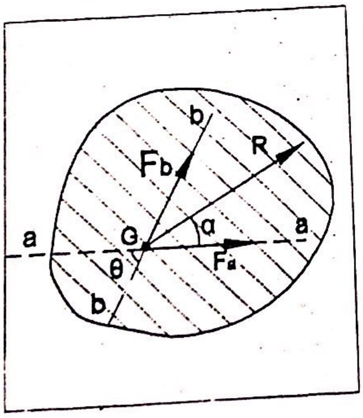
Chapter 2 - COPLANAR FORCES

Coplanar Forces

(I) Resolution of a force:



It is always necessary to replace a given fore by its vector components which act in specified directions. This is called as resolution of a force. A given force can be resolved into two or more vector components.

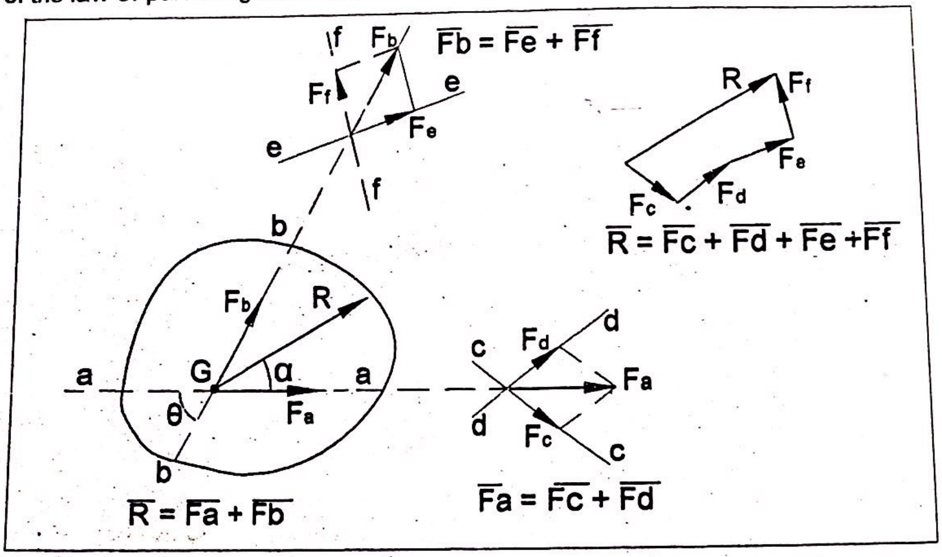
To resolve the given force 'R', into two components 'Fa' and 'Fb' along the lines (a-a) and (b-b), we use the law of parallelogram of forces. Thus in the above figure,

$$R=\sqrt{F_a^2+F_b^2+2.F_a.F_b.\cos\theta} \text{ ——— } (1)$$

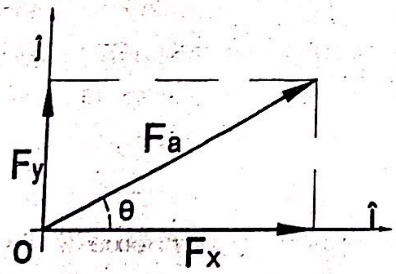
$$\tan\alpha=\left(\frac{F_b.\sin\theta}{F_a+F_b.\sin\theta}\right) \text{ ——— } (2)$$

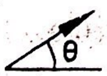
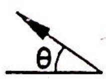
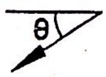
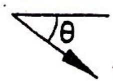
solving the above two equations, the components Fa and Fb along the two given lines can be obtained.

A single force can be resolved into many vector components by repeated use of the law of parallelogram of forces. This is explained in the following figure.



Rectangular components:



Polar Form	Rectangular Form
	F = Fx î +Fy ĵ
	F = - Fx î +Fy ĵ
	F = - Fx î -Fy ĵ
	F = Fx î -Fy ĵ

This is the most common resolution of a force vector into two mutually perpendicular component vectors. This can also be considered as a special case of the law of parallelogram of forces where the angle between the two component vectors is 90°. In this case $\overline{Fx}$ and $\overline{Fy}$ are the vector components of a force vector $\overline{F}$.

Fx and Fy are called as x and y scalar components force F. Depending upon the quadrant into which force 'F' points, the components Fx and Fy can be taken as positive or negative.

Thus, a force can be expressed in two different forms (i) Rectangular form using Cartesian frame of reference (ii) Polar form using polar frame of reference

Conversion from polar to rectangular (p to R)

