6th Exercise Class For Mathematical Analysis III

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Last time's HW

1.7

Calculate the Fourier series: $f(x) = \frac{1-r^2}{1-2r\cos x+r^2}\,(|r| < 1)$.

Proof.

There are two ways to prove the above thing equals to $1 + \sum_{n\geq 1} 2r^n \cos(nx)$. You can just calculate $(1 - 2r\cos x + r^2)(1 + \sum_{n\geq 1} 2r^n \cos(nx))$ and compare both sides' Fourier coefficients and obtain a recursion formula for the Fourier coefficients a_n of $f(x)$. This is true because $f(x)$ is a very good function. You can prove the convergence of its Fourier series as well as the uniqueness.

Another way is to use some complex analysis. Since $\cos x = \frac{e^{ix}+e^{-ix}}{2}$, we have

$$\begin{aligned}
\frac{1-r^2}{1-2r\cos x+r^2} &= \frac{1-r^2}{(1-re^{ix})(1-re^{-ix})} = \frac{(1-r^2)e^{ix}}{(1-re^{ix})(e^{ix}-r)} \\
&= \frac{1}{1-re^{ix}} + \frac{re^{-ix}}{1-re^{-ix}} = \sum_{n\geq 0}(re^{ix})^n + \sum_{n\geq 1}(re^{-ix})^n \\
&= 1 + \sum_{n\geq 1} 2r^n \cos(nx)
\end{aligned}$$

$\square$

Last time's HW

Proof.

This is true because the expansion $\frac{1}{1-x} = \sum_{n\geq 0} x^n$ is true not only for real x, but is also true for complex x whenever $|x| < 1$. Therefore for $\frac{re^{-ix}}{1-re^{-ix}} = \frac{1}{e^{ix}/r-1}$, we cannot expand it in the latter way with respect to $\frac{e^{ix}}{r}$ since we have $|\frac{e^{ix}}{r}| = \frac{1}{r} > 1$. □

Remark: Question.12.6. Here it uses identity to relate the original function and its Fourier series. So besides calculate its Fourier series, you also need to prove its convergence here. And you'd better not use the results in previous section concerning the convergence of series to prove the convergence of Fourier series. Because you will also need to prove the relation between the Fourier series and the original function. Only when $f(x)$ is continuous and periodic, you can use the pointwise convergence to prove the Fourier series' convergence. Here mention the original function is piecewisely differentiable or piecewisely monotone is sufficient.

Last time's HW

10

If $f(x)$ satisfies $\int_{-\infty}^{+\infty} |f(x)|\mathrm{d}x < +\infty$ and $\exists x_0, \delta > 0$ s.t. either the following (1) or (2) is true:
(1) f(x) can be expressed as a difference between two monotone functions on $[x_0 - \delta, x_0 + \delta]$.
(2) $f(x_0 + 0)$ and $f(x_0 - 0)$ exist, $\int_0^\delta \frac{f(x_0+t)-f(x_0+0)}{t}\mathrm{d}t$ and $\int_0^\delta \frac{f(x_0-t)-f(x_0-0)}{t}\mathrm{d}t$.
Then prove:

$$\lim_{\alpha\to+\infty} \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x_0+t)\,\frac{\sin \alpha t}{t}\mathrm{d}t = \frac{f(x_0+0)+f(x_0-0)}{2}$$

Proof.

Since $\int \frac{\sin \alpha t}{t} dt = \int \frac{\sin t}{t} dt = \pi$, we just need to prove $\frac{1}{\pi}\int_0^{+\infty}\left(\frac{f(x_0+t)+f(x_0-t)}{2} - \frac{f(x_0+0)+f(x_0-0)}{2}\right)\frac{\sin \alpha t}{t}\mathrm{d}t \to 0$. So we just need to prove $\int \frac{f(x_0+t)-f(x_0+0)}{t}\sin(\alpha t)dt \to 0$, the minus side is similar. By Riemann's localization theorem, we can restrict our attention to $[x_0 - \delta, x_0 + \delta]$. If (1) is true, by Dirichlet's test we have the above results.
If (2) is true, the above results can be derived from Riemann-Lebesgue Lemma or Dini's test. $\square$

Last time's HW

12

- Calculate the Fourier sereis of $f(x)=\cos(\alpha x)(x\in(-\pi,\pi),0<\alpha<1)$. Calculate the summation function.
- Prove the identity $\frac{\pi}{\sin\alpha\pi}=\frac{1}{\alpha}+\sum_{n=1}^{+\infty}(-1)^n\frac{2\alpha}{\alpha^2-n^2}$.
- Let $\alpha=\frac{x}{\pi}$, use (2) to prove $\int_0^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$.

