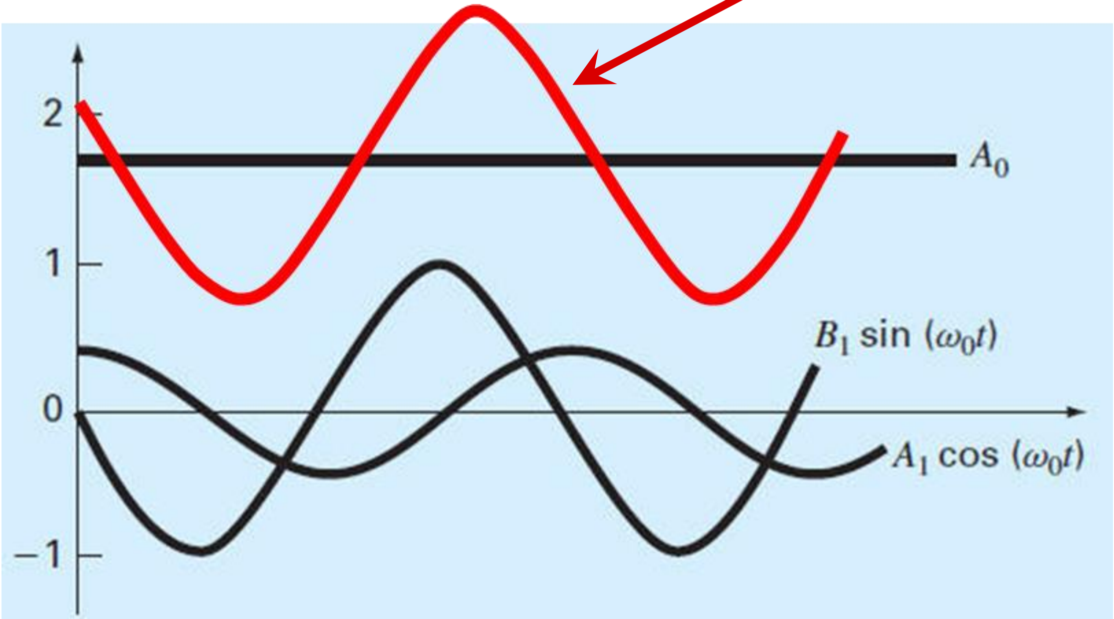
Alternative Representation

$$C_1\cos(\omega_0 t + \theta) = C_1[\cos(\omega_0 t + \theta)\cos(\theta) - \sin(\omega_0 t + \theta)\sin(\theta)]$$

$$f(t) = A_0 + A_1\cos(\omega_0 t) + B_1\sin(\omega_0 t)$$

$f(t) = A_0 + A_1\cos(\omega_0 t) + B_1\sin(\omega_0 t)$

$$A_1 = C_1\cos(\theta)$$
$$B_1 = -C_1\sin(\theta)$$



- The two forms are related by

$$C_1 = \sqrt{A_1^2 + B_1^2} \qquad \theta = \arctan(-B_1/A_1)$$

Least-Squares Fit of a Sinusoid

- linear least-squares model of a sinusoid function

$$y = A_0 + A_1\cos(\omega_0 t) + B_1\sin(\omega_0 t) + e$$

- Thus, our goal is to determine coefficient values that minimize

$$S_r = \sum_{i=1}^{N}\{y_i - [A_0 + A_1\cos(\omega_0 t) + B_1\sin(\omega_0 t)]\}^2 \Rightarrow Z = \begin{bmatrix} 1 & \cos(\omega_0 t) & \sin(\omega_0 t) \\ & \ldots. & \\ & \ldots.. & \end{bmatrix}$$

- The normal equations to accomplish this minimization can be expressed in matrix form as [recall Eq. (15.10)]

$$\begin{bmatrix} N & \Sigma\cos(\omega_0 t) & \Sigma\sin(\omega_0 t) \\ \Sigma\cos(\omega_0 t) & \Sigma\cos^2(\omega_0 t) & \Sigma\cos(\omega_0 t)\sin(\omega_0 t) \\ \Sigma\sin(\omega_0 t) & \Sigma\cos(\omega_0 t)\sin(\omega_0 t) & \Sigma\sin^2(\omega_0 t) \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ B_1 \end{Bmatrix} = \begin{Bmatrix} \Sigma y \\ \Sigma y\cos(\omega_0 t) \\ \Sigma y\sin(\omega_0 t) \end{Bmatrix}$$

$$Z^T Z\, x = Z^T y$$

Least-Squares Fit of a Sinusoid

- there are N observations equispaced at intervals of t and with a total record length of $T = (N - 1)\Delta t$.

