Proceedings of the 18th World Congress
The International Federation of Automatic Control
Milano (Italy) August 28 - September 2, 2011

Unit-Dual-Quaternion-Based PID Control Scheme for Rigid-Body Transformation ⋆

X. Wang [*,**] **C. Yu** [**]

[*] *College of Mechatronics and Automation, National University of Defense Technology, Changsha, China (e-mail: xkwang@nudt.edu.cn).*
[**] *Research School of Engineering, the Australian National University, Canberra, ACT 2601 Australia (e-mail: brad.yu@anu.edu.au)*

Abstract: This paper proposes a unit-dual-quaternion-based PID control scheme for rigid-body transformation in 6 degree-of-freedom, taking advantage of the unit dual quaternion for its compactness, computational effectiveness and non-singularity in describing arbitrary rigid-body transformation globally. Both the discrete form and the incremental form of the generalized PID control scheme are obtained after a new kinematic control model for rigid-body transformation based on unit dual quaternion in body-frame is derived. The proposed control scheme can deal with rotation and translation simultaneously and preserve the interconnection between rotation and translation with compact form. Simulation results based on USARSim platform and quad-rotor are provided to demonstrate the effectiveness of the proposed control scheme.

Keywords: PID control scheme, unit dual quaternion, transformation control, discrete form, incremental form

1. INTRODUCTION

With advances in technology, the science of automatic control now offers a wide spectrum of choices for advanced control schemes. However, more than 90% of all control schemes in industry are still PID (Proportional, Integral and Derivative) algorithm for its simplicity, proven functionality, applicability and ease of use (Åström and Hagglund, 2001; Ang et al., 2005). Given its unchallengeable popularity, in this study, we focus on PID control scheme, but we embed the PID algorithm into unit dual quaternion Lie-group to deal with the rigid-body transformation (composition of rotation and translation) control in 6 DOF (Degree-Of-Freedom), which includes 3 rotational DOF and 3 translational DOF.

One key question in transformation control is how to describe the motions of rigid-body; and different mathematical tools lead to different control methodologies. A conventional method is to describe rotation and translation separately, and another is using the homogeneous transformation matrix, which is still the dominant choice for most robotic systems applications today (Murray et al., 1994). Correspondingly, the approaches to deal with transformation control problem are characterized into two categories, viz., to divide into rotation control problem and translation control problem or to utilize the geometric structure of SE(3) (Bullo and Murray, 1995), in existing literatures. This paper is concerned with an alternative but more concise representation, termed unit dual quaternions. They are a natural extension of unit quaternions and composition of unit quaternions and three-dimensional translational vectors using Plücker coordinates intrinsically. The unit dual quaternion is better to represent arbitrary transformation with its non-singularity and compactness using only 8 numbers. Moreover, it also has been revealed by exiting works that among so many mathematical approaches such as homogeneous transformation, quaternion/vector pairs, Lie algebra and alike, unit dual quaternion offers the most compact and most computational efficient screw transformation formalism (Funda et al., 1990; Funda and Paul, 1990; Aspragathos and Dimitros, 1998). Until now, the unit dual quaternion has remained an elegant and useful tool in many research areas such as computer-aided geometric design (Purwar and Ge, 2005), image-based localization (Goddard, 1997), hand-eye calibration (Daniilidis, 1999) and navigation (Wu et al., 2005).

However, it is a pity that unit dual quaternions seldom play a certain role in the control of rigid-bodies, as unit quaternions do in rotation control. As mentioned by Funda in Funda et al. (1990), although dual quaternions offer a potentially significant advantage, they have been relatively neglected in practical robotic system, partly because key algorithms are too complex to be well appreciated by the robotics community. This paper will contribute an attempt with designing a key and widely used algorithm – PID control scheme to promote the understanding and application about unit dual quaternions. Wu et al. (2005) introduce a unit dual quaternion to describe a rotation succeeded by a translation (or a translation succeeded by a rotation), and derive the kinematic equations of a unit dual quaternion based on Plücker coordinate. Following this work, the kinematic control using unit dual quaternions is studied in Han et al. (2008). It is shown that unit dual quaternions form a Lie-group and define its Lie-algebra on the basis of logarithmic mapping, and then parallel (but

⋆ X. Wang is supported in part by the China Scholarship Council and NICTA Ltd; C. Yu is supported by the Australian Research Council through a Queen Elizabeth II Fellowship under DP-110100538.

not equivalent) to Bullo and Murray (1995) to derive the generalized proportional control law based on unit dual quaternion Lie-group. However, the work of Han et al. (2008) is described in spatial frame and just focuses on proportional control. In this study, we first derive a new kinematic model for rigid-body transformation in body-frame rigorously and then extend the kinematic study by combining the advantages of unit dual quaternion and PID control scheme. It can be extended to a wide spread of control schemes, from generalized proportional control to full generalized PID control scheme, which can deal with rotation and translation simultaneously and preserve the interconnection between rotation and translation with compact form compared with convenient methods. And then the related discrete form and incremental form of proposed generalized PID control scheme, which are the most common forms in modern digital control, are derived and validated by Urban Search And Rescue Simulation (USARSim) platform with quad-rotor.

2. MATHEMATICAL PRELIMINARIES

We present the basic mathematical formulations and necessary notions about the (dual) quaternion. More details can be found in, for example, Wu et al. (2005) and Han et al. (2008). Some useful notions, which are important for the control scheme design, are given in Han et al. (2008) and are restated or extended in this study.

