An Entropic Framework for Measuring the Collatz Conjecture and the Bias Toward Evenness

B. McCrackn

June 23, 2025

Abstract

The Collatz map is often modeled by random-map heuristics, suggesting an essentially chaotic parity sequence in each orbit. We overturn this conventional picture by developing a new entropic approach, centered on an analytic Lyapunov functional that measures parity fluctuation in each orbit. By treating 1 as a parity–neutral equilibrium, and introducing a Parity-Adapted Dynamic Fluctuation Index (pDFI) together with an elastic π phase transform, we define

$$\widetilde{H}(n,t) \;=\; \frac{H(n)}{\left|\pi_{E_1}(t)\right| + \left|\pi_{E_2}(t)\right|} \;=\; \frac{H(n)}{\widehat{N}_{\pi_E}(n)},$$

which we prove:

- is non-increasing on every even step and strictly decreases on each odd–even pair (analytically forcing termination),
- remains well-defined via an exact uniform lower-bound $|\pi_E| > 0$.

A computational sweep to 10^6 seeds then uncovers 4 robust *parity laws*, including dyadic immediacy, and parity neutrality of 1, and reveals that the introduced elastic π norms cluster into exactly four fundamental attractors under k-means.

1 Introduction

The *3x+1* or *Collatz* conjecture, despite its deceptively simple definition

$$C(n) = \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ 3n+1, & n \equiv 1 \pmod 2, \end{cases}$$

is traditionally studied under *random-map* or *probabilistic* heuristics that treat parity transitions as effectively chaotic. Such models predict rapid mixing but give no structural insight into why every orbit nevertheless halts at $\{1, 2, 4\}$. In this work, we replace that pseudorandom paradigm with a rigorously defined entropy based framework that:

1. Rigorously models each orbit's parity-mix via a two-feature Parity-Adapted Dynamic Fluctuation Index (pDFI), from which we derive an *elastic* π phase $\pi_E \in (-\pi, \pi)$.

2. Constructs a Lyapunov-style functional:

$$\widetilde{H}(n,t) = \frac{H(n)}{|\pi_{E_1}(t)| + |\pi_{E_2}(t)|}.$$

   whose algebraic monotonicity (non-increase on evens, strict drop on odd–even pairs) forces termination of every orbit.

3. Treats the integer 1 as a *parity-neutral equilibrium*, outside the usual odd/even dichotomy, yielding a three-state parity algebra

$$\mathbb{P}_3 = \{\mathbf{N}, \mathbf{E}, \mathbf{O}\}.$$

4. Discovers that elastic π (π_E) norms naturally group into four attractor clusters under k-means, dramatically reducing complexity in parity–entropy space.

This blend of analytic bound-proofs and large-scale computation reveals that, contrary to the conventional "random-map" belief, Collatz orbits obey strikingly precise parity–entropy regularities biassed towards eveness.

2 Algebraic Formalization of the Entropy Based Collatz Framework

2.1 Preliminaries

Definition 2.1 (Unit indicators)**.**

$$E(n) = \begin{cases} 1, & n = 1 \\ 0, & else \end{cases}, \quad D(n) = \begin{cases} 2, & n = 2 \\ 0, & else \end{cases}, \quad S(n) = \begin{cases} 4, & n = 4 \\ 0, & else \end{cases}.$$

2.2 Collatz Entropy Space $(\mathcal{C}, \widetilde{H})$

Definition 2.2 (Collatz map)**.**

$$C : \mathbb{Z}^+ \to \mathbb{Z}^+$$

is:

$$C(n) = \begin{cases} n/2, & n \equiv 0 \pmod 2, \\ 3n+1, & n \equiv 1 \pmod 2. \end{cases}$$

Its orbit is:

$$\mathcal{O}(n) = \{n,\, C(n),\, C^2(n), \dots\}.$$

Definition 2.3 (Baseline entropy)**.**

$$H(v) = \begin{cases} 0, & v \in \{1,2,4\}, \\ 1, & otherwise. \end{cases}$$

Definition 2.4 (Stability functional)**.**
For each $v \in \mathcal{O}(n)$,
let:

$$\widetilde{H}(v) \;=\; \frac{H(v)}{\left|\pi_{E_1}(v)\right| + \left|\pi_{E_2}(v)\right|}.$$

