Math 104 Online

Philip T. Gressman

November 16, 2019

Contents

1 Computing Volume

We study an application of integration: computing volumes of a variety of complicated three-dimensional objects.

Integration is the tool to use whenever a quantity can be conceived as an *accumulation of infinitesimal parts*. Volume is one of the most basic and important of such quantities. In the activities that follow, we regard volume as the accumulated size of infinitely thin slices and use this perspective to derive and apply a number of formulas for computing volume.

1.1 Volume By General Cross Sections

We use cross-sectional area to compute volume.

Example 1. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 2. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`volumes/genslicepractice.tex`

1.2 Exercises: General Slicing

Exercises computing volume by cross-sectional area.

Exercise 1 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length $L = \boxed{x - x^2}$ and area $A = \boxed{(x - x^2)^2}$.*
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of $x = \boxed{0}$ up to a maximum of $x = \boxed{1}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 2 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

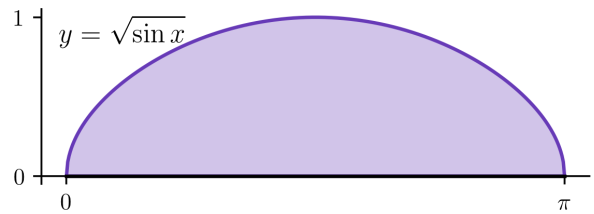
- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length $\boxed{1}$ in the x-direction and width $\boxed{z^2}$ in the y-direction.*
- *The area of a z cross section is $A(z) = \boxed{z^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

1.3 The Disk and Washer Methods

We practice setting up calculations related to the disk and washer methods.

Example 3. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*



- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

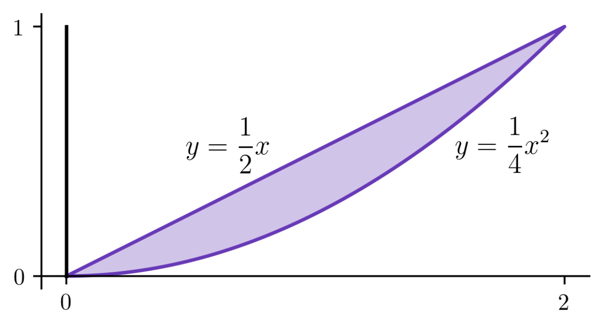
  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓

  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}} \right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 4. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*



- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

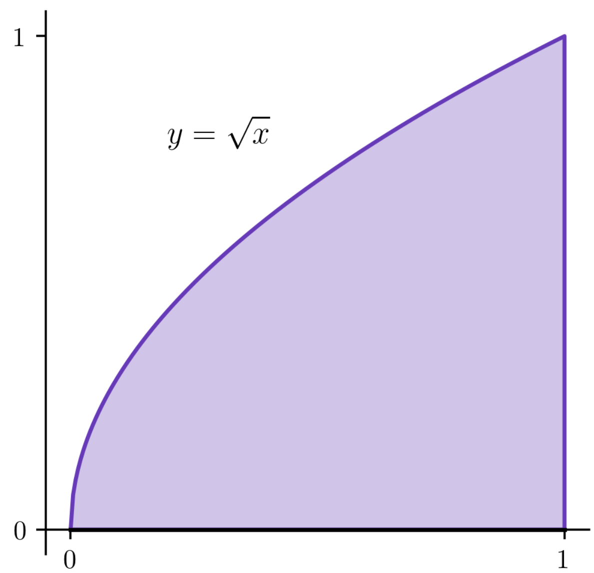
$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\,\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

volumes/washerpractice.tex

1.4 Exercises: Disks and Washers

Exercises for the disk and washer methods.

Exercise 3 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the x-axis. Use the disk method to find the volume of the solid of revolution.*

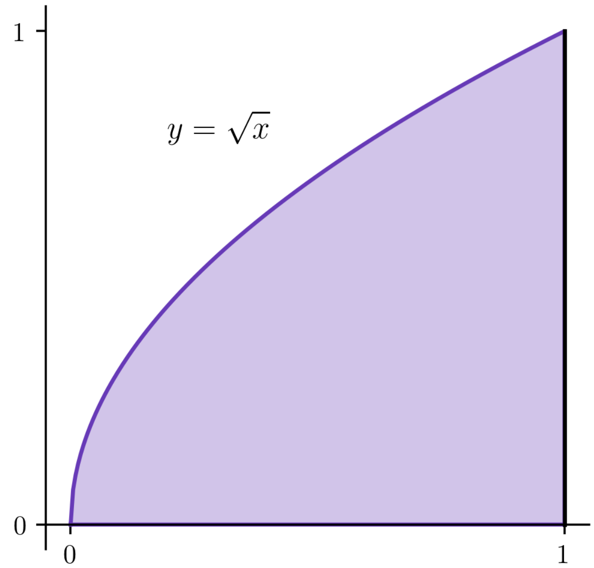


Hint: *The radius $R(x)$ will be a difference of y-values because slices are indexed by the variable x. Each slice will extend from $y = 0$ to $y = \sqrt{x}$, and so $R(x)$ must be the larger of these y-values minus the smaller of these y-values.*

$$R(x) = \boxed{\sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi(R(x))^2 dx = \boxed{\frac{\pi}{2}}$$

Exercise 4 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 1$. Use the disk method to find the volume of the solid of revolution.*

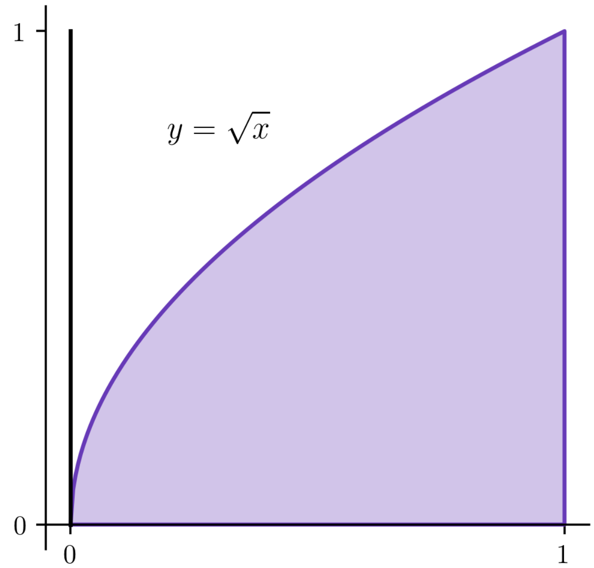


Hint: *The radius $R(y)$ will be a difference of x-values because slices are indexed by the variable y. Each slice will extend from $x = y^2$ to $x = 1$, and so $R(y)$ must be the larger of these x-values minus the smaller of these x-values*

$$R(y) = \boxed{1 - y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(y))^2 dy = \boxed{\frac{8\pi}{15}}$$

Exercise 5 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 0$. Use the washer method to find the volume of the solid of revolution.*

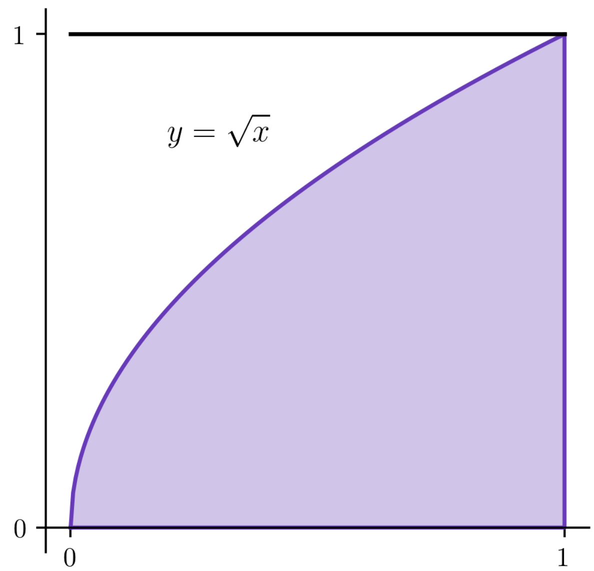


Hint: *Each radius will be a difference of x-values because slices are indexed by the variable y. The distance from the axis $x = 0$ to the line $x = 1$ is 1, and the distance from the axis $x = 0$ to $x = y^2$ is y^2.*

$$R_{\text{outer}}(y) = \boxed{1} \text{ and } r_{\text{inner}}(y) = \boxed{y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(y))^2 - (r_{\text{inner}}(y))^2\right] dy = \boxed{\frac{4\pi}{5}}$$

Exercise 6 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the axis $y = 1$. Use the washer method to find the volume of the solid of revolution.*



Hint: *Each radius will be a difference of y-values because slices are indexed by the variable x. The distance from the axis $y = 1$ to the line $y = 0$ is 1, and the distance from the axis $y = 1$ to $y = \sqrt{x}$ is $1 - \sqrt{x}$.*

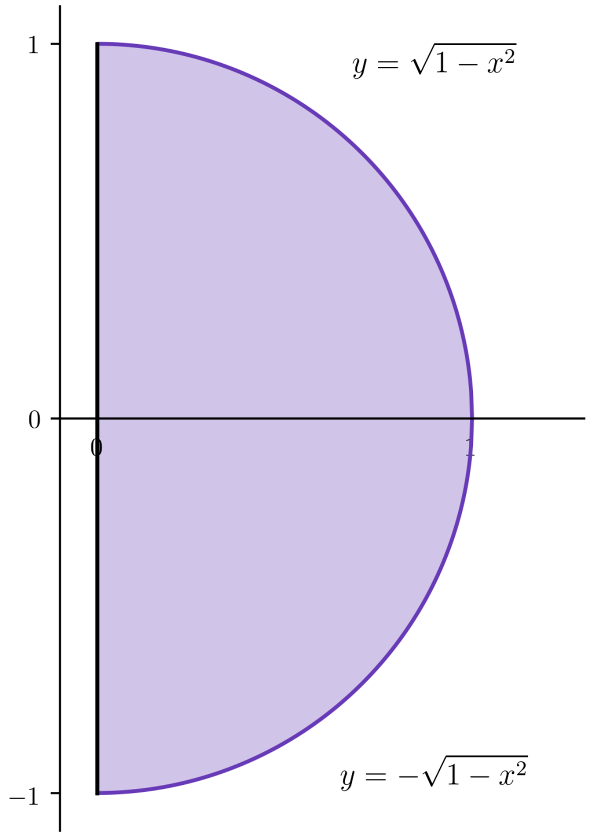
$$R_{\text{outer}}(x) = \boxed{1} \text{ and } r_{\text{inner}}(x) = \boxed{1 - \sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(x))^2 - (r_{\text{inner}}(x))^2\right] dx = \boxed{\frac{5\pi}{6}}$$

1.5 The Shell Method

We practice setting up setting up volume calculations using the shell method.

