Algebraic operads, Koszul duality and Gröbner bases: an introduction

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This lecture series aim to offer a gentle introduction to the theory of algebraic operads, starting with the elements of the theory, and progressing slowly towards more advanced themes, including (inhomogeneous) Koszul duality theory, Gröbner bases and higher structures. The course will consists of approximately 12 lectures, along with extra talks by willing participants, with the goal of introducing extra material to the course, and making them more familiar with the theory.

Contents

1 Motivation and history

Goals. The goals of this lecture is to give a broad picture of the history and pre-history of operads, and some current trends, and give a road-map for the course.

1.1 Introduction and motivation

Operads originally appeared as tools in homotopy theory, specifically in the study of iterated loop spaces (May, 1972 and Boardman and Vogt before). They appeared as *comp algebras* in Gerstenhaber's work on Hochschild cohomology and topologically as Stasheff's 'associahedra' for his homotopy characterization of loop spaces (both in 1963). The theory of operads received new inspiration from homological algebra, category theory, algebraic geometry and mathematical physics:

(1) (Stasheff, Sugawara) Study homotopy associative H-spaces, Stasheff implicitly discovers a topological ns operad K with $C_*(K) = \mathsf{As}_\infty$ and a recognition principle for A_∞-spaces.
(2) (Boardmann–Vogt) Study infinite loop spaces, build a PROP (a version of an E_∞-operad) and obtain a recognition principle for infinite loop spaces.
(3) (Kontsevich) Uses L_∞-algebras and configuration spaces to prove his deformation quantization theorem that every Poisson manifold admits a deformation quantization.
(4) (Kontsevich) The above is implied by the formality theorem: the Lie algebra of polyvector fields is L_∞-quasi-isomorphic to the Hochschild complex, and $f_1 = \mathsf{HKR}$.
(5) (Tamarkin) Approaches this result through the formality of the little disks operad D_2. Proves that the Hochschild complex of a polynomial algebra is *intrinsically formal* as a Gerstenhaber algebra.
(6) (Manifold calculus) Describes the homotopy type of embedding spaces as certain derived operadic module maps and to produces their explicit deloopings using little disk operads, due to Goodwillie–Weiss, Boavida de Brito—Weiss, Turchin, Arone–Turchin, Dwyer—Hess, Ducoulombier—Turchin.
(7) (Ginzburg–Kapranov, Fresse, Vallette, Hinich) Koszul duality for algebraic operads and cousins allows to develop a robust homotopy theory of homotopy algebras, cohomology theory, deformation theory, Quillen homology, etc.
(8) (Deligne conjecture and variants) The study of natural operations on the Hochschild complex of an associative algebra lead to a manifold of results beginning with the proof that there is an action of the little disks operad D_2 on it, and the ultimate version by Markl–Voronov, who proved that the operad of natural operations on it has the homotopy type of $C_*(D_2)$.

Algebraic operads are modeled by trees (planar or non-planar, rooted or not), and relaxing these graphs allows us to produce other type of algebraic structures:

Type	Graph	Compositions	Due to
PROPs	Any graph	Any	Adams–MacLane
Modular	Any graph	$\xi_{i,j}$, $\circ_{i,j}$	Getzler–Kapranov
Properads	Connected graphs	Any	B. Vallette
Dioperads	Trees	${}_i\circ_j$ (no genus)	W. L. Gan
Half-PROPs	Trees	$\circ_j$, ${}_i\circ$	Markl–Voronov
Cyclic operads	Trees	$\circ_{i,j}$	Getzler–Kapranov
Symmetric operads	Rooted trees	$\circ_i$	J. P. May

1.2 Exercises

The corresponding exercises to this lecture appear in **Exercise Sheet #0**.

2 Symmetric sequences, composition products, unbiased approach

Goals. We will define some related gadgets (symmetric collections, algebras, modules, endomorphism operads) necessary to introduce operads. Then, we define what an operad is (topological, algebraic, symmetric, non-symmetric). We will then give some (not so) well known examples of topological and algebraic operads.

2.1 Basic definitions

What is an operad? A group is a model of $\mathrm{Aut}(X)$ for X a set, an algebra is a model of $\mathrm{End}(V)$ for V a vector space. Equivalently, groups are the gadgets that act on objects by automorphisms, and algebras are the gadgets that act on objects by their (linear) endomorphisms. Operads are the gadgets that act on objects through operations with many inputs (and one output), and at the same time keep track of symmetries when the inputs are permuted.

The underlying objects to operads are known as *symmetric sequences*: a symmetric sequence (also known as an Σ-module or symmetric module) is a sequence of vector spaces $\mathcal{X} = (\mathcal{X}(n))_{n\geqslant 0}$ such that for each $n \in \mathbb{N}_0$ there is a right action of S_n on $\mathcal{X}(n)$. We usually consider *reduced* Σ-modules, those for which $\mathcal{X}(0) = 0$.

A map of Σ-modules is a collection of maps $(f_n : \mathcal{X}_1(n) \longrightarrow \mathcal{X}_2(n)\}_{n\geqslant 0}$, each equivariant for the corresponding group action. This defines the category ${}_\Sigma\mathsf{dgMod}$ of symmetric sequences, and whenever we think of symmetric sequences using this definition, we will say we are considering a a biased or skeletal approach to them.

In parallel, it is convenient to consider the category $\mathsf{Fin}^\times$ of finite sets and bijections. An object in this category is a finite set I, and a morphism $\sigma : I \longrightarrow J$ is a bijection. Since every finite set I with n elements is (non-canonically) isomorphic to $[n] = \{1,\dots,n\}$, the following holds:

Lemma 2.1 *The skeleton of* $\mathsf{Fin}^\times$ *is equal to the category with objects the finite sets* $[n]$ *for* $n \geqslant 0$ *and with morphisms the bijections* $[n] \longrightarrow [n]$ *(and no morphism between* $[n]$ *and* $[m]$ *if* $m \neq n$*).*

Proof. This is in Exercise Sheet #1. ◀

We set ${}_\Sigma\mathsf{Mod} = \mathsf{Fun}(\mathsf{Fin}^\times, \mathsf{Vect}^{\mathrm{op}})$, so that a Σ-module is a pre-sheaf of vector spaces $I \longmapsto \mathcal{X}(I)$ assigning to each isomorphism $\tau : I \longrightarrow J$ an isomorphism $\mathcal{X}(\tau) : \mathcal{X}(J) \longrightarrow \mathcal{X}(I)$. When we think of Σ-modules as pre-sheaves, we will say we are taking an unbiased approach, will if we specify only its values on natural numbers, we will say we are taking the biased or skeletal approach; we will come back to this later.

With this at hand, we can in turn define the *Cauchy product* of two Σ-modules $\mathcal{X}$ and $\mathcal{Y}$

$$(\mathcal{X} \otimes_\Sigma \mathcal{Y})(I) = \bigoplus_{S \sqcup T = I} \mathcal{X}(S) \otimes \mathcal{Y}(T)$$

where the right-hand is the usual tensor product of vector spaces and the sum runs through partitions of I into two disjoint sets. The symmetric product is then defined by

$$(\mathcal{X} \circ_\Sigma \mathcal{Y})(I) = \bigoplus_{\pi \vdash I} \mathcal{X}(\pi) \otimes \mathcal{Y}^{\otimes k}(\pi)$$

as the sum runs through (ordered) partitions of I. These two products will be central in what follows.

Lemma 2.2 *The category* ${}_\Sigma\mathsf{Mod}$ *with* $\circ_\Sigma$ *is monoidal with unit the species taking the value* $\Bbbk e_x$ *at the singleton sets* $\{x\}$ *and zero everywhere else. The same category is monoidal for* $\otimes_\Sigma$ *with unit the species taking the value* $\Bbbk$ *at* $\varnothing$ *and zero everywhere else.*

We will use the notation $\Bbbk$ for the base field but also for the unit for the composition product $\circ_\Sigma$, hoping it will not cause much confusion. It will be useful later to think of $\Bbbk$ as a twig or "stick".

Observe that the associator for $\circ_\Sigma$ is not too simple and involves reordering certain factors of tensor products in Vect. In particular, replacing vector spaces by graded vector spaces or complexes will create signs in the associator.