$$\frac{\Sigma \sin(\omega_0 t)}{N} = 0 \qquad \frac{\Sigma \cos(\omega_0 t)}{N} = 0$$

$$\frac{\Sigma \sin^2(\omega_0 t)}{N} = \frac{1}{2} \qquad \frac{\Sigma \cos^2(\omega_0 t)}{N} = \frac{1}{2}$$

$$\frac{\Sigma \cos(\omega_0 t)\,\sin(\omega_0 t)}{N} = 0$$

- Thus, for equispaced points the normal equations become

$$\begin{bmatrix} N & 0 & 0 \\ 0 & N/2 & 0 \\ 0 & 0 & N/2 \end{bmatrix} \begin{Bmatrix} A_0 \\ A_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} \Sigma y \\ \Sigma y \cos(\omega_0 t) \\ \Sigma y \sin(\omega_0 t) \end{Bmatrix}$$

Least-Squares Fit of a Sinusoid

- Thus, the coefficients can be determined as

$$\begin{Bmatrix} A_0 \\ A_1 \\ B_2 \end{Bmatrix} = \begin{bmatrix} 1/N & 0 & 0 \\ 0 & 2/N & 0 \\ 0 & 0 & 2/N \end{bmatrix} \begin{Bmatrix} \Sigma y \\ \Sigma y \cos(\omega_0 t) \\ \Sigma y \sin(\omega_0 t) \end{Bmatrix}$$

$$A_0 = \frac{\Sigma y}{N}$$

$$A_1 = \frac{2}{N} \Sigma y \cos(\omega_0 t)$$

$$B_1 = \frac{2}{N} \Sigma y \sin(\omega_0 t)$$

EXAMPLE 16.1

Using Sinusoids for Curve Fitting

- You will frequently have occasions to estimate intermediate values between precise data points.
- The function you use to interpolate must pass through the actual data points - this makes interpolation more restrictive than fitting.
- The most common method for this purpose is polynomial interpolation, where an $(n\text{-}1)^{th}$ order polynomial is solved that passes through n data points:

$$f(x) = a_1 + a_2x + a_3x^2 + \cdots + a_nx^{n-1}$$

- MATLAB version:

$$f(x) = p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$$

Continuous Fourier Series

- For a function with period T, a continuous Fourier series can be written

$$f(t) = a_0 + a_1\cos(\omega_0 t) + b_1\sin(\omega_0 t) + a_2\cos(2\omega_0 t) + b_2\sin(2\omega_0 t) + \cdots$$

or more concisely,

$$f(t) = a_0 + \sum_{k=1}^{\infty}[a_k\cos(k\omega_0 t) + b_k\sin(k\omega_0 t)]$$

  - $\omega_0 = 2\pi/T$: fundamental frequency
  - $2\omega_0$, $3\omega_0$, etc., : harmonics

$$a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_0 t)\,dt \qquad b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_0 t)\,dt$$

