Evolution to a small-world network with chaotic units

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PACS: 89.75.Hc Networks and genealogical trees
PACS 05.45.Xt Synchronization; coupled oscillators
PACS 87.18.Sn Neural networks

Abstract: The mutually supporting role of evolving structure and chaotic activity is studied in a complex network. An initially randomly coupled network with a chaotic activation rule is adaptively rewired according to dynamic coherence between its units. The evolving network can reach a small-world structure. While the network is assuming small-world structure, network collective activity tends to an intermittent dynamic clustering regime. The combination of spontaneous chaotic activity and adaptive rewiring thus jointly contribute in improving information propagation in the network.

In the study of complex systems, small-world networks have recently become the focus of interest because of their potentials as models for interacting structures of complex systems composed of large numbers of units [1-3]. Small-world networks are those networks of which the connectivity topology is placed between regular and completely random connectivity [1,2]. Their main advantage is that they show a high degree of clustering, like regular networks but at the same time have small path lengths like random ones. In more traditional networks, these are mutually exclusive properties. The combination of high clustering and small path length makes small-world networks optimal structures for processing information.

For a large number of real networks with small-world property, including neural networks, protein networks, the world-wide web, and even some social networks [1], the structure has emerged spontaneously, rather than by design. In this perspective, an important issue is how this could have happened. In the initial study, Watts and Strogatz created small-world networks by randomly rewiring a regular network with probability p [2]. Starting from initially regular conditions may be a useful strategy for artificial system design, it is implausible however for natural system development and evolution. Current studies in the evolution of structure of complex network are focused on adding or changing static nodes in some ways to obtain structured network. A particular kind of complex network structures, scale-free networks, can be obtained in this manner [4]. Furthermore, the topological evolution of binary Boolean networks has been studied, and the results showed that an asymmetrically connected threshold network can evolve to a network with critical connectivity [5]. Comparably little attention has been paid to the evolution of complex networks to small-world structure in which nodes have complicated dynamics. Yet, such a study may be of considerable interest, because the dynamics and the evolving network structure may have a mutually supportive, autocatalytic role.

In the current study, we present a simple dynamical model to reveal whether a small-world network can evolve from a randomly coupled network with dynamical units. In the model, the activity of each unit is chaotic, and there is the dynamical rewiring according to coherent activities of units. It is worth noticing that networks with adaptive rewiring according to coherent activity are common in gene networks, as well as in neural networks [1,6,7]. It is expected that the study about this topic may shed light on understanding the generation of complex network in real systems with dynamic units and revealing the roots of the complicated spatiotemporal dynamics of these systems.

In this study, we use coupled logistic maps as a general paradigmatic to study the evolution of a network from a random structure to small-world structure. Globally and locally coupled logistic maps have been studied extensively. Their dynamics have revealed a number of interesting phenomena of spontaneous pattern formation [8]. The logistic map $f(x) = 1 - ax^2$ is one of a generic family of one-hump maps. The dynamics of the map is controlled by the parameter *a*. In the following study, we use values of parameter *a* such that the dynamics of a single unit is chaotic. We consider the following network model which utilizing coupled logistic maps:

$$x_{n+1}^{i} = (1-\varepsilon)\ f\left(x_n^{\ i}\right) + \frac{\varepsilon}{M_i} \sum_{j} {}_{j \in B(i)} f\left(x_n^{\ j}\right) \tag{1}$$

where x_n^i is the activity of the *i*-th unit ($1 \le i \le N$) at the *n*-th time step. M_i and $B(i)$ are the number and the set of the neighbors of the unit i respectively. The neighbors of the units i are the units that have a direct connection with unit i. The connection here is bi-directional. The parameter ε is the coupling strength. At the *n*-th time step, the coherence between unit i and unit j, $d_{ij}(n)$, is defined as the absolute value of the difference between the activation values of the units, as in Eq. (2).

$$d_{ij}(n) = \left| x_n^i - x_n^j \right| \tag{2}$$

We start by constructing a randomly coupled network and study the evolution of its structure according to the following adaptive rewiring algorithm. To generate a random network, we randomly select a pair of nodes and establish a link between them, and this procedure is repeated up to a certain number of links L_c. We assure that the initial network is fully connected by choosing a value for L_c which satisfies the following inequation [9]:

$$L_c >> \frac{N}{2}\ln(N) \tag{3}$$

where N is the total number of the units in the network. Throughout this paper *N=300,* $L_c = 5200$ are adopted, large enough to guarantee that the random network is fully connected.