We are now ready to define the prototypical symmetric sequence that carries the structure of an algebraic operad.

Definition 2.3 The *endomorphism operad* of a space V is the symmetric sequence End_V where for each $n \in \mathbb{N}$ we set $\mathrm{End}_V(n) = \mathrm{End}(V^{\otimes}, V)$. The symmetric group S_n acts on the right on $\mathrm{End}_V(n)$ so that $(f\sigma)(v) = f(\sigma v)$ for $v \in V^{\otimes n}$, where S_n acts on the left on $V^{\otimes n}$ by $(\sigma v)_i = v_{\sigma i}$. The composition maps are defined by $\gamma(f; g_1, \ldots, g_n) = f \circ (g_1 \otimes \cdots \otimes g_n)$.

The following two operations on permutations will streamline our definition of (algebraic) operads.

Two useful maps. For each $k \geqslant 1$ and each tuple $\lambda = (n_1, \ldots, n_k)$ with sum n there is a map

$$S_k \longrightarrow S_{n_1 + \cdots + n_k}$$

$$\lambda = (2,1,2), \quad \sigma = 312 \quad \rightsquigarrow \quad \lambda(\sigma) = 34512 \in S_5$$
$$(213,213,132) \in S_3 \times S_3 \times S_3 \quad \rightsquigarrow \quad 213546798 \in S_9$$

Figure 1: The useful operations

that sends $\sigma \in S_k$ to the permutation $\lambda(\sigma)$ of $[n]$ that permutes the blocks $\pi_i = \{n_1 + \cdots + n_{i-1} + 1, \ldots n_1 + \cdots + n_{i-1} + n_i\}$ according to σ. There is also a map

$$S_{n_1} \times \cdots \times S_{n_k} \longrightarrow S_{n_1 + \cdots + n_k}$$

that sends a tuple of permutations $(\sigma_1, \ldots, \sigma_k)$ to the permutation $\sigma_1 \# \cdots \# \sigma_k$ that acts like σ_i on the block π_i as above. These operations are illustrated in Figure 1. With these at hand, one can check that these composition maps satisfy the following axioms:

(1) *Associativity*: let $f \in \mathrm{End}_V(n)$, and consider $g_1, \ldots, g_n \in \mathrm{End}_V$ and for each $i \in [n]$ a tuple $h_i = (h_{i1}, \ldots, h_{in_i})$ were $n_i = \mathrm{ar}(g_i)$. Then for $f_i = \gamma(g_i; h_{i1}, \ldots, h_{in_i})$ and $g = \gamma(f; g_1, \ldots, g_n)$ we have that

$$\gamma(f; f_1, \ldots, f_n) = \gamma(g; h_1, \ldots, h_n).$$

(2) *Intrinsic equivariance*: for each $\sigma \in S_k$ and $\lambda = (\mathrm{ar}(g_1), \ldots, \mathrm{ar}(g_k))$ we have that

$$\gamma(f\sigma; g_1, \ldots, g_k) = \gamma(f; g_{\sigma 1}, \ldots, g_{\sigma k})\lambda(\sigma),$$

(3) *Extrinsic equivariance*: for each tuple of permutations $(\sigma_1, \ldots, \sigma_k) \in S_{n_1} \times \cdots \times S_{n_k}$, if $\sigma = \sigma_1 \# \cdots \# \sigma_k$, we have that

$$\gamma(f, g_1\sigma_1, \ldots, g_k\sigma_k) = \gamma(f; g_1, \ldots, g_k)\sigma.$$

(4) *Unitality:* the identity $1 \in \mathrm{End}_V(1)$ satisfies $\gamma(1; g) = g$ and $\gamma(g; 1, \ldots, 1) = g$ for every $g \in \mathrm{End}_V$.

Definition 2.4 A symmetric operad (in vector spaces) is an Σ-module $\mathcal{P}$ along with a composition map $\gamma : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$ of signature

$$\gamma : \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \longrightarrow \mathcal{P}(n_1 + \cdots + n_k)$$

along with a unit $1 \in \mathcal{P}(1)$, that satisfy the axioms above.

Variant 2.5 A non-symmetric operad is an operad whose underlying object is a collection (with no symmetric group actions). Operads in topological spaces or chain complexes require the composition maps to be morphisms (that is, continuous maps or maps of chain complexes, respectively) and, more generally, operads defined on a symmetric monoidal category require, naturally, that the composition maps be morphisms in that category.

Pseudo-operads. One can define operads through *partial composition maps*, modeling the honest partial composition map

$$f \circ_i g = f(1,\dots,1,g,1,\dots,1)$$

in End_V. These composition maps satisfy the following properties:

(1) *Associativity*: for each $f,g,h \in \mathrm{End}_V$, and $\delta = i - j + 1$, we have

$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{\mathrm{ar}(f)+j-1} g & \delta \leqslant 0 \\ f \circ_j (g \circ_\delta h) & \delta \in [1, \mathrm{ar}(g)] \\ (f \circ_\delta h) \circ_j g & \delta > \mathrm{ar}(g) \end{cases}$$

(2) *Intrinsic equivariance*: for each $\sigma \in S_k$, we have that

$$(f\sigma) \circ_i g = (f \circ_{\sigma i} g)\sigma'$$

where σ' is the same permutation as σ that treats the block $\{i, i+1, \dots, i+\mathrm{ar}(g)-1\}$ as a single element.

(3) *Extrinsic equivariance*: for each $\sigma \in S_k$, we have that

$$f \circ_i (g\sigma) = (f \circ_i g)\sigma''$$

where σ'' acts by only permuting the block $\{i, \dots, i+\mathrm{ar}(g)-1\}$ according to σ.

(4) *Unitality:* the identity $1 \in \mathrm{End}_V(1)$ satisfies $1 \circ_1 g = g$ and $g \circ_i 1 = g$ for every $g \in \mathrm{End}_V$ and $1 \leqslant i \leqslant \mathrm{ar}(g)$.

Definition 2.6 A symmetric operad (in vector spaces) is an Σ-module $\mathcal{P}$ along with partial composition map of signature

$$- \circ_i - : \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$$

and a unit $1 \in \mathcal{P}(1)$ satisfying the axioms above.

It is not hard to see (but must be checked at least once) that an operad with $\mathcal{P}(n) = 0$ for $n \neq 1$ is exactly the same as an associative algebra.

Warning! If one does not require the existence of a unit, the notion of a *pseudo-operad* by Markl (defined by partial compositions) does not coincide with the notion of an operad as defined by May.

2.2 Constructing operads by hand

One can define operads in various ways. For example, one can define the underlying collection explicitly, and give the composition maps directly:

(1) *Commutative operad.* The reduced symmetric topological (or set) operad with $\mathsf{Com}(n)$ a single point for each $n \in \mathbb{N}$, and composition maps the unique map from a point to a point.

(2) *Associative operad.* The reduced set operad with $\mathsf{As}(n) = S_n$ the regular representation and composition maps

$$S_k \times S_{n_1} \times \cdots \times S_{n_k} \longrightarrow S_{n_1 + \cdots n_k}$$

the unique equivariant map that sends the tuple of identities to the identity.

(3) *Stasheff operad.* Let K_{n+2} be the subset of I^n (the product of n copies of $I = [0,1]$) consisting of tuples $(t_1, \dots, t_{n+2})$ such that $t_1 \cdots t_k \leqslant 2^{-k}$ for $j \in [n+2]$. The boundary of K_{n+2} consists of those points such that for some $j \in [n+2]$ we have either t_j or $t_1 \cdots t_j = 2^{-j}$. It is tedious (but otherwise doable) to show that for each pair (r,s) of natural numbers and each $i \in [r]$ there exists an inclusion

$$\circ_i : K_{r+1} \times K_{s+1} \longrightarrow K_{r+s+1}$$

that defines on the sequence of spaces $\{K_{n+2}\}_{n \geqslant 0}$ the structure of a non-symmetric operad. We will see in the exercise a realization of K_n as the convex hull of points with positive integer coordinates (due to J.-L. Loday) using planar binary rooted trees, which will make the operad structure more transparent.

