Algebraic operads, Koszul duality and Gröbner bases: an introduction

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This lecture series aim to offer a gentle introduction to the theory of algebraic operads, starting with the elements of the theory, and progressing slowly towards more advanced themes, including (inhomogeneous) Koszul duality theory, Gröbner bases and higher structures. The course will consist of approximately twelve lectures, along with extra talks by willing participants, with the goal of introducing extra material to the course, and making them more familiar with the theory.

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0 Motivation and history

Goals. The goals of this lecture is to give a broad picture of the history and pre-history of operads, and some current trends, and give a road-map for the course.

0.1 Introduction and motivation

Operads (topological operads, more precisely) originally appeared as tools in algebraic topology and homotopy theory, specifically in the study of iterated loop spaces (May, 1972 and Boardman and Vogt before). They also appeared as *comp algebras* in Gerstenhaber's work on Hochschild cohomology and topologically as Stasheff's 'associahedra' for his homotopy characterization of loop spaces (both in 1963). The theory of operads, in particular topological and algebraic, saw itself very much influenced by homological algebra, category theory, algebraic geometry, rational homotopy theory and mathematical phyisics. Here we list a few examples:

(1) (Stasheff, Sugawara) Study homotopy associative H-spaces, Stasheff implicitly discovers a topological ns operad K with $C_*(K) = \mathsf{As}_\infty$ and a recognition principle for A_∞-spaces.
(2) (Boardmann–Vogt) Study infinite loop spaces, build a PROP (a version of an E_∞-operad) and obtain a recognition principle for infinite loop spaces.
(3) (Kontsevich) Uses L_∞-algebras and configuration spaces to prove his deformation quantization theorem that every Poisson manifold admits a deformation quantization.
(4) (Kontsevich) The above is implied by the formality theorem: the Lie algebra of polyvector fields is L_∞-quasi-isomorphic to the Hochschild complex, and $f_1 = \mathsf{HKR}$.
(5) (Tamarkin) Approaches this result through the formality of the little disks operad D_2. Proves that the Hochschild complex of a polynomial algebra is *intrinsically formal* as a Gerstenhaber algebra.
(6) (Manifold calculus) Describes the homotopy type of embedding spaces as certain derived operadic module maps and to produces their explicit deloopings using little disk operads, due to Goodwillie–Weiss, Boavida de Brito—Weiss, Turchin, Arone–Turchin, Dwyer—Hess, Ducoulombier—Turchin.
(7) (Ginzburg–Kapranov, Fresse, Vallette, Hinich) Koszul duality for algebraic operads and cousins allows to develop a robust homotopy theory of homotopy algebras, cohomology theory, deformation theory, Quillen homology, etc.
(8) (Deligne conjecture and variants) The study of natural operations on the Hochschild complex of an associative algebra lead to a manifold of results beginning with the proof that there is an action of the little disks operad D_2 on it, and the ultimate version by Markl–Voronov, who proved that the operad of natural operations on it has the homotopy type of $C_*(D_2)$.

Operads are modeled by trees (planar or non-planar, rooted or not), and relaxing these graphs allows us to produce other type of algebraic structures. The following table gives the reader a "taxonomy cheat sheet" for operads and their kin; we will, for better or worse, defer from diving into the curious world that lies beyond operads, but encourage the reader to do this for themselves (and find out what "wheeled structures" are, and how they fit in the table below).

Type	Graph	Compositions	Due to
PROPs	Any graph	Any	Adams–MacLane
Modular	Any graph	$\xi_{i,j}$, $\circ_{i,j}$	Getzler–Kapranov
Properads	Connected graphs	Any	B. Vallette
Dioperads	Trees	${}_i\circ_j$ (no genus)	W. L. Gan
Half-PROPs	Trees	$\circ_j$, ${}_i\circ$	Markl–Voronov
Cyclic operads	Trees	$\circ_{i,j}$	Getzler–Kapranov
Symmetric operads	Rooted trees	$\circ_i$	J. P. May

0.2 Koszul duality

Koszul duality was invented by Steward Priddy in the seventies [?Priddy1970], with the objective of streamlining computations of certain cohomology theories for classes of algebras (notably, Lie and associative algebras). One of the reasons this was (and still is) relevant is that such cohomology groups play a central role in the computation of other more complicated invariants of algebras and topological spaces: in particular, the cohomology of the Steenrod algebra famously featured in Adam's spectral sequence computing the stable homotopy groups of spheres at each prime. In Priddy's own words:

> *The purpose of this paper is to construct resolutions for a large class of algebras which includes the Steenrod algebra and the universal enveloping algebras. It is a basic problem of homological algebra to compute the cohomology algebras of various augmented algebras. Unfortunately, the canonical tool for attacking this problem —the bar resolution— is often intractable. In some instances, however, one is able to find a simpler resolution.*

Priddy developed his theory for both "inhomogeneous" and "homogeneous" quadratic algebras —those presented in coordinates by quadratic equations in their variables— and, while in the homogeneous case his formalism gave the answer immediately, the inhomogenous case required an additional step, which nonetheless simplified the existing methods considerably.

Although Koszul duality nowadays has a much broader meaning and casts an immense net in modern day algebra, representation theory, combinatorics, topology and geometry, among other areas of mathematics, in this lecture series we will follow Priddy's motivation and see it as an instance of a phenomenon in which certain algebraic objects have very economical —and thus computationally and theoretically useful— resolutions. An interested reader can consult [?KellerKoszul2003, ?Positselski2011, ?Sinha2010, ?MO329, ?holstein2021categorical] to obtain a broader view of this phenomenon, and in particular find a wide variety of answers to the question "...but what *exactly* is Koszul duality?".

Naturally, one of the reasons why Koszul duality has cemented itself in modern day mathematics is that it appears often: algebraic structures of interest have an inclination to be quadratic and, when in luck, Koszul. These can be anything from Lie, commutative or associative algebras, to Feynmann categories, dg categories, operads and their kin. In this lecture series, we will focus on algebraic operads: our goal is to introduce the reader to algebraic operads in general and to quadratic operads in particular, define what it means for such operads to be Koszul, and explore the consequences this property has on the operads and its representations.

Although, as we mentioned, we will take a rather old fashioned point of view and think of Koszul operads as those operads having a "nice resolution", we aim to give the reader a modern outlook on the current methods available to prove that an operad is Koszul, and some relatively new developments in the area from the last two (or maybe three) decades: the inhomogenous Koszul duality for (pr)operads due to Galvez-Carillo–Tonks–Vallette, which followed the original theory of Ginzburg–Kapranov, the use of filtered distributive laws of Dotsenko which followed the methods of Markl, and the general theory stating Koszul operads give rise to good notions of algebras up to homotopy, due to Vallette. Naturally, we will also focus on the classical developments, and on the effective methods of Hoffbeck and Dotsenko–Khoroshkin, which we detail in the next section.

0.3 Gröbner bases

Write introduction to Groebner bases.

References for introduction.

0.4 Exercises

A. Symmetric groups. Operads are meant to encode operations on objects *along with their symmetries*, which is done through the representation theory of the symmetric groups. The following exercises will remind you of some basic facts about them.

Exercise 1. Let $I = [n]$ so that $\mathrm{Aut}(I) = S_n$ is the symmetric group on n letters. For each ordered partition $\pi = (\pi_1, \dots, \pi_k)$, let λ be the ordered partition of n with $\lambda_i = \#\pi_i$ for $i \in [k]$. Show that the permutations of $[n]$ that preserve π determine a subgroup of S_n isomorphic to $S_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.

Exercise 2. Consider the subgroup of S_n corresponding to the ordered partition of $[n]$ given by $([1,k],[k+1,n])$, along with the inclusion $S_k \times S_{n-k} \hookrightarrow S_n$. Show that a set of representatives for the cosets of this inclusion in S_n is given by the $(k,n-k)$-*shuffles*, those permutations $\sigma \in S_n$ that preserve the linear order in $[1,k]$ and $[k+1,n]$. Conclude that there are exactly $\binom{n}{k}$ shuffles of type $(k,n-k)$ on $[n]$. Define shuffles associated to other partitions of n.

B. Categories. The language of categories and functors permeates most of modern algebra and geometry, and in particular is useful to work with operads and other combinatorial structures defines by graphs. The following will remind you of some important notions we will use during the course.

Exercise 3. A category $\mathcal{C}$ is the datum of a set of objects $\mathrm{Ob}(\mathcal{C})$, and for each $x, y \in \mathrm{Ob}(\mathcal{C})$ a set $\mathcal{C}(x,y)$ of morphisms from x to y. Moreover, we require the existence of an associative and unital composition law

$$- \circ - : \mathcal{C}(y,z) \times \mathcal{C}(x,y) \longrightarrow \mathcal{C}(x,z).$$

The latter means there are distinguished elements $1_x \in \mathcal{C}(x,x)$ for each object of $\mathcal{C}$ that induce the identity $- \circ 1_x$ an $1_x \circ -$ of any $\mathcal{C}(-,x)$ and $\mathcal{C}(x,-)$. Find examples of categories: sets, finite sets, rings, vector spaces, open subsets, posets, and others.

Exercise 4. A functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a datum that assigns to each object x of the domain an object $F(x)$ of the codomain, and to each morphism $f : x \to y$ a morphism $F(f)$ such that $F(f \circ g) = F(f) \circ F(g)$ and $F(1_x) = 1_{Fx}$ for each pair of composable arrows f and g and each object x of $\mathcal{C}_1$. Find examples of functors between the examples of categories you found above.

Exercise 5. A monoidal category is a category $\mathcal{C}$ along with the datum of a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ along with an associator and left and right units. A monoidal category is *strict* if the associator and left and right units are identities.

(1) Expand on the details of these definitions. Define what a braided monoidal category and what a symmetric monoidal category are.

(2) Exhibit monoidal structures the following categories: sets, vector spaces, linear representations of a group G, topological spaces, associative algebras, Lie algebras, and others.

Hint. In the case of Lie algebras, consider the category of Lie groups with its canonical tensor product (the cartesian product) and the functor $G \longmapsto T_e(G)$ to decide what the tensor product of two Lie algebras is.

Exercise 6. If $(\mathcal{V}, \otimes)$ is a monoidal category, we say $\mathcal{C}$ is a $\mathcal{V}$-enriched category if each hom-set $\mathcal{C}(x,y)$ is an object of $\mathcal{V}$ and there is a composition law

$$- \circ - : \mathcal{C}(y,z) \otimes \mathcal{C}(x,y) \longrightarrow \mathcal{C}(x,z).$$

which consists of morphisms in $\mathcal{V}$, and which is associative and unital. Note that an ordinary category is just a category enriched over the category of sets. A linear category is a category enriched over the category of vector spaces, an additive category is a category enriched over Abelian groups. Expand on what this means. Find about Abelian categories, and ponder over the difference: an additive category is a category with structure, while an Abelian category is a category with additional properties.

Exercise 7. A category $\mathcal{D}$ is skeletal if no two distinct objects in it are isomorphic. We say that $\mathcal{D}$ is the skeleton of $\mathcal{C}$ if:

(1) It is a full subcategory of $\mathcal{C}$: for each pair of objects $x, y \in \mathcal{D}$, we have that $\mathcal{D}(x,y) = \mathcal{C}(x,y)$.
(2) The inclusion of $\mathcal{D}$ in $\mathcal{C}$ is essentially surjective: every object of $\mathcal{C}$ is isomorphic to an object of $\mathcal{D}$.
(3) $\mathcal{D}$ is skeletal.

Show that every small category admits a skeleton, and compute the skeleton of the following categories: sets, finite sets, finite dimensional vector spaces over a field.

Exercise 8. Suppose that x is an object in a symmetric monoidal category $(\mathcal{C}, \tau)$. For each $n \in \mathbb{N}$ and each $i \in [n-1]$ define $\tau_i : x^{\otimes n} \longrightarrow x^{\otimes n}$ by

$$\tau_i = 1^{i-1} \otimes \tau \otimes 1^{n-i-1}.$$

Show that the assignment $(i, i+1) \in S_n \longmapsto \tau_i \in \mathrm{Aut}(x^{\otimes n})$ is a group homomorphism. *Note.* This produces in particular a map $S_2 \longrightarrow \mathrm{Aut}(x^{\otimes 2})$ that sends the transposition $(12) \in S_2$ to the flip $\tau_{x,x} : x \otimes x \longrightarrow x \otimes x$.

Exercise 9. A product and permutation category (abbreviated 'PROP') is a monoidal category $\mathcal{C}$ whose set of objects is $\mathbb{N} = \{0, 1, 2, \ldots\}$ and its tensor product is addition (in particular, it is strict and symmetric). Unravel the definitions:

(1) Use that $n = 1 + \cdots + 1$ to show that $\mathcal{C}(m,n)$ is a right S_n-module.
(2) Similarly, show that $\mathcal{C}(m,n)$ is also a left S_m-module.
(3) Show these two actions are compatible (i.e. they commute).
(4) Show that the product $+$ induces a *horizontal* composition rule

$$\mathcal{C}(m,n) \times \mathcal{C}(m',n') \longrightarrow \mathcal{C}(m+m',n+n').$$

(5) Interpret the usual categorical product as a *vertical* composition rule

$$\mathcal{C}(n,k) \times \mathcal{C}(m,n) \longrightarrow \mathcal{C}(m,k).$$

Consider the definition of a PROP enriched over a symmetric strict monoidal category, like Vect (these are called $\daleth$-linear PROPs). Define the category of PROPs.

Note. For each $n \in \mathrm{Ob}(\mathcal{C})$ the object $\mathcal{C}(n,n)$ is a monoid under composition that receives a map $S_n \longrightarrow \mathcal{C}(n,n)$. Under the interpretation above, the image of σ is equal to both the left and the right action of S_n on the identity map $n \to n$. In particular, the twist τ of $\mathcal{C}$ is equal to $(12)\mathrm{id}_2$, and may (or may not) be trivial.

C. Graded spaces and complexes. When studying algebraic structures like operads, it will be necessary to use some tools from homological algebra: graded spaces, chain complexes, differentials, their homology, among others. The following exercises are intended to familiarize you with the elements of homological algebra, but we will look at them in more detail during the course.

Exercise 10. A $\mathbb{Z}$-graded vector space (usually just called a graded vector space) is a vector space V with a direct sum decomposition

$$V = \bigoplus_{n \in \mathbb{Z}} V_n.$$

If $v \in V_n$ we say that v is homogeneous of degree n. Find out about the category of graded vector spaces, specifically:

- What are its (degree zero) morphisms?
- What are its (degree homogeneous) morphisms?
- What is the tensor product of two graded spaces?
- What is the natural isomorphism $V \otimes W \longrightarrow W \otimes V$?
- How does the last item relate to the 'Koszul sign rule'?

Exercise 11. A differential graded (dg) vector space, usually called a complex, is a pair (V,d) where V is a graded vector space V and $d : V \to V$ is a homogeneous map of degree -1 such that $d^2 = 0$. Repeat the previous exercise replacing gVect with Ch, the category of complexes of vector spaces.

Exercise 12. If (V,d) is a complex, then $Z(V) = \ker d$ is called its space of cycles, and $B(V) = \operatorname{im} d$ is called its space of boundaries. The quotient $Z(V)/B(V)$ is called the homology of V, and is written $H(V)$. Show that a map of complexes $f : V \to W$ induces a map $Z(V) \to Z(W)$ and in turn a map $H(f) : H(V) \to H(W)$.

Exercise 13. (Leisure) Find a book on homological algebra and read about the *snake lemma* and the *five lemma*. If you are very motivated, read about double complexes and spectral sequences.

1 Symmetric modules and algebraic operads

Goals. We will define some related gadgets (symmetric collections, algebras, modules, endomorphism operads) necessary to introduce operads. Then, we define what an operad is (topological, algebraic, symmetric, non-symmetric). We will then give some (not so) well known examples of topological and algebraic operads.

1.1 Basic definitions

What is an operad? A group is a model of $\mathrm{Aut}(X)$ for X a set, an algebra is a model of $\mathrm{End}(V)$ for V a vector space. Equivalently, groups are the gadgets that act on objects by automorphisms, and algebras are the gadgets that act on objects by their (linear) endomorphisms. Operads are the gadgets that act on objects through operations with many inputs (and one output), and at the same time keep track of symmetries when the inputs are permuted.

The underlying objects to operads are known as *symmetric sequences*: a symmetric sequence (also known as an Σ-module or symmetric module) is a sequence of vector spaces $\mathcal{X} = (\mathcal{X}(n))_{n\geqslant 0}$ such that for each $n \in \mathbb{N}_0$ there is a right action of S_n on $\mathcal{X}(n)$. We usually consider *reduced* Σ-modules, those for which $\mathcal{X}(0) = 0$.

A map of Σ-modules is a collection of maps $(f_n : \mathcal{X}_1(n) \longrightarrow \mathcal{X}_2(n)\}_{n\geqslant 0}$, each equivariant for the corresponding group action. This defines the category ${}_{\Sigma}\mathsf{Mod}$ of symmetric sequences, and whenever we think of symmetric sequences using this definition, we will say we are considering a a biased or skeletal approach to them.

In parallel, it is convenient to consider the category $\mathsf{Fin}^{\times}$ of finite sets and bijections. An object in this category is a finite set I, and a morphism $\sigma : I \longrightarrow J$ is a bijection. Since every finite set I with n elements is (non-canonically) isomorphic to $[n] = \{1,\dots,n\}$, the following holds:

Lemma 1.1 *The skeleton of* $\mathsf{Fin}^{\times}$ *is equal to the category with objects the finite sets* $[n]$ *for* $n \geqslant 0$ *and with morphisms the bijections* $[n] \longrightarrow [n]$ *(and no morphism between* $[n]$ *and* $[m]$ *if* $m \neq n$*).*

Proof. This is Exercise 7. □

We set ${}_{\Sigma}\mathsf{Mod} = \mathsf{Fun}(\mathsf{Fin}^{\times}, \mathsf{Vect}^{\mathrm{op}})$, so that a Σ-module is a pre-sheaf of vector spaces $I \longmapsto \mathcal{X}(I)$ assigning to each isomorphism $\tau : I \longrightarrow J$ an isomorphism $\mathcal{X}(\tau) : \mathcal{X}(J) \longrightarrow \mathcal{X}(I)$. When we think of Σ-modules as pre-sheaves, we will say we are taking an unbiased approach, will if we specify only its values on natural numbers, we will say we are taking the biased or skeletal approach; we will come back to this later.

With this at hand, we can in turn define the *Cauchy product* of two Σ-modules $\mathcal{X}$ and $\mathcal{Y}$

$$(\mathcal{X} \otimes_\Sigma \mathcal{Y})(I) = \bigoplus_{S \sqcup T = I} \mathcal{X}(S) \otimes \mathcal{Y}(T)$$

where the right-hand is the usual tensor product of vector spaces and the sum runs through partitions of I into two disjoint sets. The symmetric product is then defined by

$$(\mathcal{X} \circ_\Sigma \mathcal{Y})(I) = \bigoplus_{\pi \vdash I} \mathcal{X}(\pi) \otimes \mathcal{Y}^{\otimes k}(\pi)$$

as the sum runs through (ordered) partitions of I. These two products will be central in what follows.

Lemma 1.2 *The category* ${}_\Sigma\mathsf{Mod}$ *with* $\circ_\Sigma$ *is monoidal with unit the species taking the value* $\Bbbk e_x$ *at the singleton sets* $\{x\}$ *and zero everywhere else. The same category is monoidal for* $\otimes_\Sigma$ *with unit the species taking the value* $\Bbbk$ *at* $\varnothing$ *and zero everywhere else.*

We will use the notation $\Bbbk$ for the base field but also for the unit for the composition product $\circ_\Sigma$, hoping it will not cause much confusion. It will be useful later to think of $\Bbbk$ as a twig or "stick".

Observe that the associator for $\circ_\Sigma$ is not too simple and involves reordering certain factors of tensor products in Vect. In particular, replacing vector spaces by graded vector spaces or complexes will create signs in the associator.

We are now ready to define the prototypical symmetric sequence that carries the structure of an algebraic operad.

Definition 1.3 The *endomorphism operad* of a space V is the symmetric sequence End_V where for each $n \in \mathbb{N}$ we set $\mathrm{End}_V(n) = \mathrm{End}(V^{\otimes}, V)$. The symmetric group S_n acts on the right on $\mathrm{End}_V(n)$ so that $(f\sigma)(v) = f(\sigma v)$ for $v \in V^{\otimes n}$, where S_n acts on the left on $V^{\otimes n}$ by $(\sigma v)_i = v_{\sigma i}$. The composition maps are defined by $\gamma(f; g_1, \ldots, g_n) = f \circ (g_1 \otimes \cdots \otimes g_n)$.

The following two operations on permutations will streamline our definition of (algebraic) operads.

Two useful maps. For each $k \geqslant 1$ and each tuple $\lambda = (n_1, \ldots, n_k)$ with sum n there is a map

$$S_k \longrightarrow S_{n_1 + \cdots + n_k}$$

that sends $\sigma \in S_k$ to the permutation $\lambda(\sigma)$ of $[n]$ that permutes the blocks $\pi_i = \{n_1 + \cdots + n_{i-1} + 1, \ldots n_1 + \cdots + n_{i-1} + n_i\}$ according to σ. There is also a map

$$S_{n_1} \times \cdots \times S_{n_k} \longrightarrow S_{n_1 + \cdots + n_k}$$

$$\lambda = (2,1,2), \quad \sigma = 312 \quad \rightsquigarrow \quad \lambda(\sigma) = 34512 \in S_5$$
$$(213, 213, 132) \in S_3 \times S_3 \times S_3 \quad \rightsquigarrow \quad 213546798 \in S_9$$

Figure 1: The useful operations

that sends a tuple of permutations $(\sigma_1, \dots, \sigma_k)$ to the permutation $\sigma_1 \# \cdots \# \sigma_k$ that acts like σ_i on the block π_i as above. These operations are illustrated in Figure 1. With these at hand, one can check that these composition maps satisfy the following axioms:

(1) *Associativity*: let $f \in \mathrm{End}_V(n)$, and consider $g_1, \dots, g_n \in \mathrm{End}_V$ and for each $i \in [n]$ a tuple $h_i = (h_{i1}, \dots, h_{in_i})$ were $n_i = \mathrm{ar}(g_i)$. Then for $f_i = \gamma(g_i; h_{i1}, \dots, h_{in_i})$ and $g = \gamma(f; g_1, \dots, g_n)$ we have that

$$\gamma(f; f_1, \dots, f_n) = \gamma(g; h_1, \dots, h_n).$$

(2) *Intrinsic equivariance*: for each $\sigma \in S_k$ and $\lambda = (\mathrm{ar}(g_1), \dots, \mathrm{ar}(g_k))$ we have that

$$\gamma(f\sigma; g_1, \dots, g_k) = \gamma(f; g_{\sigma 1}, \dots, g_{\sigma k})\lambda(\sigma),$$

(3) *Extrinsic equivariance*: for each tuple of permutations $(\sigma_1, \dots, \sigma_k) \in S_{n_1} \times \cdots \times S_{n_k}$, if $\sigma = \sigma_1 \# \cdots \# \sigma_k$, we have that

$$\gamma(f, g_1\sigma_1, \dots, g_k\sigma_k) = \gamma(f; g_1, \dots, g_k)\sigma.$$

(4) *Unitality:* the identity $1 \in \mathrm{End}_V(1)$ satisfies $\gamma(1; g) = g$ and $\gamma(g; 1, \dots, 1) = g$ for every $g \in \mathrm{End}_V$.

Definition 1.4 A symmetric operad (in vector spaces) is an Σ-module $\mathcal{P}$ along with a composition map $\gamma : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$ of signature

$$\gamma : \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \longrightarrow \mathcal{P}(n_1 + \cdots + n_k)$$

along with a unit $1 \in \mathcal{P}(1)$, that satisfy the axioms above.

Variant 1.5 A non-symmetric operad is an operad whose underlying object is a collection (with no symmetric group actions). Operads in topological spaces or chain complexes require the composition maps to be morphisms (that is, continuous maps or maps of chain complexes, respectively) and, more generally, operads defined on a symmetric monoidal category require, naturally, that the composition maps be morphisms in that category.

Pseudo-operads. One can define operads through *partial composition maps*, modeling the honest partial composition map

$$f \circ_i g = f(1, \dots, 1, g, 1, \dots, 1)$$

in End_V. These composition maps satisfy the following properties:

(1) *Associativity*: for each $f, g, h \in \mathrm{End}_V$, and $\delta = i - j + 1$, we have

$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{\mathrm{ar}(f)+j-1} g & \delta \leqslant 0 \\ f \circ_j (g \circ_\delta h) & \delta \in [1, \mathrm{ar}(g)] \\ (f \circ_\delta h) \circ_j g & \delta > \mathrm{ar}(g) \end{cases}$$

(2) *Intrinsic equivariance*: for each $\sigma \in S_k$, we have that

$$(f\sigma) \circ_i g = (f \circ_{\sigma i} g)\sigma'$$

where σ' is the same permutation as σ that treats the block $\{i, i+1, \dots, i+\mathrm{ar}(g)-1\}$ as a single element.

(3) *Extrinsic equivariance*: for each $\sigma \in S_k$, we have that

$$f \circ_i (g\sigma) = (f \circ_i g)\sigma''$$

where σ'' acts by only permuting the block $\{i, \dots, i+\mathrm{ar}(g)-1\}$ according to σ.

(4) *Unitality:* the identity $1 \in \mathrm{End}_V(1)$ satisfies $1 \circ_1 g = g$ and $g \circ_i 1 = g$ for every $g \in \mathrm{End}_V$ and $1 \leqslant i \leqslant \mathrm{ar}(g)$.

Definition 1.6 A symmetric operad (in vector spaces) is an Σ-module $\mathcal{P}$ along with partial composition map of signature

$$- \circ_i - : \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$$

and a unit $1 \in \mathcal{P}(1)$ satisfying the axioms above.

It is not hard to see (but must be checked at least once) that an operad with $\mathcal{P}(n) = 0$ for $n \neq 1$ is exactly the same as an associative algebra.

Warning! If one does not require the existence of a unit, the notion of a *pseudo-operad* by Markl (defined by partial compositions) does not coincide with the notion of an operad as defined by May.

1.2 Constructing operads by hand

One can define operads in various ways. For example, one can define the underlying collection explicitly, and give the composition maps directly:

(1) *Commutative operad.* The reduced symmetric topological (or set) operad with $\mathsf{Com}(n)$ a single point for each $n \in \mathbb{N}$, and composition maps the unique map from a point to a point.

(2) *Associative operad.* The reduced set operad with $\mathsf{As}(n) = S_n$ the regular representation and composition maps

$$S_k \times S_{n_1} \times \cdots \times S_{n_k} \longrightarrow S_{n_1+\cdots n_k}$$

the unique equivariant map that sends the tuple of identities to the identity.

(3) *Stasheff operad.* Let K_{n+2} be the subset of I^n (the product of n copies of $I = [0,1]$) consisting of tuples $(t_1,\dots,t_{n+2})$ such that $t_1 \cdots t_k \leqslant 2^{-k}$ for $j \in [n+2]$. The boundary of K_{n+2} consists of those points such that for some $j \in [n+2]$ we have either t_j or $t_1 \cdots t_j = 2^{-j}$. It is tedious (but otherwise doable) to show that for each pair (r,s) of natural numbers and each $i \in [r]$ there exists an inclusion

$$\circ_i : K_{r+1} \times K_{s+1} \longrightarrow K_{r+s+1}$$

that defines on the sequence of spaces $\{K_{n+2}\}_{n\geqslant 0}$ the structure of a non-symmetric operad. We will see in the exercise a realization of K_n as the convex hull of points with positive integer coordinates (due to J.-L. Loday) using planar binary rooted trees, which will make the operad structure more transparent.

(4) If M is a monoid, there is an operad $\mathbb{W}_M$ with $\mathbb{W}_M(n) = M^n$ such that

$$(m_1,\dots,m_s) \circ_i (m'_1,\dots,m'_t) = (m_1,\dots,m_{i-1}, m_i m'_1,\dots,m_i m'_t, m_{i+1},\dots,m_s).$$

We call it the *word operad of* M. Its underlying symmetric collection is $\mathsf{As} \circ M$.

(5) Write $\mathrm{Aff}(\mathbb{C}) = \mathbb{C} \times \mathbb{C}^\times$ for the group of affine transformations of $\mathbb{C}$ with group law $(z,\lambda)(w,\mu) = (z+\lambda w, \lambda\mu)$. In turn, define for each finite set I the topological space

$$\mathcal{C}(I) = \{(z_i,\lambda_i) \in \mathrm{Aff}(\mathbb{C})^I : |z_i - z_j| > |\lambda_i| + |\lambda_j|\}.$$

The group law of $\mathrm{Aff}(\mathbb{C})$ allows us to define an operad structure on $\mathcal{C}(I)$ using the exact same definition as in the word operad of a monoid. The subspaces $\mathcal{D}_2^{\mathrm{fr}}(I) \subseteq \mathcal{C}(I)$ where $|z_i| + |\lambda_i| \leqslant 1$ for all $i \in I$, and where the inequality is strict unless $z_i = 0$ is called the *framed little disks operad*. The little disks operad is the suboperad where $\lambda_i = 1$ for all $i \in I$, and we write it $\mathcal{D}_2(I)$.