$$Fx = F.\cos\theta$$

$$Fy = F.\sin\theta$$

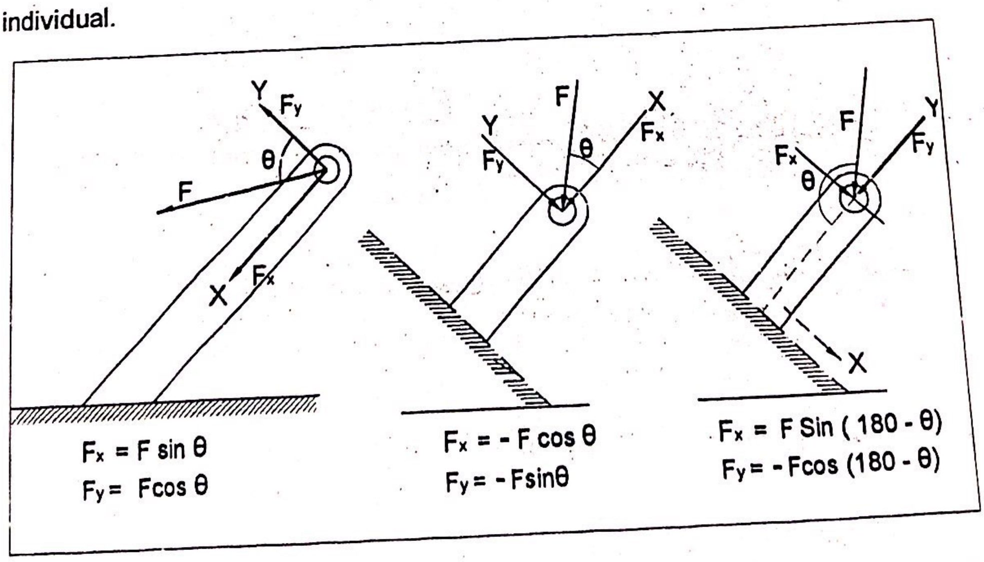
conversion from Rectangular to Polar (R to P)

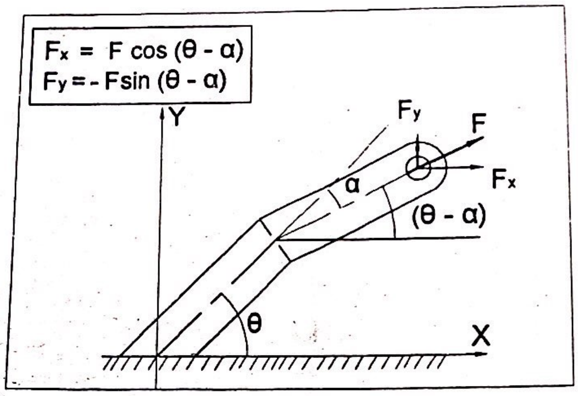
$$F = \sqrt{F_x^2 + F_y^2}$$

$$\theta = \tan^{-1}\left(\frac{F_y}{F_x}\right)$$

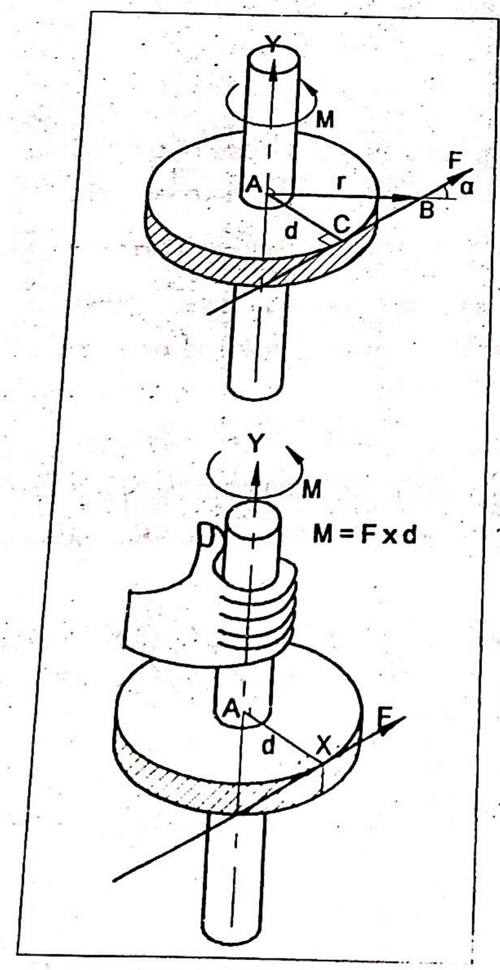
Note:

It should be noted that θ is the acute angle made by the line of action of the force with the horizontal. Hence, do not use negative quantities while calculating θ. Use only $|F_x|$ and $|F_y|$ i.e. only numerical values. Choice of reference axes: Actual problem do not come with reference axes; so their assignment is a matter of arbitrary convenience; and the choice is up to the individual.





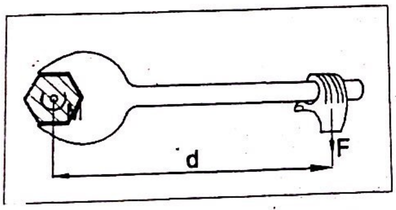
Moment of a Force:



A force acting on a rigid body can produce mainly two effects. These are (i) Tendency of translation of a body in the direction of its application (ii) Tendency of rotation of a body about an axis. The axis may be any line which neither intersects nor is parallel to the line of action of a force.

The rotation producing capacity of a force is called as moment or torque.

