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# Math 104 Online

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December 4, 2019

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## Welcome: About This Course

*This course is built in Ximera.*

The philosophy behind Ximera is to make it possible to learn mathematics *actively* online. **Wrong answers never hurt you.** You should go through these pages with paper and pencil in hand; take notes, draw pictures, do calculations. A number of exercises have hints: you are strongly encouraged (and sometimes structurally required) to make your best effort to answer an exercise before referring to the hints. Remember: **these exercises are for you to gain insight and strength by working through carefully** and are not simple boxes to check off or hoops to jump through.

## The Mechanics of Ximera

See these links for information and examples of how this system is used:

- How to Use Ximera
- How Work is Scored in Ximera

## The Structure of This Course

Each topic in this course is addressed in a two-part way. You will first encounter a page with links to videos and online texts that should be the first place you go to learn new material. **There is a wealth of high-quality, open-source material on the web, and it is not the purpose of this course to reinvent the wheel.** The main sources that you are encouraged to use are

- Calculus: Single Variable , a series of short YouTube videos by Penn professor Robert Ghrist

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Learning outcomes:

Author(s):

See How to Use Ximera at <https://ximera.osu.edu/mooculus/calculus1/ximeraTutorial/howToUseXimera>

See How Work is Scored in Ximera at <https://ximera.osu.edu/mooculus/calculus1/ximeraTutorial/howIsMyWorkScored>

See Calculus: Single Variable at <https://www.youtube.com/playlist?list=PLKc2X0QpOdMwj9zAXD5L1WpriIXIrGaNb>

- OpenStax Calculus II , a web-based, open-source, modern Calculus textbook. It is approved by the Open Textbook Initiative of the American Institute of Mathematics.

In most cases, these materials should be sufficient for you to actively engage with the material we will study. **For every topic, it is expected that you will view videos and read textbook sections on your own *before we discuss these topics in class*.** You should further work through all the included “Examples” on a topic before class. In particular, these same examples will often appear on pre-class Canvas quizzes.

After each set of examples, there is a list of exercises which are a mix of original content, questions drawn from open-source texts: OpenStax Calculus II, Calculus: Early Transcendentals , and APEX Calculus ; and old Penn calculus exam questions. These exercises are generally more challenging than the initial examples and may take more time and some thought inside the classroom as well as outside in order to complete. The exercises sections will have sample questions from quizzes and old exams when relevant. It is highly recommended that you carefully work such questions and avoid guessing answers, as they are there for you primarily to test your own understanding of how to solve problems and not simply there for you to know their answers.

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See OpenStax Calculus II at <https://openstax.org/details/books/calculus-volume-2>  
See Calculus: Early Transcendentals at [https://www.whitman.edu/mathematics/calculus\\_online/](https://www.whitman.edu/mathematics/calculus_online/)  
See APEX Calculus at <http://www.apexcalculus.com/>

## 2 Applications of Integration

*We study some important application of integrations: computing volumes of a variety of complicated three-dimensional objects, computing arc length and surface area, and finding centers of mass.*

Integration is the tool to use whenever a quantity can be conceived as an *accumulation of infinitesimal parts*. Volume is one of the most basic and important of such quantities. In the activities that follow, we regard volume as the accumulated size of infinitely thin slices and use this perspective to derive and apply a number of formulas for computing volume. Following this, we will examine other applications of integration to the computation of arc length, surface area, and centers of mass.

## 2.1 Volume By General Cross Sections

We use cross-sectional area to compute volume.

### Online Texts

- OpenStax II 2.2: Determining Volumes by Slicing<sup>9</sup>
- Ximera OSU: Accumulated Cross Sections<sup>10</sup>

### Examples

**Example 1.** The base of a solid region is bounded by the curves  $x = 0$ ,  $y = 0$ , and  $y = \sqrt{1 - x^2}$ . The cross sections perpendicular to the  $x$  axis are squares. Compute the volume of the region.

**Solution:** Lines in the  $xy$ -plane which are perpendicular to the  $x$ -axis are vertical, so the base of a typical  $x$  cross section will extend from  $y = 0$  to  $y = \sqrt{1 - x^2}$ . Since each cross section will have area

$$A(x) = \left( \sqrt{1 - x^2} - 0 \right)^2 = 1 - x^2.$$

To compute volume, we integrate  $dV = A(x)dx$  between  $x = 0$  and  $x = 1$ , since these are the most extreme values of  $x$  found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3} \Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

**Example 2.** The base of a solid region is bounded by the curves  $y = 0$ ,  $x = \sqrt{y}$ , and  $x = 1$ . The cross sections perpendicular to the  $y$ -axis are squares. Compute the volume of the region.

**Solution:** Lines in the  $xy$ -plane which are perpendicular to the  $y$ -axis are (horizontal ✓/ vertical), so the base of a typical  $y$  cross section will extend from the graph  $x = \sqrt{y}$  to the graph  $x = 1$ . The length of the base is the difference

<sup>9</sup>See OpenStax II 2.2: Determining Volumes by Slicing at <https://openstax.org/books/calculus-volume-2/pages/2-2-determining-volumes-by-slicing>

<sup>10</sup>See Ximera OSU: Accumulated Cross Sections at <https://ximera.osu.edu/mooculus/calculus2/accumulatedCrossSections/titlePage>



## Volume By General Cross Sections

of  $x$ -coordinates (since all points on a slice have the same  $y$ -coordinate), so the length of the base is  $1 - \sqrt{y}$ , giving the square an area of

$$A(y) = (1 - \sqrt{y})^2$$

(note that the answer is a function of  $y$  because different  $y$  cross sections will generally have different areas). To compute volume, we integrate  $dV = A(y)dy$  between  $y = 0$  and  $y = 1$ , since these are the most extreme values of  $y$  found in our region (note that we can find the upper value  $y = 1$  by solving for the intersection of the curves  $x = \sqrt{y}$  and  $x = 1$ ). Therefore we integrate  $A(y)dy$  to conclude

$$V = \int_0^1 (1 - \sqrt{y})^2 dy = \frac{1}{6}.$$

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volumes/01genslicepractice.tex

## 2.2 Exercises: General Slicing

Exercises computing volume by cross-sectional area.

**Exercise 1** The base of a solid region is bounded by the curves  $x = 0$ ,  $y = x^2$  and  $y = x$ . The cross sections perpendicular to the  $x$ -axis are squares. Compute the volume of the region.

- A typical square cross section has side length  $L = x - x^2$  and area  $A = (x - x^2)^2$ .
- Possible numerical values of the  $x$ -coordinates of points in the base range from a minimum value of  $x = 0$  up to a maximum of  $x = 1$ .
- To compute volume, integrate:

$$V = \int_0^1 (x - x^2)^2 dx = \frac{1}{30}.$$

**Exercise 2** Find the volume of the region in three-dimensional space defined by the inequalities

$$\begin{aligned} 0 &\leq x \leq 1, \\ 0 &\leq y \leq z^2, \\ 0 &\leq z \leq 3. \end{aligned}$$

- Cross sections perpendicular to the  $z$ -axis are (square / rectangular ✓ / triangular) with length  $1$  in the  $x$ -direction and width  $z^2$  in the  $y$ -direction.
- The area of a  $z$  cross section is  $A(z) = z^2$ .
- To compute volume, integrate:

$$V = \int_0^3 z^2 dz = 9.$$

**Exercise 3** A right circular cylinder of radius 1 and height 3 is twisted along its axis so that the disk at height  $z$  is centered on the axis  $x = \cos(2\pi z/3), y = \sin(2\pi z/3)$ , which corresponds to one full twist along the axis. Compute the volume of this twisted cylinder.

$$V = \boxed{3\pi}.$$

**Exercise 4** A certain three-dimensional region has a base in the  $xy$ -plane which is bounded above by the graph  $y = 1 - x^2$  and below by  $y = 0$ . Slices perpendicular to the  $y$ -axis are equilateral triangles whose base lies in the  $xy$ -plane as well. Compute the volume of the region.

$$V = \boxed{\frac{\sqrt{3}}{2}}.$$

## Sample Exam Questions

**Question 5** (2018 Midterm 1) Compute the volume of the region in 3-dimensional space which satisfies the inequalities

$$0 \leq x \leq (1 - z^2) \quad \text{and} \quad 0 \leq y \leq (1 + z^2) \quad \text{and} \quad 0 \leq z \leq 1.$$

**Multiple Choice:**

(a)  $\frac{2}{3}$

(b)  $\frac{3}{4}$

(c)  $\frac{4}{5}$  ✓

(d)  $\frac{5}{6}$

(e)  $\frac{6}{7}$

(f) none of these

**Question 6** (2019 Midterm 1) The inequality

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

defines an ellipse in the  $xy$ -plane whose area is  $\pi ab$  for any positive values of the constants  $a$  and  $b$ . Compute the three dimensional volume of the region defined by

$$4x^2 + z^2y^2 \leq z^2 \text{ for } 0 \leq z \leq 1.$$

(Hints won't reveal until after you choose a response.)

**Multiple Choice:**

- (a)  $4\pi z^2$
- (b)  $4\pi$
- (c)  $\frac{\pi}{4z}$
- (d)  $\frac{\pi}{4}$  ✓
- (e)  $\pi z$
- (f)  $\pi$

**Feedback(attempt):** Dividing the first inequality by  $z^2$  on both sides gives

$$\frac{\frac{x^2}{z^2}}{4} + \frac{y^2}{1} \leq 1,$$

**Hint:** which means that slices in the  $z$ -direction are ellipses with area  $\pi \frac{z}{2} \cdot 1 = \frac{\pi z}{2}$ .

**Hint:** Volume is obtained by integration:

$$V = \int_0^1 \frac{\pi z}{2} dz = \left. \frac{\pi z^2}{4} \right|_0^1 = \frac{\pi}{4}.$$

## 2.3 The Disk and Washer Methods

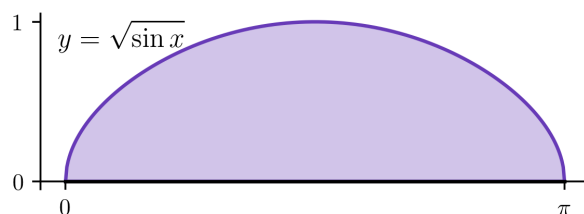
We practice setting up calculations related to the disk and washer methods.

## Online Texts

- OpenStax II 2.2 (beginning with the Disk Method)<sup>11</sup>
- Ximera OSU: The Washer Method<sup>12</sup>

## Examples

**Example 3.** Suppose the region below the graph  $y = \sqrt{\sin x}$  and above the  $x$ -axis between  $x = 0$  and  $x = \pi$  is revolved around the  $x$ -axis. Compute the volume of the resulting solid.



- Because the axis of rotation lies perfectly along the boundary of the region, the (disk  $\checkmark$ / washer) method can be used.
- The radius  $R$  is the length of a (horizontal/ vertical  $\checkmark$ ) extending from the axis to the graph  $y = \sqrt{\sin x}$ .
- Thus we know that the radius  $R$  must equal

**Multiple Choice:**

- (a)  $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x} \checkmark$
- (b)  $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

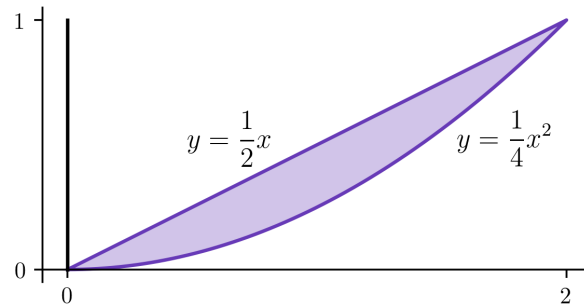
- We conclude that

$$V = \int_0^{\pi} \pi \left( \sqrt{\sin x} \right)^2 dx = 2\pi.$$

**Example 4.** Suppose the region between the graphs  $y = x/2$  and  $y = x^2/4$  is revolved around the axis  $x = 0$ . Compute the volume of the resulting solid.

<sup>11</sup>See OpenStax II 2.2 (beginning with the Disk Method) at <https://openstax.org/books/calculus-volume-2/pages/2-2-determining-volumes-by-slicing>

<sup>12</sup>See Ximera OSU: The Washer Method at <https://ximera.osu.edu/mooculus/calculus2/solidsOfRevolution/digInWasherMethod>



- Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.
- In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs  $y = x/2$  and  $y = x^2/4$ .
- **Multiple Choice:**
  - (a)  $R_{\text{outer}}(x) = x/2$  and  $r_{\text{inner}}(x) = x^2/4$
  - (b)  $R_{\text{outer}}(y) = 2\sqrt{y}$  and  $r_{\text{inner}}(y) = 2y$  ✓
- We conclude that

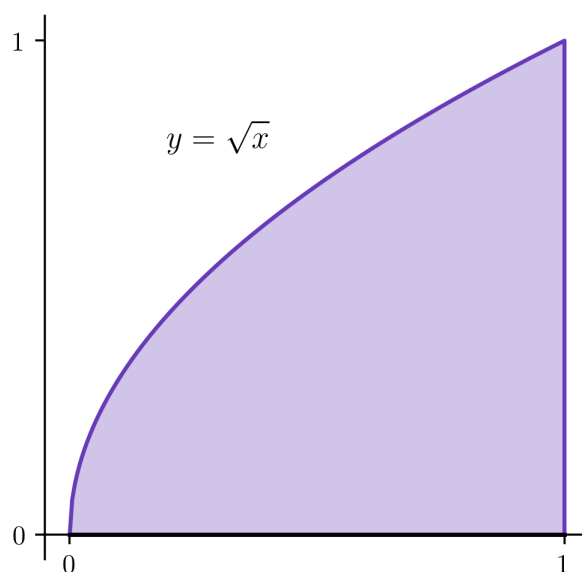
$$V = \int_0^1 \pi \left[ \left( 2\sqrt{y} \right)^2 - \left( 2y \right)^2 \right] dy = \frac{2\pi}{3}.$$

volumes/02washerpractice.tex

## 2.4 Exercises: Disks and Washers

Exercises for the disk and washer methods.

**Exercise 7** The region  $0 \leq y \leq \sqrt{x}$  with  $x \leq 1$ , shown below, is revolved around the  $x$ -axis. Use the disk method to find the volume of the solid of revolution.

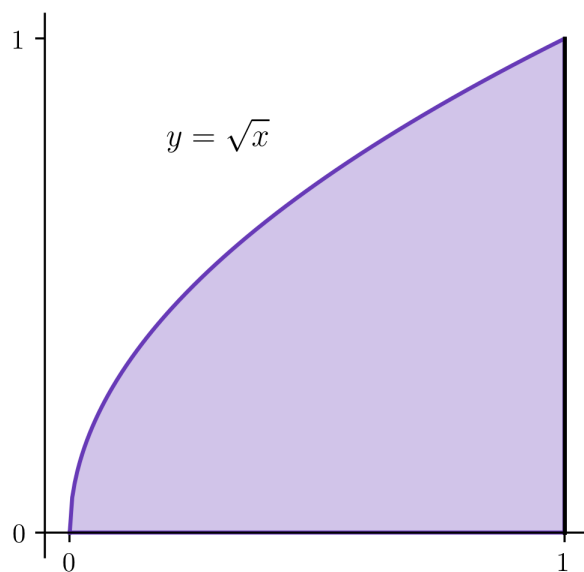


**Hint:** The radius  $R(x)$  will be a difference of  $y$ -values because slices are indexed by the variable  $x$ . Each slice will extend from  $y = 0$  to  $y = \sqrt{x}$ , and so  $R(x)$  must be the larger of these  $y$ -values minus the smaller of these  $y$ -values.

$$R(x) = \boxed{\sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi(R(x))^2 dx = \boxed{\frac{\pi}{2}}$$

**Exercise 8** The region  $0 \leq y \leq \sqrt{x}$  with  $x \leq 1$ , shown below, is revolved around the axis  $x = 1$ . Use the disk method to find the volume of the solid of revolution.



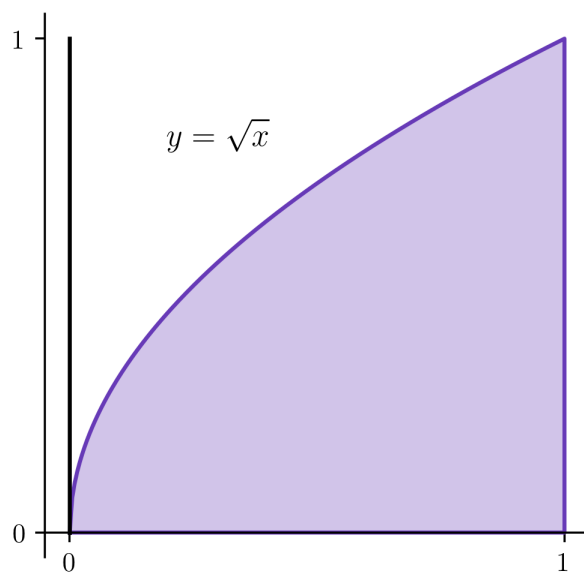
**Hint:** The radius  $R(y)$  will be a difference of  $x$ -values because slices are indexed by the variable  $y$ . Each slice will extend from  $x = y^2$  to  $x = 1$ , and so  $R(y)$  must be the larger of these  $x$ -values minus the smaller of these  $x$ -values

$$R(y) = 1 - y^2$$

$$V = \int_0^1 \pi(R(y))^2 dy = \frac{8\pi}{15}$$

**Exercise 9** The region  $0 \leq y \leq \sqrt{x}$  with  $x \leq 1$ , shown below, is revolved around the axis  $x = 0$ . Use the washer method to find the volume of the solid of revolution.



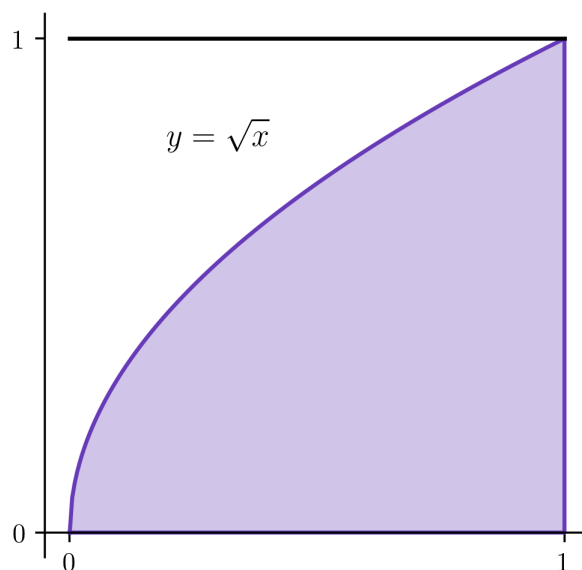


**Hint:** Each radius will be a difference of  $x$ -values because slices are indexed by the variable  $y$ . The distance from the axis  $x = 0$  to the line  $x = 1$  is 1, and the distance from the axis  $x = 0$  to  $x = y^2$  is  $y^2$ .

$$R_{\text{outer}}(y) = \boxed{1} \text{ and } r_{\text{inner}}(y) = \boxed{y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi [(R_{\text{outer}}(y))^2 - (r_{\text{inner}}(y))^2] dy = \boxed{\frac{4\pi}{5}}$$

**Exercise 10** The region  $0 \leq y \leq \sqrt{x}$  with  $x \leq 1$ , shown below, is revolved around the axis  $y = 1$ . Use the washer method to find the volume of the solid of revolution.



**Hint:** Each radius will be a difference of  $y$ -values because slices are indexed by the variable  $x$ . The distance from the axis  $y = 1$  to the line  $y = 0$  is 1, and the distance from the axis  $y = 1$  to  $y = \sqrt{x}$  is  $1 - \sqrt{x}$ .

$$R_{\text{outer}}(x) = \boxed{1} \text{ and } r_{\text{inner}}(x) = \boxed{1 - \sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi [(R_{\text{outer}}(x))^2 - (r_{\text{inner}}(x))^2] dx = \boxed{\frac{5\pi}{6}}$$

## Sample Quiz Questions

**Question 11** The region in the plane bounded on the left by the curve  $x = -y^2$ , on the right by the curve  $x = y^2 + 2y + 2$ , above by the line  $y = 0$ , and below by the line  $y = -2$  is revolved around the axis  $x = 2$ . Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)

**Multiple Choice:**

- (a)  $16\pi$
- (b)  $20\pi$
- (c)  $24\pi$  ✓

(d)  $28\pi$ (e)  $32\pi$ (f)  $36\pi$ 

**Feedback(attempt):** The axis  $x = 2$  is perpendicular to the direction of slices using the integration variable  $y$ , which indicates the washer method. The region lies to the left of the axis. One way to see this is to evaluate  $x = -y^2$  at  $y = -2$ , giving  $x = -4$ , which is to the left of the axis  $x = 2$ .

**Hint:** The integral to compute equals

$$\begin{aligned} V &= \int_{-2}^0 \pi \left( (2 - (-y^2))^2 - (2 - (y^2 + 2y + 2))^2 \right) dy \\ &= \pi \int_{-2}^0 (-4y^3 + 4) dy \\ &= \pi \left( -y^4 + 4y \right) \Big|_{-2}^0 = 24\pi. \end{aligned}$$

**Question 12** The region in the plane bounded below by the curve  $y = -2x^2 + 5x + 2$ , above by the curve  $y = -2x^2 + 2x + 2$ , on the right by the line  $x = 0$ , and on the left by the line  $x = -1$  is revolved around the axis  $y = 2$ . Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)

**Multiple Choice:**

(a)  $10\pi$  ✓(b)  $14\pi$ (c)  $18\pi$ (d)  $22\pi$ (e)  $26\pi$ (f)  $30\pi$ 

**Feedback(attempt):** The axis  $y = 2$  is perpendicular to the direction of slices using the integration variable  $x$ , which indicates the washer method. The region lies below the axis. One way to see this is to evaluate  $y = -2x^2 + 5x + 2$  at  $x = -1$ , giving  $y = -5$ , which is below the axis  $y = 2$ .

**Hint:** The integral to compute equals

$$\begin{aligned} V &= \int_{-1}^0 \pi \left( (2 - (-2x^2 + 5x + 2))^2 - (2 - (-2x^2 + 2x + 2))^2 \right) dx \\ &= \pi \int_{-1}^0 (-12x^3 + 21x^2) dx \\ &= \pi (-3x^4 + 7x^3) \Big|_{-1}^0 = 10\pi. \end{aligned}$$

**Question 13** The region in the plane given by  $\left| -\frac{x}{2} + \frac{1}{2}\sqrt{9-6x^2} \right| \leq y \leq \frac{x}{2} + \frac{1}{2}\sqrt{9-6x^2}$  and  $0 \leq x \leq \frac{2}{3}\sqrt{3}$  is revolved around the  $x$ -axis. Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)

**Multiple Choice:**

- (a)  $\frac{13}{9}\pi$  ✓
- (b)  $\frac{19}{9}\pi$
- (c)  $\frac{26}{9}\pi$
- (d)  $\frac{28}{9}\pi$
- (e)  $\frac{37}{9}\pi$
- (f)  $\frac{49}{9}\pi$

**Feedback(attempt):** If the variable  $x$  is used for slicing, then slices are perpendicular to the axis of rotation, which indicates the washer method should be used.

**Hint:** The inequalities for  $y$  give the outer and inner radii, and

$$\left( \frac{x}{2} + \frac{1}{2}\sqrt{9-6x^2} \right)^2 - \left( \left| -\frac{x}{2} + \frac{1}{2}\sqrt{9-6x^2} \right| \right)^2 = x\sqrt{9-6x^2}.$$

(Note that the absolute values go away when the radius is squared.)

**Hint:** To compute the integral

$$\int_0^{\frac{2}{3}\sqrt{3}} \pi x \sqrt{9-6x^2} dx$$

we can use the substitution  $u = 9 - 6x^2$  which implies the equality  $du = (-12x) dx$  for the differentials. This gives the equality

$$\begin{aligned}\int \pi x \sqrt{9 - 6x^2} dx &= \int \left(-\frac{\pi}{12} \sqrt{u}\right) du \\ &= -\frac{\pi}{18} u^{\frac{3}{2}}.\end{aligned}$$

Reversing the substitution gives

$$\begin{aligned}\int_0^{\frac{2}{3}\sqrt{3}} \pi x \sqrt{9 - 6x^2} dx &= \left[-\frac{\pi}{18} (9 - 6x^2)^{\frac{3}{2}}\right]_0^{\frac{2}{3}\sqrt{3}} \\ &= \left(-\frac{\pi}{18}\right) - \left(-\frac{3}{2}\pi\right) = \frac{13}{9}\pi.\end{aligned}$$

## 2.5 The Shell Method

We practice setting up setting up volume calculations using the shell method.

### (Video) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=oraebRYMvbs>

### Online Texts

- OpenStax II 2.3: Volumes of Revolution: Cylindrical Shells<sup>13</sup>
- Ximera OSU: The Shell Method<sup>14</sup>

### Examples

**Example 5.** The region defined by the inequalities  $-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$  and  $x \geq 0$  (shown below) is revolved around the  $y$ -axis. Compute the volume using the shell method.