(6) The operad of rooted trees RT has $\mathsf{RT}(n)$ the collection of rooted threes with n vertices labeled by $[n]$, and the composition $T \circ_j T'$ is obtained by inserting T' at the jth vertex of T and reattaching the children of that vertex to T' in all possible ways. For example, if then we have that

1.3 Exercises

Exercise 14. Follow the lecture notes and read about the partial definition of an operad (and what a Markl operad is). Show that a unital pseudo-operad is the same as a unital May operad.

Exercise 15. Define the category of collections in Vect using the biased approach and the unbiased approach (this requires considering *totally ordered* sets instead of sets, and their order preserving bijections. We will write them with calligraphic letters but use subscripts, so $\mathcal{X}$ has ns components $\{\mathcal{X}_n\}_{n\geqslant 1}$.

(1) Show that it supports a non-symmetric Cauchy product given by

$$(\mathcal{X} \otimes \mathcal{Y})_n = \bigoplus_{i+j=n} \mathcal{X}_i \otimes \mathcal{Y}_j.$$

(2) Use this and the unbiased approach to argue that the ns counterpart of a ‘subset of I’ is an interval: a totally ordered subset of I of the form $[i, j] = \{x \in I : i \leqslant x \leqslant j\}$.

(3) Use the previous item to define the non-symmetric composition of ns collections. Define the generating function associated to a collection, and show it behaves well with respect to the products above.

Exercise 16. Since every finite totally ordered set is, in particular, a finite set (and every order preserving function is a fortiori a function) there is a map of categories $\mathsf{FinOrd}^\times \longrightarrow \mathsf{FinSet}^\times$ which induces a map that ‘forgets the symmetries’ ${}_\Sigma\mathsf{Mod} \longrightarrow \mathsf{Coll}$. Show that there is a functor that assigns a ns sequence $\mathcal{X}$ to the sequence $\mathcal{X}_\Sigma(n) = \Bbbk S_n \otimes \mathcal{X}_n$ which is left adjoint and monoidal.

Exercise 17. Describe the associator for $\circ_\Sigma$ in the category of differential graded collections. In particular, write down the signs explicitly. Explain how this is related to the signs in the parallel composition axiom for *graded operads* that read as follows: for elements f, g and h

in an operad (of homogeneous arities) and $\delta = i - j + 1$, we have that

$$(f \circ_j g) \circ_i h = \begin{cases} (-1)^{|g||h|}(f \circ_i h) \circ_{\mathrm{ar}(f)+j-1} g & \delta \leqslant 0 \\ f \circ_j (g \circ_\delta h) & \delta \in [1, \mathrm{ar}(g)] \\ (-1)^{|g||h|}(f \circ_\delta h) \circ_j g & \delta > \mathrm{ar}(g). \end{cases}$$

Exercise 18. A (unital associative) monoid x in a monoidal category $(\mathcal{C}, \otimes, \alpha, \rho, \lambda, 1)$ is an object along with maps $\mu : x \otimes x \to x$ and $\eta : 1 \longrightarrow x$ such that μ is associative, that is $\mu(\mu \otimes 1) = \mu(1 \otimes \mu)\alpha_{x,x,x}$, and unital for η, that is $\mu(\eta \otimes 1) = \rho_x$ and $\mu(1 \otimes \eta) = \lambda_x$. Show that a Σ-operad is exactly the same as a monoid in $({}_\Sigma\mathsf{Mod}, \circ_\Sigma)$.

Exercise 19. We write End for category of endofunctors of Vect. Show that there is a *monoidal* functor $S : {}_\Sigma\mathsf{Mod} \longrightarrow \mathsf{End}$ that assigns $\mathcal{X}$ to $V \longmapsto \bigoplus_{n \geqslant 0} \mathcal{X}(n) \otimes_{\Sigma_n} V^{\otimes n}$. It is called the *Schur functor* associated to $\mathcal{X}$. The endofunctors in the essential image of S are called *analytic*.

Exercise 20. If $\mathcal{X}$ is a symmetric sequence, describe the Σ_n action on $\mathcal{X}^{\otimes n}$ where $\otimes$ is the Cauchy product. Observe that it commutes with the $\mathrm{Aut}(I)$ action on $\mathcal{X}^{\otimes n}(I)$.

Exercise 21. Define ${}_\Sigma\mathsf{Mod}(\mathcal{C})$ for any symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ (such as the category of sets, or topological spaces, or chain complexes, among others) along with its *symmetric composition product* $- \circ_\Sigma -$.

Exercise 22. Prove that non-unital Markl operads and non-unital May operads differ. To do this, consider the non-unital ns operad $\mathcal{P}$ such that $\mathcal{P}(2)$ and $\mathcal{P}(4)$ are its only non-zero components, and are both one dimensional, and define

$$\gamma : \mathcal{P}(2) \otimes \mathcal{P}(2) \otimes \mathcal{P}(2) \longrightarrow \mathcal{P}(4)$$

to be an isomorphism, and all other maps zero. Check that $\mathcal{P}$ is a May operad, and show that $\mathcal{P}$ is not a Markl operad by exploring the consequences of the equality

$$\mu(\mu, \mu) = (\mu \circ_2 \mu) \circ_1 \mu$$

in any Markl operad.

Exercise 23. Check that examples (1), (2), (4), (5) in page 10 are indeed all operads.

Exercise 24. Suppose that $T \in \mathsf{RT}(n)$ and that $T' \in \mathsf{RT}(m)$, where RT is the symmetric collection of rooted trees of Lecture 1, and let $\mathrm{In}(T,i)$ denote the set of incoming edges of T at the vertex labeled i. For each function $f : \mathrm{In}(T,i) \longrightarrow [m]$, define the tree $T \circ_i^f T'$ by replacing vertex i of T by T' and attaching the loose incoming edges of vertex i to the vertices of T' according to the map f: the edge $e \in \mathrm{In}(T,i)$ is attached to vertex $f(e) \in T'$.

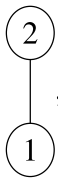
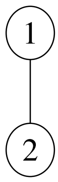
Finally, define $T \circ_i T'$ by taking the sum through all possible functions f. Show that this gives RT the structure of a unital pseudo-operad, and thus of a usual operad, with unit the tree with no edges and one vertex.

Exercise 25. Describe the operation $T \star T' = S(T, T')$ where S is the rooted tree above in terms of insertions of T' in T and regrafting of incoming edges. Show that it satisfies the following *pre-Lie identity*:

$$(T \star T') \star T'' - T \star (T' \star T'') = (T \star T'') \star T' - T \star (T'' \star T')$$

by explicitly interpreting the left hand side in terms of certain insertions of T' and T'' in T, and showing the resulting sum of trees is symmetric in T' and T''.

Exercise 26. Suppose that $\mathcal{P}$ is an operad and that $\mathcal{X} \subseteq \mathcal{P}$ is a symmetric subsequence. We say $\mathcal{X}$ generates $\mathcal{P}$ if every element of $\mathcal{P}$ is an iterated composition of elements of $\mathcal{X}$. Show that the rooted trees operad RT is generated by the symmetric subsequence given by the two labeled rooted trees with two vertices:

$S =$  , $S(12) =$ 

spanning the regular representation of S_2. Follow these steps:

(1) Suppose that T is an n-rooted tree and let J be a subset of $[n]$ corresponding to leaves of T that are the children of a vertex $i \in T$. Let T' be the tree obtained by erasing all these leaves and replacing the vertex label by a new symbol $*$, and let T'' be the rooted tree with root i and children labeled by J. Show that $T' \circ_* T'' = T$.

(2) Use the above and induction on the number of vertices to show it suffices to prove the claim for the corollas, that is, trees with one internal root vertex.

(3) Let us write T_n for the operation in $\mathsf{RT}(n)$ corresponding to a corolla with root 1, so in particular $T_2 = S$. Show that

$$T_n = T_2 \circ_1 T_{n-1} - \sum_{i=1}^{n-1} (T_{n-1} \circ T_i)\sigma_i$$

where $\sigma_i = (i+1, i+2, \ldots, n) \in S_n$ is a cycle, and use this to conclude.

Note. The operation T_n is usually denoted $\{x_1;x_2,\ldots,x_n\}$ and is called a *symmetric brace*, and the equation above is usually written in the form

$$\{x_1;x_2,\ldots,x_n\} = \{\{x_1;x_2,\ldots,x_{n-1}\};x_n\} - \sum_{i=1}^{n-1}\{x_1;x_2,\ldots,x_{i-1},\{x_i;x_n\},x_{i+1},\ldots,x_{n-1}\}.$$

Exercise 27. Let $\mathcal{X}$ be a symmetric sequence, and define the derivative $\partial\mathcal{X}$ of $\mathcal{X}$ to be symmetric sequence with $(\partial\mathcal{X})(I) = \mathcal{X}(I^*)$ where $I^* = I \sqcup \{I\}$. Note that S_I acts on I^* fixing the element I. Show that $(\partial\mathcal{X})(n)$ is isomorphic to the restriction of $\mathcal{X}(n+1)$ to $S_n = \mathrm{Fix}(n+1)$, and conclude that

$$\partial_z f_{\mathcal{X}}(z) = f_{\partial\mathcal{X}}(z).$$

Let s be the sequence of singletons and define the pointing of operation by $\mathcal{X}^\bullet = s \otimes_\Sigma \partial\mathcal{X}$. Determine the representation $\mathcal{X}^\bullet(n)$ in terms of $\mathcal{X}(n)$.

2 Free operads and presentations

Goals. We will define algebraic operads by generators and relations, and with this at hand define quadratic and quadratic-linear presentations of operads.

2.1 Trees

Operads and their kin are gadgets modeled after combinatorial graph-like objects. Operads themselves are modeled after rooted trees, so it is a good idea to have a concrete definition of what a rooted tree is. We will also consider planar rooted trees, and trees with certain decorations, so it is a good idea to digest the definitions carefully to later embellish them.

A rooted tree τ is the datum of a finite set $V(\tau)$ of vertices along with a partition $V(\tau) = \mathrm{Int}(\tau) \sqcup L(\tau) \cup R(\tau)$, where the first are the *interior* vertices, L are the leaves, and $R(\tau)$ is a singleton, called the root of τ. We also require there is a function $p : V(\tau) \smallsetminus R(\tau) \longrightarrow V(\tau)$, describing the edges of τ, with the following properties: call a vertex $v \in V(\tau)$ a child of $w \in V(\tau)$ if $v \in p^{-1}(w)$. Then:

(1) The root $r \in R(\tau)$ has exactly one child.
(2) The leaves of τ have no children.
(3) For each non-root vertex v there exist a unique sequence $(v_0, v_1, \ldots, v_k)$ such that $p(v_{i-1}) = v_i$ for $i \in [k]$ with $v_0 = v$ and $v_k = r$.

We will call a non-leaf vertex that has no children a *stump* (or an endpoint, or a cherry-top). A tree is reduced if has no stumps and all of its non-root and non-leaf vertices have at least two children. We will also call the root the (unique) output vertex τ, and the leaves the input vertices of τ.

A planar rooted tree is a rooted tree τ along with a linear order in each of the fibers of the parent function p of τ. In short, the children of each vertex are linearly ordered, so we are effectively considering a drawing of τ in the plane, where the clockwise orientation gives us the order at each vertex.

Two rooted trees τ and τ' are isomorphic if there exists a bijection $f : V(\tau) \longrightarrow V(\tau')$ that preserves the root, the input vertices and the interior vertices, so that $p' \circ f = p$ where we also write f for the induced bijection $f : V(\tau) \smallsetminus r \longrightarrow V(\tau') \smallsetminus r'$. Two planar rooted trees are isomorphic if in addition f respects the linear order at each vertex.

For example, consider the rooted tree τ with $V = \{1,2,3\} \cup \{4,5\} \cup \{0\}$, that is, three leaves, two interior vertices and the root. Then the choice of $p : [5] \to [\![5]\!]$ with $p(\{1,2\}) = 4$, $p(\{3,4\}) = 5$, $p(5) = 0$ gives a tree isomorphic to the one with with $p(\{1,2\}) = 3$,

$p(\{3,4\}) = 5$, $p(5) = 0$. On the other hand, if we consider the vertices linearly ordered by their natural order, these two planar rooted trees are no longer isomorphic.

Definition 2.1 For a finite set I, an I-labeled tree T is a pair (τ, f) where τ is a reduced rooted tree, along with a bijection $f : I \longrightarrow L(\tau)$. Two I-labeled trees T an T' are isomorphic if there exists a pair (g, σ) where g is an isomorphism between τ and τ' and σ is an automorphism of I such that $g\,|_{L(\tau)} \circ f = \sigma \circ f'$.

Suppose that (τ, f) is an I-tree and that (τ', f') is a J-tree, and that $i \in I$. We define $K = I \cup_i J = I \sqcup J \smallsetminus i$ and the K-tree $\tau \circ_i \tau'$ as follows:

(1) Its leaves are $L(\tau \circ_i \tau') = L(\tau) \sqcup L(\tau') \smallsetminus f^{-1}(i)$.
(2) Its internal vertices are $V(\tau) \sqcup V(\tau')$, with root r.
(3) The parent function q is defined by declaring that:
   - q coincides with p on $V(\tau)$,
   - $q(w) = p(f^{-1}(i))$ if w is the unique children of the root of τ',
   - q coincides with p' on $V(\tau') \smallsetminus \{r', w\}$.

(4) The leaf labeling is the unique bijection $L(\tau \circ_i \tau') \longrightarrow I \circ_i J$ extending f and f'.

2.2 Tree monomials

Let us now consider an (unbiased) reduced symmetric sequence $\mathcal{X}$ which we will think of as an *alphabet*. A tree monomial in the alphabet $\mathcal{X}$ is a pair (τ, x) where τ is a reduced rooted tree and $x : \mathrm{Int}(\tau) \longrightarrow \mathcal{X}$ is a map with the property that $x(v) \in \mathcal{X}(p^{-1}(v))$. Observe that reduced sequences and reduced trees correspond to each other, in the sense that with this definition we can only decorate a stump of τ with an element of $\mathcal{X}(\varnothing)$.

An I-labeled $\mathcal{X}$-tree T is a triple (τ, x, f) where (τ, f) is I-labeled and (τ, x) is an $\mathcal{X}$-tree. We will say that (τ, x, f) is a (symmetric) tree monomial if $\mathcal{X}$ is symmetric. If it is just a collection, we will say that (τ, x, f) is a ns tree monomial. In particular, if T is an I-labeled tree, and if $\sigma \in \mathrm{Aut}(I)$, there is another I-labeled tree $\sigma(T) = (\tau, f\sigma^{-1})$.

Suppose that $T = (\tau, x, f)$ is a tree monomial on an alphabet $\mathcal{X}$, and let us pick a vertex v of τ and a permutation σ of the set $C = p^{-1}(v)$ of children of v. We define the tree τ^σ as follows: the datum defining τ remains unchanged except p is modified to p^σ so that

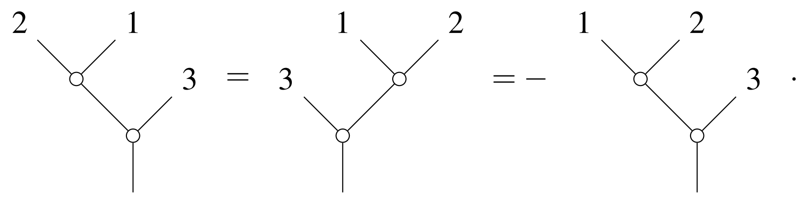
$$p^\sigma(w) = \begin{cases} p(w) & \text{if } p^2(w) \neq v \\ p(\sigma^{-1}(w')) & \text{if } p(w) = w' \in C. \end{cases}$$

Briefly, we just relabel the vertices of τ using σ. With this at hand, we define T^σ to be the tree monomial with underlying tree τ^σ and with x modified to x^σ so that

$$x^\sigma(w) = \begin{cases} \sigma x(v) & \text{if } v = w, \\ x(\sigma^{-1}(w')) & \text{if } p(w) = w' \in C. \end{cases}$$

Note that it is possible some children of v are leaves, in which case the definitions make sense if we think of leaves as decorated by the unit of $\Bbbk$.

Example 2.2 Let us consider the alphabet $\mathcal{X} = \mathcal{X}(2) = \{*\}$ where the unique operation is antisymmetric. Then we have the following equalities of symmetric tree monomials:



Let us now define for each $n \geqslant 1$ the space $\mathcal{F}_{\mathcal{X}}(I)$ as the span of all tree monomials $T = (\tau, f, x)$ on $\mathcal{X}$ with leaves labeled by I, modulo the subspace generated by all elements of the form

$$R(T, v, \sigma) = T - T^\sigma$$

where σ ranges through $\mathrm{Aut}(p^{-1}(v))$ as v ranges through the vertices of τ. In case all children of v are leaves, this is saying that the tree where x_v is replace by $\sigma(x_v)$ is equal to the tree where the leaves of T that are children of v are relabeled according to σ. We also require that tree decorations behave like tensors, so that $T = T_1 + T_2$ if the decoration of T at a vertex v is of the form $x_1 + x_2$ and for $i \in [2]$ the tree T_i coincides with T except that it is decorated by x_i at v.

2.3 The free operad

An algebraically inclined way to construct (algebraic) operads is through generators and relations. There is a forgetful functor from the category of operads to the category of collections. In general, it admits a left adjoint, which is the free operad functor.

Definition 2.3 The *free symmetric operad* on $\mathcal{X}$ is the symmetric sequence $\mathcal{F}_{\mathcal{X}}$ along with the composition law obtained by grafting of trees. More precisely, suppose that $T \in \mathcal{F}_{\mathcal{X}}(I)$ and that $T' \in \mathcal{F}_{\mathcal{X}}(J)$, and that $i \in I$. We define $T'' = T \circ_i T' \in \mathcal{F}_{\mathcal{X}}(I \cup_I J)$ by taking its underlying labeled tree to be $\tau \circ_i \tau'$, and by decorating it in the unique way which extends the decorations of T and T'.

The following lemma shows that this indeed defines an operad.

Lemma 2.4 *Tree grafting respects both I-tree isomorphisms and the relations $T \sim T^\sigma$ above, and hence is well defined on $\mathcal{F}_{\mathcal{X}}$.*

Proof. This is Exercise 29. □

We will later interpret $\mathcal{X} \longmapsto \mathcal{F}_{\mathcal{X}}$ as a *monad*, thus giving another definition of operads. The advantage of this 'monadic approach' is its flexibility, which allow us to define other operad like structures, like the ones mentioned in the introduction. In this direction, a curious reader can consider the following equivalent definition; see also Exercise 32.

Definition 2.5 The free operad generated by a symmetric collection X is defined inductively by letting $\mathcal{F}_{0,X} = \Bbbk$ be spanned by the 'twig' (tree with no vertices and one edge) in arity zero and

$$\mathcal{F}_{n+1,\mathcal{X}} = \Bbbk \oplus (\mathcal{X} \circ \mathcal{F}_{n,\mathcal{X}}),$$

and finally by setting $\mathcal{F}_{\mathcal{X}} = \varinjlim_n \mathcal{F}_{n+1,\mathcal{X}}$. The composition maps are defined by induction, and the axioms are also checked by induction.

Intuitively, the previous definition says that an element of $\mathcal{F}_{\mathcal{X}}$ is either the twig, or corolla with n vertices decorated by $\mathcal{X}$, whose leaves have on them an element of $\mathcal{F}_{\mathcal{X}}$. The final shape of $\mathcal{F}_{\mathcal{X}}$ will however depend on the symmetric structure of $\mathcal{X}$.

2.4 Exercises

Exercise 28. Let $\mathcal{X}$ be a collection such that $\underline{\mathcal{X}} = \mathcal{X}(2)$. Compute a basis of tree monomials for the free operad over $\mathcal{X}$ in case $\mathcal{X}(2)$ is:

(1) The regular representation of S_2.
(2) The sign representation of S_2.
(3) The trivial representation of S_2.

In all cases, decompose the S_3-module $\mathcal{F}_{\mathcal{X}}(3)$ into irreducible representations.

Exercise 29. Show that tree grafting respects both I-tree isomorphism and the relation $T \sim T^{\sigma}$, and hence descends to $\mathcal{F}_{\mathcal{X}}$.

Exercise 30. Suppose that $\mathcal{X}$ is an alphabet (in sets) that is finite in each arity and such that $\mathcal{X}(n) = \varnothing$ for $n = 0, 1$. Show that $\mathcal{F}_{\mathcal{X}}$ is finite in each arity.

Exercise 31. Define non-symmetric tree monomials over a ns alphabet $\mathcal{X}$ and thus define the free *non-symmetric* operad over a collection $\mathcal{X}$.

Exercise 32. Read the statement and proof of *Theorem 5.4.2* in 'Algebraic Operads' that the colimit construction briefly described in the lecture notes does give the free operad on a symmetric collection.

Exercise 33. Consider the map from ns collections to symmetric sequences that assigns $\mathcal{X}$ to $\Sigma \otimes \mathcal{X}$ such that $(\Sigma \times \mathcal{X})(n) = \Sigma_n \times \mathcal{X}(n)$ with its corresponding symmetric group action. What is the relation between the free ns operad on $\mathcal{X}$ and the free symmetric operad on $\Sigma \times \mathcal{X}$?

Exercise 34. Let V be an S_2-module, and let $\mathcal{X}$ be the symmetric collection with $\mathcal{X}(2) = V$ and zero everywhere else. Show that $\mathcal{F}_{\mathcal{X}}(3)$ consists of three copies of $V^{\otimes 2}$ and describe explicitly the action of S_3 on it.

Exercise 35. Show that the construction of the free operad we carried out during **Lecture 2** indeed defines the free operad on $\mathcal{X}$ where $i : \mathcal{X} \longrightarrow \mathcal{F}_{\mathcal{X}}$ sends an element $x \in \mathcal{X}(I)$ to the corolla whose unique internal vertex is labeled by x (and whose leaves are labeled by I).

Exercise 36. Follow the lecture notes and read about weight gradings and the canonical weight grading on a free operad.

3 Quadratic operads

Goal. Introduce weight graded gadgets, define operads by generators and relations, and introduce quadratic operads. Give plenty of examples of 'real life' quadratic operads to work on: Hilbert series, Koszul dual, bar construction.

3.1 Weight gradings and presentations

The notion of a quadratic operad is based on the observation every free operad has a canonical 'weight grading' by the number of internal vertices of a tree. Let us make this precise.

Definition 3.1 A symmetric sequence $\mathcal{X}$ is weight graded if for each finite set the component $\mathcal{X}(I)$ admits a decomposition $\mathcal{X}(I) = \bigoplus_{j\geqslant 0} \mathcal{X}^{(j)}(I)$. A symmetric operad $\mathcal{P}$ is weight graded if its underlying symmetric sequence is weight graded and its composition maps preserve the weight grading.

Thus, a weight graded operad must have composition maps of the form

$$\mathcal{P}^{(a)}(k)\otimes\mathcal{P}^{(b_1)}(n_1)\otimes\cdots\otimes\mathcal{P}^{(b_k)}(n_k)\longrightarrow\mathcal{P}^{(b)}(n)$$

where $b = b_1+\cdots+b_k$ and $n = n_1+\cdots+n_k$. In the case we consider partial composition maps, observe we have instead maps of the form

$$\circ_i : \mathcal{P}^{(a)}(m)\otimes\mathcal{P}^{(b)}(n)\longrightarrow\mathcal{P}^{(a+b)}(m+n-1).$$

The free operad $\mathcal{F}_{\mathcal{X}}$ is weight graded by the number of internal vertices of a tree (that is, we put $\mathcal{X}$ in weight one, and extend the weight to trees by counting occurrences of elements of $\mathcal{X}$. More generally, if $\mathcal{X}$ admits a weight grading, then $\mathcal{F}_{\mathcal{X}}$ inherits this weight grading: the weight of a tree monomial is the sum of the weight of the decorations of its vertices, and we write $\mathcal{F}_{\mathcal{X}}^{(n)}$ for the homogeneous component of weight $n\in\mathbb{N}_0$. If we do not specify a weight grading on $\mathcal{F}_{\mathcal{X}}$, we will always assume we are taking the canonical weight grading above.

Definition 3.2 An ideal in an operad $\mathcal{P}$ is a subcollection $\mathcal{I}$ for which both $\gamma(\mathcal{I}\circ\mathcal{P})$ and $\gamma(\mathcal{P}\circ_{(1)}\mathcal{I})$ are contained in $\mathcal{I}$. The quotient of $\mathcal{P}/\mathcal{I}$ is again an operad, called the quotient of $\mathcal{P}$ by $\mathcal{I}$. Every subcollection $\mathcal{R}$ of $\mathcal{P}$ is contained in a smallest ideal, called the *ideal generated by* $\mathcal{R}$.

The notion of ideals and of free operads allow us to define operads by generators and relations.

Definition 3.3 We write $\mathcal{F}(\mathcal{X},\mathcal{R})$ for the quotient of $\mathcal{F}_{\mathcal{X}}$ by the ideal generated by a subcollection $\mathcal{R}$ of $\mathcal{F}_{\mathcal{X}}$. We say $\mathcal{P}$ is presented by generators $\mathcal{X}$ and relations $\mathcal{R}$ if there is an isomorphism $\mathcal{F}(\mathcal{X},\mathcal{R}) \longrightarrow \mathcal{P}$.

Note that if $\mathcal{P}$ is symmetric, the definition requires that $\mathcal{I}$ be stable under the symmetric group actions, so we may sometimes specify $\mathcal{R}$ by a generating set only, and understand that $(\mathcal{R})$ is generated by the Σ-orbit of $\mathcal{R}$.

Some examples. To illustrate the definitions above, let us give three examples of algebraic operads whose associated algebras are probably well known to the reader:

(1) The associative operad is generated by a binary operation μ generating the regular representation of S_2 subject to the relation $\mu \circ_1 \mu = \mu \circ_2 \mu$.
(2) The commutative operad is generated by a binary operation which instead generates the trivial representation of S_2 and is also associative. Both of this and the previous example arise as the linearization of a set operad.
(3) The Lie operad is generated by a single binary operation β that generates the sign representation of S_2 subject to the only relation $(\beta \circ_1 \beta)(1+\tau+\tau^2) = 0$ where $\tau = (123) \in S_3$ is the 3-cycle.

We write these operads $\mathsf{As}, \mathsf{Com}$ and Lie and, following J.-L. Loday, call them the *three graces*. We have that

$$\mathsf{As}(n) = \Bbbk S_n, \quad \mathsf{Com}(n) = \Bbbk, \quad \mathsf{Lie}(n) = \operatorname{Ind}_{\mathbb{Z}/n}^{S_n} \Bbbk_\zeta$$

where $\Bbbk_\zeta$ is a character of $\mathbb{Z}/n$ for a primitive nth root of the unit. Concretely, the last equality is stating that if we fix a primitive kth root of unity ζ_k, and if we let ρ_k be the standard k-cycle of S_k, the free Lie algebra $L(V) \subseteq T(V)$ identifies in each weight degree k with those $v \in V^{\otimes k}$ such that $\rho_k v = \zeta_k v$.

Note 3.4 It is not always advantageous to define an operad by generators and relations: the operad pre-Lie can be defined explicitly in terms of labeled rooted trees and a grafting operation, as done by Chapoton–Livernet, and this ‘presentation’ is very useful in practice, for example, to show that the pre-Lie operad is Koszul.

3.2 Quadratic operads

An operad $\mathcal{P}$ is *quadratic* if it admits a presentation $\mathcal{F}(\mathcal{X},\mathcal{R})$ where $\mathcal{R} \subseteq \mathcal{F}(\mathcal{X})^{(2)}$. That is, $\mathcal{P}$ is generated by some collection of operations $\mathcal{X}$ and all its defining relations are of the form

$$\sum \lambda^i_{\mu,\nu} \mu \circ_i \nu = 0$$

where $\mathrm{ar}(\mu)+\mathrm{ar}(\nu)$ is constant. An operad is *binary quadratic* if moreover $\mathcal{X}=\mathcal{X}(2)$ or, what is the same, all the generating operations of $\mathcal{P}$ are of arity two (binary). A *quadratic-linear presentation* of an operad $\mathcal{P}$ is a presentation $\mathcal{F}(\mathcal{X},\mathcal{R})$ of $\mathcal{P}$ where $\mathcal{R}\subseteq\mathcal{X}\oplus\mathcal{F}(\mathcal{X})^{(2)}$. That is, it is a presentation of the form

$$\sum\lambda^i_{\mu,\nu}\mu\circ_i\nu+\sum\lambda_\rho\rho=0$$

where $\mathrm{ar}(\mu)+\mathrm{ar}(\nu)=\mathrm{ar}(\rho)+1$ is constant. Every operad admits a quadratic-linear presentation, albeit with possibly with infinitely many generators, We will postpone the discussion of such presentations to a later lecture.