Starting from a generated random network, our adaptive rewiring algorithm according to dynamic coherence between elements is described as follows: [I] Choose random initial activation values in the range $(-1, 1)$ for all units of the system. [II] Calculate the state of the system according to the Eq. (1), and discard an initial transient time T. [III] Then a unit i is chosen at random and the value of $d_{ij}(T+1)$ with all the other units in the current network is calculated. We obtain the unit $j = j_1$, for which the value $d_{ij_1}(T+1)$ is minimum amongst all the other units. Furthermore we obtain the unit $j = j_2$ for which $d_{ij_2}(T+1)$ is maximum amongst the neighbors of unit i. [IV] If unit j_1 is one of the neighbors of the unit i, then no change in the connections is made. Otherwise the connection between units i and j_2 is replaced by a connection between units i and j_1. [V] Go to step [I] and repeat.

This process is iterated, and after every change in the connections, the clustering coefficient C as defined in Ref. [2] is calculated. If the increasing magnitude between the previous and the current values of the clustering coefficient is smaller than a predefined value chosen as 0.001, we stop the algorithm and obtain a self-organized network. For the final state of the network we also calculate the characteristic path length L for the self-organized network and the initial random network. The characteristic path length L is defined as the number of the edges in the shortest path between two units, averaged over all pairs of units in the network [2]. Firstly, we investigate the dynamical evolution of the topology of the network. The parameters of the network are chosen as $\varepsilon = 0.5$, $a = 1.7$, and in the adaptive rewiring algorithm $T = 6000$. The clustering coefficients versus the change time are shown in Fig.1. As shown in Fig.1 that after an initially rapid increase, the clustering coefficient fluctuates around a constant 0.584, denoted by $C_f \approx 0.584$. For the final self-organized network, the characteristic path length is also calculated, and the value is $L_f = 2.66$. The distribution of the connections of the final self-organized network is calculated and shown in Fig.2. As shown in the figure, the distribution is a *Poisson* distribution. This means that the connections of the network are around an average value. This is a feature of small-world networks. For the initial random network, the clustering coefficient and characteristic path length are $C_0 = 0.058$, $L_0 = 2.3$ respectively. It is necessary to point out that our algorithm leaves the total number of units and connections unchanged, so we can meaningfully compare the clustering coefficients and characteristic path lengths of the initial network and the self-organized network. The results, $C_f >> C_0$, $L_f > L_0$, mean that the clustering coefficient of the self-organized network is much larger than the initial random network, and the average shortest path is close to the initial random network. Moreover, as shown in Fig.2, the distribution of the self-organized network is around the average value. So, it demonstrates that a small-world network is obtained by adaptive rewiring according to the chaotic network activity.

