Algebraic operads, Koszul duality and Gröbner bases: an introduction

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This lecture series aim to offer a gentle introduction to the theory of algebraic operads, starting with the elements of the theory, and progressing slowly towards more advanced themes, including (inhomogeneous) Koszul duality theory, Gröbner bases and higher structures. The course will consists of approximately twelve lectures, along with extra talks by willing participants, with the goal of introducing extra material to the course, and making them more familiar with the theory.

Contents

Draft, 5th November 2021

1 Motivation and history

Goals. The goals of this lecture is to give a broad picture of the history and pre-history of operads, and some current trends, and give a road-map for the course.

1.1 Introduction and motivation

Operads originally appeared as tools in homotopy theory, specifically in the study of iterated loop spaces (May, 1972 and Boardman and Vogt before). They appeared as *comp algebras* in Gerstenhaber's work on Hochschild cohomology and topologically as Stasheff's 'associahedra' for his homotopy characterization of loop spaces (both in 1963). The theory of operads received new inspiration from homological algebra, category theory, algebraic geometry and mathematical physics:

(1) (Stasheff, Sugawara) Study homotopy associative H-spaces, Stasheff implicitly discovers a topological ns operad K with $C_*(K) = \mathsf{As}_\infty$ and a recognition principle for A_∞-spaces.
(2) (Boardmann–Vogt) Study infinite loop spaces, build a PROP (a version of an E_∞-operad) and obtain a recognition principle for infinite loop spaces.
(3) (Kontsevich) Uses L_∞-algebras and configuration spaces to prove his deformation quantization theorem that every Poisson manifold admits a deformation quantization.
(4) (Kontsevich) The above is implied by the formality theorem: the Lie algebra of polyvector fields is L_∞-quasi-isomorphic to the Hochschild complex, and $f_1 = \mathsf{HKR}$.
(5) (Tamarkin) Approaches this result through the formality of the little disks operad D_2. Proves that the Hochschild complex of a polynomial algebra is *intrinsically formal* as a Gerstenhaber algebra.
(6) (Manifold calculus) Describes the homotopy type of embedding spaces as certain derived operadic module maps and to produces their explicit deloopings using little disk operads, due to Goodwillie–Weiss, Boavida de Brito—Weiss, Turchin, Arone–Turchin, Dwyer—Hess, Ducoulombier—Turchin.
(7) (Ginzburg–Kapranov, Fresse, Vallette, Hinich) Koszul duality for algebraic operads and cousins allows to develop a robust homotopy theory of homotopy algebras, cohomology theory, deformation theory, Quillen homology, etc.
(8) (Deligne conjecture and variants) The study of natural operations on the Hochschild complex of an associative algebra lead to a manifold of results beginning with the proof that there is an action of the little disks operad D_2 on it, and the ultimate version by Markl–Voronov, who proved that the operad of natural operations on it has the homotopy type of $C_*(D_2)$.

Algebraic operads are modeled by trees (planar or non-planar, rooted or not), and relaxing these graphs allows us to produce other type of algebraic structures:

Type	Graph	Compositions	Due to
PROPs	Any graph	Any	Adams–MacLane
Modular	Any graph	$\xi_{i,j}$, $\circ_{i,j}$	Getzler–Kapranov
Properads	Connected graphs	Any	B. Vallette
Dioperads	Trees	${}_i\circ_j$ (no genus)	W. L. Gan
Half-PROPs	Trees	$\circ_j$, ${}_i\circ$	Markl–Voronov
Cyclic operads	Trees	$\circ_{i,j}$	Getzler–Kapranov
Symmetric operads	Rooted trees	$\circ_i$	J. P. May

1.2 Exercises

The corresponding exercises to this lecture appear in **Exercise Sheet #0**.

2 Symmetric sequences, composition products, unbiased approach

Goals. We will define some related gadgets (symmetric collections, algebras, modules, endomorphism operads) necessary to introduce operads. Then, we define what an operad is (topological, algebraic, symmetric, non-symmetric). We will then give some (not so) well known examples of topological and algebraic operads.