Let us define a quadratic datum to be a pair $(\mathcal{X},\mathcal{R})$ where $\mathcal{X}$ is a symmetric sequence and $\mathcal{R}\subseteq\mathcal{F}^{(2)}_{\mathcal{X}}$. A map of quadratic data $(\mathcal{X}_1,\mathcal{R}_1)\longrightarrow(\mathcal{X}_2,\mathcal{R}_2)$ is a map $\mathcal{X}_1\to\mathcal{X}_2$ of symmetric sequences for which the induced map on free operads sends $\mathcal{R}_1$ to $\mathcal{R}_2$. The assignment $(\mathcal{X},\mathcal{R})\longrightarrow\mathcal{F}(\mathcal{X},\mathcal{R})$ defines a functor from the category of quadratic data to the category of quadratic operads.

More examples. The presentations of the associative, commutative and Lie operad above are quadratic. The following are also quadratic operads:

The Gerstenhaber operad. The symmetric operad Ger and its cousin, the Poisson operad Poiss belong to the two parameter family $\mathsf{Poiss}(a,b)$ of binary quadratic operads generated by two operations x_1x_2 and $[x_1,x_2]$ of respective degrees a and b, so that the first is commutative associative, the second is a Lie bracket, and they satisfy the Leibniz rule. With this at hand $\mathsf{Ger}=\mathsf{Poiss}(0,-1)$ while $\mathsf{Poiss}=\mathsf{Poiss}(0,0)$.

The pre-Lie operad. The operad PreLie and its quotient, the Novikov operad Nov, are quadratic binary operads generated by a single operation $x_1\circ x_2$ with no symmetries. The first one is subject to the right-symmetry condition for the associator

$$x_1\circ(x_2\circ x_3)-(x_1\circ x_2)\circ x_3=x_1\circ(x_3\circ x_2)-(x_1\circ x_3)\circ x_2.$$

The second operad is obtained by further imposing the left-permutative relation that

$$x_1\circ(x_2\circ x_3)=x_2\circ(x_1\circ x_3).$$

The permutative operad Perm is the binary operad generated by a single operation with no symmetries satisfying the last quadratic equation.

The operad of totally associative k-ary algebras. tAs_k (and its commutative counterpart). It is generated by a k-ary non-symmetric operation α subject to the relations $\alpha\circ_i\alpha=\alpha\circ_k\alpha$ for all $i\in[k]$. One can consider α to be totally symmetric, and obtain the operad of totally associative commutative k-ary algebras.

The operad of partially associative k-ary algebras. pAs^k (and its Lie counterpart). It is generated by a k-ary non-symmetric operation α of degree $k-2$ subject to the single relation

$$\sum_{i=1}^{k}(-1)^{(k-1)(i-1)}\alpha\circ_i\alpha=0.$$

One can consider a k-ary totally antisymmetric operation β of degree 1, and obtain the operad of Lie k-algebras, which is subject to the single equation

$$\sum_{\substack{A\sqcup B=[2k-3]\\|A|=k-1,|B|=k-2}}(\beta\circ_1\beta)\sigma_{A,B}=0.$$

The operad of anti-associative algebras. As^- is generated by a single operation of degree zero with no symmetries satisfying the 'anti-associative law'

$$x_1(x_2x_3)+(x_1x_2)x_3=0.$$

3.3 Exercises

Exercise 37. During Lecture 3 we introduced the associative and commutative operads through binary quadratic presentations. Show that for all $n\geqslant 1$ the space $\mathsf{Ass}(n)$ is the regular representation of S_n, and that for all $n\geqslant 1$ the space $\mathsf{Com}(n)$ is the trivial representation.

Exercise 38. Use the presentation of the Poisson operad given during Lecture 3 to show that $\dim\mathsf{Poiss}(n)\leqslant n!$ for all $n\geqslant 1$[1].

Exercise 39. Let x_1x_2 be the associative binary generator of Ass and let us consider the operations (which are symmetric and antisymmetric, respectively)

$$x_1\cdot x_2=\frac{1}{2}(x_1x_2+x_2x_1),\quad [x_1,x_2]=\frac{1}{2}(x_1x_2-x_2x_1)$$

obtained by 'polarization'. Show that the second is a Lie bracket, and that the first is a commutative (but not associative) product that satisfies the Leibniz rule for $[x_1,x_2]$, and whose associator is equal to $[x_2,[x_1,x_3]]$. This is called the *Livernet–Loday presentation* of the associative operad.

Exercise 40. During Lecture 3, we introduced to operad $\mathtt{tCom}_k$ of totally associative commutative k-ary algebras. It is generated by a single fully symmetric operation μ or arity k subject to the relations $\mu\circ_1\mu=\mu\circ_i\mu$ for each $i\in[k]$ (and all its symmetric translates).

[1] There are at least three different ways to show that equality holds.

Show that $\mathsf{tCom}_k(n)$ is either the one dimensional trivial representation or zero depending on n. What values must n take so that it is non-zero?

Exercise 41. The permutative operad Perm is generated by a single binary operation x_1x_2 with no symmetries which is associative, and such that

$$x_1(x_2x_3) = x_2(x_1x_3).$$

Show that $\mathsf{Perm}(n)$ is of dimension n and is isomorphic as a representation to $\operatorname{Ind}_{S_{n-1}}^{S_n}\mathbb{C}$ where $\mathbb{C}$ is the trivial representation.

We have defined quadratic operads as precisely those operads presented by (homogeneous) quadratic relations on some set of generators. Let us explore how to create maps between them.

Exercise 42. Suppose that $(\mathcal{X},\mathcal{R})$ and $(\mathcal{Y},\mathcal{Q})$ are quadratic data. Show that a map of sequences $f:\mathcal{X}\longrightarrow\mathcal{Y}$ induces a map on the corresponding quadratic operads if and only if the induced map $F=\mathcal{F}_f$ sends $\mathcal{R}$ to $\mathcal{Q}$.

Exercise 43. Show that:

(1) The augmentation map $\mathbb{C}S_2\longrightarrow\mathbb{C}$ (that sends 1 and (12) to 1) induces a surjective map of operads $\mathsf{Ass}\longrightarrow\mathsf{Com}$.
(2) The inclusion map $\mathbb{C}^-\longrightarrow\mathbb{C}S_2$ that assigns 1 to $1-(12)$ induces a map of operads $\mathsf{Lie}\longrightarrow\mathsf{Ass}$ and also a map of operads $\mathsf{Lie}\longrightarrow\mathsf{PreLie}$.
(3) The projection $\mathsf{Ass}\longrightarrow\mathsf{Com}$ actually factors through Perm.

In each case, what is the interpretation at the level of algebras?

4 Koszul duality I: quadratic duals

Goals. Give the definition of the Koszul dual operad of a quadratic operad, and then compute some Koszul duals.

4.1 Differential graded sequences

Homologically graded Σ-modules. A (homologically) graded vector space is a vector space V along with a direct sum decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$. We call the components of this sum the *graded (or homogeneous) components of* V, and say that an element in on of these summands is *homogeneous*. If $v \in V_n$, we say that v is *homogeneous of degree* n and write $|v| = n$.

A map $f : V \longrightarrow W$ of graded vector spaces is *homogeneous of degree* n if $f(V_j) \subseteq W_{j+n}$ for all $j \geqslant 1$. We write $\hom(V,W)$ for the space of all homogeneous maps, which is itself a graded vector space with $\hom(V,W)_n$ the space of all graded maps of degree n for each $n \in \mathbb{Z}$. In this way, we obtain the category $\mathsf{Vect}_{\mathbb{Z}}$ of graded vector spaces and graded maps.

A *differential graded (dg) vector space* is a pair (V,d) where V is a graded vector space and $d : V \longrightarrow V$ is a homogeneous map of degree -1 such that $d^2 = 0$. We usually will call (V,d) a *chain complex*. The collection of homogeneous maps $V \longrightarrow W$ is again a chain complex, with differential

$$d\varphi = d_V \varphi - (-1)^{|\varphi|}\varphi d_W .$$

A homogeneous map of degree zero such that $d(\varphi) = 0$ is called a *chain map*. It is convenient to also consider *cohomologically graded* vector spaces, by formally inverting the order of $\mathbb{Z}$ and letting $V^n = V_{-n}$ for all $n \in \mathbb{Z}$.

Monoidal structure. If V and W are graded vector spaces, we define their tensor product by setting

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$$

for all $n \in \mathbb{Z}$, and setting the symmetry map

$$\tau : V \otimes W \longrightarrow W \otimes V$$

to be $\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v$ on homogeneous elements, and extending it linearly on all of $V \otimes W$. This makes $\mathsf{Vect}_{\mathbb{Z}}$ into a symmetric monoidal category with unit the graded vector space with $V_0 = \Bbbk$ and $V_n = 0$ for $n \neq 0$. The tensor product of maps $f : V \longrightarrow V'$ and $g : W \longrightarrow W'$ acts in such a way that $f \otimes g : V \otimes V' \longrightarrow W \otimes W'$ is the map

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

In case V and W are in fact dg, their tensor product is also dg with $d_{V\otimes W} = d_V \otimes 1 + 1 \otimes d_W$.

Definition 4.1 A (homologically) graded Σ-module $\mathcal{X}$ is a Σ-module taking values in the category of graded vector spaces. Similarly, a dg Σ-module is one taking values in dg vector spaces.

The endomorphism operad functor on dg modules. Let us consider the most natural way to create dg modules from dg vector spaces, as we did in the case of usual vector spaces. Namely, we may as before consider the *endomorphism operad* of a dg vector space V by setting, for each $n \geqslant 0$,

$$\mathrm{End}_V(n) = \hom(V^{\otimes n}, V)$$

where these consists of homogeneous maps of dg vector spaces. In particular, each of these arity components is itself a dg vector space, and the (total or partial) composition maps of the resulting operad are maps of dg vector spaces.

Of particular importance to us will be the *suspension* operation on dg vector spaces. Let us write s for the unique dg vector space with $s_1 = \mathbb{C}$ and zero elsewhere, and similarly let us write s^{-1} for the unique dg vector space with $s^{-1} = \mathbb{C}$ and zero elsewhere. The *suspension* of the dg vector space V is the tensor product $s \otimes V$, which we write more simply sV, and whose basis elements we write sv for $v \in V$. Thus $|sv| = |v| + 1$ for all homogeneous $v \in V$. Similarly, we define the *desuspension* $s^{-1}V$.

Note 4.2 The differential of sV is given by $d(sv) = -sdv$. Can you explain why this is so using the Koszul sign rule?

The following lemma shows that $V \mapsto \mathrm{End}_V$ is monoidal for the *Hadamard product* of operads on the target (and the usual tensor product on the domain):

Lemma 4.3 *The map* $\Phi : \mathrm{End}_V \otimes \mathrm{End}_W \longrightarrow \mathrm{End}_{V\otimes W}$ *that assigns* $\varphi \otimes \psi \in \mathrm{End}_V(n) \otimes \mathrm{End}_W(n)$ *to the map*

$$\Phi(\varphi,\psi)(v,w) = (-1)^{\varepsilon} \varphi(v) \otimes \psi(w)$$

where $\varepsilon = \sum_{i=1}^{n} (|w_1| + \cdots + |w_{i-1}| + |\psi|)|v_i|$ *is an isomorphism of operads provided* V *and* W *are locally finite.*

Proof. This is Exercise 44 □

In particular, we see that End_{sV} is canonically isomorphic with $\mathrm{End}_s \otimes \mathrm{End}_V$, and hence that algebra structures on sV are related to algebra structures on V through the operad End_s. Let us give it a name.

4.2 The Koszul dual

Suspensions. We call End_s the suspension operad and write it $\mathscr{S}$. Note that $\mathrm{End}_s(n)$ is the sign representation of Σ_n put in degree $1-n$.

Proposition 4.4 *For each $n \geqslant 1$ let us we write ν_n for the unique map in $\mathrm{End}_s(n)$ that sends s^n to s. Then for every $m \geqslant 1$ we have that*

$$\nu_n \circ_i \nu_m = (-1)^{(i-1)(m-1)} \nu_{m+n-1}.$$

In particular, the binary operation $\nu := \nu_2$ of degree -1 generates End_s, and presents it as a quadratic operad subject to the anti-associativity relation

$$\nu \circ_1 \nu + \nu \circ_2 \nu = 0.$$

Proof. This is Exercise 45. □

If $\mathcal{P}$ is an operad, then the arity-wise tensor product $\mathscr{S} \otimes \mathcal{P}$ is called the suspension of $\mathcal{P}$ and we write it $\mathscr{S}\mathcal{P}$ or $\mathcal{P}\{1\}$. Dually, we write $\mathscr{S}^{-1}$ for the desuspension operad defined by $\mathrm{End}_{s^{-1}\Bbbk}$.

Note 4.5 As we just observed, the operad $\mathscr{S}\mathcal{P}$ has the property that $\mathscr{S}\mathcal{P}(sV) = s\mathcal{P}(V)$, so that algebras over $\mathscr{S}\mathcal{P}$ are exactly those vector spaces V such that $s^{-1}V$ is a $\mathcal{P}$-algebra. Equivalently, sV is a $\mathscr{S}\mathcal{P}$-algebra if and only if V is a $\mathcal{P}$-algebra.

Pairings. We define a pairing between $\mathcal{F}_{\mathcal{X}}$ and $\mathcal{F}_{s^{-1}\mathscr{S}^{-1}\mathcal{X}^*}$ as follows (the appearance of the suspensions will be evident later):

$$\langle \Sigma\nu^* \circ_j \Sigma\mu*, \rho \circ_i \tau \rangle = \delta_{ij}(-1)^{\varepsilon}\nu^*(\rho)\mu^*(\tau).$$

where $\varepsilon_1 = (\mathrm{ar}(\nu)-1)(|\mu|+i-1)+|\nu||\mu|$ and ε_2 counts the total number of inversions in the shuffle permutations appearing in the two tree monomials. If $\mathcal{X} = \mathcal{X}(2)$ is binary and has no homological degrees, this simplifies to

$$\langle \Sigma\nu^* \circ_i \Sigma\mu*, \rho \circ_i \tau \rangle = \begin{cases} (-1)^{\varepsilon}\nu^*(\rho)\mu^*(\tau) & i = 1 \\ -\nu^*(\rho)\mu^*(\tau) & i = 2. \end{cases}$$

where ε depends on the decoration of the leaves (it is 1 if both decorations are equal, and is -1 if exactly one is the shuffle 132.

Definition 4.6 The Koszul dual operad of a quadratic operad $\mathcal{P}$ generated by $\mathcal{X}$ subject to relations $\mathcal{R}$, is the operad $\mathcal{P}^!$ generated by $s^{-1}\mathscr{S}^{-1}\mathcal{X}^*$ and subject to the orthogonal space of relations $\mathcal{R}^\perp$ according to the pairing above.

Note 4.7 Let $\mathcal{P}$ be an operad. Then $\mathcal{P}$ is quadratic if and only if $\mathscr{S}\mathcal{P}$ is quadratic, and it is Koszul if and only if $\mathscr{S}\mathcal{P}$ is Koszul.

Some examples. Let us compute the Koszul duals of some of the quadratic operads we introduced in **Lecture 3**. For simplicity, we will consider only those with binary generators of degree zero, though one can in the same way carry out computations with generators of higher arities and varying homological degrees.

The associative operad. We saw previously that for $\underline{\mathcal{X}}$ consisting of a single operation x_1x_2 with no symmetries, the space $\mathcal{F}_{\mathcal{X}}(3)$ is twelve dimensional, spanned by the S_3-orbits of $\alpha = x_1(x_2x_3)$ and $\beta = (x_1x_2)x_3$, each of size six. We also noted that $\alpha - \beta$ spans a six dimensional submodule, complemented by the orbit of $\alpha + \beta$.
Using the pairing above, we see that

$$\langle \alpha, \alpha \rangle = 1, \quad \langle \beta, \beta \rangle = -1, \quad \langle \alpha, \beta \rangle = 0,$$

from where it follows that the dual space to the associativity relation is the corresponding associativity relation $\alpha^* - \beta^*$ in $\mathcal{X}^*$. In other words, the associative operad is Koszul self-dual:

$$\mathsf{Ass}^! = \mathsf{Ass}.$$

It is important to note how the minus sign in our definition of the pairing or, more generally, the Koszul sign we have introduced, guaranteeing that this pairing in equivariant, introduces the minus sign in the dual of $\alpha + \beta$.

The commutative and Lie operads. We have computed that if $\mathcal{X}(2)$ is the trivial representation of S_2 spanned by some commutative operation x_1x_2, then $\mathcal{F}_{\mathcal{X}}(3)$ is three dimensional, spanned by $x_1(x_2x_3)$, $(x_1x_2)x_3$ and $(x_1x_3)x_2$. Moreover, we verified that if we put

$$\alpha = x_1(x_2x_3) - (x_1x_2)x_3, \quad \beta = x_1(x_2x_3) - (x_1x_3)x_2$$

then these two element span an S_3-submodule that is complemented by the S_3-submodule generated by

$$\gamma = x_1(x_2x_3) + (x_1x_2)x_3 + (x_1x_3)x_2.$$

This is in fact an orthogonal complement as a direct computation shows, so we see that the orthogonal set of relations to the commutative associative relation is the dual of γ for the dual antisymmetric operation $[x_1,x_2]$: this is exactly the Jacobi relation

$$\gamma^* = -[x_1,[x_2,x_3]] + [[x_1,x_2],x_3] + [[x_1,x_3],x_2].$$

It follows that the Koszul dual of the commutative operad is the Lie operad, and conversely:

$$\mathsf{Com}^! = \mathsf{Lie}, \quad \mathsf{Lie}^! = \mathsf{Com}.$$

With this at hand, one can compute that the Poisson operad is self-dual: one only needs to address the Leibniz relation.

The pre-Lie and permutative operads. The Novikov operad. Recall the pre-Lie operad is generated by a single operation x_1x_2 with no symmetries, subject to the pre-Lie relation

$$(x_1x_2)x_3 - x_1(x_2x_3) - (x_1x_3)x_2 + x_1(x_3x_2).$$

One can check that the S_3-orbit V of this element is three dimensional, so let us write α_1, α_2 and α_3 for the translates of this relation in $\mathcal{F}_{\mathcal{X}}(3)$.

This orbit is complemented by the orbit W of the associativity relation $(x_1x_2)x_3 - x_1(x_2x_3)$ and the orbit U of the permutative relation $(x_1x_2)x_3 - (x_1x_3)x_2$. The first is six dimensional, as we already computed, while the second is three dimensional. It is a direct computation to check that $V^\perp$ identifies with the nine dimensional subspace $U^* \oplus W^*$.

Thus, we see that the operad of pre-Lie algebra is Koszul dual to that of permutative algebras:

$$\mathsf{PreLie}^! = \mathsf{Perm}, \quad \mathsf{Perm}^! = \mathsf{PreLie}.$$

One can use this to show that the operad controlling Novikov algebras, those pre-Lie algebras whose product is *left* permutative

$$x_1(x_2x_3) = x_2(x_1x_3)$$

is almost Koszul self-dual: we have that $\mathsf{Nov}^! = \mathsf{Nov}^{\mathrm{op}}$, by which we mean the resulting operad controls pre-Lie algebras with associator symmetric in the *first two* variables (left-symmetric) and whose pre-Lie operation is *right* permutative.

4.3 Exercises

Exercise 44. Show the map $\Phi_{V,W}$ of Lemma 4.3 is an isomorphism for V and W locally finite dimensional dg symmetric sequences.

Exercise 45. Show that the suspension operad is binary quadratic generated by a single operation ν of degree -1 that is "anti-associative", in the sense that $\nu \circ_1 \nu + \nu \circ_2 \nu = 0$.

Exercise 46. Show that:

(1) Ass is Koszul self dual.
(2) Com and Lie are Koszul dual to each other.
(3) PreLie and Perm are Koszul dual to each other.
(4) the Poisson operad is Koszul self-dual.

Exercise 47. The operad Nov of Novikov algebras is the quotient of the (right) pre-Lie operad by the left permutative relation $x_1(x_2x_3) = x_2(x_1x_3)$. Show that Nov is Koszul dual to its "opposite" operad $\mathsf{Nov}^{\mathrm{op}}$ controlling left pre-Lie algebras satisfying the right permutative relation.

Exercise 48. Show that:

(1) The Koszul dual of the operad controlling totally associative k-ary algebras is the operad controlling partially associative k-ary algebras.
(2) The Koszul dual of the operad controlling commutative totally associative k-ary algebras is the operad controlling k-ary Lie algebras.

Exercise 49. Let x_1x_2 be a binary operation and consider the two relations:

$$R = (x_1x_2)x_3 - \sum_{\sigma \in S_3} \lambda_\sigma \sigma(x_1(x_2x_3)), \qquad S = x_1(x_2x_3) - \sum_{\sigma \in S_3} \lambda_\sigma \sigma^{-1}((x_1x_2)x_3).$$

Show that the resulting quadratic operads $\mathcal{F}(x_1x_2)/(R)$ and $\mathcal{F}(x_1x_2)/(S)$ are Koszul dual to each other.

Exercise 50. Show that in the case of binary operads, the bilinear form we constructed during the lectures is S_3-invariant.

5 Shuffle operads

Goal. Introduce shuffle operads and prove that the free symmetric operad on a reduced symmetric collection is isomorphic, as a shuffle operad, to the free shuffle operad on the corresponding shuffle collection.

5.1 Shuffle operads

Recall that the category of ns collections on some category C consists of those pre-sheaves on the category of finite ordered sets and order preserving bijections with values in C: a ns collection on C is simply a list of objects of C indexed by the non-negative integers (considered as totally ordered sets of finite cardinality).

Definition 5.1 An ordered partition π of length n of a finite totally order set set is called *shuffling* if $\min \pi_i < \min \pi_{i+1}$ for each $i \in [n-1]$. Equivalently, a surjection $f : I \longrightarrow [n]$ with I a totally ordered set is called *shuffling* if $\min f^{-1}(i) < \min f^{-1}(i+1)$ for each $i \in [n-1]$.

Although totally ordered sets along with bijections form a rather dull category, this category admits a composition product, which we call the *shuffle composition product*, defined as follows, and which will turn out to be crucial for our purposes.

Definition 5.2 For each pair of ns collections $\mathcal{X}$ and $\mathcal{Y}$, we define the ns collection $\mathcal{X} \circ_{\text{Ш}} \mathcal{Y}$ so that on each totally order finite set we have that

$$(\mathcal{X} \circ_{\text{Ш}} \mathcal{Y})(I) = \bigoplus_{\substack{r \geqslant 1 \\ f : I \longrightarrow [r]}} \mathcal{X}([r]) \otimes \mathcal{Y}(f^{-1}(1)) \otimes \cdots \otimes \mathcal{Y}(f^{-1}(r))$$

where the sum runs through all $r \geqslant 1$ and all possible shuffling surjections $f : I \longrightarrow [r]$.

One can prove that this product is associative, in the same way that one proves $\circ_\Sigma$ and $\circ_{\text{ns}}$ are. In some way, the shuffle composition product interpolates between the symmetric composition product, which contains "too many" summands, and the ns composition product, which contains too few. We leave the following proposition as an exercise.

Proposition 5.3 *The category of ns collections along with the shuffle composition product is monoidal with the same unit as that of the ns composition product.* □

Note that we can also define a shuffle Cauchy product, by looking at shuffling partitions of a finite order set that have length two. Although we will not study the resulting monoidal category here, we remark it gives rise to interesting monoids, usually known as shuffle algebras.

Definition 5.4 A shuffle operad is a monoid in the category of ns collections with the shuffle composition product.

Thus, a shuffle operad consists of the datum of a ns sequence $\mathcal{P}$ along with shuffle composition maps, one for each shuffle partition π of a finite ordered set I of the form

$$\gamma_\pi : \mathcal{P}(r) \otimes \mathcal{P}(\pi_1) \otimes \cdots \otimes \mathcal{P}(\pi_r) \longrightarrow \mathcal{P}(I)$$

that satisfy suitable associativity and unitality axioms. Precisely, let us pick a finite totally ordered set I, a shuffling partition π of I, and let us assume that we pick a shuffling partition $\pi^{(i)}$ of each block of π. There is a unique way to order the collection of blocks of these to obtain a shuffling partition π' of I. For each part π_i of π and each $(g_i;\vec{h}_i) \in \mathcal{P}(\pi_i) \otimes \mathcal{P}[\pi^{(i)}]$, let us write $f_i = \gamma_{\pi^{(i)}}(g_i;\vec{h}_i)$, and let $\vec{h}$ be obtained for the tuple $(\vec{h}_1,\ldots,\vec{h}_r)$ by reordering the entries according to π'. Then

$$\gamma_\pi(f;f_1,\ldots,f_r) = \gamma_{\pi'}(\gamma_\pi(f;g_1,\ldots,g_r);\vec{h}).$$

Moreover, for each finite set I, if $\{I\}$ and I denote the corresponding partitions into one block and into singletons, we a fixed $1 \in \mathcal{P}(1)$ such that for every $v \in \mathcal{P}(I)$ we have

$$\gamma_{\{I\}}(1;v) = v, \quad \gamma_I(v;1,\ldots,1) = v.$$

Naturally, one can consider partial compositions on a shuffle operad, but carefully noting that for each i, there exist many different shuffling partitions π of the form

$$(1,\ldots,i-1,A,j_1,\ldots,j_s)$$

where $\min(A) = i$. Namely, for each $[n]$ we need simply choose a subset A of $[n] \smallsetminus [i-1]$ that contains i, and this can be done by choosing a subset of $[n] \smallsetminus [i]$ and appending i.

Definition 5.5 An ideal of a shuffle operad $\mathcal{P}$ is a ns subcollection $\mathcal{I}$ such that

$$\gamma_\pi(v_0;v_1,\ldots,v_r) \in \mathcal{I}$$

if at least one of v_i is in $\mathcal{I}$ for some $i \in [0,r]$.

As we will see later, ideals of shuffle operads are slightly more refined than those in symmetric operads. For example, the ideal generated by the left comb $(x_1x_2)x_3$ in a symmetric operad automatically contains its two translates, while in a shuffle operad, the three ideals corresponding to these three possible shuffle tree monomials are different.

5.2 Free shuffle operad

Let us now give an explicit description of the free shuffle operad on a ns collection. Since we have already defined the free symmetric and non-symmetric operad on a collection (of the appropriate kind), we already have almost all the language necessary to define it.

Definition 5.6 Let τ be a planar tree, which we draw on the plane with the counter-clockwise orientation. Begin at the left side of root edge, and transverse the “boundary” of the tree in the counter-clockwise direction. This path will meet the vertices of τ in some order, and we call this total order the *canonical planar order* of its vertices.

Observe that this also orders the edges of τ, and the leaves (which are given the usual left-to-right planar order).

Now let $\mathcal{X}$ be a ns collection and let T be a planar tree monomial with variables in $\mathcal{X}$, and let us pick a bijective labelling $\mathsf{n} : L(\tau) \longrightarrow [n]$ of the leaves of τ. This induces a labelling of the vertices of τ inductively by inductively labelling v with the minimum label appearing among its set of children.

Definition 5.7 We say a leaf labelling of a planar tree monomial T is shuffling if the induced order on the children of each of its vertices coincides with the canonical planar order. A pair (T, n) where n is a shuffling leaf labelling is called a shuffle tree monomial.

We now define the ns collection $\mathrm{Tree}_{\mathcal{X}}^{\text{Ш}}$ so that for each finite totally ordered set I the set $\mathrm{Tree}_{\mathcal{X}}^{\text{Ш}}(I)$ consists of those shuffle tree monomials on $\mathcal{X}$ with shuffling labellings by I. We write $\mathcal{F}_{\mathcal{X}}^{\text{Ш}}$ for the corresponding linear ns collection.

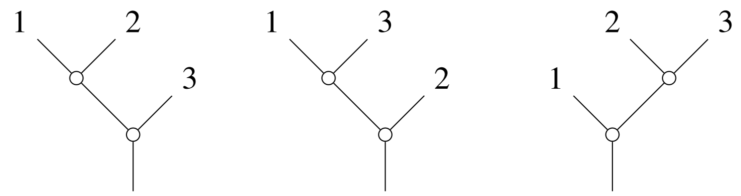


Figure 2: The three shuffle trees with three leaves on a binary generator.

Suppose that T and T' are shuffle tree monomials on $[n]$ and $[m]$, that $i \in [n]$ and that we pick a shuffling partition π of $[m+n-1]$ whose only non-singleton part is of the form

$$\{i = j_1, j_2, \ldots, j_m\}.$$

We define the tree monomial $T \circ_\pi T'$ by grating the tree T' at the leaf of T labelled by i, with its leaf labels renumbered through the unique order preserving bijection $j_i \longmapsto i$, and

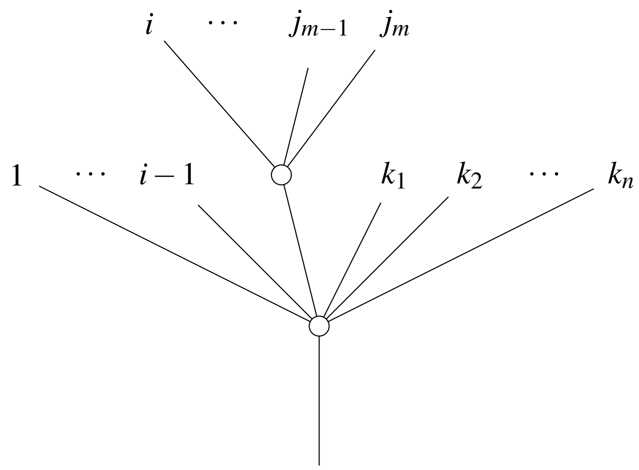


Figure 3: The two-level trees corresponding to partial compositions of shuffle operads

we renumber the leaf labels of T distinct from $1,\ldots,i-1$ using the remaining blocks of π. This defines the "partial shuffle composition" of shuffle tree monomials.

