General Slicing By Cross Sections

The general relationship between volume and cross-sectional area.

Example 1. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 2. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

Learning outcomes:

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Exercise 1 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *Possible x-coordinates of points in the base range from a minimum value of $x = \boxed{0}$ up to a maximum of $x = \boxed{1}$.*
- *A typical square cross section has side length $\boxed{x - x^2}$ and area $\boxed{(x - x^2)^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$