Algebraic operads, Koszul duality and Gröbner bases: an introduction

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This lecture series aim to offer a gentle introduction to the theory of algebraic operads, starting with the elements of the theory, and progressing slowly towards more advanced themes, including (inhomogeneous) Koszul duality theory, Gröbner bases and higher structures. The course will consists of approximately twelve lectures, along with extra talks by willing participants, with the goal of introducing extra material to the course, and making them more familiar with the theory.

Contents

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1 Motivation and history

Goals. The goals of this lecture is to give a broad picture of the history and pre-history of operads, and some current trends, and give a road-map for the course.

1.1 Introduction and motivation

Operads originally appeared as tools in homotopy theory, specifically in the study of iterated loop spaces (May, 1972 and Boardman and Vogt before). They appeared as *comp algebras* in Gerstenhaber's work on Hochschild cohomology and topologically as Stasheff's 'associahedra' for his homotopy characterization of loop spaces (both in 1963). The theory of operads received new inspiration from homological algebra, category theory, algebraic geometry and mathematical physics:

(1) (Stasheff, Sugawara) Study homotopy associative H-spaces, Stasheff implicitly discovers a topological ns operad K with $C_*(K) = \mathsf{As}_\infty$ and a recognition principle for A_∞-spaces.
(2) (Boardmann–Vogt) Study infinite loop spaces, build a PROP (a version of an E_∞-operad) and obtain a recognition principle for infinite loop spaces.
(3) (Kontsevich) Uses L_∞-algebras and configuration spaces to prove his deformation quantization theorem that every Poisson manifold admits a deformation quantization.
(4) (Kontsevich) The above is implied by the formality theorem: the Lie algebra of polyvector fields is L_∞-quasi-isomorphic to the Hochschild complex, and $f_1 = \mathsf{HKR}$.
(5) (Tamarkin) Approaches this result through the formality of the little disks operad D_2. Proves that the Hochschild complex of a polynomial algebra is *intrinsically formal* as a Gerstenhaber algebra.
(6) (Manifold calculus) Describes the homotopy type of embedding spaces as certain derived operadic module maps and to produces their explicit deloopings using little disk operads, due to Goodwillie–Weiss, Boavida de Brito—Weiss, Turchin, Arone–Turchin, Dwyer—Hess, Ducoulombier—Turchin.
(7) (Ginzburg–Kapranov, Fresse, Vallette, Hinich) Koszul duality for algebraic operads and cousins allows to develop a robust homotopy theory of homotopy algebras, cohomology theory, deformation theory, Quillen homology, etc.
(8) (Deligne conjecture and variants) The study of natural operations on the Hochschild complex of an associative algebra lead to a manifold of results beginning with the proof that there is an action of the little disks operad D_2 on it, and the ultimate version by Markl–Voronov, who proved that the operad of natural operations on it has the homotopy type of $C_*(D_2)$.

Algebraic operads are modeled by trees (planar or non-planar, rooted or not), and relaxing these graphs allows us to produce other type of algebraic structures:

Type	Graph	Compositions	Due to
PROPs	Any graph	Any	Adams–MacLane
Modular	Any graph	$\xi_{i,j}$, $\circ_{i,j}$	Getzler–Kapranov
Properads	Connected graphs	Any	B. Vallette
Dioperads	Trees	${}_i\circ_j$ (no genus)	W. L. Gan
Half-PROPs	Trees	$\circ_j$, ${}_i\circ$	Markl–Voronov
Cyclic operads	Trees	$\circ_{i,j}$	Getzler–Kapranov
Symmetric operads	Rooted trees	$\circ_i$	J. P. May

1.2 Exercises

The corresponding exercises to this lecture appear in **Exercise Sheet #0**.

2 Symmetric sequences, composition products, unbiased approach

Goals. We will define some related gadgets (symmetric collections, algebras, modules, endomorphism operads) necessary to introduce operads. Then, we define what an operad is (topological, algebraic, symmetric, non-symmetric). We will then give some (not so) well known examples of topological and algebraic operads.

2.1 Basic definitions

What is an operad? A group is a model of $\mathrm{Aut}(X)$ for X a set, an algebra is a model of $\mathrm{End}(V)$ for V a vector space. Equivalently, groups are the gadgets that act on objects by automorphisms, and algebras are the gadgets that act on objects by their (linear) endomorphisms. Operads are the gadgets that act on objects through operations with many inputs (and one output), and at the same time keep track of symmetries when the inputs are permuted.

The underlying objects to operads are known as *symmetric sequences*: a symmetric sequence (also known as an Σ-module or symmetric module) is a sequence of vector spaces $\mathcal{X} = (\mathcal{X}(n))_{n\geqslant 0}$ such that for each $n \in \mathbb{N}_0$ there is a right action of S_n on $\mathcal{X}(n)$. We usually consider *reduced* Σ-modules, those for which $\mathcal{X}(0) = 0$.

A map of Σ-modules is a collection of maps $(f_n : \mathcal{X}_1(n) \longrightarrow \mathcal{X}_2(n)\}_{n\geqslant 0}$, each equivariant for the corresponding group action. This defines the category ${}_\Sigma\mathsf{Mod}$ of symmetric sequences, and whenever we think of symmetric sequences using this definition, we will say we are considering a a biased or skeletal approach to them.

In parallel, it is convenient to consider the category $\mathsf{Fin}^\times$ of finite sets and bijections. An object in this category is a finite set I, and a morphism $\sigma : I \longrightarrow J$ is a bijection. Since every finite set I with n elements is (non-canonically) isomorphic to $[n] = \{1,\dots,n\}$, the following holds:

Lemma 2.1 *The skeleton of* $\mathsf{Fin}^\times$ *is equal to the category with objects the finite sets* $[n]$ *for* $n \geqslant 0$ *and with morphisms the bijections* $[n] \longrightarrow [n]$ *(and no morphism between* $[n]$ *and* $[m]$ *if* $m \neq n$*).*

Proof. This is in Exercise Sheet #0. $\square$

We set ${}_\Sigma\mathsf{Mod} = \mathsf{Fun}(\mathsf{Fin}^\times, \mathsf{Vect}^{\mathrm{op}})$, so that a Σ-module is a pre-sheaf of vector spaces $I \longmapsto \mathcal{X}(I)$ assigning to each isomorphism $\tau : I \longrightarrow J$ an isomorphism $\mathcal{X}(\tau) : \mathcal{X}(J) \longrightarrow \mathcal{X}(I)$. When we think of Σ-modules as pre-sheaves, we will say we are taking an unbiased approach, will if we specify only its values on natural numbers, we will say we are taking the biased or skeletal approach; we will come back to this later.

With this at hand, we can in turn define the *Cauchy product* of two Σ-modules $\mathcal{X}$ and $\mathcal{Y}$

$$(\mathcal{X} \otimes_\Sigma \mathcal{Y})(I) = \bigoplus_{S \sqcup T = I} \mathcal{X}(S) \otimes \mathcal{Y}(T)$$

where the right-hand is the usual tensor product of vector spaces and the sum runs through partitions of I into two disjoint sets. The symmetric product is then defined by

$$(\mathcal{X} \circ_\Sigma \mathcal{Y})(I) = \bigoplus_{\pi \vdash I} \mathcal{X}(\pi) \otimes \mathcal{Y}^{\otimes k}(\pi)$$

as the sum runs through (ordered) partitions of I. These two products will be central in what follows.

Lemma 2.2 *The category* ${}_\Sigma\mathsf{Mod}$ *with* $\circ_\Sigma$ *is monoidal with unit the species taking the value* $\Bbbk e_x$ *at the singleton sets* $\{x\}$ *and zero everywhere else. The same category is monoidal for* $\otimes_\Sigma$ *with unit the species taking the value* $\Bbbk$ *at* $\varnothing$ *and zero everywhere else.*

We will use the notation $\Bbbk$ for the base field but also for the unit for the composition product $\circ_\Sigma$, hoping it will not cause much confusion. It will be useful later to think of $\Bbbk$ as a twig or "stick".

Observe that the associator for $\circ_\Sigma$ is not too simple and involves reordering certain factors of tensor products in Vect. In particular, replacing vector spaces by graded vector spaces or complexes will create signs in the associator.

We are now ready to define the prototypical symmetric sequence that carries the structure of an algebraic operad.

Definition 2.3 The *endomorphism operad* of a space V is the symmetric sequence End_V where for each $n \in \mathbb{N}$ we set $\mathrm{End}_V(n) = \mathrm{End}(V^{\otimes}, V)$. The symmetric group S_n acts on the right on $\mathrm{End}_V(n)$ so that $(f\sigma)(v) = f(\sigma v)$ for $v \in V^{\otimes n}$, where S_n acts on the left on $V^{\otimes n}$ by $(\sigma v)_i = v_{\sigma i}$. The composition maps are defined by $\gamma(f; g_1, \ldots, g_n) = f \circ (g_1 \otimes \cdots \otimes g_n)$.

