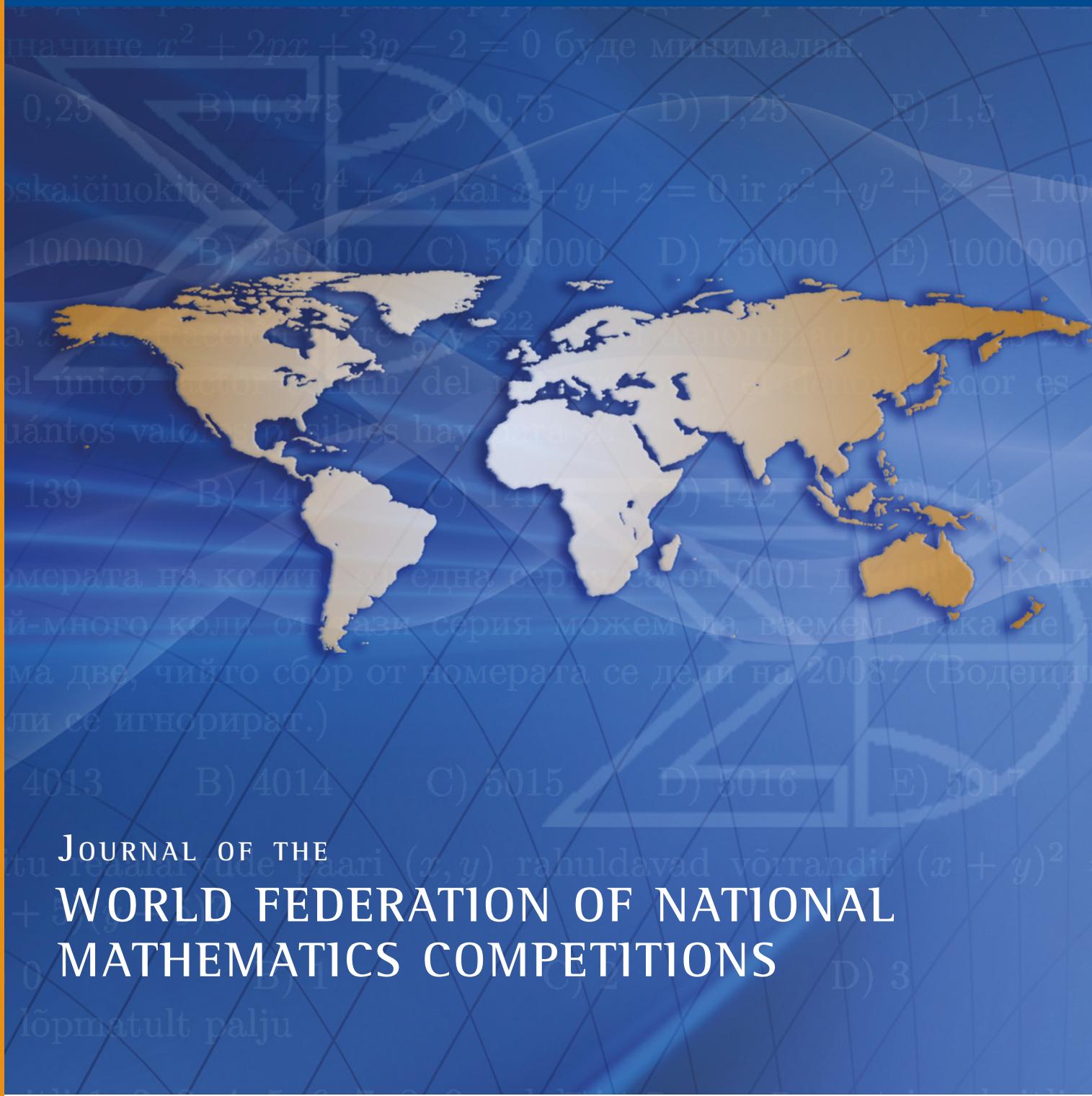


MATHEMATICS COMPETITIONS



JOURNAL OF THE

WORLD FEDERATION OF NATIONAL
MATHEMATICS COMPETITIONS



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AUSTRALIAN MATHS TRUST

MATHEMATICS COMPETITIONS

Одредити реалан параметар p , тако да збир квадрата решења једначине $x^2 + 2px + 3p - 2 = 0$ буде минималан.

- A) 0,25 B) 0,375 C) 0,75 D) 1,25 E) 1,5

Apskaičiuokite $x^4 + y^4 + z^4$, kai $x + y + z = 0$ ir $x^2 + y^2 + z^2 = 1000$.

- A) 100000 B) 250000 C) 500000 D) 750000 E) 1000000

Sea x una fracci髇 entre $\frac{1}{2}$ y $\frac{2}{3}$ que es divisible por 11. ¿Cu醤tos valores posibles hay para x ?

- A) 139 B) 140 C) 141 D) 142 E) 143

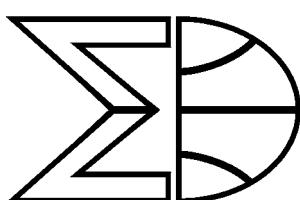
Номерата на колите са една серия от 0001 до 2008. Колко най-много коли от тази серия можем да вземем, така че да няма две, чийто сбор от номерата се дели на 2008? (Водещите нули се игнорират.)

- A) 4013 B) 4014 C) 5015 D) 5016 E) 5017

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MATHEMATICS COMPETITIONS

- A) 0 B) 1 C) 2 D) 3
E) лõpmatult palju



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The views expressed here are those of the authors and do not necessarily represent the views of the Australian Mathematics Trust.

Contents

WFNMC Committee	3
From the President	6
From the Editor	7
Milestones, Memories, Perspectives	
<i>Peter Taylor</i>	8
New and Old Problems and Conjectures related to the Chromatic Number of the Plane	
<i>Alexander Soifer</i>	18
Restarting the Mathematical Duel	
<i>Robert Geretschläger, Jaroslav Švrček, Jacek Uryga</i>	25
Another concurrency related to the Fermat point of a triangle	
<i>Michael de Villiers and Johan Meyer</i>	33
The Summer Conference of the Tournament of Towns	
<i>Nikolay Konstantinov</i>	41

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For WFNMC Standing Committees please refer to ABOUT WFNMC section of the WFNMC website <http://www.wfnmc.org/>.

From the President

Dear readers,

As I write these lines, the weather in central Europe is in the process of transitioning from a somewhat balmy autumn to the more familiar early winter chill we expect around mid-November in this part of the world. Soon, the Holiday Season will be upon us, and as the new year beckons, my thoughts inevitably turn to the coming math competition season. There is, of course, always work to do in preparing for our competitions, no matter the time of year, but the chilly weather seems to bring an especially appropriate mood to sit at the keyboard or the sketchpad, inventing competition problems and deliberating on the fine points of competition organisation.

If your circumstances allow you to have similar thoughts, may I suggest that it might be an especially suitable time of year to think about sharing your current work with the competition community at large by submitting an article to this very journal? We would love to hear all about your special ideas for original problems, the latest great idea for competition organisation in your part of the world, or your most recent research on competition mathematics. We always like to remind you that the quality of the journal depends of your submissions, and your submissions are what keep us going. Rest assured that there are many potential readers, eager to find out what you have to say!

Also, I would like to remind you to mark the date of our next congress on your calendars. The next WFNMC meeting is planned for Kuala Lumpur in July of 2026, and your participation will make it an especially memorable event. The world of Mathematics Competitions is booming, and there is so much to share. I am looking forward to your contributions and to seeing you at the congress!

Robert Geretschläger

Editor's Page

Dear Competitions enthusiasts, readers of our *Mathematics Competitions* journal!

Mathematics Competitions is the right place for you to publish and read the different activities about competitions in Mathematics from around the world. For those of us who have spent a great part of our life encouraging students to enjoy mathematics and the different challenges surrounding its study and development, the journal can offer a platform to exhibit our results as well as a place to find new inspiration in the ways others have motivated young students to explore and learn mathematics through competitions. In a way, this learning from others is one of the better benefits of the competitions environment.

Following the example of previous editors, I invite you to submit to our journal *Mathematics Competitions* your creative essays on a variety of topics related to creating original problems, working with students and teachers, organizing and running mathematics competitions, historical and philosophical views on mathematics and closely related fields, and even your original literary works related to mathematics.

Just be original, creative, and inspirational. Share your ideas, problems, conjectures, and solutions with all your colleagues by publishing them here. We have formalized the submission format to establish uniformity in our journal.

Submission Format

FORMAT: should be LaTeX, TeX, or for only text articles in Microsoft Word, accompanied by another copy in pdf. However, the authors are strongly recommended to send article in TeX or LaTeX format. This is because the whole journal will be compiled in LaTeX. Thus your Word document will be typeset again. Texts in Word, if sent, should mainly contain non-mathematical text and any images used should be sent separately.

ILLUSTRATIONS: must be inserted at about the correct place of the text of your submission in one of the following formats: jpeg, pdf, tiff, eps, or mp. Your illustration will not be redrawn. Resolution of your illustrations must be at least 300 dpi, or, preferably, done as vector illustrations. If a text is embedded in illustrations, use a font from the Times New Roman family in 11 pt.

START: with the title centered in Large format (roughly 14 pt), followed on the next line by the author(s)' name(s) in italic 12 pt.

MAIN TEXT: Use a font from the Times New Roman family or 12 pt in LaTeX.

END: with your name-address-email and your website (if applicable).

INCLUDE: your high resolution small photo and a concise professional summary of your works and titles.

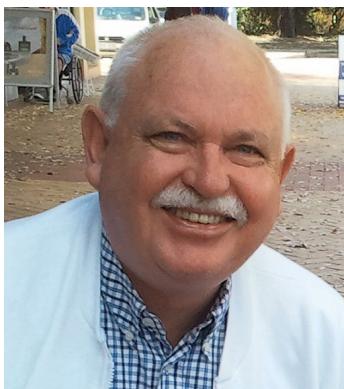
Please submit your manuscripts to María Elizabeth Losada at
director.olimpiadas@uan.edu.co

We are counting on receiving your contributions, informative, inspired and creative. Best wishes,

Maria Elizabeth Losada
EDITOR

Milestones, Memories, Perspectives

Peter Taylor



Peter Taylor was awarded a PhD from the University of Adelaide and became Professor of Mathematics at the University of Canberra, including 18 years as Executive Director of the Australian Mathematics Trust.

Introduction

I was asked to speak on the history of WFNMC at the recent ICME-15 conference in Sydney. I was present during the buildup to forming the WFNMC and have been continuously active during its life. This paper reflects what I said at ICME. It does extend the definitive paper by Kenderov [2] and it gives a number of personal perspectives on the history.

The First Competitions of the modern era

As Kenderov [2] reports, a competition was held in Romania in 1885, but most papers recognise the Eötvös Competition (more recently known by the name Kürschák) and founding of the school journal KöMal in Hungary, both 1894, are the real fore-runners of the modern era.