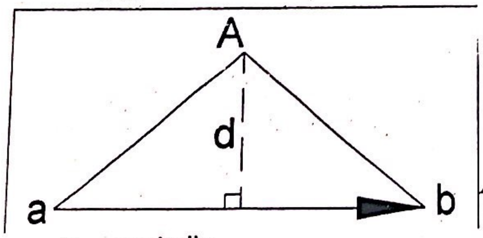
The moment of a force about a point is the product of the magnitude of the force and the perpendicular distance of that point from the line of action of the force. The point about which the rotational effect is produced is called as 'moment center'. The perpendicular distance of the axis of rotation from the line of action of the force is called as 'moment arm'. Moment of a plane force about a point is a vector quantity the direction of the moment vector is always perpendicular to the plane containing the force vector and the moment center. The sense of the moment vector is identified by the right-hand rule. The moment vector is a sliding vector with a line of action coinciding with the moment axis. Moment with anticlockwise sense of rotation are considered as positive and moments with clockwise sense of rotation are considered as negative. Unit of the moment in S.I. units is N-m. we can also use vector approach for calculation of moments. The moment of force $\overline{F}$ about point A may be represented by the cross-product expression, $\overline{M} = \overline{r} \times \overline{F}$



Where, $\overline{r}$ = position vector of ay point on the line of action of a force from the moment center. The magnitude of this cross-product is

$$M=(F\times r\times \sin\alpha)=F\times d$$

$$\therefore d=r.\sin\alpha$$



Geometrically,

$$Area\ \Delta aAb = \frac{1}{2}\times F\times d$$

$$\therefore (F\times d) = 2.A(\Delta aAb)$$

But F x d = M_A = moment of force ‘F’ about point A.

Thus, the magnitude of the moment of a force about a point is twice the area of the triangle formed by the force vector and the moment center.

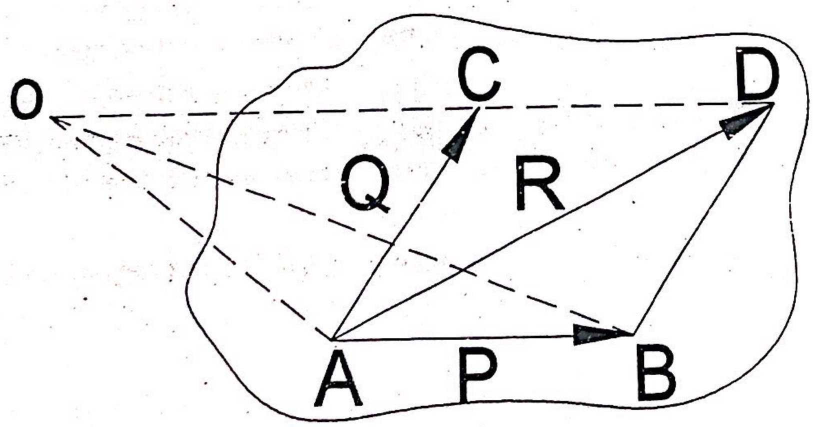
<u>Varignon's Theorem:</u>

This is also called as the principle of moments. It states that the moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.

In this figure, $\overline{R} = \overline{P} + \overline{Q}$

Then force $\overline{R}$ is called as the resultant of forces $\overline{P}$ and $\overline{Q}$. And Forces $\overline{P}$ and $\overline{Q}$ are called as the components of the force $\overline{R}$. Then by the Varignon's theorem of moments, the moment of the resultant force about any point in the plane of the force is equal to the sum of the moments of the individual forces about the same point.

To prove this theorem, let us make use of the geometrical interpretation of the moment of a force about a point.



Moment of force $\overline{R}$ about point O = 2.Area(ΔAOD)

$= 2.Area(\Delta AOD + \Delta ACD)$

But $A(\Delta ACD) = A(\Delta ABD) = A(\Delta AOB)$

∴ Moment of force $\overline{R}$ about point O

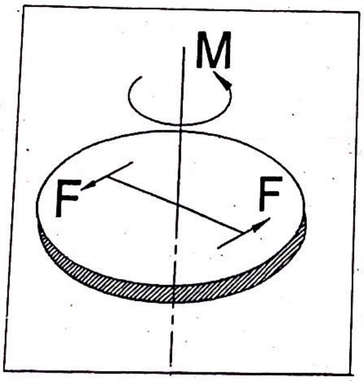
$= 2.\ Area(\Delta AOC + \Delta AOB)$

$= 2.\ Area(\Delta AOC) + 2.Area(\Delta AOB)$

=moment of forces $\overline{Q}$ about point O + moment of force $\overline{P}$ about point O

Varignon's theorem of moments is useful in locating the position of the point of application of the resultant force, in case of non-concurrent coplanar force systems.

Couples:



Two forces having same magnitude, parallel lines of action but acting in opposite directions are said to form a couple. The perpendicular distance between the lines of actions of the two forces forming the couple is called as the 'moment arm' of the couple.

The two equal and opposite forces with moment arm will not tend to introduce any translator motion on the body. The only effect of a couple acting on the body is the tendency to rotate the body about an axis perpendicular to the plane of the couple. The resultant force of the couple is always zero.

The magnitude of the couple moment is the product of the magnitude of the force and the moment arm. The sense of the couple can be clockwise or anticlockwise. Anti-clockwise couple moments are considered as positive and clockwise couple moments are considered as negative.

The moment of a couple is a constant quantity irrespective of the axis about which it is taken. Thus, moment of a couple is a free vector. There is no specific point of application for the couple. Due to this we can change the position of the couple on the body to any other point on the same body without changing its effect on the body.

Two couples are said to be equivalent, if they have same magnitude & same sense irrespective of the forces forming the couples.

