Optimization

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Unconstrained Optimization

- Given f(**x**). Find its (local) maximum/minimum points.
- Following one of the necessary conditions, we can differentiate the given function wrt input variables and set them to be zero.
- Use the result to find the corresponding values of f(**x**)

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Examples

- $f(x) = x^2$
- $f(x) = x^2 + 1$
- $f(x) = x^2 + x$
- $f(x1, x2) = x_1^2 + x_2^2$

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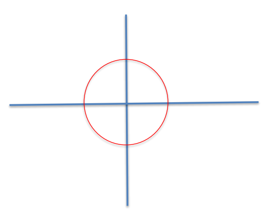
Constrained Optimization

- Find minimum value of f(x1,x2)= x1 + x2
  Subject to the condition that $x_1^2 + x_2^2 = 4$
- Latter is a constraint and hence represents a boundary (in 2-d space)



<0,2>

<2,0>

<1.414, 1.414>

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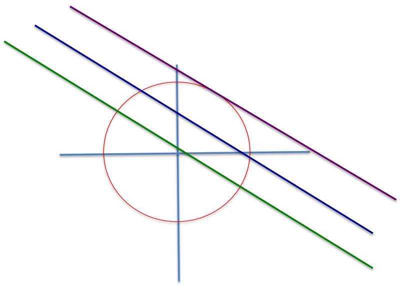
Using function and constraint



Where is it tangential? Remember line is given by $x_1+x_2 = b$

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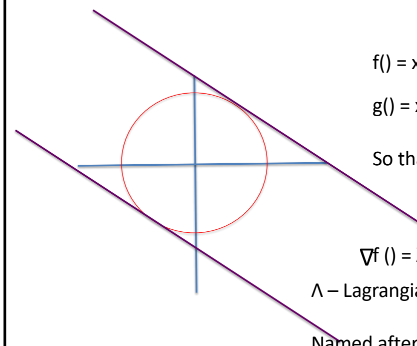
Where the slopes meet



$f() = x_1 + x_2$

$g() = x_1^2 + x_2^2 -4$

So that g() = 0

$\nabla f () = \lambda \ \nabla g( )$

Λ – Lagrangian multipliers

Named after Joseph Louis Lagrange

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Method for Constrained Optimization

- Minimize $f(x_1,\ldots,x_k)$ subject $g(x_1,\ldots,x_k) = 0$
- Create $L(x_1,\ldots,x_k, \lambda) = f(x_1,\ldots,x_k) - \lambda\ g(x_1,\ldots,x_k)$
- Differentiate wrt $x_1,\ldots,x_k$ as well as λ and set each to zero.
- Last differentiation just yields the constraint.

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Constrained Optimization 1 constraint

- Minimize $f(x_1,\ldots,x_k)$ subject to
  $g(x_1,\ldots,x_k) = 0$
- Then new objective function is
  - $L(x_1,\ldots,x_k,\lambda) = f(x_1,\ldots,x_k) - \lambda\ g(x_1,\ldots,x_k)$
  - $dL()/dx_1=0$, … , $dL()/dx_n=0$
  - $dL()/d\lambda=0$
- k+1 variables and k+1 equations.

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Constrained Optimization

- Minimize $f(x_1,\ldots,x_k)$ subject to
  $g_1(x_1,\ldots,x_k) = 0, \ldots\ g_m(x_1,\ldots,x_k) = 0$
- Then new objective function is
  - $L(x_1,\ldots,x_k,\lambda) = f(x_1,\ldots,x_k) - \Sigma\ \lambda_i\ g_i(x_1,\ldots,x_k)$
  - $dL()/dx_1=0$, … , $dL()/dx_n=0$
  - $dL()/d\lambda_1=0$,… , $dL()/d\lambda_m=0$

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Example

- Min $x_1+ x_2$ subject to $x_1^2 + x_2^2 = 4$
- $f(x_1,x_2) = x_1 + x_2$
- $g(x_1,x_2) = x_1^2 + x_2^2 - 4$
- $L(x_1,x_2,\lambda) = x_1 + x_2 - \lambda(x_1^2 + x_2^2 - 4)$

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Example 1

- $L(x_1,x_2,\lambda) = x_1 + x_2 - \lambda(x_1^2 + x_2^2 - 4)$
- $dL()/dx_1 = 0 \Rightarrow 1 -2\ \lambda\ x_1=0$ (I)
- $dL()/dx_2 = 0 \Rightarrow 1 -2\ \lambda\ x_2=0$ (II)
- $dl()/d\lambda = 0 \Rightarrow x_1^2 + x_2^2 - 4=0$ (III)
- From (1) and (II), $x_1 = 1/2\lambda = x_2$
- Using (IV) $1/(4\lambda^2) + 1/(4\lambda^2) -4 = 0$

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Example 1 Continued

- $x_1 = 1/2\lambda = x_2$
- $1/(4\lambda^2) + 1/(4\lambda^2) -4 = 0$
- $\lambda^2 = 1/8$
- $\lambda = \pm\ 1/(2\sqrt{2})$
- So minimum values when $x_1 = x_2 = -\sqrt{2}$.

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Constrained Optimization (Inequalties)

- Minimize $f(x_1,\ldots x_k)$ subject to
  $g_i(x_1,\ldots x_k) = 0 \quad 1 \le i \le m$
  $h_j(x_1,\ldots x_k) \le 0 \quad 1 \le j \le n$
- $L(x_1,\ldots x_k, \lambda_1,\ldots \lambda_m, \mu_1,\ldots \mu_n) =$
  $f(x_1,\ldots x_k) + \Sigma_i\ \lambda_i g_i(x_1,\ldots x_k) + \Sigma_j\ \mu_j h_j(x_1,\ldots x_k)$

  Notice "+"

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KKT Conditions

- Setup new objective function:
- $L(x_1,\ldots x_k, \lambda_1,\ldots \lambda_m, \mu_1,\ldots \mu_n) =$
  $f(x_1,\ldots x_k) + \Sigma_i\ \lambda_i g_i(x_1,\ldots x_k) + \Sigma_j\ \mu_j h_j(x_1,\ldots x_k)$
  1. $dL()/dx_1=0$, … , $dL()/dx_n=0$
  2. $dL()/d\lambda_1=0$,… , $dL()/d\lambda_m=0$

  1. $\mu_j \ge 0 \quad 1 \le j \le n$ (no longer unbounded)
  2. $\mu_j h_j(x_1,\ldots x_k) = 0 \ 1 \le j \le n$ ***
  3. $h_j(x_1,\ldots x_k) \le 0 \ 1 \le j \le n$

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More on KKT

1. $\mu_j \ge 0 \quad 1 \le j \le n$
2. $\mu_j h_j(x_1,\ldots x_k) = 0 \ 1 \le j \le n$ ***
3. $h_j(x_1,\ldots x_k) \le 0 \ 1 \le j \le n$

1. μ's are no longer unbounded
2. From 2, we can say (i) $\mu_j > 0$ and $h_j(x_1,\ldots x_k) = 0$ or (ii) $\mu_j = 0$
3. Notice we don't differentiate wrt μ, we still need to impose $h_j(x_1,\ldots x_k) \le 0 \ 1 \le j \le n$

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An Example(from Tan, Steinbach and Kumar)

- Minimize $f(x,y) = (x+1)^2 + (y-3)^2$ subject to $x+y \le 2$ and $y \ge x$.
- write second constraint as $x-y \le 0$ (not $y-x \ge 0$)
- $L() = (x+1)^2 + (y-3)^2 + \mu_1(x+y-2) + \mu_2(x-y)$

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Working Out the Example

- dL/dx = 0 → $2(x-1) + \mu_1 + \mu_2 = 0$ --- (1)
- dL/dy = 0 → $2(y-3) + \mu_1 - \mu_2 = 0$ --- (2)
- $\mu_1 (x+y-2) = 0$ ----(3)
- $\mu_2 (x-y) = 0$ ---(4)
- $\mu_1 \geq 0$, $\mu_2 \geq 0$, $x+y \leq 2$, $y \geq x$. ----(5)

- 4 cases to consider from $\mu_1 \geq 0$, $\mu_2 \geq 0$ part of (5)

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Case 1

- $\mu_1 = 0$ and $\mu_2 = 0$
- (1) and (2) become $2(x-1)=0$ and $2(y-3)=0$
- That is, x=1 and y=3.
- So x+y =4.
- Not a feasible solution since $x + y \leq 2$

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Case 2

- Now $\mu_1=0$ and $\mu_2 >0$.
- (4), (1) and (2) become
- $x-y=0$, $2(x-1)+\mu_2=0$ and $2(y-3) - \mu_2 =0$
- So x=y. Therefore $2x-2+\mu_2=0$ and $2x -6 - \mu_2 =0$
- This givex x=2. Then y=2. And $\mu_2=-2$
- Again not feasible.

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Case 3

- Now $\mu_1>0$ and $\mu_2 =0$.
- Then (3), (1) and (2) become
- $x+y-2=0$, $2(x-1)+\mu_1=0$ and $2(y-3) - \mu_1 =0$
- Solutions are x=0, y=2 and $\mu_1=2$

- This is feasible!!!

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Case 4

- Now μ_1>0 and μ_2 > 0.
- Then (3), (4), (1) and (2) become
- $x+y-2=0$, $x+y\ 0$, $2(x-1)+\mu_1=0$ and $2(y-3) – \mu_1=0$
- Which gives rise to $x=1$, $y=1$, $\mu_1=2$ and $\mu_2=-2$
- Again, not feasible.

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