2.1 Basic definitions

What is an operad? A group is a model of $\mathrm{Aut}(X)$ for X a set, an algebra is a model of $\mathrm{End}(V)$ for V a vector space. Equivalently, groups are the gadgets that act on objects by automorphisms, and algebras are the gadgets that act on objects by their (linear) endomorphisms. Operads are the gadgets that act on objects through operations with many inputs (and one output), and at the same time keep track of symmetries when the inputs are permuted.

The underlying objects to operads are known as *symmetric sequences*: a symmetric sequence (also known as an Σ-module or symmetric module) is a sequence of vector spaces $\mathcal{X} = (\mathcal{X}(n))_{n\geqslant 0}$ such that for each $n \in \mathbb{N}_0$ there is a right action of S_n on $\mathcal{X}(n)$. We usually consider *reduced* Σ-modules, those for which $\mathcal{X}(0) = 0$.

A map of Σ-modules is a collection of maps $(f_n : \mathcal{X}_1(n) \longrightarrow \mathcal{X}_2(n)\}_{n\geqslant 0}$, each equivariant for the corresponding group action. This defines the category ${}_{\Sigma}\mathsf{dgMod}$ of symmetric sequences, and whenever we think of symmetric sequences using this definition, we will say we are considering a a biased or skeletal approach to them.

In parallel, it is convenient to consider the category $\mathsf{Fin}^{\times}$ of finite sets and bijections. An object in this category is a finite set I, and a morphism $\sigma : I \longrightarrow J$ is a bijection. Since every finite set I with n elements is (non-canonically) isomorphic to $[n] = \{1,\dots,n\}$, the following holds:

Lemma 2.1 *The skeleton of* $\mathsf{Fin}^{\times}$ *is equal to the category with objects the finite sets* $[n]$ *for* $n \geqslant 0$ *and with morphisms the bijections* $[n] \longrightarrow [n]$ *(and no morphism between* $[n]$ *and* $[m]$ *if* $m \neq n$*).*

Proof. This is in Exercise Sheet #0. ◀

We set ${}_{\Sigma}\mathsf{Mod} = \mathsf{Fun}(\mathsf{Fin}^{\times}, \mathsf{Vect}^{\mathrm{op}})$, so that a Σ-module is a pre-sheaf of vector spaces $I \longmapsto \mathcal{X}(I)$ assigning to each isomorphism $\tau : I \longrightarrow J$ an isomorphism $\mathcal{X}(\tau) : \mathcal{X}(J) \longrightarrow \mathcal{X}(I)$. When we think of Σ-modules as pre-sheaves, we will say we are taking an unbiased approach, will if we specify only its values on natural numbers, we will say we are taking the biased or skeletal approach; we will come back to this later.

With this at hand, we can in turn define the *Cauchy product* of two Σ-modules $\mathcal{X}$ and $\mathcal{Y}$

$$(\mathcal{X} \otimes_\Sigma \mathcal{Y})(I) = \bigoplus_{S \sqcup T = I} \mathcal{X}(S) \otimes \mathcal{Y}(T)$$

where the right-hand is the usual tensor product of vector spaces and the sum runs through partitions of I into two disjoint sets. The symmetric product is then defined by

$$(\mathcal{X} \circ_\Sigma \mathcal{Y})(I) = \bigoplus_{\pi \vdash I} \mathcal{X}(\pi) \otimes \mathcal{Y}^{\otimes k}(\pi)$$

as the sum runs through (ordered) partitions of I. These two products will be central in what follows.

Lemma 2.2 *The category* ${}_\Sigma\mathsf{Mod}$ *with* $\circ_\Sigma$ *is monoidal with unit the species taking the value* $\Bbbk e_x$ *at the singleton sets* $\{x\}$ *and zero everywhere else. The same category is monoidal for* $\otimes_\Sigma$ *with unit the species taking the value* $\Bbbk$ *at* $\varnothing$ *and zero everywhere else.*

We will use the notation $\Bbbk$ for the base field but also for the unit for the composition product $\circ_\Sigma$, hoping it will not cause much confusion. It will be useful later to think of $\Bbbk$ as a twig or “stick”.

Observe that the associator for $\circ_\Sigma$ is not too simple and involves reordering certain factors of tensor products in Vect. In particular, replacing vector spaces by graded vector spaces or complexes will create signs in the associator.

We are now ready to define the prototypical symmetric sequence that carries the structure of an algebraic operad.