First Olympiads and other Exclusive Activities

The first use of the word Olympiad in our context seems to be the 1934 Leningrad Olympiad, which also used oral solutions by students. In 1935 a Moscow Olympiad started, with the more traditional written solutions, and between then until the 1950s various Olympiads, local and national, started in the Soviet bloc countries. Generally, one would call an Olympiad an exclusive competition, as it requires knowledge acquired beyond the classroom.

Finally in 1959 Romania hosted the very first International Mathematical Olympiad (IMO). This was attended only by countries in the Soviet bloc. The first Western country to enter the IMO was Finland in 1965. Then in 1967 France, Italy, Sweden and UK attended, and gradually more and more countries added until today, when over 100 countries each enter a team of 6.

There are also regional Olympiads, involving several countries nearby one another. These include the Ibero-American, Pan African, Baltic Way, Balkan and the Asian-Pacific Olympiads. Countries in all regions have their national Olympiads but also enter a regional one if it exists in their region

to give extra training and further insight to selection for the national IMO teams.

There are many other local Olympiads around the world, but there is one other major international competition, which is exclusive, in the sense that it is at a level beyond the classroom and that is the Tournament of Towns, which started in the Soviet Union in 1980. The main reason it started seems to be that the All Union Soviet Olympiad did not give enough entries to the big cities such as Moscow, Leningrad and Kiev. In fact it seems to have started with just those cities, but had such an attractive format that it quickly spread to other towns in the Soviet Bloc.

By 1988 it had only been participated within that bloc, and there was an interest in spreading it to the West. And by 1988 Australia had such developed competition infrastructure that when I attended ICME-6 in Budapest (while Peter O'Halloran was hosting IMO in Canberra) I was approached by two Bulgarian mathematicians, Jordan Tabov and Petar Kenderov, neither of whom I had known, to see if I could help develop the Tournament in Australia. It needed what the Russians describe as a Mathematics Circle, that is a group of identified talented students work with an experienced mathematician on a regular basis to develop their knowledge.

We in fact did have a Circle in Canberra. Two Australian National University mathematicians, Mike Newman and Laci Kovacs (the latter having been a Hungarian who as a student had been a winner in the Eötvös Competition) had for years run classes on Friday nights for talented students. They had found this popular as there were students who were good at mathematics, bored by the syllabus and looking for more. On my return to Canberra I discussed this with Mike and he suggested he and Laci had been doing this for a long time and they would be happy to hand this over to me and my colleague Malcolm Brooks, using the Tournament as a focus. For the sake of continuity Mike stayed on for some time and it was highly successful, we entered a Canberra group in late 1988, very successfully.

The Tournament became known to the Australian Mathematical Olympiad Committee and Circles with focus on the Tournament started in various other Australian towns. It also started to take off in other countries, including Germany.

I visited the Moscow manager of the Tournament, Nikolay Konstantinov, in 1990 and told him that for it to be internationally successful, I would need all the past questions so we could publish them with solutions in English. He went to his filing cabinet and produced such a full set, but only in Russian. I could not find a colleague eager to translate these problems. I started to teach myself Russian, with the help of the mathematics dictionary produced by the American Mathematical Society and the Soviet Academy of Science. I also enrolled in Russian 1 at my University and found that I could at least translate mathematics. I introduced Andy Liu to the Tournament and he was enthusiastic when he saw the nature of the problems, often set in everyday life, and involved structural thinking, not just technical knowledge.

So I was able to produce the problems in English and with the help of colleagues and Andy (who did the heaviest problem solving) our Australian Mathematics Trust was able to publish all this work in the next two years. I gained great pleasure when translating but my favourite discovery of this type was the following, which had been set in 1984.

On the Island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colours meet, they both simultaneously change colour to the third colour (e.g. if a grey and brown chameleon meet each other they both change to crimson). Is it possible that they will eventually all be the same colour?

(Composed by V.G. Ilichev.)

This problem clearly does not need high level technical knowledge, but requires good structural thinking.

First Inclusive Competitions

I define an inclusive competition as one held in schools, with a syllabus, written or unwritten, familiar within the classroom, although they test also the ability of a student to use the mathematics they know to solve problems in unforeseen situations.

USA

Gábor Szegő had been a winner of the Eötvös Competition in 1912 and George Pólya (then at Stanford with Szegő) saw the value in competitions. In 1946 Pólya and Szegő founded the Stanford University Competitive Examination in Mathematics. In its first year 322 students from 60 schools in California entered. The competition grew to having typically 1200 students from 150 schools in 3 western states. However the competition was terminated in 1965 when Stanford shifted its emphasis to postgraduate study. Pólya however continued his activity in this area by publishing problem material in books and journals.

In 1950 a competition started in New York. This grew and became the national competition, which has had various names, but is under the administration of the Mathematical Association of America. It is the first step towards the training and selection of members of the USA IMO team. As with other countries, the next step would operate with a name including “Intermediate”, and these eventually started, using training beyond the classroom, in the 1980s.

Canada

In 1963 Canada started an inclusive competition. This competition became de facto the national competition of Canada, the first step in identifying members of the Canadian IMO team. This competition is administered out of the University of Waterloo, and is also used to offer scholarships at that University.

Australia

At high school I was inspired by two very good mathematics teachers. But there was a local mathematics competition there which was not held at my school. Furthermore two schools nearby to one another did participate and this was combined with extracurricular sessions with one of the teachers.

I undertook my undergraduate and postgraduate studies at the University of Adelaide. At the end of an ordinary degree those students wishing to progress to postgraduate complete an honours year. My year had a record number of students, many of whom seemed more experienced than me, and had in fact taken part in the local competition and enrichment while attending one of those two nearby schools.

After completing my PhD I was appointed as a Lecturer in the institution which is now the University of Canberra in 1972. It was a new institution and one of my new colleagues was a Senior Lecturer, Peter O'Halloran. He was to be the first academic at this institution to take study leave and this was to include 6 months at the mathematics faculty at the University of Waterloo in Canada. This experience was covered by a Canberra newspaper.

MATHS MAN FOR CANADA

Canberra College of Advanced Education senior lecturer in mathematics Mr Peter O'Halloran will begin a 12-month study tour in Canada and Europe next week.

He will be the first staff member of the college to take a 12 month tour under its "staff development scheme" — which equates with the sabbatical leave granted to university professors and lecturers.

Mr O'Halloran will spend the first six months of his tour as consultant to a Canadian glass-making company, and then will become a visiting professor at Canada's Waterloo University for 4½ months, before going to Europe.

"The college differs from universities in that it emphasises the work and vocation of students", he said today.

"Staff members are encouraged to have experience in industry, commerce, or whatever the case may be, so that is why I will be spending so much time with the glassmaking company.

"Waterloo University is foremost in the world in what is called combinatorial mathematics — the type of mathematics associated with computers.

"This is my field.

"The university has about 30 courses



● PETER O'HALLORAN

just in this field, and so it is important for me to have a look at them.

"The use of computers is rising, and this area of mathematics is expanding".

Mr O'Halloran will make the trip with his wife Marjorie — who is the Arts Council representative in Aranda — and their four children.

He returned in 1973 with tales of the two competitions he had seen, in Canada and the US. He said he wanted to develop an Australian version. He had a lot of coffees with me over the forthcoming months and with my background he had no difficulty in recruiting me. He also recruited Warren Atkins and Jo Edwards and ultimately at a meeting at his home (he had become the President of the local maths teachers professional organisation, so knew who the best teachers were) of us and some teachers in early 1976 we resolved to run a competition in Canberra later in that year.

We had not been that confident, as the word "competition" had unfortunate connotations related to pressure, but to our surprise every high school in the city entered and we realised we were on to something. So we decided to pilot the scheme in some other parts of Australia in 1977 and the competition became truly national in 1978 as the Australian Mathematics Competition in 1978. The competition grew rapidly. Within a few years we had annual entries of over 500,000, or 1 in

every 3 high school students in the country.

And the competition spread to other countries in the Indo-Pacific region, including New Zealand, Singapore, Malaysia, Hong Kong, Taiwan, Indonesia, Philippines and Brunei. The competition was also held through the Pacific and was translated into French to enable participation in New Caledonia and French Polynesia.

As Peter O'Halloran had also developed Australia's participation in IMO, and we had developed a strong competition infrastructure, I assume this was why Jordan Tabov and Petar Kenderov approached me in respect of Tournament of Towns, as discussed above, at ICME-6 in Budapest in 1988.

UK South Africa Europe Brazil

We had colleagues in other countries. John Webb started one in South Africa and Tony Gardiner started them in the UK. And after visiting Canberra in 1991 Andrei Deledicq in Paris started the kangourou in Paris, which was to become the largest and most international of all the inclusive competitions. And another very large competition started in Brazil.

WFNMC

1984: Adelaide ICME-5

We had developed then relations with several other countries organising competitions, and with ICME-5 coming up in Adelaide in 1984, Peter O'Halloran realised there would be a number of relevant mathematicians present together and organised a meeting there. So the meeting was held with 20 mathematicians present and the meeting founded WFNMC as a professional organisation.

Peter was elected as President and the Vice Presidents included Walter Mientka of the US and Ron Dunkley of Canada. The first activity would be a Newsletter, to be edited by Warren Atkins. After a few editions, this Newsletter became a refereed Journal, which Warren edited until the mid 2000s.

I should note the minutes of this original meeting does not name the 20 mathematicians present, but as I recall it, other than those named in the paragraph above, mathematicians present there who are still active included Maria Falk de Losada, George Berzsenyi and Mark Saul.

1988: Budapest ICME-6

The next meeting of WFNMC was the next ICME-6, held in Budapest in 1988. Peter was hosting IMO in Canberra concurrently, although I was able to attend, as well as a host of WFNMC members, as the photo at that conference of the group demonstrates.



1990: Conference

In 1990 Ron Dunkley made a remarkable initiative in hosting the first WFNMC Conference at the University of Waterloo. This was possibly the most exciting conference I ever attended. WFNMC had become a mature organisation, it became the norm that WFNMC would always host the conference in the even-numbered year between ICME Conferences. Photos of attending groups, such as the one above, where available, can be found on the WFNMC web site at <http://www.wfnmc.org/photoslarge.html>.

1991: Awards

The next major activity of WFNMC was the introduction of awards. In the first year, 1991, three Hilbert Awards were presented for the best articles in the journal. Later the criteria were changed to general contribution to competitions, Hilbert being international, and the Erdős Award similarly for contribution within a country. Later this distinction was too difficult to judge and later the two awards were merged as the Erdős Award. Information, including past winners, can be found at <http://www.wfnmc.org/awards.html>.

1992: Quebec ICME-7

In 1988 Competitions had become a topic at ICMEs for a Topic Study Group, and this was continued at Quebec.

