Numerical methods with Julia

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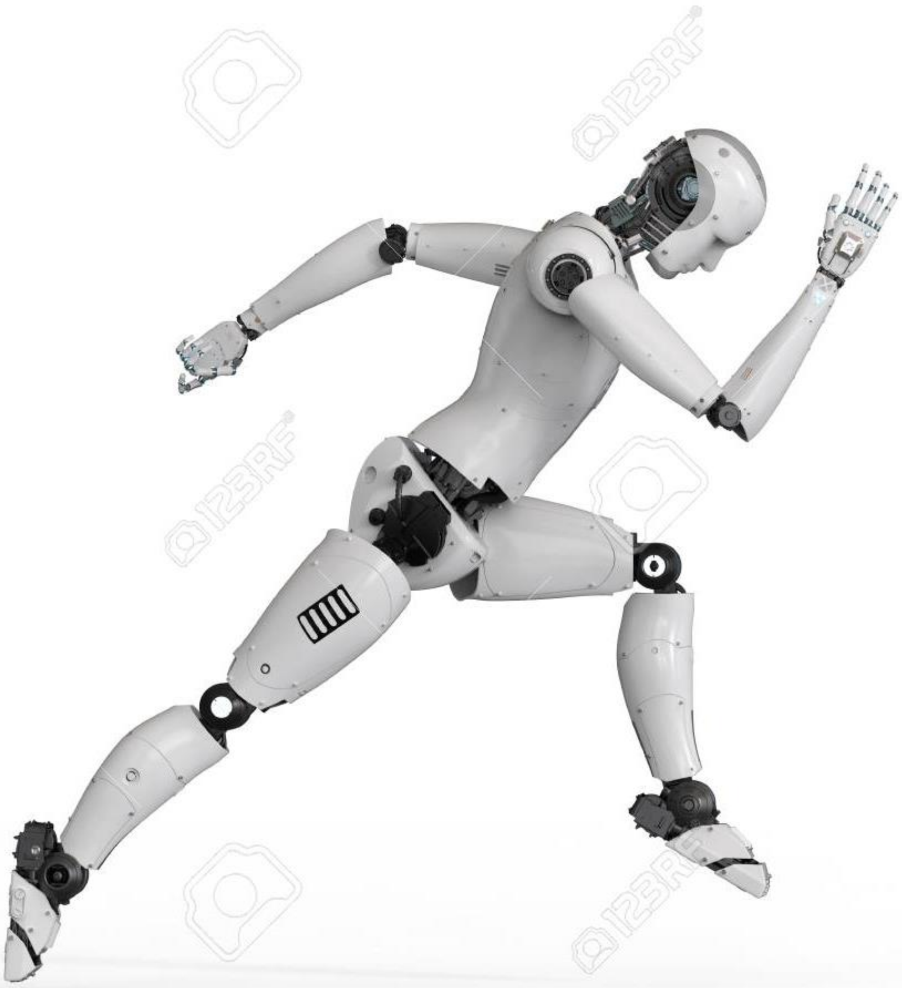


Why Do We Need Numerical Analysis?

Presentation outline

- Why Do We Need Numerical Analysis In Everyday Life?
- Application areas
- Basic examples
- Julia implemantation
- Conclusions

Application areas

The goal of numerical analysis can be formulated to give an approximate but accurate solution to the advanced problem.

01

Interpolation

Curve-fitting to the given set of points.

02

Extrapolation

Interpolation, where evaluated values are outside the range of points.

03

Regression

Statistical process that helps to understand the relationship between explained and explanatory variables. Widely used for forecasting and predicting in the field of machine learning.

04

Approximation

Finding unknown functions best suited to the data (e.g. measurements of physical phenomena).

05

Systems of linear equations

Solving systems of linear equations in the case of a larger number of equations and unknowns.