Example 5. *The region defined by the inequalities $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ and $x \geq 0$ (shown below) is revolved around the y-ais. Compute the volume using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

Multiple Choice:

(a) $h(x) = \sqrt{1-x^2}$

(b) $h(x) = -\sqrt{1-x^2}$

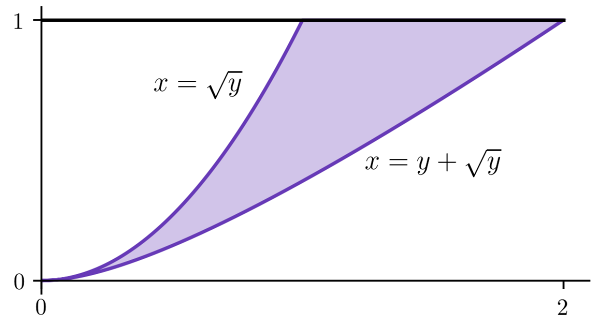
(c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\,dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$.)

Example 6. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*



- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$
  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$
  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

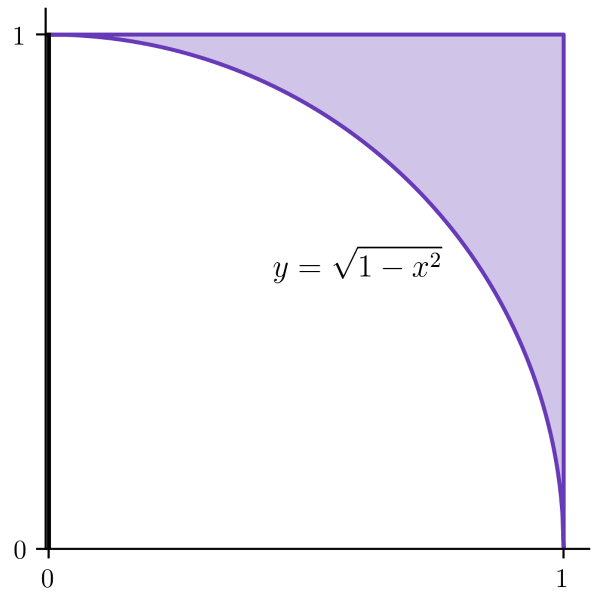
$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\, dy = \boxed{\frac{\pi}{3}}.$$

`volumes/shellpractice.tex`

1.6 Exercises: Shell Method

Exercises for using the shell method.

Exercise 7 *The region defined by the inequalities $\sqrt{1-x^2} \leq y \leq 1$ for $0 \leq y \leq 1$ is revolved around the y-axis. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis of rotation. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

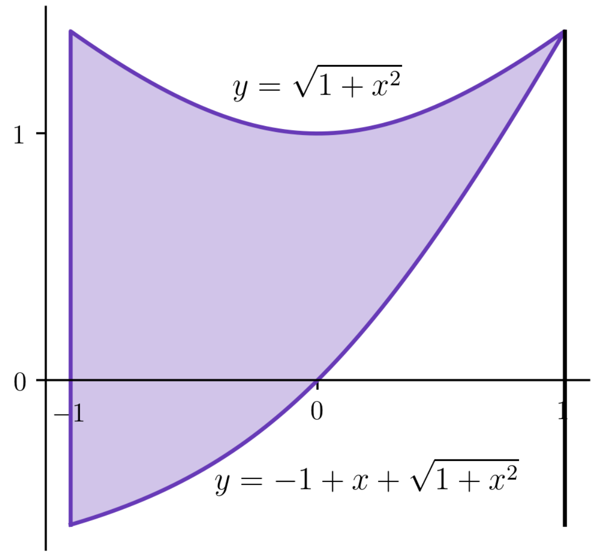
$$h(x) = \boxed{1 - \sqrt{1-x^2}}.$$

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi x(1-\sqrt{1-x^2})}\, dx = \boxed{\frac{\pi}{3}}.$$

(Note: to compute the integral, split it into two parts and make the substitution $u = 1 - x^2$ for one of them.)

Exercise 8 *The region in the plane bounded above by the graph* $y = \sqrt{1+x^2}$, *below by* $y = -1 + x + \sqrt{1+x^2}$, *and on the left by* $x = 0$ *is revolved around the axis* $x = 1$. *Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is* x, *the radius of a shell is the (horizontal ✓/ vertical) distance from an* x*-slice to the axis* $x = 0$. *Thus*

$$r(x) = \boxed{1} - \boxed{x}.$$

- *The height of an* x*-slice is equal to*

$$h(x) = \boxed{-1 + x}.$$

- *The volume is equal to the integral of* $2\pi rh$, *so*

$$V = \int_{\boxed{-1}}^{\boxed{1}} \boxed{2\pi(1-x)^2}\, dx = \boxed{\frac{16\pi}{3}}.$$

1.7 Synthesis: Choose Your Method

We practice choosing a method for computing volume when none is specified.

Problem 9 *Type* 2. $\boxed{2}$, $\underset{\text{given}}{\boxed{2}}$, $\boxed{\frac{1}{2}}$, $\int_a^b f(x)\ dx$,

$$-\frac{1}{\frac{12}{1}}\sum_{n=1}^{\boxed{\infty}} n$$

1.8 Exercises: Choose Your Method

Exercises choosing a method for computing volume.

Exercise 10 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 11 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

- *Cross sections perpendicular to the z-axis are (square / rectangular* ✓ */ triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

2 Computing Arc Length and Surface Area

We study a second application of integration: arc length and surface area.

Example 7. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \left.\frac{x^3}{3}\right|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 8. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

2.1 Arc Length

We practice setting up and executing arc length calculations.

Example 9. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 10. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`arclengths/arclengthpractice.tex`

2.2 Exercises: Arc Length

We practice computing arc length.

Example 11. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 12. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

2.3 Surface Area

We practice setting up integrals for the surface area of surfaces of revolution.

Example 13. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 14. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

arclengths/surfacepractice.tex

2.4 Exercises: Surface Area

Various exercises related to the computation of areas of surfaces of revolution.

Example 15. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 16. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

3 Centers of Mass and Centroids

We study a third application of integration: centers of mass and centroids.

Example 17. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 18. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

3.1 Centers of Mass and Centroids

We practice setting up calculations for centers of mass and centroids.

Example 19. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \left.\frac{x^3}{3}\right|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 20. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

centroids/centroidpractice.tex

3.2 Exercises: Centers of Mass and Centroids

Various questions relating to centers of mass and centroids.

Example 21. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 22. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

4 Integration Techniques

We begin a study of techniques for computing integrals.

Example 23. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 24. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

4.1 Substitution and Tables

We review substitution and the use of integral tables.

Example 25. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 26. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

techniques/substitutionpractice.tex

4.2 Exercises: Substitution and Tables

Various exercises relating to substitution and the use of integral tables.

Example 27. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 28. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

4.3 Trigonometric Integrals

We learn various techniques for integrating certain combinations of trigonometric functions.

Example 29. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 30. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

`techniques/trigonometricpractice.tex`

4.4 Exercises: Trigonometric Integral

Various exercises relating to the integration of trigonometric functions.

Exercise 12 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 13 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$0 \leq x \leq 1,$$
$$0 \leq y \leq z^2,$$
$$0 \leq z \leq 3.$$

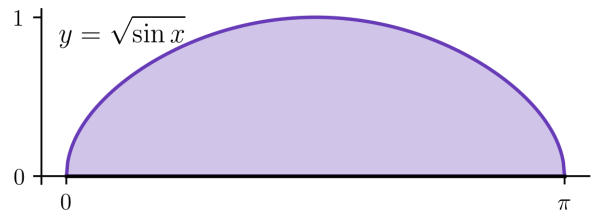
- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

4.5 Trigonometric Substitions

We practice executing trigonometric substitutions.