The following two operations on permutations will streamline our definition of (algebraic) operads.

Two useful maps. For each $k \geqslant 1$ and each tuple $\lambda = (n_1, \ldots, n_k)$ with sum n there is a map

$$S_k \longrightarrow S_{n_1 + \cdots + n_k}$$

$$\lambda = (2,1,2), \quad \sigma = 312 \quad \rightsquigarrow \quad \lambda(\sigma) = 34512 \in S_5$$
$$(213,213,132) \in S_3 \times S_3 \times S_3 \quad \rightsquigarrow \quad 213546798 \in S_9$$

Figure 1: The useful operations

that sends $\sigma \in S_k$ to the permutation $\lambda(\sigma)$ of $[n]$ that permutes the blocks $\pi_i = \{n_1 + \cdots + n_{i-1} + 1, \ldots n_1 + \cdots + n_{i-1} + n_i\}$ according to σ. There is also a map

$$S_{n_1} \times \cdots \times S_{n_k} \longrightarrow S_{n_1+\cdots+n_k}$$

that sends a tuple of permutations $(\sigma_1, \ldots, \sigma_k)$ to the permutation $\sigma_1 \# \cdots \# \sigma_k$ that acts like σ_i on the block π_i as above. These operations are illustrated in Figure 1. With these at hand, one can check that these composition maps satisfy the following axioms:

(1) *Associativity*: let $f \in \mathrm{End}_V(n)$, and consider $g_1, \ldots, g_n \in \mathrm{End}_V$ and for each $i \in [n]$ a tuple $h_i = (h_{i1}, \ldots, h_{in_i})$ were $n_i = \mathrm{ar}(g_i)$. Then for $f_i = \gamma(g_i; h_{i1}, \ldots, h_{in_i})$ and $g = \gamma(f; g_1, \ldots, g_n)$ we have that

$$\gamma(f; f_1, \ldots, f_n) = \gamma(g; h_1, \ldots, h_n).$$

(2) *Intrinsic equivariance*: for each $\sigma \in S_k$ and $\lambda = (\mathrm{ar}(g_1), \ldots, \mathrm{ar}(g_k))$ we have that

$$\gamma(f\sigma; g_1, \ldots, g_k) = \gamma(f; g_{\sigma 1}, \ldots, g_{\sigma k})\lambda(\sigma),$$

(3) *Extrinsic equivariance*: for each tuple of permutations $(\sigma_1, \ldots, \sigma_k) \in S_{n_1} \times \cdots \times S_{n_k}$, if $\sigma = \sigma_1 \# \cdots \# \sigma_k$, we have that

$$\gamma(f, g_1\sigma_1, \ldots, g_k\sigma_k) = \gamma(f; g_1, \ldots, g_k)\sigma.$$

(4) *Unitality:* the identity $1 \in \mathrm{End}_V(1)$ satisfies $\gamma(1; g) = g$ and $\gamma(g; 1, \ldots, 1) = g$ for every $g \in \mathrm{End}_V$.

Definition 2.4 A symmetric operad (in vector spaces) is an Σ-module $\mathcal{P}$ along with a composition map $\gamma : \mathcal{P} \circ \mathcal{P} \longrightarrow \mathcal{P}$ of signature

$$\gamma : \mathcal{P}(k) \otimes \mathcal{P}(n_1) \otimes \cdots \otimes \mathcal{P}(n_k) \longrightarrow \mathcal{P}(n_1 + \cdots + n_k)$$

along with a unit $1 \in \mathcal{P}(1)$, that satisfy the axioms above.

Variant 2.5 A non-symmetric operad is an operad whose underlying object is a collection (with no symmetric group actions). Operads in topological spaces or chain complexes require the composition maps to be morphisms (that is, continuous maps or maps of chain complexes, respectively) and, more generally, operads defined on a symmetric monoidal category require, naturally, that the composition maps be morphisms in that category.

Pseudo-operads. One can define operads through *partial composition maps*, modeling the honest partial composition map

$$f \circ_i g = f(1, \dots, 1, g, 1, \dots, 1)$$

in End_V. These composition maps satisfy the following properties:

(1) *Associativity*: for each $f, g, h \in \mathrm{End}_V$, and $\delta = i - j + 1$, we have

$$(f \circ_j g) \circ_i h = \begin{cases} (f \circ_i h) \circ_{\mathrm{ar}(f)+j-1} g & \delta \leqslant 0 \\ f \circ_j (g \circ_\delta h) & \delta \in [1, \mathrm{ar}(g)] \\ (f \circ_\delta h) \circ_j g & \delta > \mathrm{ar}(g) \end{cases}$$

(2) *Intrinsic equivariance*: for each $\sigma \in S_k$, we have that

$$(f\sigma) \circ_i g = (f \circ_{\sigma i} g)\sigma'$$

where σ' is the same permutation as σ that treats the block $\{i, i+1, \dots, i+\mathrm{ar}(g)-1\}$ as a single element.

(3) *Extrinsic equivariance*: for each $\sigma \in S_k$, we have that

$$f \circ_i (g\sigma) = (f \circ_i g)\sigma''$$

where σ'' acts by only permuting the block $\{i, \dots, i+\mathrm{ar}(g)-1\}$ according to σ.

(4) *Unitality:* the identity $1 \in \mathrm{End}_V(1)$ satisfies $1 \circ_1 g = g$ and $g \circ_i 1 = g$ for every $g \in \mathrm{End}_V$ and $1 \leqslant i \leqslant \mathrm{ar}(g)$.

Definition 2.6 A symmetric operad (in vector spaces) is an Σ-module $\mathcal{P}$ along with partial composition map of signature

$$- \circ_i - : \mathcal{P}(m) \otimes \mathcal{P}(n) \longrightarrow \mathcal{P}(m+n-1)$$

and a unit $1 \in \mathcal{P}(1)$ satisfying the axioms above.

It is not hard to see (but must be checked at least once) that an operad with $\mathcal{P}(n) = 0$ for $n \neq 1$ is exactly the same as an associative algebra.

Warning! If one does not require the existence of a unit, the notion of a *pseudo-operad* by Markl (defined by partial compositions) does not coincide with the notion of an operad as defined by May.

2.2 Constructing operads by hand

One can define operads in various ways. For example, one can define the underlying collection explicitly, and give the composition maps directly:

(1) *Commutative operad.* The reduced symmetric topological (or set) operad with $\mathsf{Com}(n)$ a single point for each $n \in \mathbb{N}$, and composition maps the unique map from a point to a point.

(2) *Associative operad.* The reduced set operad with $\mathsf{As}(n) = S_n$ the regular representation and composition maps

$$S_k \times S_{n_1} \times \cdots \times S_{n_k} \longrightarrow S_{n_1+\cdots n_k}$$

the unique equivariant map that sends the tuple of identities to the identity.

(3) *Stasheff operad.* Let K_{n+2} be the subset of I^n (the product of n copies of $I = [0,1]$) consisting of tuples $(t_1,\ldots,t_{n+2})$ such that $t_1\cdots t_k \leqslant 2^{-k}$ for $j \in [n+2]$. The boundary of K_{n+2} consists of those points such that for some $j \in [n+2]$ we have either t_j or $t_1\cdots t_j = 2^{-j}$. It is tedious (but otherwise doable) to show that for each pair (r,s) of natural numbers and each $i \in [r]$ there exists an inclusion

$$\circ_i : K_{r+1} \times K_{s+1} \longrightarrow K_{r+s+1}$$

that defines on the sequence of spaces $\{K_{n+2}\}_{n\geqslant 0}$ the structure of a non-symmetric operad. We will see in the exercise a realization of K_n as the convex hull of points with positive integer coordinates (due to J.-L. Loday) using planar binary rooted trees, which will make the operad structure more transparent.

(4) If M is a monoid, there is an operad $\mathbb{W}_M$ with $\mathbb{W}_M(n) = M^n$ such that

$$(m_1,\ldots,m_s)\circ_i(m_1',\ldots,m_t') = (m_1,\ldots,m_{i-1},m_im_1',\ldots,m_im_t',m_{i+1},\ldots,m_s).$$

We call it the *word operad of M*. Its underlying symmetric collection is $\mathsf{As}\circ M$.

(5) Write $\mathrm{Aff}(\mathbb{C}) = \mathbb{C}\times\mathbb{C}^\times$ for the group of affine transformations of $\mathbb{C}$ with group law $(z,\lambda)(w,\mu) = (z+\lambda w,\lambda\mu)$. In turn, define for each finite set I the topological space

$$\mathcal{C}(I) = \{(z_i,\lambda_i) \in \mathrm{Aff}(\mathbb{C})^I : |z_i - z_j| > |\lambda_i| + |\lambda_j|\}.$$

The group law of $\mathrm{Aff}(\mathbb{C})$ allows us to define an operad structure on $\mathcal{C}(I)$ using the exact same definition as in the word operad of a monoid. The subspaces $\mathcal{D}_2^{\mathrm{fr}}(I) \subseteq \mathcal{C}(I)$ where $|z_i| + |\lambda_i| \leqslant 1$ for all $i \in I$, and where the inequality is strict unless $z_i = 0$ is called the *framed little disks operad*. The little disks operad is the suboperad where $\lambda_i = 1$ for all $i \in I$, and we write it $\mathcal{D}_2(I)$.

