10907 Pattern Recognition

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Exercise 4 — Support Vector Machine

Deadline **15.11** Upload .ZIP on Adam.

The exercises are done in groups of 2 students. Only upload 1 version of the exercise on Adam and specify your group partner.

Upload in .zip format containing 1 .pdf file with your answers to the questions and the code folder files (do NOT include the data folder). Create the .zip file with the `createSubmission.py` script.

In this series you will implement and compare linear and non-linear support vector machines. The classifiers will be applied to four problems: A linear separable toy problem, a non-linear separable toy problem, handwritten character recognition (OCR) and images of *ships* versus *not ships*.

Data:

Each `mat`-file contains the training and the test set for a problem, named `NAME_(train|test)`. The sets are encoded as $(d+1) \times N$-matrix, where N is the total number of data points and d the dimension of the feature vector. The first column contains the label $y \in \{-1, 1\}$. For the non-linear separable toy example, the data is available in the `.mat` files. The dimensionality of the data is structured the same as in the `mat`-files, except that the data and the labels are stored in separate files.
The provided skeleton code already loads the data for you and reshapes it into an appropriate format.

- `toy` ($d = 2$): A very small (x,y) toy data set. Use it for development and to study the behaviour of your linear classifiers. It can be easily visualized.
- `zip13` ($d = 256$): Handwritten digits automatically scanned by the U.S. Postal Service. The set is reduced to binary classification scenarios: the digits 1 and 3 are provided. The digits are normalised to a grey scale 16×16 grid.
- `zip38` ($d = 256$): As above, but with the digits 3 and 8.
- `ship_no_ship` ($d = 32 \times 32 \times 3 = 3072$): An extract from the popular CIFAR dataset.

If you are interested in the complete USPS data set, it can be obtained from
`http://statweb.stanford.edu/~tibs/ElemStatLearn/data.html`

1 Linear SVM

Implement and test a Support Vector Machine classifier. The following TODO sections will first provide mathematic notation to help to solve the programming parts.

Todo 1 (Linear Support Vector Machine - CVXOPT interface) *Start with the quadratic programming problem given in the lecture script. To solve it, the function `cvxopt.solvers.qp()` will be used. Use the full documentation online to get more details:*
`http://cvxopt.org/userguide/coneprog.html#quadratic-programming`.

The primal problem of the SVM as seen in the lecture:

$$\underline{\omega} = \underset{\underline{\omega}}{\operatorname{argmin}} \frac{1}{2}\|\underline{\omega}\|^2 \text{ subject to } y_i(\underline{\omega}^T \underline{x}_i + \omega_0) \geq 1 \text{ , } \forall i$$

And with the corresponding dual problem:

$$\underline{\lambda} = \underset{\underline{\lambda}}{\operatorname{argmax}} \left(\sum_{i}^{N} \lambda_i - \frac{1}{2} \sum_{i,j}^{N} \lambda_i \lambda_j y_i y_j \underline{x}_i^T \underline{x}_j \right) \text{ subject to } \sum_{i=1}^{N} \lambda_i y_i = 0 \text{ , } \lambda_i \geq 0 \text{ , } \forall i$$

In vector format, the dual problem looks as follows:

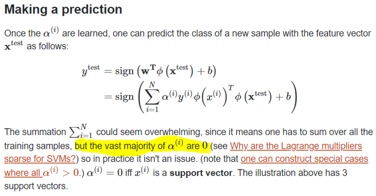
$$\underline{\lambda} = \underset{\underline{\lambda}}{\operatorname{argmin}} \left(\frac{1}{2} \underline{\lambda}^T \underline{H} \underline{\lambda} - \underline{1}^T \underline{\lambda} \right) \text{ subject to } \sum_{i=1}^{N} \lambda_i y_i = 0 \text{ , } \lambda_i \geq 0, \forall i$$

Where H is a $N \times N$ matrix ($N = \#ofSamples$) created from the given data x and their respective labels y as seen in the expanded dual problem: $y_i y_j \underline{x}_i^T \underline{x}_j$.
The problem needs to be translated into the interface provided by `CVXOPT` with the function `cvxopt.solvers.qp(P, q[, G, h[, A, b[, solver[, initvals]]]])`.

$$\underline{x} = \underset{\underline{x}}{\operatorname{argmin}} \left(\frac{1}{2} \underline{x}^T \underline{P} \underline{x} + \underline{q}^T \underline{x} \right) \text{ subject to } \underline{A}\underline{x} = \underline{b} \text{ , } \underline{G}\underline{x} \leq \underline{h}$$

Construct the respective matrices and vectors by using the CVXOPT matrix `cvx.matrix()`. The solution can be optioned through the x variable:

```
import cvxopt as cvx
cvx.solvers.options['show_progress'] = False
solution = cvx.solvers.qp(P, q, G, h, A, b)
lambdas = solution['x']
```



Making a prediction

Once the $\alpha^{(i)}$ are learned, one can predict the class of a new sample with the feature vector $\mathbf{x}^{\text{test}}$ as follows:

$$\begin{aligned} y^{\text{test}} &= \operatorname{sign}\left(\mathbf{w}^{\mathbf{T}} \phi\left(\mathbf{x}^{\text{test}}\right) + b\right) \\ &= \operatorname{sign}\left(\sum_{i=1}^{N} \alpha^{(i)} y^{(i)} \phi\left(x^{(i)}\right)^{T} \phi\left(\mathbf{x}^{\text{test}}\right) + b\right) \end{aligned}$$

The summation $\sum_{i=1}^{N}$ could seem overwhelming, since it means one has to sum over all the training samples, but the vast majority of $\alpha^{(i)}$ are 0 (see Why are the Lagrange multipliers sparse for SVMs?) so in practice it isn't an issue. (note that one can construct special cases where all $\alpha^{(i)} > 0$.) $\alpha^{(i)} = 0$ iff $x^{(i)}$ is a **support vector**. The illustration above has 3 support vectors.