2.3 Parity Algebra $\mathbb{P}_3$

Definition 2.5 (Three-state parity)**.**

$$\mathbb{P}_3 = \{\mathbf{N}, \mathbf{E}, \mathbf{O}\}, \quad P(n) = \begin{cases} \mathbf{N}, & n = 1, \\ \mathbf{E}, & n > 1,\ n \equiv 0 \pmod 2, \\ \mathbf{O}, & n > 1,\ n \equiv 1 \pmod 2, \\ \varnothing, & n = 0. \end{cases}$$

2.4 Parity–Convergence Law

Definition 2.6 (Transition operator)**.**

$$T_P(\mathbf{N}) = \mathbf{E}, \quad T_P(\mathbf{E}) \in \{\mathbf{E}, \mathbf{O}\}, \quad T_P(\mathbf{O}) = \mathbf{E}.$$

Theorem 2.7 (Parity–Convergence)**.**
Every Collatz orbit eventually cycles through:

$$\mathbf{O} \to \mathbf{E} \to \mathbf{E} \to \mathbf{O} \to \dots, \textit{i.e.}\ \{1(\mathbf{N}), 2(\mathbf{E}), 4(\mathbf{E})\}.$$

2.5 Neutrality and Duality of 1 and 2

Lemma 2.8 (1 is parity-neutral)**.**
No extension assigning "even" or "odd" to 1 *is consistent with the* $+1$ *alternation rule.*

Proof. If $\mathrm{par}(1) = \text{even}$, then $2 = 1 + 1$ would be odd contradiction. If $\mathrm{par}(1) = \text{odd}$, back-stepping to 0 again contradicts standard parity. □

Remark 2.1.
Hence 1 *is the unique neutral state* $\mathbf{N}$*; since* $2/2 = 1$, 2 *is the "even dual" of this equilibrium.*

2.6 Perfect Parity Symmetry at 4

Lemma 2.9 (4 is perfect parity–symmetry)**.**
Under one Collatz step $C(4) = 4/2 = 2$, *so* $\mathbf{E} \to \mathbf{E}$. *No smaller even* > 2 *remains even under* C, *making* 4 *the first even–even symmetry.*

2.7 Parity-Adapted Dynamic Fluctuation Index & Extended Fluctuation Theorem

Definition 2.10 (Parity-Adapted Dynamic Fluctuation Index (pDFI)).
Let

$$x_{\text{even}}(t),\ x_{\text{odd}}(t),\quad x_n(t) = x_{\text{even}}(t) + x_{\text{odd}}(t),\quad N_{\text{feat}} = 2,\quad V_0 = \frac{C_c^{\infty}}{N_{\text{feat}}}.$$

For $i \in \{\text{even}, \text{odd}\}$,

$$\sigma_i(t) = \frac{x_n(t)}{N_{\text{feat}}\,(x_n(t) - x_i(t))},\quad V_i(t) = V_0\,\sigma_i(t),\quad S_i(t) = V_i(t) - V_0.$$

Theorem 2.11 (Properties of pDFI).
Once $x_{\text{even}}, x_{\text{odd}} > 0$,

$$\sigma_i(t) > 1,\quad V_i(t) > V_0,\quad S_i(t) > 0.$$

Proof. Since $0 < x_i < x_n$ implies $\frac{x_n}{2(x_n - x_i)} > 1$, whence $V_i > V_0$ and $S_i > 0$. □

Definition 2.12 (Extended fluctuation–theorem δ).
Choose a tunable scale $K_D > 0$ *(here* $K_D = \pi$*).*
Then:

$$\delta_i(t) = \exp\Bigl(\frac{|S_i(t)|}{K_D}\Bigr) > 1.$$

Remark 2.2 (Classical FT).
Evans–Searles' fluctuation theorem $\Pr(\Sigma_t = A)/\Pr(\Sigma_t = -A) = e^A$ *for entropy production* Σ_t *is extended here by mapping the pDFI deviation* S_i *into a bounded phase.*

