

Mining Networks

Chapter 5 — Community Detection

Bogumił Kamiński, Paweł Prałat, and François Théberge

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Department of Mathematics, Toronto Metropolitan University
File: DS8014-Chapter05



Experiments

Jupyter notebooks and datasets that accompany the textbook/slides are available on GitHub:

<https://github.com/ftheberge/GraphMiningNotebooks>

[Python_Notebooks_Second_Edition](#)

[Julia_Notebooks_2nd](#)



Tutorials for Python notebooks can be found on YouTube:

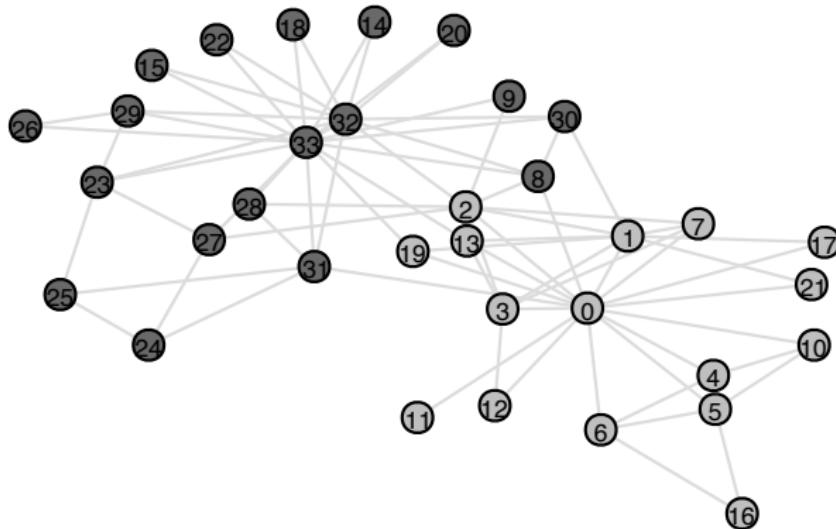
<https://www.youtube.com/@MiningComplexNetworks>

Overview

1. Basic Properties of Communities
2. Synthetic Models with Community Structure
3. Graph Modularity
4. Hierarchical Clustering
5. A Few Other Methods
6. Experiments

Introduction

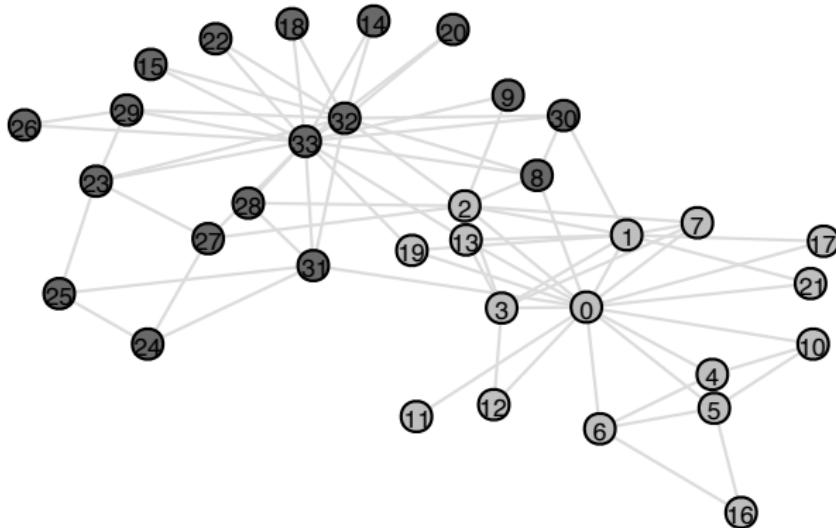
Zachary's Karate Club: 0 – instructor, 33 – administrator



A network has **community structure** if its set of nodes can be split into a number of subsets such that each subset is **densely** internally connected.

Introduction

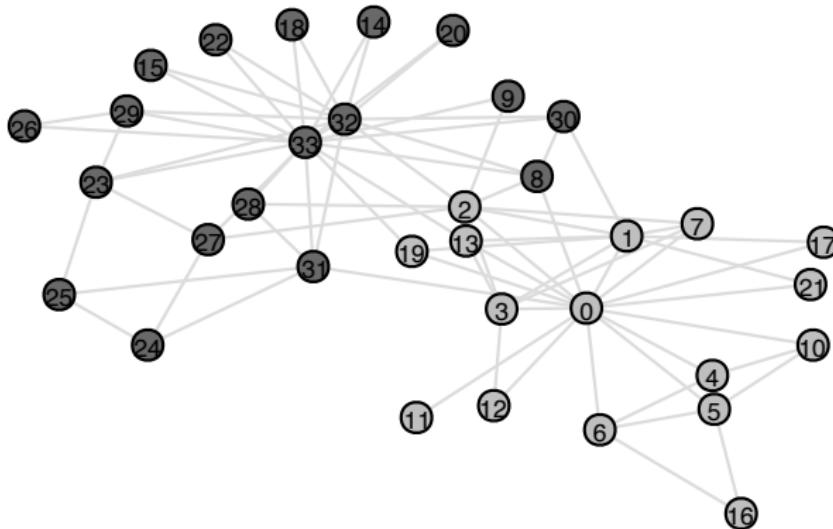
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social networks: communities based on common location of their users, their interests, occupation, gender, age, etc.

Introduction

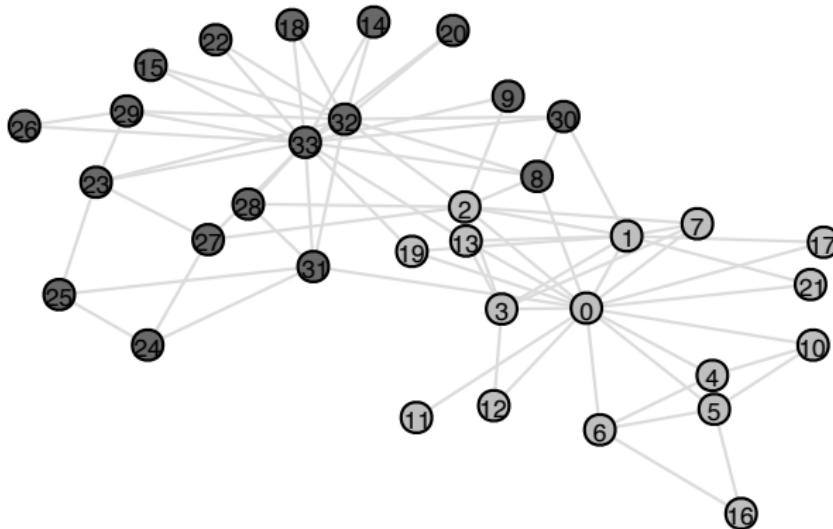
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web graph: web pages that belong to the same community are on a similar topic.

Introduction

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protein-protein interaction networks: proteins that belong to the same community are often associated with a particular biological function within the organism.

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Communities allow us to...

- see a “**big picture**” (large scale map with individual communities represented as meta-nodes),
- better **understand** the **function** of the system represented by the network,
- **classify** the nodes based on the position they have in their own clusters and how they are connected to other clusters: **roles** and **importance**,
- ...

Basic Properties of Communities

Ground Truth

Collaboration network:

nodes – researchers, edges – collaborations.

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We do not know where are mathematicians (**ground truth**) but we want to **uncover** this **hidden**, underlying attributes of nodes based on a **graph** that is visible to us—**reversed engineering process**.

Ground Truth

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Ground Truth

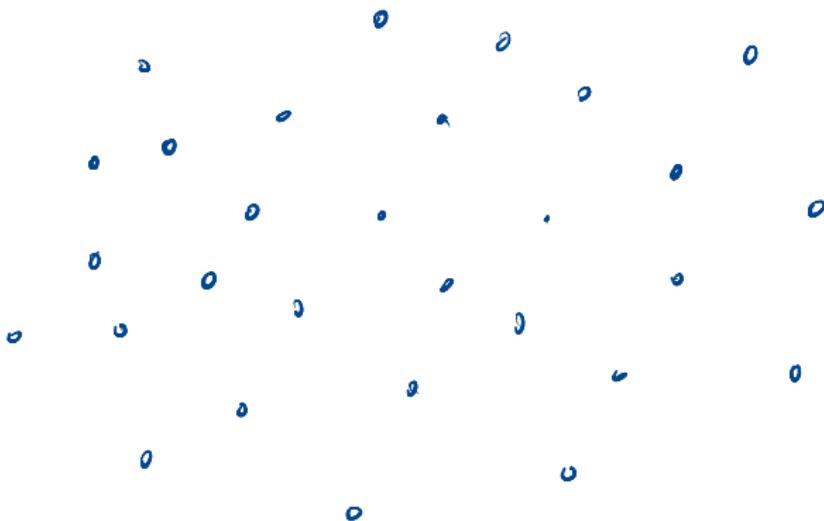
The **ground truth** could simply be a partition of nodes (the number of parts can be arbitrary).