`Warning: The` x `matrix in CVXOPT should not be confused with the data matrix. The` x `matrix used in CVXOPT is instead the` λ `vector from the dual problem.`

Remember only to use the λ's that are larger than 0 (in practice use a value close to zero: 1e-5). Refer to the lecture slides for information about how to compute the $\underline{w}$ vector and the bias term w_0. The w_0 value can be computed by using the mean of all support vectors for stability.

Todo 2 (SVM training implementation) *Use the above information about the CVXOPT library to implement the `train` function in the SVM class found in the `svm.py` file.*

Todo 3 (SVM linear classification) *Implement the linear classification function `classifyLinear()` in the SVM class. The formula for the linear classifier (primal form):*

$$f(\underline{x}) = \underline{\omega}^T \underline{x} + \omega_0$$

Call the above classification function from the `printLinearClassificationError()`. Compute and print the classification error.

Todo 4 (Experiments) *Apply your classifier to the linear toy example. Skeleton code is given in the file `ex4-SVM_1_LinearToy.py`.*

Todo 5 (Soft-margin) *Extend the linear solution with a soft margin (controlled by the C parameter).*
Remember from the lecture slides that the contraint $0 \leq \lambda_i, \forall i$ needs to be further constrained to

$$0 \leq \lambda_i \leq C$$

Try running the `ex4-SVM_1_LinearToy.py` with different `C` values and observe what happens. C values: `1,10,100,1000,None`

Hint: To add the soft-margin, extend the G and the h matrices.

NOTE: the bias w_0 should only be approximated from the support vectors, i.e. the data points on the margins. When using the slack-variables, data points not on the margin will also have $\lambda > 0$. For simplicity, you can however approximate the bias from all $\lambda > 0$.

2 Kernel SVM

Todo 6 (Kernel functions and kernel matrix) *Extend the `SVM` class by implementing the local kernel functions.*
Implement the linear, the polynomial and the RBF kernel function

- *Linear kernel:* $k(\underline{x}, \underline{x'}) = \underline{x}^T \underline{x'}$
- *Polynomial kernel:* $k(\underline{x}, \underline{x'}) = (\underline{x}^T \underline{x'} + 1)^p$
- *RBF (Gaussian) kernel:* $k(\underline{x}, \underline{x'}) = \exp\left(-\frac{\|x-x'\|^2}{2\sigma^2}\right)$

The `__computeKernel__` function also needs to be implemented. This computes the complete kernel matrix and takes in the kernel function to be used as an argument together with the kernel parameter.

Todo 7 (Non-linear SVM) *Expand the `SVM` class with a `classifyKernel()` function and the `printKernelClassificationError` similar to the linear case.*
Formula for the non-linear classifier (dual form):

$$f(\underline{x}) = \omega_0 + \sum_{i=0}^{N_s} \lambda_i y_i K(\underline{x}_i, \underline{x})$$

Because of the non-linearity of the classification there are no w coefficients anymore. Think of how the decision surface is now defined and how a single data point is classified.

Todo 8 (Non-linear Experiments) *Apply your classifier to the non-linear separable toy example. Main function found in the file `ex4-SVM_2_KernelToy.py`. Try out different kernels and kernel parameters to see what gives the best solution boundary.*

Todo 9 (MNIST and CIFAR Experiments) *Train your non-linear SVM for the `zip13`, `zip38` and the `CIFAR` datasets. The main files to run are the files: `ex4-SVM_3_ZIP13.py`, `ex4-SVM_3_ZIP38.py` and ex4-SVM_3_Ship_noShip.py. Visualization of correct and incorrect classified images are done in the `visualizeClassification()` function. Compare the classification train and test error by using different kernels (linear, polynomial, RBF).*

Todo 10 (Linear vs Non-linear speed comparison) *Use the linear separable dataset to perform a speed comparison of the linear svm and a kernel svm with a linear kernel. Implement this in the `ex4-SVM_4_SpeedComparison.py` file. Train the 2 SVM classifiers, and measure how much time the `classifyLinear` and `classifyKernel` calls take.*

3 Theory / coding questions [10p]

1. What criterion does the SVM rely on when finding the optimal hyperplane location? [1P]
2. Include the SVM plot for the linearly separable toy example case with $C = None$ and $C = 10$. [1P]
3. What is a support vector? And, how do you determine the support vectors in your implementation? [1P]
4. Compare the testing and training errors of the linear SVM in the linearly separable case when $C = None$ and $C = 10$. [1P].
5. What is the role of the slack parameter C? [1P]
6. List benefits of using the SVM compared to: the perceptron classifier and the least squares classifier. [1P]
7. In the nonlinear SVM that uses the RBF kernel, what is the influence of Sigma on the decision surface? Are you more likely to see islands forming around single data inputs as sigma increases or decreases and why? [0.5P]
8. Include the SVM decision surface plot for the kernel toy example with $C = None$, $kernel =' rbf'$, $kernelpar = 0.5$. [1P]
9. `SpeedComparison`, note down the average time for each classifier (linear SVM vs Kernel SVM with a linear kernel) by running it 1000 times [0.5P]
10. Which classifier (linear SVM or Kernel SVM with a linear kernel) is faster? And why? [1P]
11. Image classification (ZIP13/ZIP38/Ship) find a configuration that works well for all 3 datasets. Specify the configuration parameters (C, kernel, kernel parameter) and the training + test error for all 3 datasets.[1P]