06

Differential and integral equations

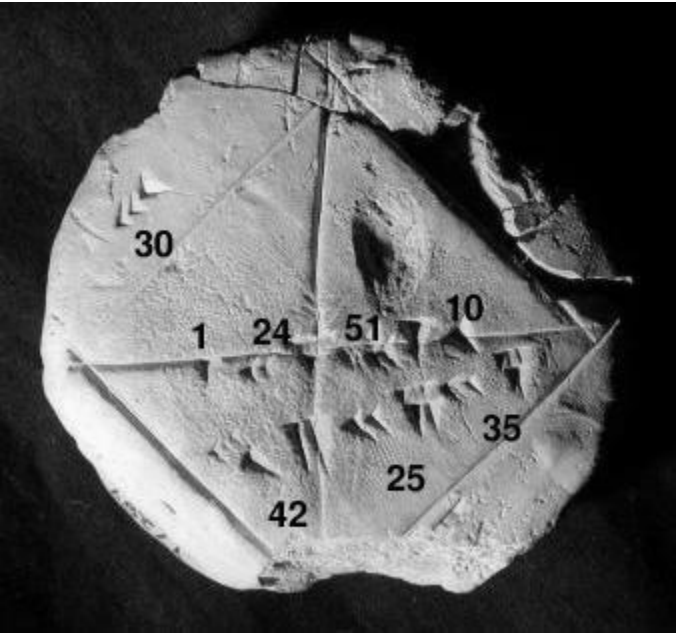
Finding solutions of partial differential equations, ordinary differential equations and integral equations used in most mathematical models.

Julia Packages

Problem type	Julia packages
Plotting	Plots
Linear system / least squares	LinearSolve
Sparse matrix	SparseArrays
Interpolation	DataInterpolations, ApproxFun
Polynomial manipulations	Polynomials
Rootfinding	NonlinearSolve
Finite differences	FiniteDifferences, FiniteDiff
Integration	Quadgk, HCubature
Optimization	Optimization
Ordinary Differential Equations	DifferentialEquations
Finite Element Method	Gridap
Automatic Differentiation	ForwardDiff, Enzyme
Fast Fourier Transform	FFTW

Packages needed during this course:

- LinearAlgebra
- SpecialMatrices
- Plots
- DataFrames



History

Babylonian clay tablet YBC 7289 (Yale Babylonian Collection, c. 1800–1600 BC) with annotations.

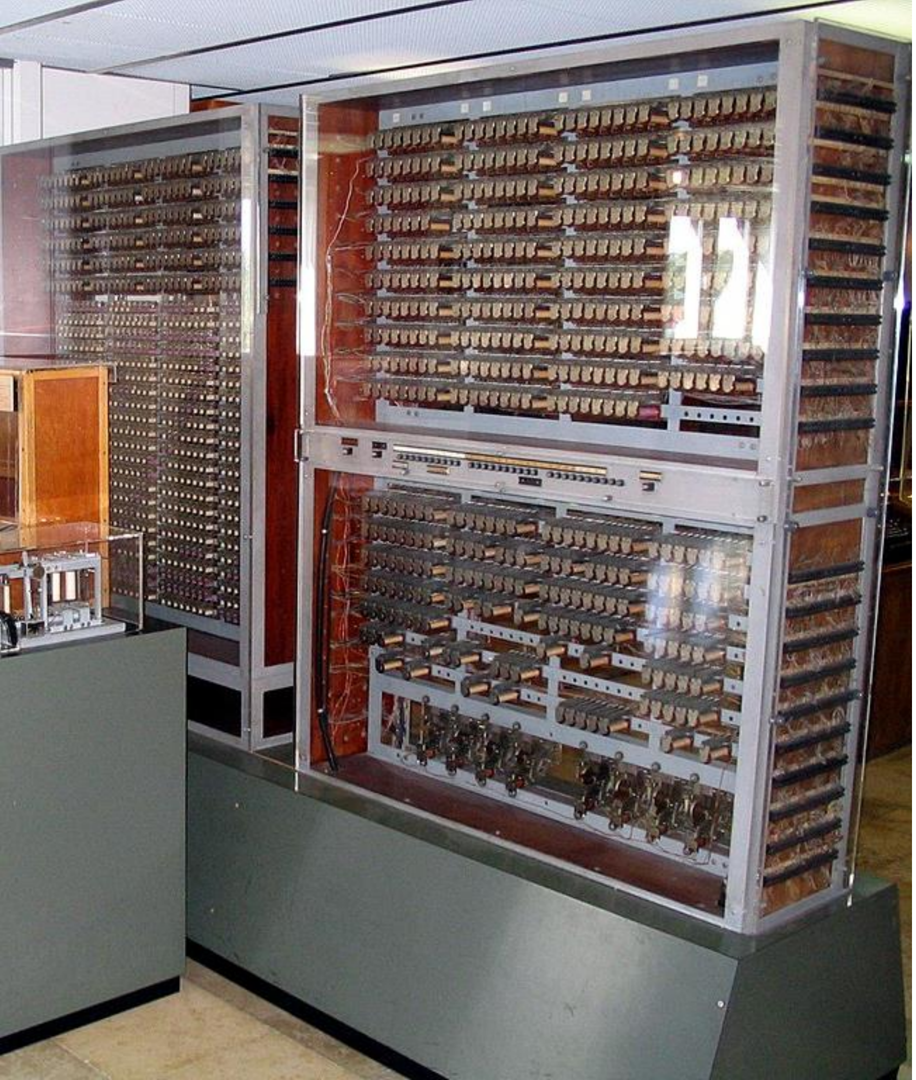
The approximation of the square root of 2:

$$1 + 24/60 + 51/602 + 10/603 = 1.41421296...$$

Today's calculators:

$$\sqrt{2} \approx 1.41421356237$$

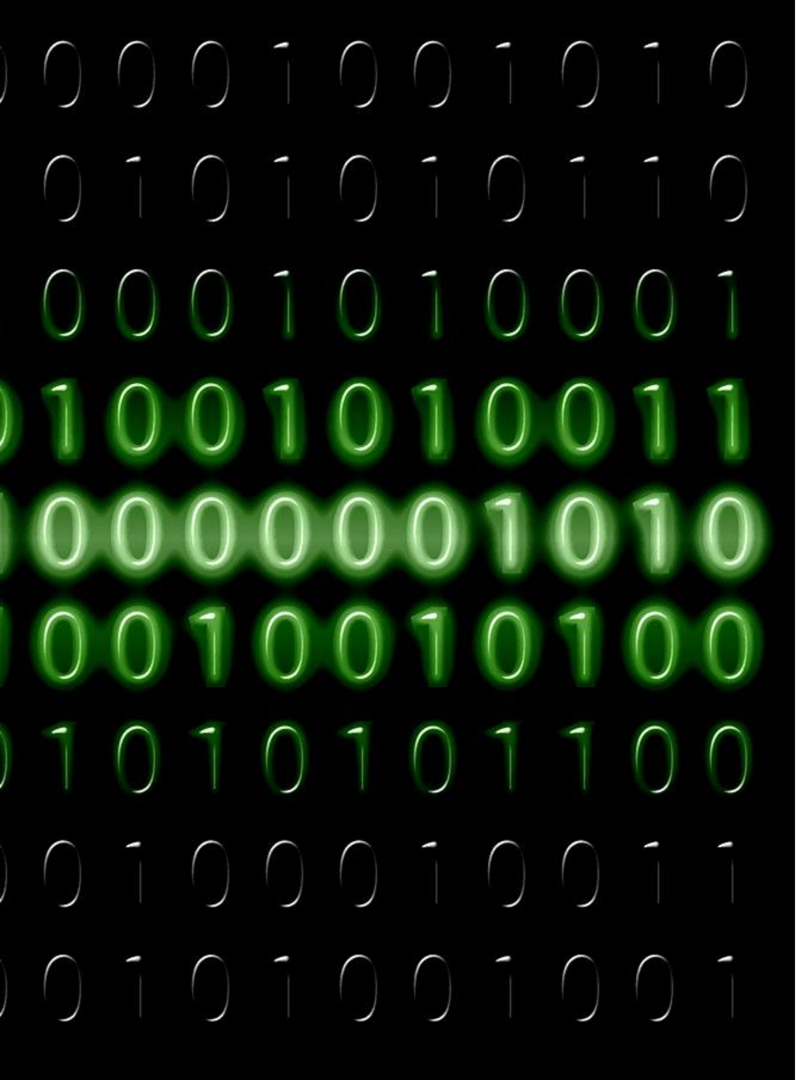
Numerical analysis in examples



Example 1
How small is zero?