(6) The operad of rooted trees RT has $\mathsf{RT}(n)$ the collection of rooted threes with n vertices labeled by $[n]$, and the composition $T \circ_j T'$ is obtained by inserting T' at the jth vertex of T and reattaching the children of that vertex to T' in all possible ways. For example, if then we have that

2.3 Exercises

The corresponding exercises to this lecture appear in **Exercise Sheet #1**. During the exercise sessions, we will review pseudo-operads and their partial insertions, and the category of operads.

3 Free operads and presentations

Goals. We will define algebraic operads by generators and relations, and with this at hand define quadratic and quadratic-linear presentations of operads.

3.1 Planar and non-planar trees

Operads and their kin are gadgets modeled after combinatorial graph-like objects. Operads themselves are modeled after rooted trees, so it is a good idea to have a concrete definition of what a rooted tree is. We will also consider planar rooted trees, and trees with certain decorations, so it is a good idea to digest the definitions carefully to later embellish them.

A rooted tree τ is the datum of a finite set $V(\tau)$ of vertices along with a partition $V(\tau) = \mathrm{Int}(\tau) \sqcup L(\tau) \cup R(\tau)$, where the first are the *interior* vertices, L are the leaves, and $R(\tau)$ is a singleton, called the root of τ. We also require there is a function $p : V(\tau) \smallsetminus R(\tau) \longrightarrow V(\tau)$, describing the edges of τ, with the following properties: call a vertex $v \in V(\tau)$ a child of $w \in V(\tau)$ if $v \in p^{-1}(w)$. Then:

(1) The root $r \in R(\tau)$ has exactly one child.
(2) The leaves of τ have no children.
(3) For each non-root vertex v there exist a unique sequence $(v_0, v_1, \ldots, v_k)$ such that $p(v_{i-1}) = v_i$ for $i \in [k]$ with $v_0 = v$ and $v_k = r$.

We will call a non-leaf vertex that has no children a *stump* (or an endpoint, or a cherry-top). A tree is reduced if has no stumps and all of its non-root and non-leaf vertices have at least two children. We will also call the root the (unique) output vertex τ, and the leaves the input vertices of τ.

A planar rooted tree is a rooted tree τ along with a linear order in each of the fibers of the parent function p of τ. In short, the children of each vertex are linearly ordered, so we are effectively considering a drawing of τ in the plane, where the clockwise orientation gives us the order at each vertex.

Two rooted trees τ and τ' are isomorphic if there exists a bijection $f : V(\tau) \longrightarrow V(\tau')$ that preserves the root, the input vertices and the interior vertices, so that $p' \circ f = p$ where we also write f for the induced bijection $f : V(\tau) \smallsetminus r \longrightarrow V(\tau') \smallsetminus r'$. Two planar rooted trees are isomorphic if in addition f respects the linear order at each vertex.

For example, consider the rooted tree τ with $V = \{1,2,3\} \cup \{4,5\} \cup \{0\}$, that is, three leaves, two interior vertices and the root. Then the choice of $p : [5] \to [\![5]\!]$ with $p(\{1,2\}) = 4$, $p(\{3,4\}) = 5$, $p(5) = 0$ gives a tree isomorphic to the one with with $p(\{1,2\}) = 3$,

$p(\{3,4\}) = 5$, $p(5) = 0$. On the other hand, if we consider the vertices linearly ordered by their natural order, these two planar rooted trees are no longer isomorphic.

Definition 3.1 For a finite set I, an I-labeled tree T is a pair (τ, f) where τ is a reduced rooted tree, along with a bijection $f : I \longrightarrow L(\tau)$. Two I-labeled trees T an T' are isomorphic if there exists a pair (g, σ) where g is an isomorphism between τ and τ' and σ is an automorphism of I such that $g\,|_{L(\tau)} \circ f = \sigma \circ f'$.

Suppose that (τ, f) is an I-tree and that (τ', f') is a J-tree, and that $i \in I$. We define $K = I \cup_i J = I \sqcup J \smallsetminus i$ and the K-tree $\tau \circ_i \tau'$ as follows:

(1) Its leaves are $L(\tau \circ_i \tau') = L(\tau) \sqcup L(\tau') \smallsetminus f^{-1}(i)$.
(2) Its internal vertices are $V(\tau) \sqcup V(\tau')$, with root r.
(3) The parent function q is defined by declaring that:
   - q coincides with p on $V(\tau)$,
   - $q(w) = p(f^{-1}(i))$ if w is the unique children of the root of τ',
   - q coincides with p' on $V(\tau') \smallsetminus \{r', w\}$.

(4) The leaf labeling is the unique bijection $L(\tau \circ_i \tau') \longrightarrow I \circ_i J$ extending f and f'.

Let us now consider an (unbiased) reduced symmetric sequence $\mathcal{X}$ which we will think of as an *alphabet*. An tree monomial in the alphabet $\mathcal{X}$ is a pair (τ, x) where τ is a reduced rooted tree and $x : \mathrm{Int}(\tau) \longrightarrow \mathcal{X}$ is a map with the property that $x(v) \in \mathcal{X}(p^{-1}(v))$. Observe that reduced sequences and reduced trees correspond to each other, in the sense that with this definition we can only decorate a stump of τ with an element of $\mathcal{X}(\varnothing)$.

An I-labeled $\mathcal{X}$-tree T is a triple (τ, x, f) where (τ, f) is I-labeled and (τ, x) is an $\mathcal{X}$-tree. We will say that (τ, x, f) is a (symmetric) tree monomial if $\mathcal{X}$ is symmetric. If it is just a collection, we will say that (τ, x, f) is a ns tree monomial. In particular, if T is an I-labeled tree, and if $\sigma \in \mathrm{Aut}(I)$, there is another I-labeled tree $\sigma(T) = (\tau, f\sigma^{-1})$.

Suppose that $T = (\tau, x, f)$ is a tree monomial on an alphabet $\mathcal{X}$, and let us pick a vertex v of τ and a permutation σ of the set $C = p^{-1}(v)$ of children of v. We define the tree τ^σ as follows: the datum defining τ remains unchanged except p is modified to p^σ so that

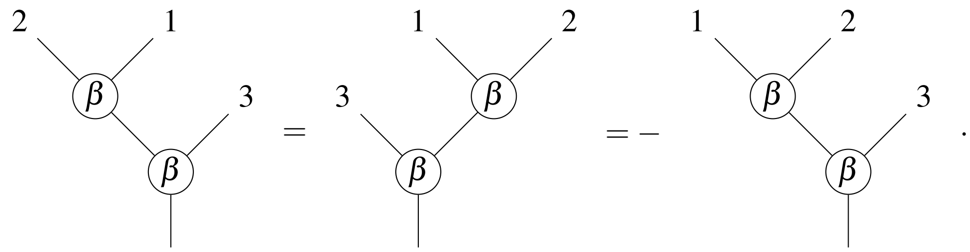
$$p^\sigma(w) = \begin{cases} p(w) & \text{if } p^2(w) \neq v \\ p(\sigma^{-1}(w')) & \text{if } p(w) = w' \in C. \end{cases}$$

Briefly, we just relabel the vertices of τ using σ. With this at hand, we define T^σ to be the tree monomial with underlying tree τ^σ and with x modified to x^σ so that

$$x^\sigma(w) = \begin{cases} \sigma x(v) & \text{if } v = w, \\ x(\sigma^{-1}(w')) & \text{if } p(w) = w' \in C. \end{cases}$$

Note that it is possible some children of v are leaves, in which case the definitions make sense if we think of leaves as decorated by the unit of $\Bbbk$.

Example 3.2 Let us consider the alphabet $\mathcal{X} = \mathcal{X}(2) = \{\beta\}$ where the unique operation is antisymmetric. Then we have the following equalities of symmetric tree monomials:



Let us now define for each $n \geqslant 1$ the space $\mathcal{F}_\mathcal{X}(I)$ as the span of all tree monomials $T = (\tau, f, x)$ on $\mathcal{X}$ with leaves labeled by I, modulo the subspace generated by all elements of the form

$$R(T, v, \sigma) = T - T^\sigma$$

where σ ranges through $\mathrm{Aut}(p^{-1}(v))$ as v ranges through the vertices of τ. In case all children of v are leaves, this is saying that the tree where x_v is replace by $\sigma(x_v)$ is equal to the tree where the leaves of T that are children of v are relabeled according to σ. We also require that tree decorations behave like tensors, so that $T = T_1 + T_2$ if the decoration of T at a vertex v is of the form $x_1 + x_2$ and for $i \in [2]$ the tree T_i coincides with T except that it is decorated by x_i at v.

