Master's in Mathematical Finance

Numerical Methods in Finance (2024/2025)

Computational Work

Nyström Method for Integro-Differential Equations

Given the functions $f, \alpha, \beta \in C^0[a,b]$ and $K \in C^0([a,b] \times [a,b])$, we aim to determine a function $y \in C^2[a,b]$ that satisfies the integro-differential equation

$$\begin{cases} y''(x) + \alpha(x)y'(x) + \beta(x)y(x) + \displaystyle\int_a^b K(x,t)\, y(t)\, dt = f(x), \quad x \in]a,b[\\ y(a) = y(b) = 0 \end{cases} \tag{1}$$

The Nyström method involves choosing a finite set of points $\{x_1, \cdots, x_n\} \in [a,b]$, at which the equation is naturally verified, obtaining the following semi-discretized equation

$$y''(x_i) + \alpha(x_i)y'(x_i) + \beta(x_i)y(x_i) + \int_a^b K(x_i,t)\, y(t)\, dt = f(x_i), \quad i = 1, \cdots, n \tag{2}$$

If we approximate $y'(x_i)$ and $y''(x_i)$ using finite differences, and approximate the integral by a sum of the type

$$\int_a^b K(x_i,t)\, y(t)\, dt \approx \sum_{j=1}^{n} w_j K(x_i, x_j)\, y(x_j), \quad i = 1, \cdots, n \tag{3}$$

we can obtain the approximate values $y_1, \cdots, y_n$ by solvinga linear system.

(A) Create a module in a programming language of your own choosing that, given the interval $[a,b]$, the right-hand side f, the coefficients α, β, the kernel $K(x,t)$, and the quadrature rule to be used, provides the numerical solution to the problem. You can assume that the points are equally spaced, with spacing $h > 0$. One possible choice of the coefficients w_j is given by $w = (\frac{h}{2}, h, h, \cdots, h, \frac{h}{2})$.

(B) Obtain a specofoc example where it is possible to determine the exact solution of the integro-differential equation and use it to test the convergence order of the method for various choices of the finite difference scheme and the quadrature rule.

(C) Now assume that the kernel, K, can also depend on the unknown, y, and that $\alpha(x) = \beta(x) = 0$. Write the nonlinear system that must be solved to determine a numerical approximation of the integro-differential equation

$$\begin{cases} y''(x) + \displaystyle\int_a^b K(x,t,y(t))\, y(t)\, dt = f(x), \quad x \in]a,b[\\ y(a) = y(b) = 0 \end{cases} . \tag{4}$$

(D) Taking $a = 0, b = 1, f(x) = x$ and $K(x,t,y) = \dfrac{e^{x+t}}{1+y^2}$, obtain numerical approximations of the solution to the equation (4), using the Newton method.