Proof.

The Fourier series can be derived: $f\sim\frac{2\sin\alpha\pi}{\pi}\left[\frac{1}{2\alpha}+\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}\alpha}{n^2-\alpha^2}\cos nx\right]$. The convergence can also be proved. Then let $x=0$ we can get the second identity. The problem is how to use (2) to derive the integral $\int_0^{\infty}\frac{\sin x}{x}dx=\frac{\pi}{2}$. In fact, let $\alpha=\frac{x}{\pi}$ and we get $\frac{1}{\sin x}=\frac{1}{x}+\sum_{n=1}^{+\infty}(-1)^n\frac{2x}{x^2-n^2\pi^2}=\frac{1}{x}+\sum_{n=1}^{+\infty}(-1)^n(\frac{1}{x-n\pi}+\frac{1}{x+n\pi})$ So we have $1=\frac{\sin x}{x}+\sum_{n=1}^{+\infty}(-1)^n(\frac{\sin x}{x-n\pi}+\frac{\sin x}{x+n\pi})$. Finaly integrate both sides from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and notice $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin x}{x-n\pi}dx=\int_{-n\pi-\frac{\pi}{2}}^{-n\pi+\frac{\pi}{2}}\frac{\sin(x+n\pi)}{x}dx=(-1)^n\int_{-n\pi-\frac{\pi}{2}}^{-n\pi+\frac{\pi}{2}}\frac{\sin(x)}{x}dx$,
$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin x}{x+n\pi}dx=\int_{n\pi-\frac{\pi}{2}}^{n\pi+\frac{\pi}{2}}\frac{\sin(x-n\pi)}{x}dx=(-1)^n\int_{n\pi-\frac{\pi}{2}}^{n\pi+\frac{\pi}{2}}\frac{\sin(x)}{x}dx$ □

Review: Integral with parametric variable

- Motivation: In functional series, we consider things like $\sum_{k\geq 0} u_k(x)$. But in parametric integral we just replace the summation with integration and consider $\sum_{k\in\mathbf{R}} u_k(x)$ and regard k as a parameter. So all the results in the functional series have their correspondence in parametric integral. More formally, let $I(x) = \int_c^d f(x,y)dy$ where (x,y) is defined in $[a,b]\times[c,d]$. Then $I(x)$ can also be regard as a function w.r.t. x.
- Then basic questions arise: when will $I(x)$ be continuous, integrable and differentiable?
- If $f \in C[a,b]\times[c,d]$, then $I(x)$ exists and is continuous.
- If $f \in C[a,b]\times[c,d]$, we can exchange the integration order, i.e., $\int_a^b \mathrm{d}x \int_c^d f(x,y)\mathrm{d}y = \int_c^d \mathrm{d}y \int_a^b f(x,y)\mathrm{d}x$
- If $f, \partial_x f \in C[a,b]\times[c,d]$, then $f(x)$ is differentiable and $I'(x) = \int_c^d \partial_x f(y)dy$
- More generally, if $I(x) = \int_c^{\varphi(x)} f(x,y)dy$, we have similar results where $I'(x) = \int_c^{\varphi(x)} \partial_x f(y)dy + f(x,\varphi(x))\varphi'(x)$.
- In fact, the continuity, integration and differentiation all tries to put the outside limit inside the integration and then use the property of the original function to derive the porperty of the integral $I(x)$.

Review: General integral with parametric variable

- For general integral, if $\int_c^{+\infty} f(x, y)dy < \infty$ is true for $\forall x \in E$, we say $I(x) = \int_c^{+\infty} f(x, y)dy$ is defined on E.
- The uniformness is treated towards x, i.e., to test whether it's true or can be controlled $\forall x \in E$. We have corresponding version of Cauchy's criterion in this sense.
- Absolutely convergence. Absolutely and uniformly convergence.
- Like in the functional series, we have similar theorems: Weiestrass, Dirichlet, Abel, Dini, with similar forms and a little modification.
- Another linkage with functional series: $\int_c^{+\infty} f(x, y)dy$ converge uniformly if and only if $\forall\{t_n\}, t_n \to \infty$, let $F_k(x) = \int_c^{t_k} f(x, y)dy < \infty$ and $\int_c^{t_k} f(x, y)dy$, then $F_k(x) \rightrightarrows \int_c^{+\infty} f(x, y)dy$ for all $x \in E$.
- If $E = [a, b]$ and $f \in C[a, b] \times [c, \infty)$, then we have the same results (I(x) continuous and integration can be exchanged) as the regular integral. Plus we need the uniform convergence of $I(x)$: $\int_c^{+\infty} f(x, y)dy \rightrightarrows I(x)$ w.r.t x.