Example 31. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*



- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

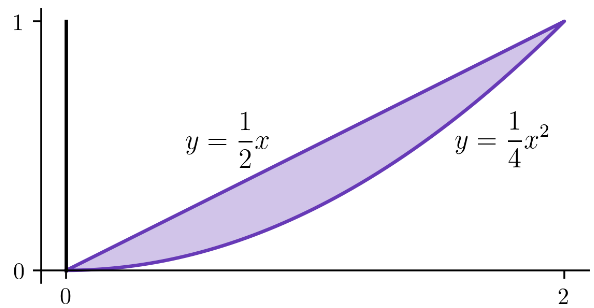
  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓

  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}}\right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 32. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*



- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓
- *We conclude that*

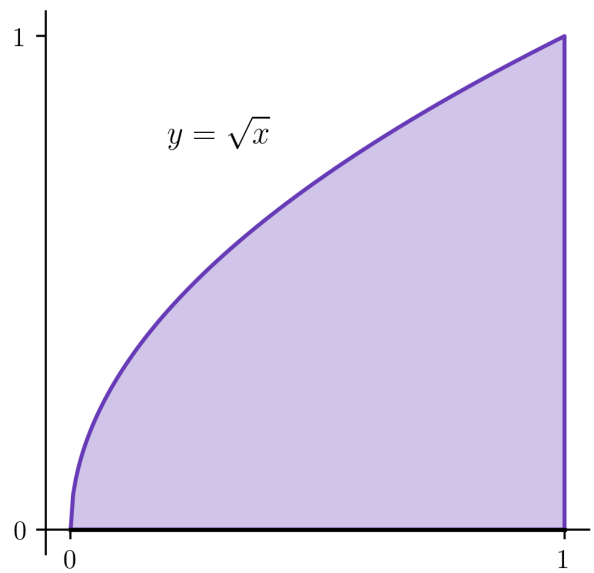
$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\,\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

`techniques/trigsubpractice.tex`

4.6 Exercises: Trigonometric Substitions

Various exercises relating to trigonometric substitutions.

Exercise 14 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the x-axis. Use the disk method to find the volume of the solid of revolution.*

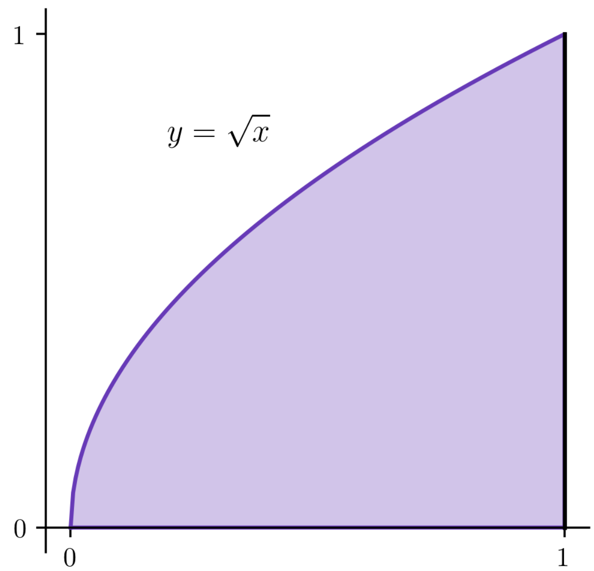


Hint: *The radius $R(x)$ will be a difference of y-values because slices are indexed by the variable x. Each slice will extend from $y = 0$ to $y = \sqrt{x}$, and so $R(x)$ must be the larger of these y-values minus the smaller of these y-values.*

$$R(x) = \boxed{\sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi(R(x))^2 dx = \boxed{\frac{\pi}{2}}$$

Exercise 15 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the axis $x = 1$. Use the disk method to find the volume of the solid of revolution.*

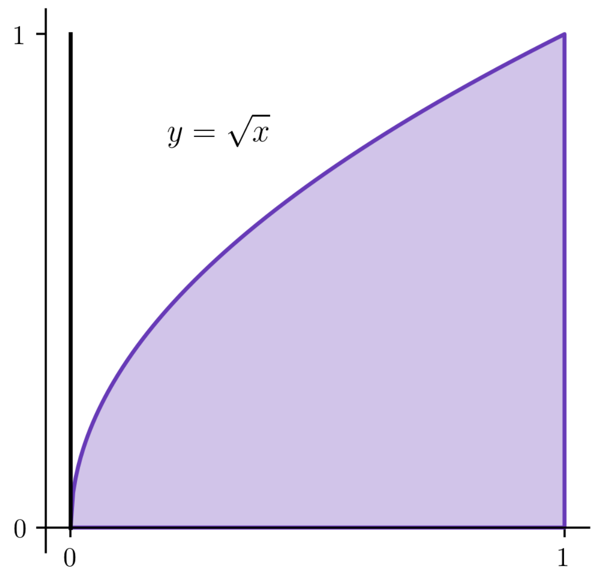


Hint: *The radius $R(y)$ will be a difference of x-values because slices are indexed by the variable y. Each slice will extend from $x = y^2$ to $x = 1$, and so $R(y)$ must be the larger of these x-values minus the smaller of these x-values*

$$R(y) = \boxed{1 - y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(y))^2 dy = \boxed{\frac{8\pi}{15}}$$

Exercise 16 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 0$. Use the washer method to find the volume of the solid of revolution.*

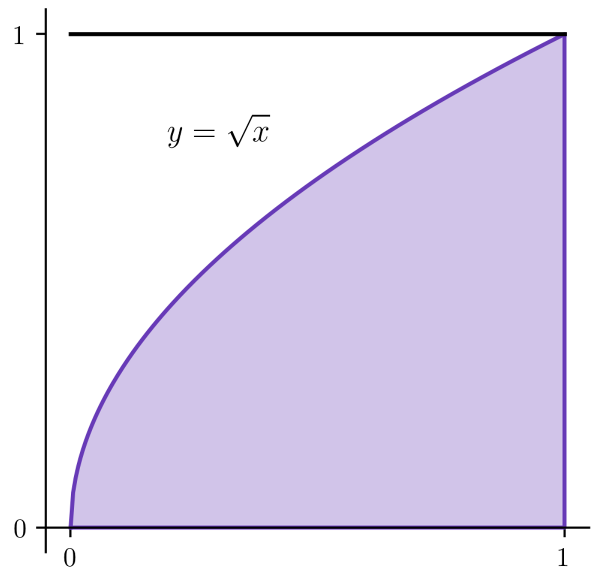


Hint: *Each radius will be a difference of x-values because slices are indexed by the variable y. The distance from the axis $x = 0$ to the line $x = 1$ is 1, and the distance from the axis $x = 0$ to $x = y^2$ is y^2.*

$$R_{\text{outer}}(y) = \boxed{1} \text{ and } r_{\text{inner}}(y) = \boxed{y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi\left[(R_{\text{outer}}(y))^2 - (r_{\text{inner}}(y))^2\right] dy = \boxed{\frac{4\pi}{5}}$$

Exercise 17 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $y = 1$. Use the washer method to find the volume of the solid of revolution.*



Hint: *Each radius will be a difference of y-values because slices are indexed by the variable x. The distance from the axis $y = 1$ to the line $y = 0$ is 1, and the distance from the axis $y = 1$ to $y = \sqrt{x}$ is $1 - \sqrt{x}$.*

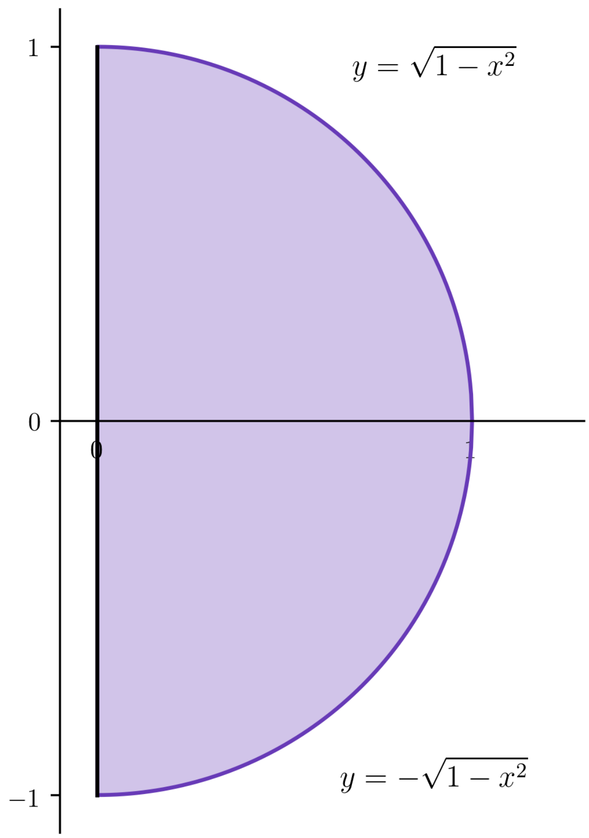
$$R_{\text{outer}}(x) = \boxed{1} \text{ and } r_{\text{inner}}(x) = \boxed{1 - \sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(x))^2 - (r_{\text{inner}}(x))^2 \right] dx = \boxed{\frac{5\pi}{6}}$$

4.7 Partial Fractions

We study the technique of partial fractions and its application to integration.

Example 33. *The region defined by the inequalities $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ and $x \ge 0$ (shown below) is revolved around the y-ais. Compute the volume using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

Multiple Choice:

(a) $h(x) = \sqrt{1-x^2}$

(b) $h(x) = -\sqrt{1-x^2}$

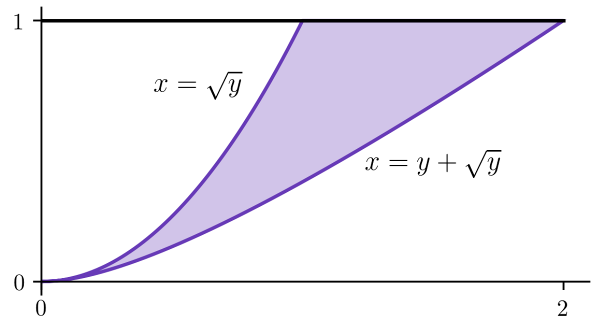
(c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\,dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$.)

