Differential Equations with Julia

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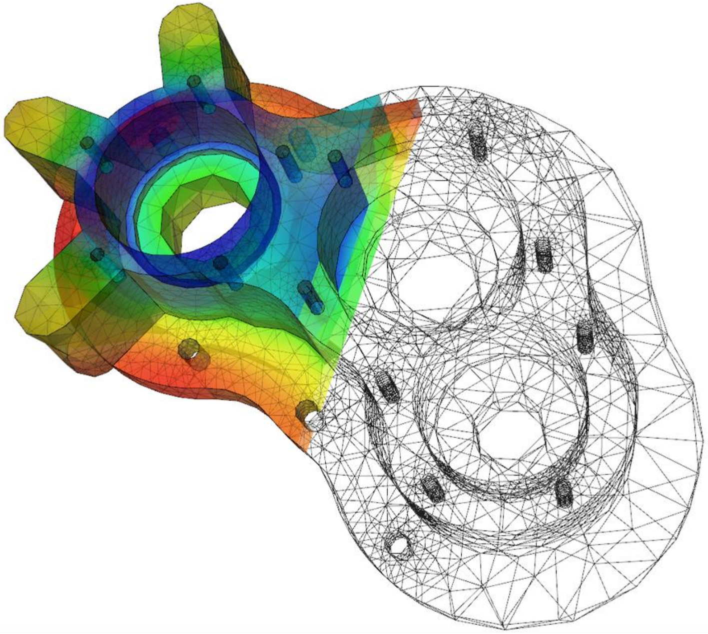

Differential equations

Application of differential equations*

1. **Physics:**
   - **Classical Mechanics:** Describing the motion of objects using equations like Newton's second law.
   - **Electrodynamics:** Maxwell's equations describe the behavior of electric and magnetic fields.
   - **Quantum Mechanics:** Schrödinger's equation is a fundamental differential equation in quantum mechanics.
2. **Engineering:**
   - **Electrical Engineering:** Circuits and systems analysis involve differential equations, especially in transient and frequency domain analyses.
   - **Mechanical Engineering:** Vibrations, fluid dynamics, and heat transfer are often modeled using differential equations.
   - **Civil Engineering:** Structural analysis, fluid flow in pipes, and other phenomena involve differential equations.
3. **Biology and Medicine:**
   - **Population Dynamics:** Modeling the growth or decline of populations.
   - **Physiology:** Modeling the behavior of biological systems, such as the spread of diseases or drug absorption in the body.
4. **Economics:**
   - **Macroeconomics:** Modeling economic growth, inflation, and unemployment.
   - **Microeconomics:** Modeling supply and demand dynamics.

* by ChatGDP

Application of differential equations*

5. **Chemistry:**
    - **Chemical Kinetics:** Describing the rate of chemical reactions.
    - **Transport Phenomena:** Diffusion and reaction processes in chemical systems.
6. **Computer Science:**
    - **Computer Graphics:** Simulating physical phenomena like fluid flow, smoke, or fire.
    - **Machine Learning:** Some models, such as neural networks, involve solving differential equations during training.
7. **Environmental Science:**
    - **Ecology:** Modeling interactions between species in ecosystems.
    - **Climate Modeling:** Describing the dynamics of the Earth's atmosphere.
8. **Finance:**
    - **Option Pricing Models:** Black-Scholes equation and other models in finance involve differential equations.
9. **Control Systems:**
    - **Control Theory:** Analyzing and designing control systems for engineering applications.
10. **Telecommunications:**
    - **Signal Processing:** Analyzing and processing signals, often involving differential equations.
11. **Mathematics:**
    - **Pure Mathematics:** Differential equations are studied as a field in their own right.

* by ChatGDP

Julia Packages

Problem type	Julia packages
Plotting	Plots
Linear system / least squares	LinearSolve
Sparse matrix	SparseArrays
Interpolation	DataInterpolations, ApproxFun
Polynomial manipulations	Polynomials
Rootfinding	NonlinearSolve
Finite differences	FiniteDifferences, FiniteDiff
Integration	Quadgk, HCubature
Optimization	Optimization
Ordinary Differential Equations	**DifferentialEquations**
Finite Element Method	Gridap
Automatic Differentiation	ForwardDiff, Enzyme
Fast Fourier Transform	FFTW

Packages needed during this course:

- DifferentialEquations
- Plots

Optional:

- DiffEqProblemLibrary

DifferentialEquations Package

- Discrete equations (function maps, discrete stochastic (Gillespie/Markov) simulations)
- Ordinary differential equations (ODEs)
- Split and Partitioned ODEs (Symplectic integrators, IMEX Methods)
- Stochastic ordinary differential equations (SODEs or SDEs)
- Stochastic differential-algebraic equations (SDAEs)
- Random differential equations (RODEs or RDEs)
- Differential algebraic equations (DAEs)