A *quaternion* is an extension of a complex number to $\mathbb{R}^4$. Formally, a quaternion q can be defined as $q = a + bi + cj + dk$, where $i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j$. A convenient shorthand notation is $q = [s, \boldsymbol{v}]$, where s is a scalar (called *scalar part*), and $\boldsymbol{v}$ is a three-dimensional vector (called *vector part*).

A *dual number* is defined as $\hat{a} = a + \epsilon b$ with $\epsilon^2 = 0$, but $\epsilon \neq 0$, where a and b are real numbers, called *real part* and *dual part*, respectively, and ϵ is a nilpotent. The *dual vectors* are a generalization of dual numbers whose real and dual parts are both three-dimensional vectors.

Definition 1. (Dot Product). Let $\hat{v} = \boldsymbol{v}_r + \epsilon \boldsymbol{v}_d$ and $\hat{k} = \boldsymbol{k}_r + \epsilon \boldsymbol{k}_d = (k_{r1}, k_{r2}, k_{r3})^T + \epsilon(k_{d1}, k_{d2}, k_{d3})^T$ be dual vectors. Then

$$\hat{k} \cdot \hat{v} = K_r \boldsymbol{v}_r + \epsilon K_d \boldsymbol{v}_d$$

where $K_r = diag(k_{r1}, k_{r2}, k_{r3})$ and $K_d = diag(k_{d1}, k_{d2}, k_{d3})$. For convenience and simplicity, the symbol '$\cdot$' is sometimes omitted in the following.

Definition 2. (Partial Order). Let $\hat{v}_1 = \boldsymbol{v}_{r1} + \epsilon \boldsymbol{v}_{d1}$, $\hat{v}_2 = \boldsymbol{v}_{r2} + \epsilon \boldsymbol{v}_{d2}$ be dual vectors. If $\boldsymbol{v}_{r1} - \boldsymbol{v}_{r2} \geq (>)(0,0,0)^T$ and $\boldsymbol{v}_{d1} - \boldsymbol{v}_{d2} \geq (>)(0,0,0)^T$, then we say $\hat{v}_1 \geq (>)\hat{v}_2$.

Definition 3. Given a dual vector $\hat{v}$, if $\hat{v} \geq (>)(0,0,0)^T + \epsilon(0,\ 0,\ 0)^T$, then we say $\hat{v} \geq (>)\mathbf{0}$.

A *dual quaternion* is the quaternion with dual number components, i.e., $\hat{q} = [\hat{s}, \hat{\boldsymbol{v}}]$, where $\hat{s}$ is a dual number and $\hat{\boldsymbol{v}}$ is a dual vector. A three-dimensional (dual) vector can also be treated equivalently as a (dual) quaternion with vanishing real part, called *(dual) vector quaternion.* If not otherwise stated, a (dual) vector is treated equivalently as its corresponding (dual) vector quaternion, for example, $v = [0, \boldsymbol{v}]$ or $\hat{v} = [0, \hat{\boldsymbol{v}}]$. A dual quaternion also can be treated as a dual number with the quaternion components, which is $\hat{q} = q_r + \epsilon q_d$, where q_r and q_d are quaternions. The *conjugate* of a dual quaternion is $\hat{q}^* = q_r^* + \epsilon q_d^*$. For two dual quaternions $\hat{q}_1$ and $\hat{q}_2$, the *multiplication* is $\hat{q}_1 \circ \hat{q}_2 = q_{r1} \circ q_{r2} + \epsilon(q_{r1} \circ q_{d2} + q_{d1} \circ q_{r2})$, where '$\circ$' represents the multiplication of (dual) quaternions. If $\|\hat{q}\|^2 = 1 + \epsilon 0$, then the dual quaternion is called *unit dual quaternion.* The *multiplicative inverse* of a dual quaternion $\hat{q}$ is $\hat{q}^{-1} = (1/\|\hat{q}\|^2) \circ \hat{q}^*$. For a unit dual quaternion, we have $\hat{q}^{-1} = \hat{q}^*$.

The unit quaternion can be used to describe rotation. For the frame rotation about a unit axis $\boldsymbol{n}$ with an angle $|\theta|$, there is a unit quaternion

$$q = [\cos(\frac{|\theta|}{2}), \sin(\frac{|\theta|}{2})\boldsymbol{n}] \tag{1}$$

relating a fixed vector expressed in the original frame r^o with the same vector expressed in the new frame r^n by

$$r^n = q^* \circ r^o \circ q,$$

where r^o and r^n are both vector quaternions. Similarly, a unit dual quaternion can be used to represent transformation (rotation and translation simultaneously). Suppose that there is a rotation q succeeded by a translation p [1], then the whole transformation can be represented using the unit dual quaternion as follows (Wu et al., 2005):

$$\hat{q} = q + \frac{\epsilon}{2} q \circ p. \tag{2}$$

A unit quaternion q serves as a rotation, taking coordinates of a point from one frame to another. On the other hand, every attitude of a rigid body that is free to rotate relative to a fixed frame can be identified with a unique unit quaternion q. Analogous to the rotational case, a unit dual quaternion $\hat{q}$ serves as both a specification of the configuration (consisting of attitude and position) of a rigid body and a transformation taking the coordinates of a point from one frame to another.

Unit quaternions form a Lie-group over multiplication with the conjugate being the inverse, denoted by Q_u. The same goes for unit dual quaternions, which form a Lie-group over dual quaternion multiplication, denoted by DQ_u. The logarithm of a unit quaternion given by (1) is $\ln q = \frac{\theta}{2}$, where $\theta = |\theta|\boldsymbol{n}$ is a vector quaternion. The logarithmic mapping of a unit dual quaternion is as follows.

