Dantzig Wolfe Decomposition

Operations Research

Anthony Papavasiliou

Block Structure of Primal

$$\begin{array}{cccc} A & & & \\ T_1 & W & & \\ T_2 & & W & \\ \vdots & & & \\ T_K & \cdots & & W \end{array}$$

L-Shaped method: ignore constraints of future stages

Block Structure of Dual

$$\begin{array}{ccccc} \boxed{A^T} & \boxed{T_1^T} & \boxed{T_2^T} & \cdots & \boxed{T_K^T} \\ & \boxed{W^T} & & & \\ & & \boxed{W^T} & & \\ & & & \ddots & \\ & & & & \boxed{W^T} \end{array}$$

Dantzig-Wolfe decomposition: ignore variables

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The Problem

$$
\begin{aligned}
z^\star &= \min c_1^T x_1 + c_2^T x_2 \\
\text{s.t. } & A_1 x_1 + A_2 x_2 = b \\
& B_1 x_1 = d_1 \\
& B_2 x_2 = d_2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

- $x_1 \in \mathbb{R}^{n_1}$, $x_2 \in \mathbb{R}^{n_2}$
- $b \in \mathbb{R}^m$, $d_1 \in \mathbb{R}^{m_1}$, $d_2 \in \mathbb{R}^{m_2}$
- $A_1 x_1 + A_2 x_2 = b$ are **complicating/coupling constraints**

Note: This will be the form of the dual of the 2-stage stochastic program (see slide 3)

Minkowski's Representation Theorem

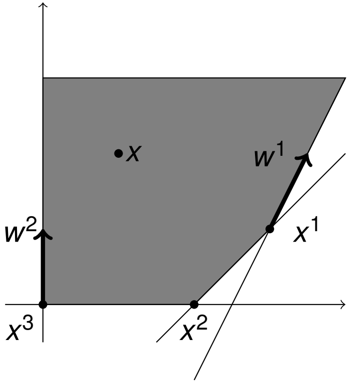
Every polyhedron P can be represented in the form

$$P = \{x \in \mathbb{R}^n : x = \sum_{j \in J} \lambda^j x^j + \sum_{r \in R} \mu^r w^r,$$
$$\sum_{j \in J} \lambda^j = 1, \lambda \in \mathbb{R}_+^{|J|}, \mu \in \mathbb{R}_+^{|R|}\}$$

where

- $\{x^j, j \in J\}$ are the extreme points of P
- $\{w^r, r \in R\}$ are the extreme rays of P

Graphical Illustration of Minkowski's Representation Theorem



- x^1, x^2, x^3: extreme points
- w^1, w^2: extreme rays
- $x = \lambda x^2 + (1-\lambda)x^3 + \mu w^2$, $0 \leq \lambda \leq 1$, $\mu \geq 0$

The Feasible Region of the Subproblems

We represent $B_1 x_1 = d_1$ as

$$\sum_{j \in J_1} \lambda_1^j x_1^j + \sum_{r \in R_1} \mu_1^r w_1^r, \lambda_1^j \geq 0, \mu_1^r \geq 0, \sum_{j \in J_1} \lambda_1^j = 1$$

and $B_2 x_2 = d_2$ as

$$\sum_{j \in J_2} \lambda_2^j x_2^j + \sum_{r \in R_2} \mu_2^r w_2^r, \lambda_2^j \geq 0, \mu_2^r \geq 0, \sum_{j \in J_2} \lambda_2^j = 1$$

Transform the full master problem using

- $x_1 = \sum_{j \in J_1} \lambda_1^j x_1^j + \sum_{r \in R_1} \mu_1^r w_1^r$
- $x_2 = \sum_{j \in J_2} \lambda_1^j x_2^j + \sum_{r \in R_2} \mu_2^r w_2^r$

For example,

$$A_1 x_1 + A_2 x_2 = b$$

becomes

$$\sum_{j \in J_1} \lambda_1^j A_1 x_1^j + \sum_{r \in R_1} \mu_1^r A_1 w_1^r + \sum_{j \in J_2} \lambda_2^j A_2 x_2^j + \sum_{r \in R_2} \mu_2^r A_2 w_2^r = b$$