3.2 The free operad

An algebraically inclined way to construct (algebraic) operads is through generators and relations. There is a forgetful functor from the category of operads to the category of collections. In general, it admits a left adjoint, which is the free operad functor.

Definition 3.3 The *free symmetric operad* on $\mathcal{X}$ is the symmetric sequence $\mathcal{F}_\mathcal{X}$ along with the composition law obtained by grafting of trees. More precisely, suppose that $T \in \mathcal{F}_\mathcal{X}(I)$

and that $T' \in \mathcal{F}_{\mathcal{X}}(J)$, and that $i \in I$. We define $T'' = T \circ_i T' \in \mathcal{F}_{\mathcal{X}}(I \cup_I J)$ by taking its underlying labeled tree to be $\tau \circ_i \tau'$, and by decorating it in the unique way which extends the decorations of T and T'.

The following lemma shows that this indeed defines an operad.

Lemma 3.4 *Tree grafting respects both I-tree isomorphisms and the relations $T \sim T^{\sigma}$ above, and hence is well defined on $\mathcal{F}_{\mathcal{X}}$.*

Proof. This is in **Exercise Sheet #2**. □

We will later interpret $\mathcal{X} \longmapsto \mathcal{F}_{\mathcal{X}}$ as a *monad*, thus giving another definition of operads. The advantage of this 'monadic approach' is its flexibility, which allow us to define other operad like structures, like the ones mentioned in the introduction. In this direction, a curious reader can consider the following equivalent definition:

Definition 3.5 The free operad generated by a symmetric collection X is defined inductively by letting $\mathcal{F}_{0,X} = \Bbbk$ be spanned by the 'twig' (tree with no vertices and one edge) in arity zero and

$$\mathcal{F}_{n+1,\mathcal{X}} = \Bbbk \oplus (\mathcal{X} \circ \mathcal{F}_{n,\mathcal{X}}),$$

and finally by setting $\mathcal{F}_{\mathcal{X}} = \varinjlim_n \mathcal{F}_{n+1,\mathcal{X}}$. The composition maps are defined by induction, and the axioms are also checked by induction.

Intuitively, the previous definition says that an element of $\mathcal{F}_{\mathcal{X}}$ is either the twig, or corolla with n vertices decorated by $\mathcal{X}$, whose leaves have on them an element of $\mathcal{F}_{\mathcal{X}}$. The final shape of $\mathcal{F}_{\mathcal{X}}$ will however depend on the symmetric structure of $\mathcal{X}$.

3.3 Exercises

The corresponding exercises to this lecture appear in **Exercise Sheet #2**.

4 Quadratic operads

Goal. Introduce weight graded gadgets, define operads by generators and relations, and introduce quadratic operads. Give plenty of examples of 'real life' quadratic operads to work on: Hilbert series, Koszul dual, bar construction.

4.1 Weight gradings and presentations

The notion of a quadratic operad is based on the observation every free operad has a canonical 'weight grading' by the number of internal vertices of a tree. Let us make this precise.

Definition 4.1 A symmetric sequence $\mathcal{X}$ is weight graded if for each finite set the component $\mathcal{X}(I)$ admits a decomposition $\mathcal{X}(I) = \bigoplus_{j\geqslant 0} \mathcal{X}^{(j)}(I)$. A symmetric operad $\mathcal{P}$ is weight graded if its underlying symmetric sequence is weight graded and its composition maps preserve the weight grading.

Thus, a weight graded operad must have composition maps of the form

$$\mathcal{P}^{(a)}(k) \otimes \mathcal{P}^{(b_1)}(n_1) \otimes \cdots \otimes \mathcal{P}^{(b_k)}(n_k) \longrightarrow \mathcal{P}^{(b)}(n)$$

where $b = b_1 + \cdots + b_k$ and $n = n_1 + \cdots + n_k$. In the case we consider partial composition maps, observe we have instead maps of the form

$$\circ_i : \mathcal{P}^{(a)}(m) \otimes \mathcal{P}^{(b)}(n) \longrightarrow \mathcal{P}^{(a+b)}(m+n-1).$$

The free operad $\mathcal{F}_{\mathcal{X}}$ is weight graded by the number of internal vertices of a tree (that is, we put $\mathcal{X}$ in weight one, and extend the weight to trees by counting occurrences of elements of $\mathcal{X}$. More generally, if $\mathcal{X}$ admits a weight grading, then $\mathcal{F}_{\mathcal{X}}$ inherits this weight grading: the weight of a tree monomial is the sum of the weight of the decorations of its vertices, and we write $\mathcal{F}_{\mathcal{X}}^{(n)}$ for the homogeneous component of weight $n \in \mathbb{N}_0$. If we do not specify a weight grading on $\mathcal{F}_{\mathcal{X}}$, we will always assume we are taking the canonical weight grading above.

Definition 4.2 An ideal in an operad $\mathcal{P}$ is a subcollection $\mathcal{I}$ for which both $\gamma(\mathcal{I} \circ \mathcal{P})$ and $\gamma(\mathcal{P} \circ_{(1)} \mathcal{I})$ are contained in $\mathcal{I}$. The quotient of $\mathcal{P}/\mathcal{I}$ is again an operad, called the quotient of $\mathcal{P}$ by $\mathcal{I}$. Every subcollection $\mathcal{R}$ of $\mathcal{P}$ is contained in a smallest ideal, called the *ideal generated by* $\mathcal{R}$.

The notion of ideals and of free operads allow us to define operads by generators and relations.

Definition 4.3 We write $\mathcal{F}(\mathcal{X},\mathcal{R})$ for the quotient of $\mathcal{F}_{\mathcal{X}}$ by the ideal generated by a subcollection $\mathcal{R}$ of $\mathcal{F}_{\mathcal{X}}$. We say $\mathcal{P}$ is presented by generators $\mathcal{X}$ and relations $\mathcal{R}$ if there is an isomorphism $\mathcal{F}(\mathcal{X},\mathcal{R}) \longrightarrow \mathcal{P}$.

Note that if $\mathcal{P}$ is symmetric, the definition requires that $\mathcal{I}$ be stable under the symmetric group actions, so we may sometimes specify $\mathcal{R}$ by a generating set only, and understand that $(\mathcal{R})$ is generated by the Σ-orbit of $\mathcal{R}$.

The classics. The associative operad is generated by a binary operation μ generating the regular representation of S_2 subject to the only relation

$$\mu \circ_1 \mu = \mu \circ_2 \mu.$$

Similarly, the commutative operad is generated by a binary operation which instead generates the trivial representation of S_2. Both of these examples arise as the linearization of a set operad.

The Lie operad is generated by a single binary operation β that generates the sign representation of S_2 subject to the only relation

$$(\beta \circ_1 \beta)(1+\tau+\tau^2) = 0$$

where $\tau = (123) \in S_3$ is the 3-cycle. We write these operads $\mathsf{As}, \mathsf{Com}$ and Lie and, following J.-L. Loday, call them the *three graces*. We have that

$$\mathsf{As}(n) = \Bbbk S_n, \quad \mathsf{Com}(n) = \Bbbk, \quad \mathsf{Lie}(n) = \mathrm{Ind}_{\mathbb{Z}/n}^{S_n} \Bbbk_\zeta$$

where $\Bbbk_\zeta$ is a character of $\mathbb{Z}/n$ for a primitive nth root of the unit.

Note 4.4 It is not always advantageous to define an operad by generators and relations: the operad pre-Lie can be defined explicitly in terms of labeled rooted trees and a grafting operation, as done by Chapoton–Livernet, and this ‘presentation’ is very useful in practice, for example, to show that the pre-Lie operad is Koszul.

4.2 Quadratic operads

An operad $\mathcal{P}$ is *quadratic* if it admits a presentation $\mathcal{F}(\mathcal{X},\mathcal{R})$ where $\mathcal{R} \subseteq \mathcal{F}(\mathcal{X})^{(2)}$. That is, $\mathcal{P}$ is generated by some collection of operations $\mathcal{X}$ and all its defining relations are of the form

$$\sum \lambda_{\mu,\nu}^i \mu \circ_i \nu = 0$$

where $\mathrm{ar}(\mu)+\mathrm{ar}(\nu)$ is constant. An operad is *binary quadratic* if moreover $\mathcal{X}=\mathcal{X}(2)$ or, what is the same, all the generating operations of $\mathcal{P}$ are of arity two (binary). A *quadratic-linear presentation* of an operad $\mathcal{P}$ is a presentation $\mathcal{F}(\mathcal{X},\mathcal{R})$ of $\mathcal{P}$ where $\mathcal{R}\subseteq\mathcal{X}\oplus\mathcal{F}(\mathcal{X})^{(2)}$. That is, it is a presentation of the form

$$\sum\lambda^i_{\mu,\nu}\mu\circ_i\nu+\sum\lambda_\rho\rho=0$$

where $\mathrm{ar}(\mu)+\mathrm{ar}(\nu)=\mathrm{ar}(\rho)+1$ is constant. Every operad admits a quadratic-linear presentation, albeit with possibly with infinitely many generators, We will postpone the discussion of such presentations to a later lecture.