The impact of the choice of parameters in Eq. 1 on the evolution of network structure was studied. Results in which the parameter a is chosen to be $a=1.7$, in combination with a number of different values for coupling strength ε are shown in Fig. 3(a); results in which parameter $\varepsilon = 0.5$ in combination with a number of different values for parameter a are shown in Fig. 3(b). As shown in these results, too low values for the coupling strength parameters fail to increase the clustering coefficients; too high values of the a parameter lead to sub-optimal solutions. Further computational studies show that whether or not an initial randomly coupled network can evolve into a small-world network depends on its spatiotemporal dynamics. Recall that our simulations start out from an initial random network. For randomly coupled chaotic logistic maps, as investigated in [10], the spatiotemporal dynamics of a randomly and sparsely coupled network is different from globally coupled map lattices. Exact synchronization and the formation of identical dynamical states of the elements cannot be found in randomly coupled chaotic maps. Correspondingly, there are only fuzzy synchronization and fuzzy clusters. For randomly coupled logistic chaotic maps, the behaviors of the system change from turbulent state to fuzzy dynamical clusters phase as the coupling strength is increased, and then to fuzzy synchronization phase. The phase of the fuzzy dynamical clusters is separated from the fuzzy synchronous phase by a region of intermittent regimes. As in [10], we investigate the spatiotemporal dynamics of the initial random network by using the distribution of the pairing coherence $d_{ij}(t)$ at different time moment. Using this measure it has been found that in order for our network to evolve from an initial randomly coupled to a small-world structure, the parameter values a and ε must be chosen such that the system starts out from the intermittent phase or the fuzzy synchronization phase.

While the initial random network evolves to a small-world network as shown in Fig.1, the interesting question is, what kind of dynamics the network evolves to. We investigate the dynamics of the self-organized small-world network with random initial conditions. After the transient time is discarded, the distribution of $d_{ij}(t)$ with a small binsize 0.01 at different times t are calculated. The results in Fig.4 reveal different shapes at different times for these distributions. We observe that they switch intermittently between a fuzzy single-cluster (Fig. 4(a)) and a fuzzy two-cluster state (Fig. 4(d)). The time evolution of $d_{ij}(t)$ for a given pair is shown in Fig. 5 to alternate between a larger and a smaller magnitude, and it alternates between "up" and "down" states. An interesting intermittent dynamic clustering behavior can therefore be concluded for the self-organized small-world network.

A functional advantage of the intermittent dynamics is that it enables the system to switch between synchronized states without interference. Interestingly, it has been shown recently that switching between different states can be utilized to dynamically encode information [11], and has been found in cat striate cortex [12]. To check the variability of the activity, we obtain the distribution of dynamic cluster sizes. All members of a cluster have $d_{ij} < 0.016$. The distribution in the time interval *(50000,160000)* is shown in Fig.6. The distribution has a power-law scaling segment followed by an exponential decay tail. The exponent of the power-law part is -2.65. The dynamic clustering size shows a high variability. The phenomena observed in our studies have been investigated to be robust. For a range of parameter settings we have found that as long as the system evolves to a small-world network, the dynamic behaviors of the network self-organizes into an intermittent dynamic clustering state.

Another indication for the functional advantage of the self-organized small-world network can be obtained by the study of information propagation in the network. In order to understand how the information spreads along the network, let us consider two replicas of the same system, $X(n) = \{x_n{}^i\}$ and $X'(n) = \{x_n'^i\}$, that initially differ only by a small perturbation Δp in a single site i of the network. The propagation of information (or perturbation) to another site j can be measured by detecting the time t_j at which the difference for the unit j between the two replicas of the system, $\Delta x_n^j = \left| x_n^j - x_n'^j \right|$, exceeds a pre-assigned threshold d_{th} for the first time.

We can obtain the average time value for all the other units: $t_v(i) = \dfrac{\sum_{j=1, j\neq i}^{N} t_j}{N-1}$ for a perturbation to the unit i and proceed to calculate the average information propagation time $T_v = \dfrac{\sum_{i=1}^{N} t_v(i)}{N}$ for the all units. We calculate T_v for the initial network (the randomly coupled network) and the obtained self-organized small-world network. By choosing a pre-arranged threshold value larger than the original perturbation, we focus on amplifying responses. With a small perturbation $\Delta p = 0.001$, and pre-assigned threshold $d_{th} = 0.03$ we obtain for the initial network $T_v = 43.35$, and for the final small-world network $T_v = 29.25$. We observe that the rewiring has reduced the propagating time. So the evolution of the network's topology according to the co-operation between the dynamical units can improve its information propagation function.