Example 34. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*



- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$
  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$
  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

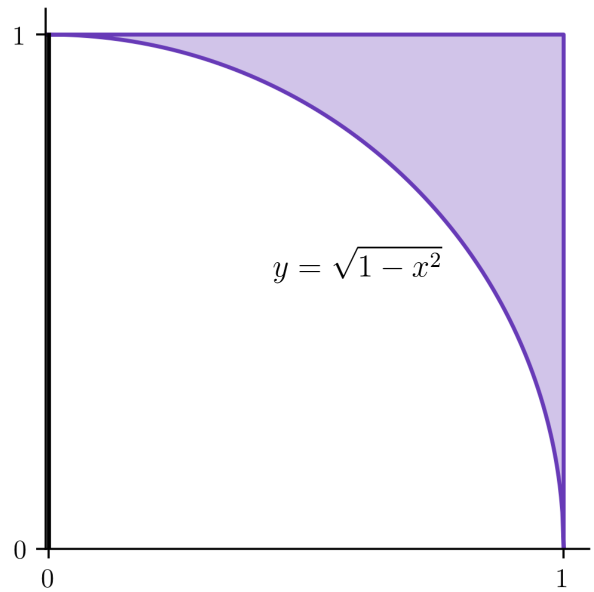
$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\, dy = \boxed{\frac{\pi}{3}}.$$

`techniques/partialfractionspractice.tex`

4.8 Exercises: Partial Fractions

Various exercises relating to partial fractions and integration.

Exercise 18 *The region defined by the inequalities $\sqrt{1-x^2} \le y \le 1$ for $0 \le y \le 1$ is revolved around the y-axis. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis of rotation. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

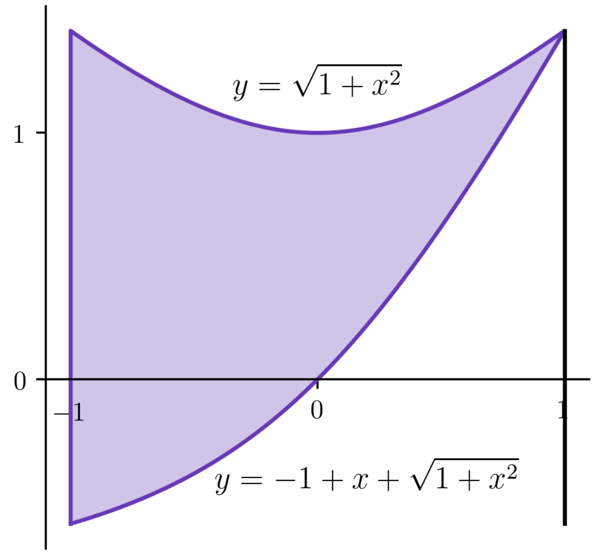
$$h(x) = \boxed{1 - \sqrt{1-x^2}}.$$

- *The volume is equal to the integral of $2\pi r h$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi x(1 - \sqrt{1-x^2})}\, dx = \boxed{\frac{\pi}{3}}.$$

(Note: to compute the integral, split it into two parts and make the substitution $u = 1 - x^2$ for one of them.)

Exercise 19 *The region in the plane bounded above by the graph $y = \sqrt{1+x^2}$, below by $y = -1+x+\sqrt{1+x^2}$, and on the left by $x = 0$ is revolved around the axis $x = 1$. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{1} - \boxed{x}.$$

- *The height of an x-slice is equal to*

$$h(x) = \boxed{-1+x}.$$

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{-1}}^{\boxed{1}} \boxed{2\pi(1-x)^2}\, dx = \boxed{\frac{16\pi}{3}}.$$

5 Additional Applications

We study additional topics relating to applications of integration.

Example 35. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 36. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

5.1 Numerical Integration

We study the problem of numerically approximating the value of an integral.

Example 37. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 38. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

applications/numericalpractice.tex

5.2 Exercises: Numerical Integration

Various exercises relating to numerical integration.

Example 39. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 40. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

5.3 Orders of Growth

We study the use of orders of growth to compute limits, in preparation for improper integrals.

Example 41. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 42. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

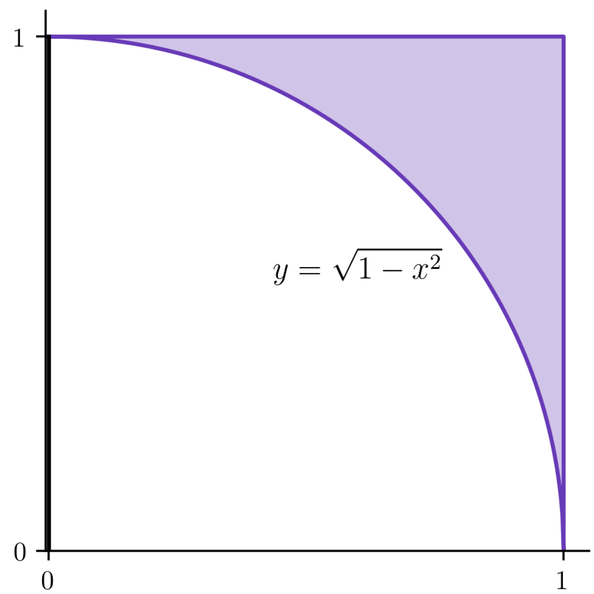
$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

applications/ordergrowthpractice.tex

5.4 Exercises: Orders of Growth

Various exercises relating to orders of growth.

Exercise 20 *The region defined by the inequalities $\sqrt{1-x^2} \leq y \leq 1$ for $0 \leq y \leq 1$ is revolved around the y-axis. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis of rotation. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

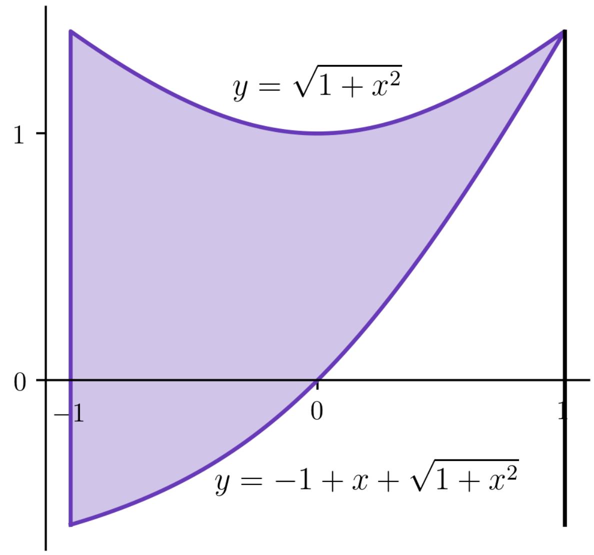
$$h(x) = \boxed{1 - \sqrt{1-x^2}}.$$

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi x(1 - \sqrt{1-x^2})}\, dx = \boxed{\frac{\pi}{3}}.$$

(Note: to compute the integral, split it into two parts and make the substitution $u = 1 - x^2$ for one of them.)

Exercise 21 *The region in the plane bounded above by the graph $y = \sqrt{1+x^2}$, below by $y = -1+x+\sqrt{1+x^2}$, and on the left by $x = 0$ is revolved around the axis $x = 1$. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{1} - \boxed{x}.$$

- *The height of an x-slice is equal to*

$$h(x) = \boxed{-1+x}.$$

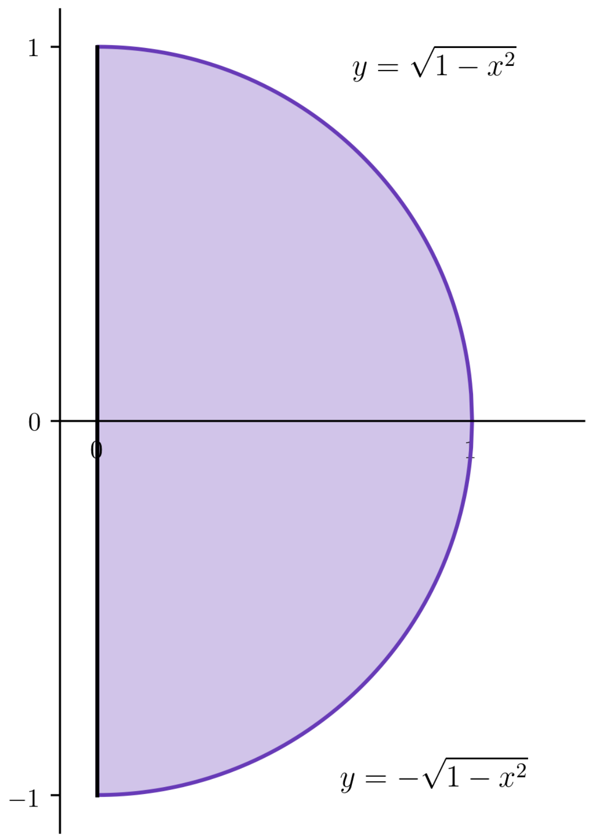
- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{-1}}^{\boxed{1}} \boxed{2\pi(1-x)^2}\, dx = \boxed{\frac{16\pi}{3}}.$$

5.5 Improper Integrals

We study the concept of improper integrals.

Example 43. *The region defined by the inequalities* $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ *and* $x \geq 0$ *(shown below) is revolved around the y-ais. Compute the volume using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis* $x = 0$*. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

Multiple Choice:

(a) $h(x) = \sqrt{1-x^2}$

(b) $h(x) = -\sqrt{1-x^2}$

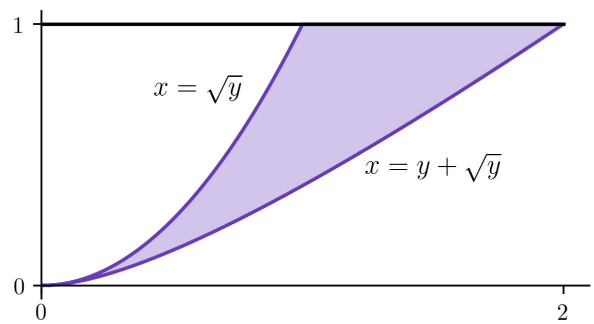
(c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\,dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$*.)*

Example 44. *The region between the curves* $x = \sqrt{y}$ *and* $x = y + \sqrt{y}$ *from* $y = 0$ *to* $y = 1$ *is revolved around the axis* $y = 1$*. Compute the volume of the resulting solid.*



- *When the slicing variable is* y*, the radius of a shell is the (horizontal/ vertical ✓) distance from a* y*-slice to the axis* $y = 1$*. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a* y*-slice is equal to*

Multiple Choice:

(a) $h(y) = \sqrt{y}$

(b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$

(c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\,dy = \boxed{\frac{\pi}{3}}.$$

`applications/improperpractice.tex`

5.6 Exercises: Improper Integrals

Various exercises relating to improper integrals.