Partition

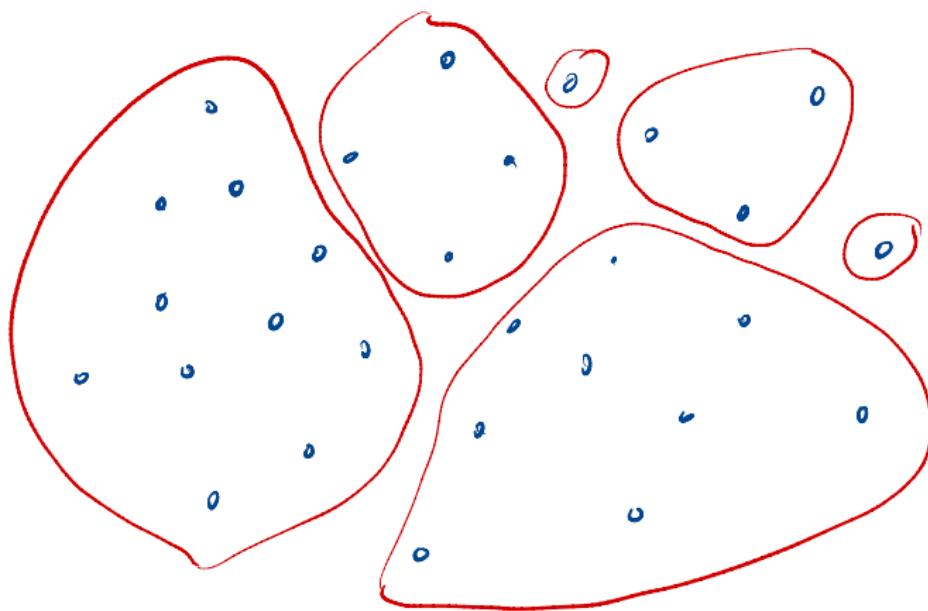
Let V be any finite **set** (in our scenario, a set of nodes). A **partition** of V is a grouping of its elements into non-empty subsets, in such a way that every element is included in exactly one subset.

In other words, V_1, V_2, \dots, V_ℓ is a **partition** if and only if $V_1 \cup V_2 \cup \dots \cup V_\ell = V$, $V_i \neq \emptyset$ for any $1 \leq i \leq \ell$, and $V_i \cap V_j = \emptyset$ for any $1 \leq i < j \leq \ell$.

Ground Truth

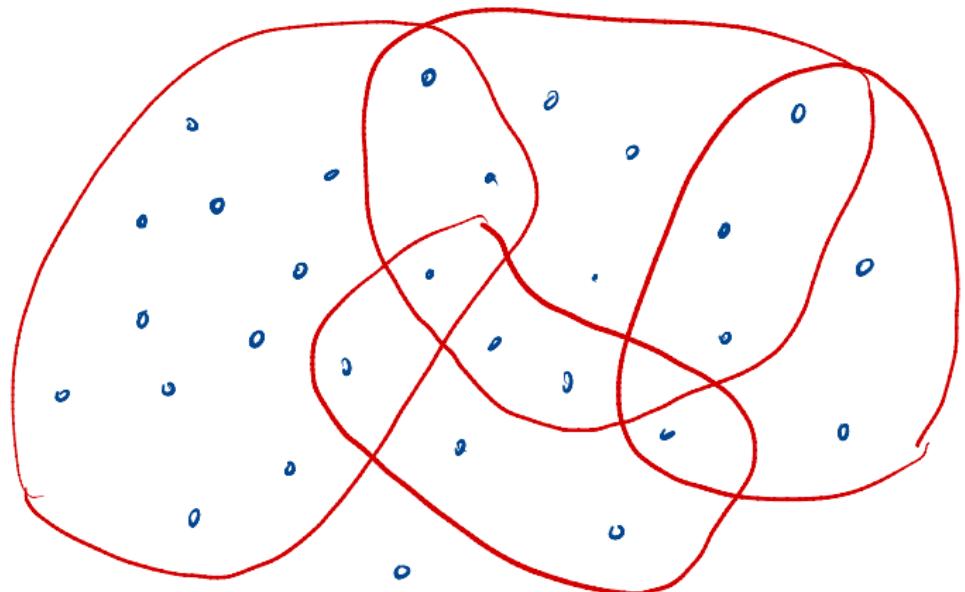


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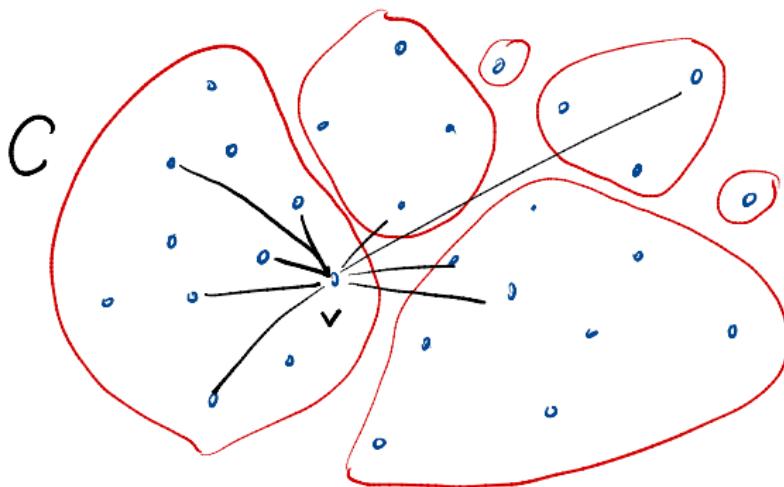
Partition

Ground Truth



Overlapping Communities and/or Outliers — see
supplementary material

Communities



Consider any $v \in C$.

internal degree: $\deg^{int}(v) = |N(v) \cap C|$

external degree: $\deg^{ext}(v) = |N(v) \setminus C| = \deg(v) - \deg^{int}(v)$

Communities

Strong/Weak Community

Set $C \subseteq V$ forms a **strong community** if each node in C has more neighbours in C than outside of C , that is, for each $v \in C$ we have

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(strong \Rightarrow weak but weak $\not\Rightarrow$ strong)

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What about Bob? Should we expect that more than 50% of his friends to be working in his field of research?

No! In fact, it might be the case that there are less than 50 people around the world working in his field!

Communities

The number of **internal neighbours** of a given node should be proportional to the **size of the community** this node belongs to. But the probability that a node is adjacent to another member of its community should be **larger** than the probability of being adjacent to a random node in the whole graph.

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Community

Let $C \subseteq V$ ($C \neq \emptyset$ and $C \neq V$). C is a **community** if each node in C has proportionally more neighbours in C than outside of C , that is, for each $v \in C$ we have

$$\frac{\deg^{int}(v)}{|C|} > \frac{\deg^{ext}(v)}{|V \setminus C|}.$$

Node Roles

Partition $\mathcal{A} = \{A_1, A_2, \dots, A_\ell\}$ of V ; (ground truth or a partition returned by some community detection algorithm).

$\deg_{A_i}(v) = |N(v) \cap A_i|$ (number of neighbours of v in part A_i),

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Normalized within-module degree

The **normalized within-module degree** of a node v (with respect to \mathcal{A}) is defined as follows:

$$z(v) = \frac{\deg^{int}(v) - \mu(v)}{\sigma(v)},$$

where $\mu(v)/\sigma(v)$ are the **mean/standard deviation** of $\deg^{int}(u)$ over all nodes in the part v belongs to.

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$|z(v)|$ is **large and $z(v)$ is negative**: v is **loosely** connected to other peers.

If $z(v) > 2.5$, then node v is classified as a **hub**; otherwise, it is a **non-hub**.

Node Roles

Another important aspect: how **edges** (both internal and external) are **distributed** between all **parts** of the partition \mathcal{A} .

Participation coefficient

The **participation coefficient** of a node v (with respect to \mathcal{A}) is defined as follows:

$$p(v) = 1 - \sum_{i=1}^{\ell} \left(\frac{\deg_{A_i}(v)}{\deg(v)} \right)^2.$$

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If v has neighbours exclusively in one part, then $p(v) = 0$.

If the neighbours of v are homogeneously distributed among all parts, then $p(v) \approx 1 - 1/\ell \approx 1$.

Node Roles

Non-hubs are partitioned into the following four classes:

- **ultra-peripheral nodes** (almost all edges are internal) if $p(v) < 0.05$,
- **peripheral nodes** (most edges are internal) if $0.05 \leq p(v) < 0.62$,
- **connector nodes** (most edges are external) if $0.62 \leq p(v) < 0.80$,
- **kinless nodes** (neighbours are homogeneously distributed) if $p(v) \geq 0.80$.

