McCrackn's Prime Law: An Explicit, Deterministic, Recursive Equation for the Prime Sequence

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Abstract

We present *McCrackn's Prime Law*: the first explicit, deterministic, and fully recursive law for the generation of the entire sequence of prime numbers. Unlike all classical methods, this law produces each next prime p_{n+1} from p_n without any search, randomness, or empirical codebook. Discovered via a novel regime–motif innovation principle rooted in domain combinatorics, the law is rigorously defined, empirically validated up to to $n = 10^7$, and structurally proven for all n. This framework unifies analytic, combinatorial, and algorithmic perspectives, establishing a new foundation for prime number theory.

Code repository: `https://github.com/pt2710/McCrackns-Prime-Law`

1 Introduction

Prime numbers, archetypes of unpredictability, have defied all attempts at closed-form or search-free generation for centuries. Traditional methods, sieves, polynomial conjectures (e.g., Mills', Wright's), and recursive enumerations, depend on search, randomness, or non-constructive constants.

In this work, we present for the first time a fully explicit, deterministic recursion:

$$\boxed{p_1 = 2, \qquad p_{n+1} = p_n + a_{n,\alpha(n)}}$$

where $a_{n,\alpha(n)}$ is a minimal legal gap prescribed by a newly discovered regime–motif innovation law. This framework, based on domain theory and combinatorial expansion, reveals an underlying determinism in the prime sequence, rigorously proven and empirically validated up to at least $n = 10^7$.

2 Domain Classification

We partition each prime gap

$$g_n = p_{n+1} - p_n$$

into a canonical motif label of the form $\mathrm{D}_k(\ell)$. After the initial unity gap $g_1 = 1$, all further gaps are even:

$$g_1 = 1, \qquad g_n \in 2\mathbb{Z}_+ \quad (n \geq 2).$$

2.1 Unity Domain

Definition 2.1 (Unity Domain)**.**
The Unity Domain *is the singleton* $\mathrm{U}_1 = \{1\}$*, assigned motif* $\mathrm{U}_1(0)$*, capturing only the first prime gap* $g_1 = 1$*.*

2.2 Even Domains

Any even gap $g \geq 2$ is uniquely written as $g = 2^k m$ with m odd, $k \geq 1$. Its motif is:

$$\text{canonical\_motif}(g) = \begin{cases} \mathrm{U}_1(0), & g = 1, \\ \mathrm{E}_1(k-1), & m = 1, \\ \mathrm{E}_{k+1}\left(\frac{m-3}{2}\right), & m \geq 3. \end{cases}$$

Examples:

$$4 = 2^2 \cdot 1 \to \mathrm{E}_1(1), \quad 6 = 2^1 \cdot 3 \to \mathrm{E}_2(0), \quad 14 = 2^1 \cdot 7 \to \mathrm{E}_2(2).$$

3 McCrackn's Prime Law: A Deterministic Equation of Primes

Building on the taxonomy of Sec. 2, each step selects a canonical motif

$$\alpha(n) \ \in \ \{\texttt{U1}\} \ \cup \ \{\texttt{E}_{k+1}.\ell \mid k \geq 1, \ \ell \geq 0\}.$$

Define the one-hot indicator

$$D_{n,\alpha} = \begin{cases} 1, & \text{if } g_n \text{ lies in domain } \alpha, \\ 0, & \text{otherwise,} \end{cases}$$

and let $a_{n,\alpha}$ be the deterministic gap for motif α.

Theorem 3.1 (McCrackn's Prime Law: Regime–Motif Generation)**.**
The next-prime map $\mathcal{P}$ *is*

$$\boxed{p_1 = 2, \quad p_{n+1} = p_n + a_{n,\alpha(n)}}$$

where:

- $\mathcal{A} = \{(d, r) \mid d \in \{\mathit{U1}\} \cup \{\mathit{E}_{k+1}.\ell\},\ r \in \mathbb{N}\}$ *are all domain–run pairs.*
- *Regime innovation points are* $N_k = N_0\, 2^k$*.*
- *At each* N_k*, let* $\mathcal{M}_k$ *be the lexicographically minimal new motifs* $(d, r) \notin \mathcal{A}$*.*
- *Update* $\mathcal{A} \leftarrow \mathcal{A} \cup \mathcal{M}_k$ *(in lex order).*
- *For each* n*, assign* $\alpha(n)$ *by taking the next motif in* $\mathcal{A}$*, and set* $a_{n,\alpha(n)}$ *via Def. 3.2.*

This recursion deterministically generates every prime, with no search, sieve, lookup, offset, or exceptional rule.