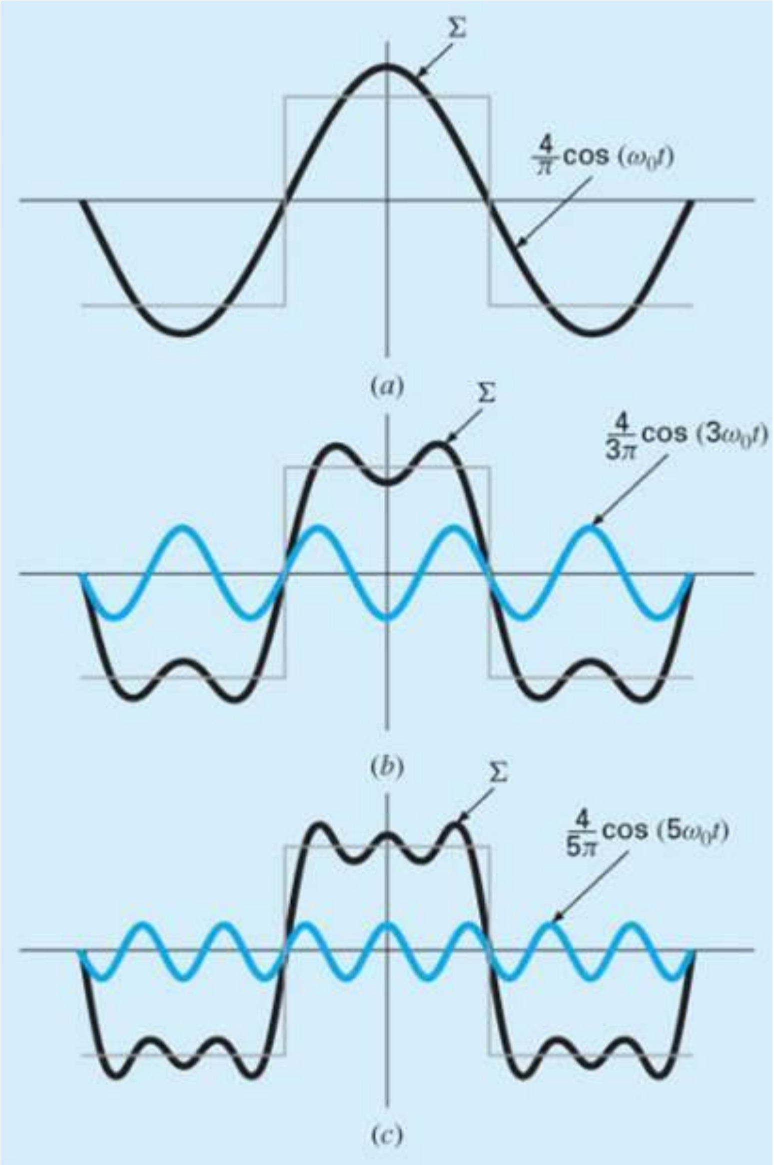
for $k = 1, 2, \ldots$ and

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

Continuous Fourier Series

EXAMPLE 16.2

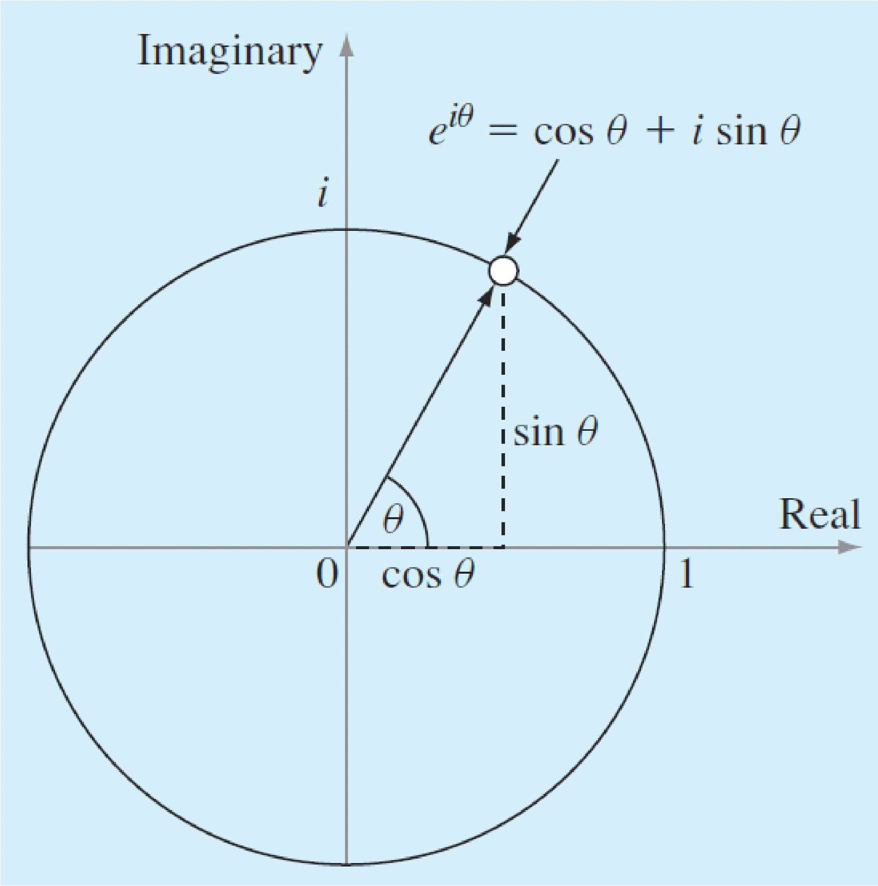
Continuous Fourier Series Approximation



Fourier Series by Euler's Formula

- Fourier series can also be expressed in a more compact form using complex notation.
  - Euler's formula

$$e^{\pm ix} = \cos x \pm i \sin x$$



- Fourier series is expressed as

$$f(t) = \sum_{k=-\infty}^{\infty} \tilde{c}_k e^{ik\omega_0 t}$$

where the coefficients are

$$\tilde{c}_k = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-ik\omega_0 t}\, dt$$

Fourier Integral (Transform)

- The transition from a periodic to a nonperiodic function can be effected by allowing the period to approach infinity. In other words, as T becomes infinite, the function never repeats itself and thus becomes aperiodic.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$$

- and the coefficients become a continuous function of the frequency variable ω, as in

