DATA 605: Assignment 8

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Problem 1

The price of one share of stock in the Pilsdorff Beer Company (see Exercise 8.2.12) is given by Y_n on the nth day of the year. Finn observes that the differences $X_n = Y_{n+1} - Y_n$ appear to be independent random variables with a common distribution having mean $\mu = 0$ and variance $\sigma^2 = 1/4$. If $Y_1 = 100$, estimate the probability that Y_{365} is

A: $\geq$ 100

$$P(Y_{365} - Y_1 \geq 0)$$

```
y.1 <- 100
mu <- 0
var <- 1/4
sd <- sqrt(var)

y.365 <- 100
x.a <- (y.365 - y.1) / sqrt(365-1)
a.result <- pnorm(x.a, mean = mu, sd = sd, lower.tail=F)
a.result
```

```
## [1] 0.5
```

B: $\geq$ 110

$$P(Y_{365} - Y_1 \geq 10)$$

```
y.365 <- 110
x.b <- (y.365 - y.1) / sqrt(365-1)
b.result <- pnorm(x.b, mean = mu, sd = sd, lower.tail=F)
b.result
```

```
## [1] 0.1472537
```

C: $\geq$ 120

$$P(Y_{365} - Y_1 \geq 20)$$

```
y.365 <- 120
x.c <- (y.365 - y.1) / sqrt(365-1)
c.result <- pnorm(x.c, mean = mu, sd = sd, lower.tail=F)
c.result
```

```
## [1] 0.01801584
```

Problem 2

Calculate the expected value and variance of the binomial distribution using the moment generating function.

Answer

Expected Value

Binomial distribution: probability mass function

$$P_X(j) = \binom{n}{x} p^x q^{n-x} \ with \ q = 1 - p$$

Moment generating function definition

$$M(t) = E(e^{tX}) = \sum_{x \epsilon S} e^{tx} f(x)$$

Substitute the binomial distribution function into mgf

$$\begin{aligned} &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} \\ &= \sum_{x=0}^{n} (pe^t)^x \binom{n}{x} q^{n-x} \\ &= (q + pe^t)^n \end{aligned}$$

The final equality represents the expansion of the binomial. Now, differentiate the moment generating function with respect to t using the function-of-a-function rule, results in

$$\begin{aligned} \frac{dM_x(t)}{dt} &= n(q + pe^t)^{n-1} pe^t \\ &= npe^t (q + pe^t)^{n-1} \end{aligned}$$

Evaluate with $t = 0$ results in expected value ...

$$E(x) = np(q + p)^{n-1} = np$$

Variance

To find the second moment, use the product rule

$$\frac{duv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

expanding and substituting

$$\begin{aligned}\frac{d^2M_x(t)}{dt^2} &= npe^t\{(n-1)(q+pe^t)^{n-2}pe^t\} + (q+pe^t)^{n-1}\{npe^t\} \\ &= npe^t(q+pe^t)^{n-2}\{(n-1)pe^t + (q+pe^t)\} \\ &= npe^t(q+pe^t)^{n-2}\{q+npe^t\}\end{aligned}$$

Evaluate with $t = 0$, results in

$$E(x^2) = np(q+p)^{n-2}(q+np) = np(q+np)$$

Now, we can calculate the Variance

$$V(x) = E(x2) - \{E(x)\}^2 = np(q+np) - n^2p^2 = npq$$

Or, substituting for $q = 1 - p$, variance is ...

$$V(x) = np(1-p)$$

Definitely took help from: https://www.le.ac.uk/users/dsgp1/COURSES/MATHSTAT/5binomgf.pdf

Problem 3

Calculate the expected value and variance of the exponential distribution using the moment generating function.

Answer

Exponential distribution function is

$$f(x) = \lambda e^{-\lambda x}$$

with

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Moment generating function is

$$M_x(t) = E(e^tX) = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$$

Now, integrating by method of substitution

$$
\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{tx} \lambda e^{-\lambda x} dx \\
&= \lambda \int_{0}^{\infty} e^{(t-\lambda)x} dx \\
&= \frac{\lambda}{t-\lambda} \; provided \; that \; |t| < \lambda
\end{aligned}
$$

Define theorem, if X has mgf $M_X(t)$, then

$$E(X^n) = M_X^{(n)}(0) where M_X^{(n)}(0) = \frac{d^n}{dt^n} M_X(t)|_0$$

That means, the n-th moment is equal to the n-th derivative of the mgf evaluated at $t = 0$.

Now using above theorem, evaluate the first and second moments

First Moment

$$E(X) = M_X^{(1)}(0) = \frac{\lambda}{(\lambda - t)^2}|_{t=0} = \frac{1}{\lambda}$$

Second Moment

$$E(X^2) = M_X^{(2)}(0) = \frac{2\lambda}{(\lambda - t)^3}|_{t=0} = \frac{2}{\lambda^2}$$

Expected Value

From first moment, expected value is

$$E(X) = \frac{1}{\lambda}$$

Variance

Based on first and second moment, variance is

$$var(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Definitely took help from: http://www.maths.qmul.ac.uk/~bb/MS_Lectures_5and6.pdf