<sup>13</sup>See OpenStax II 2.3: Volumes of Revolution: Cylindrical Shells at <https://openstax.org/books/calculus-volume-2/pages/2-3-volumes-of-revolution-cylindrical-shells>

<sup>14</sup>See Ximera OSU: The Shell Method at <https://ximera.osu.edu/mooculus/calculus2/solidsOfRevolution/digInShellMethod>



- When the slicing variable is  $x$ , the radius of a shell is the (horizontal  $\checkmark$ /vertical) distance from an  $x$ -slice to the axis  $x = 0$ . Thus

$$r(x) = \boxed{x} - \boxed{0}.$$

- The height of an  $x$ -slice is equal to

**Multiple Choice:**

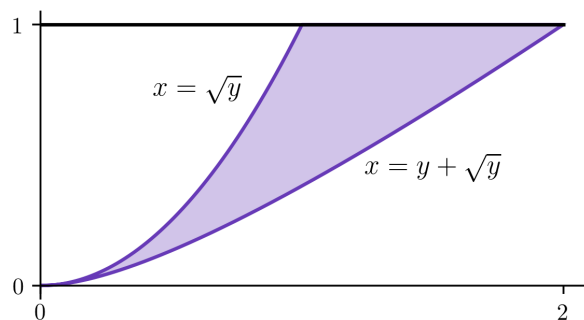
- (a)  $h(x) = \sqrt{1-x^2}$
- (b)  $h(x) = -\sqrt{1-x^2}$
- (c)  $h(x) = \sqrt{1-x^2} - (-\sqrt{1-x^2}) = 2\sqrt{1-x^2} \checkmark$

- The volume is equal to the integral of  $2\pi rh$ , so

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x \sqrt{1-x^2}} dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution  $u = 1 - x^2$ .)

**Example 6.** The region between the curves  $x = \sqrt{y}$  and  $x = y + \sqrt{y}$  from  $y = 0$  to  $y = 1$  is revolved around the axis  $y = 1$ . Compute the volume of the resulting solid.



- When the slicing variable is  $y$ , the radius of a shell is the (horizontal/vertical ✓) distance from a  $y$ -slice to the axis  $y = 1$ . Thus

$$r(y) = \boxed{1} - \boxed{y}.$$

- The “height” of a  $y$ -slice is equal to

**Multiple Choice:**

- (a)  $h(y) = \sqrt{y}$
- (b)  $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$
- (c)  $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$  ✓

- The volume is equal to the integral of  $2\pi rh$ , so

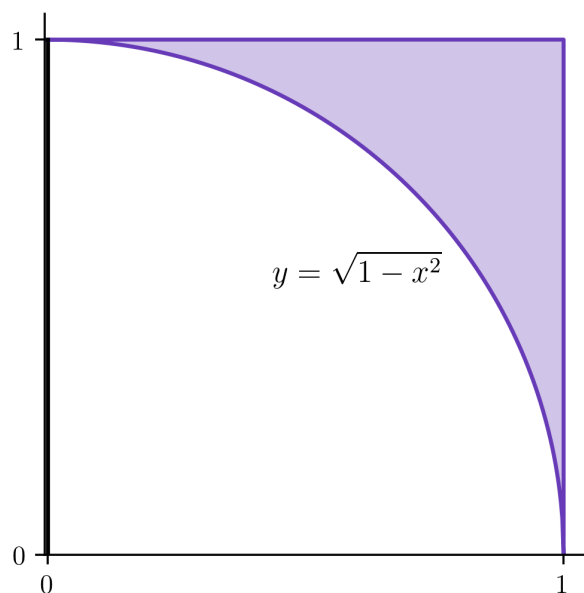
$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1 - y)} dy = \boxed{\frac{\pi}{3}}.$$

volumes/03shellpractice.tex

## 2.6 Exercises: Shell Method

Exercises for using the shell method.

**Exercise 14** The region defined by the inequalities  $\sqrt{1-x^2} \leq y \leq 1$  for  $0 \leq x \leq 1$  is revolved around the  $y$ -axis. Compute the volume of the resulting solid using the shell method.



- When the slicing variable is  $x$ , the radius of a shell is the (horizontal √/vertical) distance from an  $x$ -slice to the axis of rotation. Thus

$$r(x) = \boxed{x} - \boxed{0}.$$

- The height of an  $x$ -slice is equal to

$$h(x) = \boxed{1 - \sqrt{1-x^2}}.$$

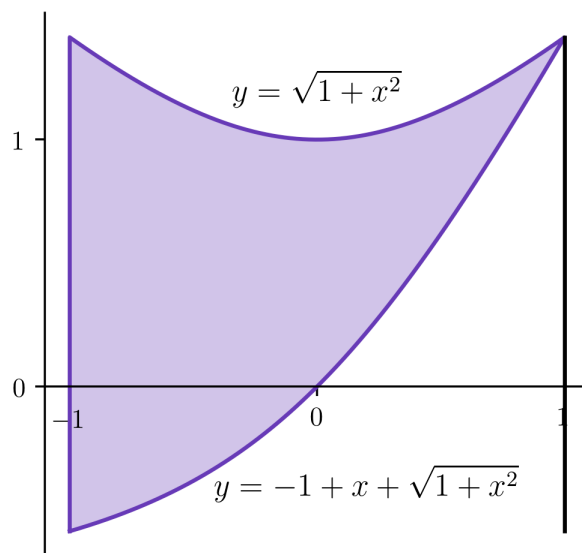
- The volume is equal to the integral of  $2\pi rh$ , so

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi x(1 - \sqrt{1-x^2})} dx = \boxed{\frac{\pi}{3}}.$$

(Note: to compute the integral, split it into two parts and make the substitution  $u = 1 - x^2$  for one of them.)



**Exercise 15** The region in the plane bounded above by the graph  $y = \sqrt{1+x^2}$ , below by  $y = -1+x+\sqrt{1+x^2}$ , and on the left by  $x = 0$  is revolved around the axis  $x = 1$ . Compute the volume of the resulting solid using the shell method.



- When the slicing variable is  $x$ , the radius of a shell is the (horizontal  $\checkmark$ /vertical) distance from an  $x$ -slice to the axis  $x = 0$ . Thus

$$r(x) = \boxed{1} - \boxed{x}.$$

- The height of an  $x$ -slice is equal to

$$h(x) = \boxed{-1+x}.$$

- The volume is equal to the integral of  $2\pi rh$ , so

$$V = \int_{\boxed{-1}}^{\boxed{1}} \boxed{2\pi(1-x)^2} dx = \boxed{\frac{16\pi}{3}}.$$

**Exercise 16** The region in the plane  $y = \sqrt{x}$ ,  $y = 0$ , and  $x = 1$  is revolved around the  $y$ -axis. Use the shell method to compute the volume.

$$V = \boxed{\frac{4\pi}{5}}.$$

Exercises: Shell Method

**Exercise 17** The same region as above (bounded by  $y = \sqrt{x}$ ,  $y = 0$ , and  $x = 1$ ) is revolved around the axis  $x = 1$ . Use the shell method to compute the volume.

$$V = \boxed{\frac{8\pi}{15}}.$$

**Exercise 18** The same region as above (bounded by  $y = \sqrt{x}$ ,  $y = 0$ , and  $x = 1$ ) is revolved around the  $x$ -axis. Use the shell method to compute the volume.

**Hint:** The “height” of a shell is  $1 - y^2$  in this case.

$$V = \boxed{\frac{\pi}{2}}.$$

**Exercise 19** For the same region as above (bounded by  $y = \sqrt{x}$ ,  $y = 0$ , and  $x = 1$ ), use the shell method to compute the volume when revolved around the axis  $y = 1$ .

$$V = \boxed{\frac{5\pi}{6}}.$$

## Sample Quiz Questions

**Question 20** The region in the plane bounded below by the curve  $y = -x^2$ , above by the curve  $y = x^2 + 2x + 2$ , on the right by the line  $x = 0$ , and on the left by the line  $x = -2$  is revolved around the axis  $x = -2$ . Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)

**Multiple Choice:**

- (a)  $4\pi$
- (b)  $6\pi$
- (c)  $8\pi$  ✓
- (d)  $10\pi$

(e)  $12\pi$ (f)  $14\pi$ 

**Feedback(attempt):** The axis  $x = -2$  is parallel to the direction of slices using the integration variable  $x$ , which indicates the shell method. The region lies to the right of the axis, which must be the case because the interval  $-2 \leq x \leq 0$  lies to the right of the axis  $x = -2$ .

**Hint:** The integral to compute equals

$$\begin{aligned} V &= \int_{-2}^0 2\pi(x - (-2))((x^2 + 2x + 2) - (-x^2)) \, dx \\ &= \pi \int_{-2}^0 (4x^3 + 12x^2 + 12x + 8) \, dx \\ &= \pi (x^4 + 4x^3 + 6x^2 + 8x) \Big|_{-2}^0 = 8\pi. \end{aligned}$$

**Question 21** The region in the plane bounded on the left by the curve  $x = -y^2 + 4y + 1$ , on the right by the curve  $x = y^2 + 2y + 1$ , and below by the line  $y = -1$  is revolved around the axis  $y = -1$ . Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)

**Multiple Choice:**

(a)  $\pi$  ✓(b)  $5\pi$ (c)  $9\pi$ (d)  $13\pi$ (e)  $17\pi$ (f)  $21\pi$ 

**Feedback(attempt):** The axis  $y = -1$  is parallel to the direction of slices using the integration variable  $y$ , which indicates the shell method. The lower endpoint of integration will be  $y = -1$ ; the upper endpoint can be determined by setting  $-y^2 + 4y + 1 = y^2 + 2y + 1$  and choosing the solution which is greater than  $-1$ . This gives the range  $-1 \leq y \leq 0$ . The region lies above the axis, which must be the case because the interval  $-1 \leq y \leq 0$  lies above the axis  $y = -1$ .

**Hint:** The integral to compute equals

$$\begin{aligned} V &= \int_{-1}^0 2\pi(y - (-1))((y^2 + 2y + 1) - (-y^2 + 4y + 1)) \, dy \\ &= \pi \int_{-1}^0 (4y^3 - 4y) \, dy \\ &= \pi (y^4 - 2y^2) \Big|_{-1}^0 = 1\pi. \end{aligned}$$

**Question 22** The region in the plane between the  $x$ -axis and the graph

$$y = \frac{1}{2\sqrt{\frac{x^2}{3} + 1}}$$

in the range  $0 \leq x \leq 3$  is revolved around the axis  $x = 0$ . Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)

**Multiple Choice:**

- (a)  $\frac{3}{2}\pi$
- (b)  $\frac{8}{5}\pi$
- (c)  $\frac{5}{3}\pi$
- (d)  $2\pi$
- (e)  $3\pi$  ✓
- (f)  $5\pi$

**Feedback(attempt):** If the variable  $x$  is used for slicing, then slices are parallel to the axis of rotation, which indicates the shell method should be used. The radius of a shell is  $x$ . The height of a shell is exactly  $\frac{1}{2\sqrt{\frac{x^2}{3} + 1}}$ .

**Hint:** The volume of the region is therefore given by

$$\int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2 + 3}} dx.$$

**Hint:** To compute the integral we can use the substitution  $u = x^2 + 3$  which implies the equality  $du = (2x) dx$  for the differentials. This gives the equality

$$\begin{aligned} \int \frac{\sqrt{3}\pi x}{\sqrt{x^2 + 3}} dx &= \int \frac{\sqrt{3}\pi}{2\sqrt{u}} du \\ &= \sqrt{3}\pi\sqrt{u}. \end{aligned}$$

Reversing the substitution gives

$$\begin{aligned} \int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2 + 3}} dx &= \left[ \sqrt{3}\pi\sqrt{x^2 + 3} \right]_0^3 \\ &= (6\pi) - (3\pi) = 3\pi. \end{aligned}$$

## 2.7 Synthesis: Choose Your Method

We practice choosing a method for computing volume when none is specified.

### Online Texts

- Ximera: Comparing Washers and Shells<sup>15</sup>

### Examples

**Example 7.** (Spring 2016 Final Exam) Find the volume of the solid generated by revolving the region in the first quadrant bounded by  $y = 1 - x^2$ ,  $y = 0$ , and  $x = 0$  about the line  $x = 2$ .

- Using  $x$  as the slicing variable, slices are (parallel ✓/ perpendicular) to the axis  $x = 2$ . This alignment corresponds to the (disk or washer/ shell ✓) method. (Expand to continue.)
- The height of a shell is

**Multiple Choice:**

- (a) 2
- (b)  $2 - x$
- (c) 0
- (d)  $x - 2$
- (e)  $1 - x^2$  ✓
- (f)  $1 - x^2 - 2$

and the radius is

**Multiple Choice:**

- (a) 2
- (b)  $2 - x$  ✓
- (c) 0

---

<sup>15</sup>See Ximera: Comparing Washers and Shells at <https://ximera.osu.edu/mooculus/calculus2/solidsOfRevolution/digInComparingWasherAndShellMethod>

- (d)  $x - 2$
- (e)  $1 - x^2$
- (f)  $1 - x^2 - 2$

Therefore

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi(2-x)(1-x^2)} dx = \boxed{\frac{13\pi}{6}}.$$

**Example 8.** (Fall 2012 Final Exam) The region in the  $xy$ -plane bounded by  $y = (x-1)^{1/4}$  and the  $x$ -axis for  $1 \leq x \leq 2$  is rotated about the  $x$ -axis. What is the volume of the resulting solid of revolution?

- Using  $x$  as the slicing variable, slices are (parallel / perpendicular ✓) to the axis  $x = 2$ . This alignment corresponds to the (disk or washer ✓ / shell ) method. (Expand to continue.)
- The outer radius of a disk is

**Multiple Choice:**

- (a) 2
- (b)  $y^4 + 1$
- (c) 0
- (d)  $(x-1)^{1/4}$  ✓

and the inner radius is

**Multiple Choice:**

- (a) 2
- (b)  $y^4 + 1$
- (c) 0 ✓
- (d)  $(x-1)^{1/4}$

Therefore

$$V = \int_{\boxed{1}}^{\boxed{2}} \boxed{\pi(x-1)^{1/2}} dx = \boxed{\frac{2\pi}{3}}.$$

**Example 9.** Find the volume of the solid generated by revolving the region in the first quadrant bounded on the left by  $y = \sqrt{x}$ , on the right by  $y = x - 2$ , and below by  $y = 0$  about the axis  $y = 0$ .

Synthesis: Choose Your Method

- To carry out the calculation as efficiently as possible, the  $(x/y \checkmark)$ -variable should be used as the variable for slicing. (Expand to continue.)
- Using  $y$  as the slicing variable, slices are (parallel  $\checkmark$ / perpendicular) to the axis  $y = 0$ . This alignment corresponds to the (disk or washer/ shell  $\checkmark$ ) method. (Expand to continue.)
- The height of a shell is

**Multiple Choice:**

- (a)  $y + 2$
- (b)  $y^2$
- (c)  $x - 2$
- (d)  $\sqrt{x}$
- (e)  $y^2 - y - 2$
- (f)  $y + 2 - y^2$   $\checkmark$

and the radius is

**Multiple Choice:**

- (a)  $y$   $\checkmark$
- (b)  $y^2$
- (c)  $x - 2$
- (d)  $\sqrt{x}$
- (e)  $y^2 - y - 2$
- (f)  $y + 2 - y^2$

Therefore

$$V = \int_{\boxed{0}}^{\boxed{2}} \boxed{2\pi y(y + 2 - y^2)} dy = \boxed{\frac{16\pi}{3}}.$$

volumes/04choosepractice.tex

## 2.8 Exercises: Choose Your Method

*Exercises choosing a method for computing volume.*

**Exercise 23** The region in the plane bounded by  $y = e^{-x/2}$  and the  $x$ -axis for  $0 \leq x \leq \ln 2$  is rotated about the  $x$ -axis. The volume of the resulting solid of revolution is

$$V = \boxed{\frac{\pi}{2}}.$$

(Hints won't be revealed until after you choose a response.)

**Feedback(attempt):** If  $x$  is used as the slicing variable, then slices are vertical and consequently perpendicular to the axis of rotation.

**Hint:** Furthermore one side of the region lies along the axis, so the disk method is appropriate in this case.

**Hint:** The distance from the axis to the upper edge of the region is  $e^{-x/2}$ , so

$$\begin{aligned} V &= \int_0^{\ln 2} \pi \left( e^{-x/2} \right)^2 dx = \pi \int_0^{\ln 2} e^{-x} dx \\ &= -\pi e^{-x} \Big|_{x=0}^{\ln 2} = \pi(-e^{-\ln 2} + e^0) = \pi \left( -\frac{1}{2} + 1 \right) = \frac{\pi}{2}. \end{aligned}$$

**Exercise 24** The region in the plane bounded on the right by the curve  $x = 2 - y^2$ , on the left by the curve  $x = y^2$ , and on the bottom by  $y = 0$  is revolved around the  $y$ -axis. Compute the volume of the resulting solid.

$$V = \boxed{\frac{8\pi}{3}}.$$

**Exercise 25** Compute the volume of the solid of revolution obtained by rotating the region between  $x = 0$ ,  $y = 0$ , and  $x = \sqrt{2 + 3y^2 - 5y^4}$  around the  $y$ -axis.

$$V = \boxed{2\pi}.$$



**Exercise 26** The region between the graph of  $y = 1 - x^2$  and the  $x$ -axis is rotated around the line  $y = 1$ . What is the volume of the resulting solid?

$$V = \boxed{\frac{8\pi}{5}}.$$

**Exercise 27** Find the volume obtained by rotating the region between the graph  $x = \frac{1}{2} \sin(y^2)$  and the  $y$ -axis for  $0 \leq y \leq \sqrt{\pi}$  about the  $x$ -axis.

$$V = \boxed{\pi}.$$

## Sample Exam Questions

**Question 28** Calculate the volume of the solid obtained by rotating the area between the graphs of  $y = \frac{1}{\sqrt{x^2 - 1}}$  and the  $x$ -axis for  $1 < x < \sqrt{5}$  around the  $y$ -axis.

**Multiple Choice:**

- (a)  $\pi$
- (b)  $4\pi$  ✓
- (c)  $6\pi$
- (d)  $8\pi$
- (e)  $3\pi$
- (f)  $2\pi$

**Question 29** Let  $f(x)$  be a continuous function that satisfies  $f(0) = 0$  and  $f(x) > 0$  for  $x > 0$ . For every  $b > 0$ , when the region between the graph of  $y = f(x)$ , the  $x$ -axis, and the line  $x = b$  is rotated around the  $x$ -axis, the volume of the resulting solid is  $18\pi b^2$ . What is  $f(x)$ ? (Hints will not be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $9x$
- (b)  $3x^2$
- (c)  $6\sqrt{x}$  ✓
- (d)  $27x^{3/2}$
- (e)  $9x^2$
- (f)  $\sqrt{3x}$

**Feedback(attempt):** By the disk method, we have that

$$\int_0^b \pi(f(x))^2 dx = 18\pi b^2$$

for each  $b > 0$ . Solve this equation for  $b$ .

**Hint:** Differentiate both sides with respect to  $b$ ; use the Fundamental Theorem of Calculus to differentiate the left-hand side.

**Question 30** Find the volume of the solid generated by revolving the region bounded above by  $y = \sec x$  and bounded below by  $y = 0$  for  $0 \leq x \leq \pi/3$  about the  $x$ -axis.

**Multiple Choice:**

- (a)  $\pi$
- (b)  $2\pi$
- (c)  $\pi\sqrt{3}$  ✓
- (d)  $3\pi$
- (e)  $4\pi$
- (f) none of these

## 2.9 Arc Length

We practice setting up and executing arc length calculations.

## (Video) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=fT1LVQdF-70>

## Online Texts

- OpenStax II 2.4 Arc Length<sup>16</sup>
- Ximera OSU: Arc Length<sup>17</sup>
- Community Calculus: Arc Length<sup>18</sup>

## Examples

**Example 10.** Compute the arc length of the curve

$$x = -\frac{1}{8}y^{-2} - \frac{1}{4}y^4 - 2$$

between the endpoints  $y = 1/\sqrt{2}$  and  $y = 1$ .

- First we compute the derivative:

$$\frac{dx}{dy} = \boxed{\frac{1}{4}y^{-3} - y^3}.$$

- Next, we write the arc length element:

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{1 + \left(\boxed{\frac{1}{4}y^{-3} - y^3}\right)^2} dy.$$

- The key to integrating is to first fully simplify the quantity inside the square root:

$$\begin{aligned} 1 + \left(\boxed{\frac{1}{4}y^{-3} - y^3}\right)^2 &= 1 + \frac{1}{16}y^{-6} + 2\left(\frac{1}{4}y^{-3}\right)(-y^3) + y^6 \\ &= 1 + \frac{1}{16}y^{-6} - \frac{1}{2} + y^6 \end{aligned}$$

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<sup>16</sup>See OpenStax II 2.4 Arc Length at <https://openstax.org/books/calculus-volume-2/pages/2-4-arc-length-of-a-curve-and-surface-area>

<sup>17</sup>See Ximera OSU: Arc Length at <https://ximera.osu.edu/mooculus/calculus2/lengthOfCurves/titlePage>

<sup>18</sup>See Community Calculus: Arc Length at [https://www.whitman.edu/mathematics/calculus\\_online/section09.09.html](https://www.whitman.edu/mathematics/calculus_online/section09.09.html)

**Observation 1.** When the middle term in a binomial (i.e., FOIL) expansion is  $-1/2$  adding one **always** gives a perfect square. In this case, the algebraic fact is

$$1 + \frac{1}{16}y^{-6} - \frac{1}{2} + y^6 = \frac{1}{16}y^{-6} + \frac{1}{2} + y^6 = \left( \boxed{\frac{1}{4}y^{-3} + y^3} \right)^2.$$

The general phenomenon is that what was formerly the square of a difference—in this case,  $\left(\frac{1}{4}y^{-3} - y^3\right)^2$ —becomes the square of the corresponding sum.

Therefore

$$\begin{aligned} L &= \int_{1/\sqrt{2}}^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \int_{1/\sqrt{2}}^1 \sqrt{1 + \left(\boxed{\frac{1}{4}y^{-3} - y^3}\right)^2} dy \\ &= \int_{1/\sqrt{2}}^1 \sqrt{1 + \frac{1}{16}y^{-6} - \frac{1}{2} + y^6} dy = \int_{1/\sqrt{2}}^1 \sqrt{\left(\boxed{\frac{1}{4}y^{-3} + y^3}\right)^2} dy \\ &= \int_{1/\sqrt{2}}^1 \boxed{\frac{1}{4}y^{-3} + y^3} dy = \left( \boxed{-\frac{1}{8}y^{-2} + \frac{1}{4}y^4} \right) \Big|_{1/\sqrt{2}}^1 \\ &= \left( -\frac{1}{8} + \frac{1}{4} \right) - \left( -\frac{1}{4} + \frac{1}{16} \right) = \frac{5}{16}. \end{aligned}$$

arclengths/05arclengthpractice.tex

**2.10 Exercises: Arc Length***We practice computing arc length.***Exercise 31** Find the arc length of the function on the given interval:  $f(x) = \sqrt{8x}$  on  $[-1, 1]$ 

$$L = \boxed{6}.$$

**Exercise 32** Find the arc length of the function on the given interval:  $f(x) = \ln(\cos x)$  on  $[0, \pi/4]$ . (You may use the fact that  $\int \sec x \, dx = \ln|\sec x + \tan x| + C$ .)

$$L = \boxed{\ln(1 + \sqrt{2})}.$$

**Exercise 33** Set up the integral to compute the arc length of the function on the given interval:  $f(x) = x^2$  on  $[0, 1]$ .

$$L = \int_{\boxed{0}}^{\boxed{1}} \boxed{\sqrt{1 + 4x^2}} \, dx$$

**Question 34** Let  $y = \frac{x^4}{16} + \frac{1}{2x^2}$ . Find the arc length for  $1 \leq x \leq \sqrt{2}$ .

$$L = \boxed{\frac{7}{16}}.$$

## Sample Quiz Questions

**Question 35** Compute the arc length of the curve

$$y = \frac{3}{4}x^{-2} + \frac{1}{24}x^4 - 1$$

between the endpoints  $x = \sqrt{3}$  and  $x = \sqrt{6}$ . (Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\frac{5}{12}$
- (b)  $\frac{7}{12}$
- (c)  $\frac{3}{4}$
- (d)  $\frac{11}{12}$
- (e)  $\frac{13}{12}$
- (f)  $\frac{5}{4}$  ✓

**Feedback(attempt):** Applying the formula for arc length gives that

$$L = \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{1 + \left(-\frac{3}{2}x^{-3} + \frac{1}{6}x^3\right)^2} dx$$

**Hint:**

$$\begin{aligned} &= \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{1 + \frac{9}{4}x^{-6} - \frac{1}{2} + \frac{1}{36}x^6} dx = \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{\left(\frac{3}{2}x^{-3} + \frac{1}{6}x^3\right)^2} dx \\ &= \int_{\sqrt{3}}^{\sqrt{6}} \left(\frac{3}{2}x^{-3} + \frac{1}{6}x^3\right) dx = \left(-\frac{3}{4}x^{-2} + \frac{1}{24}x^4\right) \Big|_{\sqrt{3}}^{\sqrt{6}} \\ &= \left(-\frac{1}{8} + \frac{3}{2}\right) - \left(-\frac{1}{4} + \frac{3}{8}\right) = \frac{5}{4}. \end{aligned}$$

Note that you must always take the positive square root in going from line two to line three. In particular, if you get a negative answer, you have likely taken the negative square root.