We may as well define the "total shuffle composition" of a tree T_0 with trees $T_1,\ldots,T_n$ along a shuffling partition $\pi = (\pi_1,\ldots,\pi_n)$ with T_i having as many leafs as π_i for each $i \in [n]$ Concretely, we consider for each such i the unique order preserving bijection between π_i and the labels of T_i, and graft T_i at the input of T_0 labelled by $\min \pi_i$.

Proposition 5.8 *The shuffle composition of shuffle tree monomials is again a shuffle tree monomial.*

Proof. This is Exercise 51. The idea is to note that the local increasing condition is not broken, and this is clear on each T_i since we simply relabelled their leafs with an isomorphic totally order set, while it is not broken on T_0 since we grafted the T_is using a shuffling partition. □

With this at hand, we can state and prove the main result in this section.

Proposition 5.9 *The ns collection $\mathcal{F}^{\text{Ш}}_{\mathcal{X}}$ with its corresponding shuffle composition is the* free shuffle operad *generated by $\mathcal{X}$, where the inclusion $\mathcal{X} \longrightarrow \mathcal{F}^{\text{Ш}}_{\mathcal{X}}$ sends an element in $\mathcal{X}$ to the corresponding corolla with its unique shuffling leaf labelling.* □

5.3 Forgetful functor

Since every finite totally order set I is in particular a finite set I^{f} after forgetting the order, we have a functor $\mathcal{X} \longmapsto \mathcal{X}^{\mathsf{f}}$ that assigns a symmetric collection $\mathcal{X}$ to the ns collection $\mathcal{X}^{\mathsf{f}}$

such that

$$\mathcal{X}^{\mathsf{f}}(I) = \mathcal{X}(I^{\mathsf{f}})$$

for each finite order set I. We call this the *forgetful functor* from symmetric to ns collections. The following will be central in what follows.

Proposition 5.10 *The forgetful functor* ${}_{\Sigma}\mathsf{Mod} \longrightarrow {}_{\mathrm{ns}}\mathsf{Mod}$ *is strong monoidal for the corresponding symmetric and shuffle composition products when restricted to* reduced *collections, in the sense that for each pair* $\mathcal{X}$ *and* $\mathcal{Y}$ *with* $\mathcal{Y}$ *reduced there is a natural isomorphism*

$$(\mathcal{X} \circ_{\Sigma} \mathcal{Y})^{\mathsf{f}} \longrightarrow \mathcal{X}^{\mathsf{f}} \circ_{\text{Ш}} \mathcal{Y}^{\mathsf{f}}.$$

Proof. Let us begin by proving that if $\mathcal{Y}$ is a reduced symmetric sequence then $\mathcal{Y}^{\otimes n}$ is a free S_n-module for every $n \geqslant 1$. This is of course true for $n = 1$. For $n > 1$, it suffices to exhibit an S_n-basis. For each finite totally ordered set I, let us consider the components of $\mathcal{Y}^{\otimes n}(I^{\mathsf{f}})$, and note that since $\mathcal{Y}$ is reduced they are of the form

$$\mathcal{Y}(\pi_1) \otimes \mathcal{Y}(\pi_n)$$

where π is a partition of I into n blocks with at least one element. For each such partition π of I, there exists a unique permutation $\sigma \in S_n$ such that $(\sigma\pi)_i = \pi_{\sigma^{-1}(i)}$ is shuffling, and this proves that $\mathcal{Y}^{\otimes n}(I^{\mathsf{f}})$ is isomorphic to the free S_n-module generated by $(\mathcal{Y}^{\mathsf{f}})^{\otimes_{\text{Ш}} n}(I)$. It follows that for each $n \geqslant 1$ we have a natural isomorphism

$$\mathcal{X}(n) \otimes_{S_n} \mathcal{Y}^{\otimes n}(I^{\mathsf{f}}) \longrightarrow \mathcal{X}^{\mathsf{f}}(n) \otimes (\mathcal{Y}^{\mathsf{f}})^{\otimes_{\text{Ш}} n}(I)$$

which gives us the desired isomorphism $(\mathcal{X} \circ_{\Sigma} \mathcal{Y})^{\mathsf{f}} \longrightarrow \mathcal{X}^{\mathsf{f}} \circ_{\text{Ш}} \mathcal{Y}^{\mathsf{f}}$. $\square$

Corollary 5.11 *For each reduced symmetric collection* $\mathcal{X}$*, there is a natural isomorphism of shuffle operads*

$$(\mathcal{F}^{\Sigma}_{\mathcal{X}})^{\mathsf{f}} \longrightarrow \mathcal{F}^{\text{Ш}}_{\mathcal{X}^{\mathsf{f}}}.$$

Moreover, if I *is an ideal in* $\mathcal{F}^{\Sigma}_{\mathcal{X}}$ *then* I^{f} *is an ideal in* $\mathcal{F}^{\text{Ш}}_{\mathcal{X}^{\mathsf{f}}}$ *and the resulting quotient shuffle operads are naturally isomorphic via the induced map*

$$(\mathcal{F}^{\Sigma}_{\mathcal{X}}/I)^{\mathsf{f}} \longrightarrow \mathcal{F}^{\text{Ш}}_{\mathcal{X}^{\mathsf{f}}}/I^{\mathsf{f}}.$$

In particular, shuffle tree monomials on $\mathcal{X}^{\mathrm{f}}$, when considered with their non-planar tree structure, give us a basis of the free symmetric operad on $\mathcal{X}$, and we can study any presentation of a symmetric operad through the resulting presentation of the corresponding shuffle operad.

5.4 Exercises

Exercise 51. Show that the shuffle composition of shuffle tree monomials is again a shuffle tree monomial.

Exercise 52. Use the definition of shuffle trees to compute a basis of $\mathrm{Tree}^{Ш}_{\mathcal{X}}(4)$ in case $\mathcal{X}$ consists of a single symmetric or antisymmetric operation. What happens if the operation is not symmetric?

Exercise 53. Explain how $\mathcal{X}^f \circ_{Ш} \mathcal{Y}^f$ fails to identify with $(\mathcal{X} \circ_\Sigma \mathcal{Y})^f$ in case $\mathcal{Y}$ is not reduced.

Exercise 54. Go through the definition of the shuffle compositions γ_π for shuffle tree monomials, and show that it maps shuffle tree monomials to shuffle tree monomials.

Exercise 55. Give an example of a shuffle operad that is not obtained from a symmetric operad through the forgetful functor. *Suggestion:* ideals coming from symmetric operads must be stable under the (now non-existent group action). Can you find a shuffle ideal that is “not very symmetric”?

Exercise 56. Write down a presentation of the following as shuffle operads: the commutative operad, the Lie operad, the associative operad, and the operad of 3-ary totally commutative associative algebras.

Exercise 57. Repeat the theme of the last four exercises with any other (quadratic) operad of your choice.

6 Monomial orders

6.1 Some reminders

In the following, we will anchor ourselves in the rewriting theory that exists for associative monoids in sets and the corresponding theory for associative algebras. Since we are not assuming the reader is familiar with this, let us give a brief recollection of the basics.

Definition 6.1 An associative monoid is a set M along with an associative multiplication $\mu : M \times M \longrightarrow M$. Given a set X, we write $\langle X \rangle$ for the free monoid on X, which is given by the set $\bigsqcup_{n \geqslant 1} X^n$ of all *words the alphabet* X with product the isomorphism $X^n \times X^m \cong X^{m+n}$ for each $m, n \geqslant 1$.

We are interested in finding bases of free objects by ideals and, to do this, we will resort to ordering our free objects. This will allow us to give a (terminating) algorithm whose input will be a set of relations and an ordering, and whose output (among other things) will be a basis of our quotient object.

Definition 6.2 An ordered monoid is a pair $(M, \prec)$ where M is a monoid and $\prec$ is a total order on M that satisfies the following three conditions:

(1) It is a well-order: every non-empty subset of M has a minimum.
(2) The product map of M is increasing in both of its arguments for $\prec$.

A *monomial order* on the alphabet X is, by definition, the structure of an ordered monoid on the free monoid $\langle X \rangle$ generated by X.

Explicitly, the last condition requires that if $m_1, m_2, m_3 \in M$ and if $m_1 \prec m_2$ then it follows that $m_3 m_1 \prec m_3 m_2$ and $m_1 m_3 \prec m_2 m_3$. If the alphabet X is given a total order, then we can produce a monomial order on it as follows:

Definition 6.3 Let $\prec$ be a total order on X. The graded lexicographic order on $\langle X \rangle$ induced by $\prec$, which we write $\prec_\ell$, is such that $w \prec_\ell w'$ if and only if

(1) The word w is shorter than w', or else
(2) We have $w = w_1 x w_2$ and $w' = w_1 y w_2'$ with $x \prec y$ in X.

It is important to note that the lexicographic order defined only by the second condition is *not* a well-order, and it is not increasing for the concatenation product: for example, if $x \prec y$ then $x \prec x^2$ but $x^2 y \prec xy$.

Lemma 6.4 *The graded lexicographic order is a monomial order on X for any choice total order $\prec$.*

Proof. It is clear that the resulting order is total, for either two words are of distinct length, or they are of the same length and differ at and entry, or else they are equal. To see the order behaves well with respect to the concatenation product, we observe that the function $w \longmapsto \mathrm{Length}(m)$ is additive for the concatenation product, so if w is longer than w', then ww'' will be longer than $w'w''$ and, similarly, $w''w$ will be longer than $w''w'$. If w and w' have the same length, then it is clear that $w''w' \prec w''w$ if and only if $w'w'' \prec ww''$ if and only if $w' \prec w$. To see that the order is a well order, let us consider a collection W of words. Then, in particular, there exists a least natural number n such that W contains words of length n but not of $n-1$. In this case, it follows that the minimum of W, if it exists, must be contained in the set X^n, and this set is well ordered by the lexicographical order if X it itself well ordered: we can find the minimum by induction on n. $\square$

We now recall from **Lecture 1** the definition of the *word operad of a monoid M*.

Definition 6.5 Let M be an associative monoid. The symmetric operad $\mathbb{W}_M$ is defined by $\mathbb{W}_M(n) = M^n$ for each $n \geqslant 1$, and its partial composition product is defined for each $s,t \geqslant 1$ and each $i \in [s]$ by the rule

$$(m_1,\dots,m_s) \circ_i (m'_1,\dots,m'_t) = (m_1,\dots,m_{i-1},m_i m'_1,\dots,m_i m'_t,m_{i+1},\dots,m_s).$$

The reader should verify that $\mathbb{W}_M$ is isomorphic, as a symmetric sequence, to the composition product $\mathrm{Ass} \circ M$, where we consider M a symmetric sequence concentrated in arity 1.

6.2 Two statistics

Of particular interest to us is the case $\mathcal{X}$ is a reduced symmetric sequence in sets, and we let $\underline{\mathcal{X}} = \bigsqcup_{n \geqslant 1} \mathcal{X}(n)$ be the underlying alphabet of $\mathcal{X}$. We will use the notation $\mathcal{X}^*$ for the free monoid $\langle \underline{\mathcal{X}} \rangle$. By definition, there exists a unique map of shuffle operads $\pi : \mathcal{F}^{\text{Ш}}_{\mathcal{X}} \longrightarrow \mathbb{W}_{\mathcal{X}^*}$ extending the map $\mathcal{X} \longrightarrow \mathbb{W}_{\mathcal{X}^*}$ that assigns $x \in \mathcal{X}(n)$ to the element $(x,\dots,x) \in \mathbb{W}_{\mathcal{X}^*}(n)$.

Definition 6.6 For each shuffle tree monomial T, we call $\pi(T)$ the *path sequence of T*.

The path sequence of a shuffle tree monomial T can be computed in a straight-forward way, as the following lemma shows. The useful observation that the previous definition allows us to make is that the path sequence statistic is compatible with shuffle compositions of tree monomials, in the sense the path sequence of a composition of tree monomials equals the compositions of the corresponding path sequences of these tree monomials.

Lemma 6.7 *Let $\mathcal{X}$ be reduced. The path sequence of T is the tuple in $\mathbb{W}_{\mathcal{X}^*}(n)$ where n is the number of leaves of T, obtained by recording at the ith entry the word in $\mathcal{X}$ read by travelling from the root of T to the leaf labelled by i.*

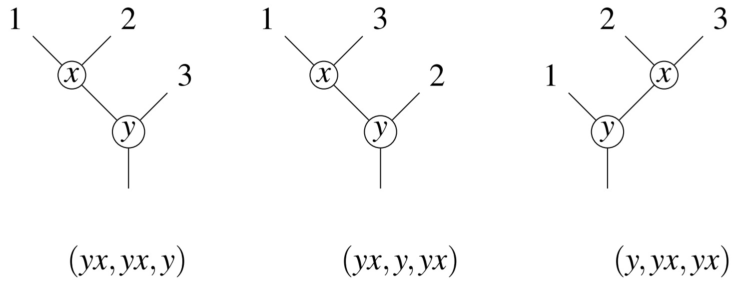


Figure 4: An example of the computation of path sequences.

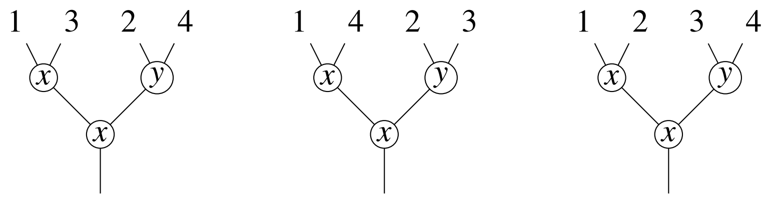


Figure 5: Three different shuffle three monomials with the same path sequence, but different permutation sequence.

More generally, in case $\mathcal{X}$ has 0-ary variables, we must look at all *endpoints* of a tree monomial. Since we will not be interested in non-reduced alphabets, we let the curious reader explore this modification on their own. It is useful to remark in situations like this that $\underline{\mathcal{X}}$ is obtained through a disjoint union of the components of $\mathcal{X}$: the path sequence of $x \circ_1 y$ for x and y unary is (xy), and the path sequence of $x \circ_1 y'$ for x unary and y' nullary is 'also' (xy'), but these are *distinct* in the free monoid $\mathcal{X}^*$.

Proof. This is Exercise 58. $\square$

Let us now consider the unique map of shuffle operads $\sigma : \mathcal{F}^{\text{ш}}_{\mathcal{X}} \longrightarrow \mathsf{Ass}$ extending the map $\mathcal{X} \longrightarrow \mathsf{Ass}$ that assigns $x \in \mathcal{X}(n)$ to the identity $1 \in \mathsf{Ass}(n) = S_n$.

Definition 6.8 If T is a shuffle tree monomial. we call $\sigma(T)$ the (leaf) permutation sequence of T. We call the pair $(\pi(T), \sigma(T))$ the path-permutation data of T.

As before, this statistic of T has a simpler description, that can be read off directly from T, and the previous definition tells us that the leaf permutation sequence of a tree monomial behaves well with respect to shuffle compositions.

Lemma 6.9 *The permutation sequence of T is obtained by reading the leaf labelling of T from left to right and recording it as a permutation in "two line notation".*

The main result of this section tells us that it suffices for us to order sequences of words in the alphabet $\mathcal{X}$ and permutations in order to order shuffle tree monomials.

Theorem 6.10 *The map $\mathcal{F}^{\text{ш}}_{\mathcal{X}} \longrightarrow \mathbb{W}_{\mathcal{X}^*} \times \mathsf{Ass}$ of the free shuffle operad on $\mathcal{X}$ into the Hadamard product of $\mathbb{W}_{\mathcal{X}^*}$ and Ass induced by π and σ is injective. In other words, the path-permutation datum of a shuffle tree monomial determines it uniquely.*

Let us call the map in the statement of the theorem the *path-permutation inclusion*.

Proof. We will sketch a proof, and ask the reader to fill in the details as an exercise; we proceed by induction on the total length of the path sequence of a tree monomial so that, for example, the path sequences appearing in Figure 4 have all length five. First, let us show that the path sequence determines the planar structure of our tree monomial uniquely:

If the length is zero, then the path sequence π is empty, and we are simply considering the trivial tree monomial. Let us consider now some positive length ℓ and search, among all words w appearing in π, that which has the largest possible length and smallest possible coordinate, let us say this coordinate is i.

If w ends in a 0-ary variable of $\mathcal{X}$, this means the ith leaf of T ends at a stump, and we can remove it, and continue by induction. If not, then w ends with some variable $x \in \mathcal{X}(k)$, and the way we have chosen it implies that the ith leaf (in the planar order) is the first child of x, and that all other children of x are also leaves. It follows that w and the next $k-1$ words in π all end with x, and that π is obtained as a non-symmetric composition with $(x,\ldots,x)$. By pruning x from π, we can proceed by induction.

Now that we know the path sequence recovers the planar structure of T uniquely, let us pick some path-permutation datum (π,σ). Then, reorder the entries of π using σ^{-1} to recover the planar structure of T, and then label its leafs according to σ, to recover the whole shuffle structure. □

6.3 Ordered shuffle operads

We can now proceed to define ordered shuffle operads.

Definition 6.11 A set shuffle operad $\mathcal{P}$ is order if for each $n \geqslant 0$ the component $\mathcal{P}(n)$ is well-ordered and if shuffle compositions are increasing in each of its arguments: for each $n \geqslant 1$, all elements $(T_0;T_1,\ldots,T_n) \in \mathcal{P}(k) \times \mathcal{P}(n_1) \times \cdots \times \mathcal{P}(n_k)$ and all shuffling partitions of $[n_1+\cdots+n_k]$, we have that

$$\gamma_\pi(T_0;T_1,\ldots,T_i,\ldots,T_n) \prec \gamma_\pi(T_0;T_1,\ldots,T_i',\ldots,T_n)$$

whenever $T_i \prec T_i'$ for some $i \in [0,n]$ as elements of $\mathcal{P}(n_1+\cdots+n_k)$.

In particular, we can apply this definition in the case $\mathcal{P}$ is the free set shuffle operad on some alphabet $\mathcal{X}$. As promised, let use the injection (π, σ) to endow tree monomials with well-orders.

Proposition 6.12 *Let $(M, \prec)$ be an ordered monoid. The word operad on $\mathbb{W}_M$ is an ordered operad through the lexicographical order of words.*

Proof. This is Exercise 60. $\square$

In particular, we can consider the case in which $M = \mathcal{X}^*$ is endowed with the graded lexicographical order induced by a total order on $\underline{\mathcal{X}}$, which implies the following corollary.

Corollary 6.13 *Suppose that $\underline{\mathcal{X}}$ is given a total order, and that we give the free monoid $\mathcal{X}^*$ the induced graded lexicographical order. Then the word operad $\mathbb{W}_{\mathcal{X}^*}$ is an ordered shuffle operad with the lexicographical order.*

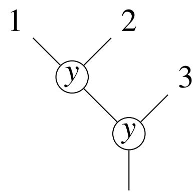
We leave it as an exercise to the reader to show that the associative operad is an ordered shuffle operad if we use on it the lexicographic order on permutations (seen as strings of numbers, in one line notation). All our work is now, done:

Definition 6.14 Let $\mathcal{X}$ be an alphabet and suppose that we give the monoid $\mathcal{X}^*$ a monomial order $\prec$. The *path-permutation extension* of $\prec$ is the unique order on $\mathcal{F}_{\mathcal{X}}^{\text{Ш}}$ induced by the path-permutation inclusion, where we use the the induced lexicographic order on $\mathbb{W}_{\mathcal{X}^*}$ first, and the lexicographic order on Ass second.

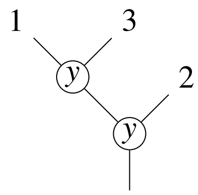
Naturally, one can switch the roles of the two factors of the path-permutation inclusion to get the *permutation-path extension* of a monomial order on $\mathcal{X}^*$. We will explore other variations in the exercises.

Definition 6.15 Let us fix a total order $\prec$ on $\underline{\mathcal{X}}$, and let us consider the induced graded lexicographic order on $\mathcal{X}^*$, where we first compare the length of a word, and then use the lexicographic order induced by the total order. The path-permutation extension on $\mathcal{F}_{\mathcal{X}}^{\text{Ш}}$ is called the *graded path-permutation lexicographic order* induced by $\prec$.

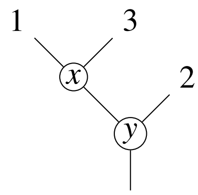
For example, let us consider the case in which $\mathcal{X}$ is binary and contains exactly two operations x and y. The next figure shows the `grapathpermlex` order induced by $x < y$ on all possible twelve tree monomials with three leaves on $\mathcal{X}$; largest elements appear first, from top left to bottom right.



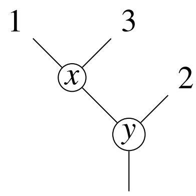
(yy, y, y)



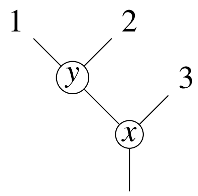
(yy, y, yy)



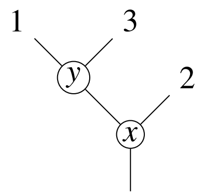
(yx, yx, x)



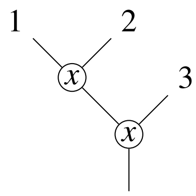
(yx, x, yx)



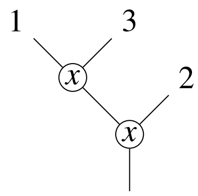
(xy, xy, x)



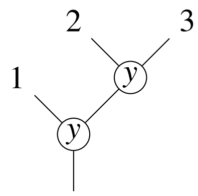
(xy, x, xy)



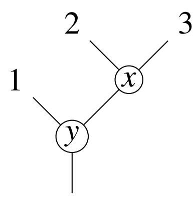
(xx, xx, x)



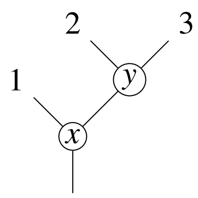
(xx, x, xx)



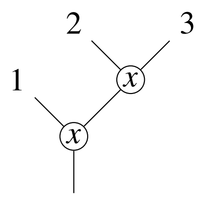
(y, yy, yy)



(y, yx, yx)



(x, xy, xy)



(x, xx, xx)

6.4 Exercises

Exercise 58. Show that the path sequence of a tree monomial, as defined using the universal property of the free shuffle operad, coincides with its combinatorial definition obtained by reading the entries of the tree from the root to the leaves.

Exercise 59. Let X be a finite set and let us give $\langle X\rangle$ the graded lexicographical order with respect to a fixed total order on X. Show this is a monomial order.

Exercise 60. Suppose $(M,\prec)$ is an ordered monoid and we let us give the shuffle operad $\mathbb{W}_M$ the induced lexicographical order. Show that the resulting order is a monomial order.

Exercise 61. Consider the ns collection $\mathcal{X}$ with $\underline{\mathcal{X}} = \mathcal{X}(2)$ a singleton. Show that we can always find a monomial order that singles out one of the three shuffle tree monomial basis elements of $\mathcal{F}^{\text{Ш}}_{\mathcal{X}}(3)$ as the largest.

Exercise 62. Consider the ns collection $\mathcal{X}$ with $\underline{\mathcal{X}} = \mathcal{X}(2) = \{x,y\}$, and the "mixed" shuffle tree monomials in $\mathcal{F}^{\text{Ш}}_{\mathcal{X}}(3)$ that have x and y (one at the top, the other at the bottom). Explore what leading terms you can obtain by choosing different induced orders on $\mathcal{F}^{\text{Ш}}_{\mathcal{X}}$.

7 Gröbner bases

Goal. Define the long division algorithm for shuffle tree polynomials. Prove Gröbner bases for shuffle operads exist and reduced Gröbner bases are unique.

7.1 Tree insertion

Definition 7.1 Let T' and T be tree monomials over some fixed alphabet $\mathcal{X}$. We say that T' divides T if the underlying tree τ of T contains a subtree τ_0 isomorphic to the underlying tree τ' of T', whose induced shuffling labelling and decorations coincide with that of T'.

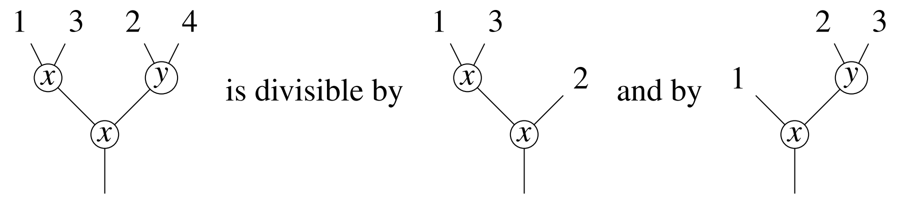


Figure 6: A "fork" and its divisors of weight two. Notice the induced labelling of the right comb, coming from the shuffling labelling $1 < 2 < 4$.

The following lemma asserts that the combinatorial notion of divisibility coincides with the algebraically inclined notion of divisibility, that of belonging to the ideal generated by the divisor.

Lemma 7.2 *A tree monomial T is divisible by another tree monomial T' if and only if it can be obtained from T' by iterated shuffle compositions with other tree monomials.*

Proof. It is clear that if T is obtained from T' by iterated shuffle compositions with tree monomials, then T is divisible by T'. Conversely, suppose T' divides T. If the root of T' is not that of T, then we can write T as a composition of several tree monomials, one which is divisible by T' and which shares the root with it, so we may assume this is the case. One this is done, we see that T is in fact obtained by grafting tree monomials at the leaves of T', and completes the proof. $\square$

Definition 7.3 Suppose that T' is a divisor of T, and let us assume that T' has ℓ' leaves and T has ℓ leaves. We define the insertion operation

$$\square_{T'}^{T}(-) : \mathcal{F}_{\mathcal{X}}^{\text{III}}(\ell') \longrightarrow \mathcal{F}_{\mathcal{X}}^{\text{III}}(\ell)$$

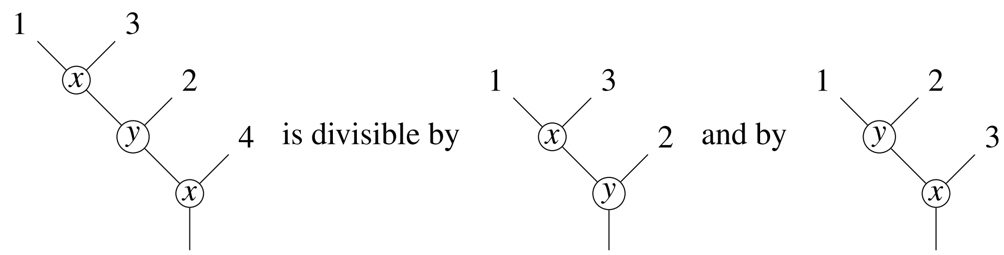


Figure 7: A right comb with two divisors that have the same underlying planar structure but different induced labelling.

that replaces the divisor T' of T by any other shuffle tree monomial with ℓ' leaves in T, and extend it linearly, making sure that leaf labels are respected.

Lemma 7.4 *Let $\mathcal{V}$ be a subset of the free shuffle operad on $\mathcal{X}$. Then the ideal generated by $\mathcal{V}$ is explicitly obtained as the linear span of all insertions $\square_{T'}^{T}(f)$ as T, T' range through pairs (T', T) with T' a divisor of T and $f \in \mathcal{V}(\ell')$.*

Proof. By construction, $(\mathcal{V})$ is the linear span of all possible shuffle compositions where at least one summand is contained in $\mathcal{V}$. Since shuffle compositions are multilinear, we can assume that all terms appearing in such shuffle compositions (except, possibly, for that in $\mathcal{V}$) are tree monomials, in which case the resulting shuffle composition coincides with an insertion operation. $\square$

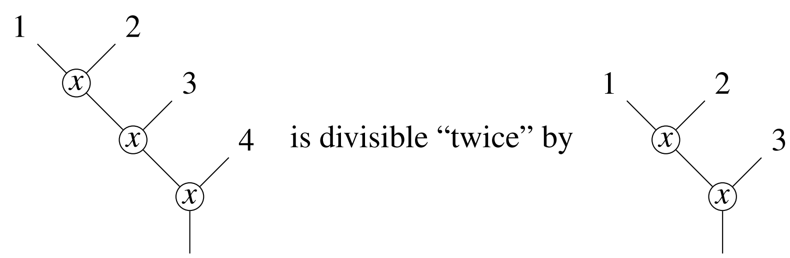


Figure 8: A right comb with two divisors that have the same induced shuffle tree structure, but happen at different places of the tree.