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

Discrete Fourier Transform

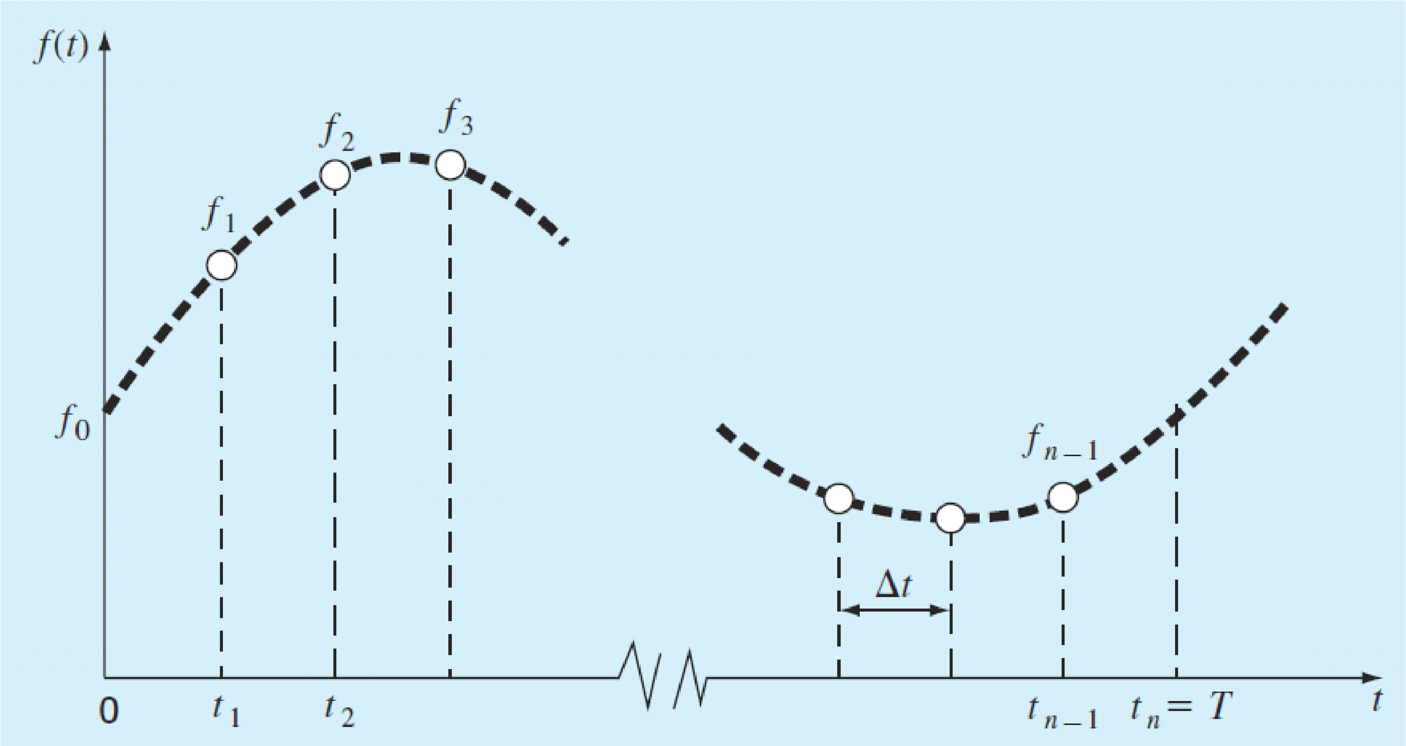
- In engineering, functions are often represented by a finite set of discrete values.
- A time interval from 0 to T can be divided into n equispaced subintervals with widths of $\Delta t = T/n$
- Thus, f_j designates a value of the continuous function $f(t)$ taken at t_j.
- A discrete Fourier transform

$$F_k = \sum_{j=0}^{n-1} f_j e^{-ik\omega_0 j} \qquad \text{for } k = 0 \text{ to } n-1 \qquad \omega_0 = 2\pi/n$$

- The inverse Fourier transform

$$f_j = \frac{1}{n}\sum_{k=0}^{n-1} F_k e^{ik\omega_0 j} \qquad \text{for } j = 0 \text{ to } n-1 \qquad \omega_0 = 2\pi/n$$