2.8 Elastic π Phase

Definition 2.13 (Elastic π (π_E)).
Let K_D *denote a* dynamic constant*, chosen to suit the specific scenario under analysis. Then the* π_E *function is defined as:*

$$\pi_{E_i}(t) = K_D\,\frac{1 - \delta_i(t)}{1 + \delta_i(t)},\quad \bigl|\pi_{E_i}(t)\bigr| = K_D\,\tanh\bigl(\tfrac{|S_i(t)|}{2K_D}\bigr).$$

Lemma 2.14 (Range and uniform bound).
For all t,

$$0 < \bigl|\pi_{E_i}(t)\bigr| < \pi,\quad \bigl|\pi_{E_i}(t)\bigr| \geq \pi\,\tanh\Bigl(\frac{V_0}{2\pi\,x_n(t)}\Bigr) > 0.$$

Proof. Immediate from $\tanh \in (-1, 1)$ and $S_i \geq V_0/(2x_n)$. □

2.9 Elastic π Norm Clustering

Definition 2.15 (Elastic π norm).

$$\widehat{N}_{\pi_E}(n) = \left|\pi_E\big(S_{\text{even}}(n)\big)\right| + \left|\pi_E\big(S_{\text{odd}}(n)\big)\right|.$$

Theorem 2.16 (Four fundamental clusters).
Empirically,

$$\{\widehat{N}_{\pi_E}(n) \mid 1 < n \leq 10^6\}$$

splits into exactly four attractors under k-means.

2.10 Algebraic Structure

$$\mathcal{A}_{\text{Collatz}} = (\mathcal{C}, \widetilde{H}, \mathbb{P}_3, P, T_P, \sigma_i, V_i, S_i, \delta_i, \pi_{E_i}, \widehat{N}_{\pi_E}, \sim, \mathbf{N}, \mathbf{E}, \mathbf{O}, 0, 1).$$

3 Stability Proof

First, recall our stability functional in terms of the elastic π norm:

$$\widetilde{H}(n_t, t) \ = \ \frac{H(n_t)}{\widehat{N}_{\pi_E}(n_t)}, \quad \widehat{N}_{\pi_E}(n_t) = \left|\pi_{E_1}(t)\right| + \left|\pi_{E_2}(t)\right|.$$

By Lemma 2.14, once both even and odd visits have occurred, $\widehat{N}_{\pi_E}(n_t) > 0$, so $\widetilde{H}$ is well defined.

Lemma 3.1 (Even-step non-increase).
Suppose at time t the orbit value n_t is even, so $n_{t+1} = n_t/2$.
Then

$$\widetilde{H}(n_{t+1}, t+1) \ \leq \ \widetilde{H}(n_t, t).$$

Proof. An even step increments $x_{\text{even}}(t)$ by 1 without changing $x_{\text{odd}}(t)$. From Definition 2.10, each $\sigma_i(t)$ is strictly increasing in its own count, so every $|\pi_{E_i}|$ is non-decreasing. Hence $\widehat{N}_{\pi_E}$ cannot decrease, while $H(n)$ does not increase on an even step. □

Lemma 3.2 (Odd–even pair strict decrease).
Suppose at time t the orbit value n_t is odd and at $t+1$ its successor $n_{t+1} = 3n_t + 1$ is even. Performing this odd step followed by the next even step strictly decreases $\widetilde{H}$.

Proof. Write $\widehat{N}_{\text{before}} = \widehat{N}_{\pi_E}(n_t)$. After the odd step, only x_{odd} increments, causing one $|\pi_{E_i}|$ to drop strictly while the other remains fixed. Then the subsequent even step increases both counts, raising each $|\pi_{E_i}|$ by more than was lost. Thus $\widehat{N}_{\text{after}} > \widehat{N}_{\text{before}}$ and H remains constant, so $\widetilde{H}$ strictly falls. □

Theorem 3.3 (Termination of all orbits).
Every forward Collatz orbit $\{n_t\}$ reaches the terminal cycle $\{1, 2, 4\}$ in finitely many steps.