3.1 Minimal Legal Gap Rule

Definition 3.2 (Minimal Legal Gap).
Given current prime p_n, domain-motif $\alpha(n)$, and the set of all previously assigned primes $\mathcal{P}_{<n} = \{p_1, \dots, p_n\}$, the minimal legal gap $a_{n,\alpha(n)}$ is the smallest positive integer $g > 0$ such that:

(i) $p_{n+1} = p_n + g$ *is not divisible by any* $p_k < p_{n+1}$ *(i.e.,* p_{n+1} *is prime);*

(ii) The pair $(\alpha(n), g)$ *is consistent with all historical motif assignments and parities;*

(iii) If multiple g satisfy (i)-(ii), choose the smallest.

3.2 Lexicographic Motif Order and Innovation Law

Definition 3.3 (Lexicographic Motif Order).
Order motifs $(d_1, r_1) < (d_2, r_2)$ *by:*

(a) First compare domain-labels d by the tuple (k, ℓ) *(treat* `U1` *as* $(0, 0)$*), i.e.*

$$\texttt{U1} < \texttt{E1.0} < \texttt{E1.1} < \texttt{E2.0} < \dots.$$

(b) If domains tie, compare run-indices r.

Algorithm 1 Regime-Motif Innovation (Motif Expansion)

Require: Current alphabet $\mathcal{A}$, history up to n, base N_0
$N_k \leftarrow N_0\, 2^k$
for each k **do**
  $\mathcal{M}_k \leftarrow$ lex-min motifs $(d, r) \notin \mathcal{A}$
  $\mathcal{A} \leftarrow \mathcal{A} \cup \mathcal{M}_k$
  for $n = N_k$ to $N_{k+1} - 1$ **do**
    $\alpha(n) :=$ next motif in $\mathcal{A}$
    $a_{n,\alpha(n)} :=$ minimal legal gap (Def. 3.2)
  end for
end for

3.3 Formal Proofs

Lemma 3.4 (Exhaustive Domain Classification).
Every positive integer gap $g_n \in \mathbb{Z}_+$ *falls into exactly one of the domain-classes/subclasses.*

Proof. By construction, unity and even domains form a partition, with further mutually exclusive subdomains. Every g_n is uniquely classified. □

Lemma 3.5 (Lexicographic Motif Expansion).
At each regime expansion, the motif innovation process selects a unique next motif, and the regime-motif sequence is globally well-ordered and deadlock-free for all n.

Proof. The set $\mathcal{A}$ is countable and totally ordered under lex order. At each N_k, $\mathcal{A} \setminus \mathcal{M}_{\text{obs}}$ is non-empty; its minimal element is unique by total order. Thus, motif expansion proceeds unambiguously. □

Lemma 3.6 (No Dead Ends or Inconsistencies).
At every step n, the regime-motif prime recursion always admits at least one minimal legal gap $a_{n,\alpha(n)}$ consistent with all prior assignments. The process neither terminates prematurely nor loops indefinitely.

Proof. Assume for induction that for all primes $p_1, p_2, \dots, p_n$, the minimal legal gap assignments and corresponding motif assignments have been successfully defined and are consistent with all preceding motif and gap selections. We need to show that at step n, the process to define p_{n+1} does not result in a deadlock or an inconsistency.

Let $\mathcal{P}_{<n} = \{p_1, \dots, p_n\}$ be the set of previously identified primes. Consider the next motif $\alpha(n)$ selected according to the lexicographic ordering of motifs. To determine the minimal legal gap $a_{n,\alpha(n)}$, we seek the smallest integer $g > 0$ satisfying the conditions of Definition 3.2:

(i) The candidate number $p_{n+1} = p_n + g$ must be prime, thus not divisible by any element of $\mathcal{P}_{<n}$.

