01_linear_regression_simple

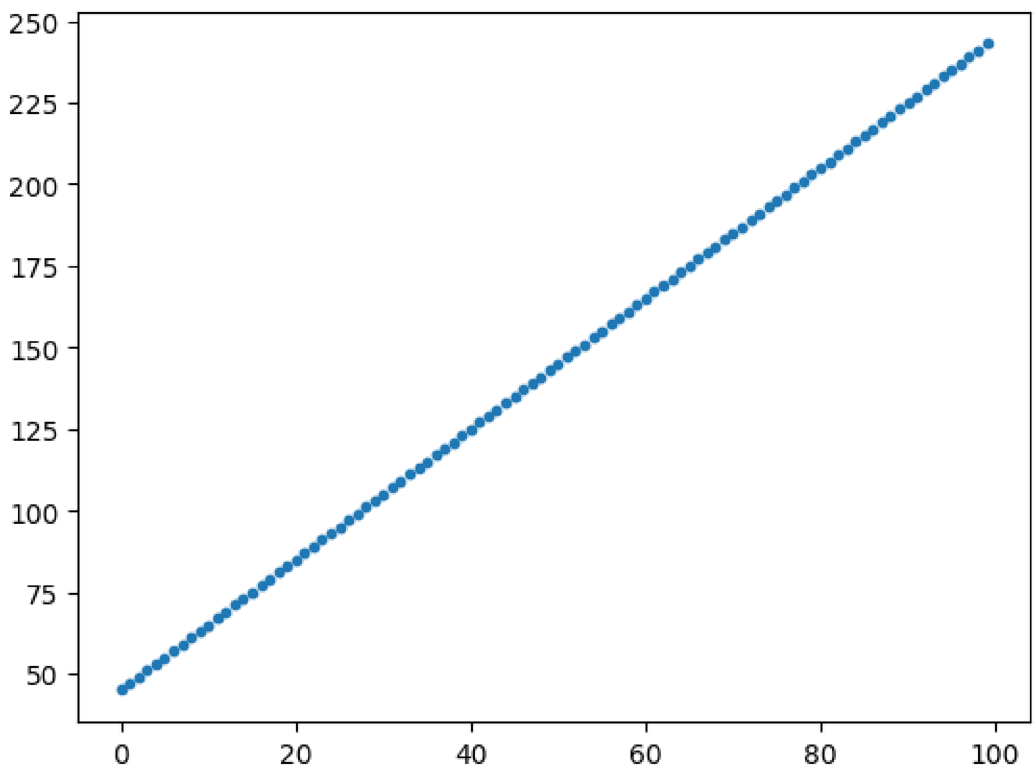
March 10, 2025

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

[2]:
```
def linear_function(a, x, b):
    return a*x + b
```

[3]:
```
a = 2
x = np.arange(100)
b = 45
y = linear_function(a, x, b)
```

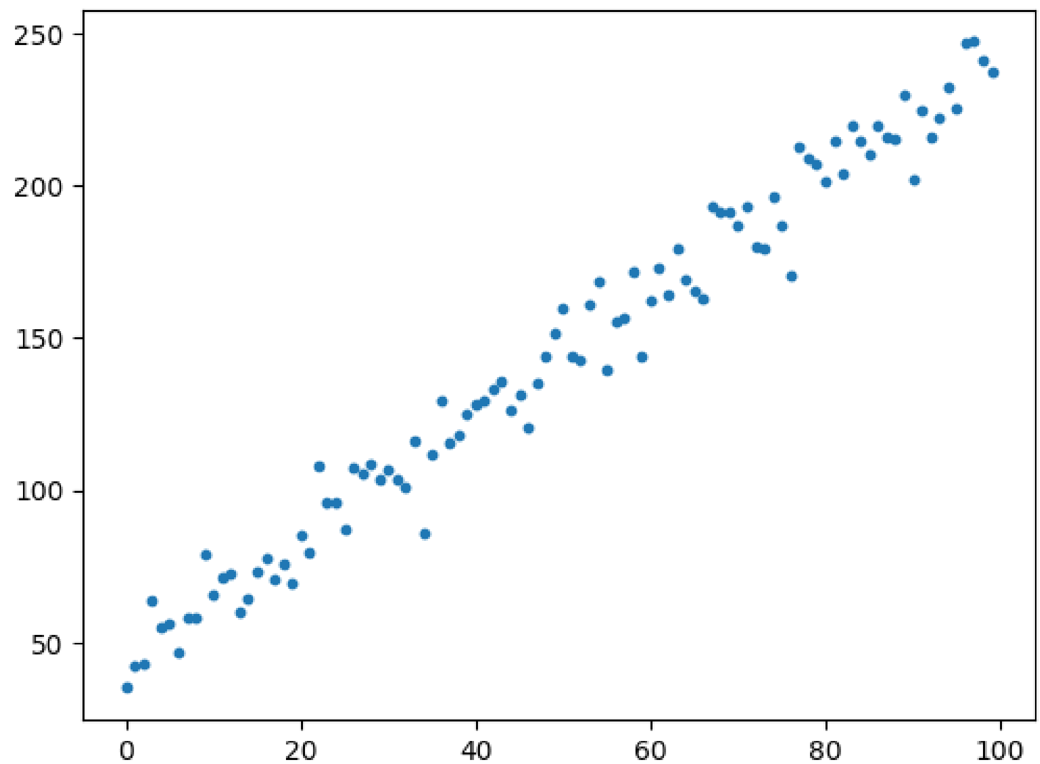
[4]:
```
plt.scatter(x, y, s=10)
plt.show()
```



[5]:
```
# https://medium.com/@ms_somanna/
↪guide-to-adding-noise-to-your-data-using-python-and-numpy-c8be815df524

noise = np.random.normal(0, 10, len(x))
y_noisy = y + noise
```

[6]:
```
plt.scatter(x, y_noisy, s=10)
plt.show()
```



1 Derivation of Least Squares Estimates for β_1 and β_0

We want to find β_1 and β_0 that minimize the **Residual Sum of Squares (RSS)** for a simple linear regression model:

$$y = \beta_1 x + \beta_0$$

1.1 Step 1: Define the Residual Sum of Squares (RSS)

The RSS is given by:

$$RSS = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}(y_i - (\beta_1 x_i + \beta_0))^2$$

Our goal is to minimize RSS with respect to β_0 and β_1.