Definition 2.3 The *endomorphism operad* of a space V is the symmetric sequence End_V where for each $n \in \mathbb{N}$ we set $\mathrm{End}_V(n) = \mathrm{End}(V^{\otimes}, V)$. The symmetric group S_n acts on the right on $\mathrm{End}_V(n)$ so that $(f\sigma)(v) = f(\sigma v)$ for $v \in V^{\otimes n}$, where S_n acts on the left on $V^{\otimes n}$ by $(\sigma v)_i = v_{\sigma i}$. The composition maps are defined by $\gamma(f; g_1, \ldots, g_n) = f \circ (g_1 \otimes \cdots \otimes g_n)$.

The following two operations on permutations will streamline our definition of (algebraic) operads.

Two useful maps. For each $k \geqslant 1$ and each tuple $\lambda = (n_1, \ldots, n_k)$ with sum n there is a map

$$S_k \longrightarrow S_{n_1 + \cdots + n_k}$$

$$\lambda = (2,1,2), \quad \sigma = 312 \quad \rightsquigarrow \quad \lambda(\sigma) = 34512 \in S_5$$
$$(213,213,132) \in S_3 \times S_3 \times S_3 \quad \rightsquigarrow \quad 213546798 \in S_9$$

Figure 1: The useful operations

that sends $\sigma \in S_k$ to the permutation $\lambda(\sigma)$ of $[n]$ that permutes the blocks $\pi_i = \{n_1 + \cdots + n_{i-1} + 1, \ldots n_1 + \cdots + n_{i-1} + n_i\}$ according to σ. There is also a map

$$S_{n_1} \times \cdots \times S_{n_k} \longrightarrow S_{n_1 + \cdots + n_k}$$

that sends a tuple of permutations $(\sigma_1, \ldots, \sigma_k)$ to the permutation $\sigma_1 \# \cdots \# \sigma_k$ that acts like σ_i on the block π_i as above. These operations are illustrated in Figure 1. With these at hand, one can check that these composition maps satisfy the following axioms:

(1) *Associativity*: let $f \in \mathrm{End}_V(n)$, and consider $g_1, \ldots, g_n \in \mathrm{End}_V$ and for each $i \in [n]$ a tuple $h_i = (h_{i1}, \ldots, h_{in_i})$ were $n_i = \mathrm{ar}(g_i)$. Then for $f_i = \gamma(g_i; h_{i1}, \ldots, h_{in_i})$ and $g = \gamma(f; g_1, \ldots, g_n)$ we have that

$$\gamma(f; f_1, \ldots, f_n) = \gamma(g; h_1, \ldots, h_n).$$

(2) *Intrinsic equivariance*: for each $\sigma \in S_k$ and $\lambda = (\mathrm{ar}(g_1), \ldots, \mathrm{ar}(g_k))$ we have that

$$\gamma(f\sigma; g_1, \ldots, g_k) = \gamma(f; g_{\sigma 1}, \ldots, g_{\sigma k})\lambda(\sigma),$$

(3) *Extrinsic equivariance*: for each tuple of permutations $(\sigma_1, \ldots, \sigma_k) \in S_{n_1} \times \cdots \times S_{n_k}$, if $\sigma = \sigma_1 \# \cdots \# \sigma_k$, we have that

$$\gamma(f, g_1\sigma_1, \ldots, g_k\sigma_k) = \gamma(f; g_1, \ldots, g_k)\sigma.$$

(4) *Unitality:* the identity $1 \in \mathrm{End}_V(1)$ satisfies $\gamma(1; g) = g$ and $\gamma(g; 1, \ldots, 1) = g$ for every $g \in \mathrm{End}_V$.

Definition 2.4 A symmetric operad (in vector spaces) is an Σ-module $\mathcal{P}$ along with a composition map $\gamma : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$ of signature

$$\gamma : \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \longrightarrow \mathcal{P}(n_1 + \cdots + n_k)$$

along with a unit $1 \in \mathcal{P}(1)$, that satisfy the axioms above.

Variant 2.5 A non-symmetric operad is an operad whose underlying object is a collection (with no symmetric group actions). Operads in topological spaces or chain complexes require the composition maps to be morphisms (that is, continuous maps or maps of chain complexes, respectively) and, more generally, operads defined on a symmetric monoidal category require, naturally, that the composition maps be morphisms in that category.