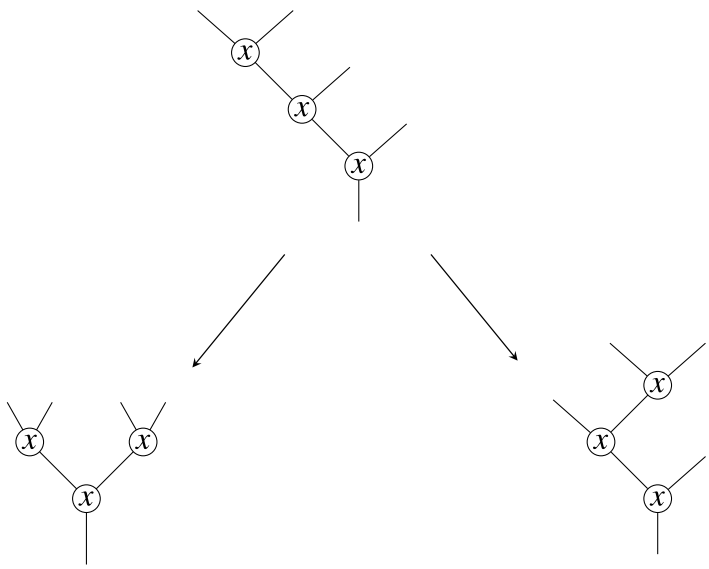


Figure 9: Two possible results of substituting a left comb by a right comb in the leftmost tree.

7.2 Long division

Suppose that we fix a tree monomial order on the free shuffle operad $\mathcal{F}^{\text{Ш}}_{\mathcal{X}}$ and $f \in \mathcal{F}^{\text{Ш}}_{\mathcal{X}}(n)$ is a tree *polynomial*. The support of f is the (finite) set of tree monomials that appear in f with non-zero coefficient. We say the tree monomial T is the *leading monomial of* f if T is the largest monomial that appears in the support of f, and we write the corresponding summand in f by $\mathrm{LM}(f)$. This summand is accompanied by a coefficient, which we call the leading coefficient of f and write $\mathrm{LC}(f)$. Thus, any f can be written in the form

$$f = \mathrm{LC}(f)\,\mathrm{LM}(f) + f_0$$

where all monomials appearing in f_0 are smaller than $\mathrm{LM}(f)$. We call $\mathrm{LC}(f)\,\mathrm{LM}(f)$ the leading term of f and write it $\mathrm{LT}(f)$. We begin with a preparatory result.

Proposition 7.5 *Suppose that T' is a divisor of T and let $f \in \mathcal{F}^{\text{Ш}}_{\mathcal{X}}(\ell')$. Then the leading term of the insertion $\square^{T}_{T'}(f) = \square^{T}_{T'}(f)$ is equal to $\square^{T}_{T'}(\mathrm{LT}(f))$.*

Proof. For tree polynomials $f_0, f_1, \ldots, f_n$ and any shuffling partition π, we have that

$$\mathrm{LT}(\gamma_\pi(f_0; f_1, \ldots, f_n)) = \gamma_\pi(\mathrm{LT}(f_0); \mathrm{LT}(f_1), \ldots, \mathrm{LT}(f_n)).$$

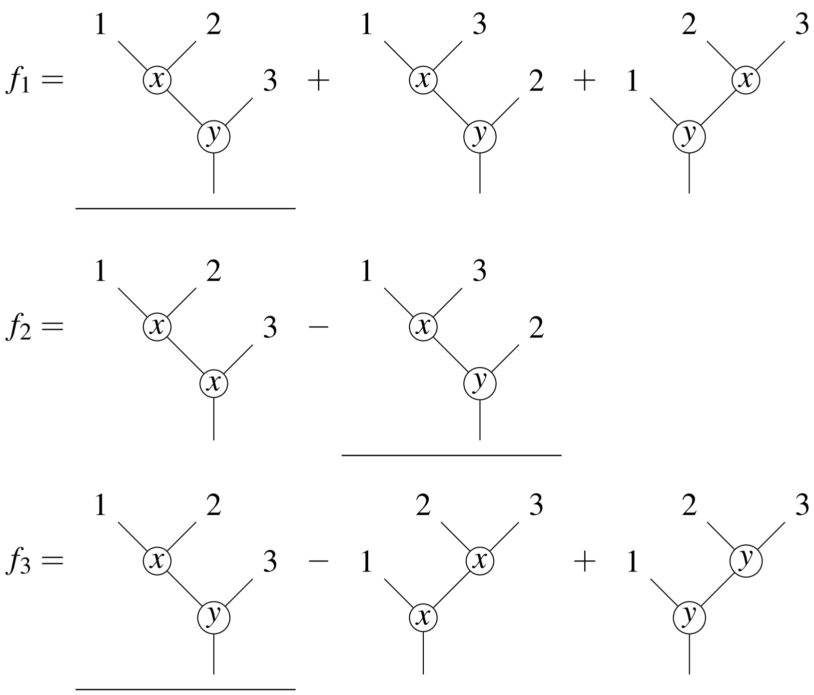


Figure 10: Leading terms underlined in some tree polynomials, for the `grpathpermlex` order induced by $y > x$.

This is clear, since the shuffling compositions are by definition (strictly) increasing for $\prec$. Since we know that $\square^{T}_{T'}(f)$ is obtained from f be iterating shuffle compositions, this proves the statement of the proposition. $\square$

Let $\mathcal{V}$ be a subset of the free shuffle operad on $\mathcal{X}$. A tree monomial T is reduced with respect to $\mathcal{V}$ if it is not divisible by any of the leading terms of polynomials appearing in it. A polynomial is reduced with respect to $\mathcal{V}$ if it is a linear combination of tree monomials that are reduced. We say $\mathcal{V}$ is self-reduced if each $v \in \mathcal{V}$ is reduced with respect to $\mathcal{V} \smallsetminus v$, and that is is linearly self-reduced if no leading term of one element divides the leading term of another.

Definition 7.6 (Reduction) Suppose that f and g are polynomials of the same arity, and that f is divisible by the leading term of g, in other words, suppose that $\mathrm{LT}(f) = \square^{T}_{T'}(\mathrm{LT}(g))$ for some tree monomial T and a divisor T'. In this case, the reduction of f with respect to g, which we write $r_g(f)$, is defined by

$$r_g(f) = f - \frac{\mathrm{LC}(f)}{\mathrm{LC}(g)} \square^{T}_{T'}(g).$$

The following lemma tells us that the reduced term $r_g(f)$ behaves like a "remainder by division", in the sense it is either zero of "smaller than f".

Lemma 7.7 *For all f and g such that $r_g(f)$ is defined, either $r_g(f) = 0$, or else we have that* $\mathrm{LT}(r_g(f)) \prec \mathrm{LT}(f)$.

Proof. If $r_g(f)$ is non-zero, we have that its leading coefficient is equal to the leading coefficient of $f - \frac{\mathrm{LC}(f)}{\mathrm{LC}(g)} \square^T_{T'}(g)$. But the leading term of the second term is, by Proposition 7.5, equal to

$$\frac{\mathrm{LC}(f)}{\mathrm{LC}(g)} \square^T_{T'}(\mathrm{LT}(g)) = \mathrm{LC}(f)\mathrm{LM}(f) = \mathrm{LT}(f).$$

It follows that the terms appearing in $r_g(f)$ are all strictly smaller than the leading term of f, as we wanted. $\square$

Long division algorithm. Let us now define the long division algorithm for shuffle operads. Its input is a polynomial f and a finite set $\mathcal{V}$, both in $\mathcal{F}_{\mathcal{X}}$, and its output is a reduced element $\overline{f}$ with respect to $\mathcal{V}$ such that $f - \overline{f} \in (\mathcal{V})$ and $\mathrm{LT}(\overline{f}) \preceq \mathrm{LT}(f)$. We can verbosely describe the algorithm as follows:

(1) If our input polynomial f is zero, just return zero. If not, ensure $\mathcal{V}$ is linearly self-reduced, using the linear self-reduction Algorithm **??**.
(2) If there is an element v in $\mathcal{V}$ whose leading term divides the leading term of f, pick that with the largest leading term.[2]
(3) Let f' be the remainder of division of f by v using the reduction procedure of Definition 7.6. Recursively call the algorithm to compute the result of long division of f' by V.
(4) If not, then $\mathrm{LT}(f)$ is $\mathcal{V}$-reduced, so let f' be the result of long division of $f - \mathrm{LT}(f)$ by $\mathcal{V}$, and return $\mathrm{LT}(f) + f'$.

The following is what this algorithm looks like in pseudo-code:

Lemma 7.8 *The long division algorithm terminates, and its output is a reduced element $\overline{f}$ with respect to $\mathcal{V}$ such that $f - \overline{f} \in (\mathcal{V})$ and* $\mathrm{LT}(\overline{f}) \preceq \mathrm{LT}(f)$.

Proof. At each step, the leading monomial of f is decreased Lemma 7.7, so the fact that $\prec$ is a well order guarantees our algorithm terminates. It also guarantees that the output will satisfy $\mathrm{LT}(\overline{f}) \preceq \mathrm{LT}(f)$. Let us suppose, for the sake of a contradiction, that the output of the algorithm is not always reduced. Among such problematic polynomials f, let us pick

[2]This can be done since we already ensured $\mathcal{V}$ is linearly self reduced.

Algorithm 1 Long division algorithm

INPUT: A polynomial f and a finite set $\mathcal{V}$ of tree polynomials.
OUTPUT: A $\mathcal{V}$-reduced polynomial $\overline{f}$ with $f - \overline{f} \in (\mathcal{V})$ and $\mathrm{LT}(\overline{f}) \preceq \mathrm{LT}(f)$.

```
procedure LongDivision(TreePolynomial,TreePolynomials)
    if TreePolynomial = 0 then return 0
    else
        Dividend ← TreePolynomial
        Divisors ← LinearSelfReduce(TreePolynomials)
        Factors ← {v ∈ Divisors : LM(v) divides LT(g)}
        if Factors ≠ ∅ then
            LargestFactor ← Largest(Factors)
            Dividend ← Reduce(Dividend, LargestFactor)
            Dividend ← LongDivision(Dividend, Divisors)
        LeadDividend ← LT(Dividend)
        Dividend ← Dividend − LeadDividend
        Dividend ← LongDivison(Dividend, Divisors)
    return LeadDividend + Dividend
```

one f with the smallest leading term, which is possible since $\prec$ is a well order. If $\mathrm{LT}(f)$ is not reduced, then the first step of our algorithm applies long division to $r_g(f) = f'$ for some $g \in \mathcal{V}$, and by Lemma 7.7, this is either zero or has a smaller leading term than f', so it must be reduced. If $\mathrm{LT}(f)$ is reduced, then the second step of the algorithm applies long division to $f - \mathrm{LT}(f)$, which has smaller leading term than f, so the output is again reduced. Finally, note that at each step of the algorithm we subtract an element of $\mathcal{V}$, so the coset of f is not modified, which concludes our proof. □

Proposition 7.9 *Suppose that $\mathcal{I}$ is an ideal in the free shuffle operad generated by $\mathcal{X}$. Then those shuffle monomials that are reduced with respect to $\mathcal{I}$ form a basis for the quotient operad $\mathcal{F}_{\mathcal{X}}/\mathcal{I}$.*

Proof. Let us first show that these monomials span the quotient operad. This follows, for the long division algorithm guarantees we can always replace a non-reduced polynomial f by a reduced one without affecting its coset. To see they are linearly independent, suppose that f is a polynomial reduced with respect to I, and that $f \in \mathcal{I}$. Then we see that $\mathrm{LT}(f) \in \mathrm{LT}(\mathcal{I})$. But this can only happen if $f = 0$ (or else f would not even be linearly reduced with respect to $\mathcal{I}$). □

In practice, we have little control over the multitude of leading terms that may appear in the elements of $\mathcal{I}$, and Gröbner bases are designed to regain this control.

Self-reduction algorithm. Suppose that $\mathcal{V}$ is a finite subset of $\mathcal{F}_{\mathcal{X}}^{\text{Ш}}$. The following algorithm takes as input this generating set, and outputs a self-reduced subset $\mathcal{V}'$ that generates the same ideal as $\mathcal{V}$. This is what this algorithm looks like in pseudo-code, though we are being slightly imprecise: $\mathcal{V}$ is not a matrix, so we cannot feed it to our linear self reduction algorithm as is: we pick a total order on $\mathcal{V}$, and then use the corresponding matrix written in the shuffle tree monomial basis.

Algorithm 2 Self-reduction algorithm

INPUT: A finite set of polynomials $\mathcal{V}$ in a free shuffle operad.
OUTPUT: A finite self-reduced set of polynomials generating the same ideal as $\mathcal{V}$.

```
procedure SELFREDUCE(Polynomials)
    ToReduce ← LINEARSELFREDUCE(Polynomials)
    if ToReduce is self reduced then return ToReduce
    else
        Largest ← LARGEST(ToReduce)
        ToReduce ← SELFREDUCE(ToReduce ∖ Largest)
        NewElement ← LONGDIVISION(Largest, ToReduce)
        ToReduce ← ToReduce ∪ NewElemet
    return SELFREDUCE(ToReduce)
```

Proposition 7.10 *The self-reduction algorithm terminates for each finite set $\mathcal{V}$ and returns a self-reduced set $\mathcal{V}'$ with $(\mathcal{V}) = (\mathcal{V}')$.*

Proof. This is Exercise 64. □

7.3 Existence and uniqueness

Lemma 7.11 *Let $\mathcal{I}$ be an ideal of $\mathcal{F}_{\mathcal{X}}$. The subspace*

$$\mathrm{LT}(\mathcal{I}) = \langle T \in \mathcal{F}_{\mathcal{X}} : T = \mathrm{LT}(f) \textit{ for some } f \in \mathcal{I} \rangle.$$

spanned by leading terms of elements of $\mathcal{I}$ is again an ideal of $\mathcal{F}_{\mathcal{X}}$.

Proof. By construction $\mathrm{LT}(\mathcal{I})$ is a subspace of $\mathcal{F}_{\mathcal{X}}$, so it suffices we prove it is an ideal. By multilinearity, it suffices to show it is stable under compositions with respect to tree monomials if at least one term is already in $\mathrm{LT}(\mathcal{I})$. To do this, note that if T is the leading term of some $f \in \mathcal{I}$, then by Proposition 7.5, for any tree monomials $T_0, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n$, the leading term of the composition

$$\gamma_\pi(T_0; T_1, \ldots, T_{i-1}, f, T_{i+1}, \ldots, T_n) \in \mathcal{I}$$

is precisely $\gamma_\pi(T_0;T_1,\dots,T_{i-1},T,T_{i+1},\dots,T_n)$, which proves this belongs to $\mathrm{LT}(\mathcal{I})$. □

Definition 7.12 Let $\mathcal{I}$ be an ideal of $\mathcal{F}_\mathcal{X}$. We say that a subset $\mathcal{G}$ of $\mathcal{I}$ is a *Gröbner basis of* $\mathcal{I}$ (with respect to our fixed monomial order) if the set of leading monomials of $\mathcal{G}$ generate the ideal of leading terms of $\mathcal{I}$. A Gröbner basis which is self-reduced is called reduced.

Lemma 7.13 *Let $\mathcal{I}$ be an ideal and let $\mathcal{G}$ be a Gröbner basis of $\mathcal{I}$. Then $\mathcal{G}$ generates $\mathcal{I}$.*

Proof. Suppose that there is some $f \in \mathcal{I}$ that is not generated by $\mathcal{G}$, and let us pick one with the least possible leading term. Since $\mathcal{G}$ generates the ideal of leading terms of $\mathcal{I}$, we can reduce the leading term of f with respect to $\mathcal{G}$ without modifying its coset in $\mathcal{I}$, and obtain an element that is generated by $\mathcal{G}$. But then f itself is generated by $\mathcal{G}$, which is a contradiction. □

Proposition 7.14 *A set $\mathcal{G}$ is a Gröbner basis if the cosets of monomials reduced with respect to it form a basis of the quotient operad. In this case, the result of long division of a polynomial by $\mathcal{G}$ is independent of the choices or the order in which we perform the reductions.*

Proof. To begin, observe that the cosets of monomials that are reduced with respect to $\mathcal{G}$ form a basis of the quotient operad precisely when every coset of $\mathcal{I}$ contains a unique element that is reduced with respect to $\mathcal{G}$.

By the long division algorithm, it follows that every coset contains at least one element that is reduced with respect to $\mathcal{G}$, so it suffices we prove that this element is unique if and only if $\mathcal{G}$ is a Gröbner basis.

Thus, first suppose that $\mathcal{G}$ is a Gröbner basis, but that there exist two $\mathcal{G}$-reduced monomials that have the same coset modulo $\mathcal{I}$. This means there exists a $\mathcal{G}$-reduced polynomial in $\mathcal{I}$, which means that its leading term is $\mathcal{G}$-reduced, which is impossible.

Conversely, Suppose that $\mathcal{G}$ is not a Gröbner basis. It follows that there is an element $f \in \mathcal{I}$ which is reduced with respect to $\mathcal{G}$. If we let $\overline{f}$ be the result of the long division of f by $\mathcal{G}$, we see we obtain a non-trivial linear combination of reduced monomials belonging to $\mathcal{I}$, so that there is not a unique reduced representative for the zero coset.

Finally, suppose that for some f, two different choices of order of reductions yield two different outputs. Then, then there exist a coset $f+I$ contains two different elements that are reduced with respect to $\mathcal{G}$, hence reduced monomials are linearly dependent, a contradiction. □

Theorem 7.15 *Every ideal admits a unique reduced Gröbner basis.*

Proof. We begin by proving uniqueness, which will in fact tell us how to prove these exist. Thus, suppose $\mathcal{G}$ is a Gröbner basis of an idea $\mathcal{I}$, so that $\mathrm{LM}(\mathcal{G})$ generates $\mathrm{LM}(\mathcal{I})$. If $\mathcal{G}$ is also reduced, then $\mathrm{LM}(\mathcal{G})$ coincides with the set

$$\mathcal{M} = \{T \in \mathrm{LM}(\mathcal{I}) : T \text{ is not divisible by any other element of } \mathrm{LM}(\mathcal{I}).\}$$

of all minimal elements of $\mathrm{LM}(\mathcal{I})$ partially ordered with respect to divisibility. To see this, not that if $T \in \mathcal{M}$ then this must be divisible by at least one element g of $\mathcal{G}$, and this can only happen if $\mathrm{LM}(g) = T$. Conversely, if T is a leading monomial in $\mathrm{LM}(\mathcal{G})$ then it is certainly a leading monomial of $\mathrm{LM}(\mathcal{I})$. If T' is any other leading monomial of $\mathrm{LM}(\mathcal{I})$ that divides T, then there is $T'' \in \mathrm{LM}(\mathcal{G})$ that divides T', and hence T. But since $\mathcal{G}$ is reduced, this happens only if $T'' = T$, and hence no other leading term divides T.

In addition, the fact that $\mathcal{G}$ is reduced guarantees that for each $T \in \mathrm{LM}(\mathcal{G})$ there exists a unique element $g \in \mathcal{G}$ such that $g = T - h$ and h is reduced with respect to $\mathcal{I}$. It follows that h must be equal to the unique element in the coset $T + \mathcal{I}$ that is reduced with respect to $\mathcal{I}$.

To prove existence, we consider the set $\mathcal{M}$ above, and let $\mathcal{G}$ consist of those elements of the form $T - h$ where $T \in \mathcal{M}$ and h is the unique element in the coset $T + \mathcal{I}$ that is reduced with respect to $\mathcal{I}$. By our definition of $\mathcal{M}$, the set $\mathcal{G}$ is self-reduced, so it suffices we show that it is a Gröbner basis. To do this, notice that every element of $\mathrm{LM}(\mathcal{I})$ is divisible by an element of $\mathcal{M}$: if not, the smallest element which is not divisible by some element of $\mathcal{M}$ is either not divisible by any other element of $\mathrm{LM}(\mathcal{I})$, which makes it an element of $\mathcal{M}$, or otherwise is divisible by some smaller element of $\mathrm{LM}(\mathcal{I})$, and hence actually does have a divisor from $\mathcal{M}$. Thus, $\mathrm{LM}(\mathcal{G})$ generates $\mathrm{LM}(\mathcal{I})$, as we wanted. □

It is important to point out that our proof above is highly non-constructive: we are considering the poset of leading terms of $\mathcal{I}$ under divisibility, which we admits a (possibly infinite) set of minimal elements, and arguing these constitute the reduced Gröbner basis of $\mathcal{I}$. In the next lecture, we will learn how to begin with any generating set of $\mathcal{I}$, and complete it to a (possibly infinite) reduced Gröbnber basis.

Let us conclude this lecture with a perhaps technical lemma that describes how the insertion operation works when it is iterated. It will be useful when we give a proof of one of the handful of important results in these lectures.

Lemma 7.16 *Let us consider the situation where T, T_1, T_1' and T_2 are tree monomials, and that T_1 divides T and T_2 divides T_1'. Assume that T_1' and T_1 have the same arity, so that we may put $T_3 = \square_{T_1}^{T}(T_1')$. Then T_2 divides T_3 and, as operations on tree monomials, we have that*

$$\square_{T_1}^{T}(\square_{T_2}^{T_1'}(-)) = \square_{T_2}^{T_3}(-).$$

In particular, if $T_1 = T_1'$ so that $T_3 = T$, this simplifies to

$$\square^{T}_{T_1}(\square^{T_1}_{T_2}(-)) = \square^{T}_{T_2}(-).$$

We leave the proof as Exercise 65, and strongly suggest the reader to draw a picture, which may well be enough to convince themselves of the validity of the lemma.

7.4 Exercises

Exercise 63. Prove the claim made in the last paragraph above: the ideal of leading terms of $\mathcal{I}$, partially ordered by divisibility, admits a possibly infinite set of minimal elements. *Hint:* divisibility refines our choice of total order $\prec$: if T divides T', then $T \prec T'$.

Exercise 64. Translate the self-reduction algorithm into prose. and prove Proposition 7.10.

Exercise 65. Give a proof of Lemma 7.16.

More exercises for Section 8

8 Computing Gröbner bases

Goal. Define overlapping ambiguities and S-polynomials of overlapping ambiguities. State and prove Bergman's Diamond Lemma. Give Buchberger's algorithm for computing Gröbner bases.

8.1 S-polynomials

As a motivating example, consider the operad with a single binary operation x_1x_2 and no symmetries, which is *anti-associative*, that is,

$$\underline{(x_1x_2)x_3} + x_1(x_2x_3) = 0.$$

At the same time, let us choose the usual `grapathpermlex` order so that the leading term of the relation above is the underlined one. The following computation shows that, among the leading terms of elements in the ideal generated by this relation, all trees of weight three appear, which might be at first unexpected:

$$\begin{aligned} \underline{((x_1x_2)x_3))x_4} + (x_1(x_2x_3))x_4 - \underline{((x_1x_2)x_3)x_4} - ((x_1x_2)(x_3x_4)) &= \\ (x_1(x_2x_3))x_4 - ((x_1x_2)(x_3x_4)) &= \\ (x_1(x_2x_3))x_4 - ((x_1x_2)x_3)x_4 &= 2(x_1(x_2x_3))x_4. \end{aligned}$$

If the characteristic is not two, then this term is non-zero and, up to a sign, equal to the other four tree monomials of weight three. Thus, it follows that this quadratic operad is in fact three dimensional! Let us introduce the device that captures this behaviour.

Definition 8.1 Let g_1, g_2 be two shuffle polynomials over an alphabet $\mathcal{X}$. We say that the monomials $\mathrm{LM}(g_1)$ and $\mathrm{LM}(g_2)$ for an overlap ambiguity if they have a *small common multiple*, that is, there exists tree monomial T properly divisible by $\mathrm{LM}(g_1)$ and $\mathrm{LM}(g_2)$, such that T is the result of merging these along an overlap. The element

$$S_T(g_1,g_2) = \square^T_{T_1}(g_1) - \square^T_{T_2}(g_2)$$

is called the S-polynomial of this overlapping ambiguity.

Revisiting the example above, we notice that the polynomial (non-symmetric, in this case), has an overlapping ambiguity with *itself* —a feature that already exists in the world of non-commutative associative algebras— and that the resulting polynomial is reduced with respect to its leading term. Thus, this overlapping ambiguity is detecting a "hidden" leading term in the ideal of leading terms of this relation.

As a second example, we could have considered the usual associative operad, where the plus sign above becomes a minus. In this case, one can check that the resulting S-polynomial is zero: the computation ends with two right combs cancelling each other. We will see later that this has very important implications for the associative operad: it shows we obtain a quadratic Gröbner basis, and hence that this operad is *Koszul*. For the moment, let us consider a useful definition.

Definition 8.2 Let $\mathcal{I}$ be an ideal generated by some set $\mathcal{G}$, and for an element $f \in \mathcal{I}$, let us consider a representation as a linear combination of insertions of elements of $\mathcal{G}$, of the form

$$f = \sum_{i \in I} \lambda_i \square_{T_i}^{S_i}(g_i)$$

where $T_i = \mathrm{LM}(g_i)$ for all $i \in I$. We call the element $S = \max\{S_i : i \in I\}$ the parameter of this representation.

Note that any S-polynomial $S_T(g_1, g_2)$ admits a representation of parameter T (although the two obvious terms carrying where this tree monomial appears will cancel): the parameter of a representation of f need not coincide with the leading term of f. We say a representation of an S-polynomial $S_T(g_1, g_2)$ is non-trivial if its parameter is smaller than T.

8.2 Diamond Lemma

The following result is one of the most useful ways one can verify a subset of an ideal is a Gröbner basis.

Theorem 8.3 (Diamond Lemma) *For a self-reduced set of generators $\mathcal{G}$ of an ideal $\mathcal{I}$, the following statements are equivalent:*

(1) The set $\mathcal{G}$ is a Gröbner basis of $\mathcal{I}$.
(2) Every S-polynomial reduces to zero modulo $\mathcal{G}$.
(3) Every S-polynomial admits a non-trivial representation.
(4) Every $f \in \mathcal{I}$ admits a representation with parameter $\mathrm{LM}(f)$.

Proof. The implications $(4) \implies (1) \implies (2) \implies (3)$ are straightforward, and we leave them as a guided Exercise 66. The hardest part of the proof is showing that (3) implies (4), which we go through in detail here. To prove it, let us assume that (3) holds, but that (4) does not. Then, there exists some $f \in \mathcal{I}$ so that every representation of f has parameter larger than $\mathrm{LM}(f)$, and let us pick a representation giving a counterexample with the following properties:

(1) The parameter T of the representation is minimum.

(2) Among representations with parameter T, we choose one with $k = \{i \in I : S_i = T\}$ minimum.

We now consider those divisors $T_1, T_2, \dots, T_k$ of T appearing in the representation and focus on the last two divisors T_{k-1} and T_k: notice that $k > 1$, for else there is not room for cancellations to occur so that the leading monomial of f is smaller than T.

We claim that we can always arrange it so that we obtain a new representation of f that breaks either the first or the second condition. To do this, we will consider the relative position of the divisors T_{k-1} and T_k in T.

Case 1: one divisor is contained in the other. In this situation, this means that $g_k = g_{k-1}$ since $\mathcal{G}$ is reduced, which implies we can merge these two terms into one. If their coefficients sum to zero, then either we get a representation with smaller parameter if $k = 2$, or with smaller k in general, which cannot be.

Case 2: the divisors are disjoint. This situation is similarly simple to the first case. In this case, we have a well-defined bilinear operation of "double insertion"

$$\square^T_{T_k,T_{k-1}}(-,-)\mathcal{F}_{\mathcal{X}}(\operatorname{ar} T_{k-1}) \otimes \mathcal{F}_{\mathcal{X}}(\operatorname{ar} T_k) \longrightarrow \mathcal{F}_{\mathcal{X}}(\operatorname{ar} T)$$

which replaces occurrences of T_k and T_{k-1} in T simultaneously. Let us write $g_k = \mathrm{LT}(g_k) + g'_k$, and similarly with the other, and let us note that the following chain of equalities hold:

$$\begin{aligned}
\square^T_{T_k}(g_k) &= \square^T_{T_k,T_{k-1}}(g_k, \mathrm{LT}(g_{k-1})) \\
&= \square^T_{T_k,T_{k-1}}(g_k, g_{k-1} - g'_{k-1}) \\
&= \square^T_{T_k,T_{k-1}}(g_k, g_{k-1}) - \square^T_{T_k,T_{k-1}}(g_k, g'_{k-1}) \\
&= \square^T_{T_k,T_{k-1}}(\mathrm{LT}(g_k) + g'_k, g_{k-1}) - \square^T_{T_k,T_{k-1}}(g_k, g'_{k-1}) \\
&= \square^T_{T_k,T_{k-1}}(\mathrm{LT}(g_k), g_{k-1}) + \square^T_{T_k,T_{k-1}}(g'_k, g_{k-1}) - \square^T_{T_k,T_{k-1}}(g_k, g'_{k-1}) \\
&= \square^T_{T_{k-1}}(g_{k-1}) + \square^T_{T_k,T_{k-1}}(g'_k, g_{k-1}) - \square^T_{T_k,T_{k-1}}(g_k, g'_{k-1}).
\end{aligned}$$

Note that this is saying there is an "obvious" relation between the two possible replacements we can make into T, but otherwise is not saying anything more profound. The takeaway is that we are able to replace the sum $\lambda_{k-1}\square^T_{T_{k-1}}(g_{k-1}) + \lambda_k\square^T_{T_k}(g_k)$ with the sum

$$(\lambda_k + \lambda_{k-1})\square^T_{T_{k-1}}(g_{k-1}) + \lambda_k(\square^T_{T_k,T_{k-1}}(g'_k, g_{k-1}) - \square^T_{T_k,T_{k-1}}(g_k, g'_{k-1}))$$

where the last term can be expanded, by using Lemma 7.16 and Proposition 7.5, into sums of terms with leading monomial smaller than T. Thus, either the parameter of our representation decreases, which happens if $k = 2$ and the coefficients add up to zero, or the parameter remains unmodified but k decreases.