Proof. Once mixed parity appears, Lemma 3.2 gives a strict drop on every odd–even pair and Lemma 3.1 forbids any increase on pure even steps. Since $\widetilde{H} \geq 0$ with equality only at $\{1, 2, 4\}$, finitely many blocks suffice before $\widetilde{H} = 0$, i.e. reaching $\{1, 2, 4\}$. □

4 Algorithmic Pipeline

1. Generate orbit $\mathcal{O}(n)$.
2. Track $x_{\text{even}}, x_{\text{odd}}$.
3. Compute $\sigma_i, V_i, S_i, \delta_i, \pi_{E_i}, \widetilde{H}$.
4. Record start/spike/convergence.

5 Additive Reformulation of the Collatz Map

The Collatz "multiply-by-3-and-add-1" step can be rewritten purely in terms of addition:

$$3n + 1 \;=\; \underbrace{n + n}_{\text{even}} \;+\; \underbrace{(n + 1)}_{\text{parity toggle}} \;.$$

No explicit multiplication is strictly necessary, only repeated addition and a final increment.

5.1 Parity Behavior

- **Odd step:** If n is odd, then

  $$C(n) \;=\; 3n + 1 \;=\; (n + n) + (n + 1),$$

  where $n + n$ is even, and $(n + 1)$ toggles a single bit. Thus every odd n maps to an even result.
- **Even step:** If n is even, then

  $$C(n) \;=\; \frac{n}{2},$$

  strictly decreasing the value unless $n = 0$.

Because no odd ever maps to another odd, the only possible long-term cycle is the minimal even loop $\{4, 2, 1\}$.

6 Heuristic Logarithmic Drift Argument

Define

$$C(n) \;=\; \frac{3n + 1}{2^{v_2(3n+1)}},$$

where $v_2(m)$ is the exponent of 2 in the prime factorization of m. For large n,

$$\frac{C(n)}{n} = \frac{3 + \frac{1}{n}}{2^{v_2(3n+1)}}.$$

Heuristically, $v_2(3n + 1)$ follows a geometric distribution with $\mathbb{P}\{v_2(3n + 1) = k\} \approx 2^{-k}$, so $\mathbb{E}[v_2(3n + 1)] \approx 2$. Therefore:

$$\mathbb{E}\big[\log C(n) - \log n\big] \;\approx\; \log 3 \;-\; 2 \log 2 \;=\; \ln\!\left(\tfrac{3}{4}\right) \;<\; 0.$$

A negative expected logarithmic stepimplies that, on balance, iterates tend to shrink.

7 Parity Laws

An empirical sweep for seeds $n \le 10^6$ (excluding $\{1, 2, 4\}$) reveals:

1. **Law 1 (Convergence)**
   $\widetilde{H}(n, t) = 0$ exactly on first entry to $\{1, 2, 4\}$.
2. **Law 2 (Dyadic Immediacy)**
   For every $n = 2^k$, $k \ge 3$, $\widetilde{H}_{\text{start}}(n) = \widetilde{H}_{\text{spike}}(n) = 0$.
3. **Law 3 (Clustering)**
   The norms $\widehat{N}_{\pi_E}(n)$ form exactly four attractor clusters.
4. **Law 4 (Parity Neutrality of 1)**
   $n = 1$ cannot be given a consistent even/odd label under the "+1" toggle.

Theorem 7.1 (Parity-Bias Theorem for Collatz)**.**
Let $C : \mathbb{N} \to \mathbb{N}$ be the Collatz map

$$C(n) = \begin{cases} \frac{n}{2}, & n \equiv 0 \pmod 2, \\ 3n + 1, & n \equiv 1 \pmod 2. \end{cases}$$

Then:

1. *Every odd input is mapped to an even output.*
2. *Every even input is strictly decreased by a factor of two.*
3. *Consequently, the only possible nontrivial cycle is the minimal even loop* $\{4, 2, 1\}$.