Let us define a quadratic datum to be a pair $(\mathcal{X},\mathcal{R})$ where $\mathcal{X}$ is a symmetric sequence and $\mathcal{R}\subseteq\mathcal{F}^{(2)}_{\mathcal{X}}$. A map of quadratic data $(\mathcal{X}_1,\mathcal{R}_1)\longrightarrow(\mathcal{X}_2,\mathcal{R}_2)$ is a map $\mathcal{X}_1\to\mathcal{X}_2$ of symmetric sequences for which the induced map on free operads sends $\mathcal{R}_1$ to $\mathcal{R}_2$. The assignment $(\mathcal{X},\mathcal{R})\longrightarrow\mathcal{F}(\mathcal{X},\mathcal{R})$ defines a functor from the category of quadratic data to the category of quadratic operads.

More examples. The presentations of the associative, commutative and Lie operad above are quadratic. The following are also quadratic operads:

The Gerstenhaber operad. The symmetric operad Ger and its cousin, the Poisson operad Poiss belong to the two parameter family $\mathsf{Poiss}(a,b)$ of binary quadratic operads generated by two operations x_1x_2 and $[x_1,x_2]$ of respective degrees a and b, so that the first is commutative associative, the second is a Lie bracket, and they satisfy the Leibniz rule. With this at hand $\mathsf{Ger}=\mathsf{Poiss}(0,-1)$ while $\mathsf{Poiss}=\mathsf{Poiss}(0,0)$.

The pre-Lie operad. The operad PreLie and its quotient, the Novikov operad Nov, are quadratic binary operads generated by a single operation $x_1\circ x_2$ with no symmetries. The first one is subject to the right-symmetry condition for the associator

$$x_1\circ(x_2\circ x_3)-(x_1\circ x_2)\circ x_3=x_1\circ(x_3\circ x_2)-(x_1\circ x_3)\circ x_2.$$

The second operad is obtained by further imposing the left-permutative relation that

$$x_1\circ(x_2\circ x_3)=x_2\circ(x_1\circ x_3).$$

The permutative operad Perm is the binary operad generated by a single operation with no symmetries satisfying the last quadratic equation.

The operad of totally associative k-ary algebras. tAs_k (and its commutative counterpart). It is generated by a k-ary non-symmetric operation α subject to the relations $\alpha\circ_i\alpha=\alpha\circ_k\alpha$ for all $i\in[k]$. One can consider α to be totally symmetric, and obtain the operad of totally associative commutative k-ary algebras.

The operad of partially associative k-ary algebras. pAs^k (and its Lie counterpart). It is generated by a k-ary non-symmetric operation α of degree $k-2$ subject to the single relation

$$\sum_{i=1}^{k}(-1)^{(k-1)(i-1)}\alpha\circ_i\alpha = 0.$$

One can consider a k-ary totally antisymmetric operation β of degree 1, and obtain the operad of Lie k-algebras, which is subject to the single equation

$$\sum_{\substack{A\sqcup B=[2k-3]\\ |A|=k-1,|B|=k-2}}(\beta\circ_1\beta)\sigma_{A,B} = 0.$$

The operad of anti-associative algebras. As^- is generated by a single operation of degree zero with no symmetries satisfying the 'anti-associative law'

$$x_1(x_2x_3)+(x_1x_2)x_3 = 0.$$

4.3 Exercises

The corresponding exercises to this lecture appear in **Exercise Sheet #3**.

5 Koszul duality I

Goals. Give the definition of the Koszul dual operad of a quadratic operad, and then compute some Koszul duals. Give the definition of the Koszul complexes associated to a quadratic operad, and define Koszul operads.

5.1 Differential graded sequences

Homologically graded Σ-modules. A (homologically) graded vector space is a vector space V along with a direct sum decomposition $V = \bigoplus_{n\in\mathbb{Z}} V_n$. We call the components of this sum the *graded (or homogeneous) components of* V, and say that an element in on of these summands is *homogeneous*. If $v \in V_n$, we say that v is *homogeneous of degree n* and write $|v| = n$.

A map $f : V \longrightarrow W$ of graded vector spaces is *homogeneous of degree n* if $f(V_j) \subseteq W_{j+n}$ for all $j \geqslant 1$. We write $\hom(V,W)$ for the space of all homogeneous maps, which is itself a graded vector space with $\hom(V,W)_n$ the space of all graded maps of degree n for each $n \in \mathbb{Z}$. In this way, we obtain the category $\mathsf{Vect}_{\mathbb{Z}}$ of graded vector spaces and graded maps.

A *differential graded (dg) vector space* is a pair (V,d) where V is a graded vector space and $d : V \longrightarrow V$ is a homogeneous map of degree -1 such that $d^2 = 0$. We usually will call (V,d) a *chain complex*. The collection of homogeneous maps $V \longrightarrow W$ is again a chain complex, with differential

$$d\varphi = d_V\varphi - (-1)^{|\varphi|}\varphi d_W.$$

A homogeneous map of degree zero such that $d(\varphi) = 0$ is called a *chain map*. It is convenient to also consider *cohomologically graded* vector spaces, by formally inverting the order of $\mathbb{Z}$ and letting $V^n = V_{-n}$ for all $n \in \mathbb{Z}$.

Monoidal structure. If V and W are graded vector spaces, we define their tensor product by setting

$$(V\otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$$

for all $n \in \mathbb{Z}$, and setting the symmetry map

$$\tau : V\otimes W \longrightarrow W \otimes V$$

to be $\tau(v\otimes w) = (-1)^{|v||w|} w\otimes v$ on homogeneous elements, and extending it linearly on all of $V\otimes W$. This makes $\mathsf{Vect}_{\mathbb{Z}}$ into a symmetric monoidal category with unit the graded vector space with $V_0 = \Bbbk$ and $V_n = 0$ for $n \neq 0$. The tensor product of maps $f : V \longrightarrow V'$ and $g : W \longrightarrow W'$ acts in such a way that $f \otimes g : V \otimes V' \longrightarrow W \otimes W'$ is the map

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

In case V and W are in fact dg, their tensor product is also dg with $d_{V \otimes W} = d_V \otimes 1 + 1 \otimes d_W$.

Definition 5.1 A (homologically) graded Σ-module $\mathcal{X}$ is a Σ-module taking values in the category of graded vector spaces. Similarly, a dg Σ-module is one taking values in dg vector spaces.

The endomorphism operad functor on dg modules. Let us consider the most natural way to create dg modules from dg vector spaces, as we did in the case of usual vector spaces. Namely, we may as before consider the *endomorphism operad* of a dg vector space V by setting, for each $n \geqslant 0$,

$$\mathrm{End}_V(n) = \hom(V^{\otimes n}, V)$$

where these consists of homogeneous maps of dg vector spaces. In particular, each of these arity components is itself a dg vector space, and the (total or partial) composition maps of the resulting operad are maps of dg vector spaces.

Of particular importance to us will be the *suspension* operation on dg vector spaces. Let us write s for the unique dg vector space with $s_1 = \mathbb{C}$ and zero elsewhere, and similarly let us write s^{-1} for the unique dg vector space with $s^{-1} = \mathbb{C}$ and zero elsewhere. The *suspension* of the dg vector space V is the tensor product $s \otimes V$, which we write more simply sV, and whose basis elements we write sv for $v \in V$. Thus $|sv| = |v| + 1$ for all homogeneous $v \in V$. Similarly, we define the *desuspension* $s^{-1}V$.

Note 5.2 The differential of sV is given by $d(sv) = -sdv$. Can you explain why this is so using the Koszul sign rule?

The following lemma shows that $V \mapsto \mathrm{End}_V$ is monoidal for the *Hadamard product* of operads on the target (and the usual tensor product on the domain):

Lemma 5.3 *The map* $\Phi : \mathrm{End}_V \otimes \mathrm{End}_W \longrightarrow \mathrm{End}_{V \otimes W}$ *that assigns* $\varphi \otimes \psi \in \mathrm{End}_V(n) \otimes \mathrm{End}_W(n)$ *to the map*

$$\Phi(\varphi, \psi)(v, w) = (-1)^{\varepsilon} \varphi(v) \otimes \psi(w)$$

where $\varepsilon = \sum_{i=1}^{n} (|w_1| + \cdots + |w_{i-1}| + |\psi|)|v_i|$ *is an isomorphism of operads.*

Proof. This is in Exercise Sheet #4. $\square$

In particular, we see that End_{sV} is canonically isomorphic with $\mathrm{End}_s \otimes \mathrm{End}_V$, and hence that algebra structures on sV are related to algebra structures on V through the operad End_s. Let us give it a name.