Moments and couples both give turning, twisting or rotational effects. But there is a significant difference between a moment of a force and a couple. The couple is a pure tuning effect which may be moved anywhere in its own plane, without change of its effect on the body. The definition of a moment requires a statement of the reference axis about which this effect occurs whereas couple does not require a reference axis for its definition. Thus couple is a free vector. The magnitude to the moment of a force depends upon the position of the axis about which moment is taken whereas the magnitude of the couple moment is always constant regardless of the axis about which moment is taken.

Force Systems:

In the previous articles we have studied the properties of force, moment and couples. In most of the common engineering problems the bodies are always subjected to group of forces and couples. This is called as a force system. Depending upon their features the force systems are classified as under,

Resultant of a force system:

Most problems in mechanics deal with a system of forces and it is generally necessary to reduce the given force system to its simplest form in describing its action. This is called as composition of forces.

The resultant of the given force system is the simplest force system that can replace the given force system without changing its external effect on the body. Thus the resultant of the given force system implies the net external action on the body. The force acting on a body can have the following actions on the body.

i) translator action ii) rotary action

iii) translator as well as rotary action

iv) continuation of the state of rest or the state of uniform rectilinear motion.

Accordingly the resultant of the given force system can be,

i) a single force or
ii) a single couple or
iii) a force-couple system or
iv) neither a force nor a couple

if the resultant of the given force system is neither a force nor a moment and the value of the resultant force as well as the resultant couple is zero, then we may say that the force system acting on the body is balanced and the body on which it acts Is in balanced and the body on which it acts is in equilibrium. If the resultant of the given force system is not zero then the body will be subjected to linear or angular acceleration.Thus, the determination of the resultant is basic to both statics and dynamics.

For any system of coplanar forces the process of obtaining the resultant can be summarized in equation form as under,

$$\overline{R}=\overline{F_1}+\overline{F_2}+\overline{F_3}+\ldots\ldots\ldots+\overline{F_n}=\sum_{1}^{n}\overline{F}$$

$$\overline{R}=R_x\hat{i}+R_y\hat{j}$$

$$R_x=\sum Fx,\quad Ry=\sum Fy,\quad R=\sqrt{(\sum Fx)^2+(\sum Fy)^2}$$

$$\theta=\tan^{-1}\left(\frac{Ry}{Rx}\right)=\tan^{-1}\left(\frac{\sum Fy}{\sum Fx}\right)$$

These equations give us the magnitude and direction of the resultant force. To locate the point of application of the resultant force on the body under consideration, we use principle of moment (i.e. Varignon's theorem of moments). For this purpose a convenient reference point O is selected. Now,

Mo= $\sum M=\sum(F.d)$ = sum of the moments of the individual forces about point O.

Let, R.d = the moment of the resultant force about point O.

Then, R.d=Mo ----------(Where, d= perpendicular distance of the line of action of the resultant force from the reference point O.)

If the resultant force R for a given force system is zero, the check for the resultant moment Mo. If the resultant moment Mo is not zero then the given force system is having the resultant as a couple. If the resultant moment Mo is also zeros then the given force system is balanced and the body on which it acts is in equilibrium.

Equilibrium:

Whenever a body is subjected to number of external forces and couples simultaneously and due to the supports or neighboring bodies or constraints the body is subjected to external reactions which are canceling the effects of external forces and couples, then the body is said to be in equilibrium. The total force system acting on the body consisting of external applied forces and external developed reactions is then called as balanced force system. For a balanced force system the magnitude of the resultant force as well as the resultant moment is zero. Due to this whenever a body is in equilibrium it is neither subjected to translation nor subjected to rotation. These are called as physical conditions of equilibrium.

Mathematically, $\overline{R} = \sum \overline{F}$ and $\overline{Mo} = \sum \overline{M} = 0$. These are called as **equations of equilibrium.** These requirements are both necessary and sufficient conditions for equilibrium. The six scalar equations obtained thereby are the analytical conditions of equilibrium or the equations of equilibrium.

For **coplanar force systems,** the above set of equations reduces to three independent equations. These are $\sum Fx = 0,\quad \sum Fy = 0,\quad \sum Mz = Mo = 0$

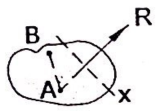
For a complete equilibrium in two dimensions all the above three equations must be satisfied independently i.e. one may hole good without the other.

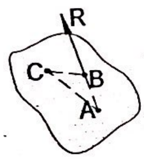
Alternative equilibrium Equations:

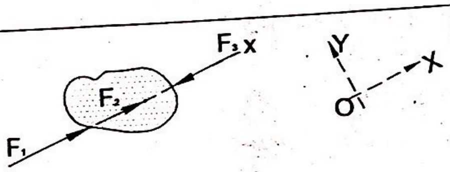
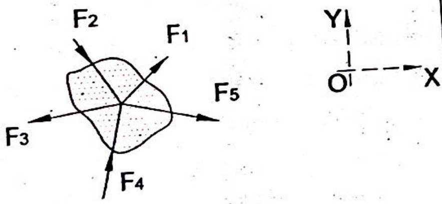
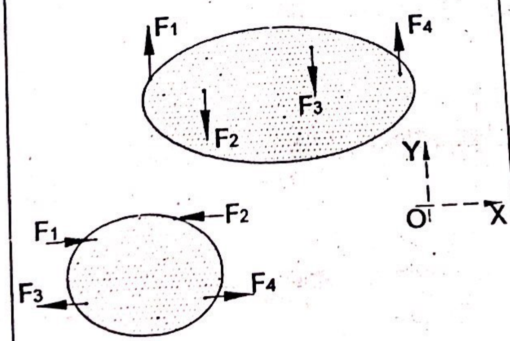
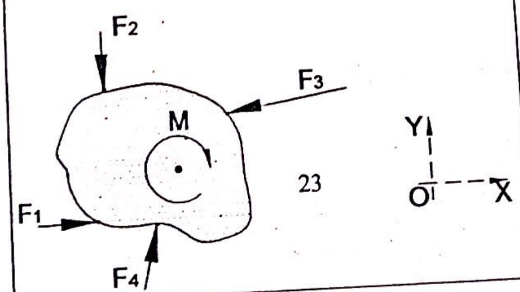
There are two additional ways in which we may express the general conditions for the equilibrium of coplanar forces.