(4) If M is a monoid, there is an operad $\mathbb{W}_M$ with $\mathbb{W}_M(n) = M^n$ such that

$$(m_1, \dots, m_s) \circ_i (m'_1, \dots, m'_t) = (m_1, \dots, m_{i-1}, m_i m'_1, \dots, m_i m'_t, m_{i+1}, \dots, m_s).$$

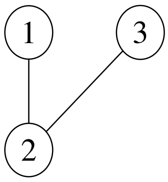
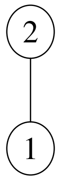
We call it the *word operad of M*. Its underlying symmetric collection is $\mathsf{As} \circ M$.

(5) Write $\mathrm{Aff}(\mathbb{C}) = \mathbb{C} \times \mathbb{C}^\times$ for the group of affine transformations of $\mathbb{C}$ with group law $(z, \lambda)(w, \mu) = (z + \lambda w, \lambda\mu)$. In turn, define for each finite set I the topological space

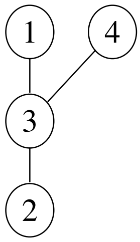
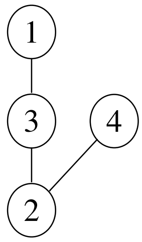
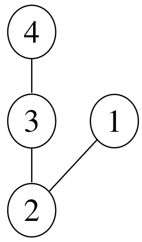
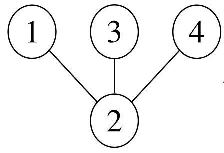
$$\mathcal{C}(I) = \{(z_i, \lambda_i) \in \mathrm{Aff}(\mathbb{C})^I : |z_i - z_j| > |\lambda_i| + |\lambda_j|\}.$$

The group law of $\mathrm{Aff}(\mathbb{C})$ allows us to define an operad structure on $\mathcal{C}(I)$ using the exact same definition as in the word operad of a monoid. The subspaces $\mathcal{D}_2^{\mathrm{fr}}(I) \subseteq \mathcal{C}(I)$ where $|z_i| + |\lambda_i| \leqslant 1$ for all $i \in I$, and where the inequality is strict unless $z_i = 0$ is called the *framed little disks operad*. The little disks operad is the suboperad where $\lambda_i = 1$ for all $i \in I$, and we write it $\mathcal{D}_2(I)$.

(6) The operad of rooted trees RT has $\mathsf{RT}(n)$ the collection of rooted threes with n vertices labeled by $[n]$, and the composition $T \circ_j T'$ is obtained by inserting T' at the jth vertex of T and reattaching the children of that vertex to T' in all possible ways. For example, if

$T =$  and $S =$ 

then we have that

$T \circ_2 S =$  $+$  $+$  $+$  .

2.3 Exercises

The corresponding exercises to this lecture appear in **Exercise Sheet #2**.

3 Free operads and presentations

Goals. We will define algebraic operads by generators and relations, and with this at hand define quadratic and quadratic-linear presentations of operads.

3.1 Planar and non-planar trees

Operads and their kin are gadgets modeled after combinatorial graph-like objects. Operads themselves are modeled after rooted trees, so it is a good idea to have a concrete definition of what a rooted tree is. We will also consider planar rooted trees, and trees with certain decorations, so it is a good idea to digest the definitions carefully to later embellish them.

A rooted tree τ is the datum of a finite set $V(\tau)$ of vertices along with a partition $V(\tau) = \operatorname{Int}(\tau) \sqcup L(\tau) \cup R(\tau)$, where the first are the *interior* vertices, L are the leaves, and $R(\tau)$ is a singleton, called the root of τ. We also require there is a function $p : V(\tau) \smallsetminus R(\tau) \longrightarrow V(\tau)$, describing the edges of τ, with the following properties: call a vertex $v \in V(\tau)$ a child of $w \in V(\tau)$ if $v \in p^{-1}(w)$. Then:

(1) The root $r \in R(\tau)$ has exactly one child.
(2) The leaves of τ have no children.
(3) For each non-root vertex v there exist a unique sequence $(v_0, v_1, \ldots, v_k)$ such that $p(v_{i-1}) = v_i$ for $i \in [k]$ with $v_0 = v$ and $v_k = r$.

We will call a non-leaf vertex that has no children a *stump* (or an endpoint, or a cherry-top). A tree is reduced if has no stumps and all of its non-root and non-leaf vertices have at least two children. We will also call the root the (unique) output vertex τ, and the leaves the input vertices of τ.

A planar rooted tree is a rooted tree τ along with a linear order in each of the fibers of the parent function p of τ. In short, the children of each vertex are linearly ordered, so we are effectively considering a drawing of τ in the plane, where the clockwise orientation gives us the order at each vertex.

Two rooted trees τ and τ' are isomorphic if there exists a bijection $f : V(\tau) \longrightarrow V(\tau')$ that preserves the root, the input vertices and the interior vertices, so that $p' \circ f = p$ where we also write f for the induced bijection $f : V(\tau) \smallsetminus r \longrightarrow V(\tau') \smallsetminus r'$. Two planar rooted trees are isomorphic if in addition f respects the linear order at each vertex.

For example, consider the rooted tree τ with $V = \{1,2,3\} \cup \{4,5\} \cup \{0\}$, that is, three leaves, two interior vertices and the root. Then the choice of $p : [5] \to [\![5]\!]$ with $p(\{1,2\}) = 4$, $p(\{3,4\}) = 5$, $p(5) = 0$ gives a tree isomorphic to the one with with $p(\{1,2\}) = 3$,

$p(\{3,4\}) = 5$, $p(5) = 0$. On the other hand, if we consider the vertices linearly ordered by their natural order, these two planar rooted trees are no longer isomorphic.

Definition 3.1 For a finite set I, an I-labeled tree T is a pair (τ, f) where τ is a reduced rooted tree, along with a bijection $f : I \longrightarrow L(\tau)$. Two I-labeled trees T an T' are isomorphic if there exists a pair (g,σ) where g is an isomorphism between τ and τ' and σ is an automorphism of I such that $g\mid_{L(\tau)} \circ f = \sigma \circ f'$.

Suppose that (τ, f) is an I-tree and that (τ', f') is a J-tree, and that $i \in I$. We define $K = I \cup_i J = I \sqcup J \smallsetminus i$ and the K-tree $\tau \circ_i \tau'$ as follows:

(1) Its leaves are $L(\tau \circ_i \tau') = L(\tau) \sqcup L(\tau') \smallsetminus f^{-1}(i)$.
(2) Its internal vertices are $V(\tau) \sqcup V(\tau')$, with root r.
(3) The parent function q is defined by declaring that:
   - q coincides with p on $V(\tau)$,
   - $q(w) = p(f^{-1}(i))$ if w is the unique children of the root of τ',
   - q coincides with p' on $V(\tau') \smallsetminus \{r', w\}$.

(4) The leaf labeling is the unique bijection $L(\tau \circ_i \tau') \longrightarrow I \circ_i J$ extending f and f'.

Let us now consider an (unbiased) reduced symmetric sequence $\mathcal{X}$ which we will think of as an *alphabet*. An tree monomial in the alphabet $\mathcal{X}$ is a pair (τ, x) where τ is a reduced rooted tree and $x : \mathrm{Int}(\tau) \longrightarrow \mathcal{X}$ is a map with the property that $x(v) \in \mathcal{X}(p^{-1}(v))$. Observe that reduced sequences and reduced trees correspond to each other, in the sense that with this definition we can only decorate a stump of τ with an element of $\mathcal{X}(\varnothing)$.

An I-labeled $\mathcal{X}$-tree T is a triple (τ, x, f) where (τ, f) is I-labeled and (τ, x) is an $\mathcal{X}$-tree. We will say that (τ, x, f) is a (symmetric) tree monomial if $\mathcal{X}$ is symmetric. If it is just a collection, we will say that (τ, x, f) is a ns tree monomial. In particular, if T is an I-labeled tree, and if $\sigma \in \mathrm{Aut}(I)$, there is another I-labeled tree $\sigma(T) = (\tau, f\sigma^{-1})$.