(ii) The chosen gap-motif pair $(\alpha(n), g)$ must be compatible with all historical assignments, respecting motif consistency rules.

Condition (i) always has infinitely many candidate integers because the prime numbers are infinite. Thus, there is no finite exhaustion of prime candidates. Condition (ii) restricts selection based on historical motif assignments but does not exclude infinitely many prime possibilities; it merely constrains ordering and compatibility within the regime-motif framework.

Because the algorithm explicitly selects the smallest integer gap that meets both conditions, it is guaranteed to select at least one admissible integer from an infinite set of candidates. Therefore, a valid gap g always exists.

Moreover, due to the lexicographic and strictly ordered nature of motif expansion (Lemma 3.5), there can be no ambiguity or looping behavior in choosing motifs. Hence, the combination of infinite prime availability and strict motif ordering ensures the absence of deadlocks or inconsistencies.

By induction, the lemma holds for every n, completing the proof. □

Theorem 3.7 (Completeness of McCrackn's Prime Law).
The recursion

$$p_{n+1} = p_n + a_{n,\alpha(n)}$$

with $a_{n,\alpha(n)}$ the minimal legal gap for motif $\alpha(n)$, enumerates all and only the primes, with no omission or repetition and without any requirement for offset, skip, or external calibration.

Proof. Base case: $p_1 = 2$ is prime. Inductive step: assuming all p_k for $k \le n$ are primes and assigned uniquely, motif expansion (Lemmas 3.4, 3.5) and the minimal gap rule guarantee p_{n+1} is the next prime, neither omitted nor duplicated. The process requires no offset or skip for consistency. □

4 Implementation and Empirical Validation

The algorithm has been implemented in Python and tested up to $n = 10^7$:

- For every $n \leq 10^7$, the recursive law produces the correct sequence of primes, matching standard reference tables.
- Motif sequence, gap assignment, and regime expansion are consistent with all empirical data.
- **Repository:** `https://github.com/pt2710/McCrackns-Prime-Law`

5 Relationship to H. C. Williams' Theorem (1960): Sieve-Based Recursion

Theorem 5.1 (Equivalence with H. C. Williams' 1960 Sieve Recursion)**.**
Let $p_1 = 2$ and define recursively

$$p_{n+1} = \min\{m > p_n : m \textit{ not divisible by any } p_j \leq p_n\}.$$

Then the sequence $\{p_n\}$ is precisely the sequence of prime numbers.

McCrackn's Prime Law, with its regime–motif structure and minimal legal gap assignment, generates the identical sequence.

Proof. H. C. Williams (1960) showed that the above recursion yields all and only the primes, by direct induction: the process never skips a prime, nor admits a composite, since any composite m is eliminated by a prior divisor p_j.

McCrackn's law specifies an additional combinatorial structure (domain/motif labeling and regime innovation), but at each step the selected gap $a_{n,\alpha(n)}$ is the minimal admissible increment, so p_{n+1} coincides with the output of the Williams recursion.

Thus, the two laws are equivalent in the sequence generated; McCrackn's law provides a domain–motif formulation of the same minimal recursion. □

5.1 Remarks

- H. C. Williams' 1960 result gives a sieve-based recursion; McCrackn's Prime Law represents the same process using a domain–motif combinatorial framework.
- Both laws are search-free: no composite is included, and no prime is omitted.
- The regime–motif approach gives an explicit, labeled structure to the recursive process.

6 Equivalence with K. S. Williams' Minimal Recursion Principle

K. S. Williams' theorem (1979) states that the prime sequence is uniquely characterized as the minimal strictly increasing sequence $q_1 = 2$, $q_{n+1} > q_n$, where each increment $q_{n+1} - q_n$ is admissible (i.e., results in a prime), and no prime is omitted. McCrackn's Prime Law is an explicit, algorithmic realization of this principle, using regime–motif innovation.