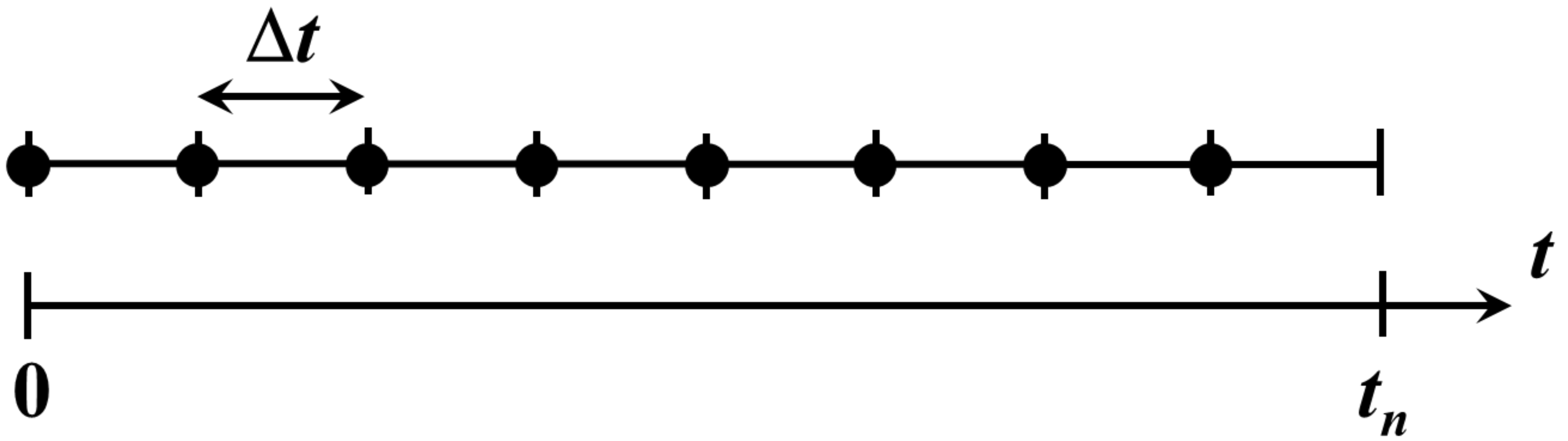
Discrete Fourier Transform



The sampling points of the discrete Fourier series.