Suppose that $T = (\tau, x, f)$ is a tree monomial on an alphabet $\mathcal{X}$, and let us pick a vertex v of τ and a permutation σ of the set $C = p^{-1}(v)$ of children of v. We define the tree τ^σ as follows: the datum defining τ remains unchanged except p is modified to p^σ so that

$$p^\sigma(w) = \begin{cases} p(w) & \text{if } p^2(w) \neq v \\ p(\sigma^{-1}(w')) & \text{if } p(w) = w' \in C. \end{cases}$$

Briefly, we just relabel the vertices of τ using σ. With this at hand, we define T^σ to be the tree monomial with underlying tree τ^σ and with x modified to x^σ so that

$$x^\sigma(w) = \begin{cases} \sigma x(v) & \text{if } v = w, \\ x(\sigma^{-1}(w')) & \text{if } p(w) = w' \in C. \end{cases}$$

Note that it is possible some children of v are leaves, in which case the definitions make sense if we think of leaves as decorated by the unit of $\Bbbk$.

Give example of T^σ with and without leaves involved.

Let us now define for each $n \geqslant 1$ the space $\mathcal{F}_\mathcal{X}(I)$ as the span of all tree monomials $T = (\tau, f, x)$ on $\mathcal{X}$ with leaves labeled by I, modulo the subspace generated by all elements of the form

$$R(T, v, \sigma) = T - T^\sigma$$

where σ ranges through $\mathrm{Aut}(p^{-1}(v))$ as v ranges through the vertices of τ. In case all children of v are leaves, this is saying that the tree where x_v is replace by $\sigma(x_v)$ is equal to the tree where the leaves of T that are children of v are relabeled according to σ.

Definition 3.2 The *free symmetric operad* on $\mathcal{X}$ is the symmetric sequence $\mathcal{F}_\mathcal{X}$ along with the composition law obtained by grafting of trees. More precisely, suppose that $T \in \mathcal{F}_\mathcal{X}(I)$ and that $T' \in \mathcal{F}_\mathcal{X}(J)$, and that $i \in I$. We define $T'' = T \circ_i T' \in \mathcal{F}_\mathcal{X}(I \cup_I J)$ by taking its underlying labeled tree to be $\tau \circ_i \tau'$, and by decorating it in the unique way which extends the decorations of T and T'.

Exercise 1. Show that tree grafting respects the relations above, and hence is well defined on $\mathcal{F}_\mathcal{X}$. We will later interpret $\mathcal{X} \longmapsto \mathcal{F}_\mathcal{X}$ as a *monad*, thus giving another definition of operads. The advantage of this monadic approach is its flexibility, which allow us to define other operad like structures, like the ones mentioned in the introduction.

Exercise 2.

Complete.

3.2 The free operad

An algebraically inclined way to construct (algebraic) operads is through generators and relations. There is a forgetful functor from the category of operads to the category of collections. In general, it admits a left adjoint, which is the free operad functor.

Definition 3.3 The free operad generated by a symmetric collection X is defined inductively by letting $\mathcal{F}_{0,X} = \Bbbk$ be spanned by the 'twig' (tree with no vertices and one edge) in arity zero and

$$\mathcal{F}_{n+1,\mathcal{X}} = \Bbbk \oplus (\mathcal{X} \circ \mathcal{F}_{n,\mathcal{X}}),$$

and finally by setting $\mathcal{F}_{\mathcal{X}} = \varinjlim_n \mathcal{F}_{n+1,\mathcal{X}}$. The composition maps are defined by induction, and the axioms are also checked by induction.

Expand on definition of free operad. Give explicit definition using trees decorated by $\mathcal{X}$.

Intuitively, the definition says that an element of $\mathcal{F}_{\mathcal{X}}$ is either the twig, or corolla with n vertices decorated by $\mathcal{X}$, whose leaves have on them an element of $\mathcal{F}_{\mathcal{X}}$. The final shape of $\mathcal{F}_{\mathcal{X}}$ will however depend on the symmetric structure of $\mathcal{X}$. For example, if $\mathcal{X} = \mathcal{X}(2) = S_2$ is the regular representation, then $\mathcal{F}_{\mathcal{X}}(k)$ consists of planar binary rooted trees with k leaves decorated by a permutation of $[k]$. If, however, $\mathcal{X} = \mathcal{X}(2)$ is the trivial (or sign) representation of S_2, then $\mathcal{F}_X(k)$ will consist of shuffle binary trees.

Note 3.4 The free conilpotent cooperad $\mathcal{F}^c_{\mathcal{X}}$ can be defined in a completely analogous fashion: it has the same underlying symmetric sequence as $\mathcal{F}_{\mathcal{X}}$ (much like TV and T^cV have the same underlying vector space) and it has decomposition maps obtained by 'degrafting' trees.

The free operad $\mathcal{F}_{\mathcal{X}}$ is weight graded by the number of internal vertices of a trees (that is, we put $\mathcal{X}$ in weight one, and extend the weight to trees by counting occurrences of elements of $\mathcal{X}$. More generally, if $\mathcal{X}$ admits a weight grading, then $\mathcal{F}_{\mathcal{X}}$ inherits this weight grading, and we write $\mathcal{F}^{(n)}_{\mathcal{X}}$ for the homogeneous component of weight $n \in \mathbb{N}_0$. If we do not specify a weight grading on $\mathcal{F}_{\mathcal{X}}$, we will always assume we are taking the canonical weight grading above.

Definition 3.5 An ideal in an operad $\mathcal{P}$ is a subcollection $\mathcal{I}$ for which $\gamma(\mathcal{I} \circ \mathcal{P}) + \gamma(\mathcal{P} \circ_{(1)} \mathcal{I}) \subseteq \mathcal{I}$. The quotient of $\mathcal{P}/\mathcal{I}$ is again an operad, called the quotient of $\mathcal{P}$ by $\mathcal{I}$. Every subcollection $\mathcal{R}$ of $\mathcal{P}$ is contained in a smallest ideal, called the *ideal generated by* $\mathcal{R}$.

The notion of ideals and of free operads allow us to define operads by generators and relations.

Definition 3.6 We write $\mathcal{F}(\mathcal{X}, \mathcal{R})$ for the quotient of $\mathcal{F}_{\mathcal{X}}$ by the ideal generated by a subcollection $\mathcal{R}$ of $\mathcal{F}_{\mathcal{X}}$. We say $\mathcal{P}$ is presented by generators $\mathcal{X}$ and relations $\mathcal{R}$ if there is an isomorphism $\mathcal{F}(\mathcal{X}, \mathcal{R}) \longrightarrow \mathcal{P}$.

Note that if $\mathcal{P}$ is symmetric, the definition requires that $\mathcal{I}$ be stable under the symmetric group actions, so we may sometimes specify $\mathcal{R}$ by a generating set as a symmetric sequence only, and understand that $(\mathcal{R})$ is generated by the Σ-orbit of $\mathcal{R}$.

The classics. The associative operad is generated by a binary operation μ generating the regular representation of S_2 subject to the only relation

$$\mu \circ_1 \mu = \mu \circ_2 \mu.$$

Similarly, the commutative operad is generated by a binary operation which instead generates the trivial representation of S_2. Both of these examples arise as the linearization of a set operad.

The Lie operad is generated by a single binary operation β that generates the sign representation of S_2 subject to the only relation

$$(\beta \circ_1 \beta)(1+\tau+\tau^2) = 0$$

where $\tau = (123) \in S_3$ is the 3-cycle. We write these operads As, Com and Lie and, following J.-L. Loday, call them the *three graces*. We have that

$$\mathsf{As}(n) = \Bbbk S_n, \quad \mathsf{Com}(n) = \Bbbk, \quad \mathsf{Lie}(n) = \operatorname{Ind}_{\mathbb{Z}/n}^{S_n} \Bbbk_\zeta$$

where $\Bbbk_\zeta$ is a character of $\mathbb{Z}/n$ for a primitive nth root of the unit.

Note 3.7 It is not always advantageous to define an operad by generators and relations: the operad pre-Lie can be defined explicitly in terms of labeled rooted trees and a grafting operation, as done by Chapoton–Livernet, and this ‘presentation’ is very useful in practice, for example, to show that the pre-Lie operad is Koszul.