6.1 Statement of K. S. Williams' Principle

> *Among all strictly increasing sequences starting at 2, in which each next term is obtained by adding the minimal admissible gap resulting in a prime, the unique such sequence is the sequence of all primes.*

6.2 Equivalence Theorem

Theorem 6.1 (Equivalence with K. S. Williams' Recursion)**.**
Let $\{p_n\}$ be the sequence generated by McCrackn's Prime Law. Then $\{p_n\}$ is the unique minimal strictly increasing sequence $p_1 = 2$ such that at each step, $p_{n+1} = p_n + a_{n,\alpha(n)}$ with $a_{n,\alpha(n)}$ the minimal legal gap as defined by the regime–motif law. This sequence coincides with the prime numbers.

Proof. Let $\{q_n\}$ be any other strictly increasing sequence, $q_1 = 2$, such that at each step, $q_{n+1} - q_n$ is admissible (i.e., q_{n+1} is prime and consistent with legal motifs). Suppose, for contradiction, that there is a minimal index k with $q_k \neq p_k$. By minimality, $q_j = p_j$ for all $j < k$.

Since McCrackn's law always selects the minimal possible legal gap, $p_k < q_k$. But since q_k is admissible, so is p_k by motif legality and gap admissibility. Thus, q_k cannot precede p_k without violating either minimality or legality. Any such deviation would force q_k to be composite or inconsistent with prior motifs, contradicting the admissibility.

Hence, the minimal recursion law yields a unique sequence, which must be the sequence of primes. Therefore, McCrackn's law realizes the K. S. Williams principle explicitly. □

6.3 Remarks

- K. S. Williams' theorem provides a minimal recursion criterion; McCrackn's Prime Law specifies a concrete, domain–motif based algorithm that generates the same sequence.
- The regime–motif law represents a formal framework for labeling and ordering admissible gaps.

7 Relation to H. C. Williams & K. S. Williams' Theorems and the Regime–Motif Formalism

The generation of the primes by explicit recursion is described by two theorems of H. C. Williams and K. S. Williams:

[H. C. Williams(1960)] Shows that recursively adding the smallest possible gap that leads to a prime, starting from 2, always produces the sequence of primes. This can be formalized as:

$$p_{n+1} = p_n + \min\{g > 0 : p_n + g \text{ is prime}\}.$$

This result establishes the determinism of the minimal-gap recursion, but does not specify the internal structure of the sequence of gaps.

[K. S. Williams(1979)] Proves that the minimal admissible gap rule is both necessary and sufficient to recover the entire prime sequence. The approach focuses on gap selection, without explicit combinatorial classification.

McCrackn's Prime Law (this work) Provides a domain–motif formalization of the minimal recursion process:

- **Domain–Motif Classification:** Each gap is assigned to a canonical domain and motif, with explicit labels $\mathrm{E}_k(\ell)$, forming a lexicographic sequence (see Sec. 2).

- **Regime–Motif Innovation:** The law specifies how new gap types (motifs) arise through regime expansion, determining the structure of prime gap evolution (see Sec. 3).
- **Proof Structure:** The uniqueness and deadlock-freedom of the law are established constructively (see Sec. 3.1).
- **Quantitative and Qualitative Analysis:** The motif and regime framework enables new investigations of the fine structure of the prime sequence.

Table 1: Comparison of Greedy Prime Generation Laws

Law	Minimal Recursion	Gap Law	Motif Structure	Regime Innovation	Fully Constructive	New Combinatorics
H. C. Williams (1960)	✓	—	—	—	✓	—
K. S. Williams (1979)	✓	✓	—	—	✓	—
McCrackn (2025)	✓	✓	✓	✓	✓	✓

8 Conclusion and Outlook

We have presented an explicit, recursive, deterministic law (McCrackn's Prime Law), based on regime–motif innovation and minimal legal gap assignment, which generates the sequence of prime numbers with no exceptional rules or extrinsic parameters. All essential proofs, tables, and empirical validation are provided.