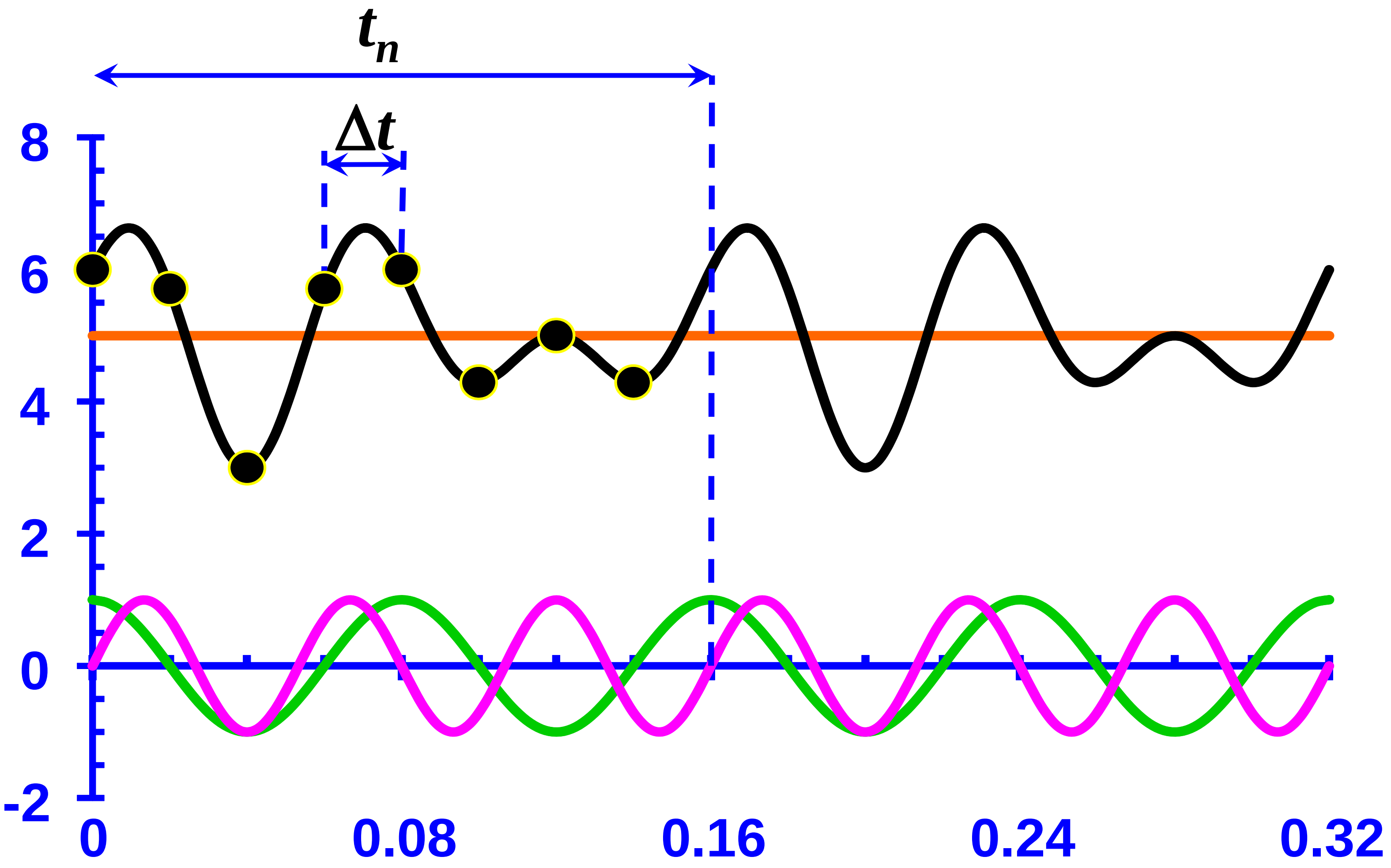
SAMPLING, 1

- The frequencies you can detect with the DFT depend on how frequently (Δt) and how long (t_n) you sample the time series.
- The highest frequency that can be measured in a signal (f_{max}), called the *Ny quist frequency,* is half the sampling frequency ($0.5f_s$).
- The lowest frequency (f_{min}) you can detect is the inverse of the total sample length ($1/t_n$).