3.3 Quadratic operads

An operad $\mathcal{P}$ is *quadratic* if it admits a presentation $\mathcal{F}(\mathcal{X},\mathcal{R})$ where $\mathcal{R} \subseteq \mathcal{F}(\mathcal{X})^{(2)}$. That is, $\mathcal{P}$ is generated by some collection of operations $\mathcal{X}$ and all its defining relations are of the form

$$\sum \lambda^i_{\mu,\nu} \mu \circ_i \nu = 0$$

where $\operatorname{ar}(\mu) + \operatorname{ar}(\nu)$ is constant. An operad is *binary quadratic* if moreover $\mathcal{X} = \mathcal{X}(2)$ or, what is the same, all the generating operations of $\mathcal{P}$ are of arity two (binary). A *quadratic-linear presentation* of an operad $\mathcal{P}$ is a presentation $\mathcal{F}(\mathcal{X},\mathcal{R})$ of $\mathcal{P}$ where $\mathcal{R} \subseteq \mathcal{X} \oplus \mathcal{F}(\mathcal{X})^{(2)}$. That is, it is a presentation of the form

$$\sum \lambda^i_{\mu,\nu} \mu \circ_i \nu + \sum \lambda_\rho \rho = 0$$

where $\mathrm{ar}(\mu)+\mathrm{ar}(\nu)=\mathrm{ar}(\rho)+1$ is constant.

Some examples. The presentations of the associative, commutative and Lie operad above are quadratic. The following are also quadratic operads:

(1) The Gerstenhaber operad Ger (and its cousin, the Poisson operad Poiss).
(2) The pre-Lie operad PreLie and its quotient, the Novikov operad Nov.
(3) The operad of totally associative k-ary algebras tAs_k (and its Lie counterpart).
(4) The operad of partially associative k-ary algebras pAs_k (and its commutative counterpart).
(5) The operad of anti-associative algebras As^-.
(6) The operad controlling Lie algebras with two compatible brackets.

Expand on the definition of these operads.

On other other hand, every operad admits a quadratic-linear presentation, and in this case the condition for a quadratic-linear presentation to be 'good enough' is more subtle. We will postpone the discussion of such presentations to Section 12.1.

3.4 Algebras over operads

Operads are important not in and of themselves but through their representations, more commonly called *algebras over operads*. In fact, one can usually 'create' an operad by declaring what kind of algebras it governs. If the algebra has certain operations of certain arities, these define the generators of the operad, and the relations these operators must satisfy give us the relations presenting our operad.

Definition 3.8 A $\mathcal{P}$-algebra structure on a vector space V is the datum of a map of operads $\mathcal{P} \longrightarrow \mathrm{End}_V$.

Alternatively, one can consider the situation when $\otimes$ is closed and has a right adjoint hom (the internal hom) so that what we want are maps

$$\gamma_{V,n} : \mathcal{P}(n) \otimes_{S_n} V^{\otimes n} \longrightarrow V$$

declaring how each $\mu \in \mathcal{P}(n)$ acts as an operation $\mu : V^{\otimes n} \longrightarrow V$. It follows that a $\mathcal{P}$-algebra structure on V is the same as the datum of maps as in 3.4 that satisfy the following conditions:

(1) *Associativity:* let $\nu \in \mathcal{P}(n)$ and $\nu_i \in \mathcal{P}(k_i)$ for $i \in [n]$, and pick $w_i \in V^{\otimes k_i}$. Set $v_i = \gamma_{V,k_i}(\nu_i, w_i) \in V$ and $\mu = \gamma_{\mathcal{P}}(\nu;\nu_1,\dots,\nu_n)$. Then

$$\gamma_{V,k_1+\cdots+k_n}(\mu;w_1,\dots,w_n) = \gamma_{V,n}(\nu;v_1,\dots,v_n).$$

(2) *Equivariance:* for $\nu \in \mathcal{P}(n)$, $v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}$ and $\sigma \in S_n$, we have that

$$\gamma_{V,n}(\nu\sigma, v_{\sigma 1} \otimes \cdots \otimes v_{\sigma n}) = \gamma_{V,n}(\nu; v_1 \otimes \cdots \otimes v_n).$$

(3) *Unitality:* if $1 \in \mathcal{P}(1)$ is the unit, then $\gamma_{V,1}(1, v) = v$.

Complete. Recognition principle. Gerstenhaber algebras and Hochschild complex.

3.5 Exercises

4 Some homological algebra

4.1 Twisting cochains

4.2 (Co)bar construction

Let $\mathcal{P}$ be a non-unital operad, and define a chain complex in symmetric sequences as follows: its underlying collection is $\mathcal{F}^c_{s\mathcal{P}}$ (the c stands for conilpotent cooperad) and s is a suspension sign, so that trees with n vertices live in homological degree n. There is a differential in $\mathcal{F}^c_{s\mathcal{P}}$ which is the unique coderivation extending the map

$$\mathcal{F}^c_{s\mathcal{P}} \longrightarrow s\mathcal{P}$$

that vanishes everywhere on the domain except on $\mathcal{F}^{c,(2)}_{s\mathcal{P}} = (s\mathcal{P}) \circ_{(1)} (s\mathcal{P})$ where it equals

$$\partial(sv_1 \circ_i sv_2) = (-1)^{|v_1|-1} s(v_1 \circ_i v_2).$$

Definition 4.1 We call $(\mathcal{F}(s\mathcal{P}), \partial, \Delta)$ the *bar construction* of $\mathcal{P}$, and write it $\mathsf{B}(\mathcal{P})$.

Thus, elements of $B(\mathcal{P})$ are just rooted trees T for which each vertex $v \in T$ is decorated by an element $\mathcal{P}(I_v)$ where I_v is the set of incoming edges of v. The differential ∂ acts by contracting one edge at a time and composing along this edge (a kind of graph complex differential).

This interpretation allows us to give a 'hands on' definition of the bar construction. Concretely, for each tree T define

$$\mathcal{P}(T)^- = \bigotimes_{v \in T} \mathcal{P}(I_v) \otimes \det(T).$$

Then the degree n component of $\mathsf{B}(\mathcal{P})$ is the direct sum of all $\mathcal{P}(T)^-$ as T runs through isomorphism classes of rooted trees with n internal vertices. Suppose that e is an edge in T directed from vertex v to vertex w in T. There is a composition map

$$\mathcal{P}(I_v) \otimes \mathcal{P}(I_w) \longrightarrow \mathcal{P}(I_v \cup I_w \setminus e) \subseteq \mathcal{P}(T/e)$$

and the the differential is induced by the resulting maps

$$d_e : \mathcal{P}(T)^- \longrightarrow \mathcal{P}(T/e)^- \quad e \in E(T).$$

Definition 4.2 The bar complex $\mathsf{B}(\mathcal{P})$ inherits a weight grading from $\mathcal{P}$ (if it admits any). In this case, the syzygy grading on $\mathsf{B}(\mathcal{P})$ is a *cohomological* grading defined by

$$\mathsf{B}^s(\mathcal{P}) = \bigoplus_{s+d=w} \mathsf{B}_d(\mathcal{P})_{(w)}.$$

That is, an element of homological degree d and weight w is in syzygy degree $w-d$.

When $\mathcal{P}$ is quadratic, it is automatically graded by placing its generators in weight 1, and thus $\mathsf{B}(\mathcal{P})$ has its syzygy grading. By construction an element of homological degree d in $\mathsf{B}(\mathcal{P})$ has weight at least d (unless $\mathcal{P}$ is dg, but that is another story), so that $\mathsf{B}_d(\mathcal{P}))_{(w)}$ can only be non-zero for $w \geqslant d$.

Lemma 4.3 *Let $\mathcal{P}$ be a weight graded operad. Then:*

(1) It is generated in weight one if and only if $H_1(\mathsf{B}(\mathcal{P}))_{(w)} = 0$ unless $w = 1$.
(2) It is one generated and quadratic if and only if $H_2(\mathsf{B}(\mathcal{P}))_{(w)} = 0$ unless $w = 2$.