## Sample Exam Questions

**Question 36** (2017 Midterm 1) Compute the length of the curve  $x = \frac{1}{8}(y^2 + 2y) - \ln(y + 1)$  between  $y = 0$  and  $y = 2$ .

**Multiple Choice:**

- (a)  $1 + \ln 3$  ✓
- (b)  $2 + \ln 6$
- (c)  $3 + \ln 9$
- (d)  $4 + \ln 12$
- (e)  $5 + \ln 15$
- (f) *none of the above*

**Question 37** Find the arc length of the following curve between  $x = -1$  and  $x = 1$ :

$$y = 3 \cosh \frac{x}{3}.$$

(Note:  $\cosh x = (e^x + e^{-x})/2$ .)

**Multiple Choice:**

- (a)  $\frac{e}{3} - \frac{1}{3e}$
- (b)  $\frac{e}{2} - \frac{1}{2e}$
- (c)  $e - \frac{1}{e}$
- (d)  $2e - \frac{2}{e}$
- (e)  $3e - \frac{3}{e}$  ✓
- (f) *none of the above*

**Question 38** A certain curve  $y = f(x)$  in the plane has the property that its length between the endpoints  $x = 0$  and  $x = a$  is equal to

$$\int_0^a \sqrt{1 + \sin^2 t} \, dt$$

for every value of  $a > 0$ . Assuming the curve passes through the points  $(0, 0)$  and  $(\frac{\pi}{2}, 1)$ , what is  $f(\frac{\pi}{4})$ ?

**Multiple Choice:**

- (a)  $\frac{1}{2}$
- (b)  $\frac{1}{\sqrt{2}}$
- (c)  $1 - \frac{1}{\sqrt{2}}$  ✓
- (d) 0
- (e)  $-\frac{1}{\sqrt{2}}$
- (f) none of these

**Question 39** Find the length of the part of the curve  $y = \frac{3}{16}e^{2x} + \frac{1}{3}e^{-2x}$  for  $0 \leq x \leq \ln 2$ .

**Multiple Choice:**

- (a)  $\frac{13}{16}$  ✓
- (b)  $\frac{11}{16}$
- (c)  $\frac{3}{8}$
- (d)  $\frac{9}{8}$
- (e)  $\frac{29}{64}$
- (f)  $\frac{3}{4}$



**Question 40** Find the length of the part of the curve  $y = \frac{x^4}{4} + \frac{1}{8x^2}$  for  $1 \leq x \leq 2$ .

**Multiple Choice:**

- (a)  $\frac{13}{16}$  ✓
- (b)  $\frac{11}{16}$
- (c)  $\frac{7}{8}$
- (d)  $\frac{13\sqrt{2}}{16}$
- (e)  $\frac{11\sqrt{2}}{16}$
- (f)  $\frac{7\sqrt{2}}{8}$

## 2.11 Surface Area

*We practice setting up integrals for the surface area of surfaces of revolution.*

### Online Texts

- OpenStax II 2.4: Surface Area<sup>19</sup>
- Ximera OSU: Surface Area<sup>20</sup>
- Community Calculus 9.10: Surface Area<sup>21</sup>

<sup>19</sup>See OpenStax II 2.4: Surface Area at <https://openstax.org/books/calculus-volume-2/pages/2-4-arc-length-of-a-curve-and-surface-area>

<sup>20</sup>See Ximera OSU: Surface Area at <https://ximera.osu.edu/mooculus/calculus2/surfaceArea/titlePage>

<sup>21</sup>See Community Calculus 9.10: Surface Area at [https://www.whitman.edu/mathematics/calculus\\_online/section09.10.html](https://www.whitman.edu/mathematics/calculus_online/section09.10.html)

## Examples

**Example 11.** Suppose the graph  $y = 2\sqrt{x}$  between  $x = 0$  and  $x = 1$  is revolved around the  $x$ -axis. Compute the area of the resulting surface.

- The distance from the  $x$ -axis to the point  $(x, 2\sqrt{x})$  is  $2\sqrt{x}$ .
- For this curve, the arc length element satisfies  $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx =$   
 $\sqrt{1 + \frac{1}{x}} dx$

$$\begin{aligned}
 A &= \int_0^1 2\sqrt{x} \, ds \\
 &= \int_0^1 2\sqrt{x+1} \, dx \\
 &= \frac{4}{3} (2\sqrt{2} - 1).
 \end{aligned}$$

arclengths/06surfacepractice.tex

## 2.12 Exercises: Surface Area

Various exercises related to the computation of areas of surfaces of revolution.

**Exercise 41** Find the surface area of the solid formed by revolving  $y = 2x$  on  $[0, 1]$  about the  $x$ -axis.

$$A = \boxed{2\pi\sqrt{5}}.$$

**Exercise 42** Find the surface area of the solid formed by revolving  $y = x^2$  on  $[0, 1]$  about the  $y$ -axis.

**Hint:** To compute the integral, you will need to make a substitution like  $u = 1 + 4x^2$  or something similar.

$$A = \boxed{\frac{(5\sqrt{5} - 1)\pi}{6}}.$$

**Exercise 43** Find the surface area of the solid formed by revolving  $y = x^3$  on  $[0, 1]$  about the  $x$ -axis.

**Hint:** To compute the integral, you will need to make a substitution like  $u = 1 + 9x^4$  or something similar.

$$L = \boxed{\frac{(10\sqrt{10} - 1)\pi}{27}}.$$

## Sample Exam Questions

**Question 44** Give an integral formula for the area of the surface generated by revolving the curve  $y = \ln x$  between  $x = 1$  and  $x = 2$  about the  $y$ -axis. Explain your answer. You do not need to evaluate the integral.

**Multiple Choice:**

- (a)  $\int_1^2 2\pi\sqrt{x^2+1} \, dx$  ✓
- (b)  $\int_1^2 2\pi(\ln x) \frac{\sqrt{x^2+1}}{x} \, dx$
- (c)  $\int_1^2 \frac{2\pi}{x} \sqrt{1+(\ln x)^2} \, dx$
- (d)  $\int_1^2 \frac{1}{2\pi\sqrt{x^2+1}} \, dx$
- (e)  $\int_1^2 2\pi(\ln x) \frac{x}{\sqrt{x^2+1}} \, dx$
- (f)  $\int_1^2 \frac{2\pi x}{\sqrt{1+(\ln x)^2}} \, dx$

**Question 45** The curve  $y = \frac{x^2}{8}$  between  $x = 0$  and  $x = 3$  is revolved around the  $y$ -axis. Compute the surface area of the resulting surface.

**Multiple Choice:**

- (a)  $\frac{31\pi}{6}$
- (b)  $\frac{41\pi}{6}$
- (c)  $\frac{61\pi}{6}$  ✓
- (d)  $\frac{71\pi}{6}$
- (e)  $\frac{91\pi}{6}$
- (f) none of the above

## 2.13 Centers of Mass and Centroids

We practice setting up calculations for centers of mass and centroids.

## (Video) Calculus: Single Variable

**Note:** It is important to get as much of an intuitive sense as you can about what the double integrals (which appear around the 3:30 mark) mean, but do not worry about precisely what they represent. YouTube link: <https://www.youtube.com/watch?v=0LeYrY4AZXk>

## Online Texts

- OpenStax II 2.6: Centroids and Centers of Mass<sup>22</sup>
- Community Calculus 9.6: Center of Mass<sup>23</sup>

## Examples

**Example 12.** Compute the centroid of the region bounded by the inequalities

$$0 \leq x \leq 2 \quad \text{and} \quad -\frac{9}{2}x^2 + x \leq y \leq \frac{9}{2}x^2 + x.$$

- The term “centroid” refers to the geometric center of a region. Practically speaking, this means we may assume constant density (e.g., density 1).
- First we compute the “mass” of the region, which in this case is simply the area between curves:

$$M = \int_0^2 \left[ \left( \frac{9}{2}x^2 + x \right) - \left( -\frac{9}{2}x^2 + x \right) \right] dx = 24.$$

- Next we compute the moments about the  $y$  and  $x$  axes. This always involves multiplying the integrand above by  $\tilde{x}$  and  $\tilde{y}$ , respectively (note the reversal), where  $(\tilde{x}, \tilde{y})$  are the coordinates of the geometric center of a typical slice.
- Using  $x$  as the slicing variable, slices are (horizontal / vertical ✓) and consequently the  $x$ -coordinate of the geometric center of a slice is just  $x$  (but note that this would be different if  $y$  were the slicing variable). Thus

$$M_y = \int_0^2 x \left[ 9x^2 \right] dx = 36.$$

<sup>22</sup>See OpenStax II 2.6: Centroids and Centers of Mass at <https://openstax.org/books/calculus-volume-2/pages/2-6-moments-and-centers-of-mass>

<sup>23</sup>See Community Calculus 9.6: Center of Mass at [https://www.whitman.edu/mathematics/calculus\\_online/section09.06.html](https://www.whitman.edu/mathematics/calculus_online/section09.06.html)

- The  $y$ -coordinate of the geometric center of a slice will be the average of  $y$ -coordinates at the top and bottom of a slice. Therefore

$$\tilde{y} = \boxed{x}.$$

Thus

$$M_x = \int_{\boxed{0}}^{\boxed{2}} \boxed{9x^3} dx = \boxed{36}.$$

- Thus

$$\begin{aligned}\bar{x} &= \frac{M_y}{M} = \boxed{\frac{3}{2}}, \\ \bar{y} &= \frac{M_x}{M} = \boxed{\frac{3}{2}}.\end{aligned}$$

**Example 13.** Compute the centroid of the region given by  $-\frac{3}{2}y^2 + 2y \leq x \leq \frac{3}{2}y^2 + 2y$  between  $y = 0$  and  $y = 2$ .

- In this example, we should use  $y$  as the slicing variable, so the roles of  $x$  and  $y$  are largely switched in comparison to the previous example.
- First compute the mass:

$$M = \int_0^2 \left[ \left( \boxed{\frac{3}{2}y^2 + 2y} \right) - \left( \boxed{-\frac{3}{2}y^2 + 2y} \right) \right] dy = \boxed{8}.$$

- In this case,  $\tilde{y} = y$  since slices are (horizontal  $\checkmark$ / vertical), so

$$M_x = \int_0^2 y \boxed{3y^2} dy = \boxed{12}.$$

- Likewise,  $\tilde{x}$  is the average of  $x$ -coordinates of endpoints of a slice. Thus

$$\tilde{x} = \boxed{2y}.$$

Therefore

$$M_y = \int_{\boxed{0}}^{\boxed{2}} \boxed{6y^3} dy = \boxed{24}$$

- To conclude,

$$\begin{aligned}\bar{x} &= \frac{M_y}{M} = \boxed{3}, \\ \bar{y} &= \frac{M_x}{M} = \boxed{\frac{3}{2}}.\end{aligned}$$

centroids/07centroidpractice.tex

## 2.14 Exercises: Centers of Mass and Centroids

Various questions relating to centers of mass and centroids.

**Exercise 46** Find the centroid of the region bounded above by  $y = x$  and below by  $y = x^2$ .

$$\bar{x} = \boxed{\frac{1}{2}} \text{ and } \bar{y} = \boxed{\frac{2}{5}}.$$

**Exercise 47** Find the centroid of the region bounded above by  $y = 4 - x^2$  and below by  $y = 0$ .

$$\bar{x} = \boxed{0} \text{ and } \bar{y} = \boxed{\frac{8}{5}}.$$

**Exercise 48** A thin plate in the plane defined by  $x^2 \leq y \leq 1$  and  $x \geq 0$  has density  $y$  at the point  $(x, y)$ . Compute the center of mass.

**Hint:** Use  $y$  as the variable of slicing. The center of mass of a single slice  $(\tilde{x}, \tilde{y})$  is then  $(\sqrt{y}/2, y)$ .

$$\bar{x} = \boxed{\frac{1}{2}} \text{ and } \bar{y} = \boxed{\frac{5}{7}}.$$

**Exercise 49** A thin plate in the plane defined by  $x^2 \leq y \leq 2x^2$  and  $0 \leq x \leq 1$  has density  $x$  at the point  $(x, y)$ . Compute the center of mass.

**Hint:** Use  $x$  as the variable of slicing. The center of mass of a single slice  $(\tilde{x}, \tilde{y})$  is then  $(x, 3x^2/2)$ .

$$\bar{x} = \boxed{\frac{4}{5}} \text{ and } \bar{y} = \boxed{1}.$$

Exercises: Centers of Mass and Centroids

**Exercise 50** The same thin plate as above ( $x^2 \leq y \leq 2x^2$  and  $0 \leq x \leq 1$ ) now has density  $x^{-2}$  at the point  $(x, y)$ . Because the density of the plate is now higher near the origin than in the previous problem, this suggests that the center of mass will shift (away from / towards  $\checkmark$ ) the origin relative to the previous exercise.

Compute the center of mass.

$$\bar{x} = \boxed{\frac{1}{2}} \text{ and } \bar{y} = \boxed{\frac{1}{2}}.$$

**Exercise 51** Compute the centroid of a thin wire along the graph  $y = \sqrt{1 - x^2}$  between  $x = 0$  and  $x = 1$ .

**Hint:** Recall that

$$\begin{aligned} M &= \int ds \\ \bar{x} &= \frac{1}{M} \int x ds \\ \bar{y} &= \frac{1}{M} \int y ds \end{aligned}$$

where  $ds$  is the arc length element. We also know that

$$\int \frac{dx}{\sqrt{1 - x^2}} = \arcsin x + C.$$

$$\bar{x} = \boxed{\frac{2}{\pi}} \text{ and } \bar{y} = \boxed{\frac{2}{\pi}}.$$

## Sample Quiz Questions

**Question 52** Compute the centroid of the region bounded by the inequalities

$$-2 \leq x \leq 0 \quad \text{and} \quad 3x^2 - \frac{13}{2} \leq y \leq 3x^2 - \frac{11}{2}.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

(a)  $\left(-\frac{3}{2}, -3\right)$



- (b)  $(-1, -3)$
- (c)  $\left(-\frac{1}{2}, -3\right)$
- (d)  $\left(-\frac{3}{2}, -2\right)$
- (e)  $(-1, -2)$  ✓
- (f)  $\left(-\frac{1}{2}, -2\right)$

**Feedback(attempt):** The key calculations are as follows:

$$\begin{aligned} M &= \int_{-2}^0 \left[ \left(3x^2 - \frac{11}{2}\right) - \left(3x^2 - \frac{13}{2}\right) \right] dx, \\ M_y &= \int_{-2}^0 x \left[ \left(3x^2 - \frac{11}{2}\right) - \left(3x^2 - \frac{13}{2}\right) \right] dx, \\ M_x &= \frac{1}{2} \int_{-2}^0 \left[ \left(3x^2 - \frac{11}{2}\right)^2 - \left(3x^2 - \frac{13}{2}\right)^2 \right] dx. \end{aligned}$$

**Hint:**

$$\begin{aligned} M &= \int_{-2}^0 \left[ \left(3x^2 - \frac{11}{2}\right) - \left(3x^2 - \frac{13}{2}\right) \right] dx = \int_{-2}^0 [1] dx = 2, \\ M_y &= \int_{-2}^0 x \left[ \left(3x^2 - \frac{11}{2}\right) - \left(3x^2 - \frac{13}{2}\right) \right] dx = \int_{-2}^0 x [1] dx = -2, \\ M_x &= \frac{1}{2} \int_{-2}^0 \left[ \left(3x^2 - \frac{11}{2}\right)^2 - \left(3x^2 - \frac{13}{2}\right)^2 \right] dx = \int_{-2}^0 [3x^2 - 6] dx = -4, \\ \bar{x} &= \frac{M_y}{M} = -1, \\ \bar{y} &= \frac{M_x}{M} = -2. \end{aligned}$$

## Sample Exam Questions

**Question 53** Find the  $y$ -coordinate of the centroid of the region bounded by the  $x$ -axis, the  $y$ -axis, and the graph of  $y = \cos x$  for  $0 \leq x \leq \pi/2$  if the density is constant.

**Hint:** Use the identity

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

to calculate the integral of  $\cos^2 x$ .

**Multiple Choice:**

- (a)  $\frac{\pi}{18}$
- (b)  $\frac{\pi}{12}$
- (c)  $\frac{\pi}{8}$  ✓
- (d)  $\frac{\pi}{6}$
- (e)  $\frac{\pi}{4}$
- (f)  $\frac{\pi}{2}$

**Feedback(attempt):** The area of the region is given by

$$M = \int_0^{\frac{\pi}{2}} \cos x \, dx = 1$$

and

$$M_x = \int_0^{\frac{\pi}{2}} \frac{0 + \cos x}{2} \cos x \, dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

$$\frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} dx = \frac{\pi}{8}.$$

Therefore  $\bar{y} = M_x/M = \pi/8$ .

**Question 54** Find the  $y$ -coordinate of the centroid of the region in the upper half-plane (i.e., for  $y > 0$ ) bounded by the semicircle  $y = \sqrt{1 - x^2}$ . (It is easiest to use a geometric formula to find the area of the region.)

**Multiple Choice:**

- (a)  $\frac{4\pi}{3}$
- (b)  $\frac{4}{3\pi}$  ✓
- (c)  $\frac{7\pi}{3}$
- (d)  $\frac{7}{3\pi}$
- (e)  $\frac{28\pi}{9}$

(f)  $\frac{28}{9\pi}$

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## 3 Integration Techniques

*We begin a study of techniques for computing integrals.*

We return to the direct study of integration with the goal of developing a variety of tools that we can use to compute antiderivatives. It will be assumed that you already have some experience with substitution, but we will see some examples of this again, and then go on to examine integration by parts, trigonometric substitutions, and partial fractions.

### 3.1 Substitution and Tables

We review substitution and the use of integral tables.

#### (Videos) Calculus: Single Variable

**Note:** The section on the Gompertz Model (7:08–11:31) relates to ideas that we will study later but have not yet seen. It is recommended that you do your best to understand it now and then come back to it again when we study ODEs. YouTube link: [https://www.youtube.com/watch?v=q\\_CmUbctfzo](https://www.youtube.com/watch?v=q_CmUbctfzo) YouTube link: <https://www.youtube.com/watch?v=T-WgpMM813E>

#### Online Texts

- OpenStax II 1.5: Substitution<sup>24</sup> and OpenStax II 3.5: Tables and Computer Systems<sup>25</sup>
- Ximera OSU: Substitution<sup>26</sup>
- Community Calculus 8.1: Substitution<sup>27</sup>

#### Examples

**Example 14.** The region in the plane given by

$$\left| \frac{x}{2} - \frac{1}{2\sqrt{\frac{x^2}{3} + 1}} \right| \leq y \leq \frac{x}{2} + \frac{1}{2\sqrt{\frac{x^2}{3} + 1}}$$

and  $0 \leq x \leq 3$  is revolved around the  $x$ -axis. Compute the volume of the resulting solid.

<sup>24</sup>See OpenStax II 1.5: Substitution at <https://openstax.org/books/calculus-volume-2/pages/1-5-substitution>

<sup>25</sup>See OpenStax II 3.5: Tables and Computer Systems at <https://openstax.org/books/calculus-volume-2/pages/3-5-other-strategies-for-integration>

<sup>26</sup>See Ximera OSU: Substitution at <https://ximera.osu.edu/mooculus/calculus1/substitution/titlePage>

<sup>27</sup>See Community Calculus 8.1: Substitution at [https://www.whitman.edu/mathematics/calculus\\_online/section08.01.html](https://www.whitman.edu/mathematics/calculus_online/section08.01.html)

If the variable  $x$  is used for slicing, then slices are perpendicular to the axis of rotation, which indicates the washer method should be used. The inequalities for  $y$  give the outer and inner radii, and

$$\left(\frac{x}{2} + \frac{1}{2\sqrt{\frac{x^2}{3} + 1}}\right)^2 - \left(\left|\frac{x}{2} - \frac{1}{2\sqrt{\frac{x^2}{3} + 1}}\right|\right)^2 = \frac{\sqrt{3}x}{\sqrt{x^2 + 3}}.$$

(Note that the absolute values go away when the radius is squared.) This leads to the conclusion

$$V = \int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2 + 3}} dx.$$

Among the options below, the best choice for a potential substitution is

**Multiple Choice:**

- (a)  $u = \sqrt{3x}$
- (b)  $u = 1/\sqrt{x^2 + 3}$
- (c)  $u = \sqrt{x^2 + 3}$
- (d)  $u = x^2 + 3$  ✓

Using the substitution above, the differentials satisfy  $du = \boxed{2x} dx$ . This gives the equality

$$\int \frac{\sqrt{3}\pi x}{\sqrt{x^2 + 3}} dx = \int \frac{\boxed{\sqrt{3}\pi}}{\boxed{2\sqrt{u}}} du = \boxed{\sqrt{3}\pi\sqrt{u}}.$$

Reversing the substitution gives

$$\begin{aligned} \int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2 + 3}} dx &= \left[ \boxed{\sqrt{3}\pi\sqrt{x^2 + 3}} \right]_0^3 \\ &= \boxed{3\pi}. \end{aligned}$$

**Note:** We would not have to reverse the substitution if we also determined the new bounds. In this case, if  $u = x^2 + 3$  and  $x = 0$ , then  $u = \boxed{3}$ . Likewise if  $x = 3$ , then  $u = \boxed{12}$ . Thus we could also have carried out the calculation by changing bounds:

$$\int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2 + 3}} dx = \int_{\boxed{3}}^{\boxed{12}} \frac{\boxed{\sqrt{3}\pi}}{\boxed{2\sqrt{u}}} du.$$

**Example 15.** Using the table of integrals below, compute the indefinite integral

$$\int \frac{d\theta}{(1 - \theta)\sqrt{\theta^2 - 2\theta - 3}}.$$

- When dealing with quadratic expressions such as the ones appearing in this integrand, it is often necessary to complete the square before appealing to a table. In this case,

$$\theta^2 - 2\theta - 3 = (\theta - \boxed{1})^2 - \boxed{4}.$$

- Using the most appropriate entry of the table and plugging in the correct value of  $a$  gives

$$\int \frac{dx}{x\sqrt{x^2 - \boxed{4}}} = \frac{1}{\boxed{2}} \operatorname{arcsec} \frac{x}{\boxed{2}} + C.$$

- Based on the results of completing the square, we make a substitution  $x = \theta - \boxed{1}$  and conclude

$$\int \frac{d\theta}{(1 - \theta)\sqrt{\theta^2 - 2\theta - 3}} = \int \frac{dx}{\boxed{-x}\sqrt{x^2 - \boxed{4}}} = \boxed{-\frac{1}{2}} \operatorname{arcsec} \frac{\theta - 1}{2} + C.$$

**Example 16.** Use a table of integrals to compute the antiderivative below.

$$\int \frac{x^2 dx}{\sqrt{16 + x^6}}$$

- The key is to make the substitution  $u = x^{\boxed{3}}$  so that the expression  $16 + x^6$  can be understood as a quadratic function of  $u$ .
- Specifically  $du = \boxed{3x^2} dx$ , so

$$\int \frac{x^2 dx}{\sqrt{16 + x^6}} = \frac{1}{3} \int \frac{du}{\sqrt{16 + u^2}}.$$

Aside from the factor of  $1/3$ , the  $u$  integral belongs to the table:

$$\int \frac{du}{\sqrt{16 + u^2}} = \boxed{\ln |u + \sqrt{16 + u^2}|} + C$$

- We conclude that

$$\int \frac{x^2 dx}{\sqrt{16 + x^6}} = \frac{1}{3} \int \frac{du}{\sqrt{16 + u^2}} = \frac{\ln(x^3 + \sqrt{16 + x^6})}{3} + C.$$

Absolute values are not needed in the logarithm because  $x^3 + \sqrt{16 + x^6}$  can never be negative.

### A Basic Table of Integrals

(Note: To use these tables,  $k$  and  $a$  must represent *constants* in the integral you wish to compute and cannot depend on the variable of integration.)

$\int x^k dx = \frac{x^{k+1}}{k+1} + C$	$(k \neq -1)$
$\int \frac{1}{x} dx = \ln x  + C$	
$\int a^x dx = \frac{a^x}{\ln a} + C$	$(a > 0, a \neq 1)$
$\int e^x dx = e^x + C$	
$\int x^k \ln x  dx = \frac{x^{k+1}}{k+1} \ln x  - \frac{x^{k+1}}{(k+1)^2} + C$	$(k \neq -1)$
$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C$	$(a \neq 0)$
$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln x + \sqrt{a^2 + x^2}  + C$	
$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C$	$(a > 0,  x  < a)$
$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln x + \sqrt{x^2 - a^2}  + C$	$( x  >  a )$
$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{x}{a} + C$	$(a > 0, x > a)$
$\int \sin x \, dx = -\cos x + C$	
$\int \cos x \, dx = \sin x + C$	
$\int \tan x \, dx = \ln \sec x  + C$	
$\int \csc x \, dx = -\ln \csc x + \cot x  + C$	
$\int \sec x \, dx = \ln \sec x + \tan x  + C$	
$\int \cot x \, dx = \ln \sin x  + C$	
$\int \sec^2 x \, dx = \tan x + C$	
$\int \csc^2 x \, dx = -\cot x + C$	
$\int \sec x \tan x \, dx = \sec x + C$	
$\int \csc x \cot x \, dx = -\csc x + C$	

techniques/08substitutionpractice.tex

**3.2 Exercises: Substitution and Tables***Various exercises relating to substitution and the use of integral tables.*

Compute the indefinite integrals below. Since there are many possible answers (which differ by constant values), use the given instructions if needed to choose which possible answer to use.