Definition 4. (Logarithmic Mapping) Given a unit dual quaternion $\hat{q}$ defined in (2), the logarithmic mapping is defined as

$$\ln \hat{q} = \frac{1}{2}(\theta + \epsilon p), \tag{3}$$

which is a dual vector quaternion.

The space consisting of all unit (dual) quaternion logarithmic mappings is denoted by v or $\hat{v}$. Space $\hat{\boldsymbol{v}}$ is the Lie-algebra of DQ_u as well as v is the Lie-algebra of Q_u.

Given a (dual) vector quaternion $V \in v$ ($\hat{V} \in \hat{v}$) and a unit (dual) quaternion $q \in Q_u$ ($\hat{q} \in DQ_u$), the *Adjoint transformation* is defined as

$$Ad_q V = q \circ V \circ q^* \quad \text{or} \quad Ad_{\hat{q}} \hat{V} = \hat{q} \circ \hat{V} \circ \hat{q}^*.$$

The kinematic equation of a rigid-body expressed with the dual quaternion in body-frame is

[1] All the kinematic variables, such as p, ξ, ω, are expressed in the body-frame unless otherwise is stated.

$$\dot{\hat{q}} = \frac{1}{2}\hat{q} \circ \xi, \tag{4}$$

where ξ is the twist in body-frame with the following form:

$$\xi = \omega + \epsilon(\dot{p} + \omega \times p). \tag{5}$$

3. PID CONTROL SCHEME

In this section, we develop a generalized PID control scheme for our deduced kinematic control model in body-frame on the basis of DQ_u and its logarithmic mapping.

3.1 Kinematic Control Model and PID Control Scheme

Left-invariant error between unit dual quaternions $\hat{q}$ and $\hat{q}_d$ is defined as $\hat{q}_e = \hat{q}^* \circ \hat{q}_d$, which can be rewritten into the similar form of (2), i.e.,

$$\hat{q}_e = q_e + \frac{\epsilon}{2}q_e \circ p_e, \tag{6}$$

where $q_e = q^* \circ q_d$ and $p_e = p_d - Ad_{q_e^*}p$. On the basis of the left-invariant error, the kinematic control model in body-frame is deduced as Lemma 5 with proof in Appendix A.

Lemma 5. (Kinematic Control Model in Body-frame) Denote the current configuration by $\hat{q}$ and the target configuration by $\hat{q}_d$. The kinematic control model for rigid-body transformation in body-frame is expressed

$$\dot{\hat{q}}_e = \frac{1}{2}\hat{q}_e \circ \xi_e, \tag{7}$$

where

$$\xi_e = \xi_d - Ad_{\hat{q}_e^*}\xi = \omega_e + \epsilon(\dot{p}_e + \omega_e \times p_e) \tag{8}$$

is the error twist in body-frame, ξ_d is the twist of target configuration, $\omega_e = \omega_d - Ad_{q_e^*}\omega$ and $p_e = p_d - Ad_{q_e^*}p$.

The kinematic control model in Lemma 5 is taken as the control model with assumption ξ_d (or ω_d and $\dot{p}_d$) and $\hat{q}_e$ (or $\hat{q}$ and $\hat{q}_d$) being known in the following. The input of the model is ξ_e, however we assume ξ_d and $\hat{q}_e$ are known, thus the actual input is ξ, which consists of angular velocity ω and linear velocity $\dot{p}$. Considering DQ_u as a Lie-group with logarithmic mapping being its Lie-group, the generalized proportional control scheme is proposed as follows.

Theorem 6. (Generalized Proportional Control Scheme) Consider the kinematic control model (7) and (8), and let dual vector $\hat{k}_p > \mathbf{0}$. The generalized proportional control

$$\xi_e = -2\hat{k}_p \cdot \ln \hat{q}_e \tag{9}$$

exponentially stabilizes the configuration $\hat{q}$ to $\hat{q}_d$.

The proof of Theorem 6 can be obtained by the similar work of Han et al. (2008), hence we omit it. Extending this generalized proportional control scheme, we can obtained the related PID control scheme as stated in Theorem 7.

Theorem 7. (Generalized PID Control Scheme). Let $\hat{k}_p = k_{pr} + \epsilon k_{pd} > \mathbf{0}$, $\hat{k}_d = k_{dr} + \epsilon k_{dd}$ and $\hat{k}_i = k_{ir} + \epsilon k_{id}$ all be dual vectors. The generalized PID control scheme extended from (9) can be written with the following form.

$$\xi_e = -2(\hat{k}_p \ln \hat{q}_e + \hat{k}_d(\ln \hat{q}_e)' + \hat{k}_i \int \ln \hat{q}_e \, dt). \tag{10}$$

Here, $\ln \hat{q}_e$ is the proportional term, $(\ln \hat{q}_e)'$ is the derivative term and $\int \ln \hat{q}_e \, dt$ is the integral term. Obviously, when $\hat{k}_d = \hat{k}_i = \mathbf{0}$, Theorem 7 degenerates into the generalized proportional control scheme in Theorem 6.

3.2 Specification in Euclidean Space

Control scheme in Theorem 7 takes ξ_e, consisting of ω and $\dot{p}$, as inputs. However the traditional measurements and calculations are conducted in Euclidean space. So, it is necessary to transform the control scheme in Theorem 7 into Euclidean space, which means ω and $\dot{p}$ should be extracted from (10). To obtain ω and $\dot{p}$, we need the following lemmas, which can be obtained by direct computations. Hence, we omit the proofs.

