4 Part I: Constructing a hedging strategy

Consider a European call option for a stock:

- The price today is $S(0) = 45$.
- The maturity is $T = 1$ (year).
- The strike price is $K = 49$.
- The continuously compounded interest rate is $r = 0.02$.

Thus the payoff at time T is $X = (S(T) - K)_+$.

I want you to find a hedging strategy based on the assumptions to be specified below. You are given a file `training_data.csv` which contains 1000 (independently) simulated price paths from a fixed model (unknown to you). Each column is a path which is of the form

$$S(t_i), \quad i = 0, 1, \ldots, 252,$$

where $t_i = i/252$. (Note: 252 is the number of trading days in a year.) In your hedging strategy, you can only rebalance at the times t_i (i.e., once a day).

You are given that the model (under the physical probability measure $\mathbb{P}$) has the form

$$\begin{aligned} dS(t) &= S(t)\alpha(S(t)) + S(t)\sigma(S(t))dW(t), \\ dB(t) &= rB(t)dt, \end{aligned} \tag{1}$$

where $\alpha(\cdot)$ and $\sigma(\cdot)$ are deterministic functions. That is, the stock price follows a one-dimensional diffusion process.

In the given `csv` file, the price paths are simulated using the simple stochastic Euler scheme:

$$S(t_{i+1}) = S(t_i) + S(t_i)\alpha(S(t_i))\Delta t + S(t_i)\sigma(S(t_i))\sqrt{\Delta t} Z_i,$$

where $\Delta t = 1/252$ and the Z_i are i.i.d. samples from $N(0, 1)$. This is not very accurate (as it only approximates the process (1)) but may make it easier for you to estimate the functions α and σ. In your work, you may assume that the prices are produced this way.

Using the training data, you need to estimate the model and come up with a hedging strategy, i.e., a self-financing trading strategy that begins with a fixed amount of capital, and produces a portfolio value at time T that is as close as possible to the payoff $(S(T) - K)_+$. If $V(T)$ is your terminal portfolio value, your objective is to minimize

$$\mathbb{E}_{\mathbb{P}}\left[(V(T) - X)^2\right].$$

You may not be able to use this objective function directly; this is included for completeness.

There is one further complication: We assume that there is *proportional transaction cost* when trading the stock. We assume that the cost parameter is $c = 0.25\%$. This means the following. Suppose that you want to buy a share of the stock when the current price is \$50. Then you will need to pay

$$\$50 \times (1 + 0.0025) = \$50.125.$$

Similarly, if you want to sell a stock when the current price is %50, you only receive

$$\$50 \times (1 - 0.0025) = \$49.875.$$

We assume that trading the money market account does not incur transaction costs. **Your group can choose to omit the transaction cost in your analysis under a 10% penalty of the maximum score you may get.** We emphasize that this problem is open-ended; there are many possible approaches. You may not be able to find the "best" approach during the given time frame (as is always the case in practice!); try your best before the due date and produce a workable solution. You can use any techniques that are useful for the problem (not necessarily the ones mentioned in class).

Your submitted work **(due on April 17)** should be typed and should contain three parts:

- Analysis of the training data, and the design of the strategy.
- Working C++ codes for the strategy in the form of a function

  ```
  double HedgePayoff(const std::vector<double>& path)
  ```

  in the spirit of Problem 3 in Assignment 2. Here you may assume that the path is $\{S(t_i)\}_{i=0}^{252}$ where the t_i's are specified above. Of course, although the whole path is given in the argument, your strategy must be non-anticipating, i.e., your positions in the stock and money market can only depend on the history up to the current time.
- An estimate of the expected hedging error $\mathbb{E}_{\mathbb{P}}\left[(V(T) - X)^2\right]$.

During the presentation, you will explain your work to the audience.