Little disks operads and configuration spaces

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Speaker's abstract. Operads are objects that govern categories of algebras. Initially introduced in the sixties to study iterated loop spaces, they have proved useful in several areas of mathematics. In most of these applications, the little disks operads play a central role. In the first part of this talk, we will focus on one of the applications studied in the 2015 Talbot Workshop, Goodwillie–Weiss embedding calculus, which will serve as an "excuse" to introduce operads. In the second part of this talk, I will set out some of the recent developments regarding the links between the little disks operads and the real homotopy types of configuration spaces of manifolds. (Second part based on joint works with Ricardo Campos, Julien Ducoulombier, Pascal Lambrechts, and Thomas Willwacher.)

Contents

1 The embedding calculus

The motivation to construct the embedding calculus on manifolds is the computation of the homotopy type of the space $\mathrm{Emb}(M,N)$ of all embeddings $f : M \longrightarrow N$ of one manifold into another. Recall that f is an *immersion* if its derivative $T_p f : T_p M \longrightarrow T_q N$ is injective at each $p \in M$, and it is a (smooth) *embedding* if it is an immersion and a topological embedding.

The computation of such space $\mathrm{Emb}(M,N)$ of embeddings is highly non-trivial: if $M = S^1$ and $N = S^3$, then the 0th homotopy group $\pi_0 \mathrm{Emb}(S^1,S^3)$ consists of all isotopy classes of smooth knots, and determining and studying these already spans a whole area of mathematics, knot theory.

The fact the previous example is slightly involved stems from the fact that the codimension of S^1 in S^3 is two: as soon as M has codimension at least three in N, then $\mathrm{Emb}(M,N)$ is a path connected space. For example, one can unknot any embedding $S^1 \longrightarrow S^4$. Nonetheless, higher homotopy and homology groups of $\mathrm{Emb}(M,N)$ are non-trivial, and give interesting and fine invariants of M and N that depend on their smooth structure of them, and not only on their homotopy type.

The immediate problem one encounters when trying to compute with the functor $\mathrm{Emb}(-,N)$ is that it not continuous (in the categorical sense): if M is a union of submanifolds $V \cup U$ then $\mathrm{Emb}(M,N)$ is *not* equal to the pullback of the cospan

$$\mathrm{Emb}(V,N) \longleftarrow \mathrm{Emb}(U\cap V,N) \longrightarrow \mathrm{Emb}(U,N). \tag{1}$$

Concretely, if one is able to build embeddings $V \hookrightarrow N$ and $U \hookrightarrow N$ that coincide in $V \cap U$, it is not always possible to extend them to an embedding defined on $M = V \cup U$: the resulting map will be an immersion, but may fail to be injective on $U \cap V$.

The idea of Goodwillie–Weiss calculus is to *approximate* the functor $F : M \longmapsto \mathrm{Emb}(M,N)$ by a tower of functors

$$\cdots \longrightarrow F_4 \longrightarrow F_3 \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \tag{2}$$

under F that are, in a precise sense, polynomial functors. As a first step, one can consider the space of *immersions* $M \longrightarrow N$, which is polynomial of order one or, what is the same, linear, in the sense that $\mathrm{Imm}(M,N)$ is the limit of the span

$$\mathrm{Imm}(V,N) \longleftarrow \mathrm{Imm}(U\cap V,N) \longrightarrow \mathrm{Imm}(U,N). \tag{3}$$

To define what it means for a functor to be "of order at most k" for some $k \in \mathbb{N}$ requires a bit more care, an involves the combinatorics of cubical posets. We point the reader to [?Calculus] for details. Informally, the cornerstone of the Goodwillie–Weiss calculus is the following result:

Theorem 1.1 *Let $F(V) = \mathrm{Emb}(V,N)$ and suppose that M has codimension at least three in N. There exists a tower of functors over F as in (2) such that the map*

$$F(M) \longrightarrow \mathrm{holim}_k F_k(M) \tag{4}$$

restricts to a homotopy equivalence of the base point components. Moreover, the (homotopy type of the?) functors $F_k(M)$ can be computed in some way using configurations of k points of M in N. □

We will in fact take a slightly different approach, and use little disks operads and configuration spaces of points to obtain a description of the homotopy type of $\mathrm{Emb}(M,N)$.