Lemma 8. Consider $a, b \in v$, $\hat{q} = q + \frac{\epsilon}{2}q \circ p \in DQ_u$. Then

$$Ad_{\hat{q}}(a + \epsilon b) = Ad_q a + \epsilon(Ad_q b + Ad_q(p \times a)).$$

Lemma 9. Consider $a, b \in v$, $q \in Q_u$. Then

$$Ad_q(a \times b) = Ad_q a \times Ad_q b.$$

Lemma 10. Considering q_e and p_e defined in (6), we have

$$Ad_{q_e}p_e = Ad_{q_e}p_d - p.$$

Now, we are ready to extract ω and $\dot{p}$ from (10).

From (8), we obtain

$$\xi = Ad_{\hat{q}_e}(\xi_d - \xi_e). \tag{11}$$

Let $\theta_e = 2\ln q_e$, we have $2\ln \hat{q}_e = \theta_e + \epsilon p_e$. Substituting it into (10), we obtain

$$\begin{aligned} \xi_e = & -(k_{pr}\theta_e + k_{dr}\dot{\theta}_e + k_{ir}\int \theta_e \, dt) \\ & - \epsilon(k_{pd}p_e + k_{dd}\dot{p}_e + k_{id}\int p_e \, dt). \end{aligned} \tag{12}$$

Substituting (12) into (11), and using Lemma 8, we obtain

$$\xi = Ad_{q_e}m + \epsilon(Ad_{q_e}n + Ad_{q_e}(p_e \times m)), \tag{13}$$

where

$$\begin{cases} m = \omega_d + k_{pr}\theta_e + k_{dr}\dot{\theta}_e + k_{ir}\int \theta_e \, dt, \\ n = \dot{p}_d + \omega_d \times p_d + k_{pd}p_e + k_{dd}\dot{p}_e + k_{id}\int p_e \, dt. \end{cases}$$

Comparing (5) and (13), and making the real part and dual part equal, respectively, we obtain

$$\begin{cases} \omega = Ad_{q_e}m, \\ \dot{p} = Ad_{q_e}n + Ad_{q_e}(p_e \times m) - \omega \times p. \end{cases} \tag{14}$$

According to Lemma 9, and using $\omega = Ad_{q_e}m$, we have

$$\dot{p} = Ad_{q_e}n + Ad_{q_e}p_e \times \omega - \omega \times p. \tag{15}$$

Using Lemma 10, the (15) yields:

$$\dot{p} = Ad_{q_e}n + Ad_{q_e}p_d \times \omega. \tag{16}$$

So, the specification of Theorem 7 in Euclidean space is summarized as Theorem 11.

Theorem 11. Denote the current and target configurations by $\hat{q} = q + \frac{1}{2}\epsilon q \circ p$ and $\hat{q}_d = q_d + \frac{1}{2}\epsilon q_d \circ p_d$, respectively, the twist of $\hat{q}_d$ by $\xi_d = \omega_d + \epsilon(\dot{p}_d + \omega_d \times p_d)$. The generalized PID control scheme in Theorem 7 can be specified in Euclidean Space as follows.

$$\begin{cases} \omega = Ad_{q_e}m, \\ \dot{p} = Ad_{q_e}n + Ad_{q_e}p_d \times \omega, \end{cases}$$

with

$$\begin{cases} m = \omega_d + k_{pr}\theta_e + k_{dr}\dot{\theta}_e + k_{ir}\int \theta_e \, dt, \\ n = \dot{p}_d + \omega_d \times p_d + k_{pd}p_e + k_{dd}\dot{p}_e + k_{id}\int p_e \, dt, \end{cases}$$

where $q_e = q^* \circ q_d$, $\theta_e = \ln q_e$ and $p_e = p_d - Ad_{q_e^*} p$.

The control scheme in Theorem 7 provides harmony between rotation and translation, which can control attitude and position simultaneously. In Theorem 11, it clearly shows that the interconnection between rotation and translation is preserved, where $\dot{p}$ is affected by ω^d and ω. In conventional methods such as decoupled control scheme in Asama et al. (1995), the interconnection does not exist.

3.3 Discrete Form and Incremental Form

In modern digital control, the most common forms of PID control scheme are with discrete form or incremental form, hence in the following we will deduce the related discrete form and incremental form of the proposed PID control scheme.

Theorem 12. (Discrete Form). Denote actual and target configurations at time k by $\hat{q}_k = q_k + \frac{1}{2}\epsilon q_k \circ p_k$ and $\hat{q}_{d_k} = q_{d_k} + \frac{1}{2}\epsilon q_{d_k} \circ p_{d_k}$, respectively. Denote the twist of $\hat{q}_{d_k}$ by $\xi_{d_k} = \omega_{d_k} + \epsilon(\dot{p}_{d_k} + \omega_{d_k} \times p_{d_k})$ [2]. The discrete form of generalized PID control scheme in Theorem 11 is

$$\begin{cases} \omega_k = Ad_{q_{e_k}} m_k, \\ \dot{p}_k = Ad_{q_{e_k}} n_k + Ad_{q_{e_k}} p_{d_k} \times \omega_k, \end{cases} \tag{17}$$

where

$$\begin{cases} m_k = \omega_{d_k} + k_{pr}\theta_{e_k} + k_{dr}(\theta_{e_k} - Ad_{q^*_{d_{(k-1)k}}} \theta_{e_{k-1}}) \\ \qquad + k_{ir} \sum_{i=0}^{k} Ad_{q^*_{d_{ik}}} \theta_{e_i}, \\ n_k = \dot{p}_{d_k} + \omega_{d_k} \times p_{d_k} + k_{pd} p_{e_k} + k_{dd}(p_{e_k} - \\ \qquad Ad_{q^*_{d_{(k-1)k}}} p_{e_{k-1}}) + k_{id} \sum_{i=0}^{k} Ad_{q^*_{d_{ik}}} p_{e_i}, \end{cases} \tag{18}$$

and $q_{e_j} = q_j^* \circ q_{d_j}$, $\theta_{e_j} = \ln q_{e_j}$, $p_{e_j} = p_{d_j} - Ad_{q^*_{e_j}} p_j$, $q_{d_{jk}} = q^*_{d_j} \circ q_{d_k}$ with $j = k, k-1, \ldots, 0$. Note that $Ad_{q^*_{d_{jk}}} V$ translates $V \in v$ from body-frame d_j to body-frame d_k.