Propose the way of checking the accuracy of the representation of real numbers on the computer.

Find the smallest n for which $10^{-n} = 0$. Consider different data types: Float32, Float64.



Example 2
Binary system and roundings

In binary system, the number 1/10 has an infinite expansion:

$$(0.1)_{10} \approx (0.00011001100110011001100110)_2$$

PATRIOT MISSILE DEFENSE: Software Problem Led to System Failure at Dhahran, Saudi Arabia

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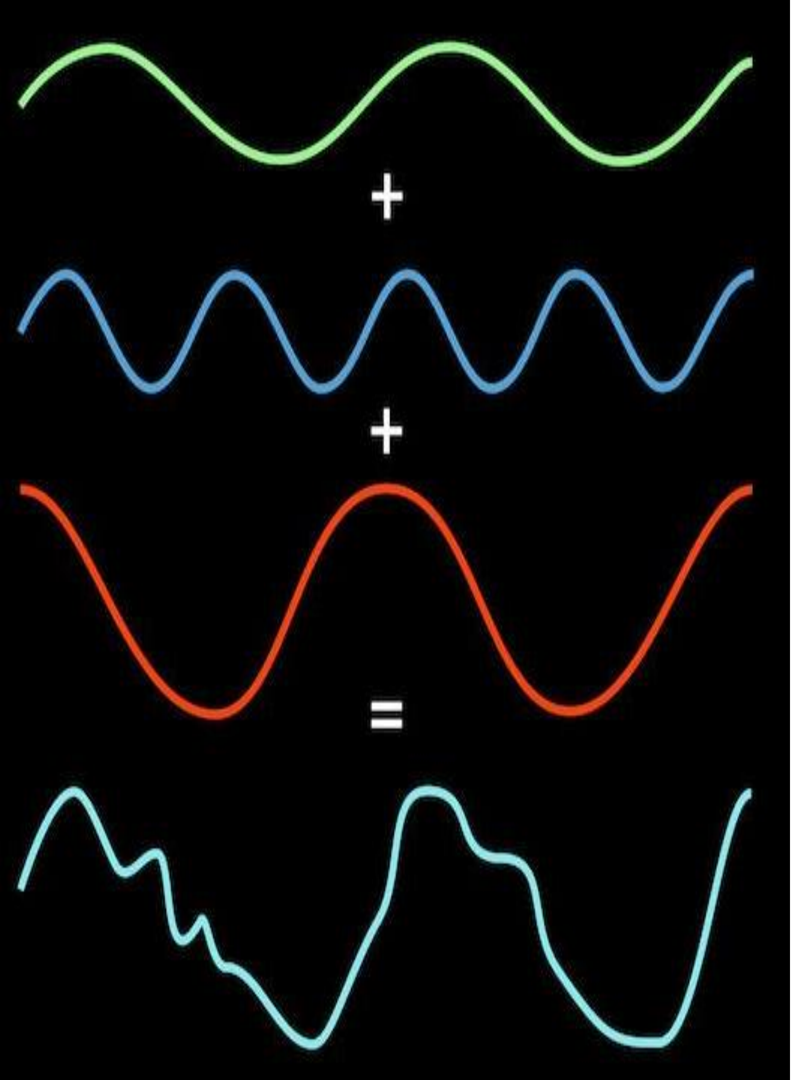
PATRIOT MISSILE DEFENSE: Software Problem Led to System Failure at Dhahran, Saudi Arabia (https://apps.dtic.mil/sti/citations/ADA344865)