Case 3: the divisors overlap. This is perhaps the most computationally heavy of the three cases. Let us assume that T_k and T_{k-1} have a small common multiple T', which of course is a divisor of T. Using Lemma 7.16, we can write for $i \in \{k-1, k\}$:

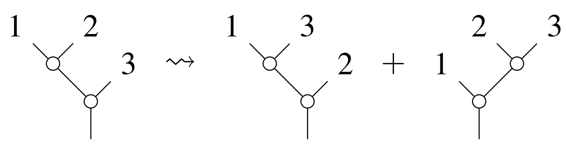
$$\lambda_i \square^{T}_{T_i}(g_i) = \lambda_i \square^{T}_{T'}(\square^{T}_{T_i}(g_i))$$

and with this write the sum $\lambda_{k-1}\square^{T}_{T_{k-1}}(g_{k-1}) + \lambda_k \square^{T}_{T_k}(g_k)$ as follows

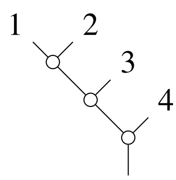
$$\begin{aligned}
&= \square^{T}_{T'}(\lambda_{k-1}\square^{T'}_{T_{k-1}}(g_{k-1}) + \lambda_k(\square^{T'}_{T_{k-1}}(g_{k-1}) - S_{T'}(g_{k-1}, g_k)))\\
&= \square^{T}_{T'}((\lambda_{k-1}+\lambda_k)\square^{T'}_{T_{k-1}}(g_{k-1}) - \lambda_k S_{T'}(g_{k-1}, g_k))\\
&= (\lambda_{k-1}+\lambda_k)\square^{T}_{T_{k-1}}(g_{k-1}) - \lambda_k \square^{T}_{T'}(S_{T'}(g_{k-1}, g_k)).
\end{aligned}$$

We have assumed that all S-polynomials admit at least one non-trivial representation, so we conclude that either we obtain a representation with a smaller parameter (which happens if $k = 2$ and the coefficients cancel) or with the same parameter, but smaller k. To see this, the reader should again make use of Lemma 7.16 and Proposition 7.5 (Exercise 67). $\square$

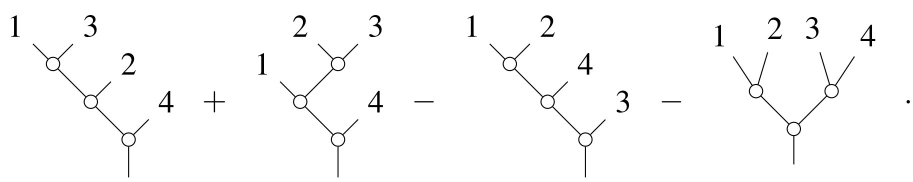
As an example, let us consider the shuffle operad $\mathrm{Lie}^{\mathrm{f}}$, given by a single binary generator and the Jacobi relation. Choosing the `grapathpermlex` that picks up the leading term as follows:



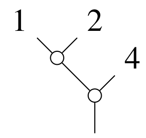
and we again obtain the following left comb with four leaves as an overlapping ambiguity:

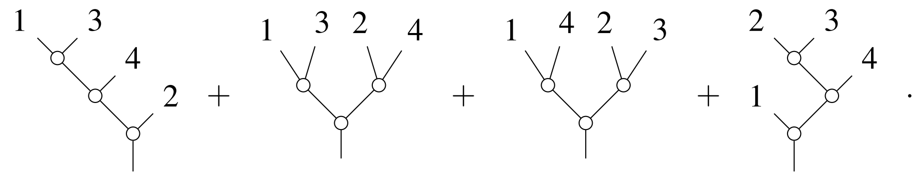


The resulting S-polynomial is the following sum of tree monomials:

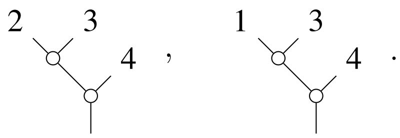


Instead of continuing to rewrite all four terms (all of them are divisible by our choice of leading term) which will create a total of eight terms, let us do this with the first two summands first, and then with the last two summands. We will see that the end results are the same, which means that this S-polynomial rewrites to zero through the Jacobi identity.

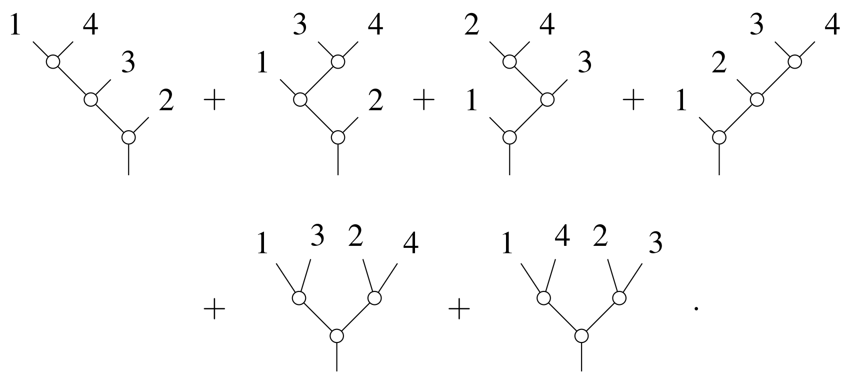
The first two terms are divisible by occurrences of a divisor  and applying the Jacobi identity we obtain the polynomial



We observe that the middle terms are reduced, and that only the last term and first term are not, since they have corresponding divisors



Applying the Jacobi identity one more time, we arrive at the six-term reduced polynomial



We leave it as Exercise 68 to carry out the reduction algorithm on the other two terms (the ones carrying minus signs): if all goes well, the reader will arrive at the same six term reduced polynomial as the one above, thus confirming that the Jacobi identity does provide us with a one element Gröbner basis for $\mathsf{Lie}^{\mathsf{f}}$. We will see later that this guarantees that our presentation of $\mathsf{Com}^{\mathsf{f}}$ is also a Gröbner basis, but for the reverse order.

8.3 Buchberger's Algorithm

The Diamond Lemma gives us a recipe to (attempt to) fix any generating set to a Gröbner basis by uncovering hidden leading terms through S-polynomials: borrowing terminology from the original theory for commutative rings, we call it the Buchberger Algorithm. We present it here in the form of pseudo-code, and remark that the algorithm may not terminate (as some ideals may admite infinite reduced Gröbner bases for certain choices of monomial orders).

Algorithm 3 Buchberger's Algorithm

INPUT: a set of generators $\mathcal{V}$ for an ideal $\mathcal{I}$ in a free shuffle operad.
OUTPUT: the reduced Gröbner basis of $\mathcal{I}$, if it is finite.

```
procedure BUCHBERGERALG(Polynomials)
    newSPolynomials ← True
    ToComplete ← Polynomials
    while newSPolynomials do
        ToComplete ← SELFREDUCE(ToComplete)
        ToComplete ← SORT(ToComplete, grapathpermlex)
        SPolynomials ← ∅
        newSPolynomials ← False
        for g1, g2 ∈ ToComplete do
            LocalOverlap ← OVERLAPS(g1, g2)
            DoneOverlaps ← ∅
            while LocalOverlaps ≠ ∅ do
                Overlap ← LocalOverlaps.POP()
                DoneOverlaps.ADD(Overlap)
                LocalSPoly ← SPOLY(g1, g2, Overlap)
                RemainderSPoly ← REDUCE(LocalSPoly, ToComplete)
                if RemainderSPoly ∉ SPolynomials then
                    newSPolynomials ← True
                    SPolynomials.ADD(RemainderSPoly)
        ToComplete.UNION(SPolynomials)
    return ToComplete
```

8.4 Exercises

Exercise 66. Prove the implications $(4) \Longrightarrow (1) \Longrightarrow (2) \Longrightarrow (3)$ in the Diamond Lemma.

Exercise 67. Fill in the missing details in the proof the implicatoin $(3) \Longrightarrow (4)$ of the Diamond Lemma, which require the use of Lemma 7.16 and Proposition 7.5.

Exercise 68. Complete the reduction of the S-polynomial coming from the self-overlap of the Jacobi identity in the presentation of the operad $\mathsf{Lie}^{\mathsf{f}}$ with respect to the `grapathpermlex` order of tree monomials.

Exercise 69. Redo the previous exercise using the Haskell Operad Calculator.

Exercise 70. The dendriform operad is a ns quadratic operad generated by two operations $a : (x_1,x_2) \mapsto x_1 \prec x_2$ and $b : (x_1,x_2) \mapsto x_1 \succ x_2$ subject to the following quadratic relations:

$$
\begin{aligned}
&(x_1 \succ x_2) \prec x_3 = x_1 \succ (x_2 \prec x_3)\\
&(x_1 \prec x_2) \prec x_3 = x_1 \prec (x_2 \prec x_3 + x_2 \succ x_3),\\
&x_1 \succ (x_2 \succ x_3) = (x_1 \prec x_2 + x_1 \succ x_2) \succ x_3.
\end{aligned}
$$

Consider the `grapathlex` order for the total order of generators given $a < b$.

(1) Determine the leading terms of the three relations, and check that this set of relations is self-reduced.
(2) Determine all overlapping ambiguities between these leading terms.
(3) Choose one overlapping ambiguity and check that the corresponding S-polynomial rewrites to zero.
(4) Use the Haskell Operad Calculator to prove that the remaining S-polynomials reduce to zero.
(5) Conclude that the quadratic relations above constitute a Gröbner basis for Dend and this choice of order.
(6) Now consider the order $b < a$. What happens?

```
# Operadic Buchberger Configuration File
# --------------------------------------

# Actions
actions: normalise

# Time limit (seconds)
time limit:

# Count limit
count limit:

# Output options
output:  new final

#Field
field:

# Operad type
operad type: asymmetric unsigned

# Measure
measure: deglex perm

# Signature
signature:  a(2) b(2)

# Theory
a( b(* *) * ) - b( * a( * * ) )
b( b( * * ) * ) - b( * b( * * ) ) + b( a ( * * ) * )
a( a( * * ) * ) - a( * a( * * ) ) - a( * b ( * * ) )
```

Figure 11: A minimal configuration file to compute a Gröbner basis of Dend.

```
*Main> main

Configuration:

actions:        normalise
count limit:    0
arity limit:    6
time limit:     none
output:         newly stable rewrite rules
                final theory
field:          rationals
operad type:    unsigned asymmetric operad
measure:        degree-lexicographic permutation
signature:      a(2) b(2)
theory:

  a(b(* *) *)  -  b(* a(* *))
  b(b(* *) *)  -  b(* b(* *))  +  b(a(* *) *)
  a(a(* *) *)  -  a(* a(* *))  -  a(* b(* *))

Arity: 4   Stable rewrite rules: 3
Current critical pairs: 4   Queued critical pairs: 0

No new rewrite rules

Success! Complete theory:

  a(b(* *) *)  ->  b(* a(* *))
  a(a(* *) *)  ->  a(* b(* *))  +  a(* a(* *))
  b(b(* *) *)  ->  - b(a(* *) *)  +  b(* b(* *))
```

Figure 12: The result of running the previous code in the Haskell Operad Calculator, showing that the given relations (theory) are a Gröbner basis for the `grapathlex` order with $a < b$.

9 Bar and cobar constructions

Goals. Define the bar (and cobar) construction, define the syzygy grading and give the first definition of what it means for an operad to be Koszul.

9.1 The bar construction

We have defines a functor on quadratic data $(\mathcal{X},\mathcal{R}) \longmapsto (\mathcal{X}^\vee,\mathcal{R}^\perp)$ that gives us the Koszul dual functor $\mathcal{P} \longmapsto \mathcal{P}^!$ on quadratic operads. Let us explain how one can "promote" this construction to a functor define on all operads, and how our more down-to-earth construction can be recovered from this. Let us begin with a useful notion, which is to an operad what an coassociative coalgebra is to an associative algebra.

Definition 9.1 A symmetric cooperad is a symmetric collection $\mathcal{C}$ endowed with maps of collections $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \circ_\Sigma \mathcal{C}$ and $\varepsilon : \mathcal{C} \longrightarrow \Bbbk$ so that Δ is coassociative and counital for ε. A cooperad $\mathcal{C}$ is augmented if there exists $\eta : \Bbbk \longrightarrow \mathcal{C}$ such that $\varepsilon\eta = 1_\Bbbk$. In this case, we write $\overline{\mathcal{C}}$ for the cokernel of η.

We warn the reader that, strictly speaking, we should have defined cooperads using a variant of the composition product $\circ_\Sigma$ that takes Σ-invariants instead of coinvariants. Since we will always work over a field of characteristic zero, we will not worry about this. At any rate, unraveling the definitions, we see that a cooperad is a symmetric sequence endowed with equivariant decomposition maps of signature

$$\Delta : \mathcal{C}(n) \longrightarrow \bigoplus_{n_1+\cdots+n_k=n} \mathcal{C}(k) \otimes \mathcal{C}(n_1) \otimes \cdots \mathcal{C}(n_k)$$

that are coassociative. One can check (Exercise 74) that if $\mathcal{C}$ is a cooperad then its arity-wise dual $\mathcal{C}^*$ is an operad, and that if $\mathcal{P}$ is an arity-wise finite dimensional reduced operad then its arit-wise dual $\mathcal{P}^*$ is a cooperad.

Definition 9.2 The reduced and half-reduced decomposition maps of a cooperad $\mathcal{C}$ are the maps $\overline{\Delta}$ and $\widetilde{\Delta}$ uniquely determined by the equation

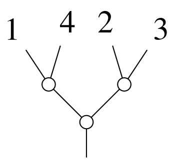
$$\Delta(v) = \overline{\Delta}(v) + (1;v) + (v;1,\ldots,1) = \widetilde{\Delta}(v) + (1;v)$$

for all arity-homogeneous $v \in \mathcal{C}$. We say that $\mathcal{C}$ is *conilpotent* if for each $v \in \mathcal{C}$ the iteration of $\widetilde{\Delta}$ on v defined by $\widetilde{\Delta}^n(v) = (1 \circ \Delta)\widetilde{\Delta}^{n-1}(v)$ stabilizes, in the sense that eventually all leaves are decorated by the identity element of $\mathcal{C}$.

As in the case of algebras and coalgebras, where for a vector space V the underlying vector space of the free associative algebra TV on V and the free *conilpotent* coassociative coalgebra coincide, the free *conilpotent* cooperad over a symmetric sequence is an object we have meet before.

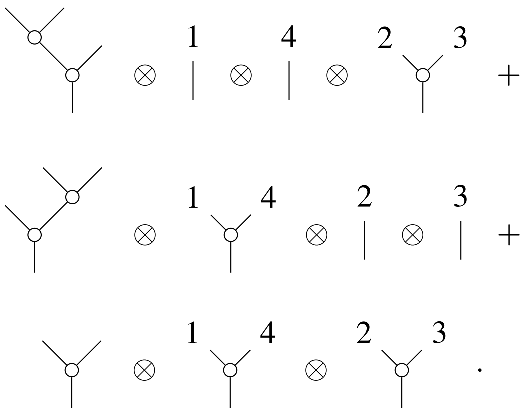
Definition 9.3 The free conilpotent cooperad over a symmetric sequence $\mathcal{X}$, which we denote by $\mathcal{F}^c_{\mathcal{X}}$, has the same underlying symmetric sequence as the free operad $\mathcal{F}_{\mathcal{X}}$, and its decomposition maps are given by "degrafing of trees".

We will look at the structure of the free conilpotent operad functor $\mathcal{F}^c$ in more detail in Exercise 72. As an example, if $x \in \mathcal{X}(2)$ is a binary symmetric operation and we consider the fork

$T =$ 

,

we have that $\overline{\Delta}(T)$ is equal to the sum



Although both $\mathcal{F}_{\mathcal{X}}$ and $\mathcal{F}^c_{\mathcal{X}}$ satisfy universal properties with respect to operad morphisms form $\mathcal{X}$ into an operad or from a conilpotent cooperad into $\mathcal{X}$, it will be convenient for us to observe they both satisfy (and are uniquely characterized by) a second universal property. To state it, we need to consider a natural generalization of the "Leibniz rule" for derivatives in the case of (co)operads.

Definition 9.4 Let $\mathcal{P}$ be an operad. A derivation of $\mathcal{P}$ is a map $d : \mathcal{P} \longrightarrow \mathcal{P}$ with the property that for any arity-homogeneous elements $\mu_0, \mu_1, \ldots, \mu_k \in \mathcal{P}$ with $\operatorname{ar} \mu_0 = k$, we have that

$$d\gamma(\mu_0; \mu_1, \ldots, \mu_k) = \sum_{i=0}^{k} \gamma(\mu_0; \mu_1, \ldots, d\mu_i, \ldots, \mu_k).$$

If $\mathcal{P}$ is homologically graded, we require that d is degree homogeneous, and then Koszul signs will appear in the display above.

The following is the universal property we were after.

Lemma 9.5 *Let $\mathcal{X}$ be a symmetric collection. The free operad $\mathcal{F}_\mathcal{X}$ on $\mathcal{X}$ satisfies the following universal property: for every map of symmetric sequences $f : \mathcal{X} \longrightarrow \mathcal{P}$ where $\mathcal{P}$ is an operad, there exists a unique derivation $F : \mathcal{F}_\mathcal{X} \longrightarrow \mathcal{P}$ which coincides with f on $\mathcal{F}_X^{(1)} = \mathcal{X}$.*
Dually, the free conilpotent cooperad $\mathcal{F}^c_\mathcal{X}$ satisfies the following universal property: for every map of symmetric sequences $f : \mathcal{C} \longrightarrow \mathcal{X}$ where $\mathcal{X}$ is an conilpotent cooperad, there exists a unique coderivation $F : \mathcal{C} \longrightarrow \mathcal{F}^c_\mathcal{X}$ which coincides with f after projecting onto $\mathcal{F}_X^{c,(1)} = \mathcal{X}$.

We use it to define one of the most important functors we will consider in this section, the *bar construction*. To do this, let us begin with an augmented operad $\mathcal{P}$, and let us form the free conilpotent cooperad $\mathcal{F}^c_{s\overline{\mathcal{P}}}$ on the naïve suspension $s\overline{\mathcal{P}}$ of its augmentation ideal. This is for the moment a homologically graded cooperad, which we will make into a dg conilpotent cooperad as follows: the composition map of $\mathcal{P}$ provides us with a map of symmetric collections

$$\mathcal{F}^c_{s\overline{\mathcal{P}}} \longrightarrow s\overline{\mathcal{P}}$$

that is zero in all tree monomials of weight different from two, and on the latter, it is equal to the unique map

$$\gamma' : \mathcal{F}^{c,(2)}_{s\overline{\mathcal{P}}} \longrightarrow s\overline{\mathcal{P}}$$

induced by the composition map of $\mathcal{P}$. The universal property of $\mathcal{F}^c$ guarantees there is a unique coderivation

$$\partial : \mathcal{F}^c_{s\overline{\mathcal{P}}} \longrightarrow \mathcal{F}^c_{s\overline{\mathcal{P}}}$$

extending γ'. In Exercise 77 we will see that $\partial^2 = 0$, but for the moment notice that ∂ has degree -1, as the map γ' does: it erases one suspension sign from a single tree monomial with exactly one internal edge whose two vertices are decorated by elements of $\overline{\mathcal{P}}$, by composing them along this unique internal edge.

Definition 9.6 Let $\mathcal{P}$ be an augmented operad. We call the dg cooperad $(\mathcal{F}^c_{s\overline{\mathcal{P}}}, \partial)$ the bar construction of $\mathcal{P}$ and write it $\mathsf{B}(\mathcal{P})$.

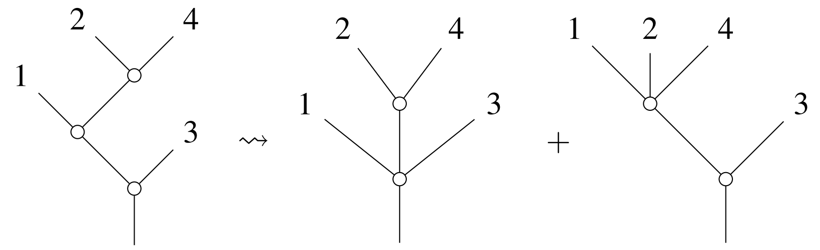


Figure 13: A boundary map in $\mathsf{B}(\mathsf{Com})$, vertices are not decorated as there is a unique k-ary variable in Com for each $k \in \mathbb{N}$.

Let us begin exploring the shape of this gadget. For starters, since its underlying symmetric collection is equal to that of a free operad, whenever $\mathcal{P}$ is reduced —which is running assumption for us— it has a basis of shuffle tree monomials decorated by elements of $\mathcal{P}$ that are *not* the unit; let us call these *bar tree monomials*. A bar tree monomial has homological degree d precisely when it has d internal vertices, as the device of using $s\overline{\mathcal{P}}$ instead of $\overline{\mathcal{P}}$ means precisely that we are turning the canonical weight grading of $\mathcal{F}$ into a homological grading. We will use the notation $|T|$ for this homological degree, as usual, and use the subscript notation $\mathsf{B}(\mathcal{P})_*$ when referring to this grading. Thus, for example, $\mathsf{B}(\mathcal{P})_2$ consists of those bar tree monomials of homological degree two.

At the same time, we will usually take $\mathcal{P}$ to be *itself* weight graded, so that bar tree monomials have a homological degree, which keeps track of how many elements of $\mathcal{P}$ appear in a tree monomial, and a total weight, which simply is obtained by adding up the weights of the elements of $\mathcal{P}$ in them. We will write $\|T\|$ for the weight of a tree monomial and use the notation $\mathsf{B}(\mathcal{P})_{(*)}$ when referring to this weight grading. Thus, for example, $\mathsf{B}(\mathcal{P})_{(2)}$ consists of those bar tree monomials of total weight degree two, which can then either have homological degree one and be decorated by one weight two element of $\mathcal{P}$, or have homological degree two and be decorated by two weight one elements of $\mathcal{P}$. Note that in general $|T| \leqslant \|T\|$.

Definition 9.7 The syzygy degree of a bar tree monomial T is the difference $\|T\| - |T|$, and we write it $\mathsf{s}(T)$. We call this the syzygy grading of $\mathsf{B}(\mathcal{P})$, and we use the superscript notation $\mathsf{B}(\mathcal{P})^*$ when referring this this cohomological grading: the differential d has degree $+1$ in the syzygy grading.

Example 9.8 Let us consider the case where $\mathcal{P} = \mathsf{Com}$ is the commutative operad, in which case $\mathcal{X} = s\overline{\mathcal{P}}$ is a symmetric sequence such that $\mathcal{X}(0) = \mathcal{X}(1) = 0$ and for which $\mathcal{X}(n)$ is one dimensional for each $n \geqslant 1$ and concentrated in degree zero and weight $n-1$. It follows that weight

$$
\begin{tikzcd}
\mathcal{P}^{\mathrm{i}}_{(4)} \arrow[r] & \mathsf{B}(\mathcal{P})_{4,(4)} \arrow[r] & \mathsf{B}(\mathcal{P})_{3,(4)} \arrow[r] & \mathsf{B}(\mathcal{P})_{2,(4)} \arrow[r] & \mathsf{B}(\mathcal{P})_{1,(4)} \\
\mathcal{P}^{\mathrm{i}}_{(3)} \arrow[r] & \mathsf{B}(\mathcal{P})_{3,(3)} \arrow[r] & \mathsf{B}(\mathcal{P})_{2,(3)} \arrow[r] & \mathsf{B}(\mathcal{P})_{1,(3)} & \\
\mathcal{P}^{\mathrm{i}}_{(2)} \arrow[r] & \mathsf{B}(\mathcal{P})_{2,(2)} \arrow[r] & \mathsf{B}(\mathcal{P})_{1,(2)} & & \\
\mathcal{P}^{\mathrm{i}}_{(1)} \arrow[r] & \mathsf{B}(\mathcal{P})_{1,(1)} & & & \\
\mathbb{k} & & & &
\end{tikzcd}
$$

syzygy

Figure 14: The bar construction of a weight graded operad $\mathcal{P}$. Horizontal arrows are differentials, leftmost arrows are inclusions. Note that the diagonals where $s+w$ is constant return the usual homological degree d.

$\mathsf{B}(\mathsf{Com})$ has basis consisting of shuffle trees where the internal vertices have all possible arities greater or equal to 2, and in this case the degree of a bar tree monomial is equal to the number of its internal vertices. Let us write S_m for the element in $\mathsf{B}(\mathsf{Com})$ of degree 1 and weight $m-1$ corresponding to the m-fold product in Com.

If we write $T_i(A,B)$ for the shuffle tree in Figure 3, where $A=\{i,j_1,\ldots,j_m\}$ and $B=\{1,\ldots,i-1,k_1,\ldots,k_n\}$, we see that $\partial T_i(A,B)=(-1)^{i-1}S_m$ where $m\geqslant 3$, so all of these elements are zero in homology. For example, if we look at $\mathsf{B}(\mathsf{Com})(3)$, we notice this complex has total dimension four, with the usual three shuffle tree monomials

$$T_1(12,3),\quad T_1(13,2),\quad \text{and}\quad T_2(1,23)$$

with three leaves in degree two (and weight two) and one single tree monomial (the corolla S_3) in degree one. Moreover, the computation above shows that generators for the homology are

$$T_1(12,3)+T_2(1,23),\qquad T_1(13,2)+T_2(1,23)$$

Similarly, one can compute as in Exercise 76 that $\mathsf{B}(\mathsf{Com})$ is of dimension 26, and that the degree two cycles are of dimension 9 (all 10 basis shuffle trees map to the only basis elements of degree 1). Moreover, one can check that all of these cycles in degree two are boundaries. Since this complex has $\chi=6$ and we have just noted it has homology only in degree three, we see this homology group is of this dimension. We will see very soon that in syzygy degree zero, the homology of $\mathsf{B}(\mathsf{Com})(n)$ is of dimension $(n-1)!$.

9.2 Koszul dual cooperad

Let us begin by observing that, in the syzygy grading, $H^0(\mathsf{B}(\mathcal{P}))$ is computed as the kernel of a linear operator, as all syzygy degrees are non-negative (so there are no boundaries in degree zero). We can in fact try to study the shape of the bar construction with a little more care, and determine precisely what this homology group is as a *cooperad* in case $\mathcal{P}$ is quadratic.

To do this, let us assume that $\mathcal{P}$ is generated by $\mathcal{X}$ (in weight one) subject to some quadratic relations $\mathcal{R}$ in $\mathcal{F}^{(2)}_{\mathcal{X}}$. Then, we see that elements of $\mathsf{B}(\mathcal{P})^0$ (that is, those that have degree equal to weight) must be precisely those of in $\mathcal{F}^c_{s\mathcal{X}}$. Note the difference! While all of $\mathsf{B}(\mathcal{P})$ is given by a free cooperad on $\mathcal{P}$, the syzygy degree zero part only retains the elements that generate $\mathcal{P}$.

Having said this, if we decompose $\mathcal{F}^c_{s\mathcal{X}}$ into its tree monomial components, and if we write $\mathcal{V}=\mathcal{F}^{(2)}_{\mathcal{X}}/\mathcal{R}$, we see that the differential ∂ maps $\mathcal{F}^c_{s\mathcal{X}}$ to $\mathcal{F}^c_{s\mathcal{X}\oplus s\mathcal{V}}$, since it at most can introduce an element of $\mathcal{P}$ of weight two, and it maps to the symmetric subsequence $\mathcal{F}^c_{s\mathcal{X},s\mathcal{V}}$ where *exactly one* internal vertex is decorated by an element of $s\mathcal{V}$.

Definition 9.9 Let $\mathcal{X}$ be a symmetric sequence and let $\mathcal{R} \subseteq \mathcal{F}^{c,(2)}_{\mathcal{X}}$. The cofree conilpotent cooperad cogenerated by $\mathcal{X}$ subject to the correlations $\mathcal{R}$ is the unique conilpotent cooperad $\mathcal{F}^c\mathcal{X},\mathcal{R})$ that is universal among those conilpotent cooperads $\mathcal{C}$ along with a map $\mathcal{C} \longrightarrow \mathcal{X}$ such that the projection of the unique map $\mathcal{C} \longrightarrow \mathcal{F}^c_{\mathcal{X}}$ to $\mathcal{C} \longrightarrow \mathcal{F}^{c,(2)}_{\mathcal{X}}$ has image in $\mathcal{R} \subseteq \mathcal{F}^{c,(2)}_{\mathcal{X}}$.