1994: Bulgaria WFNMG-2, Affiliated Organisation of ICMI, and passing of Peter O'Halloran

1994 was a rather momentous year. Bulgaria hosted the second WFNMC Conference at Pravets, and very notably they arranged for Paul Erdős to attend for the whole week, including presenting the Awards in his name. He also mixed with participants throughout.



1994 was also very significant as ICMI acknowledged WFNMC as the 4th of its Affiliated Organisations, a formal link. There are now 8 such organisations.

The year was also significant for the death of Peter O'Halloran. Peter was able to attend at Pravets but it was obvious he was very ill, and he passed away about 2 months later. Bulgarian mathematician Blagovest Sendov was appointed as President.

1996: Seville, ICME-8

In 1996 Sendov resigned because of Parliamentary commitments in Bulgaria, and Ron Dunkley had become President. Ron undertook to draft a Constitution, for the first time, and he had one adopted at ICME. Also in 1996 the WFNMC web site was established under the domain name wfnmc.org, hosted by a server of the Australian Mathematics Trust and based on source html. By now, most of the WFNMC activities were established, so I will pass through more recent history more quickly.

1998: Zhong Shan, China WFNMC-3

This successful conference, hosted by Professor Qiu Zhonggu, was held in southern China near Hong Kong.

2000: Tokyo, ICME-9

Ron Dunkley had resigned as President and I was elected. I successfully moved that the Presidency be limited to a single term of 4 years.

2002: Melbourne, WFNMC-4

I was fortunate to host this and to arrange for John Conway to attend as a guest and give a public lecture at the University of Melbourne, and he enthusiastically mixed with the participants during the week. His presence was infectious.

2002-2009: ICMI Study 16

In 2002 ICMI invited Ed Barbeau, of the University of Toronto, Canada, and me to co-chair a study entitled Challenging Mathematics In and Beyond the Classroom. This Study included Competitions, and generalised to other forms of challenge. An international Program committee was appointed and met in Modena, Italy, in 2003 and the main conference of the Study was attended by about 45 mathematics educators in Trondheim, Norway, in 2006. The final report took the form of a Springer book, Barbeau and Taylor [1]. The Study was a wider recognition of the role of competitions and similar challenges in mathematics education. In addition to the book, the web site for the Study, containing details of participants also, can also be found at <http://www.wfnmc.org/icmisi16.html>.

2004: Copenhagen, ICME-10

At the meeting Petar Kenderov was elected as President.

2006: Cambridge WFNMC-5

This Conference was memorably held at Robinson College, Cambridge University, and hosted by Tony Gardiner.

2008: Monterrey ICME-11

At this Conference Maria Falk de Losada was elected President.

2010: Riga WFNMC-6

This conference was hosted by Agnis Andzans.

2012: Seoul ICME-12

At this Conference Alexander Soifer was elected President, and for 6 years, as it was decided to hold elections at WFNMC Conferences rather than ICME conferences.

2014: Barranquilla, Colombia WFNMC-7

This conference was hosted by Maria Falk de Losada.

2016: Hamburg ICME-13

This was mainly held on the University of Hamburg campus, where Gabriele Kaiser, President of ICMI, is a Professor.

2017: Journal becomes electronic

This was the year in which the Journal became electronic, and from then on no subscriptions were taken by AMT and the Journal has been published in full on the web site.

2018: Graz WFNMC-8

This was hosted by Robert Geretschläger and Kiril Bankov was elected as President.

2020: Beijing ICME-14

This was the ICME postponed by COVID-19 until 2021, and even then it was substantially attended electronically.

2022: Sofia WFNMC-9

This was still in the pandemic era, although a number of people attended in person. It was also made available electronically. It was hosted by Kiril Bankov and Robert Geretschläger was elected President.

2024: Sydney ICME-15

This was held at the Sydney Convention Centre, a central location in Darling Harbour.

2026 WFNMC-10

This will be held in Kuala Lumpur hosted by M. Suhaimi Ramli.

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- [1] Barbeau EJ and Taylor PJ, *Challenging Mathematics in and beyond the Classroom*, The 16th ICMI Study, Springer, New York, 2009.
- [2] Kenderov, Petar S., *A Short History of the World Federation of National Mathematics Competitions, Mathematics Competitions*), 22, 2, 2009, pp. 14-31. <http://www.wfnmc.org/history.pdf>

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New and Old Problems and Conjectures related to the Chromatic Number of the Plane

Alexander Soifer



Alexander Soifer has been a professor at the University of Colorado since 1979. For the past 28 years he was a member of the Executive of WFNMC, including the presidency 2012–2018. Soifer authored ca. 400 articles and 14 books, including his latest, April 2024, “The New Mathematical Coloring Book: Mathematics of Coloring and the Colorful Life of Its Creators.”

“Old News”: My Conjectures born in 2002

Many mathematical results are surprisingly simple, as are our conjectures. Just look at the Erdős-Szekeres’ Happy End Conjecture 31.15! Contemporary media coined a self-contradictory term “Old news.” I’ll use it here for my 22-year-old conjectures, which did not lose their relevance.

Let us start with the plane, Euclidean plane. To color the plane is to assign each point of the plane a color. The minimal number of colors in coloring of the plane that avoids a monochromatic pair of points at the distance 1 from each other, is called the chromatic number of the plane. In 2002, when I created this chromatic number of the plane conjecture, the majority of my colleagues leaned toward values 4 or 5. After the 2018 de Grey’s breakthrough result, the majority of researchers joined me believing in 7.

Chromatic Number of the Plane Conjecture 64.3 (A. Soifer, 2002)

$$\chi(E^2) = 7.$$

“OK,” I hear your reply, “but then a finite unit-distance 7-chromatic graph must exist in the plane!” This is true, but it would be quite large. In 1998, Dan Pritikin published the lower bound for the number of vertices $|G|$ of such a graph: $|G| \geq 6198$. This lower bound stood for 22 years, when in 2020 Jaan Parts succeeded in improving it:

Lower Bound for a Unit-Distance 7-Chromatic Graph 64.4 (Jaan Parts, 2020). Any unit-distance 7-chromatic graph G satisfies the following inequality: $|G| \geq 6993$.

Parts achieves his result by constructing a tiling of more than 99.985698% of the Euclidean plane with 6 colors, thus setting the new record in these indoor competitions. In fact, we all expect that the size of the smallest such graph to be much-much larger than 6993.

The best we can hope to achieve in our lifetime is the lower bound of 6 for the chromatic number of the plane.

Similarly, the chromatic number of the Euclidean space is the minimum number of colors required for coloring the points of the space to forbid monochromatic pairs of points at the distance 1 apart

In 2002, I also formulated a conjecture for the 3-dimesional Euclidean space E^3 :

Chromatic Number of the Euclidean 3-Space Conjecture 64.6 (A. Soifer, 2002).

$$\chi(E^3) = 15.$$

My expectations for small dimensions led me to my old general conjecture for the chromatic number of the Euclidean n -dimensional space E^n . Conjecturing it was akin seeing the light at the end of the tunnel.

The General Chromatic Number of E^n Conjecture 64.7 (A. Soifer, 2002).

For any positive $n > 1$,

$$\chi(E^n) = 2^{n+1} - 1.$$

I believe it will take 300 years to prove this general conjecture, and less than that to disprove it. As Paul Erdős used to say, we will see!

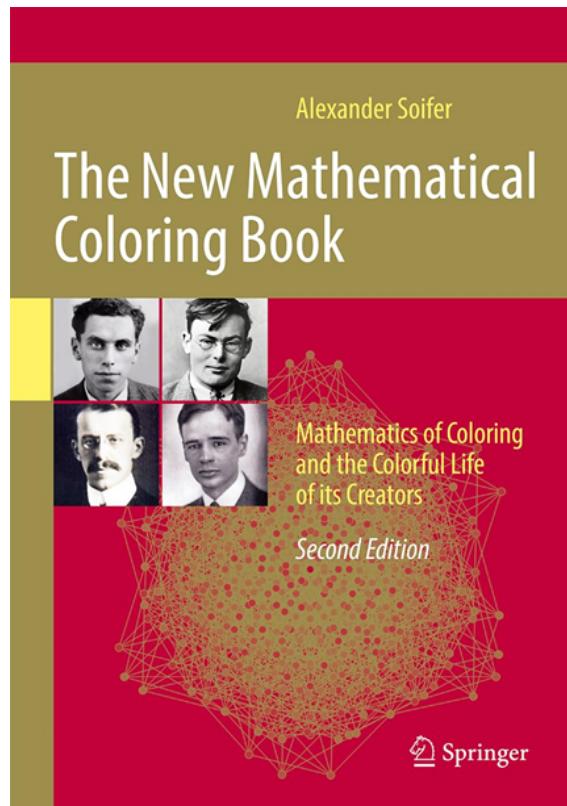
New Book, New Problems and Conjectures

I worked on The Mathematical Coloring Book for 18 years, 1990–2008. It took ca. 15 more years to produce the new, ca. 900-oversized-page, 5 cm thick, magnum opus:

The New Mathematical Coloring Book, Springer, New York, 2024 [5].

It was born in April 2024. No Kings and Shepherds came to express adoration, however, the great sorcerer Saharon Shelah called the newborn “a masterpiece.”

There is a popular proverb “The more I know, the less I know.” Taken literally, it is nonsense. The intended prose should be “The more I know, the better I realize how much more there is to know.” The problems and conjectures assembled here are a result of my new field of vision. Let me keep the numbering from the book, so that you can read more in the book on problems of your interest.



One Odd Problem

Definition 60.1 (Rosenfeld[4]). The odd-distance graph E_{odd} is the graph with the vertex set E^2 of all points in the plane, in which two vertices are adjacent if and only if the distance between them is an odd integer.

Conjecture 60.5 (Soifer, 2009). $\chi(E_{odd}) \geq \aleph_0$.

Grapevine brought me news about James Davies, a graduate student at the University of Waterloo, Canada, whom I contacted on October 1, 2022. The same day, he sent me his paper, which was to appear two days later in arXiv. In it, Davies proves in the positive my Conjecture 60.5. Finite coloring is coloring in finitely many colors.

Theorem 60.7 (Davies[3], 2022). Every finite coloring of the plane contains a monochromatic pair of points at an odd distance from each other.

On December 23, 2023, James Davies informed me that this paper has finally been accepted by Geometric and Functional Analysis journal.

Prime Numbers Enter Ramsey Theory

In his recent lecture, Davies formulated two important extensions of Theorem 60.7 obtained by James Davies of Cambridge, Rose McCarty of Princeton, and Michał Pilipczuk of the University of Warsaw:

Theorem 60.9 Let $f(x) = a_nx^n + \dots + a_0$ be a polynomial with integer coefficients and an $a_n \geq 1$. Then, every finite coloring of the plane contains a monochromatic pair of distinct points at a distance of $f(x)$ from each other for some integer x .

