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Contents

1 August 21st: D. Lewanski

A natural basis for intersection theory. Consider some numbers with $\sum d_i = 3g-3+n$, of the form $\langle \tau_{d_1}, \dots, \tau_{d_n} \rangle_g \in \mathbb{Q}$, and a corresponding generating series in x^d which we write $A_{g,n}(x)$. We assume that the numbers above are symmetric in n so that $A_{g,n}(x)$ is a symmetric polynomial in the variables x.

Questions: (1) What polynomials do we obtain? (2) Symmetric polynomials have different bases, like the elementary symmetric polynomials. Do the $\{A_{g,n}\}$ behave nicely for either of these bases?

1.1 Witten–Kontsevich

Let $\overline{\mathcal{M}}_{g,n}$ be the Deligne–Mumford–Knudsen moduli space of stable singular genus g curves with n marked points.

Each curve C in this space admits a marked point labeled i which is a smooth point, and has a cotangent bundle $T^*_{p_i}(C)$. This gives a bundle $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$ and the number $\langle \tau_{d_1}, \dots, \tau_{d_n} \rangle_g$ is in fact

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}$$

where $\psi_i = c_1(\mathcal{L}_i)$. Here $c_1(E) \in H^2(X, \mathbb{Q})$ is the Chern class of a bundle, defined [as follows].

Potential. Define the *potential function* for $\overline{M}$ as follows:

$$F(t) = \sum_{3g-3+n \geqslant 0} \langle \tau_{d_1}, \dots, \tau_{d_n} \rangle_g t^{\vec{d}} / \vec{d}!$$

Theorem 1.1 (Kontsevich) *If $U = \partial_0^2 F$ then the following Korteweg–De Vries equation holds:*

$$\partial_1 U = U \partial_0 U + \frac{1}{12} \partial_0^3 U.$$

1.2 Some closed formulas

In genus zero, we have that

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} = \binom{n-3}{d_1, \dots, d_n}$$

so that $A_{0,n} = e_1^{n-3}$. In genus one, we have that

$$\int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} = \frac{1}{24} \left(\binom{n}{d_1, \dots, d_n} - \sum_{b_1, \dots, b_n \in \{0,1\}} \binom{n-b}{d_i - b_i} (b-2)! \right)$$

where $b = \sum b_i$ (and at least two b_i are non zero), so that

$$A_{1,n} = e_1^n - \sum_{k=2}^{n} (k-2)! e_k e_1^{n-k}.$$

Theorem 1.2 *There exists an algorithm to explicitly compute $A_{g,n}$ for a fixed $g \geqslant 0$. In particular, there exists a way to compute them in the basis of elementary symmetric polynomials, of the form*

$$A_{g,n}(x) = \sum_{\lambda} C_{g,n}(\lambda) e_\lambda e_1^{n-\ell(\lambda)},$$

which stabilizes for $n \geqslant 2g-2$.

Conjecture 1 (Eynard–Lewanski–Ooms) *If we write $A_{g,n}$ in the basis of elementary symmetric polynomials, there should exist a clear pattern where some coefficients are zero:*

$$A_{g,n}(x) = \sum_{\substack{|\lambda| \leqslant 3g-3+n, \\ \lambda_i \geqslant 2, \ell(\lambda) \leqslant g}} C_{g,n}(\lambda) e_\lambda e_1^{3g-3+n-|\lambda|}$$

The conjecture has been checked for $g \leqslant 7$ and every n, and for $n \leqslant 3$ for every $g \geqslant 0$ ($n = 1$ by Witten, $n = 2$ by Dijkgraaf, $n = 3$ by Zagier).

2 August 22nd: G. Baverez

Liouville Conformal Field Theories. Gaussian free field. Gaussian multiplicative chaos.

2.1 Motivation

Let Σ be a Riemann surface with a Riemannian metric g, and let us write

$$[g] = \{e^{2\omega} g : \omega \in C^\infty(\Sigma)\}.$$

Solutions to the Liouville equation

$$K(e^{2\omega} g) = e^{-2\omega}(K_g - 2\Delta_g \omega)$$

are the minimizers of

$$S_L(\omega) = \frac{1}{2\pi} \int_\Sigma \left(|d\omega|_g^2 + K_g \omega + 2\pi\mu e^{2\sigma}\right) d\mathrm{vol}_g$$

where $e^{-2\omega}(K_g - 2\Delta_g \omega) = -2\pi\mu$. One can modify this operator as follows:

$$S_L(\omega, \gamma) = \frac{1}{2\pi} \int_\Sigma \left(|d\omega|_g^2 + QK_g \omega + 2\pi\mu e^{\gamma\sigma}\right) d\mathrm{vol}_g$$

where $Q = 2/\gamma + \gamma/2$.
Goal: make sense of the path integral

$$P = \int e^{-S_L(\omega)} D\omega,$$

which is not defined for various reasons: S_L is actually a distribution, and the space of all ωs is infinite dimensional.

2.2 White noise

Let H be a Hilbert space, and let ξ_v be random variables $\xi_h : \Omega \longrightarrow \mathbb{R}$ for $h \in H$ defined as follows: pick a basis (e_n) of H, and let ξ_{e_n} be normal Gaussian $N(0,1)$ and i.i.d., and extend this linearly to H. We have that $\mathbb{E}[\xi_v \xi_w] = \langle v, w \rangle$. We call the datum $H \longrightarrow \mathcal{F}(\Omega)$ a *white noise on H*.

If $F \in L^2(\Omega)$ is smooth and non-zero on finitely many directions, then there exist f smooth such that $F = f(\xi_1, \ldots, \xi_n)$, and we write

$$dF = \sum_{i=1}^{n} \partial_k f e_k.$$

Gaussian:

$$\mathbb{E}(e^{a\xi_u - \alpha^2/2\mathbb{E}(\xi_n^2)} F) = \mathbb{E}(f(\xi + \alpha u))$$