**Exercise 55**

$$\int (12x + 14) (3x^2 + 7x - 1)^5 dx = \boxed{\frac{1}{3}(3x^2 + 7x - 1)^6} + C$$

(Add a constant to your answer if needed so that it equals 1/3 at  $x = 0$ .)

**Exercise 56**

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \boxed{2e^{\sqrt{x}}} + C$$

(Add a constant to your answer if needed so that it equals 2 at  $x = 0$ .)

**Exercise 57**

$$\int \frac{x}{\sqrt{x+3}} dx = \boxed{\frac{2}{3}(x-6)\sqrt{x+3}} + C$$

(Add a constant to your answer if needed so that it equals 0 at  $x = 6$ .)

**Exercise 58**

$$\int \frac{\ln|x|}{x} dx = \boxed{\frac{1}{2} \ln^2|x|} + C$$

Remember absolute value in your logarithm. (Add a constant to your answer if needed so that it equals 0 at  $x = 1$ .)

**Exercise 59**

$$\int \sin x \sqrt{\cos x} dx = \boxed{-\frac{2}{3} \cos^{\frac{3}{2}}(x)} + C$$

(Add a constant to your answer if needed so that it equals  $-2/3$  at  $x = 0$ .)



**Exercise 60**

$$\int \frac{9(2x+3)}{3x^2+9x+7} dx = \boxed{3 \ln |3x^2+9x+7|} + C$$

Use absolute values as needed in logarithms. (Add a constant to your answer if needed so that it equals  $3 \ln 7$  at  $x = 0$ .)

**Exercise 61**

$$\int \frac{x}{x^4+81} dx = \boxed{\frac{1}{18} \arctan\left(\frac{x^2}{9}\right)} + C$$

(Add a constant to your answer if needed so that it equals 0 at  $x = 0$ .)

**Hint:** Make a substitution  $x^2 = 9u$ .

**Exercise 62** Evaluate the definite integral  $\int_{-2}^{-1} (x+1)e^{x^2+2x+1} dx$ .

$$\text{Value} = \boxed{(1-e)/2}$$

## Sample Exam Questions

**Question 63** Evaluate the integral  $\int_1^3 (x - \sqrt{4x^2 - 8x + 13}) dx$  using the fact that  $\int_0^4 \sqrt{x^2 + 9} dx = \frac{20 + 9 \ln 3}{2}$ . (Hints will not be displayed until you have chosen a response.)

**Multiple Choice:**

- (a)  $-\frac{4+9 \ln 3}{6}$
- (b)  $-\frac{4+9 \ln 3}{4}$  ✓
- (c)  $-\frac{4+9 \ln 3}{2}$
- (d)  $-4 - 9 \ln 3$

(e)  $-8 - 18 \ln 3$

(f)  $-12 - 27 \ln 3$

**Feedback(attempt):** First complete the square:

$$4x^2 - 8x + 13 = 4x^2 - 8x + 4 + (13 - 4) = (2x - 2)^2 + 9.$$

**Hint:** Next substitute  $u = 2x - 2$  (i.e.,  $x = \frac{u}{2} + 1$ ). Then  $du = 2dx$  and  $x = 1 \leftrightarrow u = 0$ ,  $x = 3 \leftrightarrow u = 4$ , so

$$\int_1^3 \left( x - \sqrt{4x^2 - 8x + 13} \right) dx = \int_0^4 \left( \frac{u}{2} + 1 - \sqrt{u^2 + 9} \right) \frac{du}{2}.$$

**Hint:** Now use linearity of the integral to finish:

$$\int_0^4 \frac{u}{4} du + \int_0^4 \frac{1}{2} du - \int_0^4 \frac{1}{2} \sqrt{u^2 + 9} du = \frac{u^2}{8} \Big|_0^4 + \frac{u}{2} \Big|_0^4 - \frac{20 + 9 \ln 3}{4} = 4 - \frac{20 + 9 \ln 3}{4} = -\frac{4 + 9 \ln 3}{4}.$$

### 3.3 Integration by Parts

We study the integration technique of integration by parts.

## (Video) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=xWm3b8GWHrg>

## Online Texts

- OpenStax II 3.1: Integration by Parts<sup>28</sup>
- Ximera OSU: Integration by Parts<sup>29</sup>
- Community Calculus 8.4: Integration by Parts<sup>30</sup>

<sup>28</sup>See OpenStax II 3.1: Integration by Parts at <https://openstax.org/books/calculus-volume-2/pages/3-1-integration-by-parts>

<sup>29</sup>See Ximera OSU: Integration by Parts at <https://ximera.osu.edu/mooculus/calculus2/integrationByParts/titlePage>

<sup>30</sup>See Community Calculus 8.4: Integration by Parts at [https://www.whitman.edu/mathematics/calculus\\_online/section08.04.html](https://www.whitman.edu/mathematics/calculus_online/section08.04.html)

## Examples

**Example 17.** Compute the indefinite integral

$$\int x e^{3x} dx.$$

- Because integrating  $e^{3x}$  and differentiating  $e^{3x}$  are at a similar level of difficulty, we opt to differentiate  $x$  so that its degree as a polynomial will be decreasing.
- This gives

$$\int x e^{3x} dx = \boxed{\frac{1}{3} x e^{3x}} - \int \boxed{\frac{1}{3} e^{3x}} dx = \boxed{\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x}} + C.$$

**Example 18.** Compute the indefinite integral

$$\int x^2 \ln |x| dx.$$

- Because integrating  $\ln |x|$  is much harder than differentiating it, we choose to differentiate  $\ln |x|$  and integrate  $x^2$ .
- This gives

$$\int x^2 \ln |x| dx = \boxed{\frac{x^3}{3}} \ln |x| - \int \boxed{\frac{x^2}{3}} dx = \boxed{\frac{x^3}{3} \ln |x| - \frac{x^3}{9}} + C.$$

**Example 19.** Compute the indefinite integral

$$\int x^4 e^{2x} dx.$$

- We'll do this using an organizational technique called "tabular integration" that many people find helpful when doing repeated integrations by parts.
- Make a table: one column for  $u$  and another for  $dv$ . In the first row, rewrite the functions that you will use for  $u$  and  $dv$ :

$$\begin{array}{cc} u & dv \\ x^4 & e^{2x} \end{array}$$

- Add new rows: differentiate items in the  $u$  column and integrate items in the  $dv$  column to determine what items go in the next row.

$u$	$dv$
$x^4$	$e^{2x}$
$4x^3$	$e^{2x}/2$
$12x^2$	$e^{2x}/4$
$24x$	$e^{2x}/8$
$24$	$e^{2x}/16$

Now you combine terms by matching items in the first column with items one row down. For the last item in the  $u$  column, match it with the last item in the  $dv$  column (which you'll end up using twice):

$$x^4 \cdot \frac{e^{2x}}{2}, \quad 4x^3 \cdot \frac{e^{2x}}{4}, \quad 12x^2 \cdot \frac{e^{2x}}{8}, \\ 24x \cdot \frac{e^{2x}}{16}, \quad 24 \cdot \frac{e^{2x}}{16}$$

- To finish, you alternate addition and subtraction. Give a  $+$  to the first term, a  $-$  to the second, and so on. Last but not least, put an integral on the last term as well:

$$+x^4 \frac{e^{2x}}{2} - 4x^3 \frac{e^{2x}}{4} + 12x^2 \frac{e^{2x}}{8} - 24x \frac{e^{2x}}{16} + \int 24 \frac{e^{2x}}{16} dx$$

In this way, we arrive at the formula

$$\int x^4 e^{2x} dx = \frac{1}{2}x^4 e^{2x} - x^3 e^{2x} + \frac{3}{2}x^2 e^{2x} - \frac{3}{2}x e^{2x} + \int \frac{3}{2}e^{2x} dx$$

- Sometimes the integrand of the last term is zero. This would happen when  $u$  is a polynomial if you have enough lines in your table. In that case there would be nothing else to compute. (We stopped one line short of that point on purpose in this example to show how you would handle a case where there the last term isn't zero.)
- In this case, the final answer is

$$\int x^4 e^{2x} dx = \frac{1}{2}x^4 e^{2x} - x^3 e^{2x} + \frac{3}{2}x^2 e^{2x} - \frac{3}{2}x e^{2x} + \frac{3}{4}e^{2x} + C$$

**Example 20.** Compute the indefinite integral below using tabular integration:

$$\int \cos 3x \cos 5x \, dx$$

- Let us take  $u = \cos 3x$  and  $dv = \cos 5x$  (it would have been fine to choose them the other way around as well). The table is then

$u$	$dv$
$\cos 3x$	$\cos 5x$
$-3 \sin 3x$	$(\sin 5x)/5$
$-9 \cos 3x$	$-(\cos 5x)/25$

Which gives

$$\int \cos 3x \cos 5x \, dx = \left[ \frac{1}{5} \cos 3x \sin 5x \right] - \left[ \frac{3}{25} \sin 3x \cos 5x \right] + \int \left[ \frac{9}{25} \cos 3x \cos 5x \right] dx.$$

- This example has a twist: the integral on the right-hand side is just a constant times the integral on the left-hand side. This indicates that further integrations by parts would be unfruitful because you'd end up in something like a cycle. What you can do is solve for the answer by moving both integrals to the left side of the equation and then combining like terms:

$$\left[ \frac{16}{25} \right] \int \cos 3x \cos 5x \, dx = \left[ \frac{1}{5} \cos 3x \sin 5x \right] - \left[ \frac{3}{25} \sin 3x \cos 5x \right]$$

and therefore

$$\int \cos 3x \cos 5x \, dx = \left[ \frac{5}{16} \cos 3x \sin 5x - \frac{3}{16} \sin 3x \cos 5x \right] + C.$$

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### 3.4 Exercises: Integration by Parts

Various exercises relating to integration by parts.

Compute the indefinite integrals below. Since there are many possible answers (which differ by constant values), use the given instructions if needed to choose which possible answer to use.

**Exercise 64**

$$\int x \sin x \, dx = \boxed{\sin x - x \cos x} + C$$

(Add a constant to your answer if needed so that it equals 0 at  $x = 0$ .)

**Exercise 65**

$$\int x e^{-x} \, dx = \boxed{-e^{-x} - x e^{-x}} + C$$

(Add a constant to your answer if needed so that it equals  $-1$  at  $x = 0$ .)

**Exercise 66**

$$\int x^3 e^x \, dx = \boxed{x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x} + C$$

(Write your answer so that it has no constant term.)

**Exercise 67**

$$\int x^2 \ln |x| \, dx = \boxed{\frac{1}{3} x^3 \ln |x| - \frac{x^3}{9}} + C$$

Don't forget absolute values in your logarithm. (Add a constant to your answer as necessary so that it equals  $-1/9$  at  $x = 1$ .)

**Exercise 68**

$$\int e^x \sin x \, dx = \boxed{\frac{1}{2} e^x (\sin x - \cos x)} + C$$

(Add a constant to your answer if needed so that it equals  $-1/2$  at  $x = 0$ .)

**Hint:** This is a case in which you need to treat the integration by parts as an equation and solve for the answer.

**Exercise 69**

$$\int \arcsin x \, dx = \boxed{\sqrt{1-x^2} + x \arcsin(x)} + C$$

(Add a constant to your answer if needed so that it equals 1 at  $x = 0$ .)

**Hint:** Write  $\arcsin x = 1 \cdot \arcsin x$ .

## Sample Quiz Questions

**Question 70** Compute the definite integral

$$\int_1^4 e^{3x}(x+1) \, dx.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\frac{14}{9}e^{12} - \frac{5}{9}e^3$  ✓
- (b)  $\frac{17}{9}e^{12} - \frac{5}{9}e^3$
- (c)  $\frac{17}{9}e^{12} - \frac{8}{9}e^3$
- (d)  $\frac{20}{9}e^{12} - \frac{8}{9}e^3$
- (e)  $\frac{20}{9}e^{12} - \frac{11}{9}e^3$
- (f)  $\frac{23}{9}e^{12} - \frac{11}{9}e^3$

**Feedback(attempt):** Integrate by parts, integrating the exponential and differentiating polynomials.

**Hint:**

$$\begin{aligned}
\int_1^4 e^{3x}(x+1) \, dx &= \left. \frac{e^{3x}}{3}(x+1) \right|_1^4 - \int_1^4 \frac{e^{3x}}{3} dx \\
&= \frac{5}{3}e^{12} - \frac{2}{3}e^3 - \left. \frac{e^{3x}}{9} \right|_1^4 \\
&= \frac{14}{9}e^{12} - \frac{5}{9}e^3
\end{aligned}$$

**Question 71** Compute the definite integral

$$\int_{\pi}^{2\pi} x \sin 4x \, dx.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a) 0
- (b)  $\frac{\pi}{4}$
- (c)  $-\frac{\pi}{4}$  ✓
- (d)  $\frac{3\pi}{4}$
- (e)  $-\frac{3\pi}{4}$
- (f)  $\frac{7\pi}{4}$

**Feedback(attempt):** Integrate by parts, integrating the trig functions and differentiating polynomials.**Hint:**

$$\begin{aligned}
\int_{\pi}^{2\pi} x \sin 4x \, dx &= -\frac{\cos 4x}{4}x \Big|_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \frac{\cos 4x}{4} dx \\
&= -\frac{2\pi}{4} + \frac{(-1)^4\pi}{4} - \left. \frac{\sin 4x}{16} \right|_{\pi}^{2\pi} = -\frac{\pi}{4}
\end{aligned}$$



**Question 72** Compute the indefinite integral

$$\int \arctan 5x \, dx.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $x \arctan 5x - \frac{1}{10} \ln |1 + 25x^2| + C$  ✓
- (b)  $x \arctan 5x - \frac{1}{12} \ln |1 + 25x^2| + C$
- (c)  $x \arctan 5x - \frac{1}{14} \ln |1 + 25x^2| + C$
- (d)  $x \arctan 5x + \frac{1}{10} \ln |1 + 25x^2| + C$
- (e)  $x \arctan 5x + \frac{1}{12} \ln |1 + 25x^2| + C$
- (f)  $x \arctan 5x + \frac{1}{14} \ln |1 + 25x^2| + C$

**Feedback(attempt):** Integrate by parts, integrating the coefficient 1 and differentiating arctangent.

**Hint:**

$$\begin{aligned} \int \arctan 5x \, dx &= x \arctan 5x - \int \frac{5x}{1 + 25x^2} \, dx \\ &= x \arctan 5x - \frac{1}{10} \int \frac{50x}{1 + 25x^2} \, dx \\ &= x \arctan 5x - \frac{1}{10} \ln |1 + 25x^2| + C \end{aligned}$$

**Question 73** Compute the indefinite integral

$$\int e^{3x} \cos 5x \, dx.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\frac{e^{3x}(2 \cos 5x + 5 \sin 5x)}{29} + C$
- (b)  $\frac{e^{3x}(3 \cos 5x + 5 \sin 5x)}{34} + C$  ✓
- (c)  $\frac{e^{3x}(\cos 5x + 3 \sin 5x)}{20} + C$
- (d)  $\frac{e^{3x}(\cos 5x + 2 \sin 5x)}{15} + C$
- (e)  $\frac{e^{3x}(2 \cos 5x + 7 \sin 5x)}{53} + C$
- (f)  $\frac{e^{3x}(3 \cos 5x + 7 \sin 5x)}{58} + C$

**Feedback(attempt):** Integrate by parts, integrating the exponential and differentiating cosine (or vice-versa), then solve for the antiderivative.

**Hint:**

$$\begin{aligned}
 & \int e^{3x} \cos 5x \, dx \\
 &= \frac{e^{3x}}{3} \cos 5x - \int \frac{e^{3x}}{3} (-5 \sin 5x) \, dx \\
 &= \frac{e^{3x} \cos 5x}{3} + \frac{5}{3} \int e^{3x} \sin 5x \, dx \\
 &= \frac{e^{3x} \cos 5x}{3} + \frac{5}{3} \frac{e^{3x}}{3} \sin 5x - \frac{5}{3} \int \frac{e^{3x}}{3} (5 \cos 5x) \, dx \\
 &= \frac{e^{3x}(3 \cos 5x + 5 \sin 5x)}{9} - \frac{25}{9} \int e^{3x} \cos 5x \, dx \\
 &\Rightarrow \frac{34}{9} \int e^{3x} \cos 5x \, dx = \frac{e^{3x}(3 \cos 5x + 5 \sin 5x)}{9} \\
 &\Rightarrow \int e^{3x} \cos 5x \, dx = \frac{e^{3x}(3 \cos 5x + 5 \sin 5x)}{34} + C
 \end{aligned}$$

## Sample Exam Questions

**Question 74** Compute the integral below.

$$\int_{\frac{1}{2}}^{\infty} \frac{\ln(2x)}{x^2} dx$$

**Multiple Choice:**

- (a)  $1 - \ln 2$
- (b)  $2$  ✓
- (c)  $\ln 2 - \frac{1}{2}$
- (d)  $\frac{1}{2}$
- (e)  $2 - 2 \ln 2$
- (f) *the integral diverges*

**Question 75** Compute the indefinite integral indicated below. [Hint: Write  $\frac{1}{\cos^2 \theta} = \sec^2 \theta$  and integrate by parts.]

$$\int \left( 1 + \frac{\ln |\sin \theta|}{\cos^2 \theta} \right) d\theta$$

**Multiple Choice:**

- (a)  $(\sin \theta) \ln |\sin \theta| + C$
- (b)  $(\cos \theta) \ln |\sin \theta| + C$
- (c)  $(\tan \theta) \ln |\sin \theta| + C$  ✓
- (d)  $(\csc \theta) \ln |\sin \theta| + C$
- (e)  $(\sec \theta) \ln |\sin \theta| + C$
- (f)  $(\cot \theta) \ln |\sin \theta| + C$

## 3.5 Trigonometric Integrals

*We learn various techniques for integrating certain combinations of trigonometric functions.*

### (Video) Calculus: Single Variable

YouTube link: [https://www.youtube.com/watch?v=Ltm\\_XPDXea8](https://www.youtube.com/watch?v=Ltm_XPDXea8)

## Online Texts

- OpenStax II 3.2: Trigonometric Integrals<sup>31</sup>
- Ximera OSU: Trigonometric Integrals<sup>32</sup>
- Community Calculus 8.2: Powers of Sine and Cosine<sup>33</sup>

## Examples

**Example 21.** Compute the indefinite integrals

$$\int \cos^5 3x \sin^2 3x \, dx \quad \text{and} \quad \int \cos^2 3x \sin^5 3x \, dx.$$

- When dealing with products of sines and cosines of the same quantity (in this case, the  $3x$  is the same inside both the sine and the cosine), we can use a substitution. We look for one with an odd power and build a substitution using the other. In this case, we would use the substitution ( $u = \cos 3x$  /  $u = \sin 3x$  ✓) in the first integral and ( $u = \cos 3x$  ✓ /  $u = \sin 3x$ ) in the second.
- Using the substitutions just identified, we have  $du = \boxed{3 \cos 3x} \, dx$  in the former case and  $du = \boxed{-3 \sin 3x} \, dx$  in the latter. This means the integrals become

$$\frac{1}{3} \int \cos^4 3x \sin^2 3x \, du \quad \text{and} \quad -\frac{1}{3} \int \cos^2 3x \sin^4 3x \, du.$$

- We continue to simplify, using the trig identity  $\cos^2 \theta + \sin^2 \theta = 1$  to completely eliminate all references to the variable  $x$  in the integrand:

$$\frac{1}{3} \int (\boxed{1 - u^2})^2 u^2 \, du \quad \text{and} \quad -\frac{1}{3} \int u^2 (\boxed{1 - u^2})^2 \, du.$$

- Now we calculate the integral, giving

$$\frac{1}{3} \int (\boxed{1 - u^2})^2 u^2 \, du = \boxed{\frac{u^3}{9} - \frac{2u^5}{15} + \frac{u^7}{21}} + C$$

<sup>31</sup>See OpenStax II 3.2: Trigonometric Integrals at <https://openstax.org/books/calculus-volume-2/pages/3-2-trigonometric-integrals>

<sup>32</sup>See Ximera OSU: Trigonometric Integrals at <https://ximera.osu.edu/mooculus/calculus2/trigonometricIntegrals/titlePage>

<sup>33</sup>See Community Calculus 8.2: Powers of Sine and Cosine at [https://www.whitman.edu/mathematics/calculus\\_online/section08.02.html](https://www.whitman.edu/mathematics/calculus_online/section08.02.html)

in the first case and

$$-\frac{1}{3} \int u^2 (1 - u^2)^2 du = -\frac{u^3}{9} + \frac{2u^5}{15} - \frac{u^7}{21} + C$$

in the second.

- To conclude, we reverse the substitution, so that

$$\int \cos^5 3x \sin^2 3x dx = \frac{(\sin 3x)^3}{9} - \frac{2(\sin 3x)^5}{15} + \frac{(\sin 3x)^7}{21} + C$$

and

$$\int \cos^2 3x \sin^5 3x dx = -\frac{(\cos 3x)^3}{9} + \frac{2(\cos 3x)^5}{15} - \frac{(\cos 3x)^7}{21} + C.$$

**Example 22.** Compute the indefinite integral

$$\int \sin^2 3x \cos^2 3x dx.$$

- When both powers are even, your only option is to use a trigonometric identity to reduce the power. In this case, the identities are

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad \text{and} \quad \cos^2 \theta = \frac{1 + \cos 2\theta}{2}.$$

- This means that

$$\sin^2 3x \cos^2 3x = \frac{1}{4} - \frac{(\cos 6x)^2}{4}.$$

- Because we still have only even powers, we should use power reduction again:

$$\sin^2 3x \cos^2 3x = \frac{1}{4} - \frac{1}{4} \left[ \frac{1}{2} + \frac{1}{2} \cos 12x \right] = \frac{1}{8} - \frac{1}{8} \cos 12x.$$

- Integrating this last expression gives

$$\int \sin^2 3x \cos^2 3x dx = \frac{x}{8} - \frac{1}{96} \sin 12x + C.$$

**Example 23.** Compute the indefinite integrals

$$\int \sec^2 3x \tan^2 3x dx \quad \text{and} \quad \int \sec^3 3x \tan^3 3x dx.$$

- In this case we look for either an even power of secant, which indicates a tangent substitution, or an odd number of both secant and tangent, which indicates a secant substitution. So we substitute  $u = \tan 3x$  in the first integral and  $u = \sec 3x$  in the second. To simplify and write the integrands in terms of  $u$  only, use the identity  $\sec^2 \theta = \tan^2 \theta + 1$ . This gives

$$\int \frac{1}{3} u^2 du \quad \text{and} \quad \int \frac{1}{3} u^2 (u^2 - 1) du$$

for the two integrals (don't forget the extra factor of  $1/3$  coming from the chain rule).

- Integrating and reversing the substitution gives

$$\int \sec^2 3x \tan^2 3x dx = \frac{(\tan 3x)^3}{9} + C$$

and

$$\int \sec^3 3x \tan^3 3x dx = \frac{(\sec 3x)^5}{15} - \frac{(\sec 3x)^3}{9} + C.$$

**Example 24.** When there are odd powers of secant and even (or just zero) powers of tangent, there's a trick: integrate by parts letting  $dv = \sec^2 x$  and  $u$  being everything else.

$$\begin{aligned} \int \sec^7 x dx &= \int \sec^5 x \sec^2 x dx \\ &= \sec^5 x \tan x - \int 5(\sec x)^4 \sec x (\tan x)^2 dx \\ &= \sec^5 x \tan x - 5 \int \sec^7 x dx + 5 \int \sec^5 x dx. \end{aligned}$$

Just like for other integration-by-parts examples, we can now solve for the integral in terms of a simpler integral:

$$\int \sec^7 x dx = \frac{1}{6} \left[ (\sec x)^5 \tan x + 5 \int \sec^5 x dx \right].$$

This is an example of a reduction formula, because the integral on the right-hand side is similar to the one we started with, but simpler. We could then compute the integral of  $\sec^5 x$  in terms of the integral of  $\sec^3 x$  and ultimately to  $\sec x$ , which is one which we can evaluate (or look up in a table).

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**3.6 Exercises: Trigonometric Integral***Various exercises relating to the integration of trigonometric functions.*

Compute the indefinite integrals below. Since there are many possible answers (which differ by constant values), use the given instructions if needed to choose which possible answer to use.

**Exercise 76**

$$\int \sin x \cos^4 x \, dx = \boxed{-\frac{1}{5} \cos^5(x)} + C$$

(Your answer should not include any constant term.)

**Exercise 77**

$$\int \sin^3 x \cos^3 x \, dx = \boxed{\frac{1}{6} \cos^6 x - \frac{1}{4} \cos^4 x} + C$$

(Your answer should not include any constant terms.)

**Exercise 78**

$$\int \tan^4 x \sec^2 x \, dx = \boxed{\frac{\tan^5(x)}{5}} + C$$

(Add a constant to your answer if needed so that it equals 0 at  $x = 0$ .)

