Assignment #1

Problem 2: Rigid-body transformations

1)
$$\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)^{-1}$$

$$M^{-1} = \begin{bmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -t_1 \\ 0 & 1 & 0 & -t_2 \\ 0 & 0 & 1 & -t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{21} & r_{31} & -r_{11}(t_1) - r_{21}(t_2) - r_{31}(t_3) \\ r_{12} & r_{22} & r_{32} & -r_{12}(t_1) - r_{22}(t_2) - r_{32}(t_3) \\ r_{13} & r_{23} & r_{33} & -r_{13}(t_1) - r_{23}(t_2) - r_{33}(t_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2)

$$\underset{(\Phi)}{\begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{(P1)}{\begin{bmatrix} x_1 \\ y_1 \\ z_1 \\ 1 \end{bmatrix}} = \underset{(\Phi P1)}{\begin{bmatrix} r_{11}(x_1) + r_{12}(y_1) + r_{13}\overset{(z_1)}{z} + t_1 \\ r_{21}(x_1) + r_{22}(y_1) + r_{23}(z_1) + t_2 \\ r_{31}(x_1) + r_{32}(y_1) + r_{33}(z_1) + t_3 \\ 1 \end{bmatrix}}$$

$$\Phi(P2) - \Phi(P1) = \begin{bmatrix} r_{11}(x_2) + r_{12}(y_2) + r_{13}(z_2) + t_1 - (r_{11}(x_1) + r_{12}(y_1) + r_{13}(z_1) + t_1) \\ r_{21}(x_2) + r_{22}(y_2) + r_{23}(z_2) + t_2 - (r_{21}(x_1) + r_{22}(y_1) + r_{23}(z_1) + t_2) \\ r_{31}(x_2) + r_{32}(y_2) + r_{33}(z_2) + t_3 - (r_{31}(x_1) + r_{32}(y_1) + r_{33}(z_1) + t_3) \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} r_{11}(x_2 - x_1) + r_{12}(y_2 - y_1) + r_{13}(z_2 - z_1) \\ r_{21}(x_2 - x_1) + r_{22}(y_2 - y_1) + r_{23}(z_2 - z_1) \\ r_{31}(x_2 - x_1) + r_{32}(y_2 - y_1) + r_{33}(z_2 - z_1) \\ 1 \end{bmatrix}$$

$$\Phi(P2 - P1) = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \\ 1 \end{bmatrix}$$

$$\underline{SAME} = \begin{bmatrix} r_{11}(x_2 - x_1) + r_{12}(y_2 - y_1) + r_{13}(z_2 - z_1) \\ r_{21}(x_2 - x_1) + r_{22}(y_2 - y_1) + r_{23}(z_2 - z_1) \\ r_{31}(x_2 - x_1) + r_{32}(y_2 - y_1) + r_{33}(z_2 - z_1) \\ 1 \end{bmatrix}$$

$\underline{\Phi}$ is linear in 3D

3) $\Phi = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{22} & r_{12} & 0 \\ r_{21} & r_{22} & r_{23} & 0 \\ r_{31} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- (A) is a translation matrix; translations do not distort in any way
  └ each point of the triangle is translated by the same $\begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ 1 \end{bmatrix}$

- (B) is a rotation matrix; it is orthonormal
$$(B^T = B^{-1})$$

for $\vec{v} \in \{\vec{c}, \vec{d}, \vec{e}\}$

$$\|B\vec{v}\|^2 = B\vec{v} \cdot B\vec{v} = (B\vec{v})^T B\vec{v}$$
$$= \vec{v}^T B^T B \vec{v}$$
$$= \vec{v}^T \vec{v} = \vec{v} \cdot \vec{v} = \boxed{\|\vec{v}\|^2}$$

Preserves length

for $\vec{v}, \vec{w} \in \{\vec{c}, \vec{d}, \vec{e}\}, \vec{v} \neq \vec{w}$

$$\cos \theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$
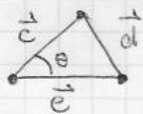
$$\cos \theta_B = \frac{B\vec{v} \cdot B\vec{w}}{\|\vec{v}\| \|\vec{w}\|} = \frac{(B\vec{w})^T B\vec{v}}{\|\vec{v}\| \|\vec{w}\|} = \frac{\vec{w}^T B^T B \vec{v}}{\|\vec{v}\| \|\vec{w}\|}$$
$$= \frac{\vec{w}^T \vec{v}}{\|\vec{v}\| \|\vec{w}\|}$$
$$= \boxed{\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}} = \cos \theta$$

Preserves angle

$$\text{Area} = \frac{1}{2} \|c\| \|d\| \sin \theta$$

$$\text{Area}_B = \frac{1}{2} \|Bc\| \|Bd\| \sin \theta$$
$$= \boxed{\frac{1}{2} \|c\| \|d\| \sin \theta}$$

preserves   preserves
length   area

↑ preserves angle