5.2 The Koszul dual

Suspensions. We call End_s the suspension operad and write it $\mathscr{S}$. Note that $\mathrm{End}_s(n)$ is the sign representation of Σ_n put in degree $1-n$.

Proposition 5.4 *For each $n \geqslant 1$ let us we write ν_n for the unique map in $\mathrm{End}_s(n)$ that sends s^n to s. Then for every $m \geqslant 1$ we have that*

$$\nu_n \circ_i \nu_m = (-1)^{(i-1)(m-1)} \nu_{m+n-1}.$$

In particular, the binary operation $\nu := nu_2$ of degree -1 generates End_s, and presents it as a quadratic operad subject to the anti-associativity relation

$$\nu \circ_1 \nu + \nu \circ_2 \nu = 0.$$

Proof. This is in Exercise Sheet #4. □

If $\mathcal{P}$ is an operad, then the arity-wise tensor product $\mathscr{S} \otimes \mathcal{P}$ is called the suspension of $\mathcal{P}$ and we write it $\mathscr{S}\mathcal{P}$ or $\mathcal{P}\{1\}$. Dually, we write $\mathscr{S}^{-1}$ for the desuspension operad defined by $\mathrm{End}_{s^{-1}\Bbbk}$.

Note 5.5 As we just observed, the operad $\mathscr{S}\mathcal{P}$ has the property that $\mathscr{S}\mathcal{P}(sV) = s\mathcal{P}(V)$, so that algebras over $\mathscr{S}\mathcal{P}$ are exactly those vector spaces V such that $s^{-1}V$ is a $\mathcal{P}$-algebra. Equivalently, sV is a $\mathscr{S}\mathcal{P}$-algebra if and only if V is a $\mathcal{P}$-algebra.

Pairings. We define a pairing between $\mathcal{F}_{\mathcal{X}}$ and $\mathcal{F}_{s^{-1}\mathscr{S}^{-1}\mathcal{X}^*}$ as follows (the appearance of the suspensions will be evident later):

$$\langle \Sigma\nu^* \circ_j \Sigma\mu*, \rho \circ_i \tau \rangle = \delta_{ij}(-1)^{\varepsilon}\nu^*(\rho)\mu^*(\tau).$$

where $\varepsilon_1 = (\mathrm{ar}(\nu)-1)(|\mu|+i-1)+|\nu||\mu|$ and ε_2 counts the total number of inversions in the shuffle permutations appearing in the two tree monomials. If $\mathcal{X} = \mathcal{X}(2)$ is binary and has no homological degrees, this simplifies to

$$\langle \Sigma\nu^* \circ_i \Sigma\mu*, \rho \circ_i \tau \rangle = \begin{cases} (-1)^{\varepsilon}\nu^*(\rho)\mu^*(\tau) & i=1 \\ -\nu^*(\rho)\mu^*(\tau) & i=2. \end{cases}$$

where ε depends on the decoration of the leaves (it is 1 if both decorations are equal, and is -1 if exactly one is the shuffle 132.

Definition 5.6 The Koszul dual operad of a quadratic operad $\mathcal{P}$ generated by $\mathcal{X}$ subject to relations $\mathcal{R}$, is the operad $\mathcal{P}^!$ generated by $s^{-1}\mathscr{S}^{-1}\mathcal{X}^*$ and subject to the orthogonal space of relations $\mathcal{R}^\perp$ according to the pairing above.

Note 5.7 Let $\mathcal{P}$ be an operad. Then $\mathcal{P}$ is quadratic if and only if $\mathscr{S}\mathcal{P}$ is quadratic, and it is Koszul if and only if $\mathscr{S}\mathcal{P}$ is Koszul.

Some examples. Let us compute the Koszul duals of some of the quadratic operads we introduced in **Lecture 3**. For simplicity, we will consider only those with binary generators of degree zero, though one can in the same way carry out computations with generators of higher arities and varying homological degrees.

The associative operad. We saw previously that for $\underline{\mathcal{X}}$ consisting of a single operation x_1x_2 with no symmetries, the space $\mathcal{F}_{\mathcal{X}}(3)$ is twelve dimensional, spanned by the S_3-orbits of $\alpha = x_1(x_2x_3)$ and $\beta = (x_1x_2)x_3$, each of size six. We also noted that $\alpha - \beta$ spans a six dimensional submodule, complemented by the orbit of $\alpha+\beta$.
Using the pairing above, we see that

$$\langle \alpha,\alpha\rangle = 1, \quad \langle \beta,\beta\rangle = -1, \quad \langle \alpha,\beta\rangle = 0,$$

from where it follows that the dual space to the associativity relation is the corresponding associativity relation $\alpha^* - \beta^*$ in $\mathcal{X}^*$. In other words, the associative operad is Koszul self-dual:

$$\mathsf{Ass}^! = \mathsf{Ass}.$$

It is important to note how the minus sign in our definition of the pairing or, more generally, the Koszul sign we have introduced, guaranteeing that this pairing in equivariant, introduces the minus sign in the dual of $\alpha+\beta$.

The commutative and Lie operads. We have computed that if $\mathcal{X}(2)$ is the trivial representation of S_2 spanned by some commutative operation x_1x_2, then $\mathcal{F}_{\mathcal{X}}(3)$ is three dimensional, spanned by $x_1(x_2x_3)$, $(x_1x_2)x_3$ and $(x_1x_3)x_2$. Moreover, we verified that if we put

$$\alpha = x_1(x_2x_3) - (x_1x_2)x_3, \quad \beta = x_1(x_2x_3) - (x_1x_3)x_2$$

then these two element span an S_3-submodule that is complemented by the S_3-submodule generated by

$$\gamma = x_1(x_2x_3) + (x_1x_2)x_3 + (x_1x_3)x_2.$$

This is in fact an orthogonal complement as a direct computation shows, so we see that the orthogonal set of relations to the commutative associative relation is the dual of γ for the dual antisymmetric operation $[x_1, x_2]$: this is exactly the Jacobi relation

$$\gamma^* = -[x_1,[x_2,x_3]] + [[x_1,x_2],x_3] + [[x_1,x_3],x_2].$$

It follows that the Koszul dual of the commutative operad is the Lie operad, and conversely:

$$\mathsf{Com}^! = \mathsf{Lie}, \quad \mathsf{Lie}^! = \mathsf{Com}.$$

With this at hand, one can compute that the Poisson operad is self-dual: one only needs to address the Leibniz relation.

The pre-Lie and permutative operads. The Novikov operad. Recall the pre-Lie operad is generated by a single operation x_1x_2 with no symmetries, subject to the pre-Lie relation

$$(x_1x_2)x_3 - x_1(x_2x_3) - (x_1x_3)x_2 + x_1(x_3x_2).$$

One can check that the S_3-orbit V of this element is three dimensional, so let us write α_1, α_2 and α_3 for the translates of this relation in $\mathcal{F}_{\mathcal{X}}(3)$.

This orbit is complemented by the orbit W of the associativity relation $(x_1x_2)x_3 - x_1(x_2x_3)$ and the orbit U of the permutative relation $(x_1x_2)x_3 - (x_1x_3)x_2$. The first is six dimensional, as we already computed, while the second is three dimensional. It is a direct computation to check that $V^\perp$ identifies with the nine dimensional subspace $U^* \oplus W^*$.

Thus, we see that the operad of pre-Lie algebra is Koszul dual to that of permutative algebras:

$$\mathsf{PreLie}^! = \mathsf{Perm}, \quad \mathsf{Perm}^! = \mathsf{PreLie}.$$

One can use this to show that the operad controlling Novikov algebras, those pre-Lie algebras whose product is *left* permutative

$$x_1(x_2x_3) = x_2(x_1x_3)$$

is almost Koszul self-dual: we have that $\mathsf{Nov}^! = \mathsf{Nov}^{\mathrm{op}}$, by which we mean the resulting operad controls pre-Lie algebras with associator symmetric in the *first two* variables (left-symmetric) and whose pre-Lie operation is *right* permutative.

5.3 Exercises

The corresponding exercises to this lecture appear in **Exercise Sheet #4**.

6 Shuffle operads

Goal. Introduce shuffle operads and prove that the free symmetric operad on a reduced symmetric collection is isomorphic, as a shuffle operad, to the free shuffle operad on the corresponding shuffle collection.

6.1 Shuffle operads

Recall that the category of ns collections on some category C consists of those pre-sheaves on the category of finite ordered sets and order preserving bijections with values in C: a ns collection on C is simply a list of objects of C indexed by the non-negative integers (considered as totally ordered sets of finite cardinality).