**Exercise 79**

$$\int \tan^3 x \sec^3 x \, dx = \boxed{\frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3}} + C$$

(Add a constant to your answer if needed so that it equals  $-2/15$  at  $x = 0$ .)

**Exercise 80**

$$\int \sin^2 x \cos^7 x \, dx = \boxed{-\frac{1}{9} \sin^9(x) + \frac{3 \sin^7(x)}{7} - \frac{3 \sin^5(x)}{5} + \frac{\sin^3(x)}{3}} + C$$

(Your answer should not include any constant terms.)

**Hint:**

$$(1 - u^2)^3 = 1 - 3u^2 + 3u^4 - u^6.$$

**Exercise 81**

$$\int \sin(5x) \cos(3x) \, dx = \boxed{\frac{1}{2} \left( -\frac{1}{8} \cos(8x) - \frac{1}{2} \cos(2x) \right)} + C$$

**Hint:** You can either use an integration by parts technique or you can use a trigonometric identity to simplify the expression  $\cos Ax \sin Bx$  (for constants  $A$  and  $B$ ) as a sum of simpler things.

(Your answer should not include any constant terms.)

**Exercise 82**

$$\int \sin^2 x \cos^2 x \, dx = \boxed{\frac{x}{8} - \frac{1}{32} \sin(4x)} + C$$

(Your answer should not include any constant terms and should equal 0 at  $x = 0$ .)

**Hint:** Use power reduction formulas.

**Exercise 83** To fully evaluate the integral from Example 24, it helps to identify the pattern. Suppose that the power 7 is replaced by an unknown positive constant  $n > 2$ . Carry out the calculation again with the unspecified exponent:

$$\begin{aligned} \int \sec^n x \, dx &= \int \sec^{\boxed{n-2}} x \sec^2 x \, dx \\ &= \sec^{\boxed{n-2}} x \tan x - \int \boxed{(n-2)} (\sec x)^{\boxed{n-3}} (\sec x \tan x) \tan x \, dx \\ &= \sec^{\boxed{n-2}} x \tan x - \boxed{(n-2)} \int \sec^n x \, dx + \boxed{n-2} \int \sec^{\boxed{n-2}} x \, dx. \end{aligned}$$

We conclude

$$\int \sec^n x \, dx = \boxed{\frac{1}{n-1}} \sec^{n-2} x \tan x + \boxed{\frac{n-2}{n-1}} \int \sec^{n-2} x \, dx.$$



Using this formula several times in a row gives the result

$$\begin{aligned}
 & \int \sec^7 x \, dx \\
 &= \frac{\sec^5 x \tan x}{6} + \frac{5}{6} \int \sec^5 x \, dx \\
 &= \frac{\sec^5 x \tan x}{6} + \frac{5}{6} \left[ \frac{\sec^3 x \tan x}{4} + \frac{3}{4} \int \sec^3 x \, dx \right] \\
 &= \frac{\sec^5 x \tan x}{6} + \frac{5 \sec^3 x \tan x}{4 \cdot 6} + \frac{5}{6} \cdot \frac{3}{4} \left[ \frac{\sec x \tan x}{2} + \frac{1}{2} \int \sec x \, dx \right] \\
 &= \frac{\sec^5 x \tan x}{6} + \frac{5 \sec^3 x \tan x}{24} + \frac{5 \sec x \tan x}{16} + \frac{5}{16} \ln |\sec x + \tan x| + C.
 \end{aligned}$$

## Sample Quiz Questions

**Question 84** Compute the value of the integral

$$\int_0^{\frac{\pi}{4}} \sin^3 2x \, dx.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\frac{1}{5}$
- (b)  $\frac{1}{3}$  ✓
- (c)  $\frac{1}{2}$
- (d) 1
- (e) 2
- (f) 3

**Feedback(attempt):** To simplify the calculation, begin with a substitution which replaces  $x$  with  $x/2$ . The question reduces to computing

$$\frac{1}{2} \int_0^{\frac{\pi}{2}} \sin^3 x \, dx.$$

This integral is compatible with the substitution  $u = \cos x$ .

Exercises: Trigonometric Integral

**Hint:** By the substitution formula, this means  $dx = -du/\sin x$ , and one must also replace  $\sin^2 x$  by  $1 - u^2$ . Furthermore, by virtue of the special angle formulas  $\cos 0 = 1$  and  $\cos \frac{\pi}{2} = 0$ , the problem is reduced to computing the integral

$$-\frac{1}{2} \int_1^0 (1 - u^2) du.$$

**Hint:** Carrying out this calculation in the usual way gives a final answer of  $\frac{1}{3}$ .

**Question 85** Compute the value of the integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \tan^{-6} x \sec^5 x \, dx.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\frac{17}{5}$
- (b)  $\frac{19}{5}$
- (c)  $\frac{23}{5}$
- (d)  $\frac{29}{5}$
- (e)  $\frac{31}{5}$  ✓
- (f)  $\frac{37}{5}$

**Feedback(attempt):** Since the power of secant is odd and the power of tangent is even, try rewriting the integral in terms of sine and cosine. This gives

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^{-6} x \cos x \, dx.$$

This integral is compatible with the substitution  $u = \sin x$ .

**Hint:** By the substitution formula, this means  $dx = du/\cos x$ . Furthermore, by virtue of the special angle formulas  $\sin \frac{\pi}{6} = \frac{1}{2}$  and  $\sin \frac{\pi}{2} = 1$ , the problem is reduced to computing the integral

$$\int_{\frac{1}{2}}^1 u^{-6} du.$$

**Hint:** Carrying out this calculation in the usual way gives a final answer of  $\frac{31}{5}$ .

**Question 86** Compute the value of the integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^{-2} x \cos^3 x \, dx.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\frac{1}{5}$
- (b)  $\frac{1}{3}$
- (c)  $\frac{1}{2}$  ✓
- (d) 1
- (e) 2
- (f) 3

**Feedback(attempt):** This integral is compatible with the substitution  $u = \sin x$ . By the substitution formula, this means  $dx = du/\cos x$ , and one must also replace  $\cos^2 x$  by  $1 - u^2$ . Furthermore, by virtue of the special angle formulas  $\sin \frac{\pi}{6} = \frac{1}{2}$  and  $\sin \frac{\pi}{2} = 1$ , the problem is reduced to computing the integral

**Hint:**

$$\int_{\frac{1}{2}}^1 u^{-2}(1 - u^2) \, du.$$

**Hint:** Carrying out this calculation in the usual way gives a final answer of  $\frac{1}{2}$ .

## Sample Exam Questions

**Question 87** Compute the integral below.

$$\int_0^{\frac{\pi}{8}} \tan^4 2x \sec^4 2x \, dx$$

**Multiple Choice:**

- (a)  $\frac{4}{9}$
- (b)  $\frac{7}{24}$
- (c)  $\frac{5}{14}$
- (d)  $\frac{9}{28}$
- (e)  $\frac{6}{35}$  ✓
- (f)  $\frac{1}{7}$

---

## 3.7 Trigonometric Substitutions

*We practice executing trigonometric substitutions.*

### (Video) Calculus: Single Variable

YouTube link: [https://www.youtube.com/watch?v=D\\_2N10LQjyk](https://www.youtube.com/watch?v=D_2N10LQjyk)

### Online Texts

- OpenStax II 3.3: Trigonometric Substitution<sup>34</sup>
- Ximera OSU: Trigonometric Substitution<sup>35</sup>
- Community Calculus 8.3: Trigonometric Substitution<sup>36</sup>

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<sup>34</sup>See OpenStax II 3.3: Trigonometric Substitution at <https://openstax.org/books/calculus-volume-2/pages/3-3-trigonometric-substitution>

<sup>35</sup>See Ximera OSU: Trigonometric Substitution at <https://ximera.osu.edu/mooculus/calculus2/trigonometricSubstitution/titlePage>

<sup>36</sup>See Community Calculus 8.3: Trigonometric Substitution at [https://www.whitman.edu/mathematics/calculus\\_online/section08.03.html](https://www.whitman.edu/mathematics/calculus_online/section08.03.html)

## Examples

**Example 25.** (c.f. APEX Calculus Example 6.4.2) Compute the indefinite integral

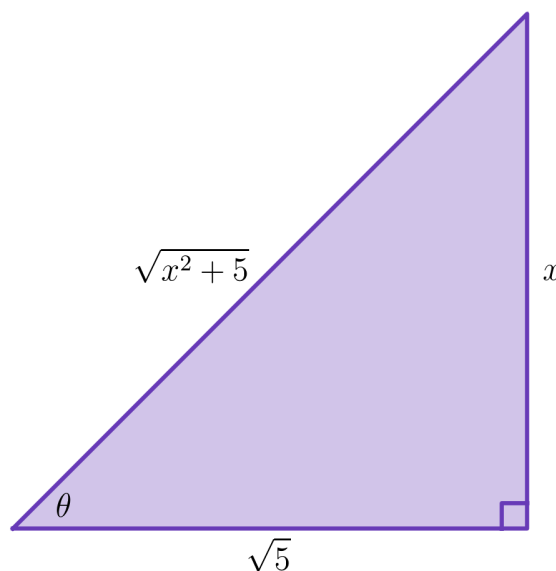
$$\int \frac{dx}{\sqrt{5+x^2}}.$$

- The structure of this integrand is adapted to a trigonometric substitution of (secant / sine / tangent  $\checkmark$ ) type, so we let  $x = \sqrt{5} \tan \theta$  (type out the word theta). This gives  $dx = \sqrt{5} \sec^2 \theta d\theta$ .
- We use the identity  $\sec^2 \theta = \tan^2 \theta + 1$ , the quantity  $5+x^2$  can be rewritten using our substitution to equal  $5 \sec^2 \theta$ . Therefore

$$\int \frac{dx}{\sqrt{5+x^2}} = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C.$$

(When writing out the indefinite integral, don't forget absolute values.)

- We construct a reference triangle compatible with our substitution. In this case, that means we need a right triangle with an angle  $\theta$  which satisfies  $x/\sqrt{5} = \tan \theta$ :



- Using this reference triangle, we get that

$$\tan \theta = \frac{x}{\sqrt{5}} \quad \sec \theta = \frac{\sqrt{x^2 + 5}}{\sqrt{5}}.$$

Using these formulas, we can write

$$\ln |\sec \theta + \tan \theta| = \ln \left| \frac{x}{\sqrt{5}} + \frac{\sqrt{x^2 + 5}}{\sqrt{5}} \right|.$$

- We conclude that

$$\int \frac{dx}{\sqrt{x^2 + 5}} = \ln |x + \sqrt{x^2 + 5}| + C.$$

(Note that the factor of  $\sqrt{5}$  in the denominator of the logarithm is not necessary when we already have an arbitrary constant  $C$  in our answer.)

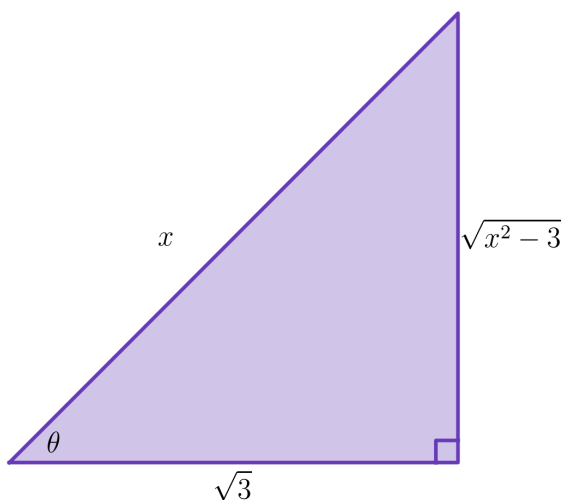
**Example 26.** Compute the indefinite integral

$$\int \frac{dx}{x^2 \sqrt{x^2 - 3}}.$$

- The structure of this integrand is adapted to a trigonometric substitution of (secant  $\checkmark$ / sine/ tangent  $\checkmark$ ) type, so we let  $x = \sqrt{3} \boxed{\tan \theta}$ . This gives  $dx = \sqrt{3} \boxed{\sec \theta \tan \theta} d\theta$ .
- We use the identity  $\sec^2 \theta = \tan^2 \theta + 1$ , the quantity  $x^2 - 3$  can be rewritten using our substitution to equal  $\boxed{3 \tan^2 \theta}$ . Therefore

$$\int \frac{dx}{x^2 \sqrt{x^2 - 3}} = \int \frac{\sqrt{3} \sec \theta \tan \theta}{3 \sec^2 \theta \sqrt{3} \tan \theta} d\theta = \frac{\sin \theta}{3} + C.$$

- We construct a reference triangle compatible with our substitution. In this case, that means we need a right triangle with an angle  $\theta$  which satisfies  $x/\sqrt{3} = \sec \theta$ :



- Using this reference triangle, we get that

$$\sin \theta = \frac{\sqrt{x^2 - 3}}{x}.$$

- We conclude that

$$\int \frac{dx}{x^2 \sqrt{x^2 - 3}} = \frac{\sqrt{x^2 - 3}}{3x} + C.$$

**Example 27.** (c.f. APEX Calculus Example 6.4.1) Compute the indefinite integral

$$\int \sqrt{9 - x^2} dx.$$

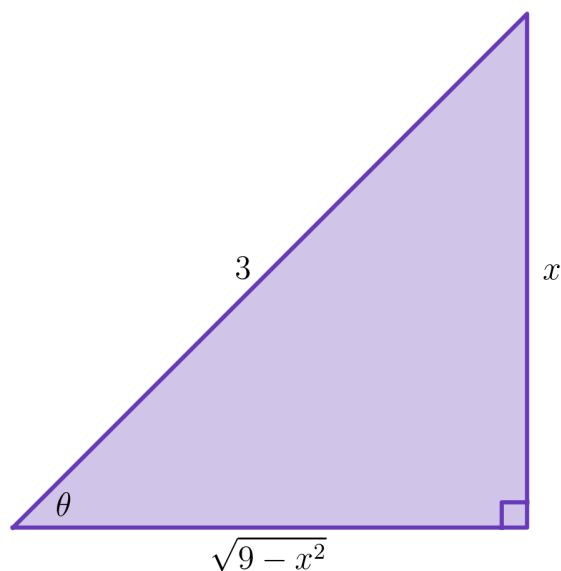
- The structure of this integrand is adapted to a trigonometric substitution of (secant / sine ✓ / tangent) type, so we let  $x = 3 \sin \theta$ . This gives  $dx = 3 \cos \theta \, d\theta$ .
- Using the identity  $\cos^2 \theta + \sin^2 \theta = 1$ , the expression  $9 - x^2$  can be rewritten using the formula above as  $(3 \cos \theta)^2$ . Therefore

$$\int \sqrt{9 - x^2} dx = \int 9(\cos \theta)^2 d\theta.$$

- We use the power reduction formula  $\cos^2 t = (1 + \cos 2t)/2$  and conclude

$$\int \sqrt{9 - x^2} dx = \frac{9}{2} \int (1 + \cos 2\theta) d\theta = \frac{9}{2} \left[ \theta + \frac{1}{2} \sin 2\theta \right] + C.$$

- Now we construct a reference triangle which is compatible with the substitution we made. In this case, this means we need a right triangle for which  $x/3 = \sin \theta$ :



- Using the reference triangle above, we have the identities

$$\sin \theta = \frac{x}{3} \quad \cos \theta = \frac{\sqrt{9 - x^2}}{3} \quad \theta = \arcsin \frac{x}{3}.$$

- The quantity  $\sin 2\theta$  is difficult to write as a function of  $x$  as it stands, so we first use the double-angle formula  $\sin 2\theta = 2 \cos \theta \sin \theta$ . We therefore conclude that

$$\int \sqrt{9 - x^2} dx = \frac{9}{2} \left( \arcsin \frac{x}{3} + \frac{x\sqrt{9 - x^2}}{9} \right) + C.$$

(Note: Ximera will interpret  $\sin^{-1} x$  as  $1/(\sin x)$ , so write  $\arcsin$  when you mean inverse sine.)



techniques/11trigsubpractice.tex

### 3.8 Exercises: Trigonometric Substitutions

Various exercises relating to trigonometric substitutions.

Compute the indefinite integrals below. Since there are many possible answers (which differ by constant values), use the given instructions if needed to choose which possible answer to use. Do not forget absolute value signs inside logarithms when they are needed.

#### Exercise 88

$$\int x^2 \sqrt{1-x^2} \, dx = \boxed{\frac{1}{8} \arcsin x - \frac{1}{8} x \sqrt{1-x^2} (1-2x^2)} + C$$

(Choose your answer to equal 0 at  $x = 0$ .)

#### Exercise 89

$$\int \frac{1}{(x^2+1)^2} \, dx = \boxed{\frac{1}{2} \left( \arctan x + \frac{x}{x^2+1} \right)} + C$$

(Choose your answer to equal 0 at  $x = 0$ .)

#### Exercise 90

$$\int \frac{x^2}{\sqrt{x^2+3}} \, dx = \boxed{\frac{1}{2} x \sqrt{x^2+3} - \frac{3}{2} \ln \left| \frac{\sqrt{x^2+3}}{\sqrt{3}} + \frac{x}{\sqrt{3}} \right|} + C$$

(Choose your answer to equal 0 at  $x = 0$ .)

### Sample Quiz Questions

**Question 91** Compute the integral

$$\int_{-2}^2 \frac{5}{(5-x^2)^{3/2}} \, dx.$$

(Hints won't be revealed until after you choose a response.)

Exercises: Trigonometric Substitutions

**Multiple Choice:**

- (a) 2
- (b) 3
- (c) 4 ✓
- (d) 5
- (e) 6
- (f) 7

**Feedback(attempt):** Begin by making the trig substitution  $x = \sqrt{5} \sin \theta$ .

**Hint:** It follows that

$$\begin{aligned}\int \frac{5}{(5-x^2)^{3/2}} dx &= \int \frac{5}{(5-(\sqrt{5} \sin \theta)^2)^{3/2}} \cdot (\sqrt{5} \cos \theta) d\theta \\ &= \int (\cos \theta)^{-2} d\theta \\ &= \int (\sec \theta)^2 d\theta = (\tan \theta) + C.\end{aligned}$$

**Hint:** To finish, use the inversion identity

$$\tan \theta = \frac{x}{\sqrt{5-x^2}}.$$

Therefore

$$\int_{-2}^2 \frac{5}{(5-x^2)^{3/2}} dx = \left. \frac{x}{\sqrt{5-x^2}} \right|_{-2}^2 = (2) - (-2) = 4.$$

**Question 92** Compute the integral

$$\int_{-1}^1 \frac{3}{(3+x^2)^{3/2}} dx.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\frac{1}{2}$
- (b) 1 ✓
- (c)  $\frac{3}{2}$

Exercises: Trigonometric Substitutions

(d) 2

(e)  $\frac{5}{2}$

(f) 3

**Feedback(attempt):** Begin by making the trig substitution  $x = \sqrt{3} \tan \theta$ .

**Hint:** It follows that

$$\begin{aligned} \int \frac{3}{(3+x^2)^{3/2}} dx &= \int \frac{3}{(3+(\sqrt{3} \tan \theta)^2)^{3/2}} \cdot (\sqrt{3} \sec^2 \theta) d\theta \\ &= \int (\sec \theta)^{-1} d\theta \\ &= \int (\cos \theta) d\theta = (\sin \theta) + C. \end{aligned}$$

**Hint:** To finish, use the inversion identity

$$\cos \theta = \frac{x}{\sqrt{3+x^2}}.$$

Therefore

$$\int_{-1}^1 \frac{3}{(3+x^2)^{3/2}} dx = \left. \frac{x}{\sqrt{3+x^2}} \right|_{-1}^1 = \left( \frac{1}{2} \right) - \left( -\frac{1}{2} \right) = 1.$$

**Question 93** Compute the integral

$$\int_4^5 \frac{16\sqrt{x^2-16}}{x^4} dx.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

(a)  $\frac{4}{125}$

(b)  $\frac{1}{25}$

(c)  $\frac{6}{125}$

(d)  $\frac{7}{125}$

(e)  $\frac{8}{125}$

(f)  $\frac{9}{125}$  ✓

**Feedback(attempt):** Begin by making the trig substitution  $x = 4 \sec \theta$ .

**Hint:** It follows that

$$\begin{aligned} \int \frac{16\sqrt{x^2 - 16}}{x^4} dx &= \int \frac{16\sqrt{(4 \sec \theta)^2 - 16}}{(4 \sec \theta)^4} \cdot (4 \sec \theta \tan \theta) d\theta \\ &= \int (\sec \theta)^{-3} (\tan \theta)^2 d\theta \\ &= \int (\sin \theta)^2 (\cos \theta) d\theta = \frac{1}{3} (\sin \theta)^3 + C. \end{aligned}$$

**Hint:** To finish, use the inversion identity

$$\sin \theta = \frac{\sqrt{x^2 - 16}}{x}.$$

Therefore

$$\int_4^5 \frac{16\sqrt{x^2 - 16}}{x^4} dx = \frac{1}{3} \frac{(x^2 - 16)^{3/2}}{x^3} \Big|_4^5 = \left( \frac{9}{125} \right) - (0) = \frac{9}{125}.$$

## Sample Exam Questions

**Question 94** Compute the value of the integral below.

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{(1 - x^2)^{\frac{3}{2}}} dx$$

**Multiple Choice:**

- (a) 0
- (b) 1 ✓
- (c) 2
- (d) 3
- (e) 4
- (f) none of these

**Question 95** Evaluate  $\int_0^3 \frac{dx}{(25-x^2)^{3/2}}$ .

**Multiple Choice:**

- (a) 0
- (b)  $\frac{1}{100}$
- (c)  $\frac{3}{100}$  ✓
- (d)  $\frac{5}{100}$
- (e)  $\frac{7}{100}$
- (f) none of these

## 3.9 Partial Fractions

*We study the technique of partial fractions and its application to integration.*

### (Video) Calculus: Single Variable

**Note:** The latter part of the Buckling Beam segment (9:35–12:55) dealing with linearization is something we are not yet ready for and may be safely skipped for now. YouTube link: <https://www.youtube.com/watch?v=QCxFfu331Sg>

### Online Texts

- OpenStax II 3.4: Partial Fractions<sup>37</sup>
- Ximera OSU: Partial Fractions<sup>38</sup>
- Community Calculus 8.5: Rational Functions<sup>39</sup>

<sup>37</sup>See OpenStax II 3.4: Partial Fractions at <https://openstax.org/books/calculus-volume-2/pages/3-4-partial-fractions>

<sup>38</sup>See Ximera OSU: Partial Fractions at <https://ximera.osu.edu/mooculus/calculus2/partialFractions/titlePage>

<sup>39</sup>See Community Calculus 8.5: Rational Functions at [https://www.whitman.edu/mathematics/calculus\\_online/section08.05.html](https://www.whitman.edu/mathematics/calculus_online/section08.05.html)

## Examples

**Example 28.** Calculate the value of the integral

$$\int_{-3}^0 \frac{2x^2 - 2x + 9}{(x-2)(x^2+9)} dx.$$

- The general form of the expansion will be

$$\frac{2x^2 - 2x + 9}{(x-2)(x^2+9)} = \frac{A}{\boxed{x-2}} + \frac{Bx+C}{\boxed{x^2+9}}$$

- Let's cover a few distinct ways of computing the coefficients  $A$ ,  $B$ , and  $C$ . One way to do it is to clear denominators by multiplying by  $(x-2)(x^2+9)$  on both sides. This gives

$$\begin{aligned} 2x^2 - 2x + 9 &= A(\boxed{x^2+9}) + Bx(\boxed{x-2}) + C(\boxed{x-2}) \\ &= (\boxed{A+B})x^2 + (\boxed{-2B+C})x + (\boxed{9A-2C}). \end{aligned}$$

Now we equate the coefficients of like powers of  $x$ :

$$\begin{aligned} 2 &= \boxed{A+B} \quad (\text{coefficients of } x^2), \\ -2 &= \boxed{-2B+C} \quad (\text{coefficients of } x), \\ 9 &= \boxed{9A-2C} \quad (\text{constant terms}). \end{aligned}$$

We solve these equations for  $A$ ,  $B$ , and  $C$  by eliminating variables: taking twice the top equation and adding the second equation gives  $2 = 2A + 2C$ . Adding this to the last equation gives  $A = \boxed{1}$ . Plugging this back in to the equation  $2 = 2A + 2C$  gives  $C = \boxed{0}$ . Plugging in our value of  $A$  into either the first or last equations above gives  $B = \boxed{1}$ . Thus we know that

$$\frac{2x^2 - 2x + 9}{(x-2)(x^2+9)} = \frac{\boxed{1}}{x-2} + \frac{\boxed{x}}{x^2+9}.$$

- Here's an entirely different approach: Go back to the formula

$$\frac{2x^2 - 2x + 9}{(x-2)(x^2+9)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+9}.$$

Multiply both sides by  $(x-2)$  and simplify as much as possible to get

$$\frac{2x^2 - 2x + 9}{x^2+9} = A + \frac{(Bx+C)(x-2)}{x^2+9}.$$

Now evaluate both sides at  $x = 2$ : We get

$$\boxed{1} = A + \boxed{0}.$$

This is closely related to what is called the “Heaviside cover-up method.” Knowing that  $A = \boxed{1}$ , we write

$$\frac{2x^2 - 2x + 9}{(x - 2)(x^2 + 9)} = \frac{1}{x - 2} + \frac{Bx + C}{x^2 + 9}.$$

Evaluating both sides at  $x = 0$  gives

$$\boxed{-\frac{1}{2}} = \boxed{-\frac{1}{2}} + \frac{C}{9},$$

which gives  $C = \boxed{0}$ . Plugging this back in gives

$$\frac{2x^2 - 2x + 9}{(x - 2)(x^2 + 9)} = \frac{1}{x - 2} + \frac{Bx}{x^2 + 9}.$$

To finish off, we can plug in  $x = 1$  to conclude

$$\boxed{-\frac{9}{10}} = \boxed{-1} + \frac{B}{10}$$

to get  $B = \boxed{1}$ .

