Solving optimization problems

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Basics…

Linear optimization

```
using JuMP, HiGHS
m = Model(optimizer_with_attributes(HiGHS.Optimizer))
@variable(m,      x₁ >= 0)
@variable(m,      x₂ >= 0)
@objective(m,     Min, 50x₁ + 70x₂)
@constraint(m,    200x₁ + 2000x₂ >= 9000    )
@constraint(m,    100x₁ +   30x₂ >=  300    )
@constraint(m,     9x₁   +   11x₂ >=   60   )
optimize!(m)
JuMP.value.([x₁,x₂])
```

Note – how to type indexes in Julia

- julia> x
- julia> x_
- julia> x_1
- julia> x_1***<TAB>***
- julia> x_1

… and Integer programming

```
using JuMP, HiGHS
m = Model(optimizer_with_attributes(HiGHS.Optimizer))
@variable(m, x₁ >= 0, Int)
@variable(m, x₂ >= 0)
@objective(m, Min, 50x₁ + 70x₂)
@constraint(m, 200x₁ + 2000x₂ >= 9000)
@constraint(m, 100x₁ +   30x₂ >=  300)
@constraint(m, 9x₁   +   11x₂ >=   60)
optimize!(m)
```

How it works - metaprogramming

```
julia> code = Meta.parse("x=5")
:(x = 5)

julia> dump(code)
Expr
  head: Symbol =
  args: Array{Any}((2,))
    1: Symbol x
    2: Int64 5

julia> eval(code)
5

julia> x
5
```

Macros – hello world...

```
macro sayhello(name)
    return :( println("Hello, ", $name) )
end

julia> macroexpand(Main,:(@sayhello("aa")))
:((Main.println)("Hello, ", "aa"))

julia> @sayhello "world!"
Hello, world!
```

Macro @variable

```
julia> @macroexpand @variable(m, x₁ >= 0)
quote
    (JuMP.validmodel)(m, :m)
    begin
        #1###361 = begin
                let
                    #1###361 = (JuMP.constructvariable!)(m, getfield(JuMP, Symbol("#_error#107")){Tuple{Symbol,Expr}}((:m, :(x₁ >= 0))), 0,
Inf, :Default, (JuMP.string)(:x₁), NaN)
                    #1###361
                end
            end
        (JuMP.registervar)(m, :x₁, #1###361)
        x₁ = #1###361
    end
end
```

Some of JuMP Solvers (a total of 50 as of today)

Solver	Julia Package	License	LP	SOCP	MILP	NLP	MINLP	SDP
Artelys Knitro	KNITRO.jl	Comm.				X	X	
BARON	BARON.jl	Comm.				X	X	
Bonmin	AmplNLWriter.jl CoinOptServices.jl	EPL	X		X	X	X	
Cbc	**Cbc.jl**	**EPL**			**X**			
CPLEX	**CPLEX.jl**	Comm.	X	X	X			
ECOS	ECOS.jl	GPL	X	X				
FICO Xpress	Xpress.jl	Comm.	X	X	X			
HiGHS	**HiGHSMathProgInterface**	**GPL**	**X**		**X**			
Gurobi	**Gurobi.jl**	Comm.	X	X	X			
Ipopt	**Ipopt.jl**	**EPL**	X			X		
MOSEK	Mosek.jl	Comm.	X	X	X	X		X
NLopt	NLopt.jl	LGPL				X		
SCS	SCS.jl	MIT	X	X				X

Full list: https://jump.dev/JuMP.jl/stable/installation/

JuMP
Transportation of good among branches

Use case scenario

The Subway restaurant chain in Las Vegas has a total of 118 restaurants in different parts of the city.

18 restaurants have adjacent huge product warehouses that keep ingredients cool and fresh, moreover fresh vegetables are delivered only to those warehouses (rather than to every restaurant) daily at 3am.

Subway has signed a contract with a transportation agency and is billed by the multiple of the weight of transported goods and the distance.

Knowing the amount of available stock at each warehouse and the expected demand at each restaurant (measured in kg), the company needs to decide how the goods should be distributed among warehouses.

