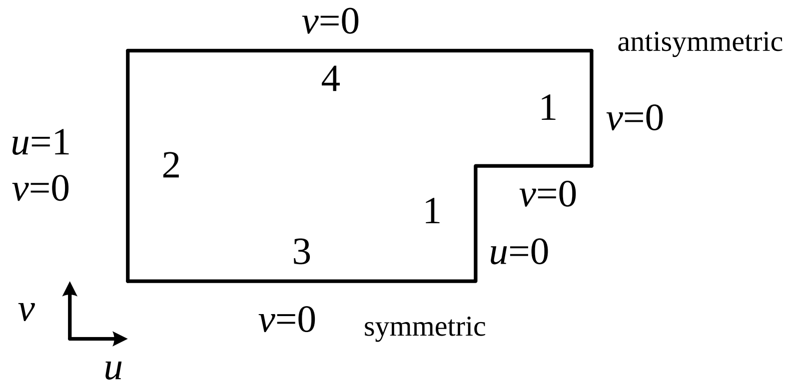
1 Ideal Fluid

The quarter model is



1.1 Velocity Potential

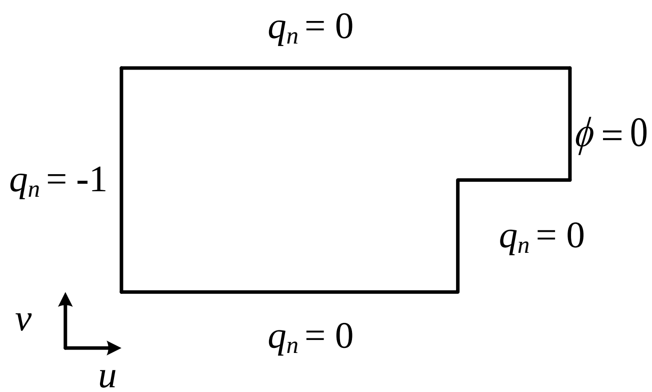
Assume the components of velocity $\vec{v} = u\hat{i} + v\hat{j}$ of the ideal fluid are

$$u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y} \tag{1.1}$$

Due to the incompressibility and irrotationality of the ideal fluid, the governing equation is

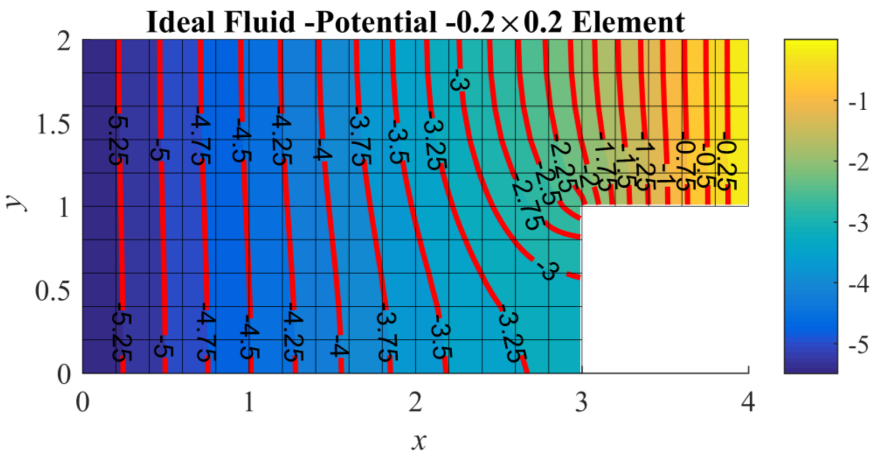
$$\nabla^2 \phi = 0 \tag{1.2}$$

Boundary conditions are



where $q_n = \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y$.

Using the 4-node quadrilateral element with the size of 0.2×0.2, the velocity potential ϕ can be obtained.



1.2 Streamline Function

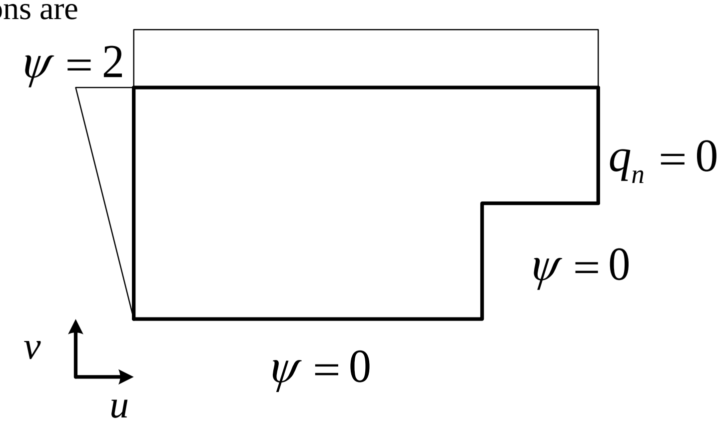
Assume the components of velocity $\vec{v} = u\hat{i} + v\hat{j}$ of the ideal fluid are