Pseudo-operads. One can define operads through *partial composition maps*, modeling the honest partial composition map

$$f \circ_i g = f(1, \ldots, 1, g, 1, \ldots, 1)$$

in End_V. These composition maps satisfy the following properties:

(1) *Associativity*: for each $f, g, h \in \mathrm{End}_V$, and $\delta = i - j + 1$, we have

$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{\mathrm{ar}(f)+j-1} g & \delta \leqslant 0 \\ f \circ_j (g \circ_\delta h) & \delta \in [1, \mathrm{ar}(g)] \\ (f \circ_\delta h) \circ_j g & \delta > \mathrm{ar}(g) \end{cases}$$

(2) *Intrinsic equivariance*: for each $\sigma \in S_k$, we have that

$$(f\sigma) \circ_i g = (f \circ_{\sigma i} g)\sigma'$$

where σ' is the same permutation as σ that treats the block $\{i, i+1, \ldots, i+\mathrm{ar}(g)-1\}$ as a single element.

(3) *Extrinsic equivariance*: for each $\sigma \in S_k$, we have that

$$f \circ_i (g\sigma) = (f \circ_i g)\sigma''$$

where σ'' acts by only permuting the block $\{i, \ldots, i+\mathrm{ar}(g)-1\}$ according to σ.

(4) *Unitality:* the identity $1 \in \mathrm{End}_V(1)$ satisfies $1 \circ_1 g = g$ and $g \circ_i 1 = g$ for every $g \in \mathrm{End}_V$ and $1 \leqslant i \leqslant \mathrm{ar}(g)$.

Definition 2.6 A symmetric operad (in vector spaces) is an Σ-module $\mathcal{P}$ along with partial composition map of signature

$$- \circ_i - : \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$$

and a unit $1 \in \mathcal{P}(1)$ satisfying the axioms above.

It is not hard to see (but must be checked at least once) that an operad with $\mathcal{P}(n) = 0$ for $n \neq 1$ is exactly the same as an associative algebra.

Warning! If one does not require the existence of a unit, the notion of a *pseudo-operad* by Markl (defined by partial compositions) does not coincide with the notion of an operad as defined by May.

2.2 Constructing operads by hand

One can define operads in various ways. For example, one can define the underlying collection explicitly, and give the composition maps directly:

(1) *Commutative operad.* The reduced symmetric topological (or set) operad with $\mathsf{Com}(n)$ a single point for each $n \in \mathbb{N}$, and composition maps the unique map from a point to a point.

(2) *Associative operad.* The reduced set operad with $\mathsf{As}(n) = S_n$ the regular representation and composition maps

$$S_k \times S_{n_1} \times \cdots \times S_{n_k} \longrightarrow S_{n_1+\cdots n_k}$$

the unique equivariant map that sends the tuple of identities to the identity.

(3) *Stasheff operad.* Let K_{n+2} be the subset of I^n (the product of n copies of $I = [0,1]$) consisting of tuples $(t_1,\ldots,t_{n+2})$ such that $t_1 \cdots t_k \leqslant 2^{-k}$ for $j \in [n+2]$. The boundary of K_{n+2} consists of those points such that for some $j \in [n+2]$ we have either t_j or $t_1 \cdots t_j = 2^{-j}$. It is tedious (but otherwise doable) to show that for each pair (r,s) of natural numbers and each $i \in [r]$ there exists an inclusion

$$\circ_i : K_{r+1} \times K_{s+1} \longrightarrow K_{r+s+1}$$

that defines on the sequence of spaces $\{K_{n+2}\}_{n\geqslant 0}$ the structure of a non-symmetric operad. We will see in the exercise a realization of K_n as the convex hull of points with positive integer coordinates (due to J.-L. Loday) using planar binary rooted trees, which will make the operad structure more transparent.

(4) If M is a monoid, there is an operad $\mathbb{W}_M$ with $\mathbb{W}_M(n) = M^n$ such that

$$(m_1,\ldots,m_s) \circ_i (m_1',\ldots,m_t') = (m_1,\ldots,m_{i-1},m_i m_1',\ldots,m_i m_t',m_{i+1},\ldots,m_s).$$

We call it the *word operad of* M. Its underlying symmetric collection is $\mathsf{As} \circ M$.