Node Roles

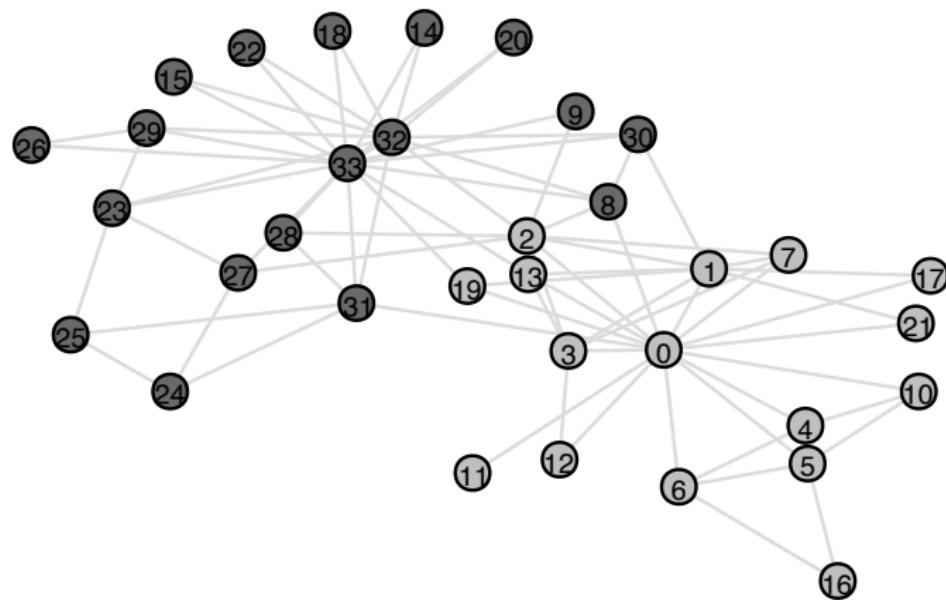
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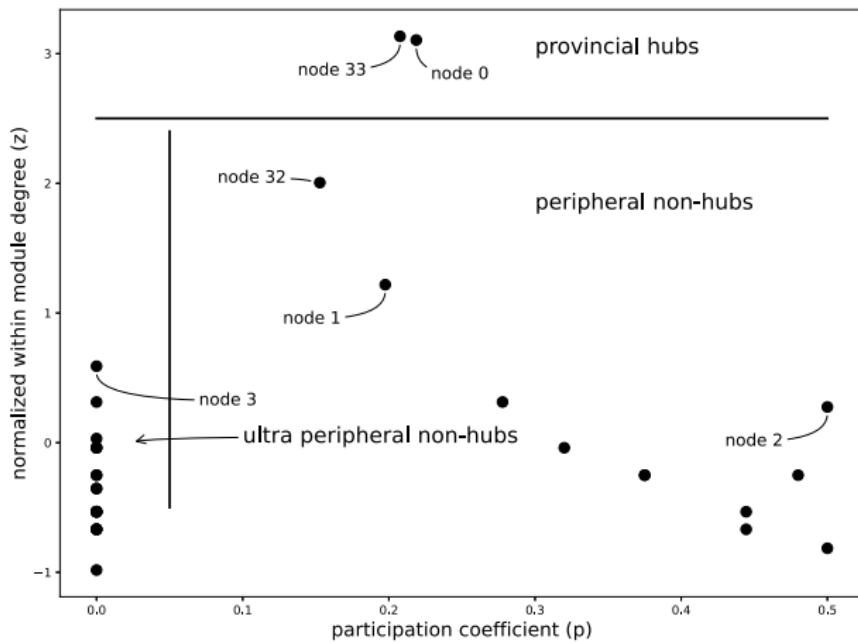
On the other hand, **hubs** are partitioned into three classes:

- **provincial hubs** if $p(v) < 0.30$,
- **connector hubs** if $0.30 \leq p(v) < 0.75$,
- **kinless hubs** if $p(v) \geq 0.75$.

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We can model “pure luck” with the **Chung-Lu** model.

Node Roles

Community Association Strength

The **community association strength** of a node v (with respect to community A_i from \mathcal{A}) is defined as follows:

$$cas(v) = \frac{\deg_{A_i}(v)}{\deg(v)} - \frac{\text{vol}(A_i) - \deg(v)}{\text{vol}(V)}.$$

Community Distribution Distance

The **community distribution distance** of a node v (with respect to \mathcal{A}) is defined as follows:

$$cdd(v) = \left(\sum_{i=1}^{\ell} \left(\frac{\deg_{A_i}(v)}{\deg(v)} - \frac{\text{vol}(A_i)}{\text{vol}(V)} \right)^2 \right)^{1/2}.$$

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Important implication: we will not be able to investigate all partitions even if, say, $n = 20$. We aim for heuristic algorithms.

Synthetic Models with Community Structure

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In order to evaluate algorithms in a fair and rigorous way, one need a good benchmark: large synthetic network with an engineered ground truth.

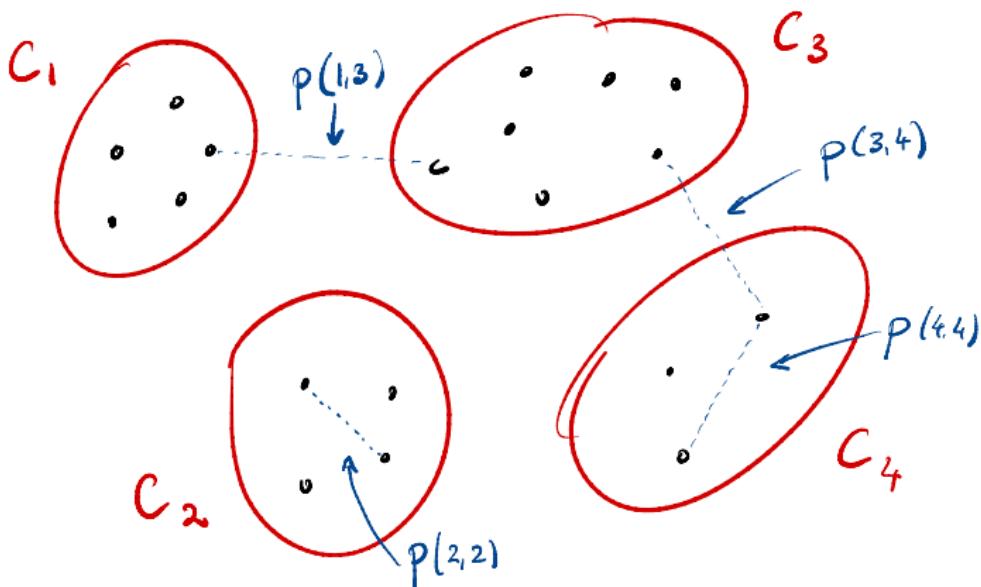
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We present a few random graph models with community structure and then show how to use them to make such evaluation.

Stochastic Block Model

Simple and natural generalization of $\mathcal{G}(n, p)$.



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Stochastic Block Model

Let $\mathbf{P} = (p(i, j))_{i, j \in [\ell]}$ be a symmetric $\ell \times \ell$ matrix with all entries in $[0, 1]$ and let $\mathbf{N} = (n_i)_{i \in [\ell]}$ be a vector of ℓ natural numbers. The stochastic block model $\mathcal{S}(\mathbf{P}, \mathbf{N})$ can be generated by starting with an empty graph on the set of nodes $[n] = \{1, 2, \dots, n\}$ with $n = \sum_{i \in [\ell]} n_i$ that is partitioned into ℓ subsets C_1, C_2, \dots, C_ℓ , called communities, with $|C_i| = n_i$, $i \in [\ell]$. For each pair of nodes $u, v \in [n]$ with $u < v$, $u \in C_i$, and $v \in C_j$, we independently introduce an edge uv in $\mathcal{S}(\mathbf{P}, \mathbf{N})$ with probability $p(i, j)$.

If $\ell = 1$, then we recover $\mathcal{G}(n, p)$.

Artificial Benchmark for Community Detection (**ABCD**)

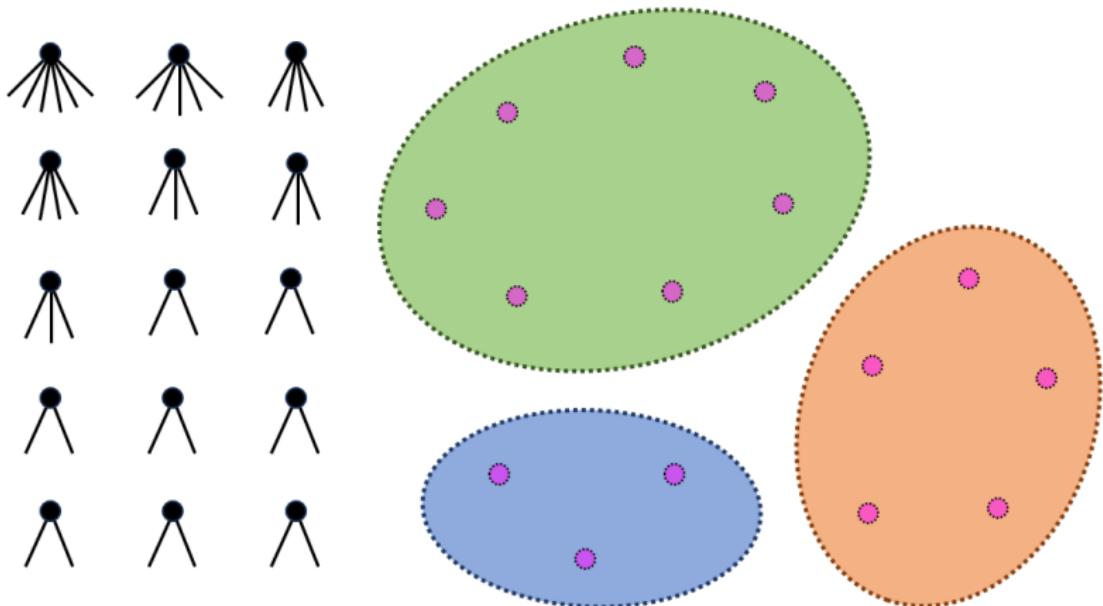
Here is an alternative.