If the reader finds the definition slightly confusing, let us remark that the quotient operad $\mathcal{F}(\mathcal{X},\mathcal{R}) = \mathcal{F}_{\mathcal{X}}/(\mathcal{R})$ is universal among those operads $\mathcal{P}$ along with a map $\mathcal{X} \longrightarrow \mathcal{P}$ with the property that the restriction of the unique map $\mathcal{F}_{\mathcal{X}} \longrightarrow \mathcal{P}$ to $\mathcal{F}^{(2)}_{\mathcal{X}}$ *factors* through $\mathcal{F}^{(2)}_{\mathcal{X}}/\mathcal{R}$: whereas the condition on the operad is that $\mathcal{R}$ is contained in the kernel of this map, the condition on conilpotent operads is that $\mathcal{R}$ is contained in the image.

Proposition 9.10 *If $\mathcal{P} = \mathcal{F}(\mathcal{X},\mathcal{R})$ then the zeroth syzygy homology groups $H^0(\mathsf{B}(\mathcal{P}))$ are equal to $\mathcal{F}^c(s\mathcal{X},s^2\mathcal{R})$. More precisely, $\mathcal{C} = H^0(\mathsf{B}(\mathcal{P}))$ is a weight graded subcooperad, with $\mathcal{C}_{(1)} = s\mathcal{X}$ and the projection $\mathcal{C} \longrightarrow s\mathcal{X}$ satisfies the universal property of the definition above for $s^2\mathcal{R}$.*

Proof. First, let us note that not only is $H^0(\mathsf{B}(\mathcal{P}))$ equal to $s\mathcal{X}$ in weight 1, but it is also equal to $s^2\mathcal{R}$ in weight 2, as this is by definition the kernel of the composition

$$\mathsf{B}(\mathcal{P})_{2,(2)} = \mathcal{F}^{(2)}_{s\mathcal{X}} \subseteq \mathsf{B}(\mathcal{P})^0 \longrightarrow s\overline{\mathcal{P}}_{(2)} \subseteq \mathsf{B}(\mathcal{P})^1$$

so that the projection $H^0(\mathsf{B}(\mathcal{P})) \longrightarrow s\mathcal{X}$ does induce a map $H^0(\mathsf{B}(\mathcal{P})) \longrightarrow \mathcal{F}^c_{\mathcal{X}}$ with the desired properties. Suppose that $\mathcal{C}$ is a cooperad with a map f to $\mathcal{X}$ for which the projection $\pi \circ F : \mathcal{C} \longrightarrow \mathcal{F}^{c,(2)}_{\mathcal{X}}$ lands in $s^2\mathcal{R}$. Since $H^0(\mathsf{B}(\mathcal{P}))$ is a subcooperad of $\mathcal{F}^c_{s\mathcal{X}}$, it suffices to show that the map $F : \mathcal{C} \longrightarrow \mathcal{F}^c_{s\mathcal{X}}$ has image in $H^0(\mathsf{B}(\mathcal{P})) = \ker \partial^0$.

By definition, F is obtained through f by iteration of the decomposition map of $\mathcal{C}$: for each $n \geqslant 1$ the component $F : \mathcal{C} \longrightarrow \mathcal{F}^{c,(n)}_{s\mathcal{X}}$ in weight n is obtained using $n-1$ instances of the decomposition map of $\mathcal{C}$. By hypothesis, every use of the decomposition map of $\mathcal{C}$ always creates quadratic summands that are in $\mathcal{R}$, so by coassociativity, iteration of Δ will produce terms that can be written to contain relations in any "big vertex" of $\mathcal{F}^c_{s\mathcal{X}}$ we prefer. This guarantees that when we apply ∂^0 the result will be zero (as this is an alternating sum of terms, each which annihilates an appropriate way of writing down the result of iterating Δ) so that F indeed lands in $H^0(\mathsf{B}(\mathcal{P}))$, like we wanted. $\square$

The reader may compare to the case of associative algebras (that is, when $\mathcal{X}$ consists exclusively of arity one operations) in which case the cofree conilpotent coalgebra C with cogenerators V and relations $R \subseteq V^{\otimes 2}$ is such that for each $n \geqslant 2$,

$$C_{(n)} = \bigcap_{i=1}^{n-1} V^{\otimes(i-1)} \otimes R \otimes V^{\otimes(n-i-1)}.$$

Definition 9.11 Let $\mathcal{P}$ be the quadratic operad associated to the quadratic datum $(\mathcal{X}, \mathcal{R})$. We call $H^0(\mathsf{B}(\mathcal{P})) = \mathcal{F}^c(s\mathcal{X}, s^2\mathcal{R})$ the Koszul dual cooperad to $\mathcal{P}$, and write it $\mathcal{P}^{\text{i}}$.

Recall that we defined $\mathscr{S} = \mathrm{End}_{s\Bbbk}$, the suspension operad. We will use $\mathscr{S}^c$ to denote the endomorphism cooperad $\mathrm{End}^c_{s\Bbbk}$, which we can think simply as the arity-wise dual of $\mathscr{S}^{-1}$. The "Koszul pairing" we used when we defined the operad $\mathcal{P}^!$ gives a hint to the following result.

Lemma 9.12 *The dual of the cooperad $\mathscr{S}^c \otimes \mathcal{P}^{\text{i}}$ is isomorphic to $\mathcal{P}^!$.*

Proof. This is Exercise 75. □

In particular, this proves that the dual operad to $H^0(\mathsf{B}(\mathsf{Com}))$ is equal to the Lie operad, which explains the computations done and suggested above. We can now conclude this lecture by introducing one of the central definitions in these notes.

Definition 9.13 (First definition) A weight graded (quadratic) operad $\mathcal{P}$ is Koszul if the inclusion of cooperads $\mathcal{P}^{\text{i}} \longrightarrow \mathsf{B}(\mathcal{P})$ is a quasi-isomorphism. In other words, we say that $\mathcal{P}$ is Koszul if the homology of its bar construction is concentrated in syzygy degree zero.

9.3 Exercises

Exercise 71. All the results in this section have corresponding dual statements for conilpotent cooperads. State and prove them. In particular, define for each weight graded cooperad $\mathcal{C}$ its cobar construction $\Omega(\mathcal{C})$, and describe the syzygy grading and the differential.

Exercise 72. Consider a binary alphabet $\mathcal{X}$ with a single operation, and compute the decomposition map of $\mathcal{F}^c_{\mathcal{X}}$ in case this operation is symmetric, antisymmetric or regular for tree monomials with four leaves.

Exercise 73. Show that a cooperad $\mathcal{C}$ is conilpotent if and only if there exists a symmetric sequence $\mathcal{X}$ and an injective map of cooperads $\mathcal{C} \longrightarrow \mathcal{F}^c_{\mathcal{X}}$.

Exercise 74. Show that if $\mathcal{C}$ is a cooperad then its arity-wise dual $\mathcal{C}^*$ is an operad, and that if $\mathcal{P}$ is an arity-wise finite dimensional reduced operad then its arit-wise dual $\mathcal{P}^*$ is a cooperad. Make sure to explain why the second set of hypotheses are needed.

Exercise 75. Prove Lemma 9.12, which states that the dual of the cooperad $\mathscr{S}^c \otimes \mathcal{P}^{\mathsf{i}}$ is isomorphic to $\mathcal{P}^!$.

Exercise 76. Compute the homology of the complex $\mathsf{B}(\mathsf{Com})(4)$ explicitly. Show it is concentrated in syzygy degree 0, where it is six dimensional, and give representatives for the cocycles giving a basis of it.

Exercise 77. Show that the differential ∂ of the bar construction of an operad $\mathcal{P}$ squares to zero if and only if the composition map of $\mathcal{P}$ is associative.

Exercise 78. Suppose that $\mathcal{P}$ is a dg operad, and let us write ∂_1 for the differential of its bar construction (considered as a non-dg operad). Recall that $s\mathcal{P}$ gets the differential $-sd_{\mathcal{P}}s^{-1} : s\overline{\mathcal{P}} \longrightarrow s\overline{\mathcal{P}}$.

(1) Show that in this case the differential of $s\mathcal{P}$ induces a differential ∂_2 on $\mathcal{F}_{s\mathcal{P}}$.
(2) Show that $\partial_1\partial_2 + \partial_2\partial_1 = 0$ is equivalent to $d_{\mathcal{P}}$ being a derivation for $\gamma_{\mathcal{P}}$.
(3) Conclude that $\partial = \partial_1 + \partial_2$ is a differential on $\mathcal{F}_{s\mathcal{P}}$.

We call the resulting cooperad the bar construction of $\mathcal{P}$ and write it $\mathsf{B}(\mathcal{P})$.

10 Koszul complexes

Goals. Define twisting morphisms, unravel the bar-cobar adjunction. Give the definition of the Koszul complexes associated to a quadratic operad, and give a second definition of Koszulness.

10.1 Twisting morphisms

We begin with some more operadic algebra, this time focusing on certain variants of the circle product that will help us state and prove some (perhaps technical) results about operads, conilpotent cooperads, and their (co)bar constructions.

Infinitesimal composites. Since we will use them often, let us define two flavours of "infinitesimal composites" between two symmetric sequences $\mathcal{X}$ and $\mathcal{Y}$. First, for a third sequence $\mathcal{Y}'$, let us write $\mathcal{X} \circ (\mathcal{Y}; \mathcal{Y}')$ for the subfunctor of $\mathcal{X} \circ (\mathcal{Y} \oplus \mathcal{Y}')$ which is linear in $\mathcal{Y}'$. In other words, we consider corollas whose root vertex is decorated by $\mathcal{X}$, and all whose leaves are decorated by an element of $\mathcal{Y}$ except for one which is decorated by an element of $\mathcal{Y}'$. This construction enjoys some sort of "mixed functoriality" with the usual composition product:

Definition 10.1 Let $f : \mathcal{X} \longrightarrow \mathcal{X}'$ and $g : \mathcal{Y} \longrightarrow \mathcal{Y}'$ be maps of symmetric sequences. We define the infinitesimal composite $f \circ' g$ to be the map

$$f \circ' g : \mathcal{X} \circ \mathcal{Y} \longrightarrow \mathcal{X}' \circ (\mathcal{Y}; \mathcal{Y}')$$

such that $(f \circ' g)(x; y_1, \ldots, y_k) = \sum_{i=1}^{k} (f(x); y_1, \ldots, g(y_i), \ldots, y_k)$.

As usual, Koszul signs will appear if our sequences are dg. We write $\mathcal{X} \circ_{(1)} \mathcal{Y}$ for $\mathcal{X} \circ (\Bbbk, \mathcal{Y})$. In other words, we consider corollas whose root vertex is decorated by $\mathcal{X}$, and all whose leaves are "empty", except for one which is decorated by an element of $\mathcal{Y}$. If f and g are as above, there is a map

$$f \circ_{(1)} g : \mathcal{X} \circ_{(1)} \mathcal{Y} \longrightarrow \mathcal{X}' \circ_{(1)} \mathcal{Y}'$$

such that $(f \circ_{(1)} g)(x; 1, \ldots, 1, y, 1, \ldots, 1) = (f(x); 1, \ldots, 1, g(y), 1, \ldots, 1)$. As we suggest, these two constructions are better behaved than composite product when it comes to linearity:

Lemma 10.2 *For any four symmetric sequences $\mathcal{X}, \mathcal{Y}, \mathcal{Y}_1, \mathcal{Y}_2$, we have a natural isomorphism*

$$\mathcal{X} \circ (\mathcal{Y}; \mathcal{Y}_1 \oplus \mathcal{Y}_2) \longrightarrow \mathcal{X} \circ (\mathcal{Y}; \mathcal{Y}_1) \oplus \mathcal{X} \circ (\mathcal{Y}; \mathcal{Y}_2).$$

In particular, if $\mathcal{Y} = \Bbbk$, we have a natural isomorphism

$$\mathcal{X} \circ_{(1)} (\mathcal{Y}_1 \oplus \mathcal{Y}_2) \longrightarrow (\mathcal{X} \circ_{(1)} \mathcal{Y}_1) \oplus (\mathcal{X} \circ_{(1)} \mathcal{Y}_2),$$

so that the functor $- \circ_{(1)} -$ is bilinear.

It is useful to remark that although $f \circ' g$ and $f \circ_{(1)} g$ look similar, they have different (co)domains and they act in different ways on summands: while the former outputs a sum of terms, the latter outputs one term only.

Infinitesimal (de)compositions. Let $\mathcal{P}$ be a dg operad and let $\mathcal{C}$ be a conilpotent dg cooperad, and let us assume (once and for all, throughout this section) that they are both (co)augmented. In this situation, we can consider morphisms of dg cooperads $\mathcal{C} \longrightarrow \mathsf{B}(\mathcal{P})$, and we claim these can be described purely in terms of linear maps $\tau : \mathcal{C} \longrightarrow \mathcal{P}$ satisfying some conditions.

Since $\mathcal{P}$ is an operad, we can then consider its *infinitesimal composition map* $\gamma_{(1)} : \mathcal{P} \circ_{(1)} \mathcal{P} \longrightarrow \mathcal{P}$ which encodes its partial compositions. Similarly, we can consider the infinitesimal decomposition map $\Delta_{(1)} : \mathcal{C} \longrightarrow \mathcal{C} \circ_{(1)} \mathcal{C}$ obtained by projecting the decomposition map of $\mathcal{C}$ onto $\mathcal{C} \circ_{(1)} \mathcal{C}$.

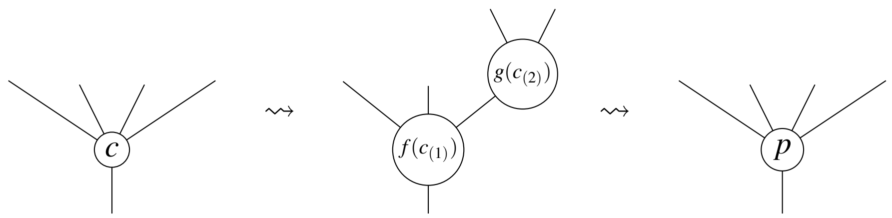


Figure 15: A schematic version of the star product: we use "Sweedler notation" for the infinitesimal decomposition of $\mathcal{C}$ and write p a generic summand $f(c_{(1)}) \circ_i g(c_{(2)})$ appearing in the final result.

Definition 10.3 Let $f, g \in \hom_\Sigma(\mathcal{C}, \mathcal{P})$ be maps of symmetric sequences. We define their star product $f \star g$ as the composition $\gamma_{(1)} \circ (f \circ_{(1)} g) \circ \Delta_{(1)}$, as depicted in Figure 15. We give $\hom_\Sigma(\mathcal{C}, \mathcal{P})$ the usual differential

$$\partial(f) = d_{\mathcal{P}} \circ f - (-1)^{|f|} f \circ d_{\mathcal{C}}.$$

One can show that $\hom_\Sigma(\mathcal{C}, \mathcal{P})$ along with the star product $- \star -$ becomes a pre-Lie algebra, which is dg if $\mathcal{C}$ or $\mathcal{P}$ are. By anti-symmetrizing $- \star -$, we thus obtain a dg Lie algebra structure on $\hom_\Sigma(\mathcal{C}, \mathcal{P})$. As in any dg Lie algebra, the degree -1 elements satisfying a particular equation play an important role and, for historical reasons, we give them a name here that different from the usual "Maurer–Cartan element":

Definition 10.4 We write $\mathsf{Tw}(\mathcal{C},\mathcal{P})$ for the set of degree -1 maps $\tau : \mathcal{C} \longrightarrow \mathcal{P}$ such that $\tau\eta = 0$, $\varepsilon\tau = 0$ and

$$\partial(\tau) + \tau \star \tau = 0,$$

and call it the set of *twisting morphisms* from $\mathcal{C}$ to $\mathcal{P}$. We call the equation in the display the *Maurer–Cartan equation* for τ.

10.2 Adjunction

The bar construction admits a dual construction for coaugmented conilpotent cooperads $\mathcal{C}$, which we call the cobar construction and write $\Omega(\mathcal{C})$. These two functors fit into a diagram

$$\mathsf{B}(-) : \mathsf{Op} \rightleftarrows \mathsf{Coop}^c : \Omega(-)$$

from the category of augmented operads (which we may allow to be dg, see Exercise 78) to the category of *conilpotent* augmented dg cooperads. The following theorem can be considered as one of the incarnations of Koszul duality between operads and conilpotent cooperads.

Theorem 10.5 *The bar and cobar functors form an adjoint pair* $\mathsf{B}(-) : \mathsf{Op} \rightleftarrows \mathsf{Coop}^c : \Omega(-)$. *More precisely, for every operad* $\mathcal{P}$ *and every conilpotent cooperad* $\mathcal{C}$*, the following are in natural bijection:*

(1) Maps of conilpotent dg cooperads $\mathcal{C} \longrightarrow \mathsf{B}(\mathcal{P})$.
(2) Maps of dg operads $\Omega(\mathcal{C}) \longrightarrow \mathcal{P}$.
(3) Twisting morphisms $\mathcal{C} \longrightarrow \mathcal{P}$.

Proof. Let us consider a map $f : \mathcal{C} \longrightarrow \mathsf{B}(\mathcal{P})$ of cooperads with $\mathcal{C}$ conilpotent. Forgetting for the moment that the codomain has a differential, we notice that there exists a unique map of symmetric collections $\mathcal{C} \longrightarrow s\overline{\mathcal{P}}$ that induces it. Equivalently, there is a degree -1 map $\tau : \mathcal{C} \longrightarrow \mathcal{P}$ such that $\varepsilon\tau = 0$ and $\tau\eta = 0$ (this second equality follows since $\mathsf{B}(\mathcal{P})$ is augmented and connected) which induces f. The fact that f commutes with the differential of $\mathsf{B}(\mathcal{P})$ is equivalent to the Maurer–Cartan equation

$$\partial\tau + \tau \star \tau = 0.$$

The degree -1 map $\tau : \mathcal{C} \longrightarrow \mathcal{P}$ in turn can be turned into a degree zero map $s^{-1}\overline{\mathcal{C}} \longrightarrow \mathcal{P}$ which, by the universal property of the free operad construction induces a map $\Omega(\mathcal{C}) \longrightarrow \mathcal{P}$, which in fact commutes with the differential, as one can check this is also equivalent to the equation above. We conclude that the two hom-sets in the statement of this theorem are in bijection with the set of symmetric sequence morphisms $\tau : \mathcal{C} \longrightarrow \mathcal{P}$ that satisfy the above Maurer–Cartan equation and such that both $\tau\eta$ and $\varepsilon\tau$ vanish, which are precisely the twisting morphisms we have defined. $\square$

Since it will be useful for us, let us make the bijection in the theorem above more explicit. To do this, let us consider the map $\pi : \mathsf{B}(\mathcal{P}) \longrightarrow \mathcal{P}$ obtained as the projection onto $s\overline{\mathcal{P}}$ and the degree -1 inclusion into $\mathcal{P}$. Then the bijection

$$\hom_{\mathsf{Coop}^c}(\mathcal{C}, \mathsf{B}(\mathcal{P})) \longrightarrow \mathsf{Tw}(\mathcal{C}, \mathcal{P})$$

is given by post-composition with π, while the inverse is given by extending the resulting degree zero map $\mathcal{C} \longrightarrow \mathcal{P}$ to a unique coderivation $\mathcal{C} \longrightarrow \mathsf{B}(\mathcal{P})$. We call π the universal twisting cochain (it is one, as per Exercise 81), and observe that it corresponds to the identity map of $\mathsf{B}(\mathcal{P})$ under the bijection above.

Dually, there is a degree -1 inclusion $\iota : \mathcal{C} \longrightarrow \Omega(\mathcal{C})$ which is also a twisting cochain —the universal twisting cochain for the conilpotent cooperad $\mathcal{C}$— and the bijection

$$\hom_{\mathsf{Op}}(\Omega(\mathcal{C}), \mathcal{P}) \longrightarrow \mathsf{Tw}(\mathcal{C}, \mathcal{P})$$

is given by pre-composition with ι. To summarize, for any map $\mathcal{C} \longrightarrow \mathcal{P}$ we have a commutative diagram

$$\begin{tikzcd}
 & \Omega(\mathcal{C}) \arrow[dr] & \\
\mathcal{C} \arrow[ur, "\iota"] \arrow[rr, "\tau"] \arrow[dr] & & \mathcal{P} \\
 & \mathsf{B}(\mathcal{P}) \arrow[ur, "\pi"'] &
\end{tikzcd}$$

where the diagonal maps are the universal twisting cochains, the anti-diagonal maps are maps of dg (co)operads and the only horizontal arrow is the corresponding twisting cochain. In other words, every twisting cochain $\mathcal{C} \longrightarrow \mathcal{P}$ factors uniquely as the composition of the universal twisting cochain ι followed by a morphism of dg operads, and as the composition of a morphism of dg conilpotent cooperads followed by the universal twisting cochain π.

10.3 Koszul complexes

Left and right modules over operads. Since it will be useful later, let us begin by noticing that operads admit representations other than their algebras (which we introduce in the Appendix). We will use this to define certain objects that are central to our lectures: the Koszul complexes of twisting cochains.

Definition 10.6 Let $\mathcal{P}$ be an operad. A right $\mathcal{P}$-module is a symmetric sequence $\mathcal{M}$ along with a map $\rho : \mathcal{M} \circ \mathcal{P} \longrightarrow \mathcal{M}$ that is unital and associative for the unit and the composition map of $\mathcal{P}$, that is, the following diagrams commute:

$$\begin{tikzcd}
(\mathcal{M} \circ \mathcal{P}) \circ \mathcal{P} \arrow[r, "\rho\circ 1"] \arrow[dd, "\alpha"'] & \mathcal{M} \circ \mathcal{P} \arrow[dr, "\rho"] & \\
& & \mathcal{M} \\
\mathcal{M} \circ (\mathcal{P} \circ \mathcal{P}) \arrow[r, "1\circ\gamma"] & \mathcal{M} \circ \mathcal{P} \arrow[ur, "\rho"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M} \circ \Bbbk \arrow[dr, "\cong"] \arrow[dd, "1\circ\eta"'] & \\
& \mathcal{M} \\
\mathcal{M} \circ \mathcal{P} \arrow[ur, "\rho"'] &
\end{tikzcd}$$

A map of right $\mathcal{P}$-modules is a morphism of symmetric sequences $f : \mathcal{M} \longrightarrow \mathcal{M}'$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M} \circ \mathcal{P} \arrow[r, "\rho"] \arrow[d, "f\circ 1"'] & \mathcal{M} \arrow[d, "f"] \\
\mathcal{M}' \circ \mathcal{P} \arrow[r, "\rho"'] & \mathcal{M}'
\end{tikzcd}$$

Finally, a free right $\mathcal{P}$-module is any module that is isomorphic to one of the form $\mathcal{M} = \mathcal{X} \circ \mathcal{P}$ where the map $\rho : \mathcal{M} \circ \mathcal{P} \longrightarrow \mathcal{M}$ is given by the composition

$$(\mathcal{X} \circ \mathcal{P}) \circ \mathcal{P} \xrightarrow{\alpha} \mathcal{X} \circ (\mathcal{P} \circ \mathcal{P}) \xrightarrow{1\circ\gamma} \mathcal{X} \circ \mathcal{P}.$$

Note. If we replace the word "right" above by "left" we obtain the corresponding definitions for *left* $\mathcal{P}$*-modules*. A word of caution is in order, however, as right modules and left modules in case of operads behave completely differently. For example, right actions are linear, in the sense that their signature is of the form

$$(m; p_1, \ldots, p_k) \longrightarrow \rho(m; p_1, \ldots, p_k)$$

and in particular there is only one argument in $\mathcal{M}$. On the other hand, left actions are not linear, in the sense that their signature is of the form

$$(p; m_1, \dots, m_k) \longrightarrow \lambda(p; m_1, \dots, m_k)$$

for multiple $m_1, \dots, m_k \in \mathcal{M}$: an element in $\mathcal{P}$ of arity k must act simultaneously on k elements of $\mathcal{M}$ at once and, since modules have no units, there is no way to linearize this. We will have the opportunity to see how this distinction will affect our development of the theory in some cases.

Definition 10.7 Let $\mathcal{M}$ be a left $\mathcal{P}$-module. An endomorphism $f : \mathcal{M} \longrightarrow \mathcal{M}$ of symmetric sequences is called a derivation if for all $m_1, \dots, m_k \in \mathcal{M}$ and $p \in \mathcal{P}(k)$ we have that $f\lambda = \lambda(1 \circ' f)$

In the case of right $\mathcal{P}$-modules, there is no difference between a derivation and a right linear map —in both cases the relevant requirement is that $f\rho = \rho(f \circ 1)$—, but we will use both names for consistency. Note that the condition above of being a derivation is linear, as f appears only once in each use of λ, so left linear maps and left derivations are slightly different. However, they are in natural bijection when we restrict ourselves to the class of free modules:

Lemma 10.8 *Let $\mathcal{P}$ be an operad and let $\mathcal{M}$ be a free left (resp. right) $\mathcal{P}$-module with basis $\mathcal{X}$. There is a natural bijection between maps of symmetric sequences $\mathcal{X} \longrightarrow \mathcal{M}$ and left (resp. right) derivations $\mathcal{M} \longrightarrow \mathcal{M}$. More precisely:*

(1) If $\mathcal{M}$ is left free, then the unique derivation $f : \mathcal{M} \longrightarrow \mathcal{M}$ which extends $\varphi : \mathcal{X} \longrightarrow \mathcal{M}$ is given by

$$f = d_{\mathcal{P}} \circ 1 + (\gamma_{(1)} \circ 1)(1 \circ' \varphi).$$

(2) If $\mathcal{M}$ is right free, then the unique derivation $f : \mathcal{M} \longrightarrow \mathcal{M}$ which extends $\varphi : \mathcal{X} \longrightarrow \mathcal{M}$ is given by

$$f = 1 \circ' d_{\mathcal{P}} + (1 \circ \gamma)(\varphi \circ 1).$$

Koszul complexes. Let $\tau : \mathcal{C} \longrightarrow \mathcal{P}$ be a twisting cochain, and let us explain how to produce a map that assigns τ two complexes $\mathcal{C} \circ_\tau \mathcal{P}$ and $\mathcal{P} \circ_\tau \mathcal{C}$, the *right (resp. left) Koszul complex* associated to τ. In the first case, let us consider the free right $\mathcal{P}$-module $\mathcal{C} \circ \mathcal{P}$, and let us consider the map $\mathcal{C} \longrightarrow \mathcal{C} \circ \mathcal{P}$ obtained as the composition $(1 \circ_{(1)} \tau)\Delta_{(1)}$. By Lemma 10.8, there is a unique derivation of right $\mathcal{P}$-modules $d_\tau^r : \mathcal{C} \circ \mathcal{P} \longrightarrow \mathcal{C} \circ \mathcal{P}$ extending the map above.

It is given by the following composition:

$$\begin{tikzcd}
 & (\mathcal{C}\circ_{(1)}\mathcal{C})\circ\mathcal{P} \arrow[r, "(1\circ_{(1)}\alpha)\circ 1"] & (\mathcal{C}\circ_{(1)}\mathcal{P})\circ\mathcal{P} \arrow[dr, "\cong"] & \\
\mathcal{C}\circ\mathcal{P} \arrow[ur, "\Delta_{(1)}\circ 1"] \arrow[dr, "d^r_\tau"'] & & & \mathcal{C}\circ(\mathcal{P};\mathcal{P}\circ\mathcal{P}) \arrow[dl, "1\circ(1;\gamma)"] \\
 & \mathcal{C}\circ\mathcal{P} & \mathcal{C}\circ(\mathcal{P};\mathcal{P}) \arrow[l, "\cong"] &
\end{tikzcd}$$

Dually, we write d^l_τ for the unique derivation of the free left $\mathcal{P}$-module $\mathcal{P}\circ\mathcal{C}$ extending the map $\mathcal{C}\longrightarrow\mathcal{P}\circ\mathcal{C}$ given by the composition $(\tau\circ 1)\Delta$.

