DATA 605: Assignment 03

Semester: Spring 2021; Professor Doc Larry

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13 February 2021

Problem Set 1

Part 1

What is the rank of the matrix A?

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 3 \\ 0 & 1 & -2 & 1 \\ 5 & 4 & -2 & -3 \end{bmatrix}$$

To find the rank of matrix A, the matrix needs to be converted to row echelon form to find the count of linearly-independent rows.

Step 1:

- Multiply R_1 by 1 and add to R_2
- Multiple R_1 by -5 and add to R_4

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 7 \\ 0 & 1 & -2 & 1 \\ 0 & -6 & -17 & -23 \end{bmatrix}$$

Step 2:

- Multiply R_2 by $\frac{1}{2}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7/2 \\ 0 & 1 & -2 & 1 \\ 0 & -6 & -17 & -23 \end{bmatrix}$$

Step 3:

- Multiply R_2 by -1 and add to R_3

- Multiple R_1 by 6 and add to R_4

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7/2 \\ 0 & 0 & -4 & -5/2 \\ 0 & 0 & -5 & -2 \end{bmatrix}$$

Step 4:

- Multiply R_3 by $-\frac{1}{4}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7/2 \\ 0 & 0 & 1 & 5/8 \\ 0 & 0 & -5 & -2 \end{bmatrix}$$

Step 5:

- Multiply R_3 by 5 and add to R_4

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7/2 \\ 0 & 0 & 1 & 5/8 \\ 0 & 0 & 0 & 9/8 \end{bmatrix}$$

Step 6:

- Multiply R_3 by $\frac{8}{9}$

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 7/2 \\ 0 & 0 & 1 & 5/8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Result: 4 linearly-independent rows, thus the rank is 4.

Part 2

Given an $m \times n$ matrix where $m > n$, what can be the maximum rank? The minimum rank, assuming that the matrix is non-zero?

Answer: Maximum rank possible is n. Minimum rank possible is 1, assuming matrix is non-zero.

Part 3

What is the rank of matrix B?

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 2 & 4 & 2 \end{bmatrix}$$

Because ...

- $-R_1 + R_2 = R_3$
- $R_1 + R_3 = R_2$

R_2 and R_3 are not linearly independent.

Thus, there is only 1 linearly-independent row, so the rank is 1.

Problem Set 2

Compute the eigenvalues and eigenvectors of the matrix A. You'll need to show your work. You'll need to write out the characteristic polynomial and show your solution.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Thus, $P_f(x) = det(F - xI_3)$, which is applied to matrix A is

$$= \begin{bmatrix} 1-x & 2 & 3 \\ 0 & 4-x & 5 \\ 0 & 0 & 6-x \end{bmatrix}$$

Taking the determinant, and then simplified

$= (1-x)((4-x)(6-x) - 0(5)) + 0 + 0$

$= (1-x)(24 - 6x - 4x + x^2 - 0) + 0$

$= (1-x)(x^2 - 10x + 24)$

$= x^2 - 10x + 24 - x^3 + 10x^2 - 24x$

$= -x^3 + 11x^2 - 34x + 24x$

Then solving for 0

$-(x-1)(x^2 - 10x + 24) = 0$

$-(x-1)(x-6)(x-4) = 0$

$-(x-1)(x-6)(x-4) = 0$

$x = 1$ or $x = 6$ or $x = 4$

Eigenvalues are 1, 4, 6

To solve for the eigenvectors

$$\begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$$

Solve for $\lambda = 1$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

- Multiply R_1 by $\frac{1}{2}$

$$\begin{bmatrix} 0 & 1 & 3/2 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

- Multiply R_1 by -3 and add to R_2

$$\begin{bmatrix} 0 & 1 & 3/2 \\ 0 & 0 & 1/2 \\ 0 & 0 & 5 \end{bmatrix}$$

- Multiply R_2 by -10 and add to R_3

$$\begin{bmatrix} 0 & 1 & 3/2 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

- Multiply R_2 by -3 and add to R_1
- Multiply R_2 by 2

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, to find the eigenvector

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- From above, $v_2 = 0$
- From above, $v_3 = 0$
- From above, $v_1 = v_1$ and then just replace with 1, $v_1 = 1$

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} for \lambda = 1$$

Solve for $\lambda = 4$

$$\begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

- Multiply R_1 by $-\frac{1}{3}$

$$\begin{bmatrix} 1 & -2/3 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

- Multiply R_1 by 5 and add to R_2

$$\begin{bmatrix} 1 & -2/3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

- Multiply R_2 by -2 and add to R_3

$$\begin{bmatrix} 1 & -2/3 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

- Add R_2 to R_1

$$\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, to find the eigenvector

$$\begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- From above, $v_1 - \frac{2}{3}v_2 = 0$
- thus, $v_1 = \frac{2}{3}v_2$
- From above, $v_3 = 0$
- From above, $v_2 = v_2$ and then just replace with 1, $v_2 = 1$

$$\mathbf{v} = \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix} for \lambda = 4$$

Solve for $\lambda = 6$

$$\begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

- Multiply R_1 by $-\frac{1}{5}$

$$\begin{bmatrix} 1 & -2/5 & -3/5 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

- Multiply R_2 by $-\frac{1}{2}$

$$\begin{bmatrix} 1 & -2/5 & -3/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

- Multiply R_2 by $\frac{2}{5}$ and add to R_1

$$\begin{bmatrix} 1 & 0 & -8/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, to find the eigenvector

$$\begin{bmatrix} 1 & 0 & -8/5 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- From above, $v_1 - \frac{8}{5}v_3 = 0$
- thus, $v_1 = \frac{8}{5}v_3$
- From above, $v_2 - \frac{5}{2}v_3 = 0$
- thus, $v_2 = \frac{5}{2}v_3$
- From above, $v_3 = v_3$ and then just replace with 1, $v_3 = 1$

$$\mathbf{v} = \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix} for\lambda = 6$$