Exercise 22 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 23 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

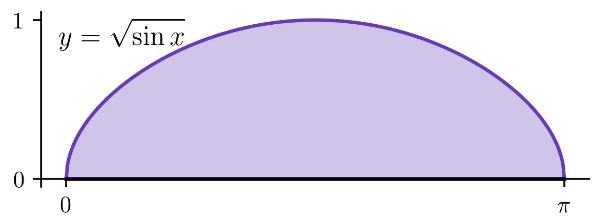
- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2} dz = \boxed{9}.$$

5.7 Probability

We study probability and its connections to integration.

Example 45. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*



- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

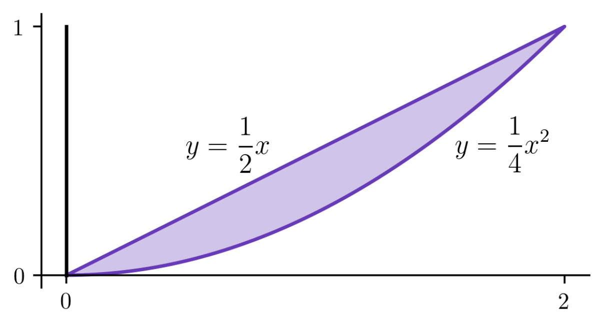
  Multiple Choice:

  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓
  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}}\right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 46. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*



- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓
- *We conclude that*

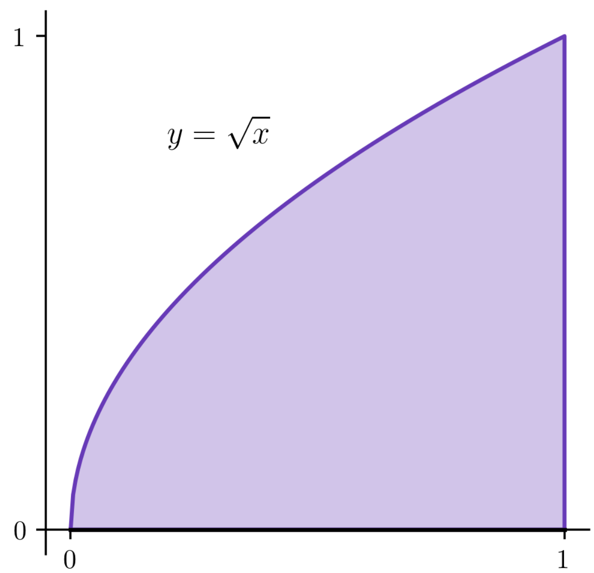
$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\,\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

`applications/probabilitypractice.tex`

5.8 Exercises: Probability

Various exercises relating to probability.

Exercise 24 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the x-axis. Use the disk method to find the volume of the solid of revolution.*

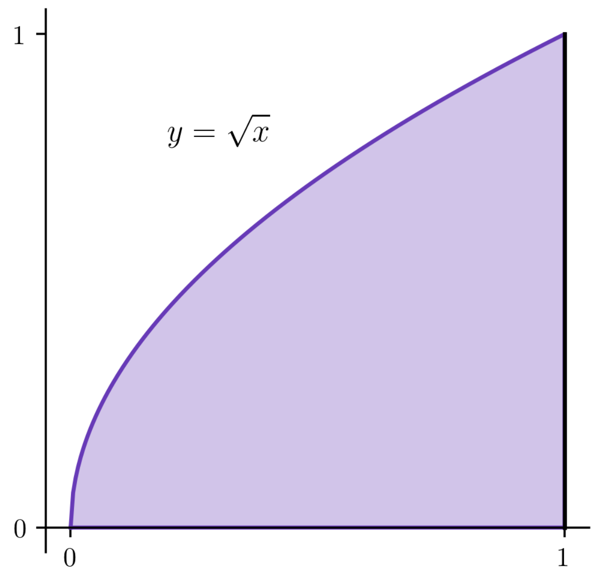


Hint: *The radius $R(x)$ will be a difference of y-values because slices are indexed by the variable x. Each slice will extend from $y = 0$ to $y = \sqrt{x}$, and so $R(x)$ must be the larger of these y-values minus the smaller of these y-values.*

$$R(x) = \boxed{\sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(x))^2 dx = \boxed{\frac{\pi}{2}}$$

Exercise 25 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the axis $x = 1$. Use the disk method to find the volume of the solid of revolution.*

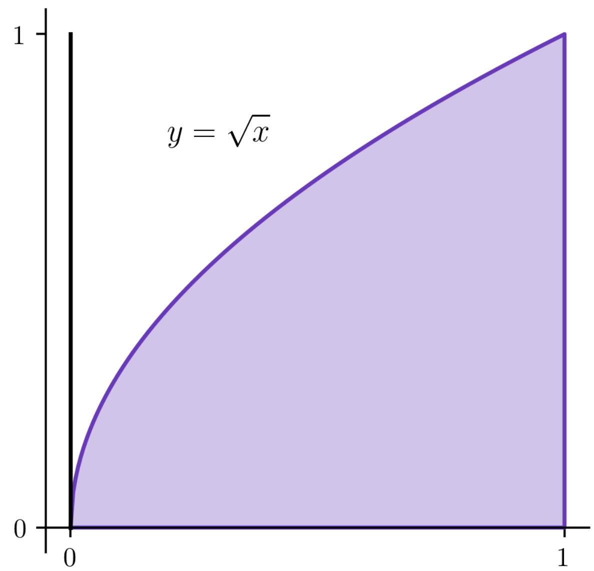


Hint: *The radius $R(y)$ will be a difference of x-values because slices are indexed by the variable y. Each slice will extend from $x = y^2$ to $x = 1$, and so $R(y)$ must be the larger of these x-values minus the smaller of these x-values*

$$R(y) = \boxed{1 - y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(y))^2 dy = \boxed{\frac{8\pi}{15}}$$

Exercise 26 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 0$. Use the washer method to find the volume of the solid of revolution.*

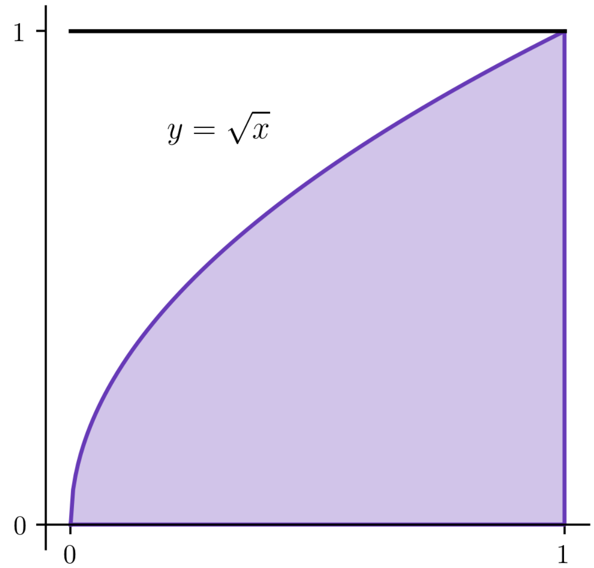


Hint: *Each radius will be a difference of x-values because slices are indexed by the variable y. The distance from the axis $x = 0$ to the line $x = 1$ is 1, and the distance from the axis $x = 0$ to $x = y^2$ is y^2.*

$$R_{\text{outer}}(y) = \boxed{1} \text{ and } r_{\text{inner}}(y) = \boxed{y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(y))^2 - (r_{\text{inner}}(y))^2 \right] dy = \boxed{\frac{4\pi}{5}}$$

Exercise 27 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $y = 1$. Use the washer method to find the volume of the solid of revolution.*



Hint: *Each radius will be a difference of y-values because slices are indexed by the variable x. The distance from the axis $y = 1$ to the line $y = 0$ is 1, and the distance from the axis $y = 1$ to $y = \sqrt{x}$ is $1 - \sqrt{x}$.*

$$R_{\text{outer}}(x) = \boxed{1} \text{ and } r_{\text{inner}}(x) = \boxed{1 - \sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(x))^2 - (r_{\text{inner}}(x))^2\right] dx = \boxed{\frac{5\pi}{6}}$$

6 Sequences and Series

We begin a study of sequences and series.

Example 47. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 48. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

6.1 Sequences

We study the mathematical concept of a sequence.

Example 49. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 50. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

series/sequencepractice.tex

6.2 Exercises: Sequences

Exercises relating to sequences.

Example 51. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 52. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

6.3 Series

We introduce the concept of a series and study some fundamental properties.

Example 53. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 54. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

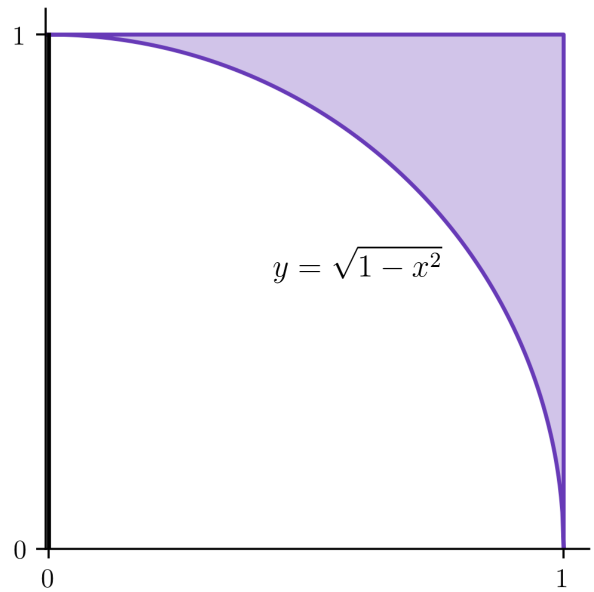
$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

series/seriespractice.tex

6.4 Exercises: Series

Exercises relating to fundamental properties of series.