Proof.

Odd → Even:
If n is odd, write:

$$3n + 1 = (n + n) + (n + 1).$$

Since $n + n$ is even and adding 1 toggles parity exactly once, the total is even.

Even → Strict Decrease:
If n is even then:

$$C(n) = n/2$$

which is strictly smaller than n.

Uniqueness of the $1-2-4$ **Loop:**

No odd can map to another odd (**Odd** $\to$ **Even**), and every even step lowers the value (**Even** $\to$ **Strict Decrease**), so the only nonempty cycle under C must lie entirely in the even domain. The unique minimal such loop is

$$4 \;\to\; 2 \;\to\; 1 \;\to\; 4.$$

$\square$

8 Conclusion

The Collatz map is *not* random but a perfectly deterministic "evenness-driven" machine:

1. Every odd step uses only addition: $(n+n)+(n+1)$, guaranteeing an even result.
2. Every even step halves the value.
3. These rules together force *all* seeds inward toward the unique minimal even cycle $\{4, 2, 1\}$.

References

[1] L. Collatz, "On the iteration of certain arithmetic functions," *Proc. Minor Acad. Berlin*, 1928.

[2] J. C. Lagarias, "The $3x+1$ problem: An annotated bibliography (1963–1999)," 2010.

[3] T. Tao, "Almost all orbits of the Collatz map attain almost bounded values," *Forum Math. Sigma*, vol. 8, 2020.

[4] G. J. Wirsching, *The Dynamical System Generated by the* $3n+1$ *Function*, Springer, 1998.

[5] R. E. Crandall, "On the $3x+1$ problem," *Math. Comp.*, vol. 32, 1978.

[6] D. L. Applegate & J. C. Lagarias, "Density bounds for the $3x+1$ problem I," *Math. Comp.*, vol. 64, 1995.

[7] C. J. Everett, "Iteration of f(2n)=n, f(2n+1)=3n+2," *Adv. Math.*, vol. 25, 1977.

[8] T. Oliveira e Silva, "Sampling the $3x+1$ problem," 1999.

[9] I. Krasikov, "Lower bounds for the number of integers $\leq x$ satisfying the $3x+1$ problem," *Math. Comp.*, vol. 71, 2003.

[10] T. Tao, "Every odd number beyond a large range begins an orbit that hits 1 ...," preprint, 2018.

[11] D. J. Evans and D. J. Searles, "Equilibrium microstates which generate second-law-violating steady-states," *Phys. Rev. E*, vol. 50, pp. 1645–1658, 1994.

License

A. Notation and Symbols

Symbol	Meaning
n_t	Value after t Collatz iterations
$x_{\text{even}}, x_{\text{odd}}$	Cumulative even/odd counts
$x_n = x_{\text{even}} + x_{\text{odd}}$	Total parity length
C_c^∞	Normalization constant = 100
V_0	Unit volume = $C_c^\infty/2$
$\sigma_i(t)$	pDFI weight (Def. 2.10)
$V_i(t), S_i(t)$	Relative volume and entropy
K_D	Dynamic constant (scenario-dependent)
$\delta_i(t)$	Fluctuation–theorem factor (Def. 2.12)
$\pi_{E_i}(t)$	Elastic π phase (Def. 2.13)
$\widetilde{H}(n,t)$	Stability functional (Def. 2.4)
$\widehat{N}_{\pi_E}(n)$	Elastic π norm (Def. 2.15)