ABCD model

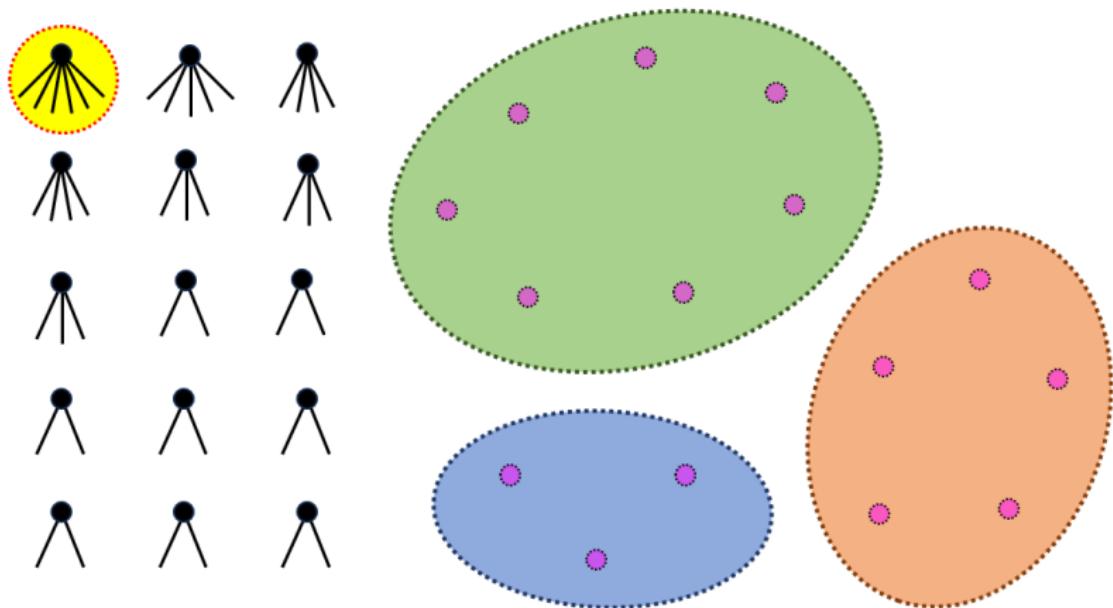
Let $\gamma \in [2, 3]$, $\tau \in [1, 2]$, and $\xi \in [0, 1]$ (ξ is called the **mixing parameter**). The **ABCD** model $\mathcal{A}(\gamma, \tau, \mu)$ generates random graphs with **community sizes** and **node degrees** following truncated **power law distributions** with **exponents** τ and, respectively, γ . The mixing parameter ξ controls the fraction of edges that are between communities.

The high level description is almost the same as for **LFR** but, of course, details differ—see the book.