Exercise 28 *The region defined by the inequalities $\sqrt{1-x^2} \le y \le 1$ for $0 \le y \le 1$ is revolved around the y-axis. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis of rotation. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

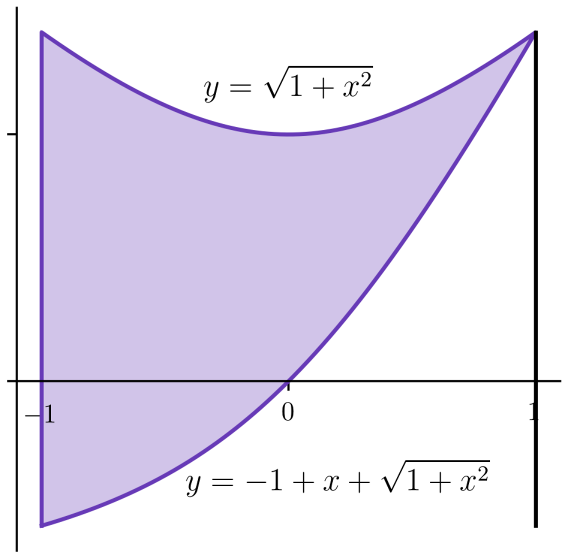
$$h(x) = \boxed{1 - \sqrt{1-x^2}}.$$

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi x(1 - \sqrt{1-x^2})}\, dx = \boxed{\frac{\pi}{3}}.$$

(Note: to compute the integral, split it into two parts and make the substitution $u = 1 - x^2$ for one of them.)

Exercise 29 *The region in the plane bounded above by the graph $y = \sqrt{1+x^2}$, below by $y = -1 + x + \sqrt{1+x^2}$, and on the left by $x = 0$ is revolved around the axis $x = 1$. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{1} - \boxed{x}.$$

- *The height of an x-slice is equal to*

$$h(x) = \boxed{-1 + x}.$$

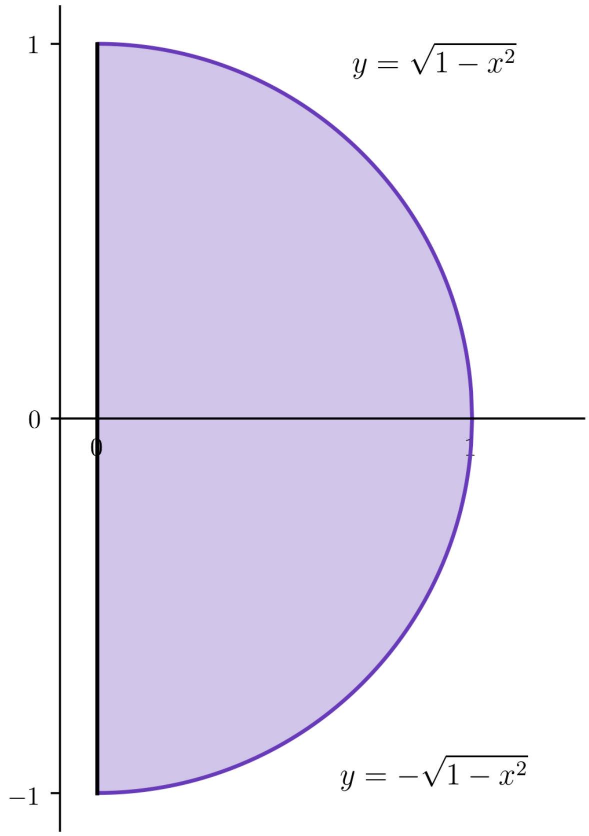
- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{-1}}^{\boxed{1}} \boxed{2\pi(1-x)^2}\, dx = \boxed{\frac{16\pi}{3}}.$$

6.5 Series Comparison Tests

We study the direct and limit comparison theorems for infinite series and practice their application.

Example 55. *The region defined by the inequalities* $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ *and* $x \geq 0$ *(shown below) is revolved around the* y*-ais. Compute the volume using the shell method.*



- *When the slicing variable is* x*, the radius of a shell is the (horizontal ✓/ vertical) distance from an* x*-slice to the axis* $x = 0$*. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an* x*-slice is equal to*

  Multiple Choice:

  (a) $h(x) = \sqrt{1-x^2}$

  (b) $h(x) = -\sqrt{1-x^2}$

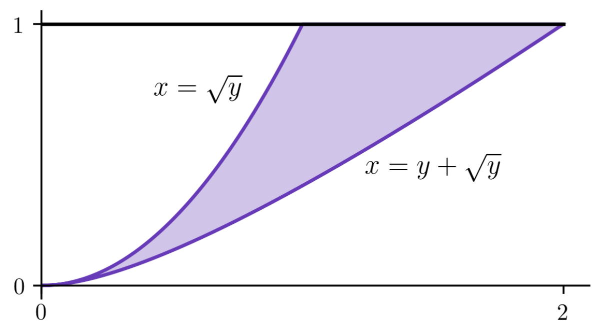
  (c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\,dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$*.)*

Example 56. *The region between the curves* $x = \sqrt{y}$ *and* $x = y + \sqrt{y}$ *from* $y = 0$ *to* $y = 1$ *is revolved around the axis* $y = 1$*. Compute the volume of the resulting solid.*



- *When the slicing variable is* y*, the radius of a shell is the (horizontal/ vertical ✓) distance from a* y*-slice to the axis* $y = 1$*. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a* y*-slice is equal to*

Multiple Choice:

(a) $h(y) = \sqrt{y}$

(b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$

(c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\,dy = \boxed{\frac{\pi}{3}}.$$

series/comparisonpractice.tex

6.6 Exercises: Series Comparison Tests

Exercises relating to the direct and limit comparison tests for series.

Exercise 30 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length $L = \boxed{x - x^2}$ and area $A = \boxed{(x - x^2)^2}$.*
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of $x = \boxed{0}$ up to a maximum of $x = \boxed{1}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 31 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

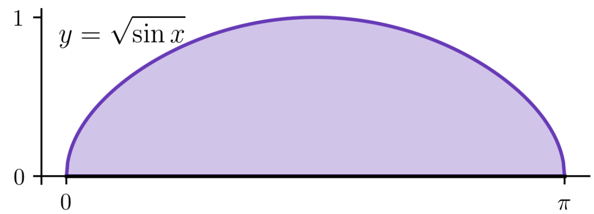
- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length $\boxed{1}$ in the x-direction and width $\boxed{z^2}$ in the y-direction.*
- *The area of a z cross section is $A(z) = \boxed{z^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

6.7 The Ratio and Root Tests

We study the ratio and root tests for infinite series and practice their application.

Example 57. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*



- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

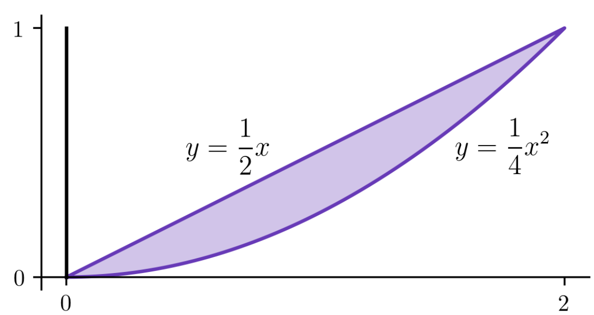
  Multiple Choice:

  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓
  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}} \right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 58. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*



- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

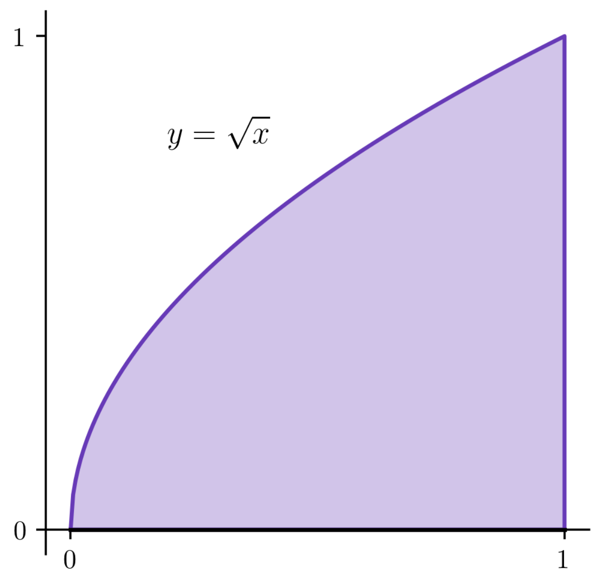
$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\,\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

series/ratiorootpractice.tex

6.8 Exercises: Ratio and Root Tests

Exercises relating to the ratio and root tests.