1.2 Step 2: Compute Partial Derivatives

To find the optimal values of β_0 and β_1, we take partial derivatives of RSS and set them to zero.

1.2.1 Partial Derivative with Respect to β_0

$$\frac{\partial RSS}{\partial \beta_0} = \sum_{i=1}^{n} 2(y_i - (\beta_1 x_i + \beta_0))(-1) = -2\sum_{i=1}^{n}(y_i - \beta_1 x_i - \beta_0)$$

Setting it to zero:

$$\sum_{i=1}^{n}(y_i - \beta_1 x_i - \beta_0) = 0$$

Rearrange:

$$\sum_{i=1}^{n} y_i = \beta_1 \sum_{i=1}^{n} x_i + n\beta_0$$

Dividing by n:

$$\bar{y} = \beta_1 \bar{x} + \beta_0$$

Thus, solving for β_0:

$$\beta_0 = \bar{y} - \beta_1 \bar{x}$$

1.2.2 Partial Derivative with Respect to β_1

$$\frac{\partial RSS}{\partial \beta_1} = \sum_{i=1}^{n} 2(y_i - (\beta_1 x_i + \beta_0))(-x_i) = -2\sum_{i=1}^{n} x_i(y_i - \beta_1 x_i - \beta_0)$$

Setting it to zero:

$$\sum_{i=1}^{n} x_i y_i - \beta_1 \sum_{i=1}^{n} x_i^2 - \beta_0 \sum_{i=1}^{n} x_i = 0$$

Substituting $\beta_0 = \bar{y} - \beta_1 \bar{x}$:

$$\sum_{i=1}^{n} x_i y_i - \beta_1 \sum_{i=1}^{n} x_i^2 - (\bar{y} - \beta_1 \bar{x}) \sum_{i=1}^{n} x_i = 0$$

Expanding:

$$\sum_{i=1}^{n} x_i y_i - \beta_1 \sum_{i=1}^{n} x_i^2 - \bar{y} \sum_{i=1}^{n} x_i + \beta_1 \bar{x} \sum_{i=1}^{n} x_i = 0$$

Rearrange:

$$\sum_{i=1}^{n} x_i y_i - \bar{y} \sum_{i=1}^{n} x_i = \beta_1 \left(\sum_{i=1}^{n} x_i^2 - \bar{x} \sum_{i=1}^{n} x_i \right)$$

Since $\sum_{i=1}^{n} x_i = n\bar{x}$, we simplify:

$$\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y} = \beta_1 \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \right)$$

Dividing both sides by $\sum_{i=1}^{n} x_i^2 - n\bar{x}^2$:

$$\beta_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

1.3 Final Formulae

Thus, we obtain the least squares estimates:

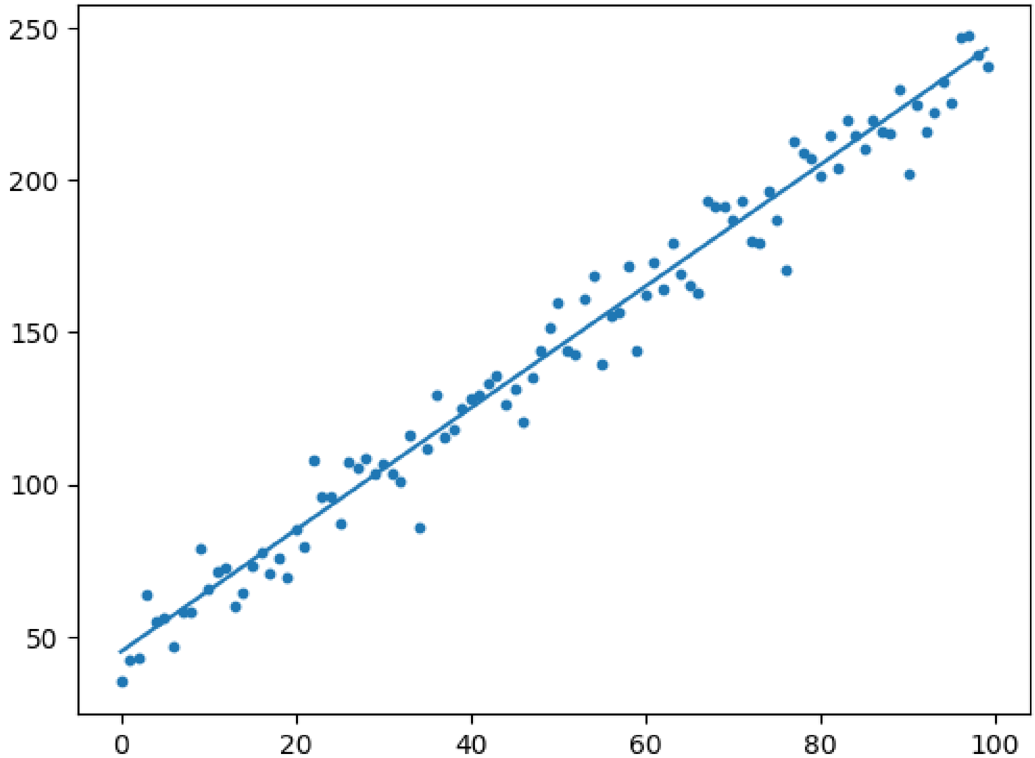
$$\beta_1 = \frac{\sum (x - \bar{x})(y - \bar{y})}{\sum (x - \bar{x})^2}$$

$$\beta_0 = \bar{y} - \beta_1 \bar{x}$$

These equations provide the optimal values of β_0 and β_1 for minimizing the RSS in simple linear regression.

[7]:
```
beta_1 = np.sum((x - np.mean(x)) * (y - np.mean(y))) / np.sum(np.power(x - np.
 ↪mean(x), 2))
beta_0 = np.mean(y) - beta_1 * np.mean(x)
```

[8]:
```
plt.scatter(x, y_noisy, s=10)
plt.plot(beta_0 + beta_1 * x)
plt.show()
```



[9]:
```
print(beta_0, beta_1)
```

```
45.0 2.0
```