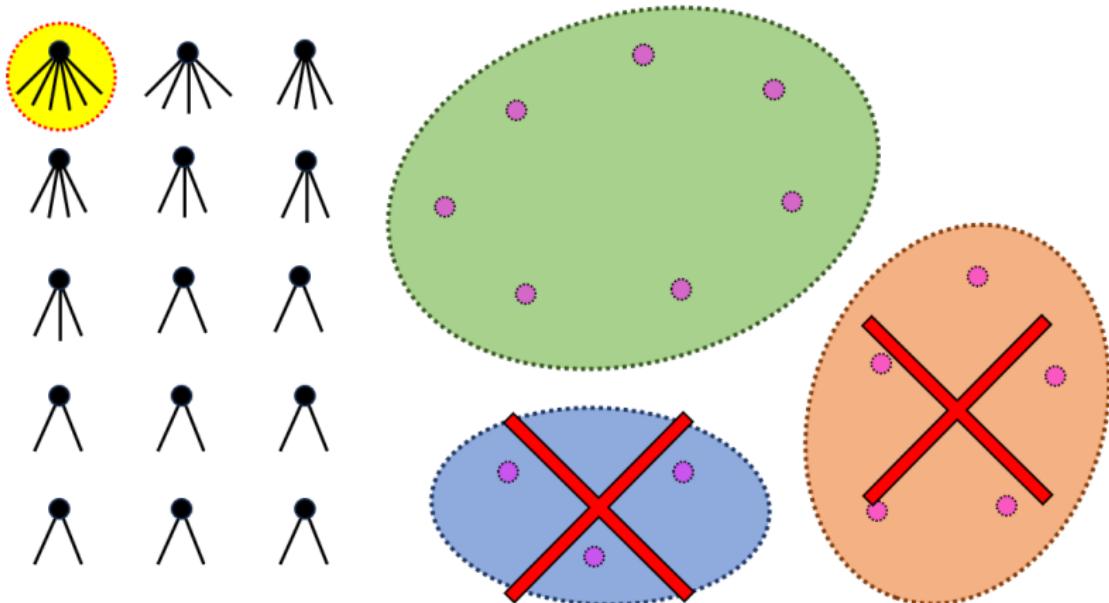
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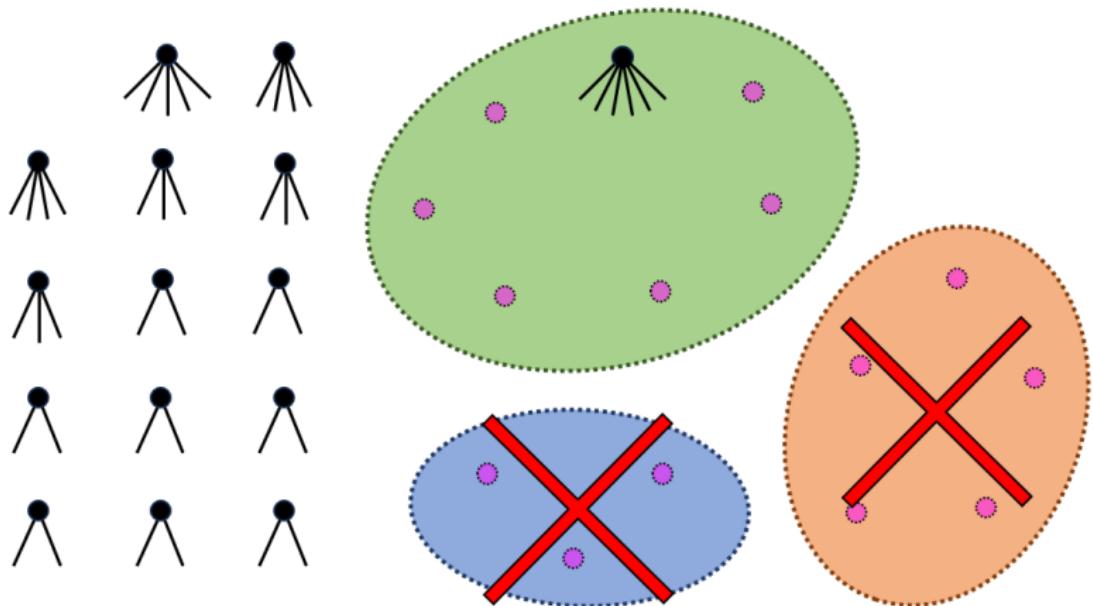
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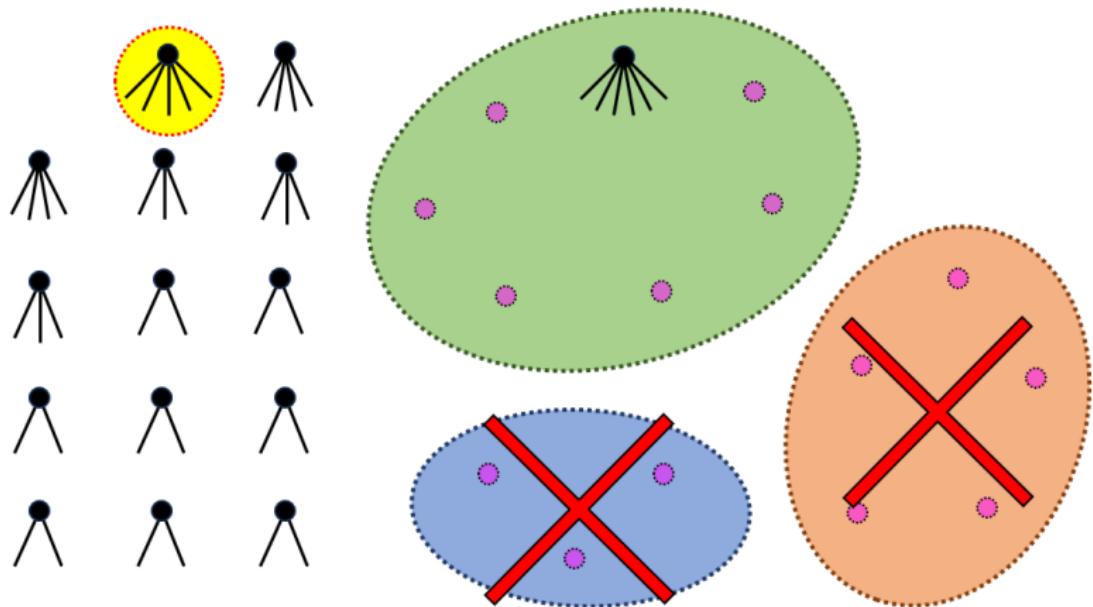
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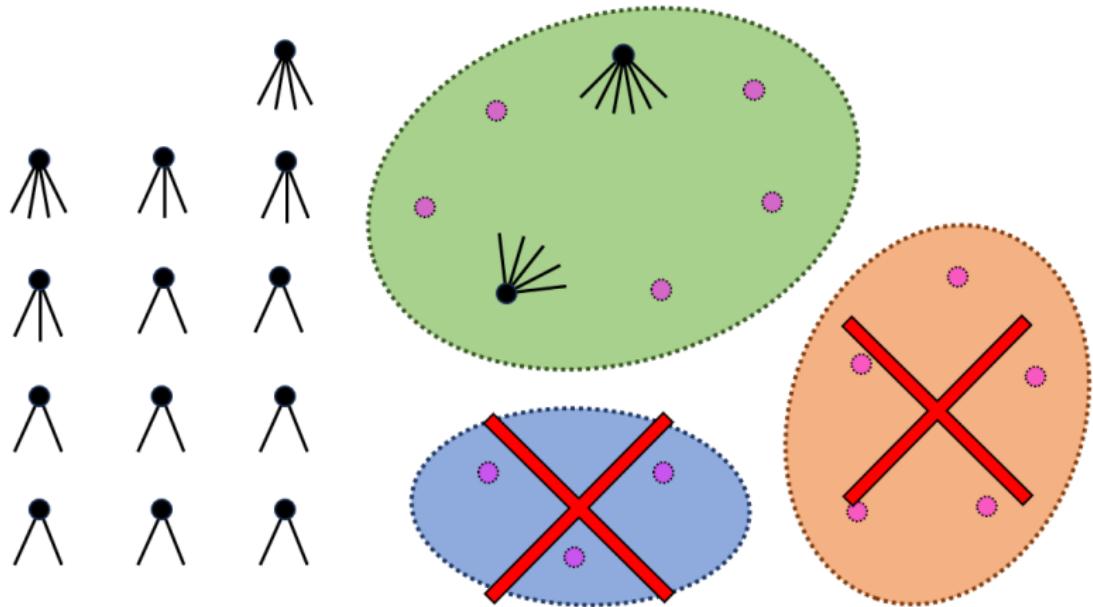
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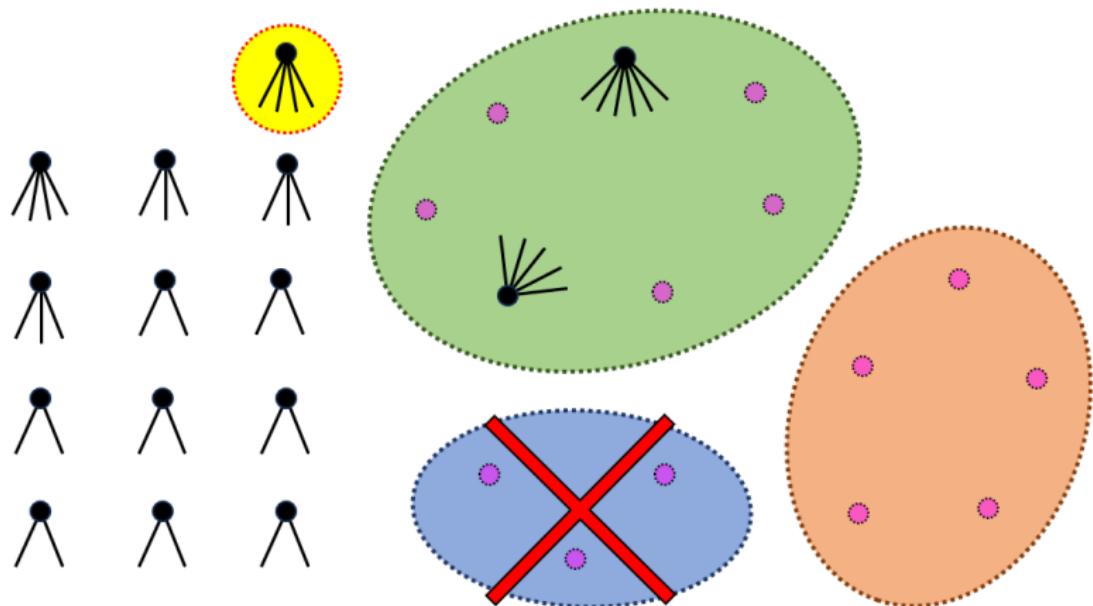
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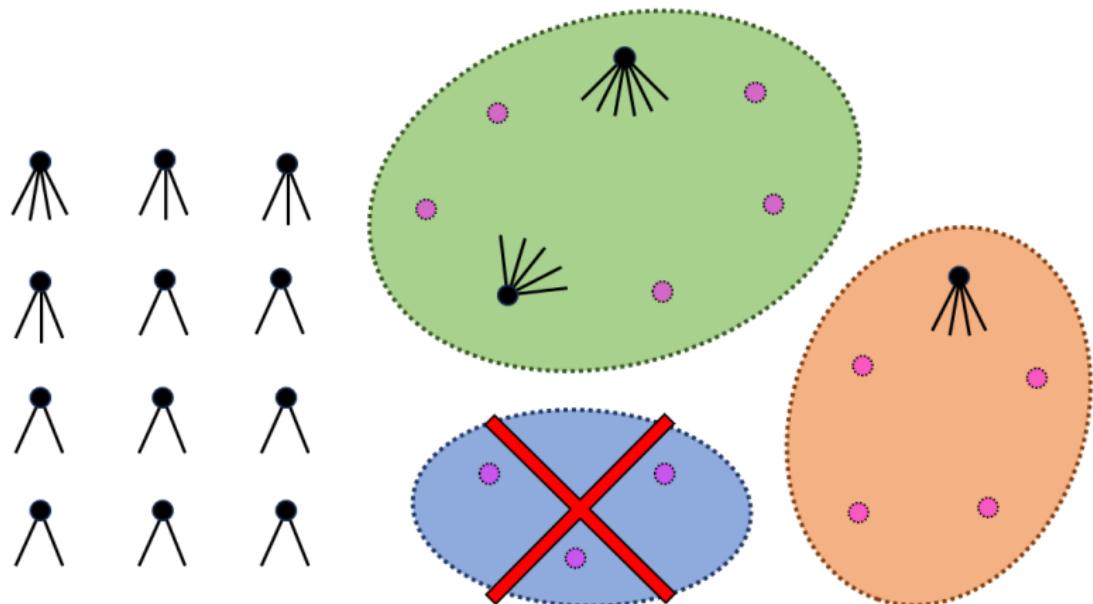
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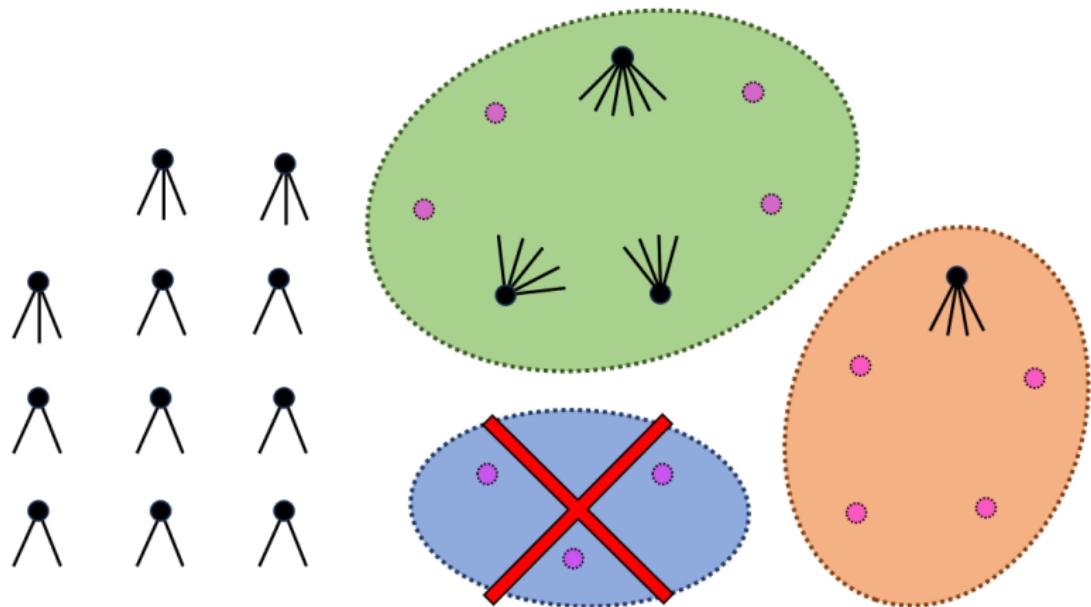
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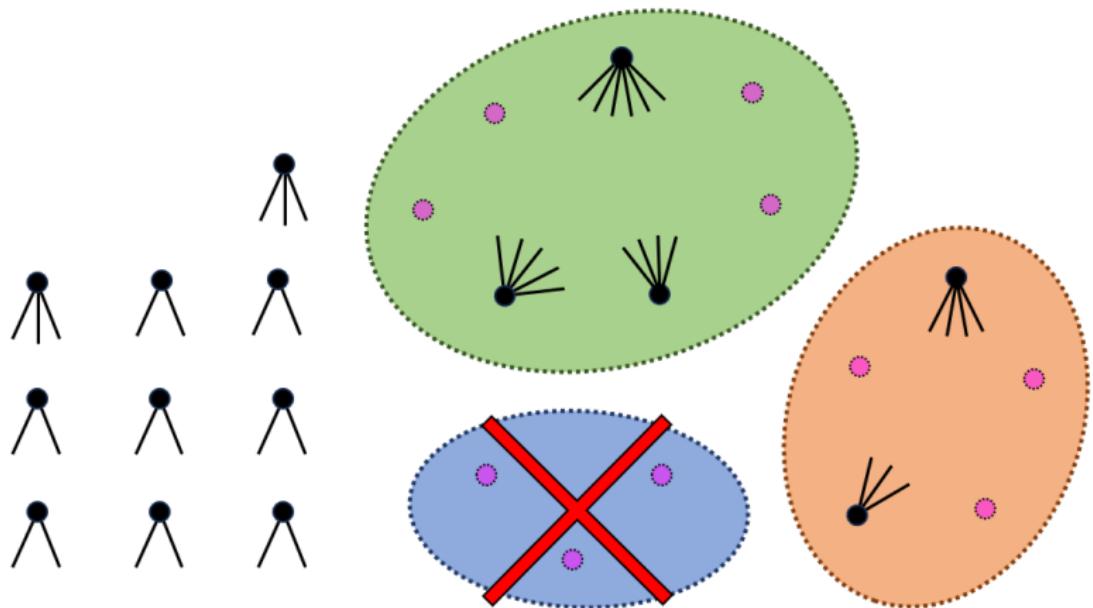
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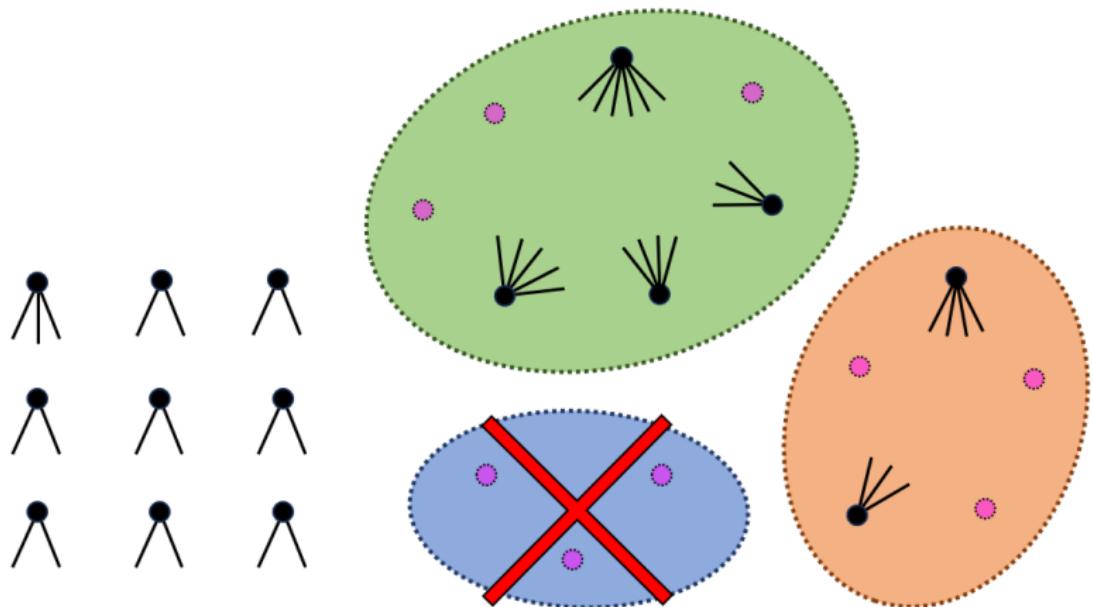
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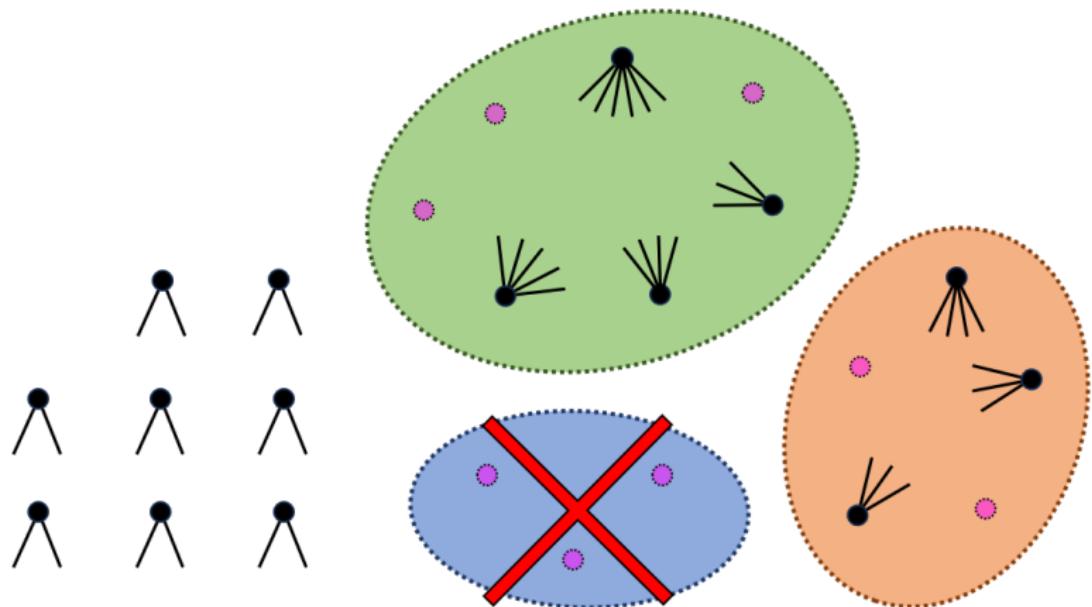
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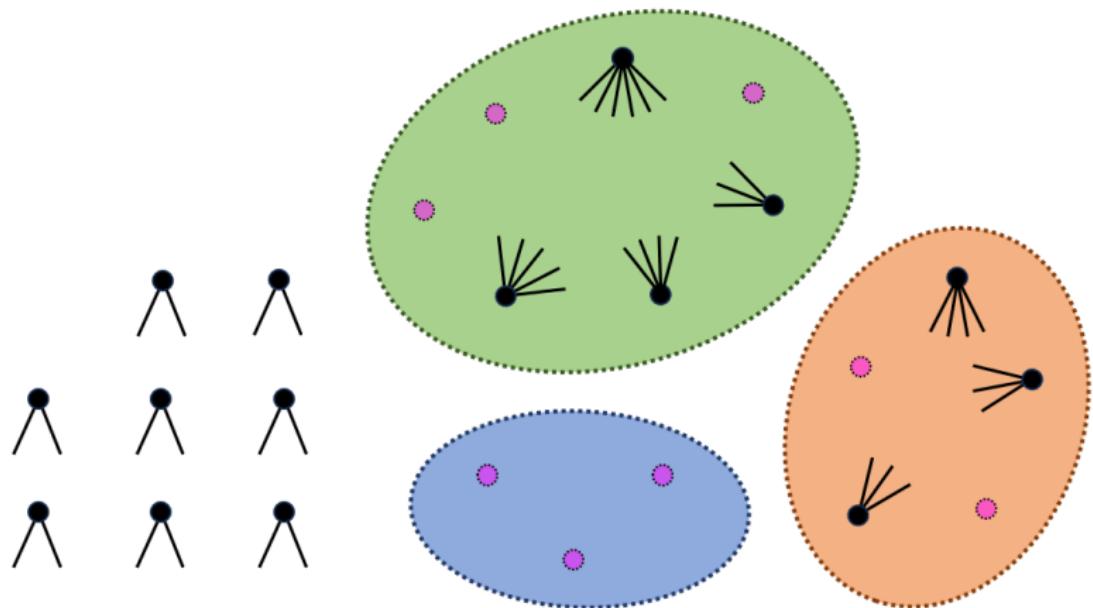
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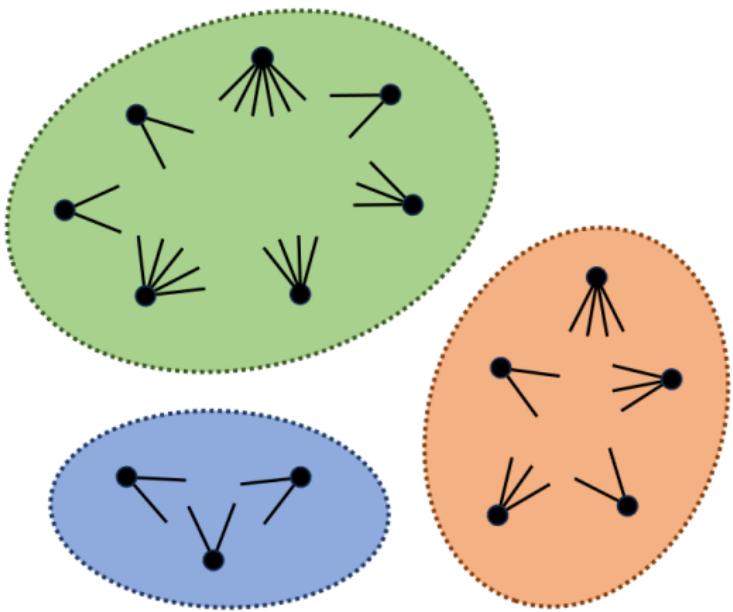
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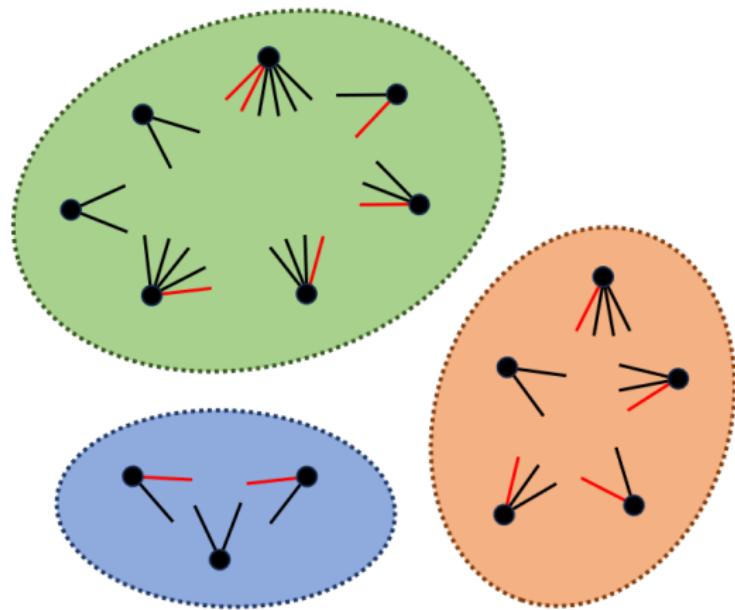