1) $\sum Fx = 0,\quad \sum M_A = 0,\quad \sum M_B = 0$ ---------(where two points A and B do not lie on a line perpendicular to X-direction.)

2) $\sum M_A = 0,\ \sum M_B = 0,\ \sum M_C = 0$ ----------(where three points A, B and C are non –collinear points.)

① 

② 

Force System	Free Body Diagram	Independant equations of equilibrium
Colliner		$\sum F_x = 0$
Concurrent at a point		$\sum F_x = 0$ $\sum F_y = 0$
Parallel		$\sum F_y = 0$ $\sum M_z = 0$ $\sum F_x = 0$ $\sum M_z = 0$
Non Concurrent Non Parallel i.e. General force System	 23	$\sum F_x = 0$ $\sum F_y = 0$ $\sum M_z = 0$

Free-Body Diagram:

In the analysis of the equilibrium of the bodies, it is necessary to know all the forces acting on the body in all respects. This can be achieved by drawing the 'free-body diagram' of the body under consideration. The free-body diagram is the most important step in solution of problems in mechanics. While drawing the free-body diagram following steps are used-

i) A neat diagram of an isolated body is to be drawn to the scale showing all the important dimensions.

ii) All the external forces acting on the body are to be shown at their respective points of application with their magnitudes as well as directions and sense.

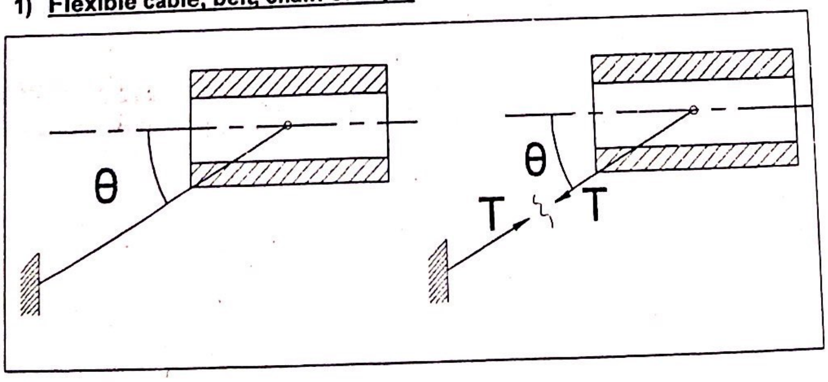
iii) The supports of body or the neighboring bodies or the constraints are to be removed and to be replaced by their appropriate reactions.

iv) The force system formed thereby consisting of all the applied forces and the external reactions is then classified and analyzed using the equations of equilibrium. Thus we develop the analytical model of an isolated mechanical system to which equations of equilibrium are applied.

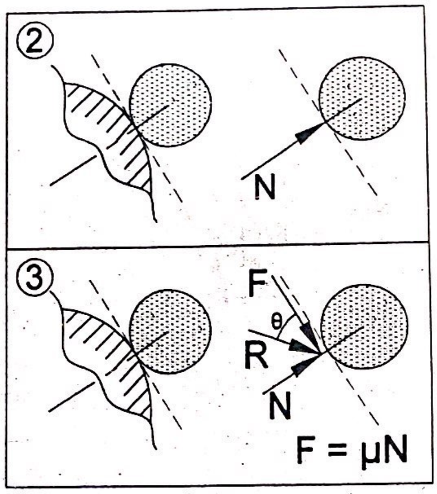
For drawing the free body diagram one must study the various types of supports and the force application on mechanical systems and their corresponding reactions.

Types of supports:

1) **Flexible cable, belt, chain or rope:**



Force exerted by the cable is always the tension away from the body in the direction of the cable. Cables, belts, chains or ropes can not be subjected to compression.



2) Smooth surfaces:

Contact force is compressive and normal to the contact surface.

3) Rough surfaces:

Due to the roughness this surface offers frictional resistance (F) as well as normal reaction N. thus the total reaction R is the resultant of F and N.