EXAMPLE 16.3

- $f(t) = 5 + \cos(2\pi(12.5)t) + \sin(2\pi(18.75)t)$

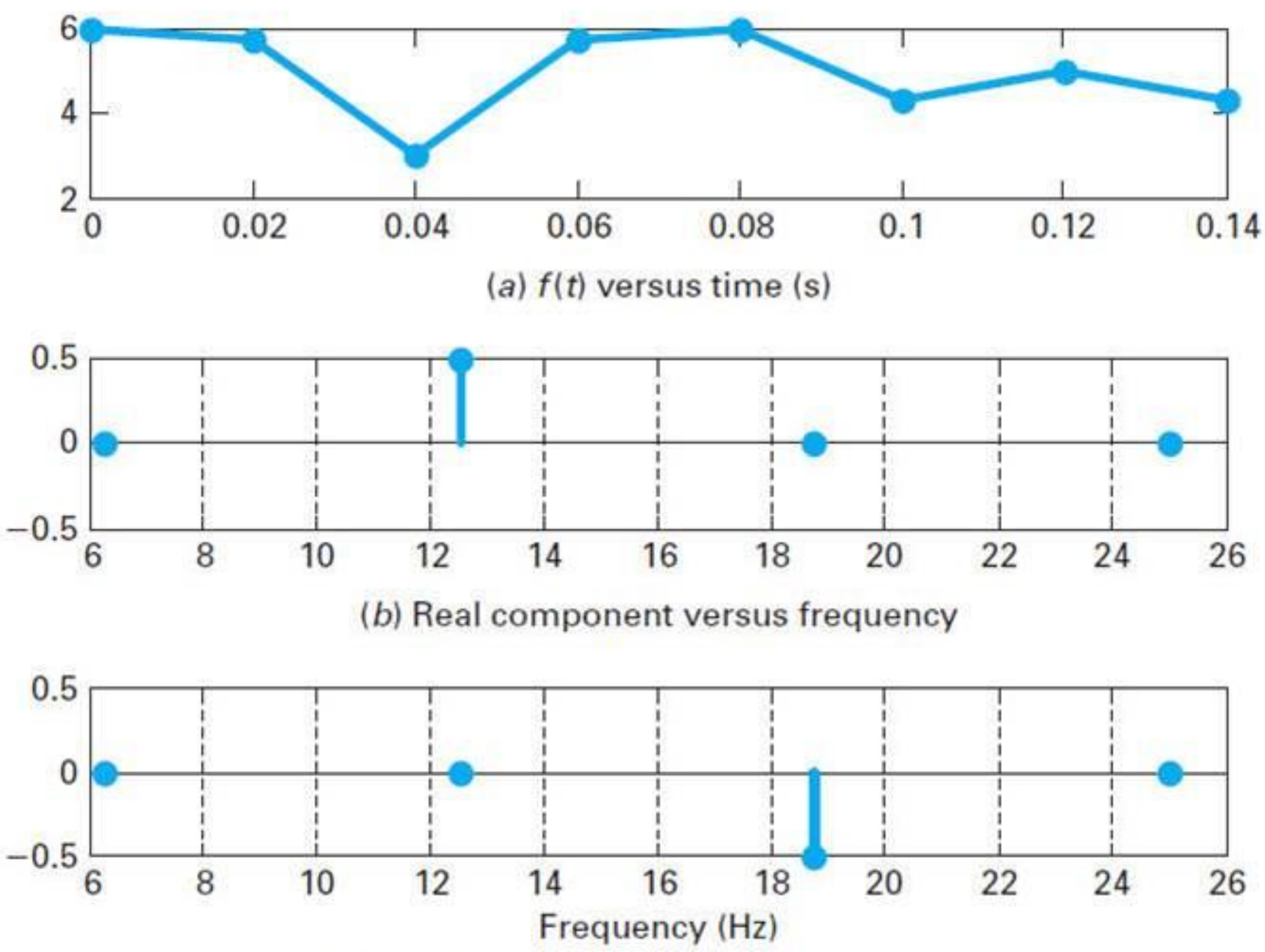


EXAMPLE, 2

$$f(t) = 5 + \cos(2\pi(12.5)t) + \sin(2\pi(18.75)t)$$
$$n = 8$$
$$\Delta t = 0.02 \text{ s}$$
$$f_s = 1/\Delta t = 1/0.02 = 50 \text{ Hz}$$
$$t_n = n/f_s = 8/50 = 0.16 \text{ s}$$
$$f_{\max} = f_s/2 = 50/2 = 25 \text{ Hz}$$
$$f_{\min} = 1/t_n = 1/0.16 = 6.25 \text{ Hz}$$
$$\Delta f = f_s/n = 50/8 = 6.25 \text{ Hz}$$

- Therefore, for this sampling strategy (taking 8 samples over 0.16 s), we can detect frequencies ranging from 6.25 to 25 Hz with a resolution of 6.25 Hz. Hence, for this simple example the FFT will exactly detect the 12.5 and the 18.75 Hz sinusoids.

$$f(t) = 5 + \cos(2\pi(12.5)t) + \sin(2\pi(18.75)t)$$



(*a*) $f(t)$ versus time (s)