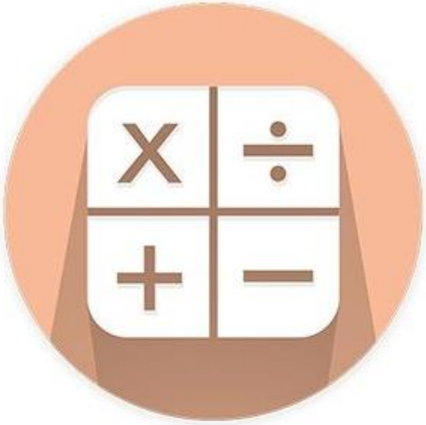
On February 25, 1991, a Patriot missile defense system operating at Dhahran, Saudi Arabia, during Operation Desert Storm failed to track and intercept an incoming Scud. This Scud subsequently hit an Army barracks, killing 28 Americans. (…) The Patriot battery at Dhahran failed to track and intercept the Scud missile because of a software problem in the systems weapons control computer. This problem led to an inaccurate tracking calculation that became worse the longer the system operated. At the time of the incident, the battery had been operating continuously for over 100 hours. By then, the inaccuracy was serious enough to cause the system to look in the wrong place for the incoming Scud. The Patriot had never before been used to defend against Scud missiles nor was it expected to operate continuously for long periods of time.



Example 3
Determining the value of sine

Use Julia to calculate these values:

$$\sin(0)$$
$$\sin(\pi)$$
$$\sin(2\pi)$$
$$\sin(20\pi)$$
$$\sin(2000\pi)$$
$$\sin(20000000\pi)$$
$$\sin(20000000000000000000\pi) = \sin(2e19 * \pi)$$



Example 4
Commutative property of addition (in floating point arithmetics)

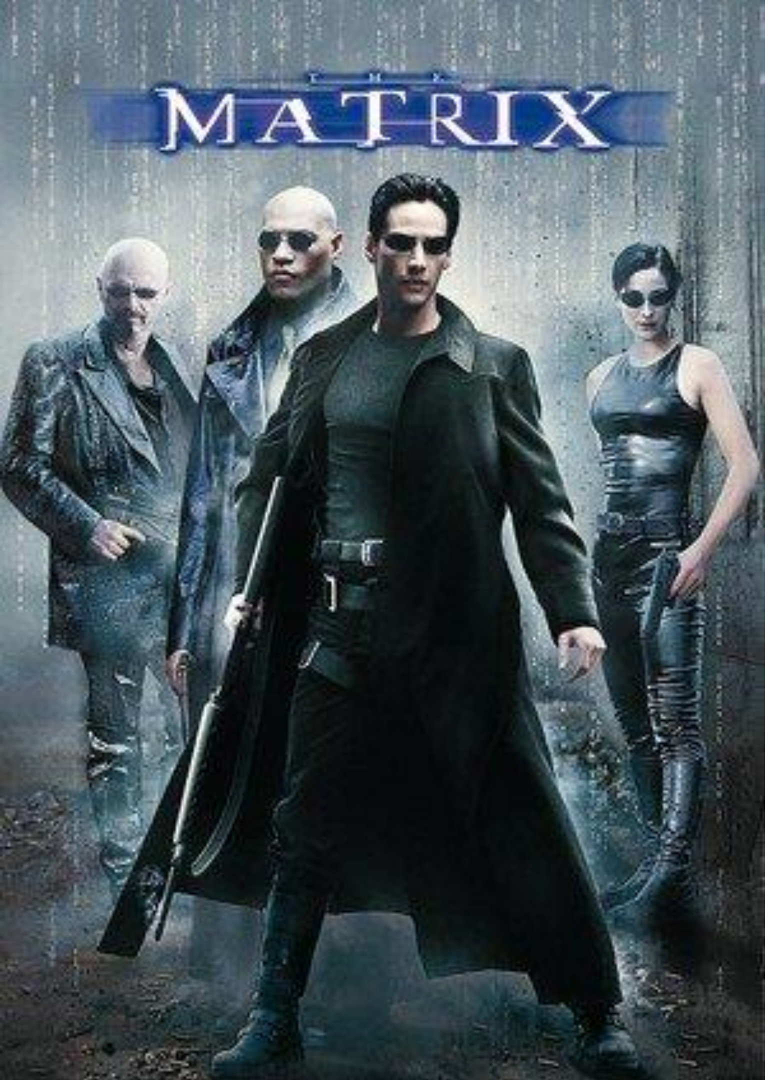
Add three following numbers:

53.54

104.44

-158.98

Use different orders of adding.