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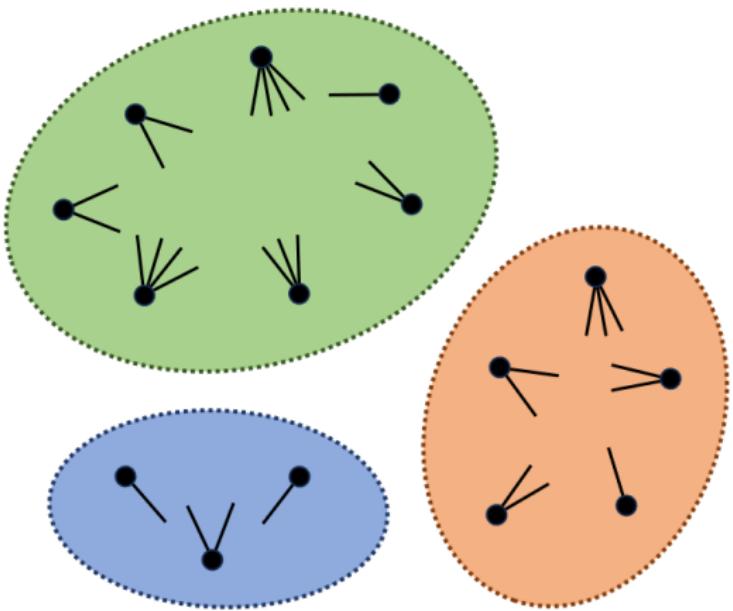