- Either way, we get

$$\frac{2x^2 - 2x + 9}{(x - 2)(x^2 + 9)} = \frac{\boxed{1}}{x - 2} + \frac{\boxed{x}}{x^2 + 9}.$$

The usual antiderivative of the right-hand side is

$$\boxed{\ln|x - 2| + \frac{1}{2} \ln|x^2 + 9|}.$$

(Don’t forget absolute values inside logarithms.) We conclude

$$\int_{-3}^0 \frac{2x^2 - 2x + 9}{(x - 2)(x^2 + 9)} dx = \boxed{(\ln 2)/2 - \ln 5}.$$

In general, expanding and equating coefficients is more **reliable but tedious**. Conversely, doing other techniques like Heaviside cover-up or evaluating at special values of  $x$  can be **more efficient but only if done wisely**.

**Example 29.** Compute the partial fractions expansion of

$$\frac{x^3 + x^2 - 1}{x^2 - 1}.$$

- Because the degree of the numerator is at least as big as the degree of the denominator, we must first prepare by doing polynomial long division. It works essentially exactly the same way as usual long division, except that we align columns by powers of  $x$  rather than by digits of a number. Take care to include coefficients of 0 when terms are absent so that different powers of  $x$  always align with different columns (note: we have left off the  $x^2 + 0x - 1$  that one would usually write to the left of the second row below and we have extended the lines to be longer than usual; both of these unusual choices are simply because of “technical difficulties” in rendering the table correctly):

$$\begin{array}{r}
 \boxed{1}x \quad + \boxed{1} \\
 \hline
 1x^3 \quad + 1x^2 \quad + 0x \quad - 1 \\
 \boxed{1}x^3 \quad + \boxed{0}x^2 \quad + \boxed{-1}x \\
 \hline
 \quad \boxed{1}x^2 \quad + \boxed{1}x \quad + \boxed{-1} \\
 \quad \boxed{1}x^2 \quad + \boxed{0}x \quad + \boxed{-1} \\
 \hline
 \quad \quad \boxed{1}x \quad + \boxed{0}
 \end{array}$$

The last line is the remainder and we write it out in much the same way we do with numerical ratios:

$$\frac{x^3 + x^2 - 1}{x^2 - 1} = \boxed{x + 1} + \frac{\boxed{x}}{x^2 - 1}$$

(the “quotient” goes in the first box and the remainder goes in the second).

- Now we perform a partial fractions expansion on just the fractional part. It will have the form

$$\frac{A}{x - 1} + \frac{B}{x + 1}.$$

- Because there are no repeated factors, we can use the cover-up method to find both coefficients. To find  $A$ , factor the denominator of the fractional part and drop the factor of  $(x - 1)$  from the denominator. This gives the expression

$$\boxed{\frac{x}{x + 1}}.$$

Now evaluate the expression at  $x = 1$  to get the value of  $A$ . In this case  $A = \text{answer } 1/2$ .

- Likewise, to get  $B$ , drop the  $(x+1)$  from the denominator instead of  $(x-1)$ . Evaluating the resulting expression at  $x = -1$  gives  $B = \boxed{-1/2}$ .
- The full expansion is now known:

$$\frac{x^3 + x^2 - 1}{x^2 - 1} = \boxed{x + 1} + \frac{\boxed{1/2}}{x - 1} + \frac{\boxed{-1/2}}{x + 1}.$$



techniques/12partialfractionspractice.tex

**3.10 Exercises: Partial Fractions***Various exercises relating to partial fractions and integration.*

Compute the indefinite integrals below. Since there are many possible answers (which differ by constant values), use the given instructions if needed to choose which possible answer to use. Do not forget absolute value signs inside logarithms when they are needed.

**Exercise 96**

$$\frac{7x+7}{x^2+3x-10} = \frac{\boxed{3}}{x-2} + \frac{\boxed{4}}{x+5}$$

$$\int \frac{7x+7}{x^2+3x-10} dx = \boxed{3 \ln|x-2| + 4 \ln|x+5|} + C$$

*(Do not include any constant terms in your antiderivative.)***Exercise 97**

$$\int \frac{7x-2}{x^2+x} dx = \boxed{9 \ln|x+1| - 2 \ln|x|} + C$$

*(Do not include any constant terms in your antiderivative.)***Exercise 98**

$$\frac{x+7}{(x+5)^2} = \frac{\boxed{1}}{x+5} + \frac{\boxed{2}}{(x+5)^2}$$

$$\int \frac{x+7}{(x+5)^2} dx = \boxed{\ln|x+5| - \frac{2}{x+5}} + C$$

*(Do not include any constant terms in your antiderivative.)***Exercise 99**

$$\int \frac{9x^2+11x+7}{x(x+1)^2} dx = \boxed{\frac{5}{x+1} + 7 \ln|x| + 2 \ln|x+1|} + C$$

*(Do not include any constant terms in your answer.)*

**Exercise 100**

$$\int \frac{x^2 + x + 1}{x^2 + x - 2} dx = \boxed{x + \ln|x - 1| - \ln|x + 2|} + C$$

(Do not include any constant terms in your answer.)

**Hint:** Don't forget polynomial long division; it is needed in this case because the degree of the numerator is at least as large as the degree of the denominator.

**Exercise 101**

$$\int \frac{x^2 + x + 5}{x^2 + 4x + 10} dx = \boxed{-\frac{3}{2} \ln|x^2 + 4x + 10| + x + \frac{\arctan\left(\frac{x+2}{\sqrt{6}}\right)}{\sqrt{6}}} + C$$

(Do not include any constant terms in your answer.)

**Hint:**

$$\frac{x^2 + x + 5}{x^2 + 4x + 10} = \boxed{1} + \frac{\boxed{-3}x + \boxed{-5}}{x^2 + 4x + 10}$$

Since the derivative of the denominator is  $2x + 4$ , we should rewrite the numerator of the big fraction to have  $x + 2$ 's if possible:

$$\frac{x^2 + x + 5}{x^2 + 4x + 10} = \boxed{1} + \frac{\boxed{-3}(x + 2) + \boxed{1}}{x^2 + 4x + 10}.$$

For expressions like

$$\int \frac{x + 2}{x^2 + 4x + 10} dx$$

we should do a substitution. For terms like

$$\int \frac{1}{x^2 + 4x + 10} dx$$

we should first complete the square:  $x^2 + 4x + 10 = (x + 2)^2 + 6$  and then make the substitution  $x + 2 = u\sqrt{6}$ .

**Exercise 102**

$$\int \frac{2x^2 + x + 1}{(x + 1)(x^2 + 9)} dx = \boxed{\frac{9}{10} \ln|x^2 + 9| + \frac{1}{5} \ln|x + 1| - \frac{4}{15} \arctan\left(\frac{x}{3}\right)} + C$$

(Do not include any constant terms in your answer.)

## Sample Quiz Questions

**Question 103** Compute the integral

$$\int_3^4 \frac{2x-3}{x^2-3x+2} dx.$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\ln 2$
- (b)  $\ln 3$  ✓
- (c)  $\ln 4$
- (d)  $\ln 5$
- (e)  $\ln 6$
- (f)  $\ln 7$

**Feedback(attempt):** First factor the denominator of the integrand:  $x^2 - 3x + 2 = (x-1)(x-2)$ . Since the roots are distinct, it is possible to use the Heaviside cover-up method.

**Hint:** The partial fractions expansion will take the form

$$\frac{A}{x-1} + \frac{B}{x-2},$$

where the coefficient  $A$  can be computed by cancelling the factor of  $x-1$  in the denominator and evaluating the result at  $x=1$ , i.e.,

$$A = \frac{2(1)-3}{(1)-2} = 1.$$

Similarly,

$$B = \frac{2(2)-3}{(2)-1} = 1,$$

which gives that

$$\frac{2x-3}{(x-1)(x-2)} = \frac{1}{x-1} + \frac{1}{x-2}.$$

**Hint:** Therefore

$$\begin{aligned} \int_3^4 \frac{2x-3}{x^2-3x+2} dx &= \int_3^4 \left( \frac{1}{x-1} + \frac{1}{x-2} \right) dx \\ &= (\ln|4-1| + \ln|4-2|) - (\ln|3-1| + \ln|3-2|) \\ &= \ln 3 + \ln 2 + \ln \frac{1}{2} + 0 = \ln 3. \end{aligned}$$

## Sample Exam Questions

**Question 104** Compute the volume of the solid of revolution obtained by revolving around the  $y$ -axis the region below the graph

$$y = \frac{1}{(x-1)^2},$$

above  $y = 0$ , and between  $x = 2$  and  $x = 3$ . (Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\pi$
- (b)  $\pi(\ln 2 + 3)$
- (c)  $\pi(2 \ln 2 + 1)$  ✓
- (d)  $\pi(2 \ln 3 + 1)$
- (e)  $\pi(3 \ln 2 + 1)$
- (f)  $\pi(3 \ln 3 + 1)$

**Feedback(attempt):** Choosing  $x$  as the variable of integration, slices will be parallel to the  $y$ -axis, indicating that the shell method should be used. The radius of a shell is  $x$  (because the axis lies to the left of the region) and the height will be  $(x-1)^{-2}$ , so

$$V = \int_2^3 \frac{2\pi x}{(x-1)^2} dx = 2\pi \int_2^3 \frac{x}{(x-1)^2} dx.$$

**Hint:** The integral can be computed by partial fractions; the expansion has the form

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

The coefficients  $A$  and  $B$  can be found by usual methods (but note that the Heaviside cover up method will *not* work in this case), but it is also possible to find them directly by carefully rewriting the numerator of the fraction in terms of  $x-1$ :

$$\frac{x}{(x-1)^2} = \frac{(x-1)+1}{(x-1)^2} = \frac{(x-1)}{(x-1)^2} + \frac{1}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2}.$$

Therefore

$$\begin{aligned} V &= 2\pi \int_2^3 \left[ \frac{1}{x-1} + \frac{1}{(x-1)^2} \right] dx = 2\pi \left[ \ln|x-1| - \frac{1}{x-1} \right]_2^3 \\ &= 2\pi \left( \ln 2 - \frac{1}{2} \right) - 2\pi(0-1) = \pi(2 \ln 2 + 1). \end{aligned}$$

**Question 105** Compute the constants  $A$  and  $B$  in the partial fractions expansion indicated below. To receive full credit, it is not necessary to compute  $C$ ,  $D$ , or  $E$ .

$$\frac{x^4 + 16}{x^4 - 16} = A + \frac{B}{x - 2} + \frac{C}{x + 2} + \frac{Dx + E}{x^2 + 4}$$

(Hints won't be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $A = -1, B = 1$
- (b)  $A = 0, B = 1$
- (c)  $A = 1, B = 1$  ✓
- (d)  $A = -1, B = -1$
- (e)  $A = 0, B = -1$
- (f)  $A = 1, B = -1$

**Feedback(attempt):** You'll need to do polynomial long division first. To compute  $B$ , you can use Heaviside cover-up.

**Hint:**

$$\frac{x^4 + 16}{x^4 - 16} = 1 + \frac{1}{x - 2} - \frac{1}{x + 2} - \frac{4}{x^2 + 4}$$

**Question 106** Evaluate  $\int_1^2 \frac{x^2 + x + 1}{x^2 + x} dx$ .

**Multiple Choice:**

- (a) 0
- (b) 1
- (c)  $1 + \ln\left(\frac{4}{3}\right)$  ✓
- (d) 2
- (e)  $2 + \ln\left(\frac{8}{3}\right)$
- (f) none of these

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### 3.11 Exercises: Cumulative

*Exercises relating to various topics we have studied.*

A key skill to practice is deciding which of the several techniques we have learned should be applied to a particular problem. In these exercises and questions, you will be responsible for choosing a method which works for you.

**Exercise 107** Compute the indefinite integral.

$$\int \ln x \, dx = \boxed{x \ln x - x} + C$$

(You should choose an answer which equals  $-1$  at  $x = 1$ ; if your answer does not satisfy this, add a constant to make it so.)

**Hint:** Integrate by parts.

**Hint:** Let  $u = \ln x$  and  $dv = 1$ .

**Exercise 108** Compute the indefinite integral.

$$\int \frac{dx}{(x^2 + 1)^2} = \boxed{\frac{1}{2} \frac{x}{x^2 + 1} + \frac{1}{2} \arctan x} + C.$$

(You should choose an answer which equals 0 at  $x = 0$ ; if your answer does not satisfy this, add a constant to make it so.)

**Hint:** Partial fractions won't help in this case because the expression is already simplified as much as possible.

**Hint:** Try a trigonometric substitution

**Hint:** Make the substitution  $x = \tan \theta$ .

**Hint:** You should arrive at the integral  $\int \cos^2 \theta \, d\theta$  after the substitution; here you'll need to use a power-reduction formula to continue.

## Sample Quiz Questions

**Question 109** The region in the plane between the  $x$ -axis and the graph

$$y = \frac{1}{2} \ln x + 1$$

in the range  $\frac{1}{5} \leq x \leq 1$  is revolved around the axis  $x = \frac{1}{10}$ . Compute the volume of the resulting solid.

**Multiple Choice:**

- (a)  $\frac{11}{25}\pi$
- (b)  $\frac{14}{25}\pi$
- (c)  $\frac{16}{25}\pi$  ✓
- (d)  $\frac{19}{25}\pi$
- (e)  $\frac{21}{25}\pi$
- (f)  $\frac{24}{25}\pi$

**Feedback(attempt):** If the variable  $x$  is used for slicing, then slices are parallel to the axis of rotation, which indicates the shell method should be used. The radius of a shell is  $x - \frac{1}{10}$ . The height of a shell is exactly  $\frac{1}{2} \ln x + 1$ . The volume of the region is therefore given by

$$\int_{\frac{1}{5}}^1 \frac{\pi}{10} (10x - 1) (\ln x + 2) dx.$$

To compute the integral, we can use integration by parts. A reasonable strategy is to integrate  $x - \frac{1}{10}$  and differentiate  $\ln x + 2$ . This gives the equality

$$\begin{aligned} \pi \int \left(x - \frac{1}{10}\right) (\ln x + 2) dx &= \pi \left( \left(\frac{x^2}{2} - \frac{x}{10}\right) (\ln x + 2) - \int \frac{1}{x} \left(\frac{x^2}{2} - \frac{x}{10}\right) dx \right) \\ &= -\pi \left(\frac{x^2}{4} - \frac{x}{10}\right) + \pi \left(\frac{x^2}{2} - \frac{x}{10}\right) (\ln x + 2). \end{aligned}$$

Therefore

$$\begin{aligned} \pi \int_{\frac{1}{5}}^1 \left(x - \frac{1}{10}\right) (\ln x + 2) dx &= \left[ -\pi \left(\frac{x^2}{4} - \frac{x}{10}\right) + \pi \left(\frac{x^2}{2} - \frac{x}{10}\right) (\ln x + 2) \right] \Big|_{\frac{1}{5}}^1 \\ &= \left(\frac{13}{20}\pi\right) - \left(\frac{\pi}{100}\right) = \frac{16}{25}\pi. \end{aligned}$$

**Question 110** Consider the region given by  $2\pi \leq x \leq \frac{5}{2}\pi$  and  $0 \leq y \leq \sin x$ . Compute the  $x$ -coordinate of the centroid (i.e., assuming constant density).

**Multiple Choice:**

- (a)  $-1 + \frac{5}{2}\pi$
- (b)  $1 + 2\pi$  ✓
- (c)  $\frac{5}{2}\pi$
- (d)  $-1 + 3\pi$
- (e)  $3\pi$
- (f)  $4\pi$

**Feedback(attempt):** The mass  $M$  will be given by the integral

$$\int_{2\pi}^{\frac{5}{2}\pi} \sin x \, dx$$

One can check that

$$\int_{2\pi}^{\frac{5}{2}\pi} \sin x \, dx = 1.$$

To compute the  $x$ -coordinate of the centroid, we also need to compute the integral

$$\int_{2\pi}^{\frac{5}{2}\pi} x \sin x \, dx$$

To compute the integral, we can use integration by parts. A reasonable strategy is to integrate  $\sin x$  and differentiate  $x$ . This gives the equality

$$\begin{aligned} \int x \sin x \, dx &= -x \cos x - \int (-\cos x) \, dx \\ &= -x \cos x + \sin x. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{2\pi}^{\frac{5}{2}\pi} x \sin x \, dx &= [-x \cos x + \sin x]_{2\pi}^{\frac{5}{2}\pi} \\ &= 1 - (-2\pi) = 1 + 2\pi. \end{aligned}$$

The correct answer is the ratio of the integrals, i.e.,

$$\bar{x} = \frac{1 + 2\pi}{1} = 1 + 2\pi.$$



## Sample Exam Questions

**Question 111** An object moves in such a way that its acceleration at time  $t$  seconds is  $(t^2 + 5t + 6)^{-1}$  meters per second squared. If the initial velocity of the object is  $2/3$  meters per second, what is the limit of its velocity as  $t \rightarrow \infty$ ?

**Multiple Choice:**

- (a)  $\ln \frac{3}{2}$  meters per second
  - (b)  $\ln 6$  meters per second
  - (c) 1 meters per second
  - (d)  $\ln \frac{4}{9}$  meters per second
  - (e)  $\ln \frac{9}{4}$  meters per second
  - (f) 0 meters per second ✓
- 

**Question 112** Find the volume of the solid generated by revolving the region bounded above by  $y = \sin x$  and bounded below by  $y = 0$  for  $0 \leq x \leq \pi$  about the line  $x = \pi$ .

**Multiple Choice:**

- (a)  $\pi^2$
  - (b)  $2\pi^2$  ✓
  - (c)  $4\pi^2$
  - (d)  $\frac{\pi^2}{2}$
  - (e)  $\frac{\pi^2}{4}$
  - (f) none of these
- 

**Question 113** Evaluate  $\int_1^2 x \ln(x^2 + 1) dx$ .

**Multiple Choice:**

- (a) 0
- (b) 1
- (c)  $\ln 2$
- (d)  $\frac{1}{2}$
- (e)  $\ln(2) - \frac{1}{2}$
- (f) none of these ✓

**Feedback(attempt):** This integral can be computed via integration by parts. If we integrate  $x$  and differentiate  $\ln(x^2 + 1)$ , we get

$$\begin{aligned}\int_1^2 x \ln(x^2 + 1) dx &= \left. \frac{x^2}{2} \ln(x^2 + 1) \right|_1^2 - \int_1^2 \frac{x^2}{2} \frac{2x}{x^2 + 1} dx \\ &= 2 \ln 5 - \frac{1}{2} \ln 2 - \int_1^2 \frac{x^3}{x^2 + 1} dx.\end{aligned}$$

The latter integral can be simplified using polynomial long division:  $\frac{x^3}{x^2 + 1} = x - \frac{x}{x^2 + 1}$ . Therefore

$$\begin{aligned}\int_1^2 x \ln(x^2 + 1) dx &= 2 \ln 5 - \frac{1}{2} \ln 2 - \int_1^2 x dx + \int_1^2 \frac{x}{x^2 + 1} dx \\ &= 2 \ln 5 - \frac{1}{2} \ln 2 - \left. \frac{x^2}{2} \right|_1^2 + \left. \frac{1}{2} \ln(x^2 + 1) \right|_1^2 \\ &= 2 \ln 5 - \frac{1}{2} \ln 2 - 2 + \frac{1}{2} + \frac{1}{2} \ln 5 - \frac{1}{2} \ln 2 \\ &= \frac{5}{2} \ln 5 - \frac{2}{2} \ln 4 - \frac{3}{2} - \frac{3}{2} = \ln \left( \frac{5^5}{4} \right) - \frac{3}{2}.\end{aligned}$$

## 4 Further Topics in Integration

We study additional topics relating to applications of integration.

In the next several units, we will be returning to two fundamental questions:

- If we are unable to evaluate an integral exactly, can we at least determine an approximate answer?

### *Further Topics in Integration*

- Is there a way to integrate functions with vertical asymptotes and/or integrate functions on infinite domains instead of finite ones?

Along the way, we will study the idea of *orders of growth*, which is a way of thinking about functions which makes it much easier to evaluate limits than is possible with l'Hôpital's rule. We will also study probability theory as an application of integration and see how these ideas all come together in this particular context.

## 4.1 Numerical Integration

*We study the problem of numerically approximating the value of an integral.*

### Online Texts

- OpenStax II 3.6: Numerical Integration<sup>40</sup>
- Community Calculus 8.6: Numerical Integration<sup>41</sup>

### Examples

**Example 30.**

**Example 31.**

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<sup>40</sup>See OpenStax II 3.6: Numerical Integration at <https://openstax.org/books/calculus-volume-2/pages/3-6-numerical-integration>

<sup>41</sup>See Community Calculus 8.6: Numerical Integration at [https://www.whitman.edu/mathematics/calculus\\_online/section08.06.html](https://www.whitman.edu/mathematics/calculus_online/section08.06.html)

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## 4.2 Exercises: Numerical Integration

Various exercises relating to numerical integration.

**Exercise 114** Consider the definite integral  $\int_{-1}^1 x^2 dx$ .

- The trapezoid rule with  $n = 4$  gives the approximation

$$\int_{-1}^1 x^2 dx \approx \boxed{\frac{3}{4}}.$$

- Simpson's rule with  $n = 4$  gives the approximation

$$\int_{-1}^1 x^2 dx \approx \boxed{\frac{2}{3}}.$$

- The exact value of the integral is

$$\int_{-1}^1 x^2 dx = \boxed{\frac{2}{3}}.$$

**Exercise 115** When estimating the integral below using Simpson's rule, what is the minimum number of intervals that would be required to guarantee that the approximation error does not exceed  $2 \times 10^{-5}$ ? (Enter the smallest value which you know is correct.)

$$\int_{-1}^0 e^{x\sqrt{6}} dx$$

$$n \geq \boxed{10}.$$

**Exercise 116** Find  $n$  such that the error in approximating the given definite integral is less than 0.0001 when using:

- The trapezoid rule:  $n \geq \boxed{\sqrt{\frac{10000\pi^3}{12}}} \approx 161$ . (Enter your answer as the exact result of your calculation; do not round or approximate.)

(b) Simpson's rule:  $n \geq \left\lceil \left( \frac{10000\pi^5}{180} \right)^{1/4} \right\rceil \approx 12$ . (Enter your answer as the exact result of your calculation; do not round or approximate.)

**Exercise 117** How many equally spaced intervals  $N$  are sufficient for the trapezoidal rule to estimate the value of the following integral with an error less than or equal to  $10^{-6}$ ? (Enter the smallest value which you know is correct.)

$$\int_{-1}^1 e^{x^2-1} dx.$$

$$N \geq \boxed{2000}.$$

**Hint:** The second derivative of the integrand is  $(4x^2 + 2)e^{x^2-1}$ . Since  $-1 \leq x \leq 1$ , it follows that

$$(4x^2 + 2)e^{x^2-1} \leq (4 \cdot 1 + 2)e^0 = 6.$$

**Hint:** By the trapezoid rule error formula, the error  $E$  satisfies

$$|E| \leq \frac{6(1 - (-1))^3}{12N^2} = \frac{4}{N^2},$$

where  $N$  is the number of intervals.

**Hint:** To be sure the error is small enough, we need

$$\frac{4}{N^2} \leq 10^{-6}$$

## 4.3 Orders of Growth

We study the use of orders of growth to compute limits, in preparation for improper integrals.

