Problem Set 1

DATA 605, Spring 2021: Assignment 2

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07 February 2021

Prompt

1. Show that $A^T A \neq AA^T$ in general. (Proof and demonstration.)
2. For a special type of square matrix A, we get $A^T A = AA^T$. Under what conditions could this be true? (Hint: The Identity matrix I is an example of such a matrix).

Part 1

Proof of $AA^T \neq A^T A$ in general.

In essence, prove matrix multiplication is not commutative. First, given a matrix A with size $m \times n$, if $m \neq n$, then AA^T would result in a $m \times m$ matrix while $A^T A$ would result in a $n \times n$ matrix. The resulting matrices are different sizes, and thus not equal.

Proof

Now to prove $A^T A \neq AA^T$ in general for a square matrix.

Given a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

With a transpose as

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Matrix multiply AA^T

$$AA^T = \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$$

Matrix multiply $A^T A$

$$A^T A = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix}$$

Thus

$$\begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} \neq \begin{bmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{bmatrix}$$

Demonstration

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

With a transpose as

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Matrix multiply AA^T

$$AA^T = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

Matrix multiply A^TA

$$A^TA = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} \neq \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$$

Validation with R

```r
a = matrix(c(1,2,3,4), 2, 2)
at = matrix(c(1,3,2,4), 2, 2)

a
```

```
##      [,1] [,2]
## [1,]    1    3
## [2,]    2    4
```

```r
at
```

```
##      [,1] [,2]
## [1,]    1    2
## [2,]    3    4
```

```r
aat <- a %*% at

ata <- at %*% a

aat
```

```
##      [,1] [,2]
## [1,]   10   14
## [2,]   14   20
```

```r
ata
```

```
##      [,1] [,2]
## [1,]    5   11
## [2,]   11   25
```

```r
(aat == ata)
```

```
##       [,1]  [,2]
## [1,] FALSE FALSE
## [2,] FALSE FALSE
```

Part 2

Under what conditions is $AA^T = A^TA$?

Answer: When matrix A is **symmetric**. Thus $A = A^T$. And then, with substitution $AA = AA$.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

With a transpose as

$$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

Matrix multiply AA^T

$$AA^T = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

Matrix multiply A^TA

$$A^TA = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

Validation with R

```r
a = matrix(c(1,2,2,4), 2, 2)
at = matrix(c(1,2,2,4), 2, 2)

a
```

```
##      [,1] [,2]
## [1,]    1    2
## [2,]    2    4
```

```r
at
```

```
##      [,1] [,2]
## [1,]    1    2
## [2,]    2    4
```

```r
aat <- a %*% at

ata <- at %*% a

aat
```

```
##      [,1] [,2]
## [1,]    5   10
## [2,]   10   20
```

```r
ata
```

```
##      [,1] [,2]
## [1,]    5   10
## [2,]   10   20
```

```r
(aat == ata)
```

```
##      [,1] [,2]
## [1,] TRUE TRUE
## [2,] TRUE TRUE
```