2 Configuration spaces of points

Not the original work of Goodwillie–Weiss but the result of a refinement of their work over the years. Span of maybe two decades.

We now aim to approximate $\mathrm{Emb}(M,N)$ using *configurations spaces*. For each $r \in \mathbb{N}$, we define the *ordered configuration space of r points in M* by

$$\mathrm{Conf}_M(r) = \{(x_1,\ldots,x_r) \in M^r :\ x_i \neq x_j \text{ for all } i \neq j\}. \tag{5}$$

They appeared initially in the study of braids and braid groups by [Name?], and later in the work of Fadell–Neuwirth and V. I. Arnol'd. Note that, more or less by definition, one has that the braid group B_r is equal to $\pi_1(\mathrm{Conf}_{D^2}(r))$.

The sequence of spaces $(\mathrm{Conf}_M(r))_{r\geqslant 0}$ forms a *symmetric sequence* in spaces, that is, it is a sequence of spaces with corresponding actions of the symmetric groups, and we will write it Conf_M. They allow us to embed $\mathrm{Emb}(M,N)$ into

$$\mathrm{Map}_\Sigma(\mathrm{Conf}_M,\mathrm{Conf}_N) = \prod_{r\geqslant 0} \hom_{\Sigma_r}(\mathrm{Conf}_M(r),\mathrm{Conf}_N(r)) \tag{6}$$

by assigning an embedding $f : M \longrightarrow N$ to the map Φ_f that assigns a configuration $(x_1,\ldots,x_r)$ in M to the configuration $(f(x_1),\ldots,f(x_r))$. There are corresponding constructions for unframed manifolds. The maps Φ_f coming from embeddings $f : M \longrightarrow N$ enjoy some additional compatibility properties:

(1) *Forgetting points:* the map Φ_f commutes with the maps induced by the forgetful maps $\pi_i : \mathrm{Conf}_M(r) \longrightarrow \mathrm{Conf}_M(r-1)$ that forget the ith point in the domain.
(2) *Continuity:* if a configuration $\vec{x_0}$ is close to a configuration $\vec{x_1}$ in M, then their images under Φ_f will be close as configurations in N.

We would like to relax these two conditions "up to homotopy". But what does this mean?

3 Operads and their modules

One can use operads to achieve this. Let us consider a richer version of configuration spaces, that give us some more wiggle room, by replacing points with disks. For each $r \in \mathbb{N}$, define

$$D_m(r) = \mathrm{Emb}_\square(D^m \sqcup \cdots \sqcup D^m, D^m) \tag{7}$$

the space of rectilinear[1] embeddings of r disjoint m-disks in another m-disk. Similarly, define

$$D_M(r) = \mathrm{Emb}(D^m \sqcup \cdots \sqcup D^m, M) \tag{8}$$

[1] We only allow for dilations and translations

and note that $\mathrm{Conf}_M(r)$ and $D_M(r)$ are homotopy equivalent spaces. The upshot is that the symmetric sequence D_m forms a (topological) operad. Namely, it comes equipped with composition maps

$$D_m(k) \times D_m(r_1) \times \cdots D_m(r_k) \longrightarrow D_m(r_1 + \cdots r_k) \tag{9}$$

that satisfy certain associativity and equivariance relations. More briefly, it forms a monoid in the monoidal category of symmetric sequences under the circle product

$$(X \circ Y)(n) = \bigsqcup_{k \geqslant 0} X(k) \times_{S_k} Y[\lambda], \tag{10}$$