In other words, $\mathcal{P}$ is one generated if and only if in weight 1, the syzygy cohomology $H^*(\mathsf{B}(\mathcal{P}))$ is concentrated in degree zero, and it is quadratic (and in particular 1-generated) if and only if in weights 1 and 2, the syzygy cohomology $H^*(\mathsf{B}(\mathcal{P}))$ is concentrated in degree zero.

Proof. We have that $\mathsf{B}_1(\mathcal{P})_{(w)} = s\mathcal{P}_{(w)}$ and that the bar differential $\mathsf{B}_1(\mathcal{P})_{(w)} \longrightarrow \mathsf{B}_0(\mathcal{P})_{(w)}$ is zero, while $\mathsf{B}_2(\mathcal{P})_{(w)} = (s\mathcal{P} \circ_{(1)} s\mathcal{P})_{(w)}$ and the differential $\mathsf{B}_2(\mathcal{P})_{(w)} \longrightarrow \mathsf{B}_1(\mathcal{P})_{(w)}$ is just a modification of the composition map of $\mathcal{P}$. It follows that $H_1(\mathsf{B}(\mathcal{P}))_{(w)} = s\mathcal{P}_{(w)}/\gamma(\mathcal{P} \circ_{(1)} \mathcal{P})_{(w)}$ consists of those elements in weight w that are not expressible as a composition of two elements in lower weights or, what is the same,

$$H_1(\mathsf{B}(\mathcal{P}))_{(w)} = s\mathcal{X}_{(w)}.$$

Thus $H_1(\mathsf{B}(\mathcal{P}))_{(w)} = 0$ unless $w = 1$ if and only if very element in weight w can be written as a composition of elements in lower weights, if and only if $\mathcal{X}$ is concentrated in weight 1. This proves the first claim.

For the second one, assume that $\mathcal{X} = \mathcal{X}_{(1)}$, and let $\mathcal{R}$ be the homogeneous ideal of relations defining $\mathcal{P}$. Since $\mathcal{P}$ is one generated, $\mathcal{R}$ must be a direct sum of homogeneous components $\mathcal{R}_{(w)}$. One can show that for each $w \geqslant 2$ we have

$$H_2(\mathsf{B}(\mathcal{P}))_{(w)} = s^2\mathcal{R}_{(w)},$$

which proves the claim. To do it, we prove that there exists a dg resolution $(\mathcal{F}_{\mathcal{X}'}, \delta)$ of $\mathcal{P}$ with $\mathcal{X}'_0 = \mathcal{X}$ and $\mathcal{X}'_1 = \mathcal{R}$, and that $\Omega\mathsf{B}(\mathcal{P})$ affords another resolution of $\mathcal{P}$. Then, one shows that these two dg resolutions give the same derived functor of indecomposables, and that this computes $H_*(\mathsf{B}(\mathcal{P}))$. ◀

4.3 Universal twisting cochains

Let $\mathcal{C}$ be a conilpotent non-unital dg-cooperad. The *cobar construction* of $\mathcal{C}$, which we write $\Omega\mathcal{C}$, is defined as follows. Its underlying sequence is the free operad $\mathcal{F}_{s^{-1}\mathcal{C}}$ on the desuspension of $\mathcal{C}$, which inherits a differential d_1 from the decomposition maps of $\mathcal{C}$, and another differential d_2 from the differential of $\mathcal{C}$. The differential of $\Omega\mathcal{C}$ is then $d_1 + d_2$.

5 Koszul duality I

Goals. We introduce the notion of a Koszul operad. We will then give the Rosetta stone of Koszul duality, and look into the ns associative operad in detail from the lens of Koszul duality theory.

5.1 Koszul duality

Definition 5.1 A quadratic operad is Koszul if one (and hence all) of the following equivalent conditions are satisfied:

(1) The cohomology group $H^s(\mathsf{B}(\mathcal{P}))$ is zero for $s > 0$.
(2) We have $\operatorname{rate}(\mathcal{P}) = 1$.
(3) The inclusion $H^0(\mathsf{B}(\mathcal{P})) \longrightarrow \mathsf{B}^*(\mathcal{P})$ is a quasi-isomorphism.

We call $H^0(\mathsf{B}(\mathcal{P}))$ the *Koszul dual cooperad to* $\mathcal{P}$ and write it $\mathcal{P}^{\mathrm{i}}$.

Add equivalence with simpler definition using twisting cochains.

Suspensions. Let $s\mathbb{k}$ be the graded Σ-module concentrated in arity 1 and degree 1, where it is one dimensional. We call $\mathrm{End}_{s\mathbb{k}}$ the suspension operad and write it $\mathcal{S}$. Note that $\mathrm{End}_{s\mathbb{k}}(n)$ is the sign representation of Σ_n put in degree $1-n$, and if we write ν_n for the map that sends s^n to s, then we have the following composition rule:

$$\nu_n \circ_i \nu_m = (-1)^{(i-1)(m-1)} \nu_{m+n-1}.$$

If $\mathcal{P}$ is an operad, then the arity-wise tensor product $\mathcal{S} \otimes \mathcal{P}$ is called the suspension of $\mathcal{P}$ and we write it $\mathcal{S}\mathcal{P}$ or $\mathcal{P}\{1\}$. Dually, we write $\mathcal{S}^{-1}$ for the desuspension operad.

Note 5.2 The operad $\mathcal{S}\mathcal{P}$ has the property that $\mathcal{S}\mathcal{P}(sV) = s\mathcal{P}(V)$, so that algebras over $\mathcal{S}\mathcal{P}$ are exactly those vector spaces V such that $s^{-1}V$ is a $\mathcal{P}$-algebra. Equivalently, sV is a $\mathcal{S}\mathcal{P}$-algebra if and only if V is a $\mathcal{P}$-algebra.

Pairings. We define a pairing between $\mathcal{F}_{\mathcal{X}}$ and $\mathcal{F}_{s^{-1}\mathcal{S}^{-1}\mathcal{X}^*}$ as follows (the appearance of the suspensions will be evident later):

$$\langle \Sigma\nu^* \circ_j \Sigma\mu*, \rho \circ_i \tau \rangle = \delta_{ij}(-1)^{(\operatorname{ar}(\nu)-1)(|\mu|+i-1)+|\nu||\mu|}\nu^*(\rho)\mu^*(\tau).$$

If $\mathcal{X} = \mathcal{X}(2)$ is binary and has no homological degrees, this simplifies to

$$\langle \Sigma\nu^* \circ_i \Sigma\mu*, \rho \circ_i \tau \rangle = \begin{cases} \nu^*(\rho)\mu^*(\tau) & i = 1 \\ -\nu^*(\rho)\mu^*(\tau) & i = 2. \end{cases}$$

Definition 5.3 The Koszul dual operad of a quadratic operad $\mathcal{P}$ is defined by $\mathscr{S}^{-1}(\mathcal{P}^{\mathrm{i}})^*$ and we write it $\mathcal{P}^!$. If $\mathcal{P}$ is generated by $\mathcal{X}$ subject to relations $\mathcal{R}$, then $\mathcal{P}^!$ is generated by $s^{-1}\mathscr{S}^{-1}\mathcal{X}^*$ subject to the orthogonal space of relations $\mathcal{R}^\perp$ according to the pairing above.

Note 5.4 Let $\mathcal{P}$ be an operad. Then $\mathcal{P}$ is quadratic if and only if $\mathscr{S}\mathcal{P}$ is quadratic, and it is Koszul if and only if $\mathscr{S}\mathcal{P}$ is Koszul. In this case

$$(\mathscr{S}\mathcal{P})^{\mathrm{i}} \cong \mathscr{S}\mathcal{P}^{\mathrm{i}},$$

For example, if $\mathsf{Poiss}(a,b)$ is the operad controlling Poisson algebra with a product of degree a and a bracket of degree b, then $\mathsf{Poiss}(a,b)^! = \mathsf{Poiss}(-b,-a)$. In particular,

$$\mathsf{Ger}^! = \mathsf{Poiss}(0,1)^! = \mathsf{Poiss}(-1,0) = \mathscr{S}^{-1}\mathsf{Ger}.$$

We will see later all the operads $\mathsf{Poiss}(a,b)$ are Koszul. Note also that $\mathsf{Ger}^! \cong \mathscr{S}^2\mathsf{Ger}$. More generally, also note that $\mathsf{e}_n = \mathsf{Poiss}(0,n-1)$ is the homology operad $H_*(D_n)$ of little disks, which satisfies

$$\mathsf{e}_n^! = \mathsf{Poiss}(1-n,0) = \mathscr{S}^{n-1}\mathsf{e}_n.$$

5.2 The associative operad

5.3 Pathological examples

Let As^- be the ns-quadratic operad generated by an anti-associative binary operation μ, that is

$$\mu \circ_1 \mu + \mu \circ_2 \mu = 0.$$

Let $\mathcal{O}_3$ be the ns-quadratic operad generated by an binary operation μ that is nilpotent of order three, in the sense that any binary tree with three (or more) occurrences of μ is zero:

$$(\mu \circ_i \mu) \circ_j \mu = 0$$

for all possible choices of i and j.