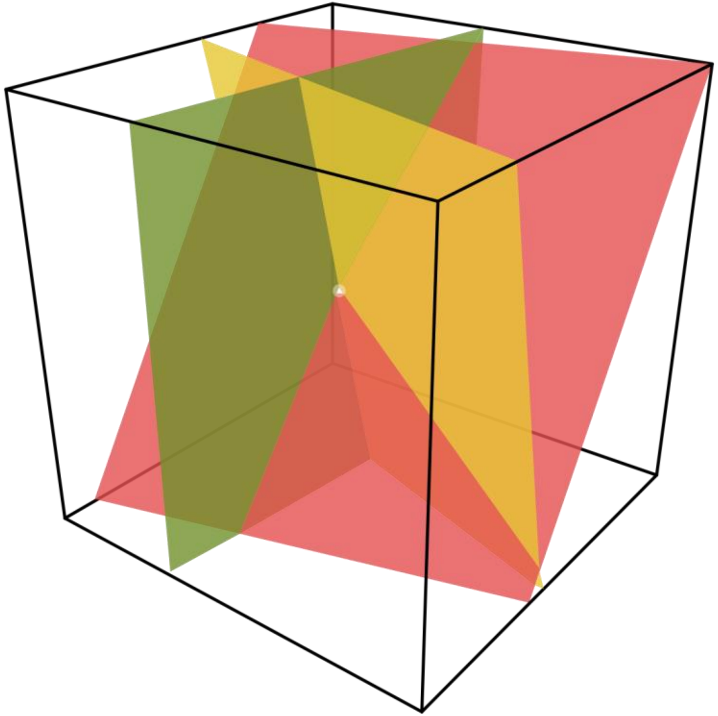


Example 5
Inverse of a matrix

Using Excel and Julia to find the inverse of a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Verify the result by calculating AA^{-1}.



Example 6
Solving system of linear equations

For the system of linear equation

$$\begin{pmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0.8642 \\ 0.1440 \end{pmatrix}$$

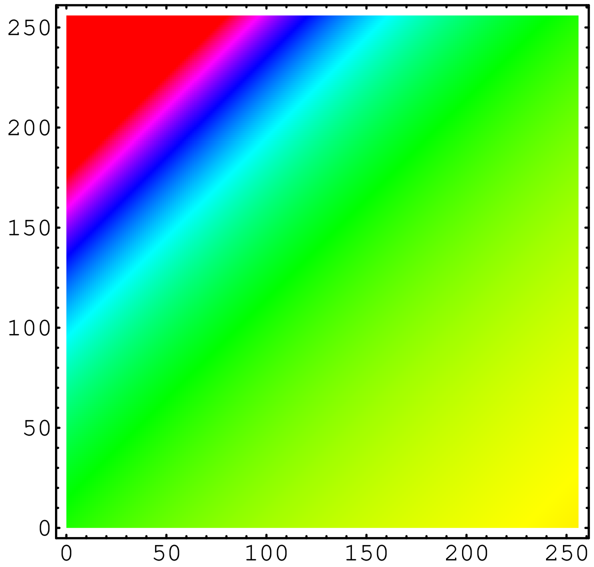
and the approximation of the solution

$$x = [0.9911, -0.4870]^T$$

find the residuum:

$$R = b - Ax$$

Is it justify to make a reasoning on accuracy of the solution based on these values?