Transportation problem statement

- Variables
  - X_{ij} – number of units transported for i-th supplier to j-th requester
  - C_{ij} – unit transportation cost between i-th supplier to j-th requester
- Cost function C

$$C = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

- Constraints:
  suppliers have maximum capacity S_i

$$\sum_{j=1}^{n} x_{ij} \leq S_i$$

  demand D_j must be met

$$\sum_{i=1}^{m} x_{ij} \geq D_j$$

Implementation in JuMP

```
m = Model(optimizer_with_attributes(HiGHS.Optimizer));
@variable(m, x[i=1:S, j=1:D])
@objective(m, Min, sum( x[i, j]*distance_mx[i, j] for i=1:S, j=1:D))
@constraint(m, x .>= 0)
for j=1:D
   @constraint(m, sum( x[i, j] for i=1:S) >= demand[j]   )
end
for i=1:S
   @constraint(m, sum( x[i, j] for j=1:D) <= supply[i]   )
end
optimize!(m)
termination_status(m)
```

JuMP
Travelling salesman problem

Use case scenario

A tourist in San Francisco and plans to visit all McDonald’s restaurant in one day

Let’s help her or him!

Spatial data
OpenStreetMap - https://www.openstreetmap.org

- Open project – “Wikipedia for maps”
- Lots, lots of data
  - Roads, Buildings, trees, …
  - Transportation systems
  - Point-of-interests (POIs) – businesses, restaurants, schools, universities…
- Formats: XML, PBF
- Multilayered structure
  - Nodes (points) ← <tag/>
  - Ways (lines, shapes) ← <tag/>
  - Relations (wider concepts) ← <tag/>

```
<osm>
  <bounds minlat="42.3609500" minlon="-71.0914900" maxlat="42.3621500" maxlon="-71.0898000"/>
  <node id="61317286" lat="42.3611637" lon="-71.0927647"/>
  <node id="61317287" lat="42.3607193" lon="-71.0937014"/>
  <node id="6898815038" lat="42.3608365" lon="-71.0894651">
    <tag k="entrance" v="yes"/>
  </node>
  ....
  <way id="17660188">
    <nd ref="182893079"/>
    <nd ref="182893081"/>
    <tag k="addr:city" v="Cambridge"/>
    <tag k="addr:housenumber" v="43"/>
    ....
   </way>
...
 <relation id="1590059">
    <member type="node" ref="9124611329" role=""/>
    <member type="way" ref="426493700" role=""/>
    <member type="way" ref="8605061" role=""/>

    <tag k="network:wikipedia" v="en:Massachusetts Bay Transp
    <tag k="operator" v="Massachusetts Bay Transportation Aut
    <tag k="public_transport:version" v="2"/>
    <tag k="ref" v="CT2"/>
    <tag k="route" v="bus"/>
    <tag k="to" v="Ruggles Station"/>
    <tag k="type" v="route"/>
  </relation>
```

Libraries for OSM data

OpenStreetMapX.jl – mainly oriented on road system

- Road system extracted as a directed graph (Graphs.jl) along with separate metadata
- Supports routing, road classes, vehicle speeds etc.
- Spatial algebra (ENU, LLA, ECEF), overlap with Geodesy.jl

OpenStreetMapXPlots.jl

- Plotting maps with Plots.jl and PyPlot.jl .

OSMtoolset.jl – https://github.com/pszufe/OSMToolset.jl

- Spatial indexes on maps
- Mass extraction of points-of-interests (POIs) from maps
- Tools for slicing/tiling large OSM *.xml files

(developed under grant National Science Centre, Poland 2021/41/B/HS4/03349)

McDonald’s in SF

In [48]:
```
using DataFrames, OSMToolset
config = DataFrame(key="brand", values="McDonald's")
dfpoi = find_poi("SF.osm"; scrape_config=ScrapePOIConfig{NoneMetaPOI}(config))
```