Proposition 10.9 *The derivation $\partial_\tau = d_{\mathcal{C}}\circ 1 + 1\circ' d_{\mathcal{P}} + d^r_\tau$ of $\mathcal{C}\circ\mathcal{P}$ is such that $\partial_\tau^2 = d^r_{\partial(\tau)+\tau\star\tau}$. In particular, it squares to zero if and only if τ is a twisting morphisms. Similarly, the derivation $\partial_\tau = d_{\mathcal{P}}\circ 1 + 1\circ' d_{\mathcal{C}} + d^l_\tau$ of $\mathcal{P}\circ\mathcal{C}$ is such that $\partial_\tau^2 = d^l_{\partial(\tau)+\tau\star\tau}$, so that it squares to zero if and only if τ is a twisting morphisms.*

Proof. The proof involves a computation, though the computation may get rather cumbersome, so we provide some details. In fact, we will prove something a bit stronger: the assignment that takes a morphism $\tau:\mathcal{C}\longrightarrow\mathcal{P}$ of symmetric sequences to the derivation $d^r_\tau:\mathcal{C}\circ\mathcal{P}\longrightarrow\mathcal{C}\circ\mathcal{P}$ is a morphism of dg Lie algebras, where we put on $\hom_\Sigma(\mathcal{C},\mathcal{P})$ the dg Lie algebra structure induced by the pre-Lie star product $-\star-$ and we give the space of derivations $\mathrm{Der}(\mathcal{C}\circ\mathcal{P})$ the usual structure of a Lie algebra through the commutator of derivations (recall that the composition of two derivations need not be a derivation, but the commutator of derivations is a derivation).

To see how this claim implies the proposition, we notice that $\partial_\tau\circ\partial_\tau = \frac{1}{2}[\partial_\tau,\partial_\tau]$ and that if we write $\partial_\tau = d_{\mathcal{C}\circ\mathcal{P}} + d^r_\tau$, then

(1) The original differential squares to zero, so that $[d_{\mathcal{C}\circ\mathcal{P}}, d_{\mathcal{C}\circ\mathcal{P}}] = 0$,
(2) The fact this is a morphism of complexes shows that $[d_{\mathcal{C}\circ\mathcal{P}}, d^r_\tau] = d^r_{\partial(\tau)}$ and,
(3) The claim above proves that $[d^r_\tau, d^r_\tau] = d^r_{[\tau,\tau]}$.

Prove the stronger claim.

This completes the proof of the proposition. □

Definition 10.10 We write $\mathcal{C}\circ_\tau\mathcal{P}$ for the complex in right $\mathcal{P}$-modules $(\mathcal{C}\circ\mathcal{P},\partial_\tau)$, and call it the *right Koszul complex associated to* τ. Analogously, we write $\mathcal{P}\circ_\tau\mathcal{C}$ for the complex in free left $\mathcal{P}$-modules $(\mathcal{P}\circ\mathcal{C},\partial_\tau)$ and call it the *left Koszul complex associated to* τ.

We are now ready to make concrete our claim that Koszul (co)operads are those that admit economic resolutions: we did not say of *what* object, but we can do that now. The only observation we need to make is the following:

Definition 10.11 Let $\mathcal{P}$ be a quadratic operad. The map $\kappa : \mathcal{P}^{\mathsf{i}} \longrightarrow \mathcal{P}$ obtained as the composition $\mathcal{P}^{\mathsf{i}} \hookrightarrow \mathsf{B}(\mathcal{P}) \longrightarrow \mathcal{P}$ is a twisting cochain.

Note that κ can be described even in simpler terms as the degree -1 map obtained as the compositions of the projection $\mathcal{P}^{\mathsf{i}} \twoheadrightarrow s\mathcal{X}$ onto weight 1, the desuspension map $s\mathcal{X} \xrightarrow{s^{-1}} \mathcal{X}$ and the inclusion $\mathcal{X} \hookrightarrow \mathcal{P}$.

Proof. This is actually a straightforward computation, since in our situation $\mathcal{P}$ has no differential and we only need to show that $\kappa \star \kappa = 0$. But $\kappa \star \kappa : \mathcal{P}^{\mathsf{i}} \longrightarrow \mathcal{P}$ is obtained as the composition

$$\mathcal{P}^{\mathsf{i}} \xrightarrow{\Delta_{(1)}} \mathcal{P}^{\mathsf{i}} \circ_{(1)} \mathcal{P}^{\mathsf{i}} \longrightarrow \mathcal{P} \circ_{(1)} \mathcal{P} \xrightarrow{\gamma_{(1)}} \mathcal{P}$$

where the map $\kappa \circ_{(1)} \kappa$ in the middle is only non-zero on elements of $s\mathcal{X} \circ_{(1)} s\mathcal{X} \subseteq \mathcal{P}^{\mathsf{i}} \circ_{(1)} \mathcal{P}^{\mathsf{i}}$, and in this case the decomposition of $\mathcal{P}^{\mathsf{i}}$ lands on $s^2\mathcal{R}$ by construction. This means that the image after using the composition map of γ lands in $\mathcal{R}$, which is zero in $\mathcal{P}$, as we claimed. $\square$

With this at hand, here is our second definition of Koszulness. By Theorem A.2 in the Appendix, the following is equivalent to our original definition of Koszulness of a quadratic operad, stating that the inclusion $\mathcal{P}^{\mathsf{i}} \longrightarrow \mathsf{B}(\mathcal{P})$ is a quasi-isomorphism of cooperads.

Definition 10.12 (Second definition) A quadratic operad $\mathcal{P}$ is Koszul if and only if the right Koszul complex $\mathcal{P}^{\mathsf{i}} \circ_{\kappa} \mathcal{P}$ is a resolution of the trivial module $\Bbbk$ in right $\mathcal{P}$-modules. Equivalently, a quadratic operad $\mathcal{P}$ is Koszul if and only if the left Koszul complex $\mathcal{P} \circ_{\kappa} \mathcal{P}^{\mathsf{i}}$ is a resolution of the trivial module $\Bbbk$ in left $\mathcal{P}$-modules.

10.4 Exercises

Exercise 79. Suppose that $\mathcal{P}$ is locally finite dimensional and that $\mathcal{P}(1) = \Bbbk$. Show that $\mathsf{B}(\mathcal{P})$ and $\Omega(\mathcal{P}^*)$ are dual to each other.

Exercise 80. Prove the claim in Theorem 10.5 that the Maurer–Cartan equation for the map $\tau : \mathcal{C} \longrightarrow \mathcal{P}$ corresponding to a morphism of dg cooperads $f : \mathcal{C} \longrightarrow \mathsf{B}(\mathcal{P})$ and to a morphism of dg operads $g : \Omega(\mathcal{C}) \longrightarrow \mathcal{P}$ is equivalent to the condition that f and g commute with the differentials.

Exercise 81. Show that the universal twisting cochain $\pi : \mathsf{B}(\mathcal{P}) \longrightarrow \mathcal{P}$ is indeed a twisting cochain, and that the Maurer–Cartan equation is equivalent to the condition that $\mathcal{P}$ is a dg operad. Do the same for the universal twisting cochain $\iota : \mathcal{C} \longrightarrow \Omega(\mathcal{C})$, and

Exercise 82. Show that $\mathcal{P} \longmapsto \pi_{\mathcal{P}}$ is natural, in the sense that for a morphism of operads $\mathcal{P} \longrightarrow \mathcal{P}'$ we have a commutative square

$$\begin{tikzcd}
\mathsf{B}(\mathcal{P}) \arrow[r] \arrow[d] & \mathcal{P} \arrow[d] \\
\mathsf{B}(\mathcal{P}') \arrow[r] & \mathcal{P}'
\end{tikzcd}$$

Do the same for ι. Show that the bijection described in Theorem 10.5 is natural.

11 Methods to prove an operad is Koszul I

11.1 Monomial operads

Monomial things are pivotal.

11.2 The numerical criterion

Hilbert and Poincaré series. Let us begin by presenting a numerical criterion that can give a *negative* answer to the question "Is this operad Koszul?". This depends on certain invariants associated to chain complexes and graded modules, which we now introduce.

Definition 11.1 Let $\mathcal{X}$ be a locally finite reduced symmetric sequence Its *Hilbert series* is the formal power-series

$$h_{\mathcal{X}}(z) = \sum_{n\geqslant 1} \dim \mathcal{X}(n) \frac{z^n}{n!}.$$

If $\mathcal{X}$ is weight graded (so that we may assume that each weight component is locally finite) we can consider the two variable Hilbert series

$$h_{\mathcal{X}}(z,u) = \sum_{w\geqslant 0} h_{\mathcal{X}(w)}(z) u^w.$$

We will mainly be interested in the case of a weight graded operad $\mathcal{P}$. When $\mathcal{P}$ is binary quadratic, the arity n and the weight w are related by $n = w+1$ and then the two variable Hilbert series carries no new information. For example, we can compute the following Hilbert series:

$$\begin{aligned}
h_{\mathsf{As}}(z) &= z + z^2 + z^3 + \cdots = \frac{z}{1-z} \\
h_{\mathsf{Lie}}(z) &= z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots = -\log(1-z) \\
h_{\mathsf{Com}}(z) &= z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots = \exp z - 1 \\
h_{\mathsf{As}^-}(z) &= z + z^2 + z^3.
\end{aligned}$$

There is a related invariant, which we now define:

Definition 11.2 Let $\mathcal{P}$ be a weight graded operad, so that $\mathsf{B}(\mathcal{P})$ is a weight graded dg cooperad. The Poincaré series of $\mathcal{P}$ is, by definition,

$$p_{\mathcal{P}}(z,u,t) = \sum_{w,d\geqslant 0,} \dim H_d(\mathsf{B}(\mathcal{P})^{(w)}(n))t^d u^w z^n/n!.$$

We can now state and prove a numerical criterion to check if an operad is *not* Koszul.

Theorem 11.3 *Let $\mathcal{P}$ be a weight graded (non-dg) operad. Then we have the following functional equation between the Hilbert and the Poincaré series of $\mathcal{P}$:*

$$h_{\mathcal{P}}(p_{\mathcal{P}}(z,u,-1),u) = p_{\mathcal{P}}(h_{\mathcal{P}}(z,u),u,-1) = z.$$

Moreover, $\mathcal{P}$ is Koszul (and in particular, quadratic) if and only if $p_{\mathcal{P}}(z,u,t) = -h_{\mathcal{P}^{¡}}(z,ut)$ in which case the previous equation simplifies to

$$h_{\mathcal{P}}(h_{\mathcal{P}^{¡}}(z,-u),u) = h_{\mathcal{P}^{¡}}(h_{\mathcal{P}}(z,u),-u) = z.$$

Proof. Let us begin by proving the first functional equation. In this case, we can consider the two augmented bar complexes of Theorem A.1, of the form

$$\mathcal{P}\circ\mathsf{B}(\mathcal{P}), \qquad \mathsf{B}(\mathcal{P})\circ\mathcal{P}.$$

That theorem states these two complexes are acyclic, which means their (weight graded) Euler characteristics are equal to the Euler characteristic of the symmetric sequence $(0,\mathbb{k},0,\ldots)$ where $\mathbb{k}$ is placed in arity one, weight zero, and homological degree zero, so that its Euler characteristic is z. On the other hand, the Euler characteristic of $\mathsf{B}(\mathcal{P})$ is $p_{\mathcal{P}}(z,u,-1)$ and that of $\mathcal{P}$ is its Hilbert series (since $\mathcal{P}$ carries no homological degrees). Thus, we obtain that

$$z = \chi(\mathcal{P}\circ\mathsf{B}(\mathcal{P})) = \chi(\mathcal{P})\circ\chi(\mathsf{B}(\mathcal{P})) = h_{\mathcal{P}}(p_{\mathcal{P}}(z,-1)),$$

while $z = p_{\mathcal{P}}(h_{\mathcal{P}}(z),-1)$ is proved in the same fashion. Suppose now that $\mathcal{P}$ is Koszul, which means that $\dim H_d(\mathsf{B}(\mathcal{P}))_{(w)} = 0$ unless $w = d$. This is true if and only if the coefficient of $z^w u^d$ in $p_{\mathcal{P}}(z,u)$ is zero unless $w = d$, in which case we have that $H_d(\mathsf{B}(\mathcal{P}))_{(d)} \simeq \mathcal{P}^{¡}_{(d)}$ and

$$p_{\mathcal{P}}(z,u,t) = \sum_{d\geqslant 0} \dim \mathcal{P}^{¡}_{(d)}(n)\frac{z^n}{n!}(tu)^d = h_{\mathcal{P}^{¡}}(z,tu).$$

Plugging this back into the equation relating the Hilbert and Poincaré series for $\mathcal{P}$ gives us the result. □

At this point it is useful to come back to the situation when $\mathcal{P}$ is binary and quadratic, in which case the functional equation simplifies even further. In this case, we notice that $h_{\mathcal{P}}(z,u) = uh_{\mathcal{P}}(zu,1)$ so we obtain the equations

$$h_{\mathcal{P}}(-h_{\mathcal{P}^{\mathrm{i}}}(-z)) = h_{\mathcal{P}^{\mathrm{i}}}(-h_{\mathcal{P}}(-z)) = z.$$

Since $\mathcal{P}^!$ and $\mathcal{P}^{\mathrm{i}}$ have the same Hilbert series (even though one is homologically graded and the other is not, we are ignoring the t variable here) we will use the functional equation above when with the operad $\mathcal{P}^!$ most of the time.

Example 11.4 The quadratic operad As^- of anti-associative algebras is not Koszul. Indeed, one can compute the first few terms of its sign-modified inverse of the Hilbert series to obtain

$$z+z^2+\frac{3}{2}z^3+\frac{5}{2}z^4+\frac{17}{4}z^5+7z^6+\frac{21}{2}z^7+\frac{99}{8}z^8+\frac{55}{16}z^9-\frac{715}{16}z^{10}+\mathrm{O}(z^{11}).$$

Since the coefficient of z^{10} is negative, we conclude that this operad cannot be Koszul.

Example 11.5 The quadratic operad Nov of Novikov algebras is not Koszul. Indeed, A. Dzhumadil'daev computed that Nov has the same Hilbert series $h(z)$ as its Koszul dual, and that

$$h(-h(-z)) = z+\frac{1}{6}z^5+\mathrm{O}(z^6) \neq z.$$

Thus, the Novikov operad is not Koszul.

11.3 Exercises

Exercise 83. Use the Haskell Calculator to show that Nov has Hilbert series

$$h_{\mathsf{Nov}}(z) = z+2\frac{z^2}{2!}+6\frac{z^3}{3!}+20\frac{z^4}{4!}+70\frac{z^5}{5!}+\mathrm{O}(z^6).$$

Show that Nov is Koszul dual to $\mathsf{Nov}^{\mathrm{op}}$ and conclude that Nov is not Koszul.

Exercise 84. Show (using Gröbner bases or otherwise) that the Hilbert series h of the operad PreLie satisfies the equation

$$h(z) = z\exp h(z).$$

Use this to compute a few terms of the series, or directly show that $\dim \mathsf{PreLie}(n) = n^{n-1}$. Compute the Hilbert series of Perm and verify that the numerical criterion holds.

```
Configuration:

actions:         normalise count
count limit:     5
arity limit:     5
time limit:      none
output:
field:           rationals
operad type:     unsigned shuffle operad
measure:         permutation reverse degree-lexicographic
signature:       x(2) y(2)
theory:
  x(x(1 3) 2)  -  x(x(1 2) 3)
  y(1 x(2 3))  -  x(y(1 2) 3)
  y(1 y(2 3))  -  x(y(1 3) 2)
  x(x(1 2) 3)  -  x(1 x(2 3))  -  x(y(1 2) 3)  +  y(x(1 3) 2)
  x(x(1 3) 2)  -  x(1 y(2 3))  -  x(y(1 3) 2)  +  y(x(1 2) 3)
  y(1 x(2 3))  -  y(y(1 3) 2)  -  y(1 y(2 3))  +  y(y(1 2) 3)


Arity: 4
Stable rewrite rules: 6
Current critical pairs: 26
Queued critical pairs: 0

Arity: 5
Stable rewrite rules: 12
Current critical pairs: 105
Queued critical pairs: 68

Stopped at arity 5.

Counting normal forms:

  arity | normal forms
     3  |       6
     4  |      20
     5  |      70

*Main>
```

Figure 16: Counting normal forms in the Novikov operad.

12 Methods to prove an operad is Koszul II

12.1 Filtrations and rewriting

In this section, we will focus on examples where $\mathcal{P}$ is a binary quadratic operad and one can compute the homology of either the left or right Koszul complex explicitly. By this, we mean that one can pin down the result directly by inspection, or use some mild homological tools, like filtrations, to simplify the problem.

Example 12.1 Let us consider the associative operad As, which we study as a ns operad, as sending a ns to the corresponding symmetric one does not break Koszulness. In this case, we can compute the Koszul complex $K = \mathsf{As}^{\text{i}} \circ_\tau \mathsf{As}$ explicitly.
Namely, $\mathsf{As}^{\text{i}}(n)$ is one dimensional for each $n \geqslant 1$ generated by a cooperation μ_n^c, and so is $\mathsf{As}(n)$, generated by the iterated associative product μ_n, so that $K(n)$ is spanned by elements of the form

$$[n_1,\ldots,n_k] = \mu_k^c(\mu_{n_1},\ldots,\mu_{n_k}).$$

Moreover, the decomposition map of As^{i} is dual up to one suspension to the (infinitesimal) composition map of $\otimes\mathsf{As}$, that is, we can explicitly compute that

$$\Delta_{(1)}(\mu_k^c) = \sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1} (-1)^{(k_2-1)(i-1)} \mu_{k_1}^c \circ_i \mu_{k_2}^c.$$

To compute the differential on a generic basis element $B(k;n_1,\ldots,n_k)$, we need only restrict to the case when $k_2 = 2$, since in this case the weight one elements are precisely those of arity 2. Computing the corresponding composition, we arrive at the following formula:

$$\partial[n_1,\ldots,n_k] = \sum_{i=1}^{k-1} (-1)^{i-1}[n_1,\ldots,n_i+n_{i+1},\ldots,n_k].$$

Thus, we see that $K(n)$ is isomorphic to the complex computing the simplicial homology of the $n-2$ simplex Δ^{n-2} spanned by vertices $v_1,\ldots,v_{n-1}$ if we consider sending the simplicial basis element $[v_{i_1},\ldots,v_{i_k}]$ where $i_1 < \cdots < i_k$ to the basis element in $K(n)$ given by $[i_1, i_2-i_1,\ldots,i_k-i_{k-1},n-i_k]$.

Let us now consider a common homological technique to prove that complexes are acyclic. Let (C,d) be a chain complex, and suppose that F is a filtration of C, that is, a family $\{F_p\}_{p\in\mathbb{Z}}$ of subcomplexes such that $F_{p-1} \subseteq F_p$ for all $p \in \mathbb{Z}$ and such that their union is C; although this is a requirement for us, some people call such filtrations *exhaustive*. For reasons that will become apparent below, we will restrict ourselves to filtrations that are bounded below, that is, we will assume that $F_p = 0$ if $p < 0$.[3]

If (C,d,F) is a complex with a non-negative filtration, we define $\mathrm{gr}_F(C)$ to be the graded complex with pth graded piece given by the quotient $\mathrm{gr}_F(C)_p = F_p/F_{p-1}$. This is indeed a complex, since d preserves each subcomplex of F.

Proposition 12.2 *Let (C,d,F) be a non-negative chain complex with a filtration that is bounded below. If $(\mathrm{gr}_F(C),d)$ is acyclic, then the same is true for C.*

Proof. Pick $c \in C$ non-zero, and suppose that $dc = 0$, so that our aim is to show that c is a boundary. By hypothesis, there exists a smallest p such that $c \in F_p$ but $c \notin F_{p-1}$, so let us proceed by induction on p. Since c is a cycle in C, it is a cycle in $\mathrm{gr}_F C$. By hypothesis, this complex is acyclic, which means that there exists a class $[c']$ such that $d[c'] = [c]$ in F_p/F_{p-1}. In other words, there exists $c_1 \in F_p$ and $c' \in F_{p-1}$ such that $c - dc_1 = c' \in F_{p-1}$. Since $dc' = 0$, we know by induction that $c' = dc_2$ for some c_2, and so $c = dc_1 + dc_2$ is a boundary, too, like we wanted. $\square$

Let us now consider the situation where $\mathcal{P}$ is a symmetric operad. Since the forgetful functor from symmetric to shuffle operads is monoidal, it preserves the bar construction, in the sense that $\mathsf{B}(\mathcal{P}^{\mathsf{f}})$ and $\mathsf{B}(\mathcal{P})^{\mathsf{f}}$ are naturally isomorphic, and we can attempt to apply filtration methods to the shuffle operad associated to $\mathcal{P}$. Thus, in what follows, we will consider instead the case $\mathcal{P}$ is a shuffle operad which is filtered, in the sense it admits a bounded below (exhaustive!) filtration by subcollections, such that for any shuffle composition we have

$$F_p \circ_{\sigma,i} F_q \subseteq F_{p+q}.$$

Lemma 12.3 *Suppose that $(\mathcal{P},F)$ is a filtered shuffle operad. Then declaring a shuffle tree monomial T in $\mathcal{P}$ to be in filtration degree p if we have that*

$$\sum_{v \in T} p(\mathsf{x}_v) \leqslant p$$

for the filtration degrees of its decorations in $\mathcal{P}$ defines a filtration $\mathsf{B}(F)$ of dg conilpotent cooperads on the shuffle bar construction $\mathsf{B}(\mathcal{P}^{\mathsf{f}})$.

Proof. The filtration will be one of cooperads without imposing any compatibility conditions with the shuffle compositions of $\mathcal{P}$. This later compatibility condition implies that the

[3] Note that we use p for the indexing letter here: this is customary, to avoid confusing it with homological degrees.

differential of the bar construction preserves the resulting subcomplexes, so that the filtration is in fact of dg cooperads. □

With this at hand, we arrive at one of the most useful criteria to check if a symmetric operad is Koszul.

Theorem 12.4 *Let $\mathcal{P}$ be a symmetric operad, and suppose that the shuffle operad $\mathcal{P}^{\mathsf{f}}$ admits a filtration (of shuffle operads) for which the associated graded operad is Koszul. Then $\mathcal{P}$ is a Koszul operad.*

Proof. The only thing we have to show is that the associated graded conilpotent cooperad to $\mathsf{B}(\mathcal{P}^{\mathsf{f}})$ with respect to the filtration $\mathsf{B}(F)$ is isomorphic to $\mathsf{B}(\mathrm{gr}_F\mathcal{P}^{\mathsf{f}})$. Indeed, once this is verified we conclude using Proposition 12.2 and our definition of Koszulness using the syzygy grading: the map $\mathcal{P}^{\mathsf{i}} \longrightarrow \mathsf{B}(\mathcal{P})$ induces a quasi-isomorphism in H^0, so it suffices we prove that $\mathsf{B}(\mathcal{P})$ has no positive cohomology.

By definition, the basis elements of $\mathrm{gr}_{\mathsf{B}(F)}\mathsf{B}(\mathcal{P}^{\mathsf{f}})$ of degree p are those shuffle tree monomials T whose decorations satisfy T

$$\sum_{v\in T} p(\mathsf{x}_v) = p.$$

At the same time, the shuffle operad $\mathrm{gr}_F\mathcal{P}^{\mathsf{f}}$ is graded and thus its bar construction inherits an extra weight grading: the weight of a tree monomial T is the sum of the weights of its decorations, so that $\mathsf{wt}(T) = p$ precisely when

$$\sum_{v\in T} \mathsf{wt}(\mathsf{x}_v) = p.$$

Thus, we have a canonical map

$$\mathrm{gr}_{\mathsf{B}(F)}\mathsf{B}(\mathcal{P}^{\mathsf{f}}) \longrightarrow \mathsf{B}(\mathrm{gr}_F\mathcal{P}^{\mathsf{f}})$$

which is an isomorphism, since the weight $\mathsf{wt}(\mathsf{x}_v)$ is precisely the least of those p such that $\mathsf{x}_v \in F_p\mathcal{P}^{\mathsf{f}}$, and if this drops, the total sum above will also drop. □

The previous result can be used together with rewriting methods, as follows. If $\mathcal{P}$ is a shuffle operad generated by a collection $\mathcal{X}$, and we have chosen an admissible monomial ordering $\prec$ on $\mathcal{F}_{\mathcal{X}}$, the projection $\mathcal{F}_{\mathcal{X}} \longrightarrow \mathcal{P}$ defines a filtration on $\mathcal{P}$, by declaring that the class of a monomial is in degree at most p if it is represented by a monomial in the free shuffle operad of weight p: notice that we are being slightly imprecise with our notation, since now p does not belong to the natural numbers, but rather to a totally order set (determined by our choice of monomial order).

The resulting filtration $F_\prec$ in $\mathcal{P}$ defines a graded shuffle operad $\mathrm{gr}_\prec \mathcal{P}$, and at the same time we have an associated operad $\mathcal{P}_{\mathsf{mon}} = \mathcal{F}_{\mathcal{X}}/(\mathrm{LT}(\mathcal{R}))$ obtained by keeping only the leading terms of relations of $\mathcal{P}$. Moreover, we have a map

$$\pi : \mathcal{P}_{\mathsf{mon}} \longrightarrow \mathrm{gr}_\prec \mathcal{P}$$

since the leading terms of $\mathcal{R}$ define relations in the codomain, and in fact this map is an isomorphism in weights one and two.

Theorem 12.5 *Let $\mathcal{P}$ be a quadratic operad and suppose that the shuffle operad $\mathcal{P}^{\mathsf{f}}$ admits a quadratic Gröbner basis with respect to some monomial order $\prec$. Then $\mathcal{P}$ is Koszul.*

Proof. By the previous theorem, it suffices to show that $\mathrm{gr}_\prec \mathcal{P}$ is Koszul. We see that this operad is isomorphic to the quotient of $\mathcal{P}$ by the initial ideal associated to $\mathcal{R}$. Thus, π is an isomorphism precisely when $\mathcal{R}$ is a Gröbner basis for $(\mathcal{R})$ with respect to $\prec$, and it suffices to show a shuffle operad with quadratic monomial relations is Koszul, which we have already done. Thus, the claim follows for $\mathcal{P}^{\mathsf{f}}$, which is what we wanted. □

12.2 Distributive laws

Write section on distributive laws.

A Technical homological results

A.1 Comparison theorem

A.2 Big Koszul complexes are acylic

Theorem A.1 *Let $\mathcal{P}$ be an operad and let $\mathcal{C}$ be a conilpotent cooperad. All four Koszul complexes associated to the canonical twisting cochains π and ι are all acyclic.*

A.3 Fundamental theorem

Theorem A.2 *Let $\mathcal{P}$ (resp. $\mathcal{C}$) be a connected weight graded dg operad (resp. conilpotent cooperad), and let $\tau : \mathcal{C} \longrightarrow \mathcal{P}$ be a twisting morphism. Then the following four assertions are equivalent:*

(1) The right Koszul complex $\mathcal{C} \circ_\tau \mathcal{P}$ is acyclic.
(2) The left Koszul complex $\mathcal{P} \circ_\tau \mathcal{C}$ is acyclic.
(3) The map $f_\tau : \mathcal{C} \longrightarrow \mathsf{B}(\mathcal{P})$ is a quasi-isomorphism.
(4) The map $f^\tau : \Omega(\mathcal{C}) \longrightarrow \mathcal{P}$ is a quasi-isomorphism.

B Further topics

Write an appendix about algebras over operads.

B.1 Algebras over operads

Operads are important not in and of themselves but through their representations, more commonly called *algebras over operads*. In fact, one can usually ‘create’ an operad by declaring what kind of algebras it governs. If the algebra has certain operations of certain arities, these define the generators of the operad, and the relations these operators must satisfy give us the relations presenting our operad.

Definition B.1 A $\mathcal{P}$-algebra structure on a vector space V is the datum of a map of operads $\mathcal{P} \longrightarrow \mathrm{End}_V$.

Alternatively, one can consider the situation when $\otimes$ is closed and has a right adjoint hom (the internal hom) so that what we want are maps

$$\gamma_{V,n} : \mathcal{P}(n) \otimes_{S_n} V^{\otimes n} \longrightarrow V$$

declaring how each $\mu \in \mathcal{P}(n)$ acts as an operation $\mu : V^{\otimes n} \longrightarrow V$. It follows that a $\mathcal{P}$-algebra structure on V is the same as the datum of maps as in B.1 that satisfy the following conditions:

(1) *Associativity:* let $\nu \in \mathcal{P}(n)$ and $\nu_i \in \mathcal{P}(k_i)$ for $i \in [n]$, and pick $w_i \in V^{\otimes k_i}$. Set $v_i = \gamma_{V,k_i}(\nu_i, w_i) \in V$ and $\mu = \gamma_{\mathcal{P}}(\nu; \nu_1, \ldots, \nu_n)$. Then

$$\gamma_{V,k_1+\cdots+k_n}(\mu; w_1, \ldots, w_n) = \gamma_{V,n}(\nu; v_1, \ldots, v_n).$$

(2) *Equivariance:* for $\nu \in \mathcal{P}(n)$, $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ and $\sigma \in S_n$, we have that

$$\gamma_{V,n}(\nu\sigma, v_{\sigma 1} \otimes \cdots \otimes v_{\sigma n}) = \gamma_{V,n}(\nu; v_1 \otimes \cdots \otimes v_n).$$

(3) *Unitality:* if $1 \in \mathcal{P}(1)$ is the unit, then $\gamma_{V,1}(1, v) = v$.

Example: recognition principle. Gerstenhaber algebras and Hochschild complex.

B.2 Rewriting theory for algebras

B.3 Homotopy algebras

C Classical theory

C.1 Non-commutative Gröbner bases

Explain the analogous results for associative algebras, and the case of the Steenrod algebra.

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