Theorem 60.10 Every finite coloring of the plane contains a monochromatic pair of points whose distance from each other is a prime number. As you can see, prime numbers make their first appearance in the coloring world on the plane

On June 5, 2023, James Davies kindly sent me a draft with these two theorems. On December 23, 2023, Davies informed me that their trio submitted this paper to *Israel Journal of Mathematics*.

Dr. James Davies is presently the Gott Research Fellow in Mathematics at Trinity Hall, Cambridge.

Forbidden Binaries and Factorials

The odd-distance graph problem reminded me of a fantastic problem, used in 2010 in the 27th Colorado (now called Soifer) Mathematical Olympiad. It was proposed by the 1990 and 1991 first prize winner and now professor at Ohio State University Matthew Kahle. There is a two-way bridge: mathematical research provides a rich source for creating original Olympiad problems, and conversely, Olympiad problems often inspire “further explorations,” open problems and exciting research work.

Colorful Integers 60.11 (M. Kahle, 2008).

A. What is the minimum number of colors necessary for coloring the set of positive integers so that any two integers which differ by any power of 2 are colored in different colors? (Observe that 1 is a power of two: $2^0 = 1$).

B. What is the minimum number of colors necessary for coloring the set of positive integers so that any two integers which differ by any factorial are colored in different colors?

Solution of 60.11.A. Clearly 3 colors are necessary, since the numbers 1, 2, 3 pairwise differ by powers of 2 and thus require three distinct colors. On the other hand, coloring the positive integers cyclically modulo 3 does the trick because under this coloring the difference between two numbers of the same color is a multiple of 3, which is never equal to a power of 2. So, 3 colors are also sufficient. \square

Solution of 60.11.B. Assume 3 colors suffice. Since $1! = 1$ and $2! = 2$, any three consecutive integers must be colored in 3 distinct colors a, b, c . Numbers 1 through 6 must be colored a, b, c, a, b, c . Accordingly, number 7 must be colored a , but this is not allowed because $7 - 1 = 3! - 1$ - a contradiction. Thus, at least four colors are needed.

Suppose for a moment that there exists a number r , (necessarily irrational), such that $n!r$ is in the interval $[1, 3] \pmod{4}$, for every positive integer n . We will determine which of the 4 colors to

use on the integer k by looking at $kr \pmod{4}$: the color-defining-intervals $[0, 1)[1, 2)[2, 3)[3, 4)(\pmod{4})$ determine the 4-coloring of the set of positive integers.

Thus defined 4-coloring satisfies the conditions of the problem. Indeed, suppose $|i - j| = n!$ for some n . By multiplying through by r , we get $|r_i - r_j| = r_n!$, which is between 1 and 3 ($\pmod{4}$). In particular, r_i and r_j belong to different color-defining-intervals modulo 4, and thus i and j received different colors.

All that is left to prove is the existence of the desired r . Read it in the book. \square

When a fabulous Problem 60.8.B gets solved, we are inspired to see better, look further, aspire to a higher ground. Inspired by his Problem 60.8.B, Matthew Kahle proposes to increase the set of forbidden monochromatic distances in the plane from a singleton $\{1\}$ to all factorials $\{1!, 2!, \dots, n!, \dots\}$.

Open Factorial Coloring Problem in the Euclidean Plane 60.14 (M. Kahle). Find the minimum number of colors $\chi_F(E^2)$ required for coloring the Euclidean plane E^2 in such a way that no two points of the same color are at a factorial distance ($n!$) apart.

We do not even know whether $\chi_F(E^2)$ is finite, so you have plenty of enjoyable research to undertake!

Of course, the dimension in this problem can be raised, and thus we find ourselves in space, in the Euclidean n -dimensional space E^n .

Open Factorial Coloring Problem in Euclidean n -Space 60.15 Find the minimum number of colors $\chi_F(E^n)$ required for coloring the Euclidean n -space E^n in such a way that no two points of the same color are at a factorial distance ($n!$) apart.

Let us not forget Problem 60.11. A simply because it was simple on the line.

Open Binary Coloring Problem in the Euclidean Plane 60.16. Find the minimum number of colors $\chi_B(E^2)$ required for coloring the Euclidean plane E^2 in such a way that no two points of the same color are at a binary distance (2^n) apart.

Open Binary Coloring Problem in the Euclidean n -Space 60.17. Find the minimum number of colors $\chi_B(E^n)$ required for coloring the Euclidean n -space E^n in such a way that no two points of the same color are at a binary distance (2^n) apart.

These two open problems, perhaps, invite you to have an infinite fun, for the answers to them could be not finite but rather infinite cardinal numbers.

Davies–McCarthy–Pilipczuk[3] include, with credit, open Problems 60.14 and 60.16, with the following comment:

While we conjecture that both of these problems should have negative answers, due to exponential growth of the forbidden distances, it appears challenging to extend current methods to solve these two problems. Of course, it would be more exciting if either of these two problems has a positive answer [i.e., a finite chromatic number].

They also include a promising conjecture by Boris Bukh (Bukh[1]):

Conjecture 60.18 (Bukh). Let the subset $A \subset \mathbb{R} > 0$ be algebraically independent. Then there is a finite coloring of E^2 containing no monochromatic pair of points whose distance is contained in A .

Perfect Squares in Ramsey Theory

William Gasarch's comments inspired me to pose the following two open problems.

Open Perfect Square Coloring Problem in the Euclidean Plane 60.19 (A. Soifer 2023). Find the minimum number of colors $\chi_S(E^2)$ required for coloring the Euclidean plane E^2 in such a way that no two points of the same color are at a perfect square distance apart.

Open Perfect Square Coloring Problem in the Euclidean n -Space 60.20 (A. Soifer 2023). Find the minimum number of colors $\chi_S(E^n)$ required for coloring the Euclidean n -space E^n in such a way that no two points of the same color are at a perfect square distance apart.

7- and 8-Chromatic Two-Distance Graphs

It is easy for me to pose the following conjecture because it is weaker than my old 2002 conjecture $\chi(E^2) = 7$. Here, as before, by “the plane” we understand the Euclidean plane. A two-distance graph $G\{1, d\}$ is a graph in which two vertices are adjacent if and only if the distance between them is 1 or d .

Conjecture 61.1 (A. Soifer, 2022). There is a 7-chromatic two-distance graph in the plane.

And now two hard open problems:

Two-Distance Open Problem 61.2 (A. Soifer, 2022). Construct an 8-chromatic two-distance graph in the plane or prove that one does not exist.

And if the answer to Problem 61.2 is positive, we would like to find the answer to the following super hard problem, or obtain partial results:



Paul Erdős & Alexander Soifer, February 22, 1996

Baton Rouge, Louisiana

Second Two-Distance Open Problem 61.3 (A. Soifer, 2022). Over all $d > 1$, find a two-distance graph $G = G\{1, d\}$ in the plane of maximum chromatic number $\chi(G) = \Psi$. What are the values of d in graphs that realize Ψ ?

Coda: Paul Erdős

One mathematician told me during Paul Erdős' working and speaking 1988 visit of me in Colorado Springs: "What is a big deal in posing open problems – solving problems is what counts." I replied: "What would a problem solver solve if somebody did not pose great problems and envisioned great conjectures?" Paul Erdős did just that more and better than anyone in the entire history of mathematics.

In my New Mathematical Coloring Book the name "Paul Erdős" appears on more than 500 pages!

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Restarting the Mathematical Duel

Robert Geretschläger, Jaroslav Švrček, Jacek Uryga



Robert Geretschläger is recently retired from active teaching of Mathematics and Descriptive Geometry after 40 years at BRG Kepler in Graz, Austria. He is still as active as ever in the Austrian Mathematical Olympiad and the Austrian Mathematical Kangaroo, as well as continuing on in his roles as treasurer of the Association Kangourou sans Frontières and President of the WFNMC.



Jaroslav Švrček works as senior lecturer at Palacký University Olomouc, Czech Republic. He is member of the Czech MO committee and Czech–Slovak MO problem committee. He works also as editor of the czech journal Matematika-Fyzika-Informatika.



Jacek Uryga graduated from the Silesian University of Technology in Gliwice with a degree in applied mathematics. He earned his doctorate at the Institute of Mathematics of the Polish Academy of Sciences and currently works at the Silesian University of Technology in the Department of Applied Mathematics. He has always been involved in high school education. He currently works at the VIII High School in Katowice and the Academic High School in Gliwice. He is also secretary of the district committee of the Polish Mathematics Olympiad.

Introduction

The Covid era will long be remembered for its interruption of all manner of activities, and mathematical competitions were no less affected by the virus than anything else. As we all know, many competitions were either forced to switch from in-person to online formats because of the pandemic, or were not able to be held at all.

One such competition, forced into hiatus by the aggressively contagious disease, was the annual Mathematical Duel. This traditional olympiad-style competition between schools in Austria, the Czech Republic and Poland was meant to be held for the 28th time in Graz, Austria in March of 2020, but this turned out to be the very week that borders were closed in this area, and travel became all but impossible for reasons of public health.

As things turned out, this little hiccup happened to coincide with changes in the financial background of the competition, and so it was not until 2024 that the tradition was able to be started again. This is the story of that restart, which appears at first glance to have been quite successful, and seems to allow a somewhat optimistic glance to the future of the competition.

1. A Short History of the Mathematical Duel

A more detailed version of the history of the Mathematical Duel is available at [1], but for anyone not familiar with the background of the competition, here is a bit of information on how it was originally started and how it developed over the years, up to its abrupt interruption by the pandemic in 2020.

The Mathematical Duel was started in 1993 as an actual “duel” between students from two schools, the Gymnázium Mikuláše Koperníka in Bílovec, Czech Republic and the I Liceum Ogólnokształcące im. Juliusza Słowackiego in Chorzów, Poland. The competition was held annually, with Bundesrealgymnasium Keplerstraße from Graz, Austria joining in 1997 and Gymnázium Jakuba Škody from Přerov, Czech Republic joining in 2008. Since then, it had been a competition between those four schools, with other schools (usually from these three countries, but also from Bulgaria, Hungary, Italy and Romania) often invited to participate as guests. (Note that a special international version of the competition is described in a bit more detail at [2].)

The competition was in typical olympiad style, and divided into three categories by age of the participants. Division A was for grades 11-12, B for grades 9-10 and C for grades 8 or younger. Typically, four students from each category would participate (although teams were sometimes incomplete due to the illness of an intended participant or for some other reason). The students wrote individual papers, comprising four problems to be solved in 150 minutes, and also a separate team competition, comprising three problems to be solved together by the group in 100 minutes. (Collections of problems from the competition are widely available. Beside the book [1] already mentioned, problem collection booklets for some of the years of the competitions have also been published: [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13].) The venue rotated annually between the four participating schools, along with the responsibility for organising the competition, accommodation, a local excursion and prizes.