Out[48]: 18×7 DataFrame

Row	elemtype	elemid	nodeid	lat	lon	key	value
	Symbol	Int64	Int64	Float64	Float64	String	String
1	node	358116917	358116917	37.7264	-122.476	brand	McDonald's
2	node	597382133	597382133	37.7066	-122.415	brand	McDonald's
3	node	1229920544	1229920544	37.7131	-122.445	brand	McDonald's
4	node	2365742146	2365742146	37.789	-122.401	brand	McDonald's
5	node	3455025884	3455025884	37.6522	-122.491	brand	McDonald's
6	node	4626983989	4626983989	37.7892	-122.408	brand	McDonald's
7	node	6959355927	6959355927	37.644	-122.404	brand	McDonald's
8	node	9980865058	9980865058	37.6438	-122.454	brand	McDonald's
9	node	11338930625	11338930625	37.6747	-122.47	brand	McDonald's
10	way	143811393	1573722786	37.669	-122.47	brand	McDonald's
11	way	159024983	1711296799	37.7236	-122.455	brand	McDonald's
12	way	231047897	2394660225	37.752	-122.418	brand	McDonald's
13	way	256462436	2621272506	37.7653	-122.408	brand	McDonald's

TSP

$$\text{Min} \sum_{f=1}^{N} \sum_{t=1}^{N} c_{ft} x_{ft}$$

Variables:

- c_{ft} – cost of travel from "f" to "t"
- x_{ft} – binary variable indicating 1 when agent travels from "f" to "t"

Each restaurant visited once

$$\sum_{t=1}^{N} x_{ft} = 1 \quad \forall f \in \{1, \ldots, N\}$$

$$\sum_{f=1}^{N} x_{ft} = 1 \quad \forall t \in \{1, \ldots, N\}$$

Restaurant cannot visit itself $x_{ff} = 0 \quad \forall f \in \{1, \ldots, N\}$

Avoid two-restaurant cycles $x_{ft} + x_{tf} <= 1 \quad \forall f, t \in \{1, \ldots, N\}$

Other cycles: /dynamically add a constraint whenever a cycle occurs/

For more details see: http://opensourc.es/blog/mip-tsp

JuMP implementation

```
m = Model(optimizer_with_attributes(HiGHS.Optimizer));
@variable(m, x[f=1:N, t=1:N], Bin)
@objective(m, Min, sum( x[i, j]*distance_mx[i,j] for i=1:N,j=1:N))
@constraint(m, notself[i=1:N], x[i, i] == 0)
@constraint(m, oneout[i=1:N], sum(x[i, 1:N]) == 1)
@constraint(m, onein[j=1:N], sum(x[1:N, j]) == 1)
for f=1:N, t=1:N
    @constraint(m, x[f, t]+x[t, f] <= 1)
end
```

Getting a cycle

```
function getcycle(m, N)
    x_val = getvalue(x)
    cycle_idx = Vector{Int}()
    push!(cycle_idx, 1)
    while true
        v, idx = findmax(x_val[cycle_idx[end], 1:N])
        if idx == cycle_idx[1]
            break
        else
            push!(cycle_idx, idx)
        end
    end
    cycle_idx
end
```

Adding a constraint…

```
function solved(m, cycle_idx, N)
    println("cycle_idx: ", cycle_idx)
    println("Length: ", length(cycle_idx))
    if length(cycle_idx) < N
        cc = @constraint(m, sum(x[cycle_idx,cycle_idx])
           <= length(cycle_idx)-1)
        println("added a constraint")
        return false
    end
    return true
end
```

Iterating over the model

```
while true
    status = solve(m)
    println(status)
    cycle_idx = getcycle(m, N)
    if solved(m, cycle_idx,N)
        break;
    end
end
```

Gurobi.jl

- Commercial software
- Free for academic use
- Integrates with JuMP via Gurobi.jl

- Supports JuMP Lazy constraints (http://www.juliaopt.org/JuMP.jl/0.18/callbacks.html)