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x} \tag{1.3}$$

Due to the incompressibility and irrotationality of the ideal fluid, the governing equation is

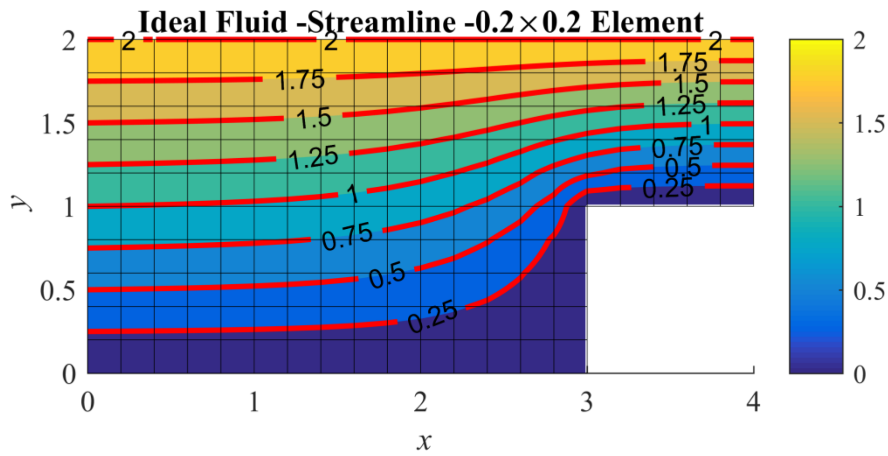
$$\nabla^2 \psi = 0 \tag{1.4}$$

Boundary conditions are



where $q_n = \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y$.

Using the 4-node quadrilateral element with the size of 0.2×0.2, the streamline function ψ can be obtained.



2 Torsion

2.1 St. Venant's Torsion Theory

Displacement field

$$\begin{cases} u(x,y,z) = -\alpha yz \\ v(x,y,z) = \alpha xz \\ w(x,y,z) = \alpha\phi(x,y) \end{cases} \tag{2.1}$$

where $\phi(x,y)$ is the warping function of the cross section, and α is the angle of twist per unit length.

The governing equation is

$$\nabla^2\phi = 0 \tag{2.2}$$

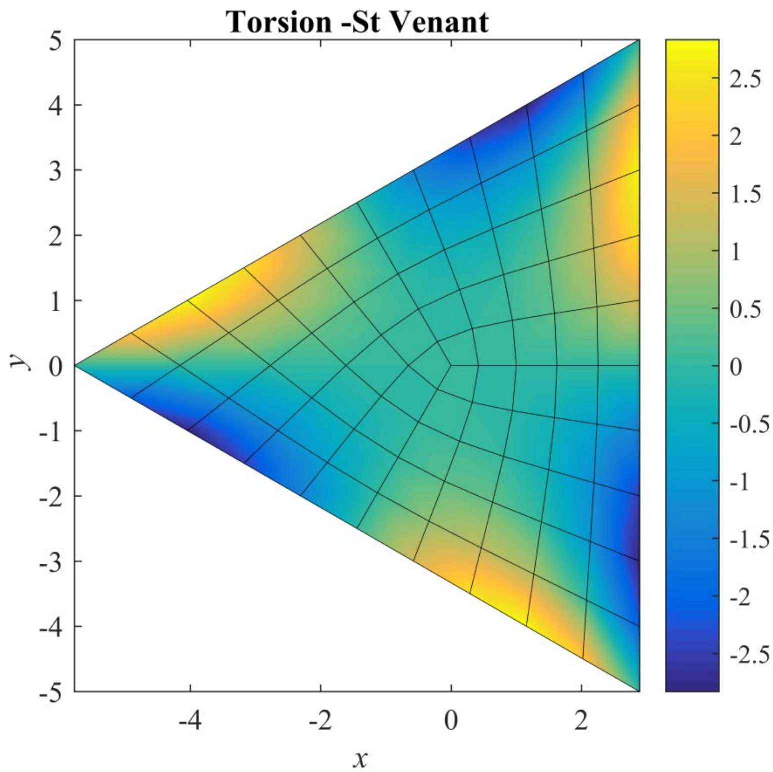
The boundary condition is

$$q_n = \frac{\partial\phi}{\partial x}n_x + \frac{\partial\phi}{\partial y}n_y = yn_x - xn_y \tag{2.3}$$

In the following example, assuming the isotropic material, Young's modulus $E = 2.06\times10^{11}$ Pa, Poisson's ratio $\nu = 0.3$, shear modulus $G = E/2(1+\nu) = 7.9231\times10^{10}$ Pa, and the applying torque $T = 2000$ N.