### (Videos) Calculus: Single Variable

**Note:** If you haven't seen Taylor series before (for arctangent, exponential, and  $1/(1-x)$ ), don't worry yet about exactly where these particular facts come from (they come up at 9:00 and again at 14:30 in the video). YouTube link: <https://www.youtube.com/watch?v=aL2FOIRH9GE>

## **Examples**

**Example 32.**

**Example 33.**

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## 4.4 Exercises: Orders of Growth

*Various exercises relating to orders of growth.*

**Exercise 118** Use orders of growth to identify dominant terms and compute limits.

- Compute the limit:

$$\lim_{x \rightarrow \infty} \frac{e^x + x^2}{3e^x + x^3} = \boxed{\frac{1}{3}}$$

The dominant terms are

**Multiple Choice:**

- (a)  $e^x$  in the numerator and  $3e^x$  in the denominator ✓
- (b)  $e^x$  in the numerator and  $x^3$  in the denominator
- (c)  $x^2$  in the numerator and  $3e^x$  in the denominator
- (d)  $x^2$  in the numerator and  $x^3$  in the denominator

- Compute the limit:

$$\lim_{x \rightarrow 0} \frac{e^x + x^2}{3e^x + x^3} = \boxed{\frac{1}{3}}$$

The dominant terms are

**Multiple Choice:**

- (a)  $e^x$  in the numerator and  $3e^x$  in the denominator ✓
- (b)  $e^x$  in the numerator and  $x^3$  in the denominator
- (c)  $x^2$  in the numerator and  $3e^x$  in the denominator
- (d)  $x^2$  in the numerator and  $x^3$  in the denominator

- Compute the limit:

$$\lim_{x \rightarrow \infty} \frac{x + x^2}{3x^4 + x^3} = \boxed{0}$$

The dominant terms are

**Multiple Choice:**

- (a)  $x$  in the numerator and  $3x^4$  in the denominator
- (b)  $x$  in the numerator and  $x^3$  in the denominator



- (c)  $x^2$  in the numerator and  $3x^4$  in the denominator ✓
- (d)  $x^2$  in the numerator and  $x^3$  in the denominator

- Compute the limit:

$$\lim_{x \rightarrow 0} \frac{x + x^2}{3x^4 + x^3} = \boxed{\infty}$$

The dominant terms are

**Multiple Choice:**

- (a)  $x$  in the numerator and  $3x^4$  in the denominator
- (b)  $x$  in the numerator and  $x^3$  in the denominator ✓
- (c)  $x^2$  in the numerator and  $3x^4$  in the denominator
- (d)  $x^2$  in the numerator and  $x^3$  in the denominator

- Compute the limit:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^3 + 9} - x}{\sqrt{x^3 + 9} + x} = \boxed{1}$$

**Multiple Choice:**

- (a)  $\sqrt{x^3 + 9}$  dominates  $x$  ✓
- (b)  $x$  dominates  $\sqrt{x^3 + 9}$

- Compute the limit:

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x^3 + 9x^2} - x}{\sqrt{x^3 - 9x^2} + x} = \boxed{1}$$

(First identify whether the square-root term is dominant or not; when it is dominant, identify the dominant term *inside* the square root and neglect the non-dominant term.)

- Compute the limit:

$$\lim_{x \rightarrow \infty} \frac{2x^3 + \sin x^5}{3x^4 + \sin x^6} = \boxed{0}.$$

- Compute the limit

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1}}{\sqrt{x}} = \boxed{\infty}.$$

**Multiple Choice:**

- (a)  $x$  dominates 1 as  $x \rightarrow 0$
- (b) 1 dominates  $x$  as  $x \rightarrow 0$  ✓

- Compute the limit:

$$\lim_{x \rightarrow \infty} \frac{(\ln x)^{100}}{x^{0.01}} = \boxed{0}.$$

**Multiple Choice:**

- (a)  $x^{0.01}$  dominates  $(\ln x)^{100}$  as  $x \rightarrow \infty$  ✓
- (b)  $(\ln x)^{100}$  dominates  $x^{0.01}$  as  $x \rightarrow \infty$

- Compute the limit:

$$\lim_{x \rightarrow 0^+} x^{0.01} (\ln x)^{100} = \boxed{\infty}$$

**Multiple Choice:**

- (a)  $x^{-0.01}$  dominates  $(\ln x)^{100}$  as  $x \rightarrow 0^+$  ✓
- (b)  $(\ln x)^{100}$  dominates  $x^{-0.01}$  as  $x \rightarrow 0^+$

- Compute the limit:

$$\lim_{x \rightarrow \infty} \frac{e^{\ln \ln x} + 1}{x^2} = \boxed{0}$$

(Use the laws of exponents to write  $x^2 = e^{2 \ln x}$ .)

## Sample Quiz Questions

**Question 119** Arrange the functions

$$x^x \quad \frac{e^x}{\ln x} \quad \ln x$$

in order from least rate of growth to greatest rate of growth as  $x \rightarrow \infty$ . Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

**Multiple Choice:**

- (a)  $x^x \ll \frac{e^x}{\ln x} \ll \ln x$
- (b)  $\ln x \ll x^x \ll \frac{e^x}{\ln x}$
- (c)  $\frac{e^x}{\ln x} \ll \ln x \ll x^x$
- (d)  $x^x \ll \ln x \ll \frac{e^x}{\ln x}$

$$(e) \frac{e^x}{\ln x} \ll x^x \ll \ln x$$

$$(f) \ln x \ll \frac{e^x}{\ln x} \ll x^x \quad \checkmark$$

**Feedback(attempt):** General Remarks:

- Higher powers of  $x$  grow faster at infinity than lower powers of  $x$ .
- As  $x \rightarrow \infty$ ,  $\ln x$  goes to infinity slower than  $x^p$  for any (presumably small) positive constant  $p$ .
- As  $x \rightarrow \infty$ ,  $e^x$  goes to infinity faster than  $x^n$  for any (presumably large) positive constant  $n$ .
- As  $x \rightarrow \infty$ ,  $x^x$  goes to infinity faster than any exponential of the form  $e^{cx}$  for any constant  $c$ .

**Question 120** Arrange the functions

$$\frac{e^{-x}}{\ln x} \quad x^3 \ln x \quad e^x$$

in order from least rate of growth to greatest rate of growth as  $x \rightarrow \infty$ . Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

**Multiple Choice:**

$$(a) \frac{e^{-x}}{\ln x} \ll x^3 \ln x \ll e^x \quad \checkmark$$

$$(b) e^x \ll \frac{e^{-x}}{\ln x} \ll x^3 \ln x$$

$$(c) x^3 \ln x \ll e^x \ll \frac{e^{-x}}{\ln x}$$

$$(d) \frac{e^{-x}}{\ln x} \ll e^x \ll x^3 \ln x$$

$$(e) x^3 \ln x \ll \frac{e^{-x}}{\ln x} \ll e^x$$

$$(f) e^x \ll x^3 \ln x \ll \frac{e^{-x}}{\ln x}$$

**Feedback(attempt):** General Remarks:

- Higher powers of  $x$  grow faster at infinity than lower powers of  $x$ .

- As  $x \rightarrow \infty$ ,  $\ln x$  goes to infinity slower than  $x^p$  for any (presumably small) positive constant  $p$ .
- As  $x \rightarrow \infty$ ,  $e^x$  goes to infinity faster than  $x^n$  for any (presumably large) positive constant  $n$ .

**Question 121** Arrange the functions

$$\left(\ln \frac{1}{x}\right)^2 \quad \frac{e^{-x}}{x^3} \ln x \quad x^3 e^x$$

in order from least rate of growth to greatest rate of growth as  $x \rightarrow 0^+$ . Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

**Multiple Choice:**

- (a)  $\left(\ln \frac{1}{x}\right)^2 \ll \frac{e^{-x}}{x^3} \ln x \ll x^3 e^x$
- (b)  $x^3 e^x \ll \left(\ln \frac{1}{x}\right)^2 \ll \frac{e^{-x}}{x^3} \ln x \checkmark$
- (c)  $\frac{e^{-x}}{x^3} \ln x \ll x^3 e^x \ll \left(\ln \frac{1}{x}\right)^2$
- (d)  $\left(\ln \frac{1}{x}\right)^2 \ll x^3 e^x \ll \frac{e^{-x}}{x^3} \ln x$
- (e)  $\frac{e^{-x}}{x^3} \ln x \ll \left(\ln \frac{1}{x}\right)^2 \ll x^3 e^x$
- (f)  $x^3 e^x \ll \frac{e^{-x}}{x^3} \ln x \ll \left(\ln \frac{1}{x}\right)^2$

**Feedback(attempt):** General Remarks:

- Lower powers of  $x$  grow faster as  $x \rightarrow 0^+$  than higher powers of  $x$ .
- As  $x \rightarrow 0^+$ ,  $-\ln x = \ln x^{-1}$  goes to  $\infty$  slower than  $x^{-p}$  for any (presumably small) positive  $p$ .
- As  $x \rightarrow 0^+$ ,  $e^x \rightarrow 1$  and so does not influence the growth rate.
- As  $x \rightarrow 0^+$ ,  $e^{-x} \rightarrow 1$  and so does not influence the growth rate.

**Question 122** Arrange the functions

$$\frac{x^3 e^x}{\ln x} \quad \frac{e^{-x}}{x^3} \quad \frac{e^{-x}}{x^3 \ln x}$$

in order from least rate of growth to greatest rate of growth as  $x \rightarrow 0^+$ . Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

**Multiple Choice:**

- (a)  $\frac{x^3 e^x}{\ln x} \ll \frac{e^{-x}}{x^3} \ll \frac{e^{-x}}{x^3 \ln x}$
- (b)  $\frac{e^{-x}}{x^3 \ln x} \ll \frac{x^3 e^x}{\ln x} \ll \frac{e^{-x}}{x^3}$
- (c)  $\frac{e^{-x}}{x^3} \ll \frac{e^{-x}}{x^3 \ln x} \ll \frac{x^3 e^x}{\ln x}$
- (d)  $\frac{x^3 e^x}{\ln x} \ll \frac{e^{-x}}{x^3 \ln x} \ll \frac{e^{-x}}{x^3} \checkmark$
- (e)  $\frac{e^{-x}}{x^3} \ll \frac{x^3 e^x}{\ln x} \ll \frac{e^{-x}}{x^3 \ln x}$
- (f)  $\frac{e^{-x}}{x^3 \ln x} \ll \frac{e^{-x}}{x^3} \ll \frac{x^3 e^x}{\ln x}$

**Feedback(attempt):** General Remarks:

- Lower powers of  $x$  grow faster as  $x \rightarrow 0^+$  than higher powers of  $x$ .
- As  $x \rightarrow 0^+$ ,  $-\ln x = \ln x^{-1}$  goes to  $\infty$  slower than  $x^{-p}$  for any (presumably small) positive  $p$ .
- As  $x \rightarrow 0^+$ ,  $e^x \rightarrow 1$  and so does not influence the growth rate.
- As  $x \rightarrow 0^+$ ,  $e^{-x} \rightarrow 1$  and so does not influence the growth rate.

## 4.5 Improper Integrals

We study the concept of improper integrals.

### (Video) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=NJKB7QRQ2yY>

## Online Texts

- OpenStax II 3.7: Improper Integrals<sup>42</sup>
- Ximera OSU: Improper Integrals<sup>43</sup>
- Community Calculus: Improper Integrals<sup>44</sup>

## Examples

**Example 34.** Compute the improper integral

$$\int_0^1 \ln x \, dx$$

if it is convergent.

- First off, we note that the integral is in fact improper because  $\ln x$  has a vertical asymptote at  $x = 0$ . We can find the antiderivative of  $\ln x$  by integration by parts:

$$\int \ln x \, dx = \boxed{x \ln x - x} + C$$

(your answer should be the usual thing given by integration by parts. In particular, it equals 0 at  $x = e$ .)

- The correct interpretation of the improper integral is

$$\int_0^1 \ln x \, dx = \lim_{b \rightarrow 0^+} \int_{\boxed{b}}^{\boxed{1}} \ln x \, dx.$$

- Using the antiderivative you found above (with  $C = 0$ ), use the Fundamental Theorem of Calculus to conclude

$$\int_0^1 \ln x \, dx = \lim_{b \rightarrow 0^+} \int_{\boxed{b}}^{\boxed{1}} \ln x \, dx = \lim_{b \rightarrow 0^+} \left[ \left( \boxed{-1} \right) - \left( \boxed{b \ln b - b} \right) \right].$$

- As  $b \rightarrow 0^+$ ,  $b \rightarrow 0$  and  $\ln b \rightarrow -\infty$ . Using knowledge of orders of growth, we know that the product  $b \ln b$

<sup>42</sup>See OpenStax II 3.7: Improper Integrals at <https://openstax.org/books/calculus-volume-2/pages/3-7-improper-integrals>

<sup>43</sup>See Ximera OSU: Improper Integrals at <https://ximera.osu.edu/mooculus/calculus2/improperIntegrals/titlePage>

<sup>44</sup>See Community Calculus: Improper Integrals at [https://www.whitman.edu/mathematics/calculus\\_online/section09.07.html](https://www.whitman.edu/mathematics/calculus_online/section09.07.html)

**Multiple Choice:**

- (a) tends to  $-\infty$  because  $\ln b$  grows faster than any negative power of  $b$  as  $b \rightarrow 0^+$ .
- (b) tends to some finite constant as  $b \rightarrow 0^+$  because  $\ln b$  has the same growth rate as  $b^{-1}$  as  $b \rightarrow 0^+$ .
- (c) tends to zero because  $\ln b$  grows slower than any negative power of  $b$  as  $b \rightarrow 0^+$ . ✓

- We take the limit as  $b \rightarrow 0^+$  to conclude

$$\int_0^1 \ln x \, dx = \boxed{-1}$$

(write N/A if the limit diverges).

**Example 35.** Without finding an antiderivative, determine whether the integral below is convergent or divergent.

$$\int_1^{\infty} \frac{dx}{x - \ln x}$$

- The dominant term in the denominator as  $x \rightarrow \infty$  will be

**Multiple Choice:**

- (a)  $x$  ✓
- (b)  $\ln x$

which suggests doing a comparison to the integral

**Multiple Choice:**

- (a)  $\int_1^{\infty} \frac{dx}{x}$  ✓
- (b)  $\int_1^{\infty} \frac{dx}{-\ln x}$

Since  $1/(x - \ln x)$  is always (less than/ greater than ✓)  $1/x$ , the simplest test which applies is the (direct comparison ✓/ limit comparison) test and it indicates (convergence/ divergence ✓/ nothing—it does not apply).

**Example 36.** Without finding an antiderivative, determine whether the integral below is convergent or divergent.

$$\int_0^1 \frac{dx}{2\sqrt{x} - x^2}.$$

- The dominant term in the denominator as  $x \rightarrow 0^+$  is

**Multiple Choice:**

- (a)  $x^2$
- (b)  $2\sqrt{x}$  ✓

which suggests doing a comparison to the integral

**Multiple Choice:**

- (a)  $\int_0^1 \frac{dx}{2\sqrt{x}}$  ✓
- (b)  $\int_0^1 \frac{dx}{-x^2}$

Since  $1/(2\sqrt{x} - x^2)$  is always (less than / greater than ✓)  $1/(2\sqrt{x})$ , the simplest test which applies is the (direct comparison / limit comparison ✓) test and it indicates (convergence ✓ / divergence / nothing—it does not apply).



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## 4.6 Exercises: Improper Integrals

Various exercises relating to improper integrals.

**Exercise 123** Evaluate the improper integral:  $\int_0^{\infty} e^{5-2x} dx = \boxed{(e^5)/2}$ .

**Exercise 124** Evaluate the given improper integral:  $\int_{-\infty}^{\infty} \frac{1}{x^2 + 9} dx = \boxed{\pi/3}$ .

**Exercise 125** Evaluate the integral:

$$\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \boxed{2(e-1)}.$$

This integral is (not improper / improper ✓) because of the behavior of the integrand near  $x = 0$ .

**Exercise 126** Evaluate the given improper integral.  $\int_3^{\infty} \frac{1}{x^2 - 4} dx = \boxed{\frac{\ln 5}{4}}$ .

**Exercise 127** Use the Direct Comparison Test or the Limit Comparison Test to determine whether the integral converges or diverges:  $\int_{10}^{\infty} \frac{3}{\sqrt{3x^2 + 2x - 5}} dx$ .  
 Answer: the integral (converges / diverges ✓) by (direct / limit ✓) comparison with the function  $\frac{1}{x^{\boxed{1}}}$ .

**Exercise 128** Use the Direct Comparison Test or the Limit Comparison Test to determine whether the integral converges or diverges:  $\int_2^{\infty} \frac{2}{\sqrt{7x^3 - x}} dx$ .  
 Answer: the integral (converges ✓ / diverges) by (direct / limit ✓) comparison with the function  $\frac{1}{x^{\boxed{3/2}}}$  (select the largest exponent for the denominator which makes the statement true).

**Exercise 129** Use the Direct Comparison Test or the Limit Comparison Test to determine whether the integral converges or diverges:  $\int_1^{\infty} e^{-x} \ln x \, dx$ . Answer: the integral (converges ✓/ diverges) by direct comparison with the function

**Multiple Choice:**

- (a)  $e^{-x}$
- (b)  $xe^{-x}$  ✓
- (c)  $e^{-x}/x$

**Exercise 130** Use the Direct Comparison Test or the Limit Comparison Test to determine whether the integral converges or diverges:  $\int_1^{\infty} e^{-x^2+3x+1} \, dx$ . Answer: the integral (converges ✓/ diverges) by direct comparison with the function

**Multiple Choice:**

- (a)  $e^{-x^2}$
- (b)  $e^{-x}$  ✓
- (c)  $e^{3x+1}$

**Exercise 131** Use the Direct Comparison Test or the Limit Comparison Test to determine whether the integral converges or diverges:  $\int_1^{\infty} \frac{x}{x^2 + \cos x} \, dx$ . Answer: the integral (converges/ diverges ✓) by (direct/ limit ✓) comparison with the function

**Multiple Choice:**

- (a)  $1/x$  ✓
- (b)  $x/\cos x$
- (c)  $1/(x^2 + \cos x)$

**Exercise 132** Use the Direct or Limit Comparison Test to determine whether the integral converges or diverges:

$$\int_0^{1/e} \frac{(\ln x)^2 - 1}{x^3 + x^2 + x} dx$$

Answer: The integral (converges/ diverges ✓) by (direct/ limit ✓) comparison with the function

**Multiple Choice:**

- (a)  $\frac{(\ln x)^2}{x^3}$
- (b)  $\frac{(\ln x)^2}{x^2}$
- (c)  $\frac{(\ln x)^2}{x}$  ✓
- (d)  $\frac{1}{x^3}$
- (e)  $\frac{1}{x^2}$

**Exercise 133** Use the Direct or Limit Comparison Test to determine whether the integral converges or diverges:

$$\int_0^{1/e} \frac{(\ln x)^2 - 1}{x + \sqrt{x} + e^{-1/x}} dx$$

Answer: The integral (converges ✓/ diverges) by direct comparison with the function

**Multiple Choice:**

- (a)  $\frac{(\ln x)^2}{x}$
- (b)  $\frac{(\ln x)^2}{\sqrt{x}}$  ✓
- (c)  $\frac{(\ln x)^2}{e^{-1/x}}$
- (d)  $\frac{1}{x}$

(e)  $\frac{1}{\sqrt{x}}$

(f)  $\frac{1}{e^{-1/x}}$

## Sample Quiz Questions

**Question 134** Which of the following improper integrals is convergent? Show how you used comparison tests to justify your answer.

$$\text{I: } \int_0^1 \frac{\sqrt{1+x^2}e^{-x}}{(\cos x)x^2} dx \quad \text{II: } \int_2^\infty \frac{e^x}{xe^x + x^2} dx \quad \text{III: } \int_2^\infty \frac{x^2}{x^4 + 1} dx$$

**Multiple Choice:**

- (a) only I converges
- (b) only II converges
- (c) only III converges ✓
- (d) I and II converge
- (e) II and III converge
- (f) I and III converge

**Feedback(attempt):** Integral I is divergent by direct comparison to the function  $\frac{1}{x^2}$ . Integral II is divergent by limit comparison to the function  $\frac{1}{x}$ . Integral III is convergent by direct comparison to the function  $\frac{1}{x^2}$ .

**Question 135** Which of the following improper integrals is convergent? Show how you used comparison tests to justify your answer.

$$\text{I: } \int_0^1 \frac{\sqrt{e^{2x} + x^3}}{x} dx \quad \text{II: } \int_0^1 \frac{x^2}{x^2\sqrt{x} + x^3} dx \quad \text{III: } \int_2^\infty \frac{x^2 \ln x}{-x + x^4} dx$$

**Multiple Choice:**

- (a) only I converges
- (b) only II converges

- (c) only III converges
- (d) I and II converge
- (e) II and III converge ✓
- (f) I and III converge

**Feedback(attempt):** Integral I is divergent by direct comparison to the function  $\frac{1}{x}$ . Integral II is convergent by direct comparison to the function  $\frac{1}{\sqrt{x}}$ . Integral III is convergent by limit comparison to the function  $\frac{\ln x}{x^2}$ .

## Sample Exam Questions

**Question 136** Only one of the following four improper integrals diverges. Choose that improper integral and justify why it diverges. (You need NOT justify why the other integrals converge.)

**Multiple Choice:**

- (a)  $\int_2^{\infty} \frac{\arctan x}{1+x^3} dx$
- (b)  $\int_2^{\infty} \frac{1}{\sqrt{x^4+x^2}} dx$
- (c)  $\int_2^{\infty} \frac{1+\sin x}{x^2} dx$
- (d)  $\int_2^{\infty} \frac{1}{\sqrt[3]{x^2-1}} dx$  ✓

## 4.7 Probability

We study probability and its connections to integration.

## (Videos) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=tBkFkwueDNI> YouTube link: <https://www.youtube.com/watch?v=Aj7EY7nCVmg>

## Online Texts

- Ximera Supplementary Material: Probability<sup>45</sup>
- Community Calculus: Probability<sup>46</sup>

## Examples

**Example 37.**

**Example 38.**

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<sup>45</sup>See Ximera Supplementary Material: Probability at <https://ximera.osu.edu/ptgmath104/math104supplement/lessons/probability>

<sup>46</sup>See Community Calculus: Probability at [https://www.whitman.edu/mathematics/calculus\\_online/section09.08.html](https://www.whitman.edu/mathematics/calculus_online/section09.08.html)

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## 4.8 Exercises: Probability

Various exercises relating to probability.

**Exercise 137** • If it exists, find the value of the constant  $c$  below which makes the function a PDF on the interval  $[e, e^2]$ . If no such constant exists, enter N/A.

$$f(x) = c \ln x$$

$$c = \boxed{e^{-2}}.$$

**Hint:** You'll need to integrate by parts.

- For the same function as above, find the value of the constant  $c$  which makes it a PDF on the interval  $[e^{-2}, e^2]$ . If no such constant exists, enter N/A.

$$c = \boxed{N/A}.$$

**Hint:** The function  $\ln x$  is negative on  $[e^{-2}, 1)$  and positive on  $(1, e^2]$ . What does that imply for the possibility of it being a PDF?

**Exercise 138** Suppose  $X$  is a random variable which represents the length of time your customers remain on hold before reaching a customer service agent. A reasonable model for such a random variable is to use an exponential distribution, i.e., to have a PDF given by

$$f(x) = ce^{-cx}$$

for some positive constant  $c$ . If the mean hold time is 2 minutes, what is the standard deviation?

$$\sigma = \boxed{2}.$$

What is the probability that a customer will have to wait more than one standard deviation beyond the mean before their call is answered?

$$P = \boxed{e^{-2}} \approx 13.5\%.$$

**Hint:** First you need to solve for  $c$ .

**Hint:** The equation will be

$$2 = \int cxe^{-cx} dx.$$

**Hint:** This gives  $c = 1/2$ .

**Hint:** The easiest formula to use here is

$$\sigma^2 = \int_0^\infty x^2 f(x) dx - \mu^2,$$

where  $\mu$  is the mean.

**Exercise 139** Suppose  $X$  is a random variable on the interval  $[1, 10]$  whose PDF is given by

$$f(x) = \frac{c}{x}$$

for some positive constant  $c$  (note: this is the PDF that arises in the phenomenon known as Benford's Law<sup>47</sup>). Compute the mean  $\mu$  and the median  $m$  associated to this PDF. Which is larger?

$$\mu = \boxed{\frac{9}{\ln 10}} \quad \text{and} \quad m = \boxed{\sqrt{10}}$$

The (mean  $\checkmark$ / median) is more sensitive to tails. Since this distribution has a tail extending (to the left/for smaller  $x$ / to the right/for larger  $x$   $\checkmark$ ), it is expected (and in fact, true) that the mean is (larger  $\checkmark$ / smaller) than the median.

**Hint:** First you need to solve for the constant  $c$ .

**Hint:**

$$c = \frac{1}{\ln 10}$$

## Sample Quiz Questions

**Question 140** Find the value of  $c$  which makes the function

$$f(x) = \frac{1}{2}e^{-x} - ce^{-2x}$$

a probability density function on the interval  $[0, \infty)$ . What is the value of the mean  $\mu$  of the corresponding random variable? (Hints will not be revealed until after you choose a response.)

**Multiple Choice:**

<sup>47</sup>See Benford's Law at [https://en.wikipedia.org/wiki/Benford%27s\\_law](https://en.wikipedia.org/wiki/Benford%27s_law)



- (a)  $c = 1, \mu = \frac{1}{2}$
- (b)  $c = -1, \mu = \frac{3}{4}$  ✓
- (c)  $c = 1, \mu = 1$
- (d)  $c = -1, \mu = \frac{5}{4}$
- (e)  $c = 1, \mu = \frac{3}{2}$
- (f)  $c = -1, \mu = \frac{7}{4}$

**Feedback(attempt):** To compute the constant  $c$ , we use the fact that the integral of a probability density function must equal 1, so

$$\mu = \int_0^\infty \left( \frac{1}{2}e^{-x} - ce^{-2x} \right) dx = 1.$$

This gives the equation

$$\frac{1}{2} - \frac{1}{2}c = 1,$$

which then implies that  $c = -1$ .

**Hint:** To compute the mean  $\mu$ , we use the formula

$$\mu = \int_0^\infty x \left( \frac{1}{2}e^{-x} + e^{-2x} \right) dx.$$

Calculating the integral gives  $\mu = 3/4$ .