Definition 6.1 An ordered partition π of length n of a finite totally order set set is called *shuffling* if $\min \pi_i < \min \pi_{i+1}$ for each $i \in [n-1]$. Equivalently, a surjection $f : I \longrightarrow [n]$ with I a totally ordered set is called *shuffling* if $\min f^{-1}(i) < \min f^{-1}(i+1)$ for each $i \in [n-1]$.

Although totally ordered sets along with bijections form a rather dull category, this category admits a composition product, which we call the *shuffle composition product*, defined as follows, and which will turn out to be crucial for our purposes.

Definition 6.2 For each pair of ns collections $\mathcal{X}$ and $\mathcal{Y}$, we define the ns collection $\mathcal{X} \circ_{\text{Ш}} \mathcal{Y}$ so that on each totally order finite set we have that

$$(\mathcal{X} \circ_{\text{Ш}} \mathcal{Y})(I) = \bigoplus_{\substack{r \geqslant 1 \\ f:I \longrightarrow [r]}} \mathcal{X}([r]) \otimes \mathcal{Y}(f^{-1}(1)) \otimes \cdots \otimes \mathcal{Y}(f^{-1}(r))$$

where the sum runs through all $r \geqslant 1$ and all possible shuffling surjections $f : I \longrightarrow [r]$.

One can prove that this product is associative, in the same way that one proves $\circ_\Sigma$ and $\circ_{\text{ns}}$ are. In some way, the shuffle composition product interpolates between the symmetric composition product, which contains "too many" summands, and the ns composition product, which contains too few. We leave the following proposition as an exercise.

Proposition 6.3 *The category of ns collections along with the shuffle composition product is monoidal with the same unit as that of the ns composition product.* □

Note that we can also define a shuffle Cauchy product, by looking at shuffling partitions of a finite order set that have length two. Although we will not study the resulting monoidal category here, we remark it gives rise to interesting monoids, usually known as shuffle algebras.

Definition 6.4 A shuffle operad is a monoid in the category of ns collections with the shuffle composition product.

Thus, a shuffle operad consists of the datum of a ns sequence $\mathcal{P}$ along with shuffle composition maps, one for each shuffle partition π of a finite ordered set I of the form

$$\gamma_\pi : \mathcal{P}(r) \otimes \mathcal{P}(\pi_1) \otimes \cdots \otimes \mathcal{P}(\pi_r) \longrightarrow \mathcal{P}(I)$$

that satisfy suitable associativity and unitality axioms. Precisely, let us pick a finite totally ordered set I, a shuffling partition π of I, and let us assume that we pick a shuffling partition $\pi^{(i)}$ of each block of π. There is a unique way to order the collection of blocks of these to obtain a shuffling partition π' of I. For each part π_i of π and each $(g_i; \vec{h}_i) \in \mathcal{P}(\pi_i) \otimes \mathcal{P}[\pi^{(i)}]$, let us write $f_i = \gamma_{\pi^{(i)}}(g_i; \vec{h}_i)$, and let $\vec{h}$ be obtained for the tuple $(\vec{h}_1, \ldots, \vec{h}_r)$ by reordering the entries according to π'. Then

$$\gamma_\pi(f; f_1, \ldots, f_r) = \gamma_{\pi'}(\gamma_\pi(f; g_1, \ldots, g_r); \vec{h}).$$

Moreover, for each finite set I, if $\{I\}$ and I denote the corresponding partitions into one block and into singletons, we a fixed $1 \in \mathcal{P}(1)$ such that for every $v \in \mathcal{P}(I)$ we have

$$\gamma_{\{I\}}(1; v) = v, \quad \gamma_I(v; 1, \ldots, 1) = v.$$

Naturally, one can consider partial compositions on a shuffle operad, but carefully noting that for each i, there exist many different shuffling partitions π of the form

$$(1, \ldots, i-1, A, j_1, \ldots, j_s)$$

where $\min(A) = i$. Namely, for each $[n]$ we need simply choose a subset A of $[n] \setminus [i-1]$ that contains i, and this can be done by choosing a subset of $[n] \setminus [i]$ and appending i.

Definition 6.5 An ideal of a shuffle operad $\mathcal{P}$ is a ns subcollection $\mathcal{I}$ such that

$$\gamma_\pi(v_0; v_1, \ldots, v_r) \in \mathcal{I}$$

if at least one of v_i is in $\mathcal{I}$ for some $i \in [0, r]$.

As we will see later, ideals of shuffle operads are slightly more refined than those in symmetric operads. For example, the ideal generated by the left comb $(x_1x_2)x_3$ in a symmetric operad automatically contains its two translates, while in a shuffle operad, the three ideals corresponding to these three possible shuffle tree monomials are different.

6.2 Free shuffle operad

Let us now give an explicit description of the free shuffle operad on a ns collection. Since we have already defined the free symmetric and non-symmetric operad on a collection (of the appropriate kind), we already have almost all the language necessary to define it.

Definition 6.6 Let τ be a planar tree, which we draw on the plane with the counter-clockwise orientation. Begin at the left side of root edge, and transverse the "boundary" of the tree in the counter-clockwise direction. This path will meet the vertices of τ in some order, and we call this total order the *canonical planar order* of its vertices.

Observe that this also orders the edges of τ, and the leaves (which are given the usual left-to-right planar order).

Now let $\mathcal{X}$ be a ns collection and let T be a planar tree monomial with variables in $\mathcal{X}$, and let us pick a bijective labelling $\mathsf{n} : L(\tau) \longrightarrow [n]$ of the leaves of τ. This induces a labelling of the vertices of τ inductively by inductively labelling v with the minimum label appearing among its set of children.

Definition 6.7 We say a leaf labelling of a planar tree monomial T is shuffling if the induced order on the children of each of its vertices coincides with the canonical planar order. A pair (T, n) where n is a shuffling leaf labelling is called a shuffle tree monomial.

We now define the ns collection $\mathrm{Tree}_{\mathcal{X}}^{\text{Ш}}$ so that for each finite totally ordered set I the set $\mathrm{Tree}_{\mathcal{X}}^{\text{Ш}}(I)$ consists of those shuffle tree monomials on $\mathcal{X}$ with shuffling labellings by I. We write $\mathcal{F}_{\mathcal{X}}^{\text{Ш}}$ for the corresponding linear ns collection.

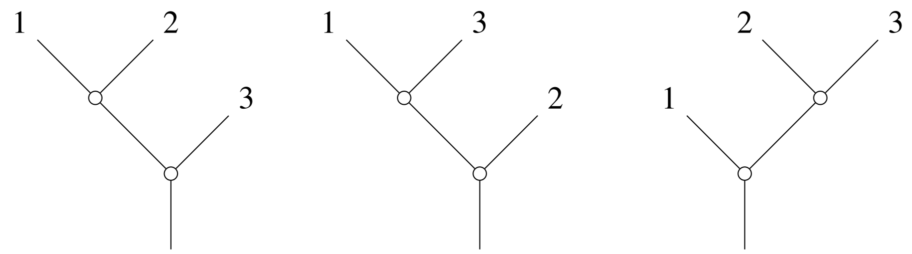


Figure 2: The three shuffle trees with three leaves on a binary generator.

Suppose that T and T' are shuffle tree monomials on $[n]$ and $[m]$, that $i \in [n]$ and that we pick a shuffling partition π of $[m+n-1]$ whose only non-singleton part is of the form

$$\{i = j_1, j_2, \ldots, j_m\}.$$

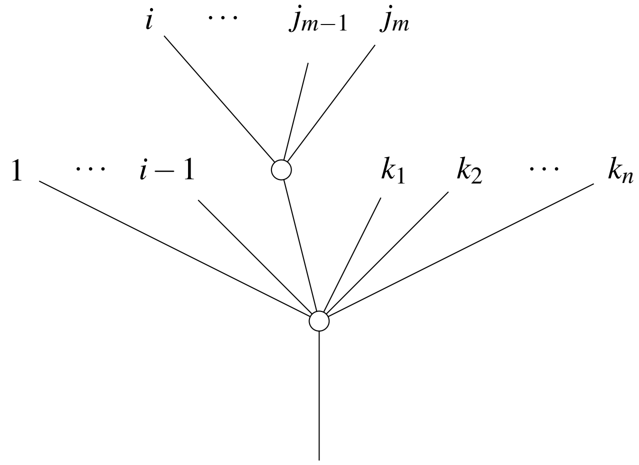


Figure 3: The two-level trees corresponding to partial compositions of shuffle operads

We define the tree monomial $T \circ_\pi T'$ by grating the tree T' at the leaf of T labelled by i, with its leaf labels renumbered through the unique order preserving bijection $j_i \longmapsto i$, and we renumber the leaf labels of T distinct from $1,\dots,i-1$ using the remaining blocks of π. This defines the "partial shuffle composition" of shuffle tree monomials.