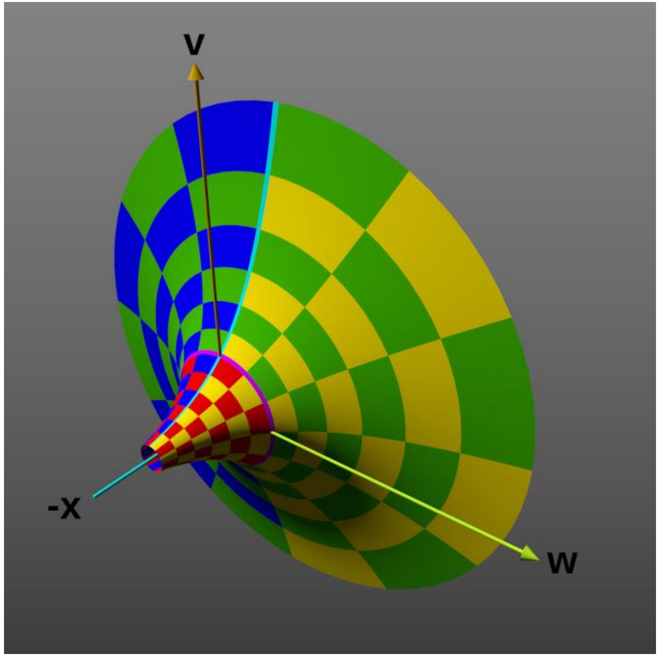


Example 7
Hilbert's matrix

Hilbert's matrix is given by the formula (for $n \geq 1$)

$$H_n = \begin{pmatrix} \frac{1}{1+1-1} & \frac{1}{1+2-1} & \cdots & \frac{1}{1+n-1} \\ \frac{1}{2+1-1} & \frac{1}{2+2-1} & \cdots & \frac{1}{2+n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n+1-1} & \frac{1}{n+2-1} & \cdots & \frac{1}{n+n-1} \end{pmatrix}$$

Solve system of linear equations with Hilbert matrix for $n \in \{5, 20\}$ and the right side vector consists of ones. Verify the result by substituting found solution to the equation.



Example 8
Calculating exp

Propose an algorithm of calculating value exp(x) based on the Taylor's algorithm

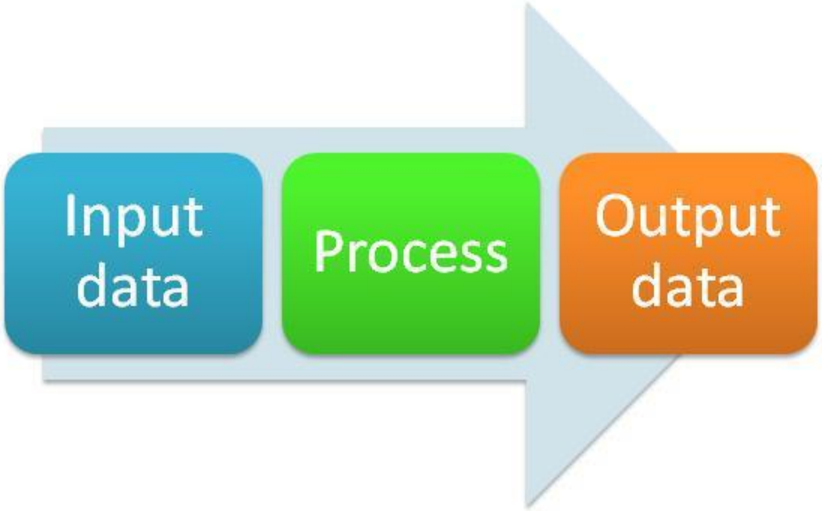
$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \underbrace{\frac{1}{(n+1)!} e^{\xi} x^{n+1}}_{R_n(x)},$$

where ξ is an intermediate point between 0 and x. As the stop criterion assume the limit of the number of iterations.

Write a Julia program and verify results for

x ∈ {100, 50, 25, 10, 1, 0, -1, -10, -25, -50, -100}.

Compare the results with the results of the build-in exp() function.



Always verify results!

Contact

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Thank you for your attention!