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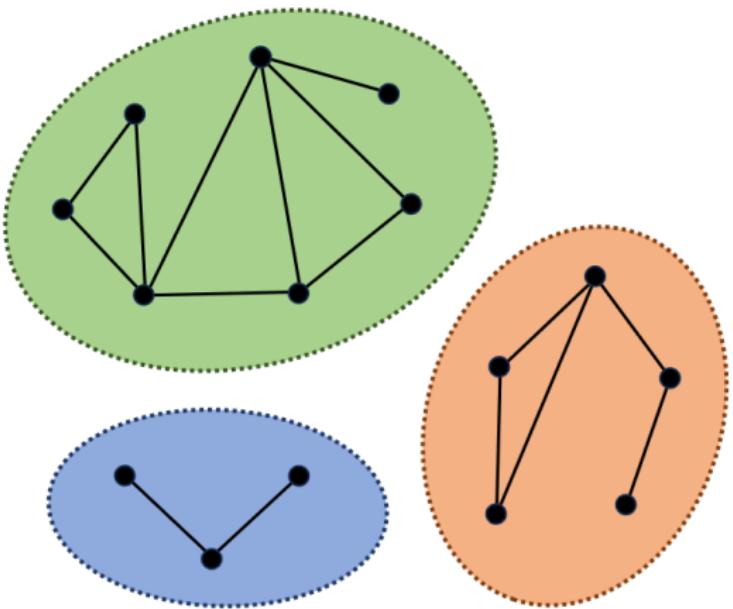
$$\xi = 0.25$$



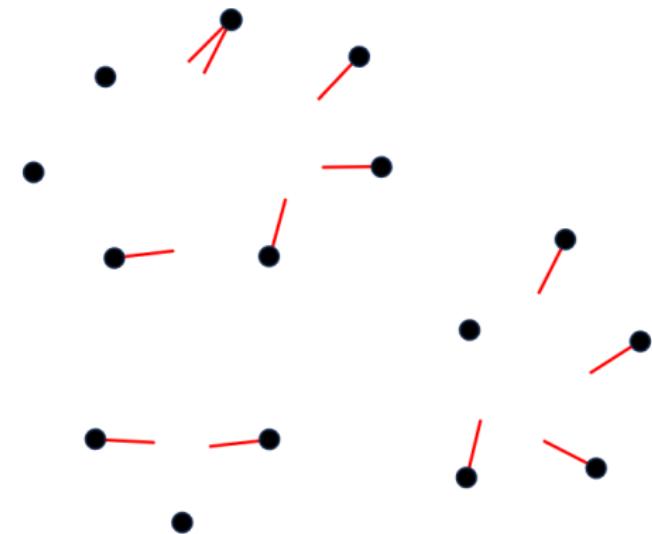
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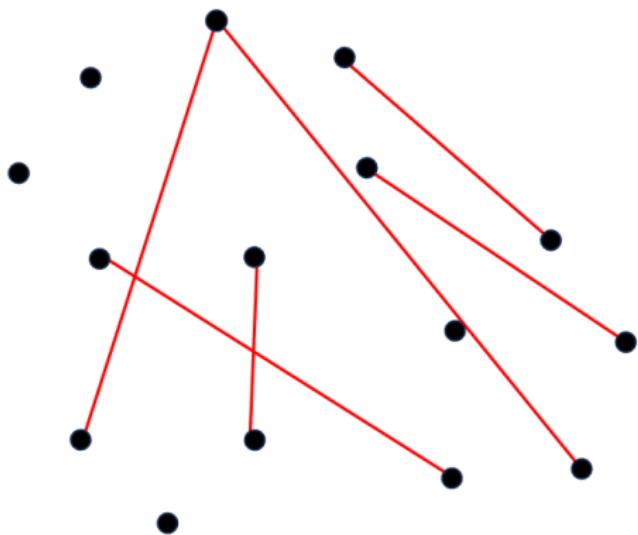
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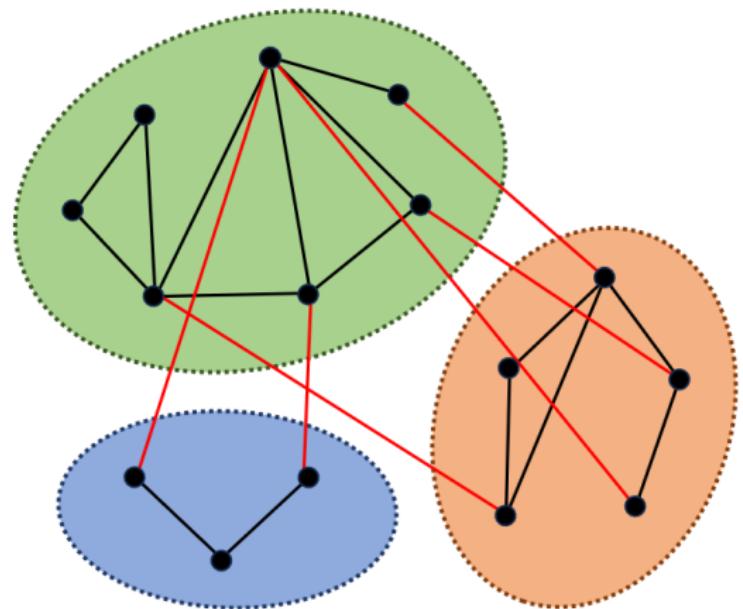
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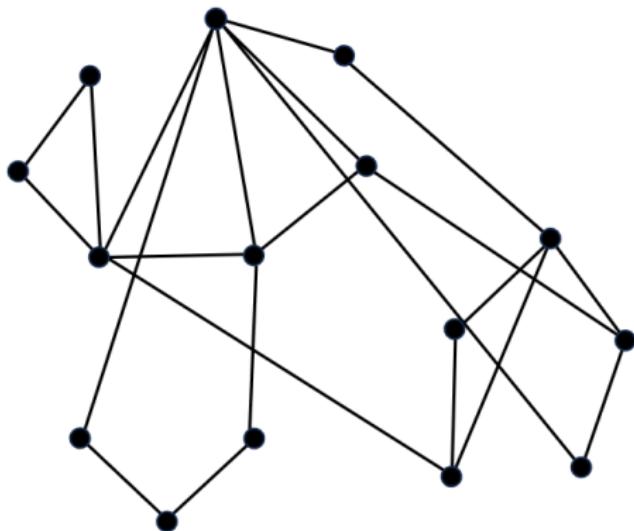
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The building blocks in the model are flexible.