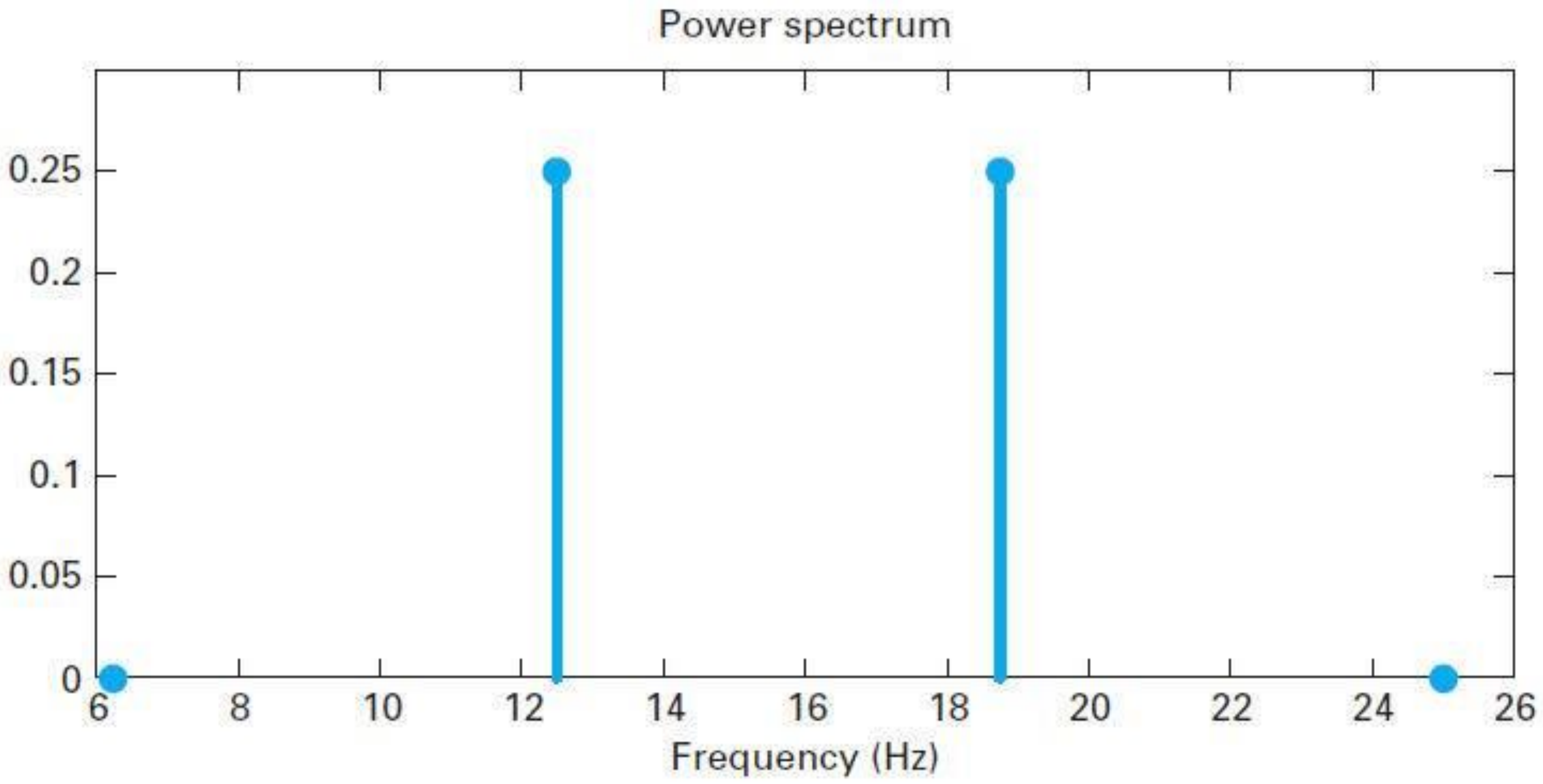
(*b*) Real component versus frequency

(*c*) Imaginary component versus frequency

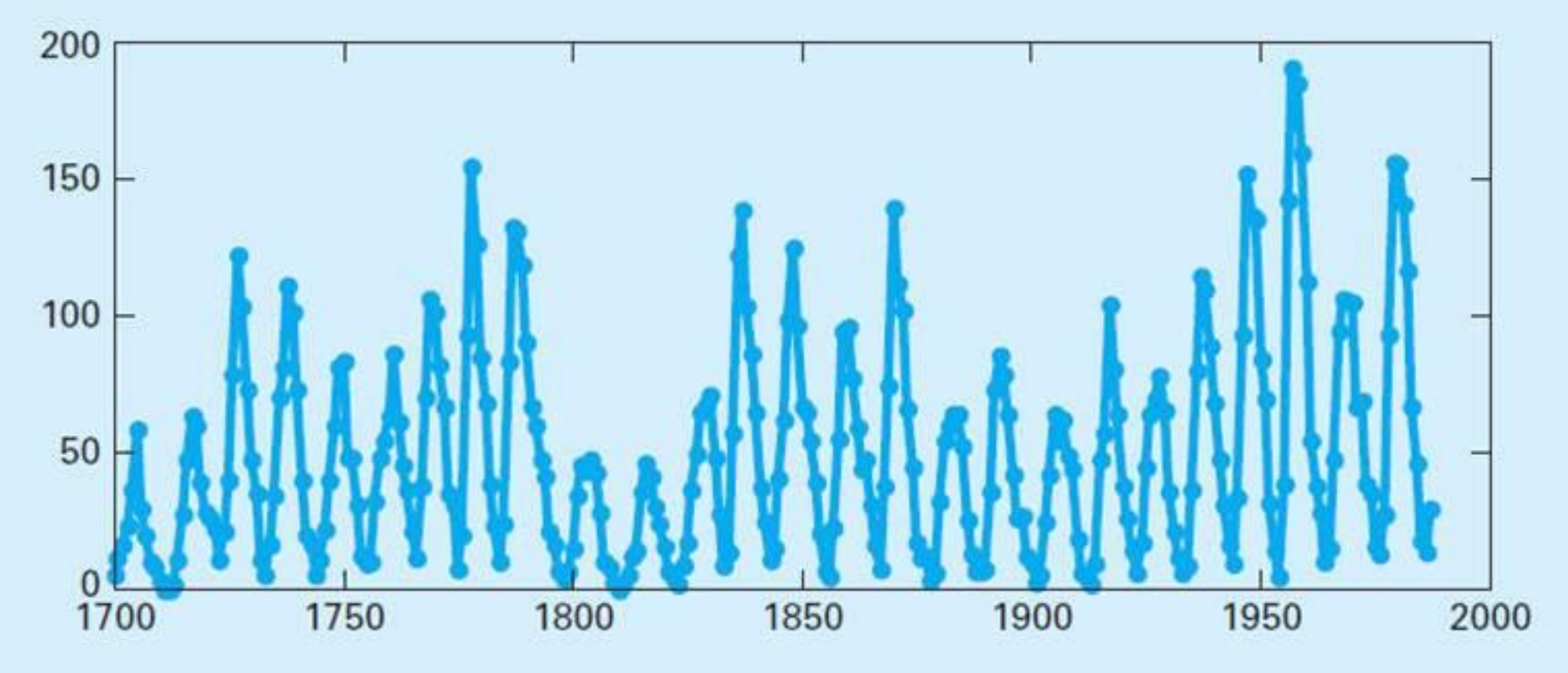
Power Spectrum

- A power spectrum consists of a plot of the power associated with each frequency component versus frequency.

$$P_k = |\tilde{c}_k|^2$$



Wolf Sunspot Number Versus Year



Power Spectrum for Sunspot Numbers