The proof of Theorem 12 is straightforward by substituting $\theta_{e_k} - Ad_{q^*_{d_{(k-1)k}}} \theta_{e_{k-1}}$, $p_{e_k} - Ad_{q^*_{d_{(k-1)k}}} p_{e_{k-1}}$, $\sum_{i=0}^{k} Ad_{q^*_{d_{ik}}} \theta_{e_i}$ and $\sum_{i=0}^{k} Ad_{q^*_{d_{ik}}} p_{e_i}$ for $\dot{\theta}_e$, $\dot{p}_e$, $\int \theta_e$ and $\int p_e$ into Theorem 11, respectively. And Further, the incremental form of the generalized PID control scheme is stated in Theorem 13.

Theorem 13. (Incremental Form). Denote actual configuration at time k by $\hat{q}_k = q_k + \frac{1}{2}\epsilon q_k \circ p_k$ and target configuration at time k by $\hat{q}_{d_k} = q_{d_k} + \frac{1}{2}\epsilon q_{d_k} \circ p_{d_k}$ with twist $\xi_{d_k} = \omega_{d_k} + \epsilon(\dot{p}_{d_k} + \omega_{d_k} \times p_{d_k})$. The incremental form of generalized PID control scheme in Theorem 12 is

$$\begin{cases} \delta\omega_k = Ad_{q_{e_k}} \delta m_k, \\ \delta\dot{p}_k = Ad_{q_{e_k}} \delta n_k + Ad_{q_{e_k}} p_{d_k} \times \omega_k \\ \qquad -Ad_{q_{e_k}} (Ad_{q^*_{d_{(k-1)k}}} p_{d_{k-1}}) \times Ad_{q^*_{(k-1)k}} \omega_{k-1}, \end{cases}$$

where

$$\begin{cases} \delta m_k = \omega_{d_k} - Ad_{q^*_{d_{(k-1)k}}} \omega_{d_{k-1}} + A_r \theta_{e_k} \\ \qquad + B_r Ad_{q^*_{d_{(k-1)k}}} \theta_{e_{k-1}} + C_r Ad_{q^*_{d_{(k-2)k}}} \theta_{e_{k-2}}, \\ \delta n_k = (\dot{p}_{d_k} - Ad_{q^*_{d_{(k-1)k}}} \dot{p}_{d_{k-1}}) + \omega_{d_k} \times p_{d_k} \\ \qquad - Ad_{q^*_{d_{(k-1)k}}} (\omega_{d_{k-1}} \times p_{d_{k-1}}) + A_d p_{e_k} \\ \qquad + B_d Ad_{q^*_{d_{(k-1)k}}} p_{e_{k-1}} + C_d Ad_{q^*_{d_{(k-2)k}}} p_{e_{k-2}}, \end{cases}$$

and $A_r = k_{pr} + k_{dr} + k_{ir}$, $B_r = -(k_{pr} + 2k_{dr})$, $C_r = k_{dr}$, $A_d = k_{pd} + k_{dd} + k_{id}$, $B_d = -(k_{pd} + 2k_{dd})$, $C_d = k_{dd}$.

The proof of Theorem 13 requires the following lemmas, which can be obtained by direct computations. Due to space limitation, the proofs are omitted herein.

Lemma 14. Consider $q_{(k-1)k}$ and $q_{e_{k-1}}$ given in Theorem 12, and $V \in v$. It is obtained

$$Ad_{q^*_{(k-1)k}} Ad_{q_{e_{k-1}}} v = Ad_{q_{e_k}} Ad_{q^*_{d_{(k-1)k}}} v.$$

Lemma 15. Denote $q_{jk} = q_j^* \circ q_k$, $q_{ij} = q_i^* \circ q_j$ and $q_{ik} = q_i^* \circ q_k$. Given $V \in v$, we have

$$Ad_{q^*_{jk}} (Ad_{q^*_{ij}} V) = Ad_{q^*_{ik}} V.$$

Proof of Theorem 13.

The incremental form of (17) in Theorem 12 is

$$\begin{cases} \delta\omega_k = \omega_k - Ad_{q^*_{(k-1)k}} \omega_{k-1}, \\ \delta\dot{p}_k = \dot{p}_k - Ad_{q^*_{(k-1)k}} \dot{p}_{k-1}. \end{cases}$$

In the sequel, we just detail the proof about $\delta\omega_k$, and $\delta\dot{p}_k$ can be obtained with the similar tricks.