- ◎ Outliers: **ABCD+o**
- ◎ Hypergraphs: **h-ABCD**
- ◎ Outliers + Overlapping: **ABCD+o²**
- ◎ Multilayered networks: **mABCD**
- ◎ Fast implementation that uses multiple threads: **ABCDe**

Graph Modularity

Definition

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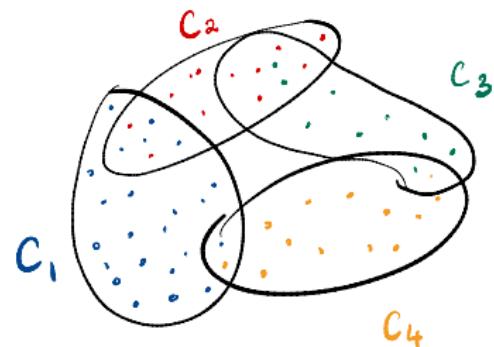
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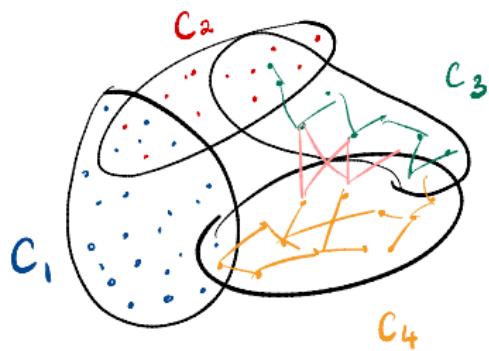
Can be easily generalized to **directed graphs**.

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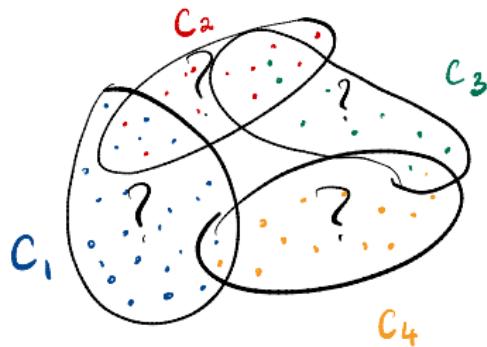
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No! Compare it to the expected fraction of edges captured by this partition in the Chung-Lu model with the (expected) degree distribution $\mathbf{d} = (\deg(v_1), \deg(v_2), \dots, \deg(v_n))$.

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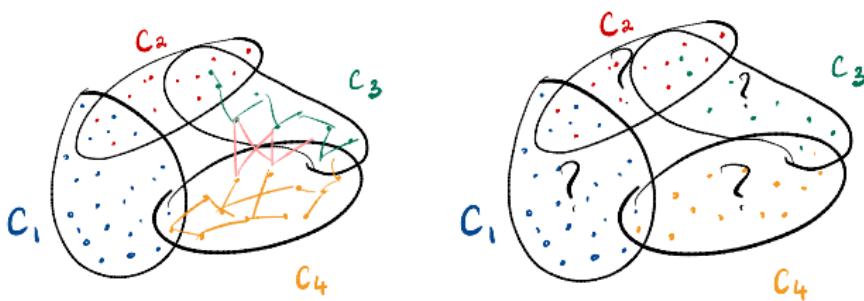
The **expected** fraction of edges is equal to

$$\begin{aligned} & \frac{1}{|E|} \left(\sum_{v_j v_k \in \binom{A_i}{2}} \frac{\deg(v_j) \deg(v_k)}{2|E|} + \sum_{v_j \in A_i} \frac{\deg^2(v_j)}{4|E|} \right) \\ &= \frac{1}{4|E|^2} \sum_{v_j \in A_i} \sum_{v_k \in A_i} \deg(v_j) \deg(v_k) \\ &= \frac{1}{4|E|^2} \left(\sum_{v_j \in A_i} \deg(v_j) \right)^2 = \frac{(\text{vol}(A_i))^2}{(\text{vol}(V))^2}. \end{aligned}$$

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Modularity function:

$$q_G(\mathcal{A}) = \sum_{A_i \in \mathcal{A}} \frac{e_G(A_i)}{|E|} - \sum_{A_i \in \mathcal{A}} \frac{(\text{vol}(A_i))^2}{(\text{vol}(V))^2}$$



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Some properties:

- $q_G(\mathcal{A}) \leq 1$,
- If $\mathcal{A} = \{V\}$, then $q_G(\mathcal{A}) = 0$,
- If $\mathcal{A} = \{\{v_1\}, \dots, \{v_n\}\}$, then $q_G(\mathcal{A}) = -\frac{\sum \deg^2(v)}{4|E|^2} < 0$,
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$$q^*(G) = \max_{\mathcal{A}} q_G(\mathcal{A})$$

- Well defined but impossible to find in practice unless G is small (then you may try **Bayan algorithm**).
- Used to guide a **heuristic** algorithms that try to maximize it.

Louvain algorithm

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Phase 1: modularity is locally optimized.

- consider nodes in a **random** order,
- for each node v , **move v** to community of **some of its neighbour** if it **improves** the modularity (can be easily and efficiently calculated!),
- if **no increase** is possible after considering all nodes, a **local maximum** value is achieved and the first phase ends.

Louvain algorithm

Phase 2: grouping nodes.

- contract nodes from each community into a single node (super-node),
- all edges within that community are replaced by a single weighted loop,
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Repeat the two phases until no improvement on the modularity function can be further achieved. The first pass is typically the most time consuming part of the algorithm.

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Solution: **Leiden** algorithm—periodically randomly breaking down communities into smaller well-connected ones.

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Solution: use **ensemble clustering** often referred to as **consensus clustering**.

Ensemble Clustering algorithm for Graphs (ECG)

Start with ℓ randomized level-1 partitions $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_\ell$ obtained by ℓ independent runs of the first phase of the Louvain algorithm and only once.

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Assign new weights to edges as follows: for each $uv \in E$,

$$w(uv) = \begin{cases} w_* + (1 - w_*) \cdot \frac{1}{\ell} \sum_{i=1}^{\ell} \alpha_i(u, v), & \text{if } uv \text{ is in the 2-core} \\ w_* & \text{otherwise,} \end{cases}$$

w_* $\in [0, 1]$ is the parameter of the algorithm, and $\alpha_i(u, v) = 1$ if u and v appear together in some part of \mathcal{P}_i ; otherwise, $\alpha_i(u, v) = 0$. (Default values are $\ell = 16$ and $w_* = 0.05$.)

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Perform Louvain (till the very end, *not* just level-1) on a weighted version of the initial graph.

Experiments

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Jupyter notebooks and datasets that accompany the textbook/slides are available on GitHub:

<https://github.com/ftheberge/GraphMiningNotebooks>

[Python_Notebooks_Second_Edition](#)

[Julia_Notebooks_2nd](#)



Tutorials for Python notebooks can be found on YouTube:

<https://www.youtube.com/@MiningComplexNetworks>

THE
END