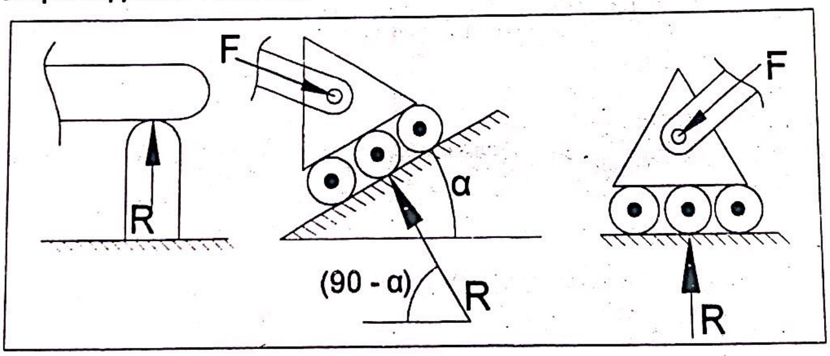
$$\overline{R} = \sqrt{F^2 + N^2}$$

$$\theta = \tan^{-1} \frac{N}{F}.$$

$F = \mu N$

where μ = coefficient of friction between the two surfaces

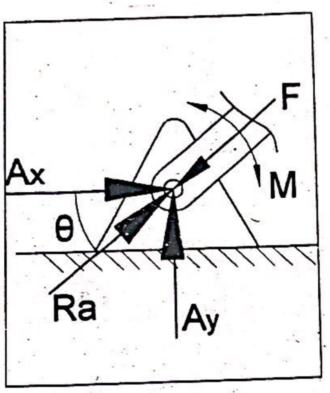
4) Simple support or roller support:



Reactions are provided to prevent motion. In case of simple support or a roller support motion is possible in all the directions except against the surface on which the roller is resting. Because of this the reaction R is compressive and perpendicular to the surface on which the roller is resting.

Number of unknown is one i.e. magnitude of the reaction.

5) Hinged support or simple pin connection:



In this case translation is completely prevented but rotation about the axis of the pin and in the plane of the forces is possible. To prevent translation the connection offers the reaction R_A in any direction. (but only in the plane of the forces) Thus magnitude as well as the direction of the reaction is unknown. Thus there are two unknowns. (R_A , θ_A) while solving problems, only for the sake of convenience we resolve the reaction into its rectangular components (A_x and A_y).

Here, A_x = Horizontal components of the reaction at A.

A_y = Vertical component of the reactions at A.

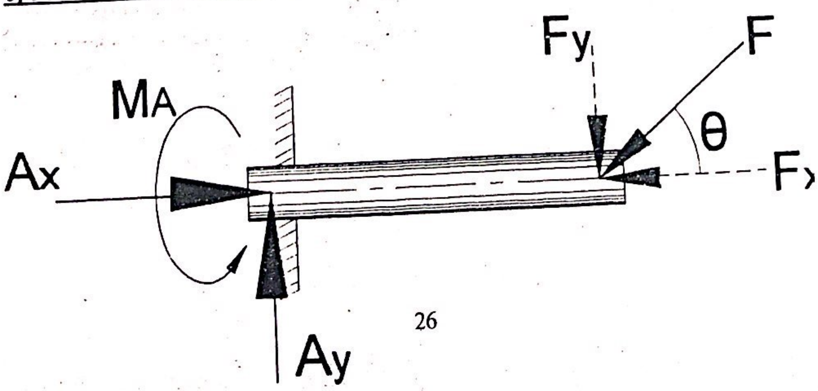
$$R_A = \sqrt{A_x^{\ 2} + A_y^{\ 2}} \quad \text{conversion from rectangular to polar form}$$

$$\theta_A = \tan^{-1}\left(\frac{Ay}{Ax}\right)$$

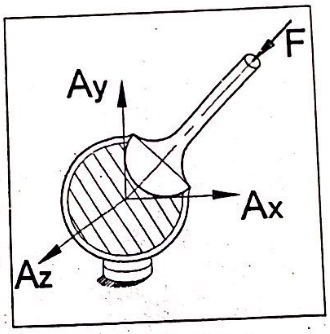
$$Ax = R_A.\cos\theta_A \quad \text{conversion from polar to rectangular form.}$$

$$Ay = R_A.\sin\theta_A$$

<u>6) Fixed support or Built-in support or Encastre:</u>



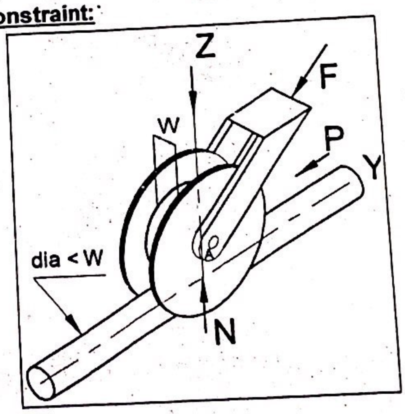
In case of a fixed support, we achieve total fixity by completely. The reaction components 'Ax' and 'Ay' are developed to prevent translation in horizontal and vertical direction. The fixed end moment 'M_A' is developed on. The fixed end moment 'M_A' is developed to prevent rotational motion. Thus, there are three unknowns.



7) Ball and Socket Joint:

A ball and socket joint is a hinge in three dimensions. In this connection translator motion in all the directions is prevented by the development of the three reaction components Ax, Ay and Az. But rotational motion about any axis passing through the center of the ball is possible. Thus there are three unknowns.

8) Roller or a wheel supports with lateral constraint:



This is a guided rolled or a wheel. A lateral force 'P' is exerted by the guide on the wheel. And the surface offers the normal reaction 'N'.