The Full Master Problem

Applying Minkowski's representation theorem we obtain:

$$z = \min \sum_{j \in J_1} \lambda_1^j c_1^T x_1^j + \sum_{r \in R_1} \mu_1^r c_1^T w_1^r + \sum_{j \in J_2} \lambda_2^j c_2^T x_2^j + \sum_{r \in R_2} \mu_2^r c_2^T w_2^r$$

$$\sum_{j \in J_1} \lambda_1^j A_1 x_1^j + \sum_{r \in R_1} \mu_1^r A_1 w_1^r + \sum_{j \in J_2} \lambda_2^j A_2 x_2^j + \sum_{r \in R_2} \mu_2^r A_2 w_2^r = b, \; (\pi)$$

$$\sum_{j \in J_1} \lambda_1^j = 1, \; (t_1)$$

$$\sum_{j \in J_2} \lambda_2^j = 1, \; (t_2)$$

$$\lambda_1^j, \lambda_2^j, \mu_1^r, \mu_2^r \geq 0$$

Thinking About the New Formulation

- This problem is equivalent to the original problem
- The decision variables are the weights of the extreme points $(\lambda_1^j, \lambda_2^j)$ and weights of the extreme rays (μ_1^r, μ_2^r)
- The number of decision variables can be *enormous* (trick: we will ignore most of them)
- The number of constraints is smaller (we got rid of $B_1 x_1 = d_1$, $B_2 x_2 = d_2$)

Columns in the New Formulation

Constraint matrix in the new formulation:

$$\sum_{j \in J_1} \lambda_1^j \begin{bmatrix} A_1 x_1^j \\ 1 \\ 0 \end{bmatrix} + \sum_{j \in J_2} \lambda_2^j \begin{bmatrix} A_2 x_2^j \\ 0 \\ 1 \end{bmatrix}$$

$$+ \sum_{r \in R_1} \mu_1^r \begin{bmatrix} A_1 w_1^r \\ 0 \\ 0 \end{bmatrix} + \sum_{r \in R_2} \mu_2^r \begin{bmatrix} A_2 w_2^r \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ 1 \\ 1 \end{bmatrix}$$

Certificate of optimality: given a basic feasible solution, all variables have non-negative reduced costs

Recall Reduced Costs

Consider a linear program in standard form

$$\min c^T x$$
$$\text{s.t. } Ax = b, (\pi)$$
$$x \geq 0$$

Given a basis B, when is it optimal?

1. $B^{-1}b \geq 0$
2. $c_B^T - \pi^T A \geq 0$

where c_B correspond to coefficients of basic variables

Reduced Costs

Given a basic feasible solution, criterion for new variable to enter is negative reduced cost

- Reduced cost of λ_1^j:

$$c_1^T x_1^j - \begin{bmatrix} \pi^T & t_1 & t_2 \end{bmatrix} \begin{bmatrix} A_1 x_1^j \\ 1 \\ 0 \end{bmatrix} = (c_1^T - \pi^T A_1) x_1^j - t_1$$

- Reduced cost of μ_1^r:

$$c_1^T w_1^r - \begin{bmatrix} \pi^T & t_1 & t_2 \end{bmatrix} \begin{bmatrix} A_1 x_1^j \\ 0 \\ 0 \end{bmatrix} = (c_1^T - \pi^T A_1) x_1^j$$

- Similarly for λ_2^j, μ_2^r

Idea of the Algorithm: Subproblems

Idea: instead of looking at reduced cost of every variable λ_1^j, λ_2^j, μ_1^r, μ_2^r (there is an enormous number) we can solve the following problems

$$z_1 = \min(c_1^T - \pi^T A_1)x_1$$
$$\text{s.t. } B_1 x_1 = d_1$$
$$x_1 \geq 0$$

$$z_2 = \min(c_2^T - \pi^T A_2)x_2$$
$$\text{s.t. } B_2 x_2 = d_2$$
$$x_2 \geq 0$$

Three Possibilities

Given the solution of subproblem 1

1. Optimal cost is $-\infty$
   - Simplex output: extreme ray w_1^r with $(c_1^T - \pi^T A_1) w_1^r < 0$
   - Conclusion: reduced cost of μ_1^r is negative
   - Action: include μ_1^r in master problem with column

$$\begin{bmatrix} A_1 w_1^r \\ 0 \\ 0 \end{bmatrix}$$

2. Optimal cost finite, less then t_1
   - Simplex output: extreme point x_1^j with $(c_1^T - \pi^T A_1) x_1^j < t_1$
   - Conclusion: reduced cost of λ_1^j is negative
   - Action: include λ_1^j in master problem with column

$$\begin{bmatrix} A_1 x_1^j \\ 1 \\ 0 \end{bmatrix}$$

3. Optimal cost is finite, no less than t_1
    - Conclusion: $(c_1^T - \pi^T A_1)x_1^j \geq t_1$ for all extreme points x_1^j, $(c_1^T - \pi^T A_1)w_1^r \geq 0$ for all extreme rays w_1^r
    - Action: terminate, we have an optimal basis