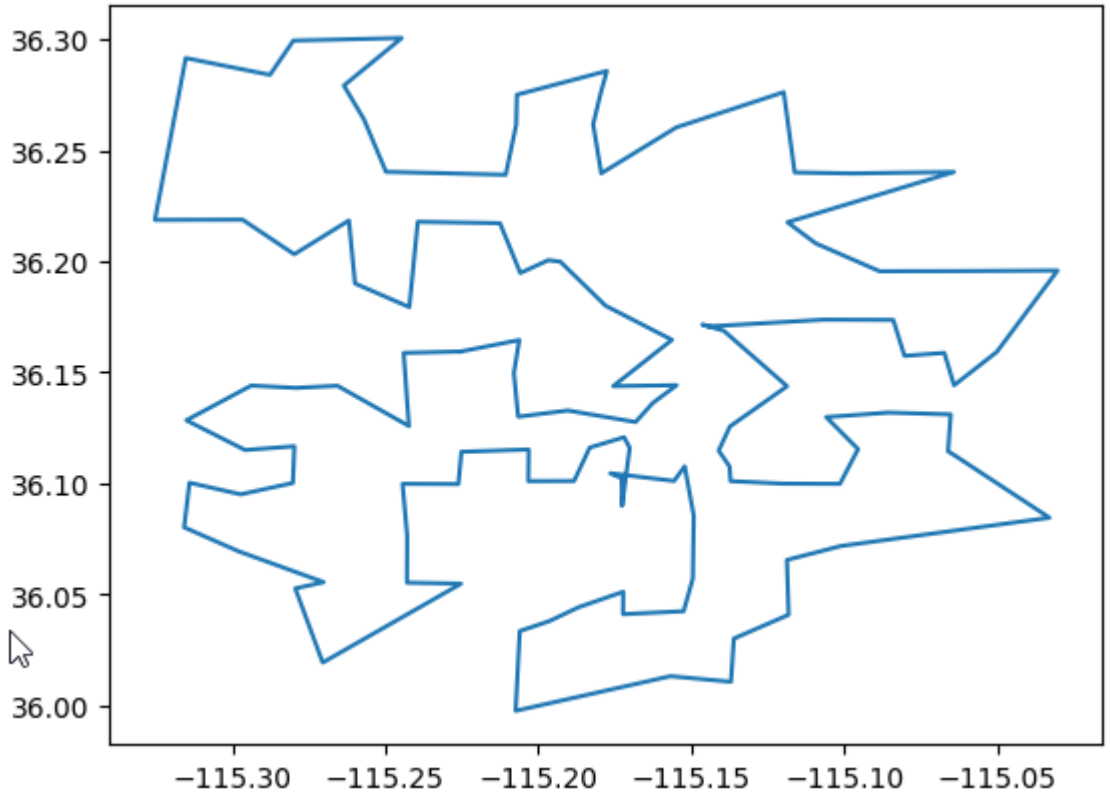
Gurobi callbacks

```
function getcycle(cb, N)
    x_val = callback_value.(Ref(cb), x)
    getcycle(x_val)
end
function callbackhandle(cb)
    cycle_idx = getcycle(cb, N)
    println("Callback! N= $N cycle_idx: ", cycle_idx)
    println("Length: ", length(cycle_idx))
    if length(cycle_idx) < N
        con = @build_constraint(sum(x[cycle_idx,cycle_idx]) <= length(cycle_idx)-1)
        MOI.submit(m, MOI.LazyConstraint(cb), con)
        println("added a lazy constraint")
    end
end
MOI.set(m, MOI.LazyConstraintCallback(), callbackhandle)
```

TravelingSalesmanHeuristics.jl

```
using TravelingSalesmanHeuristics
sol = TravelingSalesmanHeuristics.solve_tsp(
distance_mx,quality_factor =100)
```

More info:
http://evanfields.github.io/TravelingSalesmanHeuristics.jl/latest/heuristics.html



JuMP
Non-Linear Programming

Simple scenario

Estimate parameters of a quadratic form

$\boldsymbol{y}(\boldsymbol{x}_i) = \boldsymbol{x}_i^T \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \boldsymbol{x}_i$, where $\boldsymbol{x}_i = \begin{bmatrix} x_i^1 \\ x_i^2 \end{bmatrix}$

for a vector of observed values $\boldsymbol{y}$ to minimize the observed error function

$$\sum_{i=1}^{N} (y(\boldsymbol{x}_i) - y_i)^2$$

Nonlinear optimization Julia

```
m = Model(optimizer_with_attributes(Ipopt.Optimizer));

@variable(m, aa[1:2,1:2])

function errs(aa)
   sum((y .- (x * aa ) .* x * [1;1]) .^ 2)
end

@objective(m, Min, errs(aa))

optimize!(m)
```