where $Y[\lambda]$ is the S_k-module obtained as the disjoint union of all permutations of $Y(\lambda_1) \times \cdots \times Y(\lambda_k)$ for λ a partition of n. With this at hand, what we want is an equivariant associative map

$$D_m \circ D_m \longrightarrow D_m. \tag{11}$$

Moreover, the sequence D_M is a right D_m-module, in the sense there is a map $D_M \circ D_m \longrightarrow D_m$ that is compatible with the operad structure of D_m. *Note:* since $n \geqslant m$, there is an inclusion of operads $D_m \longrightarrow D_n$, and in particular the right D_n-module D_N can be viewed as a right D_m-module, which we will do in what follows.

We now observe that an embedding $f : M \longrightarrow N$ produces for us a map $D_M \longrightarrow D_N$ that is not just a map of symmetric sequences, but in fact a map right D_m-modules. In this way, it makes sense to form the mapping space $\mathrm{Map}_{D_m}(D_M, D_N)$ and we obtain a map

$$\mathrm{Emb}(M,N) \longrightarrow \mathrm{Map}_{D_m}(D_M, D_N) \tag{12}$$

by virtue of the additional compatibility properties we observed before. With this at hand, we can state the following theorem:

Theorem 3.1 (Goodwillie–Weiss, Arone–Turchin, Turchin, Boavida–Weiss, Sinha, ...) *If M has codimension at least three in N, then the map above induces a homotopy equivalence*

$$\mathrm{Emb}(M,N) \longrightarrow \mathbb{R}\mathrm{Map}_{D_m}(D_M, D_N) \tag{13}$$

where the right hand side is the derived[2] *mapping space of D_m-module maps $D_M \longrightarrow D_N$.* □

It is useful to note that the derived mapping space describes "F_∞ term" in the Goodwillie–Weiss tower for $\mathrm{Emb}(M,N)$, and that one can use truncated versions of mapping spaces to describe the finite stages of the tower.

The upshot of this result is that if we can understand the homotopy type of the configuration spaces as right modules over the little disk operad, then we can understand the homotopy type of the space of embeddings. However, there is a trade-off: we now need to determine the homotopy type of a very simple manifold into another —a disjoint union of points— but nonetheless the functor

[2] For details, the reader can consult the work of B. Fresse [?].

$M \longmapsto \mathrm{Conf}_M(r)$ is not easy to compute. In particular, it is not homotopy invariant! For example, the point has empty higher configuration spaces, but the contractible space D^2 has $\mathrm{Conf}_{D^2}(2) \simeq S^1$. More generally, $\mathrm{Conf}_{D^m}(2)$ is homotopy equivalent to S^{m-1}, through the map

$$(v,w) \longmapsto \frac{v-w}{|v-w|}. \tag{14}$$

One may suspect that the problem above is that the manifold D^2 is not closed (compact), but Longoni–Salvatore have proved that the lens spaces $L_{7,1}$ and $L_{7,2}$, which are homotopy equivalent, have non-homotopy equivalent configuration spaces. However, these are non-simply connected spaces, and the following is still an open question:

Question. *Is it true that two simply connected closed manifolds of the same homotopy type have homotopy equivalent configuration spaces?*

3.1 Bonus: deloopings

The little disks operads were initially introduced to study and identify when a space can be delooped. That is, given a space X, when is it weakly homotopy equivalent to a space of the form

$$\Omega^n(Y) = \mathrm{Map}(D^n; S^{n-1}, Y; *) \tag{15}$$

for some other space Y? Any loop space $X = \Omega Y$ is an H-space and, as such, the monoid $\pi_0(X)$ is in fact group like, in the sense it is a group under the induced product of X. Moreover, it is an algebra over the little n-disks operad: there are equivariant maps $D_n(k) \times X^k \longrightarrow X$ obtained by inserting maps $D^n \longrightarrow Y$ into a disk configuration to obtain a new map $D^n \longrightarrow Y$, which is associative in a precise sense.[3]

Conversely, one can consider such a D_n-algebra X which, in particular, comes equipped with a single homotopy class of operations $m : X^2 \longrightarrow X$ originating from $\pi_0(D_n(2))$, making $\pi_0(X)$ into an associative monoid. We say X is a *group-like* D_n-algebra if this monoid is in fact a group.