General integral with parametric variable

- If $f, \partial_x f \in C[a,b] \times [c,\infty)$, $\exists x_0, \int_c^\infty f(x,y)dy < \infty$ and $\int_c^\infty \partial_x f(x,y)dy$ uniformly converge, then
  (1) I(x) also uniformly converge.
  (2) The differentiation can be taken termwisely,
  $I'(x) = \left(\int_c^{+\infty} f(x,y)\mathrm{d}y\right)' = \int_c^{+\infty} f'_x(x,y)\mathrm{d}y.$
  This results resembles a similar results in functional series where you can control the original series using information from its differentiation but the converse way is not always possible.
- The integral with singular point can be transformed into infinite integral. Just change the variable, let $y = \frac{1}{x-c}$ if c is the singular point. Then all the results here have their correspondence in singular integral.
- For Gamma function and Beta function, just remember: $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$. Gamma function is a generalizing version of product: $\Gamma(s+1) = s!$ if $s \in \mathbf{N}$. Recursion formula: $\Gamma(s+1) = s\Gamma(s)$. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.
- $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx$ and $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$.

Exercise

1

Give an example to illustrate there's a function $f(x, y)$ on $D = [0, 1] \times [0, 1]$ such that (1) the discontinuous point of f is dense in D. (2) $\forall x \in [0, 1]$, $I(x) = \int_0^1 f(x, y)dy$ exists and continuous on $[0, 1]$.

Proof.

Just let $f(x, y) = F(y)$, where $F(y)$ has dense discontinuous point in $[0, 1]$ and $F(y)$ is integrable in $[0, 1]$. Then $\forall x$, $I(x) = \int_0^1 F(y)dy$ is a constant. And the discontinous point of f is $\{(x, y) : x \in \mathbf{R}, \text{y is a discontinuous point for F}\}$.

A typical example for this is a function you've learned in Math Analysis (i): $F(y) = \frac{1}{p}$ if $x \in \mathbf{Q}$ and $x = \frac{q}{p}$ and $= 0$ if $x \notin \mathbf{Q}$. Then we can prove it's integrable in $[0, 1]$. The integration is 0. And all the rational point is discontinuous, which is dense in $[0, 1]$. □

Exercise

2

If $f(x,y)$ is continous on $D=[a,b]\times[c,d]$. Prove $\forall(x,u)\in D$, $I(x,u)=\int_c^u f(x,y)\mathrm{d}y$ exists and continuous on D.

Proof.

Since $f(x,y)$ is continuous, we have the $I(x,u)$ exists for any $(x,u)\in D$. And we have the identity (without loss of generality we assume $u_1>u_2$):

$$\begin{aligned} I(x_1,u_1)-I(x_2,u_2) &= \int_c^{u_1} f(x_1,y)\mathrm{d}y - \int_c^{u_2} f(x_2,y)\mathrm{d}y \\ &= \int_c^{u_2} (f(x_1,y)-f(x_2,y))dy + \int_{u_2}^{u_1} f(x_1,y)dy \end{aligned}$$

To prove the continuity we just need to prove $\forall\epsilon,\exists\delta>0$ s.t. whenener $\sqrt{|x_1-x_2|^2+|u_1-u_2|^2}<\delta$, we have $|f(x_1,u_1)-f(x_2,u_2)|<\epsilon$. Since f is continuous in D, it's also uniformly continuous on D (D is compact). Then we can find δ_1 s.t. $|f(x_1,y)-f(x_2,y)|<\epsilon$ for $|x_1-x_2|<\delta_1$. So the first term $<\epsilon(u_2-c)\leq\epsilon$. The second term $\leq(u_1-u_2)M$ where $M=\max_D|f(x)|$. In conclusion we have $|I(x_1,u_1)-I(x_2,u_2)|<\epsilon+\delta M$, let $\delta<\min\{\delta_1,\frac{\epsilon}{M}\}$. Then we get $|I(x_1,u_1)-I(x_2,u_2)|<2\epsilon$. Let $\epsilon\to 0$ and we get the continuity of $I(x,u)$ on D. □

Exercise

25

Prove $F(x) = \int_1^{+\infty} \frac{\sin y}{y^x} \mathrm{d}y$ has continuous differentiation in $(0, \infty)$.