DifferentialEquations Package

- Delay differential equations (DDEs)
- Neutral, retarded, and algebraic delay differential equations (NDDEs, RDDEs, and DDAEs)
- Stochastic delay differential equations (SDDEs)
- Experimental support for stochastic neutral, retarded, and algebraic delay differential equations (SNDDEs, SRDDEs, and SDDAEs)
- Mixed discrete and continuous equations (Hybrid Equations, Jump Diffusions)
- (Stochastic) partial differential equations ((S)PDEs) (with both finite difference and finite element methods)

General workflow

Define a problem

Solve the problem

Analyze the output

Ordinary Differential Equation (ODE)

Defining a problem

Mathematical Specification of an ODE Problem:

$$\mathrm{M}\frac{dt}{du} = f(u,p,t)$$

1. Definition of the function f
2. Specification of the initial condition u_0
3. The timespan tspan for the problem

Ordinary Differential Equation (ODE)

Solving a problem

https://docs.sciml.ai/DiffEqDocs/stable/basics/common_solver_opts/#solver_options

Parameters:

- alg – algorithm; by default, alg = nothing (solve dispatches to the DifferentialEquations.jl automated algorithm selection)
- maxiters – maximum number of iterations before stopping. Defaults to 1e5.
- saveat – denotes specific times to save the solution at, during the solving phase

Ordinary Differential Equation (ODE)

Solving a problem

https://docs.sciml.ai/DiffEqDocs/stable/basics/common_solver_opts/#solver_options

Parameters:

- reltol – Relative tolerance in adaptive timestepping. This is the tolerance on local error estimates, not necessarily the global error (though these quantities are related). Defaults to 1e-3 on deterministic equations (ODEs/DDEs/DAEs) and 1e-2 on stochastic equations (SDEs/RODEs).
- abstol – Absolute tolerance in adaptive timestepping. This is the tolerance on local error estimates, not necessarily the global error (though these quantities are related). Defaults to 1e-6 on deterministic equations (ODEs/DDEs/DAEs) and 1e-2 on stochastic equations (SDEs/RODEs).

OrdinaryDiffEq algorithms

https://docs.sciml.ai/DiffEqDocs/stable/solvers/ode_solve

- Explicit Runge-Kutta Methods

Euler	OwrenZen4	RKO65	MSRK5
Midpoint	OwrenZen5	TanYam7	MSRK6
Heun	DP5	DP8	Stepanov5
Ralston	Tsit5	TsitPap8	SIR54
RK4	Anas5(w)	Feagin10	Alshina2
BS3	FRK65(w=0)	Feagin12	Alshina3
OwrenZen3	PFRK87(w=0)	Feagin14	Alshina6

- Parallel Explicit Runge-Kutta Methods
- Explicit Strong-Stability Preserving Runge-Kutta Methods for Hyperbolic PDEs

OrdinaryDiffEq algorithms

- Low-Storage Methods
- Parallelized Explicit Extrapolation Methods
- Explicit Multistep Methods
- Adams-Bashforth Explicit Methods
- Adaptive step size Adams explicit Methods
- SDIRK Methods
- Fully-Implicit Runge-Kutta Methods
- Parallel Diagonally Implicit Runge-Kutta Methods
- Rosenbrock Methods

OrdinaryDiffEq algorithms

- Rosenbrock-W Methods
- Stabilized Explicit Methods
- Parallelized Implicit Extrapolation Methods
- Parallelized DIRK Methods
- Exponential Runge-Kutta Methods
- Adaptive Exponential Rosenbrock Methods
- Exponential Propagation Iterative Runge-Kutta Methods
- Multistep Methods
- Implicit Strong-Stability Preserving Runge-Kutta Methods for Hyperbolic PDEs

OrdinaryDiffEq algorithms - good “go-to” choices

- AutoTsit5(Rosenbrock23()) handles both stiff and non-stiff equations. This is a good algorithm to use if you know nothing about the equation.
- AutoVern7(Rodas5()) handles both stiff and non-stiff equations in a way that's efficient for high accuracy.
- Tsit5() for standard non-stiff. This is the first algorithm to try in most cases.
- BS3() for fast low accuracy non-stiff.
- Vern7() for high accuracy non-stiff.
- Rodas4() or Rodas5() for small stiff equations with Julia-defined types, events, etc.
- KenCarp4() or TRBDF2() for medium-sized (100-2000 ODEs) stiff equations.
- RadauIIA5() for really high accuracy stiff equations.
- QNDF() for large stiff equations.

In-place functions

- *Instead of writing a function which outputs its solution f(u,p,t) you write a function which updates a vector that is designated to hold the solution f(du,u,p,t). By doing this, DifferentialEquations.jl's solver packages are able to reduce the amount of array allocations and achieve better performance.*
- *Convention: name functions with ! at the end.*
- *Memory-efficient but not always possible (mutation sometimes not allowed).*

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Thank you for your attention!