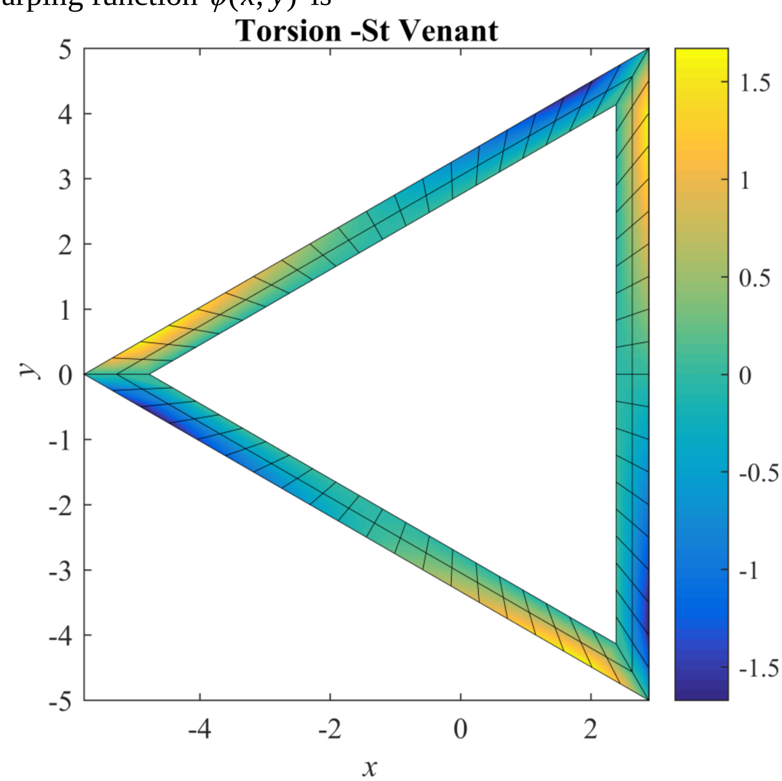
2.1.1 Solid cross section

The solution of warping function $\phi(x,y)$ is



2.1.2 Hollow cross section

The solution of warping function $\phi(x, y)$ is



2.2 Prandtl's Torsion Theory

Assume the shear stresses σ_{xz} and σ_{yz} can be expressed by a stress function $\psi(x,y)$

$$\begin{cases} \sigma_{xz} = \dfrac{\partial \psi}{\partial y} \\ \sigma_{yz} = -\dfrac{\partial \psi}{\partial x} \end{cases} \tag{2.4}$$

The governing equation is

$$\nabla^2 \psi = -2G\alpha \tag{2.5}$$

where G is the shear modulus and α is the angle of twist per unit length.

The boundary condition is

$$\frac{d\psi}{ds} = 0 \tag{2.6}$$

or equivalently, on each boundary (internal or external)

$$\psi = C_i,\ i = 1, 2, ..., n \tag{2.7}$$

where C_i are constants which are not necessarily equal, and n is the number of boundaries (internal and external).

In practice, for the external boundary, we can specify $\psi = 0$. To find C_i for the other n-1 internal boundaries, we use the single-valuedness condition of the warping displacement

$$\oint_{\Gamma^i} \phi ds = 0 \tag{2.8}$$

Substituting St. Venant's stresses and Eq.(2.4) into (2.8) leads to

$$\begin{aligned} &\oint_{\Gamma^i} \sigma_{xz} dx + \sigma_{yz} dy + G\alpha \oint_{\Gamma^i} y dx - x dy = 0 \\ &\Rightarrow -\oint_{\Gamma^i} \left(\frac{\partial \psi}{\partial x} n_x + \frac{\partial \psi}{\partial y} n_y\right) ds - 2G\alpha \iint_{\Omega^i} dx dy = 0 \\ &\Rightarrow \oint_{\Gamma^i} -\left(\frac{\partial \psi}{\partial x} n_x + \frac{\partial \psi}{\partial y} n_y\right) ds = 2G\alpha A_i \end{aligned} \tag{2.9}$$

where A_i is the area of the contour Γ^i encloses, and the path integrals are in a counterclockwise direction.