With this at hand, we can state the following result, going back to work of Beck, Boardman-Vogt, May, Segal and Stasheff. The theorem, due to J. P. May, provides a useful theorem to determine when a space can be delooped.

Theorem 3.2 *A space X is weakly equivalent to an nth loop space $\Omega^n(Y)$ if, and only if, it is a group-like D_n-algebra.* □

4 The homotopy type of configuration spaces

Computing integral homotopy types of spaces and, in particular, of configuration spaces, is remarkably difficult. A useful simplification that still allows us to obtain significant information on

[3] Which one?

the homotopy type of a space is to work rationally. Informally, one can think that we are throwing away the torsion of homotopy groups of a space.

More precisely, a simply connected topological space X is *rational* if all of the abelian groups $\pi_n(X)$ for $n \geqslant 2$ are $\mathbb{Q}$-vector spaces. A map of spaces $f : X \longrightarrow Y$ is a *rationalization* if Y is a rational space and the induced map $\pi_*(X) \otimes \mathbb{Q} \longrightarrow \pi_*(Y)$ is an isomorphism. One can show such space Y is unique up to homotopy equivalence relative to X, and we write it $X_{\mathbb{Q}}$.

We say f is a *rational homotopy equivalence* if the following equivalent conditions hold:

(1) The map $\pi_*(f) \otimes \mathbb{Q}$ is an isomorphism.
(2) The map $H_*(f, \mathbb{Q})$ is an isomorphism.
(3) The map $H^*(f, \mathbb{Q})$ is an isomorphism.

We call the weak homotopy type of $X_{\mathbb{Q}}$ the *rational homotopy type of* X. For example, the odd sphere S^{2n+1} have the rational homotopy type of a $K(2n+1, \mathbb{Q})$, while the even sphere have a slightly (but still simple) rational homotopy type. In this sense, doing homotopy theory over $\mathbb{Q}$ is much simpler, at the cost of losing some information (for example, torsion, Steenrod operations, among others). The following theorem is one of the cornerstones of rational homotopy theory, along with the seminal work of D. Quillen [?]:

Theorem 4.1 (Sullivan) *There is a Quillen adjunction*[4]

$$\Omega^*(-) : \mathsf{Top}^{\geqslant 1} \rightleftarrows \mathsf{CDGA}^{\geqslant 1} : \langle - \rangle \tag{16}$$

between the category of simply connected[5] *topological spaces of finite type up to rational equivalences and simply connected commutative dg algebras of finite type up to quasi-isomorphism.* □

The functor $\Omega^*(X)$ is analogous to the de Rham functor of forms, but instead consists of 'piecewise linear polynomial forms' on X, while the functor $\langle A \rangle$ is a 'geometric realization' functor. The upshot of this theorem is that the determination of the rational homotopy type of a space X is a purely algebraic task, and can be done by producing a cdga model of the (non-commutative) dga of cochains $C^*(X)$. For example, for each $n \geqslant 1$ the free commutative algebra $(S(x_{2n+1}), 0)$ on a single generator of degree $2n+1$ is a model for the rational homotopy type of the odd sphere S^{2n+1}, while the commutative algebra $(S(x_{2n}, y_{4n-1}), d)$ with $dy_{4n-1} = x_{2n}^2$ is a model for the rational homotopy type of the even sphere S^{2n}.