(5) Write $\mathrm{Aff}(\mathbb{C}) = \mathbb{C} \times \mathbb{C}^\times$ for the group of affine transformations of $\mathbb{C}$ with group law $(z,\lambda)(w,\mu) = (z+\lambda w, \lambda\mu)$. In turn, define for each finite set I the topological space

$$\mathcal{C}(I) = \{(z_i,\lambda_i) \in \mathrm{Aff}(\mathbb{C})^I : |z_i - z_j| > |\lambda_i| + |\lambda_j|\}.$$

The group law of $\mathrm{Aff}(\mathbb{C})$ allows us to define an operad structure on $\mathcal{C}(I)$ using the exact same definition as in the word operad of a monoid. The subspaces $\mathcal{D}_2^{\mathrm{fr}}(I) \subseteq \mathcal{C}(I)$ where $|z_i| + |\lambda_i| \leqslant 1$ for all $i \in I$, and where the inequality is strict unless $z_i = 0$ is called the *framed little disks operad*. The little disks operad is the suboperad where $\lambda_i = 1$ for all $i \in I$, and we write it $\mathcal{D}_2(I)$.

(6) The operad of rooted trees RT has $\mathsf{RT}(n)$ the collection of rooted threes with n vertices labeled by $[n]$, and the composition $T \circ_j T'$ is obtained by inserting T' at the jth vertex of T and reattaching the children of that vertex to T' in all possible ways. For example, if then we have that

2.3 Exercises

The corresponding exercises to this lecture appear in **Exercise Sheet #1**. During the exercise sessions, we will review pseudo-operads and their partial insertions, and the category of operads.

3 Free operads and presentations

Goals. We will define algebraic operads by generators and relations, and with this at hand define quadratic and quadratic-linear presentations of operads.

3.1 Planar and non-planar trees

Operads and their kin are gadgets modeled after combinatorial graph-like objects. Operads themselves are modeled after rooted trees, so it is a good idea to have a concrete definition of what a rooted tree is. We will also consider planar rooted trees, and trees with certain decorations, so it is a good idea to digest the definitions carefully to later embellish them.

A rooted tree τ is the datum of a finite set $V(\tau)$ of vertices along with a partition $V(\tau) = \mathrm{Int}(\tau) \sqcup L(\tau) \cup R(\tau)$, where the first are the *interior* vertices, L are the leaves, and $R(\tau)$ is a singleton, called the root of τ. We also require there is a function $p : V(\tau) \smallsetminus R(\tau) \longrightarrow V(\tau)$, describing the edges of τ, with the following properties: call a vertex $v \in V(\tau)$ a child of $w \in V(\tau)$ if $v \in p^{-1}(w)$. Then:

(1) The root $r \in R(\tau)$ has exactly one child.
(2) The leaves of τ have no children.
(3) For each non-root vertex v there exist a unique sequence $(v_0, v_1, \ldots, v_k)$ such that $p(v_{i-1}) = v_i$ for $i \in [k]$ with $v_0 = v$ and $v_k = r$.

We will call a non-leaf vertex that has no children a *stump* (or an endpoint, or a cherry-top). A tree is reduced if has no stumps and all of its non-root and non-leaf vertices have at least two children. We will also call the root the (unique) output vertex τ, and the leaves the input vertices of τ.

A planar rooted tree is a rooted tree τ along with a linear order in each of the fibers of the parent function p of τ. In short, the children of each vertex are linearly ordered, so we are effectively considering a drawing of τ in the plane, where the clockwise orientation gives us the order at each vertex.

Two rooted trees τ and τ' are isomorphic if there exists a bijection $f : V(\tau) \longrightarrow V(\tau')$ that preserves the root, the input vertices and the interior vertices, so that $p' \circ f = p$ where we also write f for the induced bijection $f : V(\tau) \smallsetminus r \longrightarrow V(\tau') \smallsetminus r'$. Two planar rooted trees are isomorphic if in addition f respects the linear order at each vertex.