Proof.

We just need to prove $\partial_x \frac{\sin y}{y^x}$ converges uniformly, which is just $-\frac{\sin y}{y^x} \ln y$. Notice its behavior is quite bad when $x \to 0$. Fortunately to consider continuous differentiation we don't need the uniform convergence in the whole $(0, \infty)$. We just need to consider x_0's neighborhood if we want to consider whether $F'(x)$ is continuous on x_0. Then we have $(\frac{\ln y}{y^x})' = \frac{y^{x-1} - x \ln y y^{x-1}}{y^{2x}} = \frac{1 - x \ln y}{y^{x+1}} < 0$ when y is large enough ($\ln y > \frac{1}{x_0 - \delta}$ if we consider $U(x_0, \delta)$). Meanwhile, $\frac{\ln y}{y^x} < \frac{\ln y}{y^{x_0 - \delta}} \to 0$. So by Dirichlet's test we have the uniform convergence of $\partial_x \frac{\sin y}{y^x}$ ($| \int_1^A \sin y dy| < 2$). This completes the proof. $\square$

Exercise

27

If f is continuous on $[0, \infty)$ and $f(\infty) = \lim_{x\to\infty} f(x)$ exists. Prove:

$$\int_0^{+\infty} \frac{f(bx) - f(ax)}{x} \mathrm{d}x = (f(+\infty) - f(0)) \ln \frac{b}{a} \quad (a, b > 0)$$

Remark: we can't derive $\int \frac{f(bx)}{x} dx = \int \frac{u}{u} du$ and obtain the right hand side is zero, since $\int \frac{f(bx)}{x} dx$ can be infinite.

Proof.

Consider the integral $\int_t^{\frac{1}{t}} \frac{f(bx)-f(ax)}{x} dx$, then $RHS = \lim_{t\to 0} \int_t^{\frac{1}{t}} \frac{f(bx)-f(ax)}{x} dx$. But now we can change variable and get $\int_t^{\frac{1}{t}} \frac{f(bx)}{x} dx = \int_{bt}^{\frac{b}{t}} \frac{f(u)}{u} du$. So $\int_t^{\frac{1}{t}} \frac{f(bx)-f(ax)}{x} dx = \int_{bt}^{\frac{b}{t}} - \int_{at}^{\frac{a}{t}} = \int_{\frac{a}{t}}^{\frac{b}{t}} - \int_{at}^{bt}$. But by mean value theorem, we have $\int_{\frac{a}{t}}^{\frac{b}{t}} \frac{f(u)}{u} du = f(\xi) \int_{\frac{a}{t}}^{\frac{b}{t}} \frac{1}{u} du = f(\xi) \ln \frac{b}{a} \to f(\infty) \ln \frac{b}{a}$, where $\xi \in (\frac{a}{t}, \frac{b}{t}) \to \infty$ as $t \to 0$. The remain parts about $f(0)$ follows similarly. $\square$

Exercise

28

$f(x) \in C[0, \infty)$, both $\int_0^\infty xf(x)$ and $\int_0^\infty \frac{f(x)}{x} dx$ converge. Prove $I(t) = \int_0^{+\infty} x^t f(x) \mathrm{d}x$ has continouous differentiation in $(-1, 1)$.

Proof.

Similarly consider $\partial_t x^t f(x) = f(x)x^t \ln x$. Consider using Abel's test, then we shall consider the function $x^\alpha \ln x$ where $\alpha = t \pm 1$.

First we have $(\ln x \cdot x^\alpha)' = (1 + \alpha \ln x)x^{\alpha-1}$. It's natural to consider $I(x) = \int_0^1 + \int_1^\infty$ and cope with the two parts separately. For $x > 1$, since $\ln x \cdot x^{t-1} \to 0$ as $x \to \infty$. Here we assume $\alpha = t - 1$. Then as x is large enough $\ln x \cdot x^{t-1}$ will be decreasing and tends to 0. Since we only consider whether it has continuous differentiation, only uniformly convergence near the neighborhood is of interest. Assume we are restricted to $t \in U(t_0, \delta)$. Then we can have $\ln t \cdot x^{t-1} \leq \ln t \cdot x^{t_0+\delta-1} \to 0$ and so it uniformly converges to 0. By Abel's test with $xf(x)$ as another function we get the uniformly convergence.

