DATA 605: Assignment 7

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Problem 1

Let X_1, X_2, $\ldots$, X_n be n mutually independent random variables, each of which is uniformly distributed on the integers from 1 to k. Let Y denote the minimum of the X_i's. Find the distribution of Y.

Answer

The goal is to find the distribution function $m(j) = P(Y = j)$, in which j is equal to the minimum value (aka Y). To do this, we must determine the number of ways the X_i variables can be assigned values between j and k with at least one of the X_i values being j, then divide that number by the total possible way the X_is can be assigned values between 1 and k.

Based on the prompt, each random variable X_i is uniformly distributed integers from 1 to k, thus the count of possibilities for each variable is k. Given this understanding, the total number of outcomes for the entire collection of random variables is k^n.

To find the numerator, let's start with $Y = 1$. The total number of ways of $Y = 1$ is calculated by $k^n - (k-1)^n$ because k^n represents all the possibilities and $(k-1)^n$ represents all the possibilities where none of the X_i variables are equal to 1.

Following the same pattern for $Y = 2$, the total number of possibilities in which 2 is the minimum and occurs at least once is $k^n - (k-2)^n - (k^n - (k-1)^n)$. Like above, $k^n - (k-2)^n$ is all possibilities minus all outcomes in which 2 or fewer is not a result. And then subtract that by $k^n - (k-1)^n$ to remove all the scenarios in which 1 occurs. So $k^n - (k-2)^n - (k^n - (k-1)^n)$ represents all the scenarios in which 2 as the minimum value occurs at least once.

This pattern for $Y = j$ then results in $(k-j+1)^n - (k-j)^n$, for a minimum value of j. We already determined the denominator, so the distribution function is ...

$$For 1 \leq j \leq k, m(j) = \frac{(k-j+1)^n - (k-j)^n}{k^n}$$

Problem 2

Your organization owns a copier (future lawyers, etc.) or MRI (future doctors). This machine has a manufacturer's expected lifetime of 10 years. This means that we expect one failure every ten years. (Include the probability statements and R Code for each part.).

A. Geometric

What is the probability that the machine will fail after 8 years? Provide also the expected value and standard deviation. Model as a geometric. (Hint: the probability is equivalent to not failing during the first 8 years.)

$$P(X = n) = q^{n-1} \times p$$

And given that $P(X > 8) = 1 - P(X \leq 8)$ and $P(X \leq k) = 1 - q^{k+1}$, then $P(X > k) = q^{k+1}$, or $P(X > 8) = 0.9^{8+1}$

```
# Probabiliy of failure 1/10
p <- .1
# q is 1 - p
q <- 1 - p
# Years
n <- 8

# Probability of 8 failures (in this case failure means not breaking) before a success (machine breakin
prob <- pgeom(n, p, lower.tail = FALSE)
prob
```

```
## [1] 0.3874205
```

```
pprob <- q^(n+1)
pprob
```

```
## [1] 0.3874205
```

Probability: 0.3874205.

Expected value formula: $E(X) = \frac{1}{p}$, so $E(X) = \frac{1}{.1}$

```
ev <- 1 / p
```

Expected value: 10.

Standard deviation formula: $\sigma = \sqrt{\frac{1-p}{p^2}}$, so $\sigma = \sqrt{\frac{1-0.1}{0.1^2}}$

```
sd <- sqrt((1-p)/(p^2))
```

Standard deviation: 9.486833.

B. Exponential

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as an exponential.

$P(X \leq k) = 1 - e^{\frac{-k}{\mu}}, where\ \mu = \frac{1}{\lambda}$. The failure rate is $\lambda = 0.1$ and $P(X > 8) = 1 - P(X \leq 8)$, so $P(X > 8) = 1 - (1 - e^{\frac{-8}{\frac{1}{0.1}}})$

```
lambda <- p

prob <- 1 - (1 - (exp(-n/(1/lambda))))
prob
```

```
## [1] 0.449329
```

```
pprob <- pexp(n, lambda, lower.tail=FALSE)
pprob
```

```
## [1] 0.449329
```

Probability: 0.449329.

Expected value formula: $E(X) = \frac{1}{\lambda}$, so $E(X) = \frac{1}{.1}$

```
ev <- 1 / p
```

Expected value: 10.

Standard deviation formula: $\sigma = \sqrt{(\frac{1}{\lambda})^2}$, so $\sigma = \sqrt{(\frac{1}{0.1})^2}$

```
sd <- sqrt((1/.1)^2)
```

Standard deviation: 10.

C. Binomial

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as a binomial. (Hint: 0 success in 8 years)

$b(n, p, k) = \binom{n}{k} \times p^k \times q^{n-k}$, thus $\binom{8}{0} \times 0.1^0 \times .9^{8-0}$

```
k <- 0

prob <- choose(n, k) * p^k * q^n
prob
```

```
## [1] 0.4304672
```

```
pprob <- dbinom(k, n, p)
pprob
```

```
## [1] 0.4304672
```

Probability: 0.4304672.

Expected value formula: $E(X) = n \times p$, so $E(X) = 8 \times 0.1$

```
ev <- n * p
```

Expected value: 0.8.

Standard deviation formula: $\sigma = \sqrt{np(1-p)}$, so $\sigma = \sqrt{8 \times 0.1 \times 0.9}$

```
sd <- sqrt(8 * .1 * .9)
```

Standard deviation: 0.8485281.

D. Poisson

What is the probability that the machine will fail after 8 years?. Provide also the expected value and standard deviation. Model as a Poisson.

$P(X=k) = \frac{\lambda^k}{k!}e^{-\lambda}$, thus $P(X=0) = \frac{0.1^0}{0!}e^{-.1}$ would be the probability of failure in a single interval, in this case one year. So to calculate the probability of 0 failures for eight years, $P(X=0)^8 = (\frac{0.1^0}{0!}e^{-.1})^8$

```r
prob <- ((0.1^0 / factorial(0)) * exp(-0.1))^8
prob
```

```
## [1] 0.449329
```

```r
pprob <- ppois(0, 0.1)^8
pprob
```

```
## [1] 0.449329
```

Probability: 0.449329.

Expected value formula: $E(X) = \lambda$, so $E(X) = \lambda \times 8$ as there are 8 trials.

```r
ev <- n * p
```

Expected value: 0.8.

Standard deviation formula: $\sigma = E(X)$ in Poisson distribution, so $\sigma = \sqrt{\lambda \times 8}$

```r
sd <- sqrt(ev)
```

Standard deviation: 0.8944272.