For example, consider the rooted tree τ with $V = \{1,2,3\} \cup \{4,5\} \cup \{0\}$, that is, three leaves, two interior vertices and the root. Then the choice of $p : [5] \to [\![5]\!]$ with $p(\{1,2\}) = 4$, $p(\{3,4\}) = 5$, $p(5) = 0$ gives a tree isomorphic to the one with with $p(\{1,2\}) = 3$,

$p(\{3,4\}) = 5$, $p(5) = 0$. On the other hand, if we consider the vertices linearly ordered by their natural order, these two planar rooted trees are no longer isomorphic.

Definition 3.1 For a finite set I, an I-labeled tree T is a pair (τ, f) where τ is a reduced rooted tree, along with a bijection $f : I \longrightarrow L(\tau)$. Two I-labeled trees T an T' are isomorphic if there exists a pair (g, σ) where g is an isomorphism between τ and τ' and σ is an automorphism of I such that $g\mid_{L(\tau)} \circ f = \sigma \circ f'$.

Suppose that (τ, f) is an I-tree and that (τ', f') is a J-tree, and that $i \in I$. We define $K = I \cup_i J = I \sqcup J \smallsetminus i$ and the K-tree $\tau \circ_i \tau'$ as follows:

(1) Its leaves are $L(\tau \circ_i \tau') = L(\tau) \sqcup L(\tau') \smallsetminus f^{-1}(i)$.
(2) Its internal vertices are $V(\tau) \sqcup V(\tau')$, with root r.
(3) The parent function q is defined by declaring that:

- q coincides with p on $V(\tau)$,
- $q(w) = p(f^{-1}(i))$ if w is the unique children of the root of τ',
- q coincides with p' on $V(\tau') \smallsetminus \{r', w\}$.

(4) The leaf labeling is the unique bijection $L(\tau \circ_i \tau') \longrightarrow I \circ_i J$ extending f and f'.

Let us now consider an (unbiased) reduced symmetric sequence $\mathcal{X}$ which we will think of as an *alphabet*. An tree monomial in the alphabet $\mathcal{X}$ is a pair (τ, x) where τ is a reduced rooted tree and $x : \mathrm{Int}(\tau) \longrightarrow \mathcal{X}$ is a map with the property that $x(v) \in \mathcal{X}(p^{-1}(v))$. Observe that reduced sequences and reduced trees correspond to each other, in the sense that with this definition we can only decorate a stump of τ with an element of $\mathcal{X}(\varnothing)$.

An I-labeled $\mathcal{X}$-tree T is a triple (τ, x, f) where (τ, f) is I-labeled and (τ, x) is an $\mathcal{X}$-tree. We will say that (τ, x, f) is a (symmetric) tree monomial if $\mathcal{X}$ is symmetric. If it is just a collection, we will say that (τ, x, f) is a ns tree monomial. In particular, if T is an I-labeled tree, and if $\sigma \in \mathrm{Aut}(I)$, there is another I-labeled tree $\sigma(T) = (\tau, f\sigma^{-1})$.

Suppose that $T = (\tau, x, f)$ is a tree monomial on an alphabet $\mathcal{X}$, and let us pick a vertex v of τ and a permutation σ of the set $C = p^{-1}(v)$ of children of v. We define the tree τ^σ as follows: the datum defining τ remains unchanged except p is modified to p^σ so that

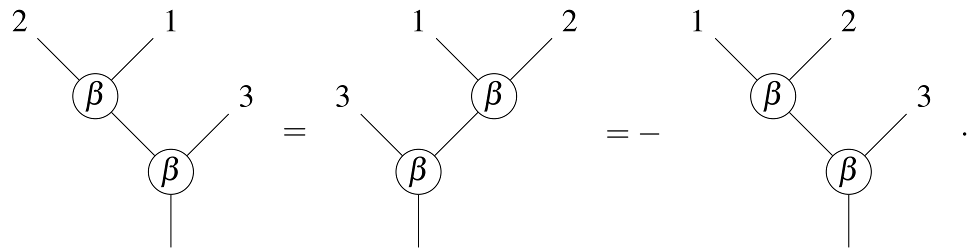
$$p^\sigma(w) = \begin{cases} p(w) & \text{if } p^2(w) \neq v \\ p(\sigma^{-1}(w')) & \text{if } p(w) = w' \in C. \end{cases}$$

Briefly, we just relabel the vertices of τ using σ. With this at hand, we define T^σ to be the tree monomial with underlying tree τ^σ and with x modified to x^σ so that

$$x^\sigma(w) = \begin{cases} \sigma x(v) & \text{if } v = w, \\ x(\sigma^{-1}(w')) & \text{if } p(w) = w' \in C. \end{cases}$$

Note that it is possible some children of v are leaves, in which case the definitions make sense if we think of leaves as decorated by the unit of $\Bbbk$.