Let, P = Ay

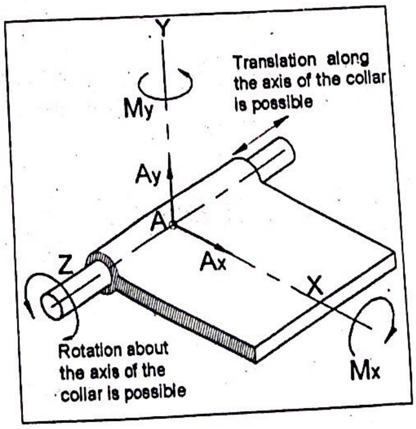
And N = Az

If the motion is possible in X – direction, then reaction 'R_A' can be expressed as,

$$\overline{R_A} = Ay\hat{j} + Az\hat{k}$$

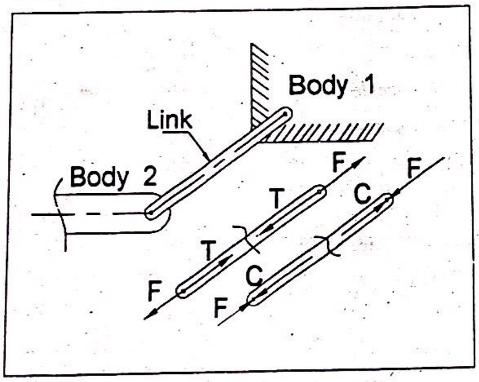
9) Collar joint or Collar Bearing :

$$\overline{R_A} = Ax\hat{j} + Ay\hat{j}$$
$$\overline{M_A} = Mx\hat{j} + My\hat{j}$$



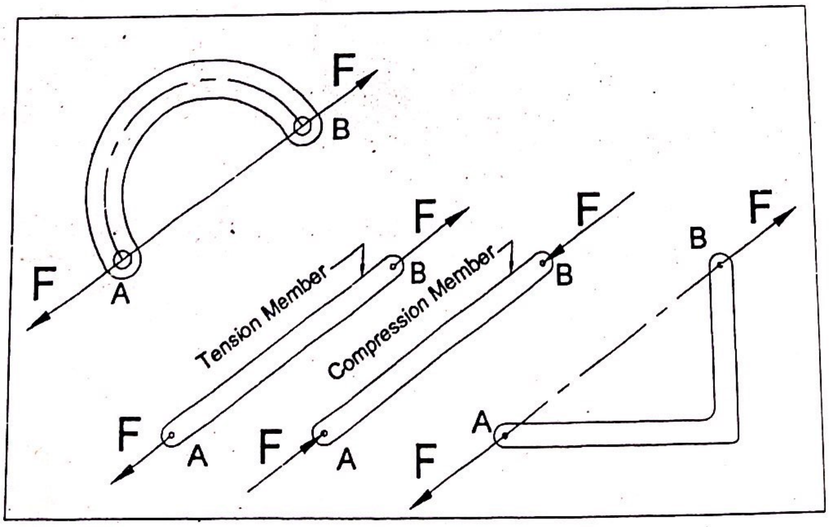
in this type of a joint translation along the axis of the collar and rotation about the axis of the collar is possible. To prevent the translation in other two directions two reaction components are developed
(say Ax and Ay) and to prevent the rotation about the other two directions two moment components are developed (say Mx and My).
Hence total number of unknowns are four.

10) Metalic link:



when two bodies are connected by a metallic link. It can be subjected to axial tension or compression. T = Axial tension in the link. C = axial compression in the link.

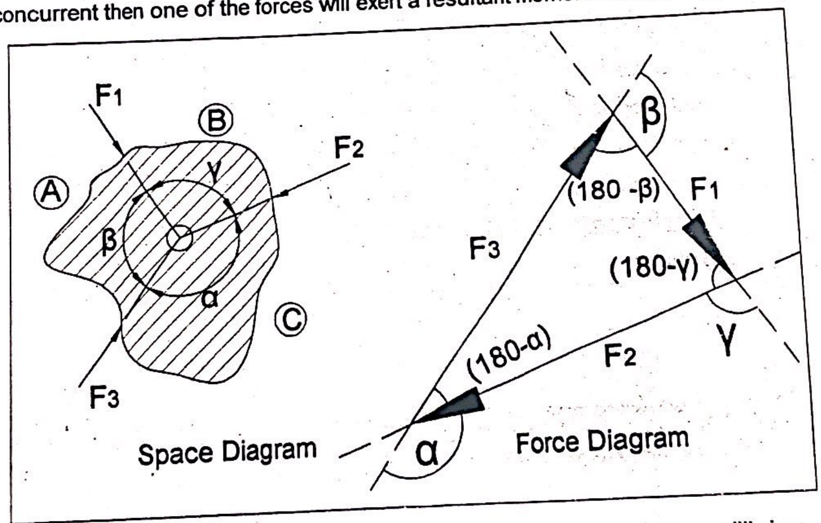
Equilibrium of two forces:



When a body is acted upon by two equal, opposite and collinear forces then it is in equilibrium. The shape of the body does not have any effect on the requirement of the equilibrium. If the body is acted upon by such a force system then it is called as a two force member. If such a body is linear in shape then it is axially loaded body subjected to axial tension or axial compression. Here weights of the member are neglected.