Use case scenario

(source: Hart et al, Pyomo-optimization modeling in python, 2017)

Simulate dynamics of disease outbreak in a small community of 300 individuals (e.g. children at school)

Three possible states of a patient:

- susceptible (S)
- infected (I)
- recovered (R)

Infection spread model :

- N – population size
- α, β – model parameters

$$I_i = \frac{\beta I_{i-1}^{\alpha} S_{i-1}}{N}$$

$$S_i = S_{i-1} - I_i$$

Optimization problem for finding parameters α and β

S - susceptible

I - infected

N – population size

α, β – model parameters

SI - time indices {1,2,3,...}

C_i - known input (the actual number of infected patients)

$$\min \sum_{i \in SI} \left(\varepsilon_i^I\right)^2$$

$$I_i = \frac{\beta I_{i-1}^{\alpha} S_{i-1}}{N} \quad \forall \; i \in SI \setminus \{1\}$$

$$S_i = S_{i-1} - I_i \quad \forall \, i \in SI \setminus \{1\}$$

$$C_i = I_i + \varepsilon_i^I$$

$$0 \leq I_i, \; Si \leq N$$

$$0.5 \leq \beta \leq 70$$

$$0.5 \leq \alpha \leq 1.5$$

Model implementation in JuMP

- Input data (disease dynamics)

```
obs_cases = vcat(1,2,4,8,15,27,44,58,55,32,12,3,1,zeros(13))
```

Full model specification in JuMP

```
m = Model(optimizer_with_attributes(Ipopt.Optimizer));
@variable(m, 0.5 <= α <= 1.5)
@variable(m, 0.05 <= β <= 70)
@variable(m, 0 <= I_[1:SI_max] <= N)
@variable(m, 0 <= S[1:SI_max]  <= N)
@variable(m, ε[1:SI_max])
@constraint(m, ε .== I_ .- obs_cases  )
@constraint(m, I_[1] == 1)
for i=2:SI_max
   @NLconstraint(m, I_[i] == β*(I_[i-1]^α)*S[i-1]/N)
end
@constraint(m, S[1] == N)
for i=2:SI_max
   @constraint(m, S[i] == S[i-1]-I_[i])
end
@NLobjective(m, Min, sum(ε[i]^2 for i in 1:SI_max))
```

JuMP
Non-Linear Programming
for estimation of model parameters

Simple scenario

Estimate parameters of a quadratic form

$\boldsymbol{y}(\boldsymbol{x}_i) = \boldsymbol{x}_i^T \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix} \boldsymbol{x}_i$, where $\boldsymbol{x}_i = \begin{bmatrix} x_i^1 \\ x_i^2 \end{bmatrix}$

for a vector of observed values $\boldsymbol{y}$ to minimize the observed error function

$$\sum_{i=1}^{N} (y(\boldsymbol{x}_i) - y_i)^2$$

Nonlinear optimization Julia

```
m = Model(optimizer_with_attributes(Ipopt.Optimizer));

@variable(m, aa[1:2,1:2])

function errs(aa)
   sum((y .- (x * aa ) .* x * [1;1]) .^ 2)
end

@objective(m, Min, errs(aa))

optimize!(m)
```

Use case scenario

(source: Hart et al, Pyomo-optimization modeling in python, 2017)

Simulate dynamics of disease outbreak in a small community of 300 individuals (e.g. children at school)

Three possible states of a patient:

- susceptible (S)
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- recovered (R)

Infection spread model :

- N – population size
- α, β – model parameters

$$I_i = \frac{\beta I_{i-1}^{\alpha} S_{i-1}}{N}$$

$$S_i = S_{i-1} - I_i$$

Optimization problem for finding parameters α and β

S - susceptible

I - infected

N – population size

α, β – model parameters

SI - time indices {1,2,3,...}

C_i - known input (the actual number of infected patients)

$$\min \sum_{i \in SI} \left(\varepsilon_i^I\right)^2$$

$$I_i = \frac{\beta I_{i-1}^{\alpha} S_{i-1}}{N} \quad \forall \; i \in SI \setminus \{1\}$$

$$S_i = S_{i-1} - I_i \quad \forall \, i \in SI \setminus \{1\}$$

$$C_i = I_i + \varepsilon_i^I$$

$$0 \leq I_i, \; Si \leq N$$

$$0.5 \leq \beta \leq 70$$

$$0.5 \leq \alpha \leq 1.5$$

Model implementation in JuMP

- Input data (disease dynamics)

```
obs_cases = vcat(1,2,4,8,15,27,44,58,55,32,12,3,1,zeros(13))
```