For $\delta\omega_k$, according to Lemma 14, we have

$$\begin{aligned} \delta\omega_k &= Ad_{q_{e_k}} m_k - Ad_{q^*_{(k-1)k}} Ad_{q_{e_{k-1}}} m_{k-1} \\ &= Ad_{q_{e_k}} m_k - Ad_{q_{e_k}} Ad_{q^*_{d_{(k-1)k}}} m_{k-1} \\ &= Ad_{q_{e_k}} (m_k - Ad_{q^*_{d_{(k-1)k}}} m_{k-1}). \end{aligned}$$

Let $m_k - Ad_{q^*_{d_{(k-1)k}}} m_{k-1} = \delta m_k$, we obtain

$$\delta\omega_k = Ad_{q_{e_k}} \delta m_k,$$

and then with the aid of Lemma 15, we obtain

$$\begin{aligned} \delta m_k &= (\omega_{d_k} - Ad_{q^*_{d_{(k-1)k}}} \omega_{d_{k-1}}) + k_{pr}(\theta_{e_k} - \\ &\quad Ad_{q^*_{d_{(k-1)k}}} \theta_{e_{k-1}}) + k_{dr}(\theta_{e_k} - Ad_{q^*_{d_{(k-1)k}}} \theta_{e_{k-1}} \\ &\quad - Ad_{q^*_{d_{(k-1)k}}} (\theta_{e_{k-1}} - Ad_{q^*_{d_{(k-2)(k-1)}}} \theta_{e_{k-2}})) \\ &\quad + k_{ir}(\sum_{i=0}^{k} Ad_{q^*_{d_{ik}}} \theta_{e_i} - Ad_{q^*_{d_{(k-1)k}}} (\sum_{i=0}^{k-1} Ad_{q^*_{d_{i(k-1)}}} \theta_{e_i})) \\ &= (\omega_{d_k} - Ad_{q^*_{d_{(k-1)k}}} \omega_{d_{k-1}}) + (k_{pr} + k_{dr} + k_{ir})\theta_{e_k} \\ &\quad - (k_{pr} + 2k_{dr}) Ad_{q^*_{d_{(k-1)k}}} \theta_{e_{k-1}} + k_{dr} Ad_{q^*_{d_{(k-2)k}}} \theta_{e_{k-2}}. \end{aligned}$$

□

4. SIMULATION

In this section, we present some simulation results demonstrating the effectiveness of the proposed control scheme with the Urban Search And Rescue Simulation (USARSim) platform and quad-rotor model. USARSim is a high-fidelity simulation of robots and environments based on the Unreal Tournament game engine. It is intended as a research tool and is the basis for the RoboCup rescue virtual robot competition. It has been expanded to support many diverse environments and robot models including

[2] The subscript k means at time k. For example, $\hat{q}_k$ means $\hat{q}$ at time k and $\hat{q}_{d_k}$ means $\hat{q}_d$ at time k. The same goes for subscripts i and j.

the DARPA urban challenge, robotic soccer, submarines, humanoids and helicopters. In this study, we use a typical space robot, quad-rotor. The position and orientation information, and the control signals are sampled at a time interval of 0.03 s.

The work space of quad-rotor is isomorphic to $SO(2)\otimes\mathbb{R}^3$, which is with 1 rotational DOF and 3 translational DOF. Taking z-axis as the rotating axis, the current and target configurations can be specified by $\hat{q} = q + \frac{\epsilon}{2} q \circ p$ with $q = (\cos\frac{\theta}{2}, 0, 0, \sin\frac{\theta}{2})$, $p = (x, y, z)^T$, and $\hat{q}_d = q_d + \frac{\epsilon}{2} q_d \circ p_d$ with $q_d = (\cos\frac{\theta_d}{2}, 0, 0, \sin\frac{\theta_d}{2})$ and $p_d = (x_d, y_d, z_d)^T$, where angles θ and θ_d relate to $SO(2)$, position (x, y, z) and (x_d, y_d, z_d) relate to $\mathbb{R}^3$. We can control the angular velocity ω and linear velocity $(\dot{x}, \dot{y}, \dot{z})^T$ in quad-rotor's body-frame, i.e., the control inputs of quad-rotor are its twist in body-frame.

The reference configuration is described by $\hat{q}_d(t)$ with $q_d(t) = (\cos\theta_d(t), 0, 0, \sin\theta_d(t))$, $p_d^s(t) = (x_d(t), y_d(t), z_d(t))^T$, where

$$\begin{cases} \theta_d(t) &= \omega_1 t, \\ x_d(t) &= r\cos(\omega_1 t) + a, \\ y_d(t) &= r\sin(\omega_1 t) + b, \\ z_d(t) &= z_0 \sin(\omega_2 t) + c. \end{cases}$$

Then, twist of $\hat{q}_d(t)$ is $\xi_d(t) = \omega_d(t) + \epsilon(\dot{p}_d(t) + omega_d(t) \times p_d(t))$ with $\omega_d(t) = (0, 0, \dot{\theta}_d(t))^T$ and $\dot{p}_d(t) = (\dot{x}_d(t), \dot{y}_d(t), \dot{z}_d(t))^T$, where

$$\begin{cases} \dot{\theta}_d(t) &= \omega_1, \\ \dot{x}_d(t) &= -r\omega_1 \sin(\omega_1 t)\cos\theta_d(t) - r\omega_1\cos(\omega_1 t)\sin\theta_d(t), \\ \dot{y}_d(t) &= r\omega_1\cos(\omega_1 t)\cos\theta_d(t) - r\omega_1\sin(\omega_1 t)\sin\theta_d(t), \\ \dot{z}_d(t) &= z_0\omega_2\cos(\omega_2 t). \end{cases}$$

Our object is to control quad-rotor to track the reference trajectory $\hat{q}_d(t)$ with $r = 10, a = 0, b = -40, c = 25, z_0 = 5, \omega_1 = \omega_2 = 0.1$. And the initial conditions are: the initial configuration $\hat{q}_0^s$ is with $q = (1, 0, 0, 0)$, $p^s = (9.75, -46.67, 20)^T$ and the target configuration $\hat{q}_d^s$ is with $q_d^s = (\cos\frac{2}{3}\pi, 0, 0, \sin\frac{2}{3}\pi)$, $p_d^s = (0, -20, 30)^T$ [3]. We apply the control scheme in Theorem 13 with PID parameters: $\hat{k}_p = (0, 0, 0.5)^T + \epsilon(1, 1, 1)^T$, $\hat{k}_d = (0, 0, 1)^T + \epsilon(2, 2, 2)^T$ and $\hat{k}_i = (0, 0, 0.001t)^T + \epsilon(0.002t, 0.002t, 0.002t)^T$ with t being the running time. The simulation results are shown in Fig. 1 to Fig. 3.