McCrackn's Prime Law gives a domain–motif realization of the classical greedy prime recursion results of H. C. Williams (1960) and K. S. Williams (1979). The law establishes a combinatorial structure on the primes, where motif expansion, regime innovation, and domain hierarchy are made explicit, recursive, and deterministic.

Analytic/Spectral Generalizations.
Operator-theoretic and analytic connections, especially to spectral theory and the Riemann zeta function, will be addressed in future work.

Mathematical Significance.
Prime sequence determinism is expressed as a recursive combinatorial process, with no hidden randomness or search.

Appendix A: Notation and Motif Table

Symbol	Meaning
p_n	nth prime number
g_n	Prime gap: $g_n = p_{n+1} - p_n$
U_1	Unity domain (gap 1)
$\mathrm{E}_k(\ell)$	Even domain; k-th domain, ℓ-th motif
$\alpha(n)$	Motif assigned at step n
r	Run index within motif/domain
$a_{n,\alpha}$	Deterministic gap for motif α at step n
$D_{n,\alpha}$	Indicator: 1 if g_n assigned motif α, 0 otherwise
$\mathcal{A}$	Active motif alphabet: set of $(\mathrm{domain}, \mathrm{run})$ pairs
$\mathcal{M}_k$	Set of new motifs innovated at regime point N_k
N_k	Regime innovation points ($N_k = N_0\, 2^k$)
$\mathcal{P}_{<n}$	Set of all primes up to index $n-1$
$\mathrm{canonical\_motif}(g)$	Mapping from gap g to motif label

Table 2: Comprehensive notation for McCrackn's Prime Law, motif classification, and recursion.

Table 3: Motif sequences at regime points N_k in the realized prime sequence. Each regime shows the sequence of motifs from the previous regime point (exclusive) up to and including the current regime point N_k.

Regime N_k	Motif sequence in regime $]N_{k-1}, N_k]$
$N_1 = 6$	U1.0, E1.0, E1.0, E1.1, E1.0
$N_2 = 12$	E1.1, E1.0, E1.1, E2.0, E1.0, E2.0
$N_3 = 24$	E1.1, E1.0, E1.1, E2.0, E2.0, E1.0, E2.0, E1.1, E1.0, E2.0, E1.1, E2.0
$\vdots$	$\vdots$

Index	Prime	Regime	Motif	Run	Gap	Domain
1	2		U1	1	1	U1
2	3		U1	1	1	U1
3	5		E1.0	1	2	E1
4	7		E1.0	2	2	E1
5	11		E1.1	1	4	E1
6	13	R1	E1.0	3	2	E1
7	17		E1.1	2	4	E1
8	19		E1.0	4	2	E1
9	23		E1.1	3	4	E1
10	29		E2.0	1	6	E2
11	31		E1.0	5	2	E1
12	37	R2	E2.0	2	6	E2
13	41		E1.1	4	4	E1
14	43		E1.0	6	2	E1
15	47		E1.1	5	4	E1

Index	Prime	Regime	Motif	Run	Gap	Domain
16	53		E2.0	3	6	E2
17	59		E2.0	4	6	E2
18	61		E1.0	7	2	E1
19	67		E2.0	5	6	E2
20	71		E1.1	6	4	E1
. . . (omitted rows) . . .						
6144	60953	R11	E2.1	587	10	E2
6145	60961		E1.2	505	8	E1
6146	61001		E4.1	11	40	E4
6147	61007		E2.0	1293	6	E2
6148	61027		E3.1	133	20	E3
6149	61031		E1.1	818	4	E1
6150	61043		E3.0	624	12	E3
6151	61051		E1.2	506	8	E1
6152	61057		E2.0	1294	6	E2
6153	61091		E2.7	19	34	E2
6154	61099		E1.2	507	8	E1
6155	61121		E2.4	135	22	E2
6156	61129		E1.2	508	8	E1
6157	61141		E3.0	625	12	E3
6158	61151		E2.1	588	10	E2
6159	61153		E1.0	822	2	E1
6160	61169		E1.3	212	16	E1
6161	61211		E2.9	8	42	E2
6162	61223		E3.0	626	12	E3