The present study illustrates that a complex, small-world network can evolve from a random network through adaptive rewiring in response to dynamic coherence in chaotic activity. Actually, small-world networks have been found in real systems with dynamic units, for example, the neural systems like c.elegans and in vitro neural networks [13,14]. Chaos has been found in different level of neural activity [15], and it has been argued that chaos serves information processing in the brain [16]. On the other hand, it has been shown that adaptive rewiring according to the coherence of neural firing activity is very important for the development of neuron circuits [7]. So, from these senses, a hypothesis or prediction from our current study is that the spontaneous activity of early stage neural firing might have the chaotic characteristics and chaotic activity has an active role in establishing the small-world network structure.

For the present study, we also demonstrate that as long as the structure of random network evolves to a small-world structure, the dynamics of the network ultimately self-organizes into an intermittent dynamic clustering regime, and the structure and the dynamics of the network can evolve together. Such systems have important functional advantages for signal propagation. For a system with intermittent coherent clusters, it can rapidly enter and exist states in which their component parts are organized in different coherent activity, so intermittent dynamics would provide the system with flexibility [17]. Therefore, the emergence of the small-world network from the initial random network according to self-organization rules has a functional justification.

References

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Figure captions

Figure 1. The clustering coefficient as a function of change time as the result of adaptive rewiring. The parameters are $\varepsilon = 0.5$, $a = 1.7$.

Figure 2 The distribution of the connections of the self-organized network.

Figure 3 (a) The cluster coefficient as a function of the change time of connections. The parameter *a* is fixed to be *a=1.7*, and it is combined with different values for the coupling strength ε; **(b)** The cluster coefficient as a function the change time of connections for different parameters. The parameter $\varepsilon = 0.5$, and different values for the parameter *a*.

Figure 4 Normalized histograms of distributions over coherence between pairs of units in the final, self-organized small-world network: **(A):** after 7900 iterations**; (B):** after 8900 iterations; **(C):** after 9200 iterations, **(D);** after 9608 iterations; **(E):** after 9800 iterations.

Figure 5 Time evolution of the distance between a pair of units in the self-organized small-world network shown in Fig.1.

Figure 6 The distribution of the size of dynamic clusters in a long time interval (logarithmic values).

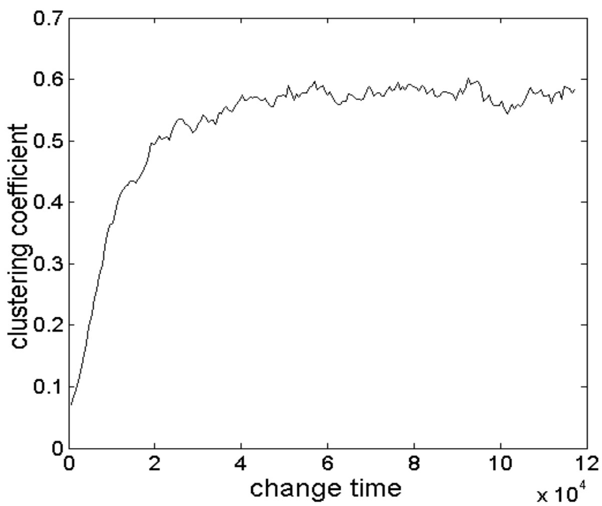
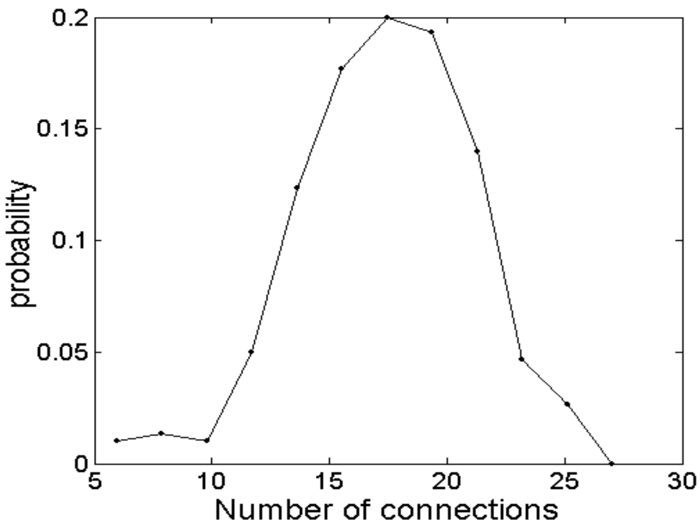
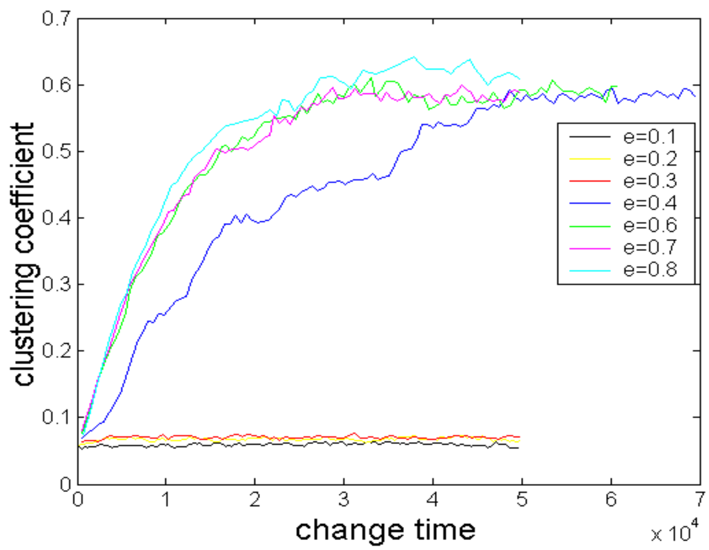