Exercise 32 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the x-axis. Use the disk method to find the volume of the solid of revolution.*

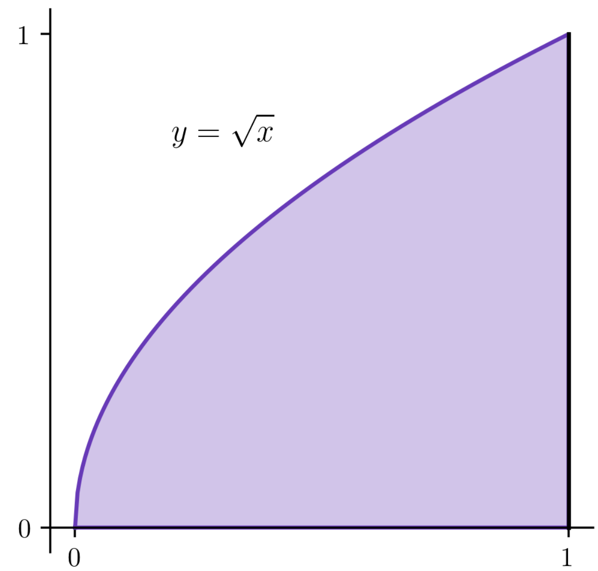


Hint: *The radius $R(x)$ will be a difference of y-values because slices are indexed by the variable x. Each slice will extend from $y = 0$ to $y = \sqrt{x}$, and so $R(x)$ must be the larger of these y-values minus the smaller of these y-values.*

$$R(x) = \boxed{\sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(x))^2 dx = \boxed{\frac{\pi}{2}}$$

Exercise 33 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the axis $x = 1$. Use the disk method to find the volume of the solid of revolution.*

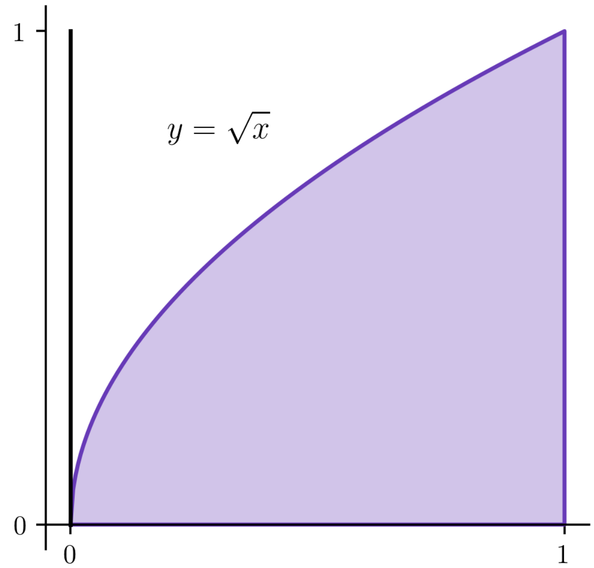


Hint: *The radius $R(y)$ will be a difference of x-values because slices are indexed by the variable y. Each slice will extend from $x = y^2$ to $x = 1$, and so $R(y)$ must be the larger of these x-values minus the smaller of these x-values*

$$R(y) = \boxed{1 - y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi(R(y))^2 dy = \boxed{\frac{8\pi}{15}}$$

Exercise 34 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 0$. Use the washer method to find the volume of the solid of revolution.*

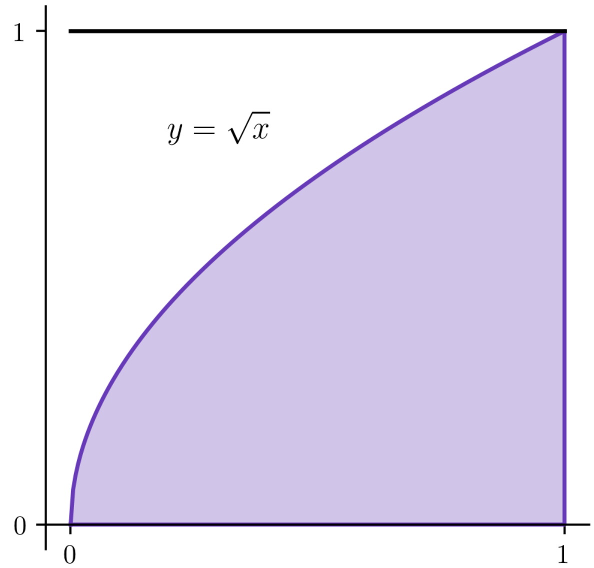


Hint: *Each radius will be a difference of x-values because slices are indexed by the variable y. The distance from the axis $x = 0$ to the line $x = 1$ is 1, and the distance from the axis $x = 0$ to $x = y^2$ is y^2.*

$$R_{\text{outer}}(y) = \boxed{1} \text{ and } r_{\text{inner}}(y) = \boxed{y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(y))^2 - (r_{\text{inner}}(y))^2 \right] dy = \boxed{\frac{4\pi}{5}}$$

Exercise 35 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $y = 1$. Use the washer method to find the volume of the solid of revolution.*



Hint: *Each radius will be a difference of y-values because slices are indexed by the variable x. The distance from the axis $y = 1$ to the line $y = 0$ is 1, and the distance from the axis $y = 1$ to $y = \sqrt{x}$ is $1 - \sqrt{x}$.*

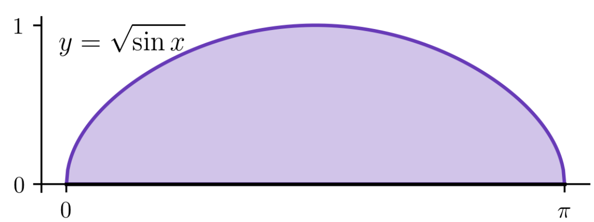
$$R_{\text{outer}}(x) = \boxed{1} \text{ and } r_{\text{inner}}(x) = \boxed{1 - \sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(x))^2 - (r_{\text{inner}}(x))^2 \right] dx = \boxed{\frac{5\pi}{6}}$$

6.9 The Integral Test

We study the integral test for infinite series and related concepts.

Example 59. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*



- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

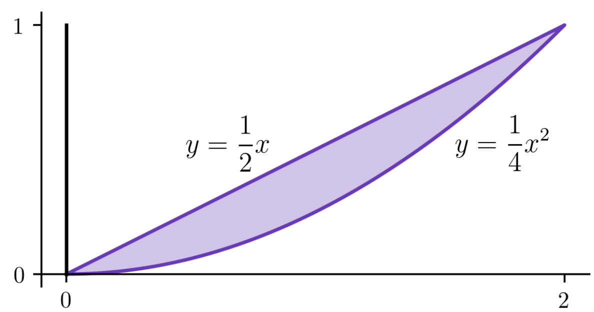
  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓

  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}}\right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 60. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*



- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

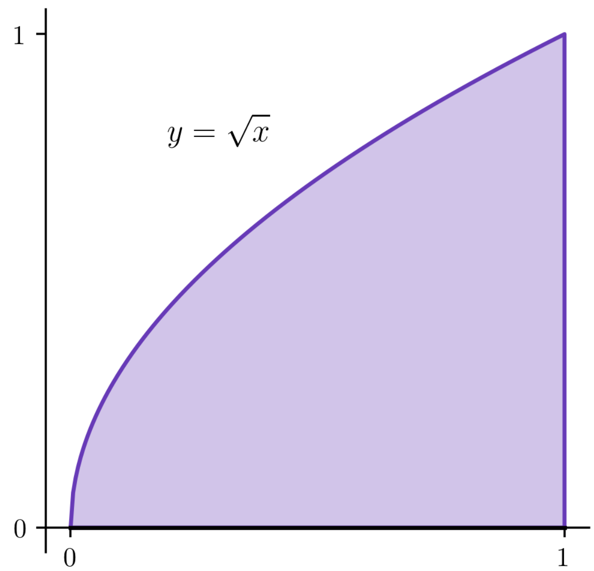
$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}}\right)^2 - \left(\boxed{2y}\right)^2\right] d\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

`series/integralpractice.tex`

6.10 Exercises: The Integral Test

Exercises relating to the integral test.

Exercise 36 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the x-axis. Use the disk method to find the volume of the solid of revolution.*

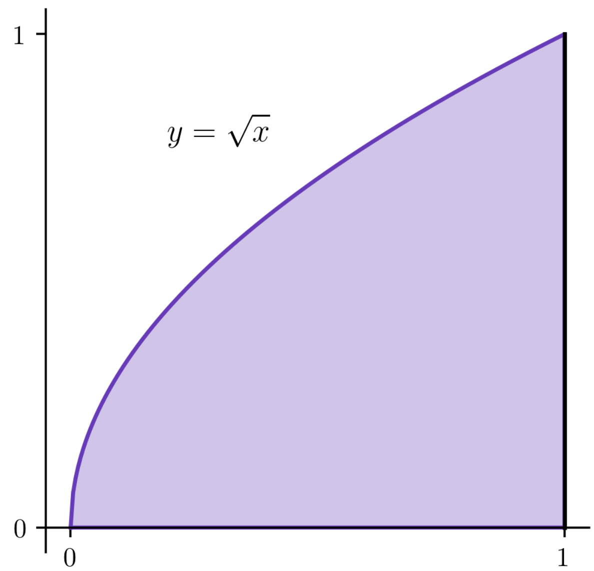


Hint: *The radius $R(x)$ will be a difference of y-values because slices are indexed by the variable x. Each slice will extend from $y = 0$ to $y = \sqrt{x}$, and so $R(x)$ must be the larger of these y-values minus the smaller of these y-values.*

$$R(x) = \boxed{\sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi(R(x))^2 dx = \boxed{\frac{\pi}{2}}$$

Exercise 37 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 1$. Use the disk method to find the volume of the solid of revolution.*