Lemma 5.5 *The operad* As^- *is three dimensional with basis* $\{1, \mu, \mu \circ_1 \mu\}$, *and* $\mathcal{O}_3$ *is four dimensional with basis* $\{1, \mu, \mu \circ_1 \mu, \mu \circ_2 \mu\}$.

Proof. The second claim is immediate. For the first, one has to show that anti-associativity implies any binary tree of weight three vanishes. This is a simple computation:

$$((x_1x_2)x_3)x_4 = -(x_1(x_2x_3))x_4 = x_1((x_2x_3)x_4) = -x_1(x_2(x_3x_4))$$

but also $((x_1x_2)x_3)x_4 = -(x_1x_2)(x_3x_4) = x_1(x_2(x_3x_4))$, so that $x_1(x_2(x_3x_4)) = -x_1(x_2(x_3x_4))$. Since all weight three trees in As^- are equal up to a sign, they must all be zero. ◀

It follows that the bar construction $\mathsf{B}(\mathsf{As}^-)$ is the free operad on a binary operation μ and a ternary operation μ_3, and that the bar construction $\mathsf{B}(\mathcal{O}_3)$ is the free operad on a binary operation μ and two ternary operations ν_3 and τ_3. The differential of the first bar construction is non-zero only on trees that contain a subtree of the form $\mu \circ_1 \mu$ or $\mu \circ_2 \mu$, and

$$d(\mu \circ_1 \mu) = \mu_3, \quad d(\mu \circ_2 \mu) = -\mu_3,$$

while the same is true for the second bar construction with the exception that

$$d(\mu \circ_1 \mu) = \nu_3, \quad d(\mu \circ_2 \mu) = \tau_3.$$

The computation of $H^*(\mathsf{As}^-)$ can be done, and one can show that

$$H_d(B(\mathsf{As}^-))_{(w)} \neq 0$$

only for $(d,w) \in \{(2k+n, 3k+n) : k,n \in \mathbb{N}\}$. It follows that if we define

$$\operatorname{rate}(\mathcal{P}) = \sup\left\{ \frac{w-1}{d-1} : H_d(\mathsf{B}(\mathcal{P}))_{(w)} \neq 0 \right\},$$

then $\operatorname{rate}(\mathsf{As}^-) = 3/2$. Thus, As^- is not Koszul (as per the definition below, which requires this ‘slope’ to be 1) but is quite close to having slope 1. In this sense, the rate of $\mathcal{P}$ is a measure of how ‘dispersed’ the bar homology is with respect to the syzygy grading.

Problem 5.6 Compute the bar cohomology $H^*(\mathcal{O}_3)$. More generally, compute the bar cohomology of any monomial operad.

6 Methods to prove Koszulness I

6.1 Distributive law methods

6.2 Koszul (co)homology

6.3 Exercises

7 Pre-Lie algebras and algebraic operads

7.1 Pre-Lie algebras associated to operads

7.2 The rooted trees operad

7.3 Exercises

8 Gröbner bases for algebraic operads I

8.1 Shuffle sequences

8.2 Shuffle operads

8.3 Exercises

9 Gröbner bases for algebraic operads II

9.1 Tree monomials

9.2 Admissible orders

9.3 Exercises

10 Methods to prove Koszulness II

10.1 Gröbner basis arguments

10.2 Filtration arguments

10.3 Exercises

11 Operads in algebraic topology 1

11.1 The little disks operad

11.2 Stasheff's operad

11.3 Recognition principles

12 Koszul duality II

12.1 Inhomogeneous duality

Suppose that $\mathcal{P}$ admits a quadratic linear presentation defined by $\mathcal{X}$ and $\mathcal{R} \subseteq \mathcal{X} \oplus \mathcal{F}_{\mathcal{X}}^{(2)}$. There is a projection $q : \mathcal{F}_{\mathcal{X}} \longrightarrow \mathcal{F}_{\mathcal{X}}^{(2)}$ and we define $q\mathcal{P} = \mathcal{F}_{\mathcal{X}}/(q\mathcal{R})$, and call it the *quadratic operad associated to* $\mathcal{P}$. We say a quadratic-linear presentation is admissible if it satisfies the following conditions:

(1) There are no superfluous generators in $\mathcal{X}$, that is, have that $\mathcal{R} \cap \mathcal{X} = 0$.
(2) No new quadratic relations can be deduced from the quadratic-linear relations, that is

$$(\mathcal{R} \circ_{(1)} \mathcal{X} + \mathcal{X} \circ_{(1)} \mathcal{R}) \cap \mathcal{F}_{\mathcal{X}}^{(2)} \subseteq \mathcal{R} \cap \mathcal{F}_{\mathcal{X}}^{(2)}.$$

When condition (1) is satisfied, there is a map $f : q\mathcal{R} \longrightarrow \mathcal{X}$ such that $\mathcal{R} = \{r - f(r) : r \in q\mathcal{R}\}$ is the graph of f. The operad $\mathcal{P}$ is filtered by weight, and we write $\mathrm{gr}(\mathcal{P})$ for the resulting operad. There is a surjection $q\mathcal{P} \longrightarrow \mathrm{gr}(\mathcal{P})$ that is an isomorphism in weights 0 and 1, but not necessarily in weight 2.

The map $f : q\mathcal{R} \longrightarrow \mathcal{X}$ induces a map $d_f : q\mathcal{P}^{\mathsf{i}} \longrightarrow \mathcal{F}_{s\mathcal{X}}^{c}$ which is the unique coderivation that correstricts on $s\mathcal{X}$ to the composition

$$\mathcal{P}^{\mathsf{i}} \xrightarrow{\pi} s^2 q\mathcal{R} \xrightarrow{s^{-1}f} s\mathcal{X}.$$

Moreover, the following holds:

(1) The coderivation d_f maps into $q\mathcal{P}^{\mathsf{i}}$ if and only if $(\mathcal{R} \circ_{(1)} \mathcal{X} + \mathcal{X} \circ_{(1)} \mathcal{R}) \cap \mathcal{F}_{\mathcal{X}}^{(2)} \subseteq q\mathcal{R}$.
(2) If condition (2) above is satisfied, then the previous condition holds, as $\mathcal{R} \cap \mathcal{F}_{\mathcal{X}}^{(2)} \subseteq q\mathcal{R}$, and $d_f^2 = 0$.

Definition 12.1 A quadratic-linear presentation of $\mathcal{P}$ is inhomogeneous Koszul if and only if it satisfies conditions (1) and (2) and if the quadratic operad $q\mathcal{P}$ is Koszul. In this case, we call $(q\mathcal{P}^{\mathsf{i}}, d_f)$ the Koszul dual conilpotent dg-cooperad of $\mathcal{P}$ and write it $\mathcal{P}^{\mathsf{i}}$.

Warning! Although not every operad is quadratic, every operad admits a inhomogeneous Koszul presentation (Exercise 3.8.10 in [1]). The problem is finding 'economical and useful' one. Concretely, if we choose $\mathcal{X} = \#\mathcal{P}$ (the symmetric sequence underlying $\#\mathcal{P}$) and quadratic-linear relations $\#\mu \circ_i \#\nu = \#(\mu \circ_i \nu)$ for every $\mu, \nu \in \mathcal{P}$ and $i \in [1, \mathrm{ar}(\mu)]$, then $\mathcal{P}^{\mathrm{i}} = \mathsf{B}(\mathcal{P})$ is the bar construction of $\mathcal{P}$ with its usual differential.