Figures 1 and 2 show that the proposed control scheme cause the actual trajectory to converges to the reference trajectory. Fig. 3 shows clearly the tracking error converges into 0 asymptotically, which means the errors on (θ, x, y, z) all converge to 0. Thus, the control scheme in Theorem 13 can track the reference configuration well.

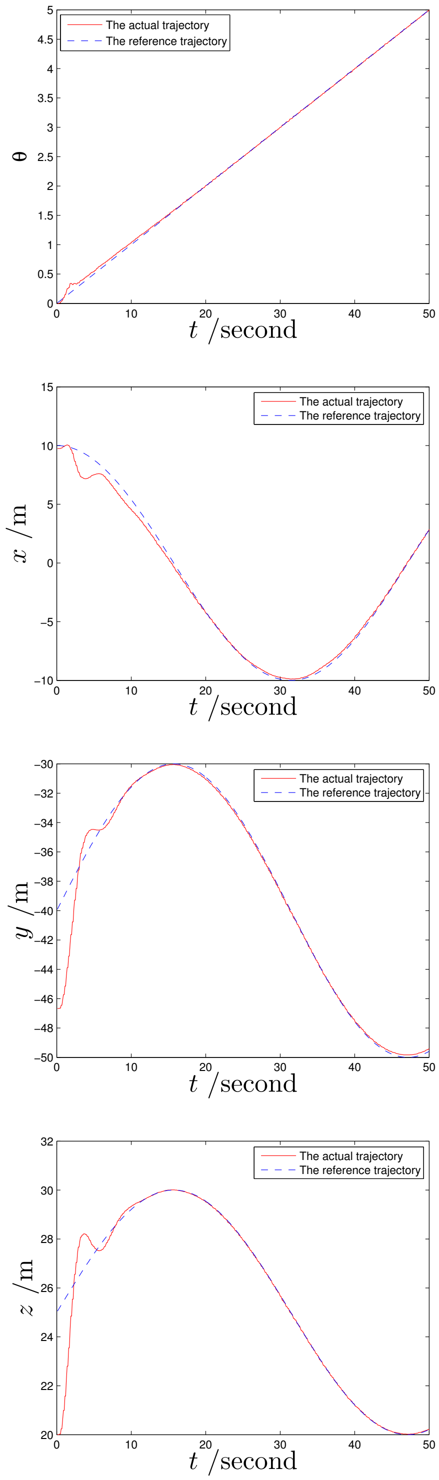

Fig. 1. Evolutions of actual and reference trajectories on (θ, x, y, z) versus time.

5. CONCLUDING REMARKS

The contribution of this work mainly lies in two aspects. Firstly, a new unit-dual-quaternion-based PID control scheme for rigid-body transformation is proposed after the new kinematic control model in body-frame is derived. Secondly, the related discrete from and incremental form, which are the most common forms in modern digital control, of PID control scheme are deduced, and then validated on USARSim platform with quad-rotor. The proposed control scheme can deal with rotation and translation simultaneously and preserve the interconnection between rotation and translation.

Tuning of the parameters used in PID control scheme is a large research area (Åström and Hagglund, 2001). In the proposed generalized PID control scheme, adjustable pa-

[3] The original configuration and the target configuration are both set in spatial-frame for convenient and clarity. In simulations, these configurations are translated into body-frame in real time.

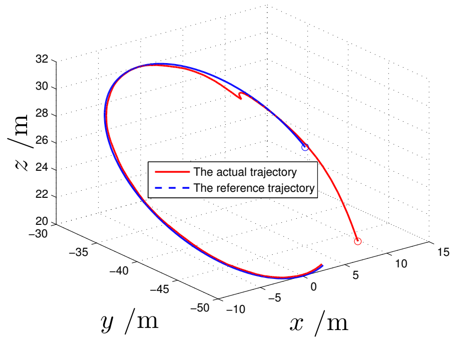

Fig. 2. Actual and reference configurations.

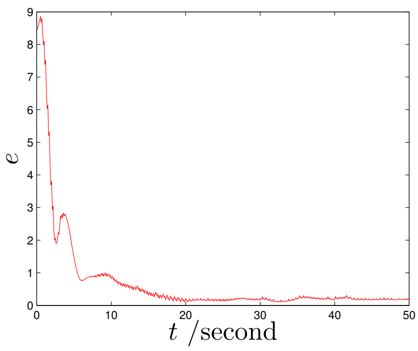

Fig. 3. Tracking errors, defined as $\sqrt{2|p_e|^2 + |\theta_e|^2}$, versus time.

rameters $\hat{k}_p$, $\hat{k}_d$ and $\hat{k}_i$ are all dual vectors. Thus, as many as 18 parameters need to be tuned. Moreover, those parameters are interacting because the control scheme does not couple rotation and translation. So it is a notable work to study the self-tuning problem of this control scheme in the future. Another issue is to deal with the rigid-body dynamic control problem with unit dual quaternion. Some preliminary work has been reported in Wang and Yu (2010), and further studies about dynamic control, such as unit-dual-quaternion-based Variable Structure Control, are to be completed.