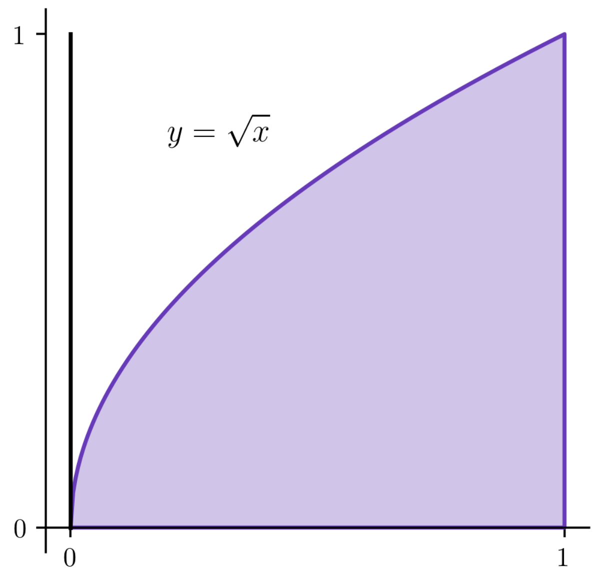


Hint: *The radius $R(y)$ will be a difference of x-values because slices are indexed by the variable y. Each slice will extend from $x = y^2$ to $x = 1$, and so $R(y)$ must be the larger of these x-values minus the smaller of these x-values*

$$R(y) = \boxed{1 - y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(y))^2 dy = \boxed{\frac{8\pi}{15}}$$

Exercise 38 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the axis $x = 0$. Use the washer method to find the volume of the solid of revolution.*

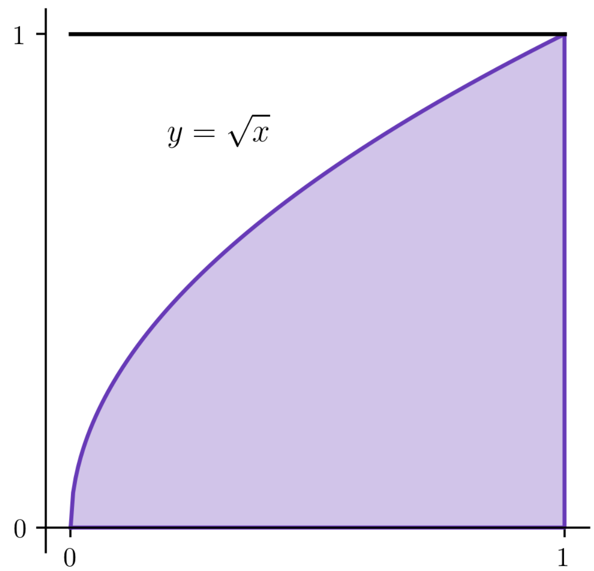


Hint: *Each radius will be a difference of x-values because slices are indexed by the variable y. The distance from the axis $x = 0$ to the line $x = 1$ is 1, and the distance from the axis $x = 0$ to $x = y^2$ is y^2.*

$$R_{\text{outer}}(y) = \boxed{1} \text{ and } r_{\text{inner}}(y) = \boxed{y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(y))^2 - (r_{\text{inner}}(y))^2\right] dy = \boxed{\frac{4\pi}{5}}$$

Exercise 39 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $y = 1$. Use the washer method to find the volume of the solid of revolution.*



Hint: *Each radius will be a difference of y-values because slices are indexed by the variable x. The distance from the axis $y = 1$ to the line $y = 0$ is 1, and the distance from the axis $y = 1$ to $y = \sqrt{x}$ is $1 - \sqrt{x}$.*

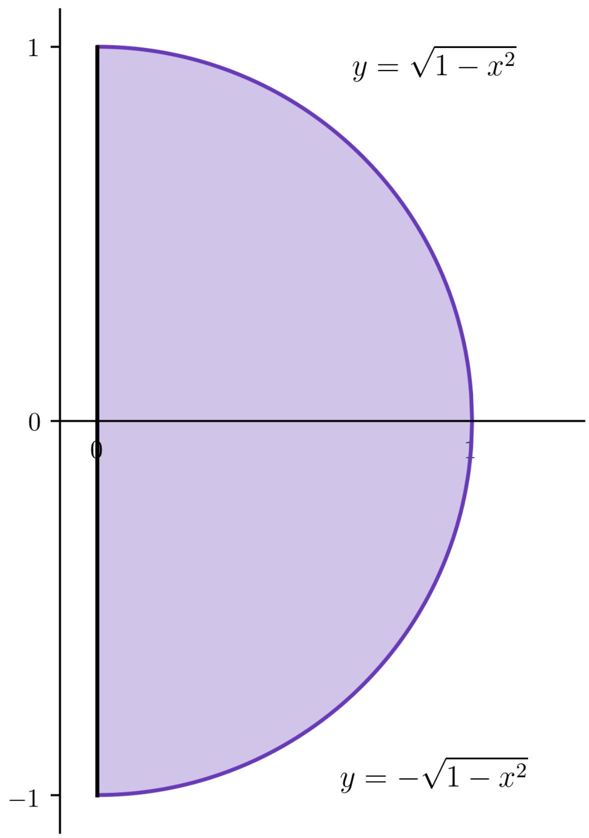
$$R_{\text{outer}}(x) = \boxed{1} \text{ and } r_{\text{inner}}(x) = \boxed{1 - \sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(x))^2 - (r_{\text{inner}}(x))^2 \right] dx = \boxed{\frac{5\pi}{6}}$$

6.11 Alternating Series

We study the notion of alternating series and related concepts.

Example 61. *The region defined by the inequalities $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ and $x \geq 0$ (shown below) is revolved around the y-ais. Compute the volume using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

Multiple Choice:

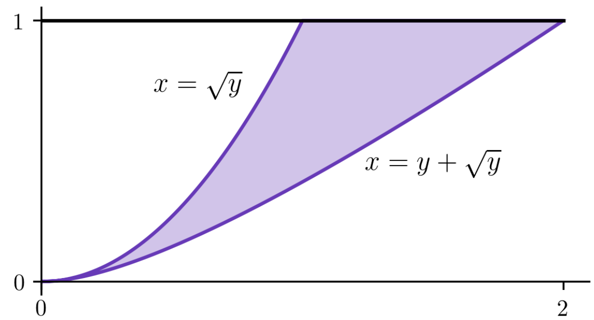
(a) $h(x) = \sqrt{1-x^2}$
(b) $h(x) = -\sqrt{1-x^2}$
(c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\, dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$.)

Example 62. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*



- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$
  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$
  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\, dy = \boxed{\frac{\pi}{3}}.$$

`series/alternatepractice.tex`

6.12 Exercises: Alternating Series

Exercises relating to alternating series and absolute or conditional convergence.

Exercise 40 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 41 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

7 Power Series

We undertake a study of an important class of infinite series.

Example 63. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 64. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

7.1 Power Series

We introduce the concept of a power series and some related fundamental properties.

Example 65. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 66. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`powerseries/powerseriespractice.tex`

7.2 Exercises: Power Series and Convergence

Exercises relating to power series and their convergence properties.

Exercise 42 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 43 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

7.3 Power Series Manipulation

We study various operations which can be performed on power series.

Example 67. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 68. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

powerseries/powerseries2practice.tex

7.4 Exercises: Power Series Manipulation

Exercises relating to the formal manipulation of power series.

Exercise 44 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 45 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$0 \leq x \leq 1,$$
$$0 \leq y \leq z^2,$$
$$0 \leq z \leq 3.$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

7.5 Taylor Series

We introduce the notion of a Taylor Series.

Example 69. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \left.\frac{x^3}{3}\right|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 70. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

powerseries/taylorseriespractice.tex

7.6 Exercises: Taylor Series

Exercises relating to Taylor series and their computation.

Exercise 46 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 47 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$0 \le x \le 1,$$
$$0 \le y \le z^2,$$
$$0 \le z \le 3.$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

7.7 Taylor Series Applications

We study the use of Taylor series for evaluating infinite series and limits.

Example 71. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 72. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

powerseries/taylorseries2practice.tex

7.8 Exercises: Taylor Series Applications

Various exercises relating to the application of Taylor Series to other problems of interest.

Exercise 48 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 49 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$0 \le x \le 1,$$
$$0 \le y \le z^2,$$
$$0 \le z \le 3.$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

8 Ordinary Differential Equations

We begin a study of first-order ordinary differential equations.

Example 73. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 74. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

8.1 ODEs: Foundations

We study the fundamental concepts and properties associated with ODEs.

Example 75. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 76. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`odes/odepractice.tex`

8.2 Exercises: ODEs

Exercises relating to fundamental properties of ODEs.

Exercise 50 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length $L = \boxed{x - x^2}$ and area $A = \boxed{(x - x^2)^2}$.*
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of $x = \boxed{0}$ up to a maximum of $x = \boxed{1}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 51 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length $\boxed{1}$ in the x-direction and width $\boxed{z^2}$ in the y-direction.*
- *The area of a z cross section is $A(z) = \boxed{z^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

8.3 Separable and Linear ODEs

We learn techniques to solve first-order linear and separable ODEs.

Example 77. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 78. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`odes/solvepractice.tex`

8.4 Exercises: Linear and Separable ODEs

Exercises related to solving linear and separable ODEs.

Exercise 52 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length $L = \boxed{x - x^2}$ and area $A = \boxed{(x - x^2)^2}$.*
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of $x = \boxed{0}$ up to a maximum of $x = \boxed{1}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 53 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length $\boxed{1}$ in the x-direction and width $\boxed{z^2}$ in the y-direction.*
- *The area of a z cross section is $A(z) = \boxed{z^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

8.5 Applications of ODEs

We study some sample applications of ODEs.

Example 79. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 80. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

`odes/applypractice.tex`

8.6 Exercises: ODE Applications

Exercises relating to the application of ODEs to solve problems.

Exercise 54 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length $L = \boxed{x - x^2}$ and area $A = \boxed{(x - x^2)^2}$.*
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of $x = \boxed{0}$ up to a maximum of $x = \boxed{1}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 55 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\leq x \leq 1, \\ 0 &\leq y \leq z^2, \\ 0 &\leq z \leq 3. \end{aligned}$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length $\boxed{1}$ in the x-direction and width $\boxed{z^2}$ in the y-direction.*
- *The area of a z cross section is $A(z) = \boxed{z^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$