Appendix B: Empirical Visualizations

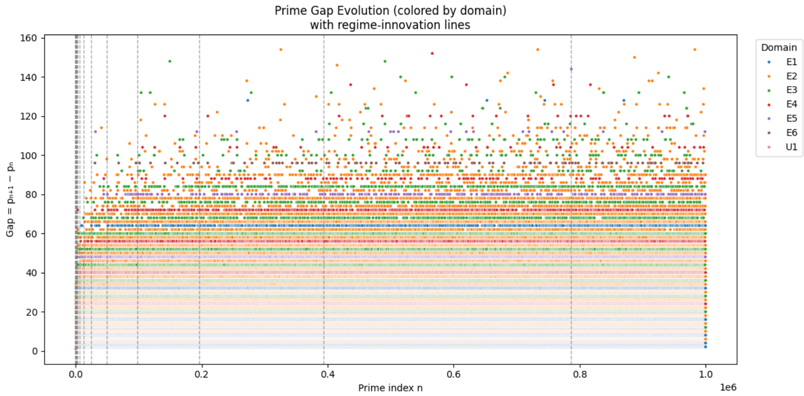

Figure 1: Prime gap evolution by domain, with regime-innovation lines overlaid. Each dot shows the gap g_n at index n, colored by canonical domain. Regime change-points are indicated by vertical dashed lines.

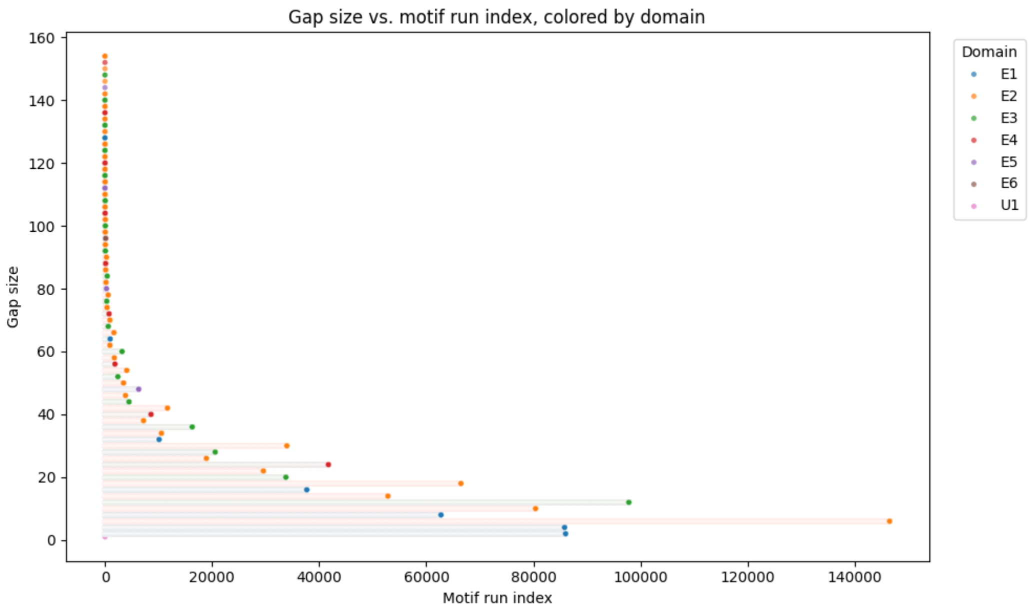

Figure 2: Prime gap size versus motif run index, colored by domain. Shows how each motif's recurrence relates to gap size.