In 2020, Graz was set to host the competition. Everything was ready to go, just as it became clear that the borders between Poland, the Czech Republic and Austria were to be closed because of the health crisis. Because of this, the 28th Duel could not take place as planned, and everything was put on hold for the time being.

2. A New Beginning

We now flash forward to early 2024. The Covid pandemic has not really gone away, but for practical purposes, people now once again go about their daily lives ignoring the various illnesses that may or may not catch up to them at some point in time. Specifically, Covid has been pushed back in the public consciousness, and is now taken about as seriously as the flu or any other potentially dangerous, but not omnipresent, health threat.

Preliminary discussions about resuscitating the Duel had started as early as 2022, but serious discussion of a continuation began in earnest at a meeting of representatives of some of the schools in January of 2024. There were many new circumstances to be considered if such a re-launch was to work. A main problem was, as can be expected in any activity of this type, the financial aspect. In the early years of the competition, students were often housed with families of participants in the hosting city, which significantly reduced costs for the schools. For whatever reason, this soon became more and more difficult to organize, with less and less families willing to host competition participants. After years of struggling to find money from year to year, the Duel was integrated as part of an Erasmus+ research project from 2015 to 2017, funded by the European Union. This resulted in a great deal of interesting research (some of which can be found at the website mathematicalduel.eu), and also led to a very welcome three-year break in the constant search for funding. Unfortunately, the extension of this research project was not supported by the Erasmus+ program, and by 2020, it had once again become necessary to find money for the competition on a year-to-year basis.

Meanwhile, the Polish partner had seen major changes between 2018 and 2020. One of the founders and co-organizer of the competition, Józef Kalinowski, withdrew from active work, while the other most active person on the Polish side, Jacek Uryga, ended his job at the high school in Chorzów and started in a new position at the Academic High School in Gliwice, founded by the Silesian University of Technology in 2018.

The new school quickly attracted young people with interests in science and natural sciences, for whom one of the most important activities is participation in scientific competitions. With all the more energy and joy, the school became involved in the reactivation of the already established competition. Unfortunately, the continuation was to go on without the participation of the Chorzów school. Not only had the main proponent of the Duel left the school, but interest in the competition had dried up to the point of non-existence.

The situation at the school in Bílovec had also changed in the meantime. When the Mathematical Duel was started in 1993, this school was one of three mathematical focus schools in the Czech Republic, and there were many students at the school with a deep interest in the subject. By the 20s of the new millennium, interest in this specialty had shrunk to the extent that the school in Bílovec was no longer offering a specialty stream of this type. For these reasons, it was decided to restart the Mathematical Duel in a somewhat reduced version.

First of all, only three schools would participate in this first new edition, namely the schools in Přerov, Graz and Gliwice. Since the school in Gliwice only offers grades 9 through 12, it was also decided to restrict the competition to categories A and B, and this reduction meant a total participation of only 24 students, in contrast to the 60 participants from five schools in the old model. Also, since there was no external money available, each school would pay their own way. In order to keep costs down, a venue was chosen in Karlov in northern Moravia, at the foot of the Praděd, the highest mountain in Moravia. This venue is a common host of similar mathematical events, and the overall atmosphere here was perfect for the purpose.

3. A Few Nice Problems from 2024

In order to illustrate the style of the competition, we include four problems from this year's competition, i.e. the 28th Mathematical Duel.

Problem 1 (Category A – Individual Competition – Number 3)

Prove that for any positive integer n , the inequality

$$\frac{1}{1^2 \cdot 2} + \frac{1}{1 \cdot 2^2} + \frac{1}{2^2 \cdot 3} + \frac{1}{2 \cdot 3^2} + \dots + \frac{1}{n^2(n+1)} + \frac{1}{n(n+1)^2} < 1.$$

holds.

Solution. For any positive integer n , we have

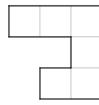
$$\frac{1}{n^2(n+1)} + \frac{1}{n(n+1)^2} = \frac{(n+1)+n}{n^2(n+1)^2} = \frac{(n+1)^2 - n^2}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}.$$

We therefore obtain

$$\begin{aligned} \frac{1}{1^2 \cdot 2} + \frac{1}{1 \cdot 2^2} + \frac{1}{2^2 \cdot 3} + \frac{1}{2 \cdot 3^2} + \dots + \frac{1}{n^2(n+1)} + \frac{1}{n(n+1)^2} \\ = \left(\frac{1}{1^2} - \frac{1}{2^2} \right) + \left(\frac{1}{2^2} - \frac{1}{3^2} \right) + \dots + \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \\ = 1 - \frac{1}{(n+1)^2} < 1, \end{aligned}$$

which proves the given inequality. \square

Problem 2 (Category A – Team Competition – Number 2)



Franz has an unlimited number of tiles of the following form:

Each tile is composed of six unit squares. They can be flipped over or moved around at will. Franz wants to tile an $m \times n$ rectangle with such tiles completely, without overlap, cutting tiles or allowing tiles to extend over the edge of the rectangle. Which of the following rectangles can be tiled in such a way?

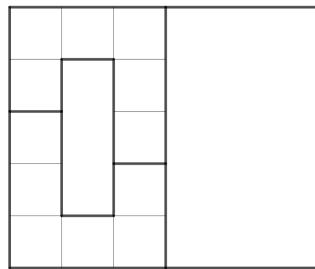
- a) 10×11 ,
- b) 6×5 ,
- c) $12 \times a$ with any integer $a \geq 6$.

Prove your claim in each case.

Solution. The answers to the three questions are as follows: a) no, b) no, c) yes.

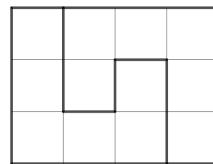
a) Since there are $10 \cdot 11 = 110$ unit squares in a 10×11 rectangle and 110 is not divisible by 6 (i.e. the number of unit squares covered by each tile), the rectangle cannot be covered completely by tiles.

b) This is not possible either. Taking a look at the side of the rectangle of length 5 (shown on the left), the only way to cover all five unit squares along the edge is the one shown in the following figure:



This leaves a 3×1 section in the interior that cannot be covered by a tile. The tiling is therefore not possible.

c) This is always possible. Two tiles can cover a 3×4 rectangle in the following way:



We can put four of these rectangles on top of each other to produce a 12×4 strip, or three of them, turned by 90° , to produce a 12×3 rectangle. Since a division of a by 3 leaves a remainder of either 0, 1 or 2, we can write a as either $a = 3k$ or $a = 3k + 1 = 3(k - 1) + 4$ or $a = 3k + 2 = 3(k - 2) + 8$.

In the first case, we can cover a $12 \times a$ rectangle by k strips of the size 12×3 . In the second, we can cover a $12 \times a$ rectangle by $k - 1$ strips of the size 12×3 and one strip of the size 12×4 (noting that $k > 1$, since $a > 6$ must hold in this case). Finally, in the third case, we can cover a $12 \times a$ rectangle by $k - 2$ strips of the size 12×3 and two strips of the size 12×4 (again noting $k > 1$, since $a > 6$ must hold). In summary, the rectangle can certainly be covered by tiles of the given type. \square

Problem 3 (Category B – Individual Competition – Number 2)

Let a and b be integers, such that $a^2 + ab + b^2$ is divisible by $a + b$. Prove that $a^4 + b^4$ is divisible by $a + b$.

Solution. Since $a^2 + ab + b^2 = a(a + b) + b^2 = a^2 + b(a + b)$ is divisible by $a + b$, the number a^2 (and similarly the number b^2) must also be divisible by $a + b$. Moreover, the numbers a^4 , b^4 and their sum $a^4 + b^4$ are therefore also divisible by $a + b$. Thus, the proof is finished. \square

Alternative solution. Rewriting the expression $a^2 + ab + b^2$ in the form

$$a^2 + ab + b^2 = (a^2 + 2ab + b^2) - ab = (a+b)^2 - ab,$$

implies that the number ab , and thus a^2b^2 , are divisible by $a+b$. We can then rewrite the sum $a^4 + b^4$ in the form

$$a^4 + b^4 = (a^2 - b^2)^2 + 2a^2b^2 = (a+b)^2(a-b)^2 + 2a^2b^2.$$

Since both summands on the right side are divisible by $a+b$, the proof is done. \square

Problem 4 (Category B – Team Competition – Number 1)

Determine all triples (x_1, x_2, x_3) of positive real numbers satisfying the following system of equations:

$$x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = x_3 + \frac{2}{x_1} = 3.$$

Solution. The equations in the system can be rewritten in the form

$$x_1 - x_2 = \frac{2(x_2 - x_3)}{x_2 x_3}, \quad x_2 - x_3 = \frac{2(x_3 - x_1)}{x_3 x_1}.$$

If $x_1 - x_2 > 0$, then $x_2 - x_3 > 0$ and so $x_3 - x_1 > 0$, since x_1, x_2 and x_3 are positive. This yields the contradiction

$$0 = (x_1 - x_2) + (x_2 - x_3) + (x_3 - x_1) > 0.$$

Similarly, the assumption $x_1 - x_2 < 0$ leads also to a contradiction. We thus have $x_1 = x_2 = x_3 = t$ and therefore

$$t + \frac{2}{t} = 3 \iff t^2 - 3t + 2 = 0 \iff (t-1)(t-2) = 0.$$

This equation is fulfilled for $t = 1$ and $t = 2$, and the given system of equations therefore has exactly two solutions

$$(x_1, x_2, x_3) \in \{(1; 1; 1), (2; 2; 2)\}.$$

\square

4. Looking to the Future

All participants agree that this revitalized version of the Mathematical Duel went well, with excellent performances by the participants. An agreement has been reached to continue on an annual schedule as before. In the next years, the competition is planned for the fall, in a change from the March dates of the previous years. This seems to fit better with the changes in the academic calendars of the participating schools. The competition in 2025 is planned for Graz, and 2026 will see Gliwice hosting for the first time.

There may well be hope for the school in Bílovec to be welcomed back to the fold, but this remains to be seen. Similarly, any hope of including guest schools again or widening the scope in some other way remains an option. Time will tell what is actually possible.

What is certain is that the high quality of the participants' solutions certainly warrants a continuation of the program. Over the years, many future IMO participants and medal winners have had a first taste of international competition at the Mathematical Duel, and there is every reason to be optimistic that this trend can continue in the coming years.

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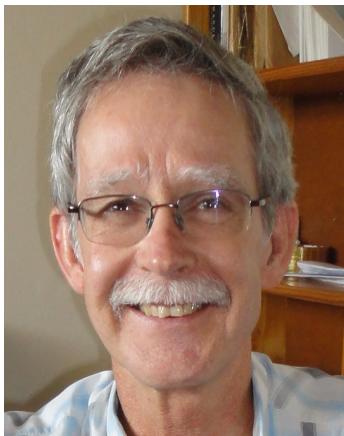
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Another concurrency related to the Fermat point of a triangle

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Johan Meyer obtained his PhD from the University of Stellenbosch, South Africa, in 1986. His research interest is mostly in Algebra, in particular rings and near-rings, and their modules. He taught mathematics at the University of Stellenbosch until June 1987, after which he was appointed as senior lecturer at the University of the Orange Free State (later renamed to the University of the Free State), in Bloemfontein, South Africa. He always had a keen interest in Olympiad-type problems and served (and still serves) on various committees in this regard (at both school level and tertiary level).

Abstract

An interesting concurrency result related to the first Fermat (isogonic) point of a triangle was recently discovered using dynamic geometry. We provide a computer proof and an algebraic proof as well as a dynamic sketch to explore.

Introduction

In the 1600's, the French mathematician, Pierre de Fermat, posed the following intriguing problem, namely, where inside a triangle should a point be placed so that it minimizes the sum of the distances to the vertices of an acute-angled triangle?

The first one to solve it was the Italian mathematician and scientist Evangelista Torricelli (famous, in particular, for his invention of the barometer). He showed that if one constructed equilateral triangles on the sides of the triangle then the required point is located at the point of concurrency of the lines connecting the outer vertices of each equilateral triangle with the opposite vertex of the base triangle. Hence, this point of concurrency is often called the Fermat-Torricelli point.

Since that time many different proofs for the result has been given, and the concurrency result has also been further generalized, and applied in different situations (e.g. see De Villiers[4]). Free downloadable worksheets with sketches for high school learners using dynamic geometry are available in De Villiers[5], and these provide an investigative introduction to the Fermat-Torricelli point for use in the classroom.

Note that when one of the angles of the base triangle becomes greater than 120° , then the optimal point for minimizing the sum of the distances to the vertices would be located at the angle greater than 120° . However, the lines connecting the outer vertices of each equilateral triangle with the opposite vertex of the base triangle still remain concurrent, though the point of concurrency lies outside the base triangle. This more general point of concurrency is usually referred to as the first ‘isogonic’ centre of a triangle (Mackay[11]). If the equilateral triangles are constructed to the interior of the triangle, then the afore-mentioned lines are concurrent at what is called the second ‘isogonic’ centre.

The following interesting result related to the first Fermat (isogonic) point of a triangle was recently discovered by us using *Geometer’s Sketchpad*¹. The beauty of using dynamic geometry is that one can experimentally very quickly verify whether a result is true before trying to find a proof. In this case we proceeded by first constructing the (first) Fermat (isogonic) point of a triangle and then the second Fermat (isogonic) points of each of the three triangles into which it is divided. To our pleasant surprise, we found that the lines connecting these points to the opposite vertices of the base triangle were concurrent. While the result was new to us personally, it is quite likely that the result appears somewhere in some paper or book, and that it is already listed in the online Encyclopedia of Triangle Centers (ETC). An online dynamic sketch for readers and students to explore is available at <http://dynamicmathematicslearning.com/another-concurrency-related-to-fermat.html>

The Theorem

Theorem. Let D be the first Fermat (isogonic) point of $\triangle ABC$ and E, F and G be the second Fermat (isogonic) points of $\triangle ABD, \triangle BCD$ and $\triangle CAD$, respectively. Then AF, BG and CE are concurrent (see Figure 1).

Computer proof. Despite the convincing power of dragging in dynamic geometry, we duplicated the construction in *Cinderella*². This was done to also check it there, since the software has a built-in ‘proof-checking’ facility. As shown in Figure 1, after constructing lines AF and BG and their intersection R , and next proceeding to construct line CE (line p in the sketch), immediately brought up a console stating that “ R ” lies on “ p ” (thus ‘proving’ the concurrency of the three lines). While it is not clear exactly how the software determines this concurrency, it appears from the manual that it is based on a technique called “randomized theorem checking” (Richter-Gebert & Kortenkamp[12] p. 48).

Presumably, the property checker of *Cinderella* is based on the mathematical theory described in Davis[3] who has pointed out that an algebraic identity can be conclusively established by a

¹*Geometer’s Sketchpad* is available for free to download from: <http://dynamicmathematicslearning.com/free-download-sketchpad.html>

²*Cinderella* is available for free to download from: <https://www.cinderella.de/tiki-index.php>

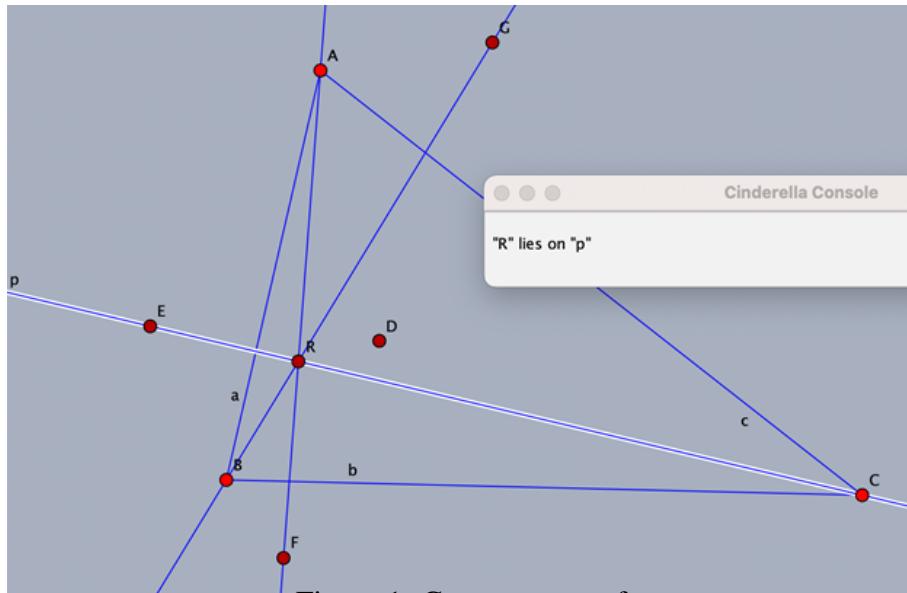


Figure 1: Computer proof

single numerical check by using algebraically independent transcendental numbers. Although computers cannot actually operate with transcendental numbers, a series of experiments selecting points at random, achieves much the same result. In other words, if experiment after experiment with randomly selected points reaffirms the same result, the probability of the result being false effectively becomes zero.

While this ‘computer proof’ with *Cinderella* provided us with further conviction of the truth of the result, we still felt the need for a proof, mainly for two reasons: 1) the given proof provides no explanation of why the result is true, and 2) the intellectual challenge of proving the result in a traditional deductive manner (compare De Villiers[6]).

The concurrency result proved to be a much harder problem than what it had seemed like initially. At first we tried using what seemed like the most natural approach, namely, the sine version of Ceva’s theorem as well as the hexagon concurrency theorem of Anghel ([1],[2]), but were unable to find a proof.

Finally, we succeeded in developing the following proof using coordinates. □

Coordinate Proof. Place the Fermat point D of $\triangle ABC$ at the centre $(0,0)$ of the coordinate system. Let us assume (without loss of generality) that $BD \leq AD \leq CD$, with $R = BD$, $S = AD$ and $T = CD$. Therefore $R \leq S \leq T$, and $\triangle ABC$ can always be rotated or reflected so that it is placed as shown in Figure 2, with a possible permutation of A, B, C so that $R \leq S \leq T$.

Since $\angle ADB = \angle BDC = \angle CDA = 120^\circ$, three concentric regular hexagons are formed as shown in the figure, with radii R, S and T . Therefore the coordinates of the vertices of $\triangle ABC$ are $A\left(\frac{-S}{2}, \frac{\sqrt{3}S}{2}\right)$, $B\left(\frac{-R}{2}, \frac{-\sqrt{3}R}{2}\right)$ and $C(T, 0)$.

Lemma. If l_1 is the line through the points $(a_1, b_1), (c_1, d_1)$, and l_2 is the line through the points $(a_2, b_2), (c_2, d_2)$, then the x -coordinate of $l_1 \cap l_2$ (assuming l_1 is not parallel to l_2) is given by

$$\frac{[(d_1 - d_2)a_2 + c_2(b_2 - d_1)]a_1 - [(b_1 - d_2)a_2 - c_2(b_1 - b_2)]c_1}{(a_1 - c_1)(b_2 - d_2) + (d_1 - b_1)(a_2 - c_2)}.$$

The proof is left to the reader as an exercise

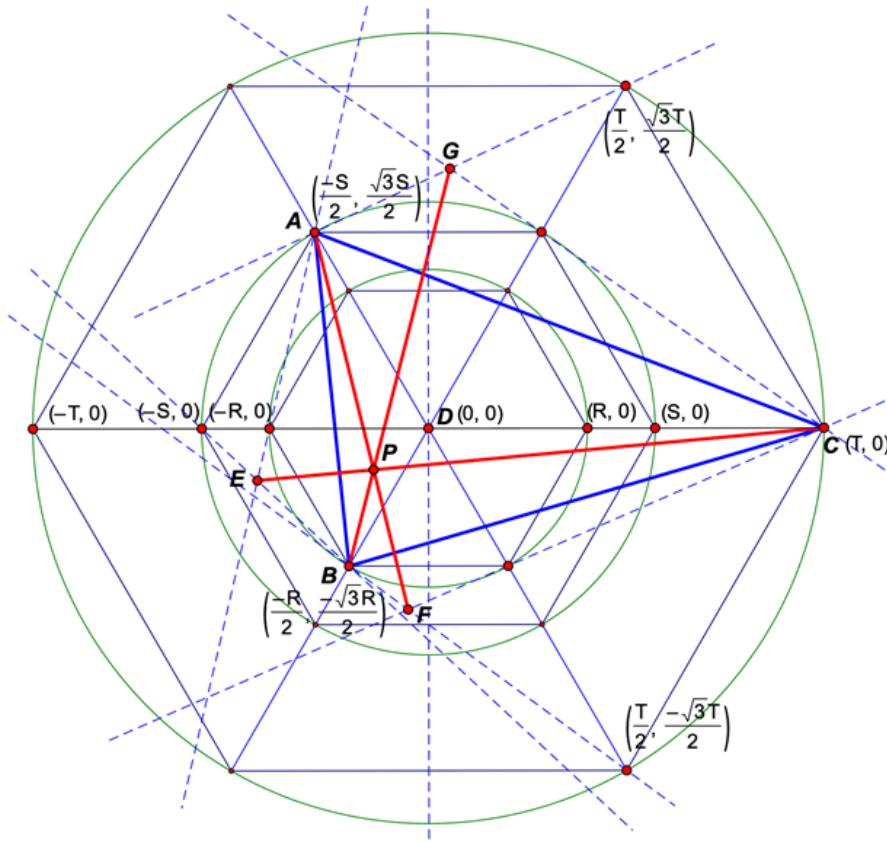


Figure 2: Coordinate proof

The y -coordinate can now similarly be determined by substituting the x -coordinate above into the equation of l_1 (or l_2).

By repeated application of this Lemma we can determine the second Fermat points G, E, F of respectively $\triangle CAD, \triangle ABD$ and $\triangle BCD$. For example, G is the intersection of the line through $C(T, 0)$ and $\left(\frac{S}{2}, \frac{\sqrt{3}S}{2}\right)$ and the line through $\left(\frac{T}{2}, \frac{\sqrt{3}T}{2}\right)$ and $A\left(-\frac{S}{2}, -\frac{\sqrt{3}S}{2}\right)$.

This gives, after simplification, $G\left(\frac{ST(2S-T)}{2S^2-2ST+2T^2}, \frac{\sqrt{3}ST^2}{2S^2-2ST+2T^2}\right)$.

Similarly, we get $E\left(\frac{-RS(R+S)}{2R^2-2RS+2S^2}, \frac{\sqrt{3}RS(R-S)}{2R^2-2RS+2S^2}\right)$ and $F\left(\frac{RT(2R-T)}{2R^2-2RT+2T^2}, \frac{-\sqrt{3}RT^2}{2R^2-2RT+2T^2}\right)$.

Now we determine the intersection points $AF \cap CE$, $AF \cap BG$ and $BG \cap CE$ by again using the Lemma. In each of the three intersections we get the same point

$$P\left(\frac{RST(2R^2S^2 - T^2(R^2 + S^2))}{2R^3S^3 + 2R^3T^3 + 2S^3T^3 + 2R^2S^2T^2}, \frac{\sqrt{3}RST^3(R^2 - S^2)}{2R^3S^3 + 2R^3T^3 + 2S^3T^3 + 2R^2S^2T^2}\right).$$

This then completes the proof of the concurrency of AF , BG and CE .

Special cases

$$\begin{aligned} R = S \ (\triangle ABC \text{ is isosceles}) &\longrightarrow P\left(\frac{(R-T)RT}{R^2-RT+2T^2}, 0\right) \\ R = S = T \ (\triangle ABC \text{ is equilateral}) &\longrightarrow P(0,0) = D. \end{aligned}$$

□

It should also be mentioned that the symbolic processing software, *Maple*, was used to assist in the algebraic manipulation and simplification of the proof above.

While this proof undoubtedly satisfied our need for personal conviction and provided us with intellectual satisfaction in conquering the challenging problem facing us, it unfortunately still does not adequately explain in a simple, elegant way why the result is true. It is therefore hoped in due course that we ourselves, or perhaps others, will succeed in finding a less brute force proof of the concurrency result that is more explanatory.

Other Interesting Properties

The configuration has several other interesting mathematical properties, some of which are:

1. $\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CG}{GA} = 1$.
2. Consider Figure 3. Extend AJ to meet BI in X , and repeat the same construction on the other two sides BC and CA of $\triangle ABC$ to locate corresponding points Y and Z . Then XC , YA and ZB are concurrent. (View this construction at the URL given earlier).
3. The circumcircles of triangles AEB , BFC and CGA are concurrent. (Similarly, the circumcircles of triangles EBF , FCG and GAE are also concurrent).
4. The respective circumcenters of triangles AEB , BFC and CGA form an equilateral triangle.
5. $\angle EFG = \angle EBA + \angle ACG$, $\angle EGF = \angle EAB + \angle BCF$ and $\angle GEF = \angle GAC + \angle CBF$.

Readers are encouraged to interactively explore these properties at the URL given earlier. While the concurrency result is probably a little too hard and therefore not suitable in our opinion for possible use in a mathematics competition or training program, the five properties given above should be accessible for talented mathematics learners at different levels.

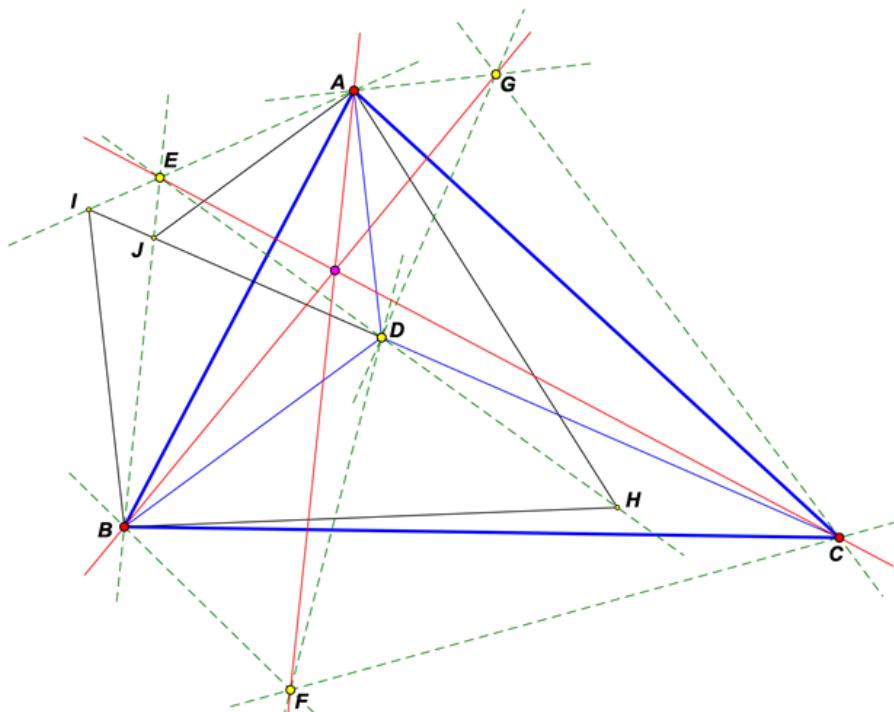


Figure 3: Proof property 1

Proof of Property 1

Consider Figure 3 which shows the relevant points as well as the constructed equilateral triangles ABH , BDI and DAJ for the location of the 2nd Fermat point E for $\triangle ABD$.

Since it is well-known that the green lines at each of E, F, G form 60° angles with each other, it follows that $AEJD$ is cyclic, since $\angle AED = 60^\circ = \angle AJD$ (angles subtended on chord AD). Therefore, if we let $\angle EAJ = x$, then $\angle EDJ = x$ (angles on chord EJ). Similarly, it follows that $EIBD$ is cyclic and that $\angle EDJ = x = \angle EBI$ (angles on chord EI). Hence, $\angle EBI = x = \angle EAJ$.

It now follows that $\angle ADE = 60^\circ - x$. But $\angle EBD = 60^\circ - x$; therefore $\angle ADE = \angle EBD$.

Apply the sine rule respectively to triangles AED and EBD to obtain the following two equations: $\frac{AE}{\sin(ADE)} = \frac{AD}{\sin(60^\circ)}$ and $\frac{ED}{\sin(EBD)} = \frac{BD}{\sin(60^\circ)}$. Divide the first equation by the second and re-arrange to obtain: $AE = \frac{ED \cdot AD}{BD}$.

Similarly, $EB = \frac{ED \cdot BD}{AD}$ and therefore $\frac{AE}{EB} = \frac{AD^2}{BD^2}$. In the same way can be shown that $\frac{BF}{FC} = \frac{BD^2}{CD^2}$ and $\frac{CG}{GA} = \frac{CD^2}{AD^2}$.

Therefore,

$$\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CG}{GA} = \frac{AD^2}{BD^2} \times \frac{BD^2}{CD^2} \times \frac{CD^2}{AD^2} = 1.$$

Note: While the property $\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CG}{GA} = 1$ may strongly remind one of Ceva's theorem, it is unfortunately not equivalent to it; in fact, Property 1 is neither necessary nor sufficient to prove AF , BG and CE concurrent. In general, according to the hexagon concurrency theorem of Anghel ([1],[2]), to prove AF , BG and CE concurrent, we need to prove $\frac{AE}{EB} \times \frac{BF}{FC} \times \frac{CG}{GA} = \frac{\sin(EBC)}{\sin(ABF)} \times \frac{\sin(FCA)}{\sin(BCG)} \times \frac{\sin(GAB)}{\sin(CAE)}$. In other words, prove that $\frac{\sin(EBC)}{\sin(ABF)} \times \frac{\sin(FCA)}{\sin(BCG)} \times \frac{\sin(GAB)}{\sin(CAE)} = 1$, but were unable to do so.

Proving the other four properties is left as an exercise to readers and students. Property 2 above follows directly from the point symmetry of the formed parallelo-hexagon $AXBYCZ$. Property 3 about the circle concurrencies is easily proven using cyclic quadrilaterals (De Villiers[7]) and Property 4 about the formed equilateral triangle is simply a variation of Napoleon's theorem (De Villiers[8]). Lastly, Property 5 follows directly from Property 1 and the application of the theorem of Egamberganov[9].

Concluding remarks

With the availability of increasingly powerful software and artificial intelligence (AI), it is perhaps prudent to ask why do we still need to deductively verify (prove) an experimentally discovered result like this when a computer or AI can produce a proof. Firstly, producing a proof can often help one to better and more deeply understand why a result is true, rather than just knowing a result is true. Secondly, while artificial intelligence and other software have already produced some impressive results, it is perhaps still cautionary to bear in mind an example given by Garaschuk[10] where five logical problems were given to ChatGPT, and while it produced plausible sounding solutions to each one, none of them were correct! Lastly, the intellectual challenge of finding a proof for oneself is what largely appeals to mathematicians, much like solving a crossword puzzle, conquering a mountain peak or running a marathon.

Web Supplement.

<http://dynamicmathematicslearning.com/another-concurrency-related-to-fermat.html>

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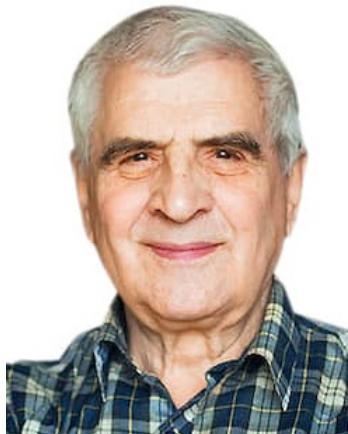
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The Summer Conference of the Tournament of Towns

Nikolay Konstantinov³

Translation by Alina Muratova, edited by Alexei Sossinsky



Nikolay Konstantinov (1932 – 2021) was a leading Soviet and Russian mathematical educator and organizer of numerous mathematics competitions for high school students. He is best known as the creator of the system of math schools and math classes and also as the creator and chief organizer of the Tournament of the Towns. For his work he was awarded the Paul Erdős award in 1992.

Preliminaries

In over 100 years of their existence, math circles and olympiads, originally conceived as an auxiliary in math teaching, have accumulated a wealth of material. Thus, they have become a noticeable separate part of math education and of the mathematical sciences. University students experienced in math circles and in olympiads outperform students who are only familiar with the school-level math curriculum. For this reason, certain communities of schoolteachers and professional mathematicians make substantial use of “olympiad mathematics” when teaching school students.

At the beginning stages of working with future mathematicians, the competitive spirit plays an important role; it is hard to pinpoint whether there are more pros or cons to that. The positive impact is undeniable: before math circles came into being, only very few people became mathematicians, whereas hundreds and thousands did not at all. Young people have the need to constantly challenge themselves. They are not embarrassed by failures, which are even necessary for choosing one’s professional path correctly.

The unfavourable effect is that not all students are interested in competing, some might be put off by having to compete, since different people have different personalities. In olympiads and other math competitions, one has to rush, which contradicts the spirit of science, and the mathematical content often takes a back seat.

Additionally, boosting the so-called “healthy ambitions” which do not become healthier by merely labeling them as such, can be detrimental to a person’s life.

³This text was written around 2006. Published in Russian in the almanac “Matematicheskoye Prosvetshchenie”, [1].

That is why once the goal of the competition (i.e. to draw beginners into problem-solving) has been attained, the students' attention should be directed to the cognitive and constructive content of science. Olympiads are not competitions of students against each other, but our collective competition against Eternity, as Sergey Markelov, one of the organisers of math olympiads and author of beautiful olympiad problems, puts it.

The following are the steps taken in that direction in some popular mathematical competitions

In the multiple-subject M. V. Lomonosov Tournament, which takes place in approximately 20 Russian cities and towns (in Moscow there were about 7,000 participants who submitted about 30,000 works)⁴, almost half of the participants are awarded with merit certificates listing their achievements in all the different types of problem sessions in which they participated, but these certificates did not indicate who was first, second, etc. The formulations never mention that the result is not the highest. Students and teachers can assess student's performance levels by themselves, but the jury abstains from that. (They act only as a jury in a courtroom: guilty or not guilty - not as a judge.) Of course, there is a distinction between those who received the certificate and those who did not, but it is lessened by the fact that all students who attend the final meeting of the Lomonosov competition receive small memorable gifts

At The Tournament of Towns, winner diplomas are given to all the competitors who score a specific amount of points (not less than 11 points in the 27th Tournament of Towns), but they also do not specify the place taken. Those who score from four to ten points are awarded bonuses by the local jury. So, there are two separating thresholds, but they are insignificant for the strongest students, for whom leveraging their ambition can be most detrimental. The rule whereby a passing grade is granted on the basis of three best problems removes the need for any rush. At the Tournament of Towns there are four rounds every year, and the best result of the four is regarded as the result for the given year. This rule (of choosing the best one, not adding up the results) also reduces stress.

If the competitive spirit prevails in working with school students, it is easy to imagine how a young person may see his or her prospects in mathematics. This is achieving higher and higher results in olympiads of higher and higher level, up to IMO (International Mathematics Olympiad). Unfortunately, that leads many students away from science. Not to mention, for the majority, this route ends with defeat, but even in case of victory, the International Olympiad does not teach things needed for academic activity.

The Summer Conference of the Tournament of Towns shows a different perspective to advanced students. Below we explain how this event is organised.

Nonetheless, if not the competition, what is it that stimulates the drive to succeed, and most importantly how will the student define success.

The ultimate matter in that period are the interesting problems to solve. The reward is the teacher's attention to the student's manner of thinking, and the main result is success in solving problems

⁴The data for the moment when the original text was written.

and the feeling of power that comes from overcoming true challenges.

Let us say that this is the second stage of the student's developing interest in mathematics, even though the boundaries of this stage are vague. For many mathematicians, this stage turns out to be the last, and it is not a bad thing, since these people contribute to the mathematical development and further advancement.

The third stage comes when the mathematician starts to wonder how his/her science is related to real life problems. "People around are in difficulty, yet we are unable to help them, although we solve difficult problems - our work does not cater for the people" in the words of the outstanding Russian mathematician A. Vitushkin.

Young mathematicians whom we work with, in most cases, have not reached stage three because of their age. Still, we do not miss the opportunity to remind them that present day mathematics influences real life quite successfully rather often.

It should be noted that circles, olympiads and summer conferences are not the most appropriate place for such reminders.

Firstly, our pupils are insufficiently educated. Emphasising the significance of real-life application of some mathematics problems may often be irrelevant at the beginning stages of training. A physics teacher I worked with, speaking of math courses, put it this way: "I wish math stayed math, and I will explain myself the things I need from it: logarithms, derivatives, integrals." I agree with this approach. To be appealing, above all, every science should resemble itself.

Secondly, at olympiads there is a time limit for working on any of the given problems.

Thirdly, the topics at mathematics olympiads and conferences cannot go far beyond the school curriculum.

At olympiads and summer conferences, students are presented with novel, previously unpublished problems, and the participants do not know the methods needed to solve these problems in advance. Nevertheless, sometimes there are problems that either have immediate real-life impact or are closely connected to problems of other sciences. I will offer two examples as illustrations.

At one of the recent Moscow mathematics olympiads, the following problem was presented: *Is it possible to place an infinite number of congruent convex polyhedrons in a layer between two planes so that no one of the polyhedrons could be shifted from the layer without shifting the others?* This problem was suggested by the Russian mathematician A. Kanel-Belov. A construction company in Australia learned about the problem and developed a technology for producing high-strength ceilings based on it.

The second example is to do with genetics. The number of generations separating two allied species from their common ancestor can be assessed by the number of upturned sections of

DNA in allied chromosomes. According to geneticists, the frequency of upturned sections is not dependent on external conditions, which is why the number of upturns can be used for this assessment. A problem based on that fact was suggested at one of the Tournament of Towns Summer Conferences.

It makes sense here to emphasize what is, basically, the role of mathematicians working with experts in related sciences or technology. The popular belief that a mathematician solves problems posed by an expert in an applied field is not quite precise. Nowadays many applied scientists are perfectly capable of solving difficult applied problems by using software packages, and often do it better than mathematicians. But there are no packages that enable one to provide the correct setting of a problem. A creatively thinking mathematician is most useful when he/she and an applied scientist dig into the problem and formulate it in a way that makes it mathematically solvable. That requires a mathematician who is not only knowledgeable but thinks creatively. New problems do not fit into the old frameworks, and knowledge becomes obsolete rapidly. That is why math education now focuses on developing creativity instead of encyclopedic knowledge, and this works in favor of developing mathematical science.

For a mathematician to be a useful participant in this partnership, he/she should not only think creatively, but also be interested in problems of related fields, which can be science, industry, social and economic problems. How should we build mathematics education to avoid narrow-mindedness if well-trained mathematicians are simply unaware of what happens around them? It is impossible to include everything in the curriculum and it is not sufficient to rely on the curiosity of individuals

In the circle for eight-graders headed now by my students and myself, we tried the following innovation: in five minutes after a math circle meeting, we would conduct an independent second circle meeting dedicated to modern problems in the natural sciences. We had sessions on biology, physics and we have geology, astronomy, and invention classes in perspective. At the moment, half of the math circle participants attend the second club.

The same goals were pursued in organising the M.V.Lomonosov multiple-subject tournament; in it, different subjects' competitions take place close to each other, and no one forces subjects on participants not interested in them. As it turns out, many people are interested in many different questions.

Summer Conference

The Summer Conference of the Tournament of Towns is held annually for a small number of participants. There are about 10,000 high school students from approximately 20 countries participating in the Tournament of the Towns annually, about 1,000 of them are awarded Winner Diplomas from the Tournament's Central Jury, and about 70 students are invited to the Summer Conference. The conference is international, the working languages are Russian and English.

It would be preferable to involve a bigger number of students in such an event, but for now this is beyond the possibilities of the organisers.

The conference lasts one week.⁵ In the first hour upon arrival, the participants are offered groups of problems printed out in advance. The presentation of the groups takes place the next day in the form of lectures, which help to understand the conditions of problems and their motivations. Then, during the entire conference, the students try to solve these problems. Each group is a whole research topic, which can include dozens of sub-problems, from fairly simple ones, needed to introduce the topic, to the last ones, which may be yet unsolved. Students are advised to choose a single group for work, trying to achieve maximum advancement in it. Maximum advancement is considered more important than the number of solved problems. Collaborative work is allowed. The half-way finish takes place at the middle of the conference, which is when half-way achievements are recorded and new questions may be added.

On the last day, there is a final seminar. Students are awarded with Merit Certificates recording on their achievements, but are not ranked. Students and their supervisors can judge their own achievements and those of the other participants since all the results are published. The jury is not involved in any ranking (just as in the Lomonosov Tournament). All participants are awarded memorable gifts and math books.

The spirit of the Summer Conference is as close as possible to the usual scientific activity of a mathematician participating in a scientific conference. The similarity is amplified by the fact that some students continue working on the chosen topic after the Summer Conference of the Tournament of Towns, communicating with their supervisors via e-mail.

The foregoing was a brief overview of initiatives by the Moscow-based organisers of circles, tournaments, and conferences aimed at a greater coherence between implementation of the latter and the higher goals of math education.

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For more information on the Tournament of Towns, the Summer Conferences or Nikolay Konstantinov contact

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⁵Usually it is a few days longer.



- 25 B) 0,375 C) 0,75 D) 1,25 E) 1,5
- aičiuokite $x^4 + y^4 + z^4$, kai $x + y + z = 0$ ir $x^2 + y^2 + z^2 = 100$.
B) 250000 C) 500000 D) 750000 E) 1000000
- c una fracción entre $\frac{8}{9}$ y $\frac{222}{223}$. Si el denominador de x es 200 y el único factor común del numerador y el denominador es 200, para que el resultado sea menor que $\frac{1}{2}$, el numerador debe ser:
- 9 B) 100 C) 145 D) 150 E) 155
- ерата на n -та фигура от единичната линия е $n^2 - 19999$. Колко е много колко е n -та фигура, ако във всяка фигура има така че двете, чийто собствен номерат са m и n , то $m^2 - n^2$ е делъд на 200? Тогава $m^2 - n^2$ ще биде щито също е делъд на 200, ако се игнорират:
- 13 B) 4014 C) 5015 D) 5016 E) 5017
- reaalaryude paari (x, y) rahuldavad võrrandit $(x + y)^2 = 5(y - 5)$?
- B) 1 C) 2 D) 3
- pmatult palju
- li 1, 2, 3, 4, 5, 6, 7, 8, 9 sadalīti trīs grupās pa trim skaitļiem. Katrai grupai $\{a, b, c\}$ aprēķināts tajā ietilpst ošo skaitļu sājums abc . Apzīmēsim lielāko no šiem reizinājumiem ar M un mazākā iespējamā M vērtību?
- B) 64 C) 66 D) 72 E) 80

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