The main theorem about inhomogeneous Koszul operads is the following:

Theorem 12.2 *If $\mathcal{P}$ admits an inhomogenous Koszul presentation then the canonical morphism $\Omega\mathcal{P}^{i} \longrightarrow \mathcal{P}$ determined by $s^{-1}\mathcal{P}^{i} \twoheadrightarrow \mathcal{X} \hookrightarrow \mathcal{P}$ is a quasi-isomorphism of operads.*

In general, one can show that the map $\Omega\mathsf{B}(\mathcal{P}) \longrightarrow \mathcal{P}$ is a quasi-isomorphism: this is the original approach of Ginzburg–Kapranov) who instead define a duality functor $\mathsf{D}(\mathcal{P})$ along with a quasi-isomorphism $\mathsf{DD}(\mathcal{P}) \longrightarrow \mathcal{P}$, and define the Koszul dual operad to $\mathcal{P}$ as $H_\Delta(\mathsf{D}(\mathcal{P}))$. The theorem above gives us a more economical resolution of $\mathcal{P}$ in case it is quadratic-linear Koszul. Note that the bar-cobar construction is, more or less, obtained as the 'least economical' resolution arising from the corresponding 'least economical' quadratic-linear presentation of an operad $\mathcal{P}$.

13 Prototype of inhomogeneous duality

Goals. We will define the operad governing BV algebras and show it admits a small inhomogeneous Koszul presentation. We will compute the homology of the corresponding dg-cooperad, and explain how it gives rise to the Gravity operad of E. Getzler. At the same time, we will explain how the homotopy quotient of BV by the circle action is the hypercommutative operad of Yu. I. Manin.

13.1 Definition and computations

The BV operad is an algebraic symmetric operad, which we write BV, generated by a binary commutative associative operation μ that we will write x_1x_2 of degree zero and a unary square-zero operation Δ of degree -1 that satisfy the following homogeneous 7-term relation:

$$\Delta(x_1x_2x_3) = x_1\Delta(x_2x_3) + x_2\Delta(x_1x_3) + x_3\Delta(x_1x_2) - x_1x_2\Delta(x_3) - x_1x_3\Delta(x_2) - x_2x_3\Delta(x_1).$$

Batalin–Vilkovisky algebras appear in several areas of mathematics:

(1) (Algebra) Vertex operator algebras, cohomology of Lie algebras, bar construction of A_∞-algebras.
(2) (Algebraic geometry) Gromov–Witten invariants and moduli spaces of curves (quantum cohomology, Frobenius manifolds), chiral algebras (geometric Langlands program),

(3) (Differential geometry) The sheaf of polyvector fields of an orientable (resp. Poisson or Calabi–Yau) manifold, the differential forms of a manifold (Hodge decomposition in the Riemannian case), Lie algebroids, Lagrangian (resp. coisotropic) intersections.
(4) (Noncommutative geometry) The Hoschchild cohomology of a symmetric algebra and the cyclic Deligne conjecture, non-commutative differential operators.
(5) (Algebraic topology) 2-fold loop spaces on topological spaces carrying an action of the circle, topological conformal field theories, Riemann surfaces, string topology.
(6) (Mathematical physics) BV quantization (gauge theory), BRST cohomology, string theory, topological field theory, Renormalization theory.

One can express the seven term relation by saying that $[\Delta,\mu]=\beta$ is a derivation for μ, where $[f,g]$ is the operadic commutator (à la Gerstenhaber) defined by

$$[f,g]=\sum_{i=1}^{\mathrm{ar}(f)} f\circ_i g-(-1)^{|f||g|}\sum_{j=1}^{\mathrm{ar}(g)} g\circ_j f.$$

This suggests defining $\beta=[\Delta,m]$, and presenting the BV-operad by quadratic-linear relations:

$$[\mu,\mu]=0,\quad \Delta^2=0,\quad [\Delta,\mu]=\beta,\quad \beta\circ_1\mu=\mu\circ_2\beta+\mu\circ_1\beta(23).$$

This presentation satisfies condition (1), but it *does not* satisfy condition (2): one can deduce that β is a Lie bracket of degree -1 and that Δ is a derivation for β from the first three equations. In other words, one can deduce that (Δ,β) defines the datum of a dg Lie algebra purely from the fact that $\Delta^2=0$ and that μ is associative.

Lemma 13.1 *The BV-operad admits a quadratic-linear presentation satisfying conditions (1) and (2) given by generators μ,β,Δ of arities 2, 2 and 1 and degrees 0, -1 and -1, respectively. The operation μ is associative commutative, β is a Lie bracket, Δ squares to zero, and*

$$\mathsf{Leib}(\Delta,\mu)=\beta,\quad \mathsf{Leib}(\beta,\mu)=0,\quad \mathsf{Leib}(\Delta,\beta)=0.$$

Theorem 13.2 *The quadratic operad $q\mathsf{BV}$ is Koszul.*

Proof. We will use the distributive law criterion of Markl, adapted to the case the operads in his result have unary operators. One can first show that $\mathsf{BV}(n)$ and its quadratic counterpart both have dimension $2^n n!$ by a Gröbner basis argument, and then that $q\mathsf{BV}$ is obtained from a distributive law between the quadratic operads Ger and $\mathsf{D}=\Bbbk[\Delta]/(\Delta^2)$, in light of the relations

$$\Delta(x_1x_2) = x_1\Delta(x_2) + \Delta(x_1)x_2, \quad \Delta[x_1,x_2] = [x_1,\Delta x_2] + [\Delta x_1,x_2].$$

One can prove this again by a dimension counting argument using the result above. With this at hand, we observe that Ger is Koszul, as it is in turn obtained from a distributive law between Com and $\mathscr{S}$Lie, which are both Koszul, and that D is Koszul (as any algebra with trivial multiplications is). We conclude that qBV is Koszul, and that we have isomorphisms of symmetric sequences

$$q\mathsf{BV} \cong \mathsf{Com} \circ \mathscr{S}\mathsf{Lie} \circ \mathsf{D}, \quad q\mathsf{BV}^{\mathsf{i}} \cong T^c(\delta) \circ \mathsf{Com}^c \circ \mathscr{S}^{-1}\mathsf{Lie}^c,$$

which will be useful to describe the dg-cooperad BV^{i}. ◀

13.2 The differential

Let us note that a generic element of $\mathsf{Com}^c \circ \mathscr{S}^{-1}\mathsf{Lie}^c$ consists of a corolla decorated by Lie words on a Lie bracket of degree 1. Any Lie word can always be written uniquely as a linear combination of words in the form $\ell = [x_1, x_{\sigma 2}, \ldots, x_{\sigma n}]$ where $\sigma \in S_n$ fixes 1, and we adopt the right bracketing convention:

$$[y_1,y_2,\ldots,y_n] = [y_1,[y_2,\ldots,y_n]].$$

We call x_1 the 'head' of ℓ. We will then write an element of $\mathsf{Com}^c \circ \mathscr{S}^{-1}\mathsf{Lie}^c$ generically by

$$\ell_1 \odot \cdots \odot \ell_n$$

where ℓ_i is a Lie word supported on π_i, with 'head' x_j with $j_i = \min \pi_i$ and such that $\min \pi_1 < \cdots < \min \pi_n$.

Theorem 13.3 *A generic element of $q\mathsf{BV}^i$ is of the form*

$$x = \delta^k \otimes \ell_1 \odot \cdots \odot \ell_n$$

and the differential d of $q\mathsf{BV}^i$ is

$$dx = \sum_{i=1}^{n} (-1)^{\varepsilon_i} \delta^{k-1} \otimes \ell_1 \odot \cdots \odot \ell_i^{(1)} \odot \ell_i^{(2)} \odot \cdots \odot \ell_n,$$

where $\ell \longmapsto \sum \ell^{(1)} \otimes \ell^{(2)}$ is the binary component of the decomposition map in $\mathscr{S}^{-1}\mathsf{Lie}^c$.

13.3 Exercises

References

[1] J.-L. Loday and B. Vallette, *Algebraic operads*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012. MR2954392

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