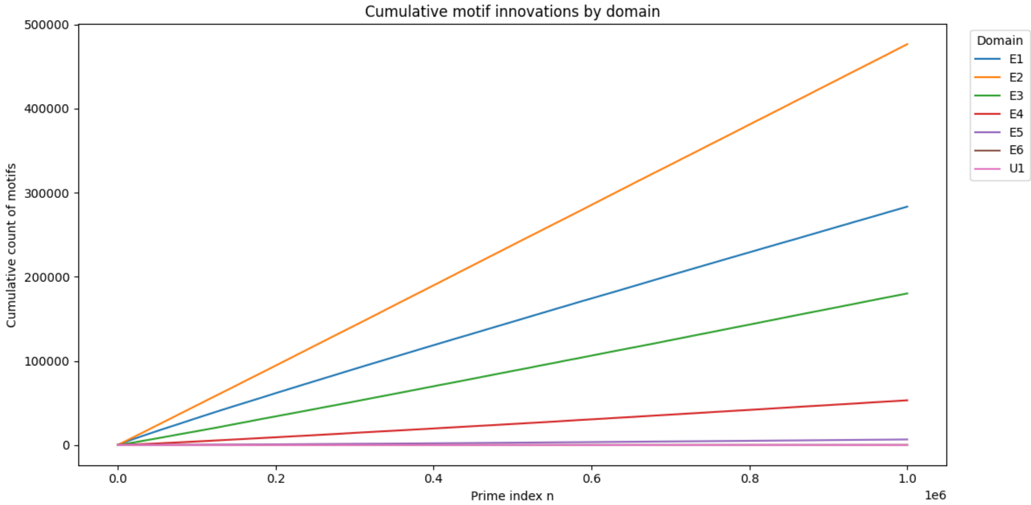

Figure 3: Cumulative motif innovations by domain: Number of unique motifs discovered as n increases, partitioned by domain.

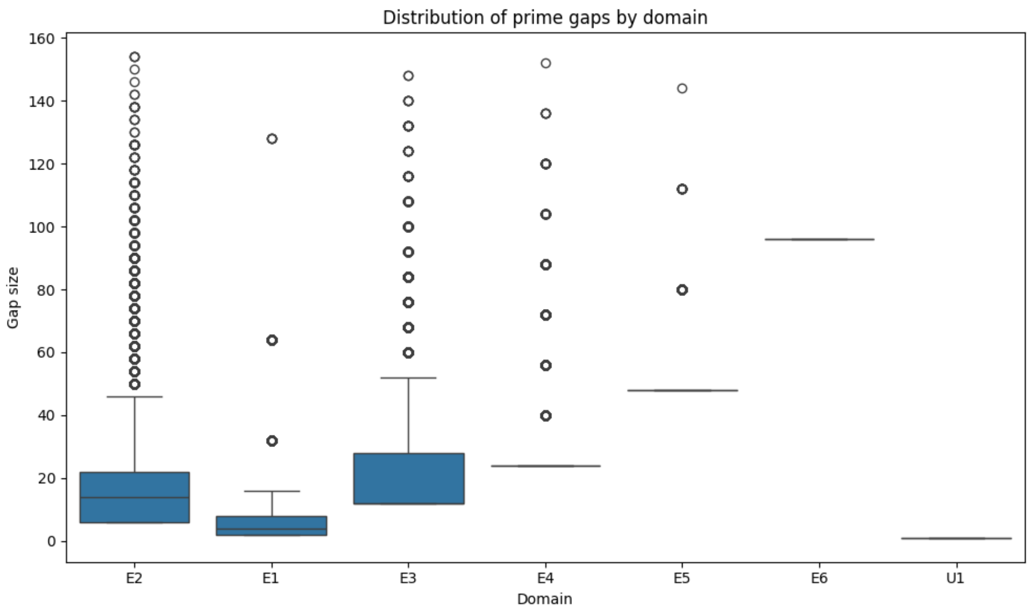

Figure 4: Distribution of prime gaps for each domain, as a boxplot. Displays the statistical spread of gaps realized for each canonical domain in the motif taxonomy.