Equilibrium of three forces:

When a body is subjected to three coplanar forces and it is equilibrium then the lines of action of these three forces must be concurrent. If they are not concurrent then one of the forces will exert a resultant moment about the point of



concurrency of the other two and the body will not remain in equilibrium. The only exception occurs when the three forces are parallel. Because of the condition of concurrency when plotted graphically the three non parallel coplanar forces in equilibrium always form a closed triangle as a force polygon.

Applying sine rule to triangle abc, we get

$$\frac{F_1}{\sin(180-\alpha)}=\frac{F_2}{\sin(180-\beta)}=\frac{F_3}{\sin(180-\gamma)}$$

$$\therefore \frac{F_1}{\sin\alpha}=\frac{F_2}{\sin\beta}=\frac{F_3}{\sin\gamma}$$

this is also called as **Lami's theorem**.

From the above discussion we can conclude that, three non-parallel coplanar forces in equilibrium are always concurrent.

As the given forces are equilibrium,

We can write, $\overline{F}_1+\overline{F}_2+\overline{F}_3=0$

From this we get three conditions, i.e.

$\overline{F}_1+\overline{F}_2=-\overline{F}_3$ ---------------------- (i)

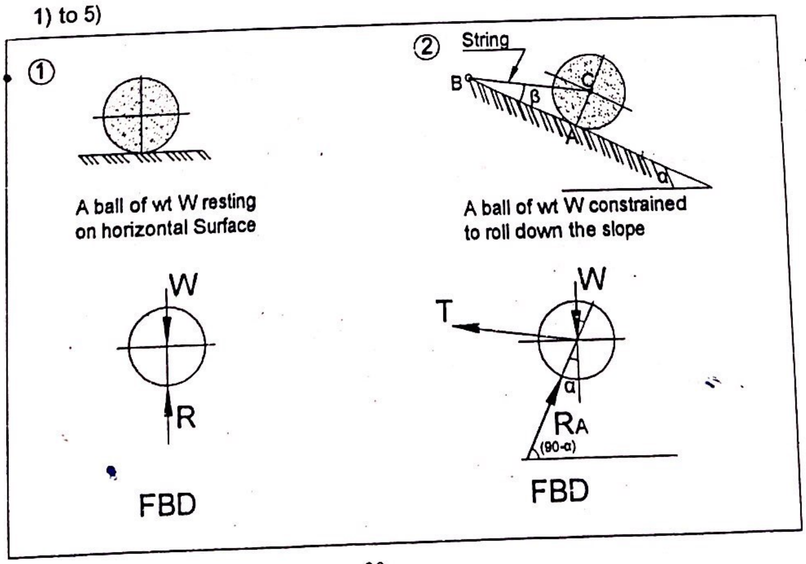
$\overline{F}_2+\overline{F}_3=-\overline{F}_1$ ---------------------- (ii)

$\overline{F}_3+\overline{F}_1=-\overline{F}_2$ ---------------------- (iii)

the above conditions are such that the resultant of any two forces must be equal, opposite and collinear to the third force. All the above three conditions are simultaneously valid only when all the three forces are concurrent any one point in the plane of the forces.

<u>Sample free Body diagrams :</u>

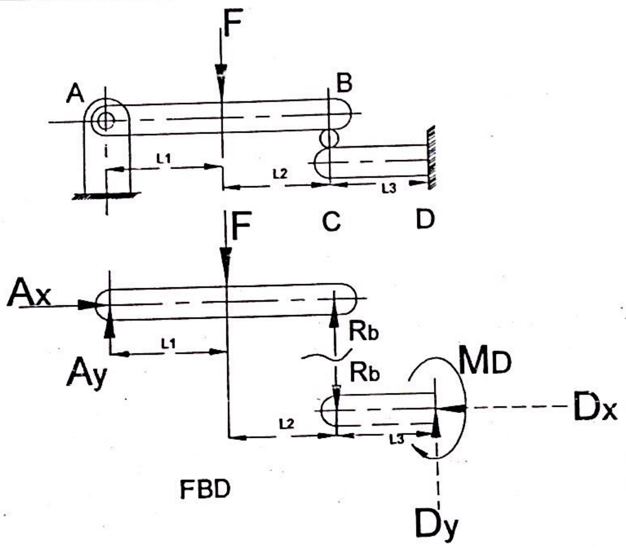
1) to 5)



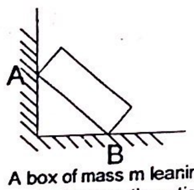
A ball of wt W resting on horizontal Surface

A ball of wt W constrained to roll down the slope

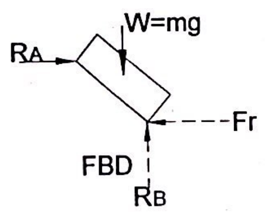
③



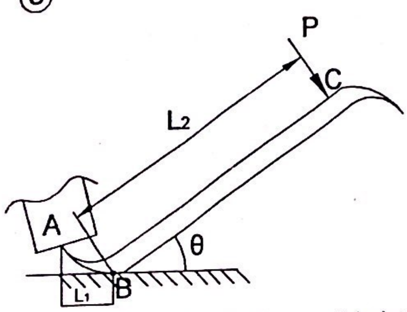
④



A box of mass m leaning against a smooth vertical wall & resting on a rough horizontal surface



⑤



A pry bar lifting a body having smooth horizontal base & is resting on a rough horizontal surface