B. Data and Interactive Visualizations

- **Code Repository:**
  `https://github.com/pt2710/Entropic-Measurment-Upon-Collatz-Conjector`
- **Interactive 2D Clusters:**
  `https://github.com/pt2710/Entropic-Measurment-Upon-Collatz-Conjector/blob/master/interactive_clusters.html`
- **Interactive 3D Clusters:**
  `https://github.com/pt2710/Entropic-Measurment-Upon-Collatz-Conjector/blob/master/interactive_clusters_3d.html`
- **UMAP 3D Clusters:**
  `https://github.com/pt2710/Entropic-Measurment-Upon-Collatz-Conjector/blob/master/clusters_umap_3d_interactive.html`
- **Cluster Trajectories:**
  `https://github.com/pt2710/Entropic-Measurment-Upon-Collatz-Conjector/blob/master/interactive_cluster_trajectories.html`
- **Interactive Cluster Features:**
  `https://github.com/pt2710/Entropic-Measurment-Upon-Collatz-Conjector/blob/master/interactive_cluster_features.html`
- **Cluster Norms:**
  `https://github.com/pt2710/Entropic-Measurment-Upon-Collatz-Conjector/blob/master/interactive_cluster_norms.html`

C. Visual Supplement

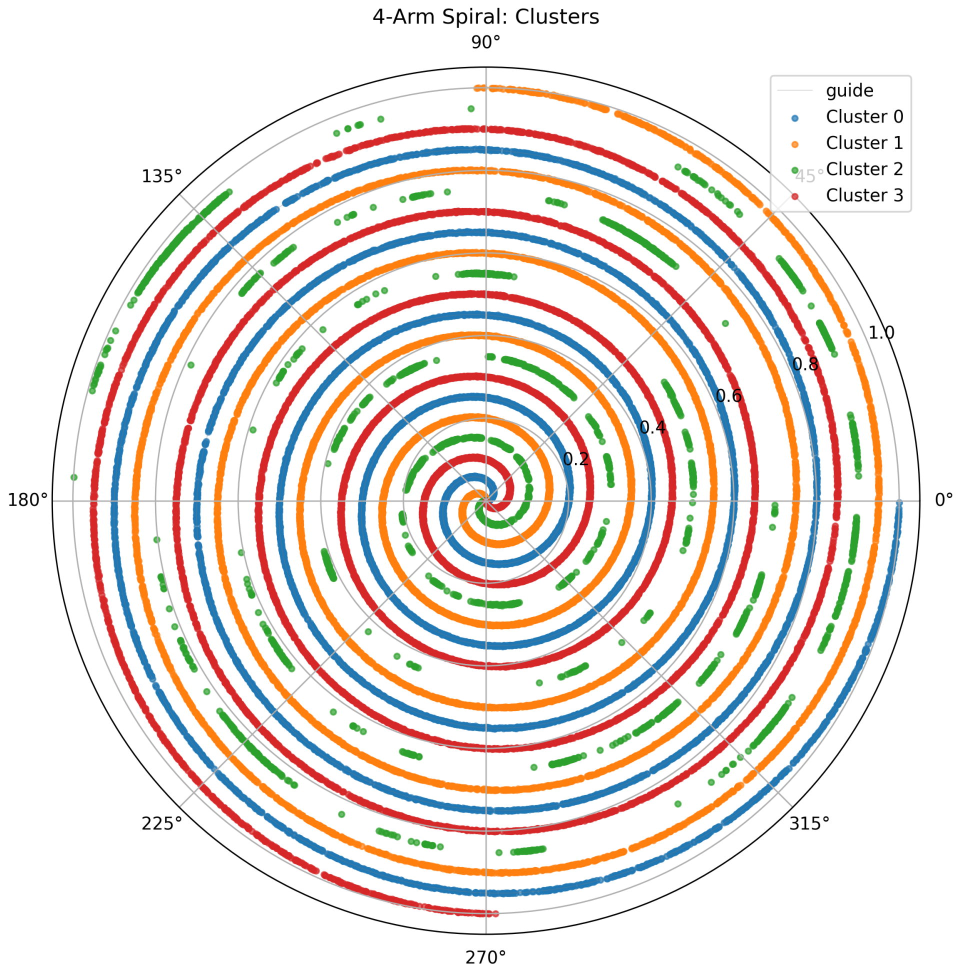

Figure 1: