15-451 Assignment 1

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Section B
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1: Recurrences

(a) Unrolling this recurrence, we get:

$$\begin{aligned} T(n) &= T(n-2) + n^3 \\ &= n^2 + (n-2)^3 + (n-4)^3 + ... + 3^3 + 1^3 \text{ Note there are } n/2 \text{ terms.} \\ &> n^2 + (n-2)^3 + (n-4)^3 + ... + (n/2)^3 \text{ by taking the top half.} \\ &> (n/2)^3 + (n/2)^3 + (n/2)^3 ... + (n/2)^3 \\ &= (n/4)(n/2)^3 \text{ since there are now } n/2 * 1/2 = n/4 \text{ terms.} \\ T(n) &> n^4/32 \end{aligned}$$

Furthermore, since the lower half that we cut off is less than $n^4/32$, $T(n) < n^4/16$.
Hence, $T(n) \in \Theta(n^4)$

(b) Unrolling this recurrence, we get:

$$\begin{aligned} T(n) &= T(n/2) + lg(n) \\ &= lg(n) + lg(n) - 1 + lg(n) - 2 + ... + 1 \text{ Note there are } lg(n) \text{ terms since } n \text{ is halved every time.} \\ &= lg(n) * lg(n) - \sum_{i=0}^{lg(n)} i \\ &= lg(n)^2 - (lg(n) * (lg(n) + 1)/2) \\ T(n) &= lg(n)^2/2 - lg(n)/2 \Rightarrow T(n) \in \Theta(lg(n)^2) \end{aligned}$$

(c) $T(n) = 7T(n-2)$
Drawing the tree shows that the work is done in the leaves. Let us count the leaves:
The number of nodes at level k is 7^k since going down multiplies the number of nodes by 7.
The depth of the tree will be $n/2$, since n is decreased by 2 each time.
$\Rightarrow$ The number of leaves is $7^{n/2} = \sqrt{7}^n$.
$\Rightarrow T(n) \in \Theta(\sqrt{7}^n)$

(d) $T(n) = \sqrt{n}T(\sqrt{n}) + n$
Drawing the tree for this recurrence reveals that an even amount of work is done at every level. This is because going down a level branches off into $\sqrt{n}$ nodes where $\sqrt{n}$ work is done. Therefore the total amount of work done in the next level is $\sqrt{n} * \sqrt{n} = n$.
Now we must find the depth of the tree.
At the kth level, the size of the node is $n^{\frac{1}{2^k}}$.
To find what k is needed for the node size to get to the base case, we do the following:
$n^{\frac{1}{2^k}} = 2^{\frac{lg(n)}{2^k}} = 2$

$= \frac{lg(n)}{2^k} = 1$
$= lglg(n) = k$
$\Rightarrow T(n) \in \Theta(nlglg(n))$ since there are $lglg(n)$ levels doing n work at each level.

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2: Recurrences, part 2.

(a) Using the master theorem, where $a = 3, b = 3/2, c = 0$, we get $T(n) \in \Theta(n^{log_{3/2}(3)})$.

(b) **WTS:** $P(k) = y_k - x_k \leq 3$. Let's use induction.

- **Base case:** $y_0 - x_0 = 0 \leq 3$ since $x_0 = y_0$
- **Induction hypothesis:** Assume $P(k')$ for all $k' <= k$.
- **Inductive step: WTS:** $P(k) \Rightarrow P(k+1)$.
  $P(k+1) = \lceil(2/3)y_k\rceil - (2/3)x_k \leq 3$
  $= (2/3)y_k - (2/3)x_k + 1 \overset{?}{\leq} 3$
  $= (2/3)(y_k - x_k) + 1 \overset{?}{\leq} 3$
  $= (2/3)(3) + 1 \overset{?}{\leq} 3$ by IH
  $= 3 \leq 3$

Therefore, $P(k)$ holds for all $k \geq 0$.

(c) We know the lower bound l is $log_{3/2}(n)$ because that's the depth if there was no rounding. From part b, we know that $u - l \leq 3$, so that $u = log_{3/2}(n) + 3$. Since the upper and lower bound differ by a constant factor, the solution to this recurrence is $\Theta(3^{log_{3/2}(n)})$.

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3: Probability and expectation.

(a) Let $X_{ij} = 1$ when $a_i > a_j$ and $i < j$.
$E[X_{ij} = 1]$ when $i < j = 1/2$, since there is $1/2$ chance that $a_i > a_j$.

$$E[I] = \sum_{i=1}^{n} \left(\sum_{j=i+1}^{n} E[X_{ij} = 1] \right)$$
$$= \sum_{i=1}^{n} \left(\sum_{j=i+1}^{n} 1/2 \right)$$
$$= \sum_{i=1}^{n} (n - i)(1/2)$$
$$= \frac{n}{2} * (n - 1) * \frac{1}{2} = \frac{n^2 - n}{4}$$

(b) The number of comparisons is between I and $n + I - 1$
$\Rightarrow$ The number of comparisons is $\Theta(I)$
$\Rightarrow$ Insertion sort is $\Theta(n^2)$.

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4: Matrix games.

(a) Since Chicago has the greatest chance of being chosen, it is the best place for Bob to choose for his concert.

The expected amount in sales is simply the sum of the expected sales at each location, by linearity of expectations. The expected sales at each location is the probability it will happen there * the anticipated sales. Therefore, the expected amount in sales is: $(0.35 * 1 + 0.2 * 1 + 0.45 * 5) * 100 = 280$.

(b) First let us prove a lemma.
Lemma 1: In order to maximize the minimum of Al's choices, each of Al's choices need to yield the same expected sales based on a random Bob's choice.
Proof: Assume for sake of contradiction that we have maximized this. WLOG, let Al's choice of A yield the minimum. Let Al's choice of B yield more than Al's choice of A. Then, we can lower the probability of Bob choosing B and increase the probability of choosing A, increasing the ticket sales since sales are highest at A when Al chooses A too.
This is a contradiction, so lemma 1 is proved.

(i) Using lemma 1, we just need to find a probability where the expected sales are the same. This probability is 1/3 for each. No matter Al's choice, the expected ticket sales is $5/3 + 1/3 + 1/3 = 7/3$.

(ii) 700/3 as shown above.

(c) (i) Using lemma 1, we solve the equation for when Al's choice of A and B are equal: $(1 - 2p) + p + 7p = 7(1 - 2p) + 2p$.
Using algebra, $p = 3/8$ and $1 - 2p = 1/4$.

(ii) $100 * ((2/8) * 7 + 3/8 + 3/8) = 250$.

(iii) A should have a lower probability in order to raise the minimum.

(d) (i) Using the same logic as lemma 1, but for a maximum, we do the same thing as in part c since the matrix is symmetric. Therefore, we get $p = 3/8$ and $1 - 2p = 1/4$.

(ii) 250.