We may as well define the "total shuffle composition" of a tree T_0 with trees $T_1,\dots,T_n$ along a shuffling partition $\pi = (\pi_1,\dots,\pi_n)$ with T_i having as many leafs as π_i for each $i \in [n]$ Concretely, we consider for each such i the unique order preserving bijection between π_i and the labels of T_i, and graft T_i at the input of T_0 labelled by $\min \pi_i$.

Proposition 6.8 *The shuffle composition of shuffle tree monomials is again a shuffle tree monomial.*

Proof. This is in **Exercise Sheet #5**. The idea is to note that the local increasing condition is not broken, and this is clear on each T_i since we simply relabelled their leafs with an isomorphic totally order set, while it is not broken on T_0 since we grafted the T_is using a shuffling partition. $\square$

With this at hand, we can state and prove the main result in this section.

Proposition 6.9 *The ns collection $\mathcal{F}^{\text{III}}_{\mathcal{X}}$ with its corresponding shuffle composition is the* free shuffle operad *generated by $\mathcal{X}$, where the inclusion $\mathcal{X} \longrightarrow \mathcal{F}^{\text{III}}_{\mathcal{X}}$ sends an element in $\mathcal{X}$ to the corresponding corolla with its unique shuffling leaf labelling.* $\square$

6.3 Forgetful functor

Since every finite totally order set I is in particular a finite set I^{f} after forgetting the order, we have a functor $\mathcal{X} \longmapsto \mathcal{X}^{\mathsf{f}}$ that assigns a symmetric collection $\mathcal{X}$ to the ns collection $\mathcal{X}^{\mathsf{f}}$ such that

$$\mathcal{X}^{\mathsf{f}}(I) = \mathcal{X}(I^{\mathsf{f}})$$

for each finite order set I. We call this the *forgetful functor* from symmetric to ns collections. The following will be central in what follows.

Proposition 6.10 *The forgetful functor* ${}_{\Sigma}\mathsf{Mod} \longrightarrow {}_{\mathrm{ns}}\mathsf{Mod}$ *is strong monoidal for the corresponding symmetric and shuffle composition products when restricted to* reduced *collections, in the sense that for each pair* $\mathcal{X}$ *and* $\mathcal{Y}$ *with* $\mathcal{Y}$ *reduced there is a natural isomorphism*

$$(\mathcal{X} \circ_{\Sigma} \mathcal{Y})^{\mathsf{f}} \longrightarrow \mathcal{X}^{\mathsf{f}} \circ_{\text{Ш}} \mathcal{Y}^{\mathsf{f}}.$$

Proof. Let us begin by proving that if $\mathcal{Y}$ is a reduced symmetric sequence then $\mathcal{Y}^{\otimes n}$ is a free S_n-module for every $n \geqslant 1$. This is of course true for $n = 1$. For $n > 1$, it suffices to exhibit an S_n-basis. For each finite totally ordered set I, let us consider the components of $\mathcal{Y}^{\otimes n}(I^{\mathsf{f}})$, and note that since $\mathcal{Y}$ is reduced they are of the form

$$\mathcal{Y}(\pi_1) \otimes \mathcal{Y}(\pi_n)$$

where π is a partition of I into n blocks with at least one element. For each such partition π of I, there exists a unique permutation $\sigma \in S_n$ such that $(\sigma\pi)_i = \pi_{\sigma^{-1}(i)}$ is shuffling, and this proves that $\mathcal{Y}^{\otimes n}(I^{\mathsf{f}})$ is isomorphic to the free S_n-module generated by $(\mathcal{Y}^{\mathsf{f}})^{\otimes_{\text{Ш}} n}(I)$. It follows that for each $n \geqslant 1$ we have a natural isomorphism

$$\mathcal{X}(n) \otimes_{S_n} \mathcal{Y}^{\otimes n}(I^{\mathsf{f}}) \longrightarrow \mathcal{X}^{\mathsf{f}}(n) \otimes (\mathcal{Y}^{\mathsf{f}})^{\otimes_{\text{Ш}} n}(I)$$

which gives us the desired isomorphism $(\mathcal{X} \circ_{\Sigma} \mathcal{Y})^{\mathsf{f}} \longrightarrow \mathcal{X}^{\mathsf{f}} \circ_{\text{Ш}} \mathcal{Y}^{\mathsf{f}}$. □

Corollary 6.11 *For each reduced symmetric collection* $\mathcal{X}$*, there is a natural isomorphism of shuffle operads*

$$(\mathcal{F}^{\Sigma}_{\mathcal{X}})^{\mathsf{f}} \longrightarrow \mathcal{F}^{\text{Ш}}_{\mathcal{X}^{\mathsf{f}}}.$$

Moreover, if I is an ideal in $\mathcal{F}^{\Sigma}_{\mathcal{X}}$ then I^{f} is an ideal in $\mathcal{F}^{\text{Ш}}_{\mathcal{X}^{\mathsf{f}}}$ and the resulting quotient shuffle operads are naturally isomorphic via the induced map

$$(\mathcal{F}^{\Sigma}_{\mathcal{X}}/I)^{\mathsf{f}} \longrightarrow \mathcal{F}^{\text{Ш}}_{\mathcal{X}^{\mathsf{f}}}/I^{\mathsf{f}}.$$

In particular, shuffle tree monomials on $\mathcal{X}^{\mathsf{f}}$, when considered with their non-planar tree structure, give us a basis of the free symmetric operad on $\mathcal{X}$, and we can study any presentation of a symmetric operad through the resulting presentation of the corresponding shuffle operad.

6.4 Exercises

7 Monomial orders

7.1 Some reminders

In the following, we will anchor ourselves in the rewriting theory that exists for associative monoids in sets and the corresponding theory for associative algebras. Since we are not assuming the reader is familiar with this, let us give a brief recollection of the basics.

Definition 7.1 An associative monoid is a set M along with an associative multiplication $\mu : M \times M \longrightarrow M$. Given a set X, we write $\langle X \rangle$ for the free monoid on X, which is given by the set $\bigsqcup_{n\geqslant 1} X^n$ of all *words the alphabet* X with product the isomorphism $X^n \times X^m \cong X^{m+n}$ for each $m,n \geqslant 1$.

We are interested in finding bases of free objects by ideals and, to do this, we will resort to ordering our free objects. This will allow us to give a (terminating) algorithm whose input will be a set of relations and an ordering, and whose output (among other things) will be a basis of our quotient object.

Definition 7.2 An ordered monoid is a pair $(M, \prec)$ where M is a monoid and $\prec$ is a total order on M that satisfies the following three conditions:

(1) It is a well-order: every non-empty subset of M has a minimum.
(2) The product map of M is increasing in both of its arguments for $\prec$.

A *monomial order* on the alphabet X is, by definition, the structure of an ordered monoid on $\langle X \rangle$.

Explicitly, the last condition requires that if $m_1, m_2, m_3 \in M$ and if $m_1 \prec m_2$ then it follows that $m_3m_1 \prec m_3m_2$ and $m_1m_3 \prec m_2m_3$. If the alphabet X is given a total order, then we can produce a monomial order on it as follows:

Definition 7.3 Let $\prec$ be a total order on X. The graded lexicographic order on $\langle X \rangle$ induced by $\prec$, which we write $\prec_\ell$, is such that $w \prec_\ell w'$ if and only if

(1) The word w is shorter than w', or else
(2) We have $w = w_1xw_2$ and $w' = w_1yw_2'$ with $x \prec y$ in X.

It is important to note that the lexicographic order defined only by the second condition is *not* a well-order, and it is not increasing for the concatenation product: for example, if $x \prec y$ then $x \prec x^2$ but $x^2y \prec xy$.

Lemma 7.4 *The graded lexicographic order is a monomial order on X for any choice total order $\prec$.*

Proof. It is clear that the resulting order is total, for either two words are of distinct length, or they are of the same length and differ at and entry, or else they are equal. To see the order behaves well with respect to the concatenation product, we observe that the function $w \longmapsto \mathrm{Length}(m)$ is additive for the concatenation product, so if w is longer than w', then ww'' will be longer than $w'w''$ and, similarly, $w''w$ will be longer than $w''w'$. If w and w' have the same length, then it is clear that $w''w' \prec w''w$ if and only if $w'w'' \prec ww''$ if and only if $w' \prec w$. To see that the order is a well order, let us consider a collection W of words. Then, in particular, there exists a least natural number n such that W contains words of length n but not of $n-1$. In this case, it follows that the minimum of W, if it exists, must be contained in the set X^n, and this set is well ordered by the lexicographical order if X it itself well ordered: we can find the minimum by induction on n. □

(1) Words operads.

(2) Associative operad.

7.2 Path sequences and leaf permutations

Part 2.

(1) Monomial orders on shuffle operads.

(2) Path-permutation extension.

(3) Graded path-permutation lexicographic order.

(4) Examples.

7.3 Exercises

A Algebras over operads