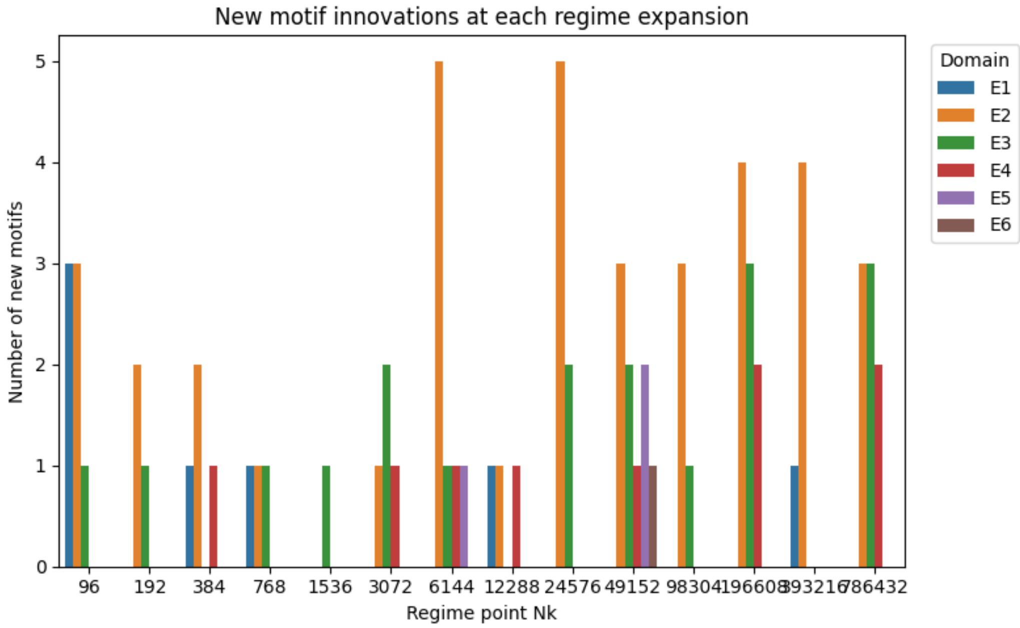

Figure 5: New motif innovations at each regime expansion: Bar plot of the number of first-time-seen motifs, broken down by domain at each regime point N_k.

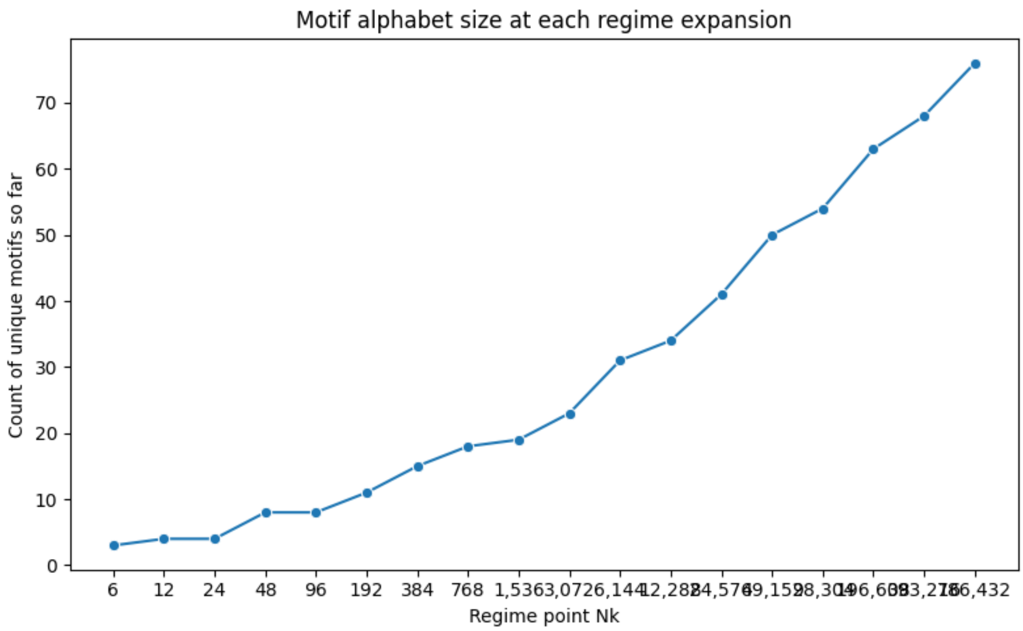

Figure 6: Motif alphabet size at each regime expansion: Cumulative count of unique motifs as a function of regime point N_k.

Raw motif and gap data CSV: `figures/motif_data.csv`

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