Same idea applies to subproblem 2

Idea: instead of solving **full master** for all variables, solve **restricted master problem** for 'worthwhile' subset of variables $\tilde{J}_1 \subset J_1$, $\tilde{J}_2 \subset J_2$, $\tilde{R}_1 \subset R_1$, $\tilde{R}_2 \subset R_2$

$$z = \min \sum_{j \in \tilde{J}_1} \lambda_1^j c_1^T x_1^j + \sum_{r \in \tilde{R}_1} \mu_1^r c_1^T w_1^r + \sum_{j \in \tilde{J}_2} \lambda_2^j c_2^T x_2^j + \sum_{r \in \tilde{R}_2} \mu_2^r c_2^T w_2^r$$

$$\sum_{j \in \tilde{J}_1} \lambda_1^j A_1 x_1^j + \sum_{r \in \tilde{R}_1} \mu_1^r A_1 w_1^r + \sum_{j \in \tilde{J}_2} \lambda_2^j A_2 x_2^j + \sum_{r \in \tilde{R}_2} \mu_2^r A_2 w_2^r = b$$

$$\sum_{j \in \tilde{J}_1} \lambda_1^j = 1, \sum_{j \in \tilde{J}_2} \lambda_2^j = 1$$

$$\lambda_1^j, \lambda_2^j, \mu_1^r, \mu_2^r \geq 0$$

Dantzig-Wolfe Decomposition Algorithm

1. Solve restricted master with initial basic feasible solution, store π, t_1, t_2
2. Solve subproblems 1 and 2. If $(c_1^T - \pi^T A_1)x \geq t_1$ and $(c_2^T - \pi^T A_2)x \geq t_2$ terminate with optimal solution:

$$
\begin{aligned}
x_1 &= \sum_{j \in \tilde{J}_1} \lambda_1^j x_1^j + \sum_{r \in \tilde{R}_1} \mu_1^r w_1^r \\
x_2 &= \sum_{j \in \tilde{J}_2} \lambda_2^j x_2^j + \sum_{r \in \tilde{R}_2} \mu_2^r w_2^r
\end{aligned}
$$

3. If subproblem i is unbounded, add μ_i^r to the master
4. If subproblem i has bounded optimal cost less than t_i, add λ_i^j to the master
5. Generate column associated with entering variable, solve master, store π, t_1, t_2 and go to step 2

Applicability of the Method

Analysis generalizes to multiple subproblems:

$$\min c_1^T x_1 + c_2^T x_2 + \cdots + c_t^T x_K$$
$$\text{s.t. } A_1 x_1 + A_2 x_2 + \cdots + A_t x_K = b$$
$$B_i x_i = d_i, i = 1, \ldots, K$$
$$x_1, x_2, \ldots, x_K \geq 0$$

Approach applies for $K = 1$, apply when *subproblem has special structure*

$$\min c^T x$$
$$\text{s.t. } Ax = b$$
$$Bx = d$$
$$x \geq 0$$

Dantzig-Wolfe Bounds

Denote:

- z_i: optimal objective function value of subproblem i, $i = 1, \ldots, K$
- $z^\star$: optimal objective function value of problem
- z: optimal objective function value of restricted master
- t_i: dual optimal multiplier of $\sum_{j \in \tilde{J}_i} \lambda_i^j = 1$ in restricted master

We get bounds at each iteration

- Upper bound:

$$z \geq z^\star$$

- Lower bound:

$$z + \sum_{i=1}^{K} (z_i - t_i) \leq z^\star$$

Proof of Upper Bound

The solution of the restricted master problem is a feasible solution to the original problem

Proof of Lower Bound ($K = 2$)

Consider the dual of the master problem:

$$
\begin{aligned}
& \max \pi^T b + t_1 + t_2 \\
& \text{s.t. } \pi^T A_1 x_1^j + t_1 \le c_1^T x_1^j, j \in J_1, (\lambda_1^j) \\
& \pi^T A_1 w_1^r \le c_1^T w_1^r, r \in R_1, (\mu_1^r) \\
& \pi^T A_2 x_2^j + t_2 \le c_2^T x_2^j, j \in J_2, (\lambda_2^j) \\
& \pi^T A_2 w_2^r \le c_2^T w_2^r, r \in R_2, (\mu_2^r)
\end{aligned}
$$

- Note that if z_1 is finite

$$
\begin{aligned}
z_1 &\leq c_1^T x_1^j - \pi^T A_1 x_1^j, \forall j \in J_1 \\
c_1^T w_1^r - \pi^T A_1 w_1^r &\geq 0, \forall r \in R_1
\end{aligned}
$$

- Same observation holds true for z_2 finite
- Conclusion: (π, z_1, z_2) is feasible for above problem
- Weak duality:

$$z^\star \geq \pi^T b + z_1 + z_2 = z + (z_1 - t_1) + (z_2 - t_2)$$

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Example 1

$$\min -4x_1 - x_2 - 6x_3$$
$$\text{s.t. } 3x_1 + 2x_2 + 4x_3 = 17$$
$$1 \le x_1 \le 2$$
$$1 \le x_2 \le 2$$
$$1 \le x_3 \le 2$$

Divide constraints as follows:

- Represent $P = \{x \in \mathbb{R}^3 | 1 \le x_i \le 2\}$ by its extreme points x^j
- Complicating constraints $Ax = b$, $A = \begin{bmatrix} 3 & 2 & 4 \end{bmatrix}$, $b = 17$

First Iteration: Master

- Initialization: pick extreme points $x^1 = (2, 2, 2)$, $x^2 = (1, 1, 2)$ with restricted master problem basic variables λ^1, λ^2
  Basis matrix:
  $$B = \begin{bmatrix} 3 \cdot 2 + 2 \cdot 2 + 4 \cdot 2 & 3 \cdot 1 + 2 \cdot 1 + 4 \cdot 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 13 \\ 1 & 1 \end{bmatrix}$$
- Restricted master:
  $$\min -22\lambda^1 - 17\lambda^2$$
  $$\text{s.t. } 18\lambda^1 + 13\lambda^2 = 17, (\pi)$$
  $$\lambda^1 + \lambda^2 = 1, (t)$$
  $$\lambda^1, \lambda^2 \geq 0$$
- Optimal solution $\lambda^1 = 0.8$, $\lambda^2 = 0.2$, optimal multipliers: $\pi = -1$, $t = -4$

First Iteration: Subproblem

- Objective function coefficients: $c^T - \pi^T A =$ $\begin{bmatrix} -4 & -1 & -6 \end{bmatrix} - (-1) \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 & -2 \end{bmatrix}$
- Subproblem:

$$\min -x_1 + x_2 - 2x_3$$
$$\text{s.t. } 1 \leq x_1 \leq 2, 1 \leq x_2 \leq 2, 1 \leq x_3 \leq 2$$

- Optimal solution: $x^3 = (2, 1, 2)$, objective function value -5 is less than $t = -4$
- Introduction of λ^3 to master with coefficients

$$\begin{bmatrix} 3 \cdot 2 + 2 \cdot 1 + 4 \cdot 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 1 \end{bmatrix}$$

Second Iteration: Master

- Restricted master problem:

$$\begin{aligned}
&\min -22\lambda^1 - 17\lambda^2 - 21\lambda^3 \\
&\text{s.t. } 18\lambda^1 + 13\lambda^2 + 16\lambda^3 = 17, (\pi) \\
&\lambda^1 + \lambda^2 + \lambda^3 = 1, (t) \\
&\lambda^1, \lambda^2, \lambda^3 \geq 0
\end{aligned}$$

- Optimal solution $\lambda^1 = 0.5$, $\lambda^3 = 0.5$, optimal multipliers: $\pi = -0.5$, $t = -13$

Second Iteration: Subproblem

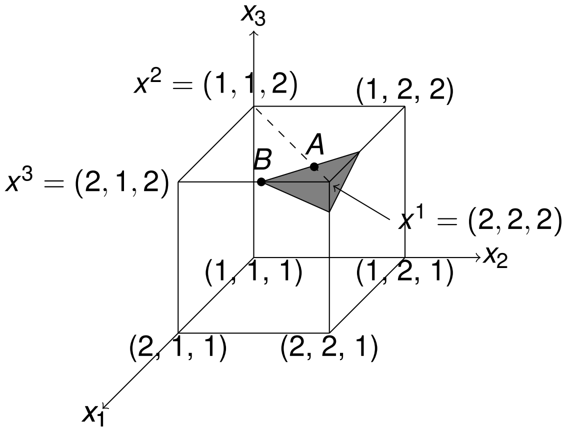
- Subproblem:

$$\min -2.5x_1 - 4x_3$$
$$\text{s.t. } 1 \leq x_1 \leq 2, 1 \leq x_2 \leq 2, 1 \leq x_3 \leq 2$$

- Optimal solution: $x^1 = (2, 2, 2)$, objective function value -13 is equal to $t = -13$
- Optimal solution is

$$x = \frac{1}{2}x^1 + \frac{1}{2}x^3 = \begin{bmatrix} 2 \\ 1.5 \\ 2 \end{bmatrix}$$

Graphical Illustration of Example 1



Explanation of Graphical Illustration

- Cube is P
- Shaded triangle is intersection of P with $3x_1 + 2x_2 + 4x_3 = 17$
- Point A: result of first basis ($\lambda^1 = 0.8, \lambda^2 = 0.2$)
- x^3: extreme point brought into master after completion of first iteration
- Point B: result of second basis ($\lambda^1 = 0.5, \lambda^3 = 0.5$)

Bounds

Recall solutions at first iteration:

- $z = -21$
- $t = -4$
- $z_1 = -5$

Bounds:

$$-21 \geq z^\star \geq -21 + (-5) - (-4) = -22$$

Indeed, $z^\star = -21.5$

Example 2

$$\min -5x_1 + x_2$$
$$\text{s.t. } x_1 \leq 8$$
$$x_1 - x_2 \leq 4$$
$$2x_1 - x_2 \leq 10$$
$$x_1, x_2 \geq 0$$

Introduce slack variable x_3:

$$\min -5x_1 + x_2$$
$$\text{s.t. } x_1 + x_3 = 8$$
$$x_1 - x_2 \leq 4$$
$$2x_1 - x_2 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Decomposition of Example 2

- Treat $x_1 + x_3 = 8$ as a coupling constraint
- $P_1 = \{(x_1, x_2) | x_1 - x_2 \leq 4, 2x_1 - x_2 \leq 10, x_1, x_2 \geq 0\}$
  - Extreme points: $x_1^1 = (6, 2)$, $x_1^2 = (4, 0)$, $x_1^3 = (0, 0)$
  - Extreme rays: $w_1^1 = (1, 2)$, $w_1^2 = (0, 1)$
- $P_2 = \{x_3 | x_3 \geq 0\}$
  - Unique extreme ray: $w_2^1 = 1$

First Iteration: Master

- Initialization: pick extreme point $x_1^1 = (6, 2)$, extreme ray $w_2^1 = 1$
- Restricted master:

$$\min -28\lambda_1^1$$
$$\text{s.t. } 6\lambda_1^1 + \mu_2^1 = 8, (\pi)$$
$$\lambda_1^1 = 1, (t_1)$$
$$\lambda_1^1, \mu_2^1 \geq 0$$

- Optimal solution $\lambda_1^1 = 1$, $\mu_2^1 = 2$, optimal multipliers: $\pi = 0$, $t_1 = -28$

First Iteration: First Subproblem

- Objective function coefficients:
$c_1^T - \pi^T A_1 = \begin{bmatrix} -5 & 1 \end{bmatrix} - (0) \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 1 \end{bmatrix}$
- Subproblem:

$$\min -5x_1 + x_2$$
$$\text{s.t. } x_1 - x_2 \leq 4, 2x_1 - x_2 \leq 10$$
$$x_1, x_2 \geq 0$$

- Optimal solution: $w_1^1 = (1, 2)$, objective function value $-\infty$

Second Iteration: Master

- Restricted master problem:

$$
\begin{aligned}
&\min -28\lambda_1^1 - 3\mu_1^1 \\
&\text{s.t. } 6\lambda_1^1 + \mu_1^1 + \mu_2^1 = 8, (\pi) \\
&\lambda_1^1 = 1, (t_1) \\
&\lambda_1^1, \mu_1^1, \mu_2^1 \geq 0
\end{aligned}
$$

- Optimal solution $\lambda_1^1 = 1$, $\mu_1^1 = 2$, $\mu_2^1 = 0$, optimal multipliers: $\pi = -3$, $t_1 = -10$

Second Iteration: Subproblems

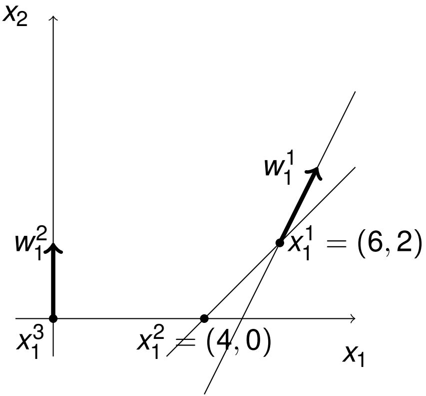
- Subproblem:

$$\min -2x_1 + x_2$$
$$\text{s.t. } x_1 - x_2 \leq 4, 2x_1 - x_2 \leq 10$$
$$x_1, x_2 \geq 0$$

- Optimal solution: $x = (8, 6)$, objective function value -10 is equal to $z_1 = -10$
- Reduced cost of μ_2^1 is 3 (non-negative)
- Optimal solution is

$$x_1^1 + 2w_1^1 = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Graphical Illustration of Example 2



- x_1^1, x_1^2, x_1^3: extreme points of P_1
- w_1^1, w_1^2: extreme rays of P_1
- Algorithm starts at $(x_1, x_2) = (6, 2)$, reaches optimal solution $(x_1, x_2) = (8, 6)$ after one iteration

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Extended Form 2-Stage Stochastic Program

Primal problem: appropriate for L-shaped method

$$\min c^T x + \sum_{k=1}^{K} p_k q_k^T y_k$$
$$\text{s.t. } Ax = b, (\rho)$$
$$T_k x + W y_k = h_k, (\pi_k)$$
$$x, y_k \geq 0$$

Dual problem: appropriate for Dantzig-Wolfe decomposition

$$\max \rho^T b + \sum_{k=1}^{K} \pi_k^T h_k$$
$$\text{s.t. } \rho^T A + \sum_{k=1}^{K} \pi_k^T T_k \leq c^T, (x)$$
$$\pi_k^T W \leq p_k q_k^T, (y_k)$$

Dantzig-Wolfe on the Dual Problem

Consider feasible region of

$$\begin{bmatrix} \pi_1^T & \cdots & \pi_K^T \end{bmatrix} \begin{bmatrix} W & \cdots & 0 \\ 0 & \cdots & 0 \\ 0 & \cdots & W \end{bmatrix} \leq \begin{bmatrix} q_1^T & \cdots & q_K^T \end{bmatrix}$$

Denote π^j, $j \in J$ as extreme points, w^r, $r \in R$ as extreme rays

$$E_j = (\pi^j)^T \begin{bmatrix} p_1 T_1 \\ \vdots \\ p_K T_K \end{bmatrix}, e_j = (\pi^j)^T \begin{bmatrix} p_1 h_1 \\ \vdots \\ p_K h_K \end{bmatrix}, \tag{1}$$

$$D_r = (w^r)^T \begin{bmatrix} p_1 T_1 \\ \vdots \\ p_K T_K \end{bmatrix}, d_r = (w^r)^T \begin{bmatrix} p_1 h_1 \\ \vdots \\ p_K h_K \end{bmatrix} \tag{2}$$

Dantzig-Wolfe Full Master Problem

$$
\begin{aligned}
z^\star = \max\ & \rho^T b + \sum_{j \in J} \lambda^j e_j + \sum_{r \in R} \mu^r d_r \\
\text{s.t. } & \rho^T A + \sum_{j \in J} \lambda^j E_j + \sum_{r \in R} \mu^r D_r \leq c^T, (x) \\
& \sum_{j \in J} \lambda^j = 1, (\theta) \\
& \lambda^j, \mu^r \geq 0
\end{aligned}
$$

Observe

The dual of the Dantzig-Wolfe full master is

$$
\begin{aligned}
&\min c^T x + \theta \\
&\text{s.t. } Ax = b \\
&E_j x + \theta \geq e_j, j \in J \\
&D_r x \geq d_r, r \in R \\
&x \geq 0
\end{aligned}
$$

This is the L-shaped full master problem

Reduced Costs

We want to bring in

- λ^j for which $e_j - E_j x - \theta > 0$
- μ^r for which $d_r - D_r x > 0$

In order to maximize reduced cost, we need to maximize

$$\sum_{k=1}^{K} (\pi_k)^T h_k - \sum_{k=1}^{K} (\pi_k)^T T_k x$$

where $\pi_k \in \mathbb{R}^{m_k}$

Dantzig-Wolfe Second-Stage Subproblems

$$
\begin{aligned}
z_k = \max\, & \pi_k^T (h_k - T_k x) \\
\text{s.t. } & \pi_k^T W \leq q_k, (y_k)
\end{aligned}
$$

The duals of the Dantzig-Wolfe subproblems are the primal L-shaped subproblems:

$$
\begin{aligned}
& \min q_k^T y_k \\
& \text{s.t. } W y_k = h_k - T_k x \\
& y_k \geq 0
\end{aligned}
$$

Summary: Dantzig-Wolfe Subproblems

Master (where $\tilde{J} \subset J$, $\tilde{R} \subset R$)

$$\max z = \rho^T b + \sum_{j=1}^{|\tilde{J}|} \lambda^j e_j + \sum_{r=1}^{|\tilde{R}|} \mu^r d_r \tag{3}$$

$$\text{s.t. } \rho^T A + \sum_{j=1}^{|\tilde{J}|} \lambda^j E_j + \sum_{r=1}^{|\tilde{R}|} \mu^r D_r \leq c^T \tag{4}$$

$$\sum_{j=1}^{|\tilde{J}|} \lambda^j = 1, \lambda^j \geq 0, \mu^r \geq 0 \tag{5}$$

Scenario subproblems:

$$\max \pi^T (h_k - T_k x^v) \tag{6}$$

$$\text{s.t. } \pi^T W \leq q^T \tag{7}$$

Algorithm

Step 0. $|\tilde{J}| = |\tilde{R}| = v = 0$

Step 1. $v = v + 1$ and solve (3) - (5). Let the solution be $(\rho^v, \lambda^v, \mu^v)$ with dual solution (x^v, θ^v)

Step 2. For $k = 1, \dots, K$, solve (6) - (7)

- If extreme ray w^v is found, set $d_{|\tilde{R}|+1} = (w^v)^T h_k$, $D_{|\tilde{R}|+1} = (w^v)^T T_k$, $|\tilde{R}| = |\tilde{R}| + 1$ and return to step 1
- If all subproblems are solvable, let

$$E_{|\tilde{J}|+1} = \sum_{k=1}^{K} p_k (\pi_k^v)^T T_k, e_{|\tilde{J}|+1} = \sum_{k=1}^{K} p_k (\pi_k^v)^T h_k$$

  - If $e_{|\tilde{J}|+1} - E_{|\tilde{J}|+1} x^v - \theta \leq 0$, then stop with $(\rho^v, \lambda^v, \mu^v)$ and (x^v, θ^v) optimal
  - If $e_{|\tilde{J}|+1} - E_{|\tilde{J}|+1} x^v - \theta^v > 0$, set $|\tilde{J}| = |\tilde{J}| + 1$ and return to step 1

Dantzig-Wolfe Bounds Revisited

- Lower bound: $z \leq z^\star$
- Upper bound: $z^\star \leq c^T x + \sum_{k=1}^{K} p_k z_k$
- Dantzig-Wolfe bounds are the same as the L-shaped bounds

Dantzig-Wolfe Versus L-Shaped Method

- Both algorithms go through the same steps
- Difference: we solve the dual problems instead of the primal problems

Table of Contents

Integer Programming Formulation

$$(IP) : \min\{c^T x : x \in X\}$$

$$X = Y \cap Z$$

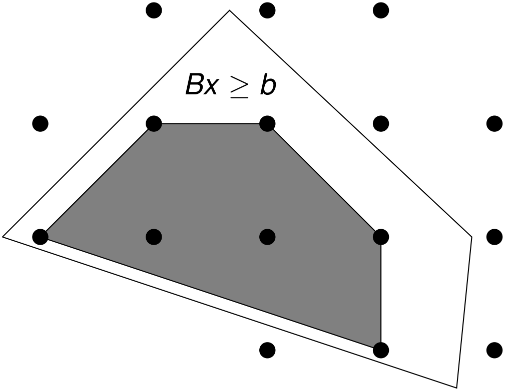
$$Y = \{Dx \geq d\}$$

$$Z = \{Bx \geq b\} \cap \mathbb{Z}^n$$

Structural assumption: $OPT(Z, c) : \{\min c^T x : x \in Z\}$ can be solved rapidly in practice

Application of Dantzig-Wolfe on Integer Program

Idea: Apply Dantzig-Wolfe to (IP) using Minkowski Representation Theorem to represent $conv(Z) = conv(\{Bx \geq b\} \cap \mathbb{Z}^n)$



$conv(Z)$ is the gray area

Dantzig-Wolfe Reformulation

$$(DWc): z^{DWc} = \min_{\lambda \geq 0} \sum_{j \in J} (c^T x^j) \lambda^j$$

$$\text{s.t. } \sum_{j \in J} (Dx^j) \lambda^j \geq d$$

$$\sum_{j \in J} \lambda^j = 1, \sum_{j \in J} x^j \lambda^j \in \mathbb{Z}^n$$

where

- x^j is the set of extreme points of $conv(Z)$,
- $conv(Z)$ is the convex hull of Z
- J is the set of extreme points of $conv(Z)$

Restricted Master Linear Program

- The linear relaxation of (DWc) is called the **Master Linear Program (MLP)**
- When we only consider a *subset* $\tilde{J} \subset J$ of the extreme points of $conv(Z)$ we get the **Restricted Master Linear Program (RMLP)**

$$
\begin{aligned}
(RMLP) : z^{RMLP} &= \min_{\lambda \geq 0} \sum_{j \in \tilde{J}} (c^T x^j) \lambda^j \\
\text{s.t. } & \sum_{j \in \tilde{J}} (D x^j) \lambda^j \geq d, (\pi) \\
& \sum_{j \in \tilde{J}} \lambda^j = 1, (\sigma)
\end{aligned}
$$

Observations

1. The reduced cost associated to λ^j is $c^T x^j - \pi^T D x^j - \sigma$
2. *Important:* $z = \min_{j \in \tilde{J}}(c^T x^j - \pi^T D x^j) = \min_{x \in Z}(c^T - \pi^T D)x = \min_{Bx \geq b, x \in \mathbb{Z}^n_+}(c^T - \pi^T D)x$ is *an easy integer program*
3. $z^{RMLP} = \sum_{j \in \tilde{J}}(c^T x^j)\lambda^j$ is an upper bound on z_{MLP} and (MLP) is solved when $z - \sigma = 0$
4. If solution λ of $(RMLP)$ is integer, z^{RMLP} is an upper bound for (IP)

Column Generation Algorithm for (MLP)

1. Initialize primal and dual bounds $UB = +\infty$, $LB = -\infty$
2. Iteration t
   - Solve $(RMLP)$ over $x^j, j \in \tilde{J}^t$, record primal solution λ^t and dual solution (π^t, σ^t)
   - Solve pricing problem $(SP^t) : z^t = \min\{(c^T - (\pi^t)^T D)x : x \in Z\}$, let x^t be an optimal solution. If $z^t - \sigma^t = 0$ set $UB = z^{RMLP}$ and stop with optimal solution to (MLP). Else, add x^t to $\tilde{J}^t$ in $(RMLP)$.
   - Compute lower bound $(\pi^t)^T d + z^t$. Update $LB = \max\{LB, (\pi^t)^T d + z^t\}$. If $LB = UB$, stop with optimal solution to (MLP)
3. Increment t, return to step 2

Relationship to Lagrange Relaxation

Relaxing ‘difficult’ constraints $Dx \geq d$, while keeping the remaining constraints $Z = \{x \in \mathbb{Z}^n_+ : Bx \geq b\}$, we get

- the **dual function**

$$g(\pi) = \min_x \{c^T x + \pi^T (d - Dx) : Bx \geq b, x \in \mathbb{Z}^n_+\} \tag{8}$$

- the **dual bound**

$$z_{LD} = \max_{\pi \geq 0} g(\pi) = \max_{\pi \geq 0} \min_{x \in Z} \{c^T x + \pi^T (d - Dx)\}$$

Reformulation of Dual Bound

$$z_{LD} = \max_{\pi \geq 0} \min_{j \in J} \{c^T x^j + \pi^T (d - Dx^j)\}$$

where

- x^j is the set of extreme points of $conv(Z)$,
- $conv(Z)$ is the convex hull of Z
- J is the set of extreme points of $conv(Z)$

Equivalently:

$$z_{LD} = \max_{\pi \geq 0, \sigma} \pi^T d + \sigma$$
$$\text{s.t. } \sigma \leq c^T x^j - \pi^T D x^j, j \in J, (\lambda^j)$$

Taking the dual:

$$z_{LD} = \min_{\lambda^j \geq 0, j \in J} \sum_{j \in J} (c^T x^j) \lambda^j \tag{9}$$

$$\text{s.t. } \sum_{j \in J} (Dx^j) \lambda^j \geq d, (\pi) \tag{10}$$

$$\sum_{j \in J} \lambda^j = 1, (\sigma) \tag{11}$$

Relationship Between Lagrange Dual Bound and LP Relaxation of Dantzig-Wolfe Reformulation

- *Observe:* The linear program (9) - (11) is the master linear program (*MLP*) of Dantzig-Wolfe
- *Conclusion:* Solving the Lagrange Relaxation (9) - (11) will give *the same bound* as solving (*MLP*) using Dantzig-Wolfe decomposition