0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
clustering coefficient
0
2
4
6
8
10
12
change time
x 10^4

Fig. 1



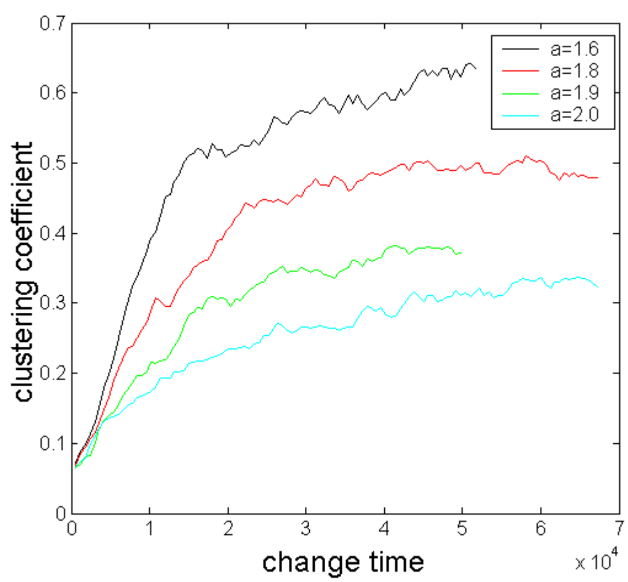
0.2
0.15
0.1
0.05
0
probability
5
10
15
20
25
30
Number of connections

Fig. 2



clustering coefficient
change time
x 10^4
e=0.1
e=0.2
e=0.3
e=0.4
e=0.6
e=0.7
e=0.8

Fig. 3(a)



clustering coefficient
change time
x 10^4
a=1.6
a=1.8
a=1.9
a=2.0

Fig. 3(b)