Example 3.2 Let us consider the alphabet $\mathcal{X} = \mathcal{X}(2) = \{\beta\}$ where the unique operation is antisymmetric. Then we have the following equalities of symmetric tree monomials:



Let us now define for each $n \geqslant 1$ the space $\mathcal{F}_\mathcal{X}(I)$ as the span of all tree monomials $T = (\tau, f, x)$ on $\mathcal{X}$ with leaves labeled by I, modulo the subspace generated by all elements of the form

$$R(T, v, \sigma) = T - T^\sigma$$

where σ ranges through $\operatorname{Aut}(p^{-1}(v))$ as v ranges through the vertices of τ. In case all children of v are leaves, this is saying that the tree where x_v is replace by $\sigma(x_v)$ is equal to the tree where the leaves of T that are children of v are relabeled according to σ. We also require that tree decorations behave like tensors, so that $T = T_1 + T_2$ if the decoration of T at a vertex v is of the form $x_1 + x_2$ and for $i \in [2]$ the tree T_i coincides with T except that it is decorated by x_i at v.

3.2 The free operad

An algebraically inclined way to construct (algebraic) operads is through generators and relations. There is a forgetful functor from the category of operads to the category of collections. In general, it admits a left adjoint, which is the free operad functor.

Definition 3.3 The *free symmetric operad* on $\mathcal{X}$ is the symmetric sequence $\mathcal{F}_\mathcal{X}$ along with the composition law obtained by grafting of trees. More precisely, suppose that $T \in \mathcal{F}_\mathcal{X}(I)$

and that $T' \in \mathcal{F}_{\mathcal{X}}(J)$, and that $i \in I$. We define $T'' = T \circ_i T' \in \mathcal{F}_{\mathcal{X}}(I \cup_I J)$ by taking its underlying labeled tree to be $\tau \circ_i \tau'$, and by decorating it in the unique way which extends the decorations of T and T'.

The following lemma shows that this indeed defines an operad.

Lemma 3.4 *Tree grafting respects both I-tree isomorphisms and the relations $T \sim T^{\sigma}$ above, and hence is well defined on $\mathcal{F}_{\mathcal{X}}$.*

Proof. This is in **Exercise Sheet #2**. ◀

We will later interpret $\mathcal{X} \longmapsto \mathcal{F}_{\mathcal{X}}$ as a *monad*, thus giving another definition of operads. The advantage of this 'monadic approach' is its flexibility, which allow us to define other operad like structures, like the ones mentioned in the introduction. In this direction, a curious reader can consider the following equivalent definition:

Definition 3.5 The free operad generated by a symmetric collection X is defined inductively by letting $\mathcal{F}_{0,X} = \mathbb{k}$ be spanned by the 'twig' (tree with no vertices and one edge) in arity zero and

$$\mathcal{F}_{n+1,\mathcal{X}} = \mathbb{k} \oplus (\mathcal{X} \circ \mathcal{F}_{n,\mathcal{X}}),$$

and finally by setting $\mathcal{F}_{\mathcal{X}} = \varinjlim_n \mathcal{F}_{n+1,\mathcal{X}}$. The composition maps are defined by induction, and the axioms are also checked by induction.

Intuitively, the previous definition says that an element of $\mathcal{F}_{\mathcal{X}}$ is either the twig, or corolla with n vertices decorated by $\mathcal{X}$, whose leaves have on them an element of $\mathcal{F}_{\mathcal{X}}$. The final shape of $\mathcal{F}_{\mathcal{X}}$ will however depend on the symmetric structure of $\mathcal{X}$.

3.3 Exercises

The corresponding exercises to this lecture appear in **Exercise Sheet #2**.