DATA 605 Assignment Week 14

CUNY Spring 2021

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Prompt

This week, we'll work out some Taylor Series expansions of popular functions.

For each function, only consider its valid ranges as indicated in the notes when you are computing the Taylor Series expansion. Please submit your assignment as a R-Markdown document.

Taylor Series Expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ...$$

Function 1

$$f(x) = \frac{1}{(1-x)}$$

Take the first few derivatives

$$f'(x) = \frac{1}{(1-x)^2}$$

$$f''(x) = \frac{2}{(1-x)^3}$$

$$f'''(x) = \frac{6}{(1-x)^4}$$

$$f^{(4)}(x) = \frac{24}{(1-x)^5}$$

Thus, we get

$$\frac{1}{(1-x)} = \frac{1}{(1-a)} + \frac{\frac{1}{(1-a)^2}}{1!}(x-a) + \frac{\frac{2}{(1-a)^3}}{2!}(x-a)^2 + \frac{\frac{6}{(1-a)^4}}{3!}(x-a)^3 + \frac{\frac{24}{(1-a)^5}}{4!}(x-a)^4 + \ ...$$

Now, we select $a = 0$

$$\frac{1}{(1-x)} = \frac{1}{(1-0)} + \frac{\frac{1}{(1-0)^2}}{1!}(x-0) + \frac{\frac{2}{(1-0)^3}}{2!}(x-0)^2 + \frac{\frac{6}{(1-0)^4}}{3!}(x-0)^3 + \frac{\frac{24}{(1-0)^5}}{4!}(x-0)^4 + \ ...$$

Simplify

$$\frac{1}{(1-x)} = 1 + \frac{1}{1!}(x) + \frac{2}{2!}(x)^2 + \frac{6}{3!}(x)^3 + \frac{24}{4!}(x)^4 + \ ...$$

Simplify some more

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + x^4 + \ ...$$

Thus, Taylor Series expansion results in

$$f(x) = \frac{1}{(1-x)} = \sum_{n=0}^{\infty} x^n$$

According to Key Idea 8.8.1, interval of convergence: $(-1, 1)$

Function 2

$$f(x) = e^x$$

Let's find some derivatives ... note: the derivative of e^x is always e^x

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$f'''(x) = e^x$$

$$f^{(4)}(x) = e^x$$

Thus, we get

$$e^x = e^a + \frac{e^a}{1!}(x-a) + \frac{e^a}{2!}(x-a)^2 + \frac{e^a}{3!}(x-a)^3 + \frac{e^a}{4!}(x-a)^4 + \ ...$$

Again, let's select $a = 0$

$$e^x = e^0 + \frac{e^0}{1!}(x-0) + \frac{e^0}{2!}(x-0)^2 + \frac{e^0}{3!}(x-0)^3 + \frac{e^0}{4!}(x-0)^4 + \ ...$$

Simplify

$$e^x = 1 + \frac{1}{1!}(x) + \frac{1}{2!}(x)^2 + \frac{1}{3!}(x)^3 + \frac{1}{4!}(x)^4 + \ ...$$

Simplify some more

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ ...$$

Thus, Taylor Series expansion results in

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

According to Key Idea 8.8.1, interval of convergence: $(-\infty, \infty)$

Function 3

$$f(x) = \ln(1 + x)$$

Take the first few derivatives

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = -\frac{1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

$$f^{(4)}(x) = -\frac{6}{(1+x)^4}$$

Thus, we get

$$\ln(1+x) = \ln(1+a) + \frac{\frac{1}{1+a}}{1!}(x-a) + \frac{-\frac{1}{(1+a)^2}}{2!}(x-a)^2 + \frac{\frac{2}{(1+a)^3}}{3!}(x-a)^3 + \frac{-\frac{6}{(1+a)^4}}{4!}(x-a)^4 + \ ...$$

Again, let's select $a = 0$

$$\ln(1+x) = \ln(1+0) + \frac{\frac{1}{1+0}}{1!}(x-0) + \frac{-\frac{1}{(1+0)^2}}{2!}(x-0)^2 + \frac{\frac{2}{(1+0)^3}}{3!}(x-0)^3 + \frac{-\frac{6}{(1+0)^4}}{4!}(x-0)^4 + \ ...$$

Simplify

$$\ln(1+x) = \ln(1) + \frac{1}{1!}(x) - \frac{1}{2!}(x)^2 + \frac{2}{3!}(x)^3 - \frac{6}{4!}(x)^4 + ...$$

Simplify some more

$$\ln(1+x) = 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \ ...$$

Simplify even more

$$\ln(1+x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ ...$$

And because the first term is zero, we can build the summation without it and thus start with $n = 1$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ ...$$

Thus, Taylor Series expansion results in

$$f(x) = \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{(n+1)} x^n}{n}$$