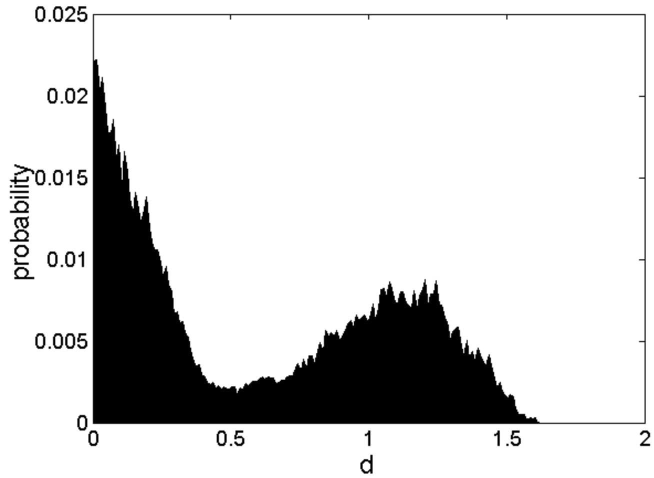

0.025
0.02
0.015
0.01
0.005
0
probability
0
0.5
1
1.5
2
d

Fig. 4(a)

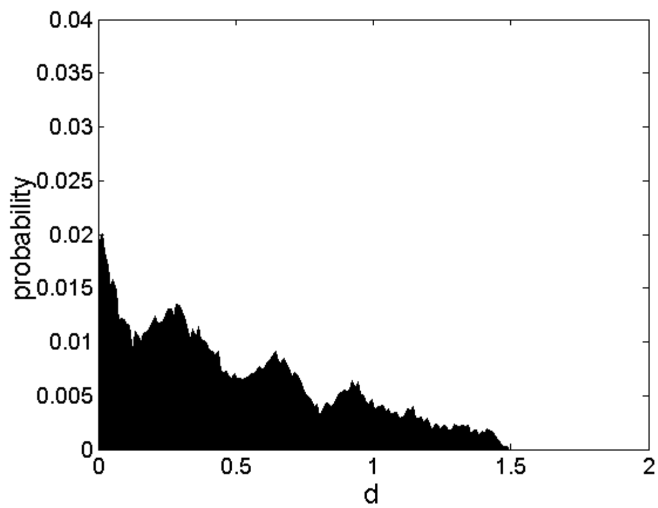

0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
probability
0
0.5
1
1.5
2
d

Fig. 4(b)

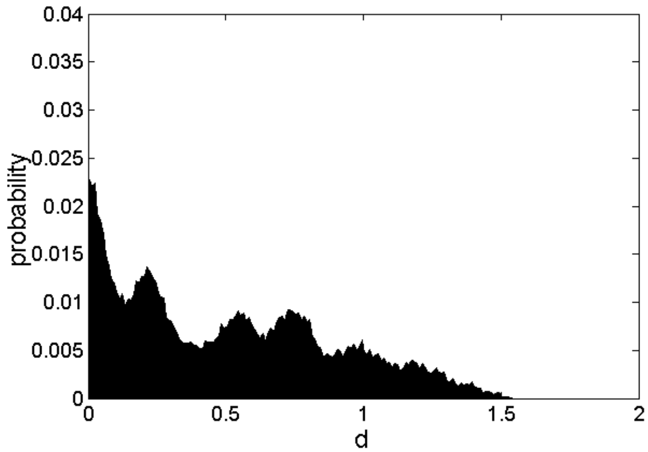

0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
probability
0
0.5
1
1.5
2
d

Fig. 4(c)

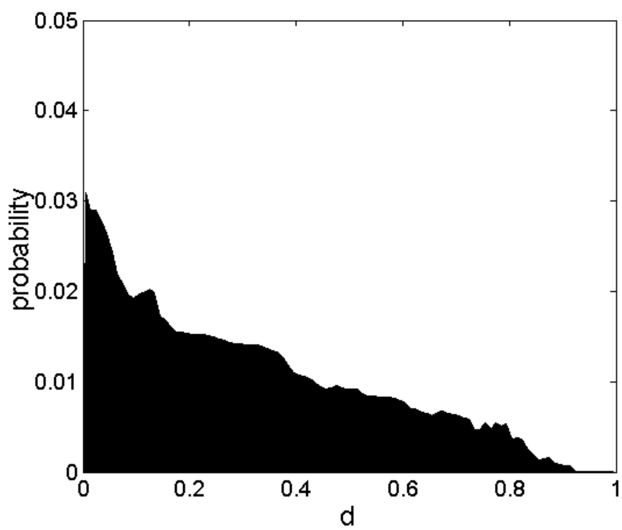

0.05
0.04
0.03
0.02
0.01
0
probability
0
0.2
0.4
0.6
0.8
1
d

Fig. 4(d)