The end goal of rational homotopy theory is reproducing such computation for more complicated spaces or, what is the same, solving the following problem: given a space X, determine a cdga $A^*(X)$ that is 'nice enough' and quasi-isomorphic to the cdga of PL-forms $\Omega^*(X)$. The model $A^*(X)$ will then give us information about the rational homotopy type of X. For example, if $A^*(X)$ is minimal, in the sense that $A^*(X) = (S^*(V), d)$ with $dV \subseteq S(V)^{\geqslant 2}$, then $V^* \cong \hom_{\mathbb{Q}}(\pi_*(X), \mathbb{Q})$.

Back to operads and modules. With this at hand, our goal is to find a model for $\Omega^*(D_m)$ and $\Omega^*(D_M)$, taking into account that the former is (almost) a cooperad —the functor Ω^* is contravariant—

[4] A special kind of adjunction, see [?] for details.

[5] There is a corresponding theory for non-simply connected spaces that require nilpotence hypotheses on fundamental groups.

and latter is a comodule over it. Since Ω^* is a good functor, these models will allow us to compute the derived mapping space in (13), at least rationally, ultimately allowing us to pin down the rational homotopy type of $\mathrm{Emb}(M,N)$.

4.1 The case of Euclidean space

Let us begin with the case $M = \mathbb{R}^m$, the building block for any other m-manifold. In this case, the computation of the cohomology groups of $\mathrm{Conf}_M(r)$ for all $r \geqslant 1$ are well understood and date back to work of V. I. Arnol'd and F. Cohen. For each r and each distinct $i, j \in [r]$, there is a map

$$\mathrm{Conf}_M(r) \longrightarrow \mathrm{Conf}_M(2) \simeq S^{m-1} \tag{17}$$

that produces for us a cohomology class of degree $m-1$ that we write ω_{ij}. Note that exchanging i and j induces the antipodal map on S^{m-1}, creating a $(-1)^{m-1}$ sign for this cohomology class. Coming from the top cohomology class of the sphere, $\omega_{ij}^2 = 0$ and, by configurations of three points, one arrives at the following 'Jacobi-like identity' for three distinct $i, j, k \in [r]$:

$$\omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij} = 0. \tag{18}$$

These is in fact a complete presentation of $H^*(D_m(r), \mathbb{Q})$:

Theorem 4.2 (Arnol'd, Cohen) *The cohomology ring $H^*(D_m(r), \mathbb{Q})$ is isomorphic to the commutative algebra generated by ω_{ij} in degree $m-1$ for $i, j \in [r]$ subject to the following three sets of relations:*

R_1: *For all distinct i, j we have that $\omega_{ij} = (-1)^{m-1}\omega_{ji}$.*
R_2: *For all distinct i, j we have that $\omega_{ij}^2 = 0$.*
R_3: *For all distinct i, j, k we have the Jacobi relation (18).*

In fact, Cohen computed the cooperadic decomposition maps of $H^*(D_m(r), \mathbb{Q})$, thus determining it completely as a Hopf cooperad. In general, this computation would not be enough to determine the rational homotopy type of D_m. However, the following formality result says that indeed this cohomology ring is quasi-isomorphic to the algebra of forms $\Omega^*(D_m)$, and thus does capture the rational homotopy type of D_m:

Theorem 4.3 (Kontsevich, Lambrechts–Volić, Tamarkin,...) *The operad D_m is formal: there are quasi-isomorphisms of Hopf cooperads*

$$H^*(D_m) \longleftarrow \cdot \longrightarrow \Omega^*(D_m). \tag{19}$$

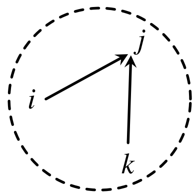
At the time, results on various flavour of formality of the operad D_m abound, see [`?Fresse`, `?Petersen`, `?Tamarkin`, `?Horel`]. One of the two most famous consequences of formality of D_2 are the solution of the Deligne conjecture (for example, as was done by Tamarkin in [`?Tamarkin2003`]) and the deformation quantization theorem of Kontsevich. For us, formality has the consequence of allowing us to restrict to cohomology when computing our derived mapping space of interest.