REFERENCES

Ang, K.H., Chong, G., and Li, Y. (2005). PID control system analysis, design, and technology. *IEEE Transactions on Control Systems Technology*, 13(4), 559–576.

Asama, H., Sato, M., L, B., and Kaetsu, H. (1995). Development of an omni-directional mobile robot with 3 DOF decoupling drive mechanism. In *Proceedings of the 1995 International Conference on Robotics and Automation*. Nagoya, Aichi, Japan.

Aspragathos, N. and Dimitros, J. (1998). A comparative study of three methods for robot kinematics. *IEEE Transactions on Systems, Man and Cybernetics Part B: Cybernetics*, 28(2), 135–145.

Åström, K.J. and Hagglund, T. (2001). The future of PID control. *Control Engineering Practicle*, 9(11), 1163–1175.

Bullo, F. and Murray, R. (1995). Proportional derivative (PD) control on the euclidean group. In *Proceedings of 1995 European Control Conference*. Rome, Italy.

Daniilidis, K. (1999). Hand-eye calibration using dual quaternions. *The International Journal of Robotics Research*, 18(3), 286–298.

Funda, J. and Paul, R. (1990). A computational analysis of screw transformations in robotics. *IEEE Transactions on Robotics and Automation*, 6(3), 348–356.

Funda, J., Taylor, R., and Paul, R. (1990). On homogeneous transformations, quaternions, and computational efficiency. *IEEE Transactions on Robotics and Automation*, 6(3), 382–388.

Goddard, J.S. (1997). *Pose and Motion Estimation from Vision Using Dual Quaternion-based Extended Kalman Filtering*. PhD thesis, The University of Tennessee.

Han, D.P., Wei, Q., and Li, Z.X. (2008). Kinematic control of free rigid bodies using dual quaternions. *International Journal of Automation and Computing*, 5(3), 319–324.

Murray, R.M., Li, Z., and Sastry, S.S. (1994). *An Mathematical Introduction to Robotic Manipulation*. CRC Press.

Purwar, A. and Ge, Q.J. (2005). On the effect of dual weights in computer aided design of rational motions. *Journal of Mechanical Design*, 127, 967–972.

Wang, X. and Yu, C. (2010). Feedback linearization regulator with coupled attitude and translation dynamics based on unit dual quaternion. In *Proceedings of 2010 IEEE Multi-Conference on Systems and Control*, 2380–2384. Pacifico Yokohoma, Japan.

Wu, Y.X., Hu, X.P., Hu, D.W., and Lian, J.X. (2005). Strapdown inertial navigation system algorithms based on dual quaternions. *IEEE Transactions on Aerospace and Electronic Systems*, 41(1), 110–132.

Appendix A. PROOF OF LEMMA 5

To prove Lemma 5, the following additional formulas are directly obtained from their definitions.

$$\begin{cases} \dot{q}_e = \frac{1}{2} q_e \circ \omega_e, \\ \dot{p}_e = \dot{p}_d - Ad_{q_e^*} p \times \omega_e - Ad_{q_e^*} \dot{p}, \\ \omega_e \times p_e = \omega_d \times p_d - Ad_{q_e^*} \omega \times p_d - \omega_e \times Ad_{q_e^*} p. \end{cases}$$

Proof. First, for (8), we obtain

$$\begin{aligned} \xi_e &= \xi_d - \hat{q}_e^* \circ \xi \circ \hat{q}_e \\ &= (\omega_d + \epsilon(\dot{p}_d + \omega_d \times p_d)) \\ &\quad -(q_e^* \circ \omega \circ q_e + \epsilon(Ad_{q_e^*}(\dot{p} + \omega \times p) + Ad_{q_e^*}\omega \times p_e)) \\ &= \omega_d - Ad_{q_e^*}\omega + \epsilon((\dot{p}_d + \omega_d \times p_d) \\ &\quad -Ad_{q_{el}^*}(\dot{p} + \omega \times p) - Ad_{q_e^*}\omega \times p_e) \\ &= \omega_e + \epsilon((\dot{p}_d - Ad_{q_e^*} p \times \omega_e - Ad_{q_e^*}\dot{p}) + (\omega_d \times p_d \\ &\quad -\omega_e \times Ad_{q_e^*} p - Ad_{q_e^*}(\omega \times p) - Ad_{q_e^*}\omega \times p_e)) \\ &= \omega_e + \epsilon(\dot{p}_e + (\omega_d \times p_d - \omega_e \times Ad_{q_e^*} p \\ &\quad -Ad_{q_e^*}(\omega \times p) - Ad_{q_e^*}\omega \times (p_d - Ad_{q_e^*} p))) \\ &= \omega_e + \epsilon(\dot{p}_e + \omega_e \times p_e). \end{aligned}$$

Then, for (7),

$$\begin{aligned} \dot{\hat{q}}_e &= \dot{\hat{q}}^* \circ \hat{q}_d + \hat{q}^* \circ \dot{\hat{q}}_d \\ &= -\frac{1}{2}\xi \circ \hat{q}^* \circ \hat{q}_d + \frac{1}{2} \cdot \hat{q}^* \circ \hat{q}_d \circ \xi_d \\ &= \frac{1}{2}\hat{q}^* \circ \hat{q}_d \circ (\xi_d - (\hat{q}^* \circ \hat{q}_d)^* \circ \xi \circ \hat{q}^* \circ \hat{q}_d) \\ &= \frac{1}{2}\hat{q}_e \circ \xi_e. \end{aligned}$$

$\square$