Then for $0 < x < 1$, since $x = 0$ might be a singular point. We can discuss in a similar way with $\alpha = t + 1$. We have $x^{t+1} \ln x \leq x^{t_0-\delta+1} \ln x \to 0$. So $x^{t+1} \ln x \rightrightarrows 0$ on $U(t_0, \delta)$. And it's decreasing when x is close to 0. Use Abel's test for singular integral and we get the uniform convergence on $[0, 1]$. The above two steps complete the proof. □

Fourier Transform

- In fact Fourier series can be thought as calculating $\int f(x)e^{inx}dx$ for $n \in \mathbf{Z}$ if you treat $cos(nx)$ and $\sin(nx)$ as the real part and imaginary part of e^{inx}.
- Fourier transform is just replace the above $n \in \mathbf{Z}$ to $\xi \in \mathbf{R}$. More formally, $\hat{f}(\xi) = \int_{\mathbf{R}} f(x)e^{-2\pi i\xi x}dx$ or in high dimension: $\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i\xi\cdot x}dx$, where $\xi \cdot x$ is the standard inner product in Euclidean space, $\xi \cdot x = \sum_i \xi_i x_i$ if ξ_i, x_i are their coordinates.
- If you think Fourier transform as a mapping from a function to another function. It actually maps square integrable funtions to square integrable funtions. And maps absolutely integrable funtions to bounded ones.
- But in general it's not defined for any function. Even some square integrable function cannot make Fourier transform from the above formula. The formula $\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-2\pi i\xi\cdot x}dx$ is only well-defined for functions satisfying $\sup_{\alpha,\beta} |x^\alpha D^\beta f(x)| < \infty$, which are functions that decay quite quickly at infinity and good enough. (Any polynomial P(x) we have $|P(x)f(x)| \leq M < \infty$, which implies $|f(x)| < \frac{M}{P(x)}$ and so it decay quite quickly when $|x| \to \infty$).
- Although the above condition may seem too restrict, in fact, it can be proved such functions is dense in the space formed by square integrable functions, if we consider limit as mean square convergence. So for a square integrable function f, we can find a sequence of "good" functions $\{\varphi_n(x)\}$ that can approach f: $\int |f - \varphi_n(x)|^2 dx \to 0$. Then the Fourier transform of $f(x)$ can be defined as $\lim_n \hat{\varphi}_n(x)$ where the latter limit can be proved to exist and be unique.

Fourier Transform: Some property

There are some nice properties that are useful for Fourier transform:

- $(\alpha f + \beta g)\hat{} = \alpha \hat{f} + \beta \hat{g}$
- $(-2\pi i x_j f)\hat{}\,(\xi) = \frac{\partial \hat{f}}{\partial \xi_j}(\xi)$.
- $(f * g)\hat{} = \hat{f} \cdot \hat{g}$, where the convolution is defined as $f * g(x) = \int f(y) g(x-y) dy$
- More general Riem-Lebesgue: $\lim_{|\xi| \to \infty} \hat{f}(\xi) = 0 \quad$ (Riemann-Lebesgue) .
- Translation. $(\tau_h f)^{T}(\xi) = \hat{f}(\xi) e^{2\pi i h \cdot \xi}$, where $\tau_h f(x) = f(x+h)$.
- Scaling. if $g(x) = \lambda^{-n} f\left(\lambda^{-1} x\right)$, then $\hat{g}(\xi) = \hat{f}(\lambda \xi)$.
- $\int_{\mathbb{R}^n} f \cdot \hat{g} = \int_{\mathbb{R}^n} \hat{f} \cdot g$.
- The Fourier tranform is a $4-$periodic mapping. $(\hat{f})\hat{} = \tilde{f}$, where $\tilde{f}(x) = f(-x)$.
- $(e^{-\pi x^2})\hat{} = e^{-\pi \xi^2}$ since they all satisfy $u' + 2\pi x u = 0$ and $u(0) = 1$. By knowledge from differential equation we know such a function is unique.