To apply this boundaries condition to the finite element model, note Eq.(2.9) also means

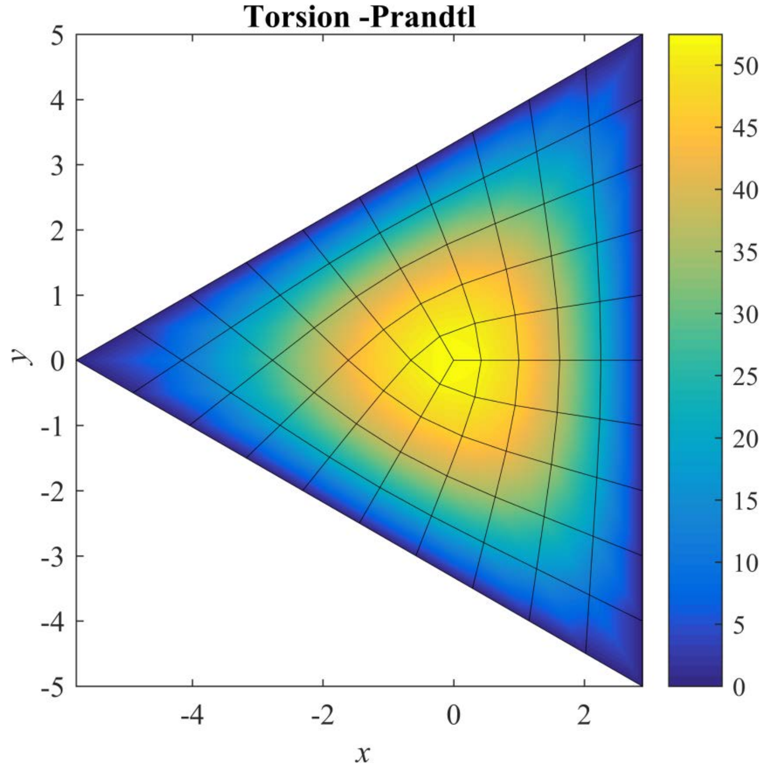
$$\oint_{\Gamma^i} q_n ds = 2G\alpha A_i \tag{2.10}$$

Therefore, for the internal boundary Γ^i, choose one node as the master node and the other nodes as the slave nodes, and let the value of ψ of slave nodes equal to that of the corresponding master node. And only apply a concentrated force which equals to $2G\alpha A_i$ to the master node. Then the stress function ψ of the whole cross section can be solved.

The parameters used here are the same as stated in St. Venant part.

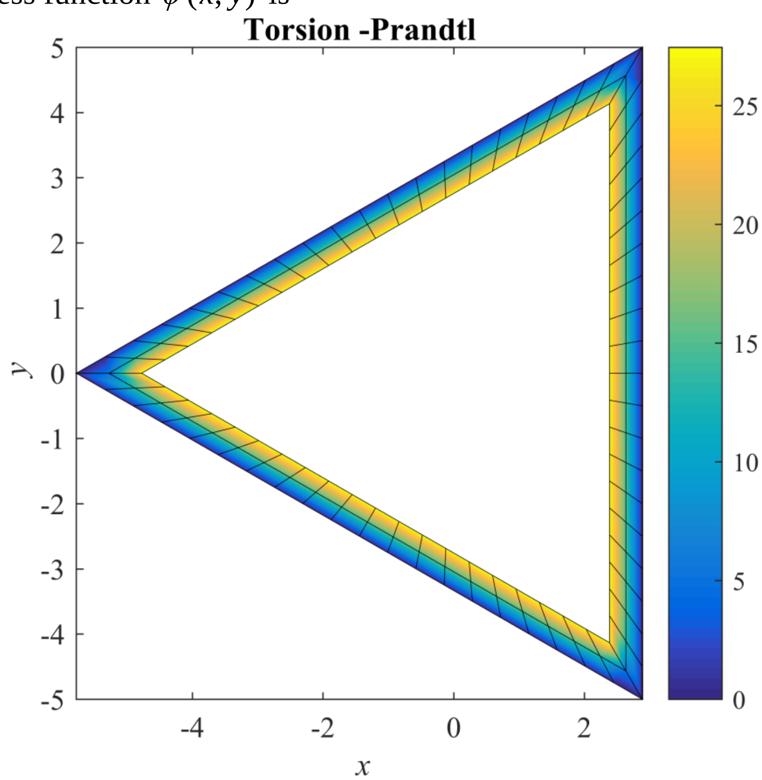
2.2.1 Solid cross section

The solution of stress function $\psi(x, y)$ is



2.2.2 Hollow cross section

The solution of stress function $\psi(x, y)$ is



2.3 Comparison

		St Venant	Prandtl	Ansys
Solid	J (m^4)	220.744	213.194	216.514
	α	1.14E-10	1.18E-10	
Hollow	J (m^4)	103.745	90.5134	101.755
	α	2.43E-10	2.79E-10	