4.2 Two approaches to formality

KONTSEVICH'S APPROACH to obtaining the formality result in the last theorem can be summarized as follows. First, one can find 'arity wise' resolutions of the commutative algebras $H^*(D_m(r))$, which involve, for example, introducing a new generator ξ_{ijk} relaxing the Jacobi identity of (18) up to homotopy:

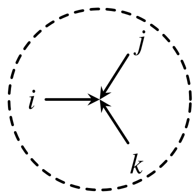
$$d\xi_{ijk} = \omega_{ij}\omega_{jk} + \omega_{jk}\omega_{ki} + \omega_{ki}\omega_{ij}. \tag{20}$$

If we interpret $\omega_{ij}\omega_{jk}$ as the directed graph (and interpret the other two terms similarly)



(21)

then we may interpret ξ_{ijk} as the directed graph



(22)

with differential the Jacobi relation, obtained by contracting one edge at a time. This idea gives rise to an object called a graph complex, filling in the 'dot' in (19). The map pointing to the left is obtained by assigning an integral to each labeled graph with some internal (black) vertices, and giving rise to an element in $\Omega^*(D_m)$. For details, the reader can refer to the book of Lambrechts and Volić [?book].

It is worth pointing out that one can show the framed little disks operad D_2^{fr} is formal, this was done by Giansiracusa–Salvatore. Later, S. Moriya and Turchin–Willwacher showed that the framed little disks operad D_{2m+1}^{fr} is not formal for $m \geqslant 2$.

TAMARKIN'S APPROACH, on the other hand, can be summarized as follows. There is an operad in groupoids called the *parenthesized braids operad*, which we write PaB[6]. The elements of PaB are parenthesized permutations (σ, π) of some finite set $[r]$, and the morphisms between (σ, π) and (σ', π') consist of the elements of B_r that join an element in σ with the same element in σ', as in the figure below. Tamarkin's observation is that the geometrical realization of this operad in groupoids is *a* little disks operad —what is now called an E_2-operad— and that to show D_2 is formal, it suffices to show his operad of parenthesized braids is formal: any two pair of E_2-operads are connected by a zig-zag of quasi-isomorphisms, so this suffices.

[6] Following Tamarkin's original notation.

Detour: It is well known that each topological space $D_2(r)$ is a $K(PB_r,1)$ where PB_r is the pure braid group on r strands. For Tamarkin, a topological B_∞-operad is a topological operad X where each $X(r)$ is a contractible space carrying a free action of the braid group B_r. The corresponding little disks operad is $E_2(X) = X/PB$, the arity-wise quotient of $X(r)$ by PB_r. Naturally, D_2 is the associated E_2-operad of its classifying space.

To show that PaB is formal, Tamarkin shows that every Drinfel'd associator produces a map of operad from $C_*(\mathsf{PaB})$ to the Chevalley–Eilenberg complex $\mathscr{C}_*(\mathfrak{t})$ of the Drinfel'd–Kohno Lie algebras

$$\mathfrak{t} = (\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3, \dots) \tag{23}$$

which form an operad in Lie algebras. Since these are the Koszul dual Lie algebras to the commutative model of Arnol'd and Cohen, this produces for the requisite quasi-isomorphisms, and thus shows that every E_2-operad is formal. Thus, Tamarkin's result can be stated as follows:

Theorem 4.4 *The E_2-operad of parenthesized braids* PaB *is formal, and each Drinfel'd associator Φ produces a quasi-isomorphism of Hopf operads*

$$f_\Phi : C_*(\mathsf{PaB}) \longrightarrow \mathscr{C}_*(\mathfrak{t}) \tag{24}$$

where the right hand side is the Chevalley–Eilenberg construction of the Drinfel'd–Kohno Lie algebras, which receives a quasi-isomorphism

$$H_*(D_2) \longrightarrow \mathscr{C}_*(\mathfrak{t}). \tag{25}$$

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