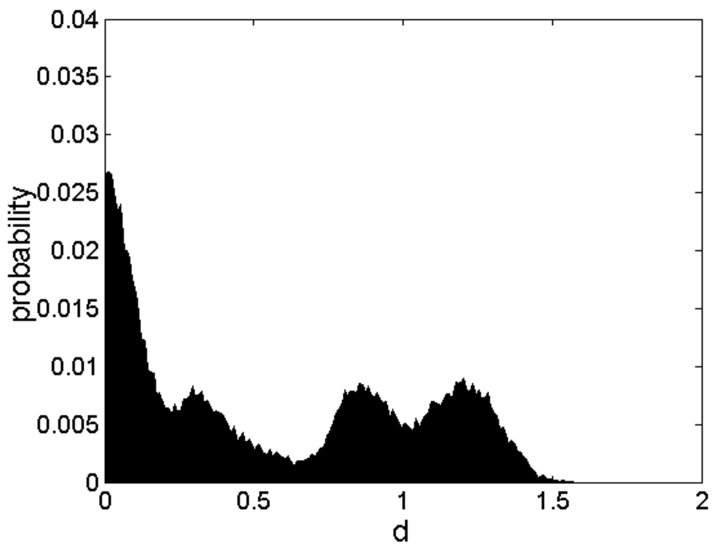


Fig. 4(E)

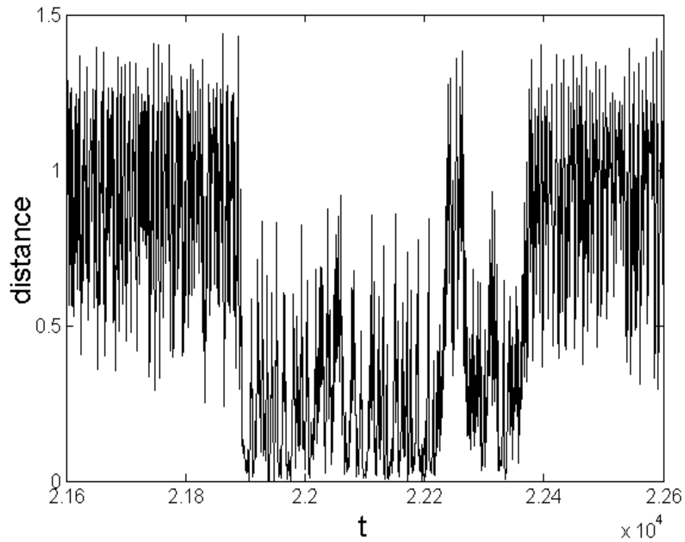


Fig. 5

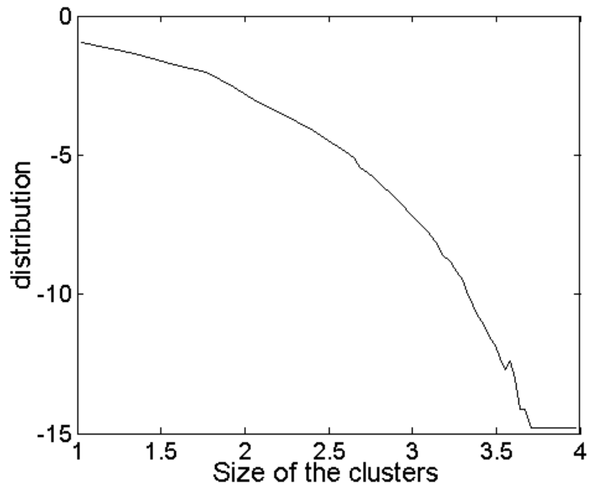


Fig. 6