Full model specification in JuMP

```
m = Model(optimizer_with_attributes(Ipopt.Optimizer));
@variable(m, 0.5 <= α <= 1.5)
@variable(m, 0.05 <= β <= 70)
@variable(m, 0 <= I_[1:SI_max] <= N)
@variable(m, 0 <= S[1:SI_max]  <= N)
@variable(m, ε[1:SI_max])
@constraint(m, ε .== I_ .- obs_cases  )
@constraint(m, I_[1] == 1)
for i=2:SI_max
   @NLconstraint(m, I_[i] == β*(I_[i-1]^α)*S[i-1]/N)
end
@constraint(m, S[1] == N)
for i=2:SI_max
   @constraint(m, S[i] == S[i-1]-I_[i])
end
@NLobjective(m, Min, sum(ε[i]^2 for i in 1:SI_max))
```

JuMP
Multi-criteria optimization for stock portfolio optimization

[additional material]

Stock portfolio optimization

- Estimate the weights vector

$$\vec{x} = \left[x_1 \; x_2 \cdots x_n \right]^T$$

  where x_i represents the share of asset i in a portfolio: $\vec{1}^T \vec{x} = 1$

- maximize the expected return $x_p = \vec{p}^T \vec{x} = \sum_{i=1}^{n} x_i p_i$
- minimize the risk

$$\sigma_p{}^2 = \vec{x}^T V \vec{x} = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \sigma_{i,j}$$

for the variance-covariance matrix:

$$V = \begin{pmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{pmatrix}$$

Source: Duan Y. C. "A multi-objective approach to portfolio optimization." *Rose-Hulman Undergraduate Mathematics Journal* 8.1 (2007): 12.

Possible approaches

- Maximize the expected return, disregarding risk
- Minimize the expected risk, disregarding return
- Maximize the return for a given level of risk
- Minimize the risk for a given level of return

- **Maximize the risk** AND **minimize the return**
  ➔ multi-criteria optimization

Stock price data

- Top 10 Fortune 500 companies
- 3 years
- Daily opening and closing prices

- Expected return
  - average daily rate of return (for simplicity calculated as a difference between opening and closing price)
- Risk
  - calculate variance-covariance matrix for daily returns

Julia implementation – data processing

```
prices = CSV.read("10_stocks_3yr.csv", allowmissing=:none)
prices.rateOfRet =
     (prices.close-prices.open) ./ prices.open
dats = unstack(prices, :date, :Name, :rateOfRet)
const avg_rets = colwise(mean, dats[2:end])
const cov_mx = cov(Matrix(dats[2:end]))
```

Julia MultiJuMP model

```
m = multi_model(Ipopt.Optimizer)
@variable(m, 0 <= x[i=1:10] <= 1)
@constraint(m,sum(x) == 1)
@variable(m, risk)
@constraint(m, risk == x'*cov_mx*x)
@variable(m, rets)
@constraint(m, rets == avg_rets' * x)
@NLexpression(m, f_risk, risk)
@NLexpression(m, f_rets, rets)
```

Solving the model

```
iv1 =  fill(0.1, 10) # Initial guess
obj1 = SingleObjective(f_risk, sense = MOI.MIN_SENSE,
                       iv = Dict{String,Any}("x[$i]" => iv1[i] for i in 1:length(iv1)))
obj2 = SingleObjective(f_rets, sense = MOI.MAX_SENSE)
md = get_multidata(m)
md.objectives = [obj1, obj2]
md.pointsperdim = 20

optimize!(m, method = NBI(false))
```

Plotting the Pareto-frontier

```
Plots.pyplot()
Plots.plot(md)
```