**Question 141** A certain random variable  $X$  takes values in the interval  $\left[2\pi, \frac{5}{2}\pi\right]$ .

If the probability density function is given by

$$A \sin x$$

for some appropriate value of the constant  $A$ , compute the expected value  $\mu$  of  $X$ . (Hints will not be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\mu = -1 + 2\pi$
- (b)  $\mu = 1 + \frac{3}{2}\pi$

(c)  $\mu = 2\pi$

(d)  $\mu = -1 + \frac{5}{2}\pi$

(e)  $\mu = 1 + 2\pi$  ✓

(f)  $\mu = \frac{5}{2}\pi$

**Feedback(attempt):** The constant  $A$  will be the reciprocal of the integral

$$\int_{2\pi}^{\frac{5}{2}\pi} \sin x \, dx$$

**Hint:** One can check that

$$\int_{2\pi}^{\frac{5}{2}\pi} \sin x \, dx = 1.$$

**Hint:** To compute the expected value  $\mu$  we also need to compute the integral

$$\int_{2\pi}^{\frac{5}{2}\pi} x \sin x \, dx$$

To compute the integral, we can use integration by parts. A reasonable strategy is to integrate  $\sin x$  and differentiate  $x$ .

**Hint:** This gives the equality

$$\begin{aligned} \int x \sin x \, dx &= -x \cos x - \int (-\cos x) \, dx \\ &= -x \cos x + \sin x. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{2\pi}^{\frac{5}{2}\pi} x \sin x \, dx &= [-x \cos x + \sin x]_{2\pi}^{\frac{5}{2}\pi} \\ &= 1 - (-2\pi) = 1 + 2\pi. \end{aligned}$$

Therefore the expected value is the ratio of the integrals, i.e.,

$$\mu = \frac{1 + 2\pi}{1} = 1 + 2\pi.$$

## Sample Exam Questions

**Question 142** A certain random variable  $X$  has values in  $(1, \infty)$  and has the property that there is some constant  $C$  such that

$$P(X > a) = C \ln \frac{a^3 + 1}{a^3}$$

for every  $a > 1$ . Compute the value of  $C$  and determine whether the expected value  $\mu$  of  $X$  is finite or infinite. [Hint: There is enough information given to compute  $C$  without calculating any integrals.]

**Multiple Choice:**

- (a)  $C = \ln 2$  and  $\mu < \infty$
- (b)  $C = 1$  and  $\mu < \infty$
- (c)  $C = (\ln 2)^{-1}$  and  $\mu < \infty$  ✓
- (d)  $C = \ln 2$  and  $\mu = \infty$
- (e)  $C = 1$  and  $\mu = \infty$
- (f)  $C = (\ln 2)^{-1}$  and  $\mu = \infty$

**Feedback(attempt):** We know that  $X$  is always greater than one, so

$$1 = P(X > 1) = C \ln \frac{1+1}{1},$$

which gives  $C = (\ln 2)^{-1}$ . If we let  $f(x)$  denote the probability density function of  $X$ , then

$$\frac{1}{\ln 2} \ln \frac{a^3 + 1}{a^3} = P(X > a) = \int_a^\infty f(x) dx.$$

Differentiating both sides with respect to  $a$  gives

$$\frac{1}{\ln 2} \left[ \frac{3a^2}{a^3 + 1} - \frac{3}{a} \right] = -f(a)$$

so

$$f(a) = \frac{1}{\ln 2} \left[ \frac{3}{a} - \frac{3a^2}{a^3 + 1} \right] = \frac{3}{a(a^3 + 1) \ln 2}.$$

The expected value of  $X$  must equal

$$\int_1^\infty \frac{3a}{a(a^3 + 1) \ln 2} da = \frac{3}{\ln 2} \int_1^\infty \frac{da}{a^3 + 1}.$$

This integral will be finite by direct comparison to the convergent integral  $\int_1^\infty a^{-3} da$ .

**Question 143** The function

$$f(x) = \begin{cases} \frac{k}{x^3} & 1 \leq x < \infty \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for a certain value of  $k$ . For that probability density function, find the probability that  $x > 2$ .

**Multiple Choice:**

- (a)  $\frac{1}{2}$
  - (b)  $\frac{1}{3}$
  - (c)  $\frac{1}{4}$  ✓
  - (d)  $\frac{2}{3}$
  - (e)  $\frac{1}{5}$
  - (f)  $\frac{1}{6}$
- 

**Question 144** For a certain real number  $k$ , the function

$$f(X) = \begin{cases} \frac{k}{X^2 + 1} & \text{if } X \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is a probability density function for a continuous random variable  $X$ . For this value of  $k$ , find the probability that  $X > 1$ .

**Multiple Choice:**

- (a) 0
  - (b)  $\frac{1}{3}$
  - (c)  $\frac{2}{3}$
  - (d) 1
  - (e)  $\frac{1}{2}$  ✓
  - (f)  $\frac{1}{4}$
-

**Question 145** Let

$$f(r) = \begin{cases} Cr^2 e^{-2r/b} & r \geq 0 \\ 0 & r < 0 \end{cases}.$$

Find  $C$  so that this is a probability density function (pdf) for the random variable  $r$ . Here  $b$  is a positive constant. This function is used to model the distance between the nucleus and the electron in a hydrogen atom. The constant  $b$  is called the *Bohr length*. Find the mean of the pdf.

**Multiple Choice:**

- (a)  $C = \frac{b^3}{4}$ , mean =  $b$
- (b)  $C = \frac{4}{b^2}$ , mean =  $b$
- (c)  $C = \frac{4}{b}$ , mean =  $b^2$
- (d)  $C = \frac{4}{b^3}$ , mean =  $\frac{3}{2}b$  ✓
- (e)  $C = \frac{4}{b^2}$ , mean =  $\frac{3}{2}b^2$
- (f)  $C = \frac{4}{b}$ , mean =  $\frac{3}{2}b^3$

## 5 Sequences and Series

*We begin a study of sequences and series.*

In the next units, we take up the topic of sequences and series. These topics involve a substantial number of new ideas and are, on the surface, very different from the topics we have studied previously. As we move forward, however, we will see a number of ways in which sequences and series relate to the single variable calculus concepts of limits, differentiation, and integration.

## 5.1 Sequences

We study the mathematical concept of a sequence.

### (Video) Calculus: Single Variable

**Note:** Don't worry yet about the Taylor series for  $e^x$  which is briefly mentioned at 5:50. We will come to this idea shortly. YouTube link: <https://www.youtube.com/watch?v=i-PkHNpNuuA>

### Online Texts

- OpenStax II 5.1: Sequences<sup>48</sup>
- Ximera OSU: Sequences<sup>49</sup>
- Community Calculus 11.1: Sequences<sup>50</sup>

### Examples

**Example 39.** The base of a solid region is bounded by the curves  $x = 0$ ,  $y = 0$ , and  $y = \sqrt{1 - x^2}$ . The cross sections perpendicular to the  $x$  axis are squares. Compute the volume of the region.

**Solution:** Lines in the  $xy$ -plane which are perpendicular to the  $x$ -axis are vertical, so the base of a typical  $x$  cross section will extend from  $y = 0$  to  $y = \sqrt{1 - x^2}$ . Since each cross section will have area

$$A(x) = \left( \sqrt{1 - x^2} - 0 \right)^2 = 1 - x^2.$$

To compute volume, we integrate  $dV = A(x)dx$  between  $x = 0$  and  $x = 1$ , since these are the most extreme values of  $x$  found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3} \Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

<sup>48</sup>See OpenStax II 5.1: Sequences at <https://openstax.org/books/calculus-volume-2/pages/5-1-sequences>

<sup>49</sup>See Ximera OSU: Sequences at <https://ximera.osu.edu/mooculus/calculus2/sequences/titlePage>

<sup>50</sup>See Community Calculus 11.1: Sequences at [https://www.whitman.edu/mathematics/calculus\\_online/section11.01.html](https://www.whitman.edu/mathematics/calculus_online/section11.01.html)

**Example 40.** The base of a solid region is bounded by the curves  $y = 0$ ,  $x = \sqrt{y}$ , and  $x = 1$ . The cross sections perpendicular to the  $y$ -axis are squares. Compute the volume of the region.

**Solution:** Lines in the  $xy$ -plane which are perpendicular to the  $y$ -axis are (horizontal  $\checkmark$ / vertical), so the base of a typical  $y$  cross section will extend from the graph  $x = \sqrt{y}$  to the graph  $x = 1$ . The length of the base is the difference of  $x$ -coordinates (since all points on a slice have the same  $y$ -coordinate), so the length of the base is  $1 - \sqrt{y}$ , giving the square an area of

$$A(y) = (1 - \sqrt{y})^2$$

(note that the answer is a function of  $y$  because different  $y$  cross sections will generally have different areas). To compute volume, we integrate  $dV = A(y)dy$  between  $y = 0$  and  $y = 1$ , since these are the most extreme values of  $y$  found in our region (note that we can find the upper value  $y = 1$  by solving for the intersection of the curves  $x = \sqrt{y}$  and  $x = 1$ ). Therefore we integrate  $A(y)dy$  to conclude

$$V = \int_0^1 (1 - \sqrt{y})^2 dy = \frac{1}{6}.$$

series/18sequencepractice.tex

## 5.2 Exercises: Sequences

*Exercises relating to sequences.*

**Exercise 146** The sequence  $a_n = 1/n^3$  has limit  $L = 0$ . Suppose  $\epsilon = 1/64$ ; find a threshold  $N$  such that

$$|a_n - L| < \epsilon$$

is guaranteed to hold for all  $n > N$  (take your value of  $N$  as small as possible).

$$N = \boxed{4}.$$

**Exercise 147** The sequence  $a_n = (2n^2 + (-1)^n)/n^2$  has limit  $L = 2$ . Suppose  $\epsilon = 1/100$ ; find a threshold  $N$  such that

$$|a_n - L| < \epsilon$$

is guaranteed to hold for all  $n > N$  (take your value of  $N$  as small as possible).

$$N = \boxed{10}.$$

**Exercise 148** Determine the  $n^{\text{th}}$  term of the given sequence.  $a_1 = 4, a_2 = 7, a_3 = 10, a_4 = 13, a_5 = 16, \dots$

$$a_n = \boxed{3n + 1}.$$

**Exercise 149** Determine the  $n^{\text{th}}$  term of the given sequence.  $a_1 = 3, a_2 = -5/2, a_3 = 7/4, a_4 = -9/8, a_5 = 11/16, \dots$

$$a_n = \boxed{\frac{(-1)^{n-1}(2n+1)}{2^{n-1}}}.$$



**Exercise 150** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_n = \left\{ (-1)^n \frac{n}{n+1} \right\}$$

The sequence (converges/ diverges ✓) to  (enter N/A if the sequence does not converge to a finite answer).

---

**Exercise 151** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_n = \frac{4n^2 - n + 5}{3n^2 + 1}$$

The sequence (converges ✓/ diverges) to  (enter N/A if the sequence does not converge to a finite answer).

---

**Exercise 152** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_n = \frac{4^n}{5^n}$$

The sequence (converges ✓/ diverges) to  (enter N/A if the sequence does not converge to a finite answer).

---

**Exercise 153** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_n = \left( 1 - \frac{3}{n} \right)^{-n}$$

The sequence (converges ✓/ diverges) to  (enter N/A if the sequence does not converge to a finite answer).

**Hint:** Take the reciprocal and compare to your reference list of commonly-occurring limits.

---

**Exercise 154** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_n = \left(1 - \frac{3}{n}\right)^{-n^2}$$

The sequence (converges/ diverges ✓) to  (enter N/A if the sequence does not converge to a finite answer).

**Hint:** If a sequence  $b_n$  tends to  $e^3$ , what will the sequence  $(b_n)^n$  do?

---

**Exercise 155** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_n = \frac{(1.1)^n}{n}$$

The sequence (converges/ diverges ✓) to  (enter N/A if the sequence does not converge to a finite answer).

**Hint:** What are the relative orders of growth of numerator versus denominator?

---

**Exercise 156** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_n = \frac{(0.9)^n}{n}$$

The sequence (converges ✓/ diverges) to  (enter N/A if the sequence does not converge to a finite answer).

---

**Exercise 157** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_n = n^{1000000}(0.9)^n$$

The sequence (converges ✓/ diverges) to  (enter N/A if the sequence does not converge to a finite answer).

---

**Exercise 158** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_n = \frac{\ln n}{n}$$

The sequence (converges ✓/ diverges) to  (enter N/A if the sequence does not converge to a finite answer).

**Exercise 159** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_n = \frac{\ln n}{n^{0.00001}}$$

The sequence (converges ✓/ diverges) to  (enter N/A if the sequence does not converge to a finite answer).

**Exercise 160** Let  $b_n$  be the sequence given by

$$b_1 = 0 \text{ and } b_{n+1} = \frac{2 + b_n}{3} \text{ for } n \geq 1$$

converges. Compute its limit.

$$\lim_{n \rightarrow \infty} b_n = \text{}.$$

**Exercise 161** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_1 = 1 \text{ and } a_{n+1} = a_n + \frac{1}{a_n} \text{ for } n \geq 1$$

The sequence (converges/ diverges ✓) to  (enter N/A if the sequence does not converge to a finite answer).

**Exercise 162** Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

$$a_1 = 1 \text{ and } a_{n+1} = a_n + \frac{1}{4}(4 - a_n^2) \text{ for } n \geq 1$$

The sequence (converges ✓/ diverges) to 2 (enter N/A if the sequence does not converge to a finite answer).

**Exercise 163** Let  $a_n$  be the sequence given by

$$a_1 = \frac{1}{4}, \text{ and } a_{n+1} = 2a_n(1 - a_n) \text{ for } n \geq 1$$

converges. Compute its limit.

$$\lim_{n \rightarrow \infty} a_n = \boxed{\frac{1}{2}}.$$

**Hint:** If  $a_n$  happens to be positive and less than  $1/2$ , then  $2(1 - a_n) > 1$ , so this forces  $2a_n(1 - a_n) > a_n$  (meaning that the term after  $a_n$  will be larger than  $a_n$ ).

**Hint:** The function  $2x(1-x)$  is nonnegative on the interval  $[0, 1]$  and has a maximum value of  $1/2$  attained at  $x = 1/2$ . This means that if  $a_n$  is anything between 0 and 1, the next term of the sequence will always be between 0 and  $1/2$ .

## Sample Quiz Questions

**Question 164** Find the limit of the sequence

$$\lim_{n \rightarrow \infty} \sqrt{\frac{2n - 2}{2n^2 - 4n + 3}}.$$

Justify your response. (Hints will not be revealed until after you choose a response.)

**Multiple Choice:**

- (a) 0 ✓
- (b)  $\frac{1}{3}$
- (c)  $\frac{1}{2}$
- (d) 1
- (e) 2

(f) 3

**Feedback(attempt):** Because the square root function is continuous, you can pass the limit through it and compute

$$\sqrt{\lim_{n \rightarrow \infty} \frac{2n - 2}{2n^2 - 4n + 3}}.$$

**Hint:** Reduce numerator and denominator to the dominant terms (in the regime  $n \rightarrow \infty$ ).

**Hint:**

$$\begin{aligned} \lim_{n \rightarrow \infty} \sqrt{\frac{2n - 2}{2n^2 - 4n + 3}} &= \lim_{n \rightarrow \infty} \sqrt{\frac{2n^{-1} - 2n^{-2}}{2 - 4n^{-1} + 3n^{-2}}} \\ &= \sqrt{\frac{\lim_{n \rightarrow \infty} 2n^{-1} - 2n^{-2}}{\lim_{n \rightarrow \infty} 2 - 4n^{-1} + 3n^{-2}}} \\ &= \sqrt{\frac{0}{2}} = 0. \end{aligned}$$

**Question 165** Determine whether the limit below exists. If it exists, find its value.

$$\lim_{n \rightarrow \infty} \frac{(-1)^{n+1}n^2 - 2^{-n-1}}{(-1)^nn^2 + 4^{-n}}.$$

Justify your response. (Hints will not be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $-1$  ✓
- (b)  $0$
- (c)  $\frac{1}{2}$
- (d)  $2$
- (e)  $3$
- (f) limit does not exist

**Feedback(attempt):** Comparing the orders of growth of the terms in the numerator, the first term dominates because  $|-1| > |1/2|$ .

**Hint:** Likewise the first term dominates in the denominator because  $|-1| > |1/4|$ .

**Hint:** Neglecting non-dominant terms leads to the limit

$$\lim_{n \rightarrow \infty} \frac{(-1)^{n+1} n^2}{(-1)^n n^2}$$

which simply equals  $-1$ .

**Question 166** Determine whether the limit below exists. If it exists, find its value.

$$\lim_{n \rightarrow \infty} \left( \frac{4n-3}{4n+1} \right)^n.$$

Justify your response. (Hints will not be revealed until after you choose a response.)

**Multiple Choice:**

- (a) 0
- (b) 1
- (c)  $e^{-1}$  ✓
- (d)  $e$
- (e)  $e^2$
- (f) limit does not exist

**Feedback(attempt):** First observe that

$$\frac{4n-3}{4n+1} = 1 - \frac{4}{4n+1} \rightarrow 1$$

as  $n \rightarrow \infty$ .

**Hint:** Next, in light of the known limit  $(1 + x/k)^k \rightarrow e^x$  as  $k \rightarrow \infty$ , manipulate exponents to see that

$$\left( 1 - \frac{4}{4n+1} \right)^n = \left( \left( 1 - \frac{4}{4n+1} \right)^{4n+1} \right)^{1/4} \left( 1 - \frac{4}{4n+1} \right)^{-1/4}.$$

**Hint:** As  $n \rightarrow \infty$ , the first term on the right-hand side tends to  $e^{-1}$  and the second term tends to 1. Thus the original sequence tends to  $e^{-1}$  as well.

**Question 167** Determine whether the limit below exists. If it exists, find its value.

$$\lim_{n \rightarrow \infty} \left( \frac{n+3}{2n-2} \right)^{n^2}.$$

Justify your response. (Hints will not be revealed until after you choose a response.)

**Multiple Choice:**

- (a) 0 ✓
- (b) 1
- (c)  $e^{-1}$
- (d)  $e$
- (e)  $e^2$
- (f) limit does not exist

**Feedback(attempt):** First observe that

$$\frac{n+3}{2n-2} = \frac{1}{2} + \frac{2}{n-1} \rightarrow \frac{1}{2}$$

as  $n \rightarrow \infty$ .

**Hint:** Since the limit is positive and less than one, raising this expression to increasingly large powers generates a sequence which converges rapidly to zero.

## Sample Exam Questions

**Question 168** Determine whether the sequence  $a_n = (-1)^{n-1} \frac{n^2}{1+n^2+n^3}$  converges or diverges. If it converges, find its limit.

**Multiple Choice:**

- (a) divergent,  $\lim_{n \rightarrow \infty} a_n = 0$
- (b) convergent,  $\lim_{n \rightarrow \infty} a_n = 1$
- (c) convergent,  $\lim_{n \rightarrow \infty} a_n = 0$  ✓

- (d) convergent,  $\lim_{n \rightarrow \infty} a_n = -1$
- (e) divergent,  $\lim_{n \rightarrow \infty} a_n = \infty$
- (f) divergent, limit doesn't exist

## 5.3 Series

We introduce the concept of a series and study some fundamental properties.

### Online Texts

- OpenStax II 5.2: Infinite Series<sup>51</sup> and OpenStax II 5.3: The Divergence Test<sup>52</sup>
- Ximera OSU: Series<sup>53</sup> and Ximera OSU: The Divergence Test<sup>54</sup>
- Community Calculus 11.2: Series<sup>55</sup>

### Examples

**Example 41.** Find a formula for the partial sums of the series

$$\sum_{n=1}^{\infty} \left[ e^{\frac{1}{n}} - e^{\frac{1}{n+1}} \right]$$

and then compute the sum of the series.

<sup>51</sup>See OpenStax II 5.2: Infinite Series at <https://openstax.org/books/calculus-volume-2/pages/5-2-infinite-series>

<sup>52</sup>See OpenStax II 5.3: The Divergence Test at <https://openstax.org/books/calculus-volume-2/pages/5-3-the-divergence-and-integral-tests>

<sup>53</sup>See Ximera OSU: Series at <https://ximera.osu.edu/mooculus/calculus2/sumsOfSequences/titlePage>

<sup>54</sup>See Ximera OSU: The Divergence Test at <https://ximera.osu.edu/mooculus/calculus2/divergenceTest/titlePage>

<sup>55</sup>See Community Calculus 11.2: Series at [https://www.whitman.edu/mathematics/calculus\\_online/section11.02.html](https://www.whitman.edu/mathematics/calculus_online/section11.02.html)



- We can start by computing the first few partial sums:

$$S_1 = (e - e^{1/2})$$

$$S_2 = (e^1 - e^{1/2}) + (e^{1/2} - e^{1/3}) = e - e^{1/3}$$

$$S_3 = (e^1 - e^{1/2}) + (e^{1/2} - e^{1/3}) + (e^{1/3} - e^{1/4}) = e - e^{1/4}$$

$$S_4 = (e^1 - e^{1/2}) + (e^{1/2} - e^{1/3}) + (e^{1/3} - e^{1/4}) + (e^{1/4} - e^{1/5}) = e - e^{1/5}$$

We observe a general pattern that is consistent with a telescoping series:

$$S_n = e - \boxed{e^{1/(n+1)}}$$

- Now we let  $n \rightarrow \infty$ :

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \left( \boxed{e - e^{1/(n+1)}} \right) = e - \lim_{n \rightarrow \infty} \frac{1}{n+1} = e - \boxed{0} = e - 1.$$

Therefore

$$\sum_{n=1}^{\infty} \left[ e^{\frac{1}{n}} - e^{\frac{1}{n+1}} \right] = \boxed{e - 1}.$$

**Example 42.** Show that each series below is a geometric series; determine which ones are convergent, and for those that are convergent, find their sum.

$$\sum_{n=0}^{\infty} \frac{3^n}{2} \quad \sum_{n=0}^{\infty} \frac{2}{3^n} \quad \sum_{n=0}^{\infty} 2^{-n-2} \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^n}{4^n}$$

- First we rewrite each geometric series in standard form: i.e., write each term in the form  $ar^n$  where  $a$  and  $r$  are fixed quantities independent of  $n$ :

$$\sum_{n=0}^{\infty} \frac{3^n}{2} = \sum_{n=0}^{\infty} \boxed{\frac{1}{2}} \left[ \boxed{3} \right]^n$$

$$\sum_{n=0}^{\infty} \frac{2}{3^n} = \sum_{n=0}^{\infty} \boxed{2} \left[ \boxed{\frac{1}{3}} \right]^n$$

$$\sum_{n=0}^{\infty} 2^{-n-2} = \sum_{n=0}^{\infty} \boxed{\frac{1}{4}} \left[ \boxed{\frac{1}{2}} \right]^n$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^n}{4^n} = \sum_{n=0}^{\infty} \boxed{-1} \left[ \boxed{\frac{-3}{4}} \right]^n$$

- A geometric series is convergent exactly when  $|r| < \boxed{1}$ . This means that the following series are convergent (select all that apply):

**Select All Correct Answers:**

(a)  $\sum_{n=0}^{\infty} \frac{3^n}{2}$

(b)  $\sum_{n=0}^{\infty} \frac{2}{3^n}$  ✓

(c)  $\sum_{n=0}^{\infty} 2^{-n-2}$  ✓

(d)  $\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^n}{4^n}$  ✓

- This means that the given series have the sums (write N/A if the series diverges, otherwise give the sum)

$$\sum_{n=0}^{\infty} \frac{3^n}{2} = \boxed{N/A} \quad \sum_{n=0}^{\infty} \frac{2}{3^n} = \boxed{3} \quad \sum_{n=0}^{\infty} 2^{-n-2} = \boxed{\frac{1}{2}} \quad \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 3^n}{4^n} = \boxed{-\frac{4}{7}}.$$

**Example 43.** Reindex each series below to begin at the specified starting value:

Series 1:

**Hint:** Inside each term, replace every occurrence of  $n$  with  $n - 1$ .

$$\sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \boxed{\frac{1}{n}}$$

Series 2:

**Hint:** Inside each term, replace every occurrence of  $n$  with  $n + 1$ .

$$\sum_{n=2}^{\infty} e^{-2n-1} = \sum_{n=0}^{\infty} e^{\boxed{-2n-5}}$$

Series 3:

**Hint:** Inside each term, replace every occurrence of  $n$  with  $n + 1$ .

$$\sum_{n=3}^{\infty} (-1)^n \cos \frac{n^2 + n}{3} = \sum_{n=2}^{\infty} \boxed{(-1)^{n+1}} \cos \frac{\boxed{(n+1)^2 + (n+1)}}{3}$$

**Example 44.** Suppose the series  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  both converge and the series

$\sum_{n=1}^{\infty} c_n$  and  $\sum_{n=1}^{\infty} d_n$  both diverge. Then

- The series  $\sum_{n=1}^{\infty} (a_n + b_n)$  (converges ✓/ diverges/ may converge or diverge).
- The series  $\sum_{n=1}^{\infty} (a_n + c_n)$  (converges/ diverges ✓/ may converge or diverge).
- The series  $\sum_{n=1}^{\infty} (c_n + d_n)$  (converges/ diverges/ may converge or diverge ✓).

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**5.4 Exercises: Series***Exercises relating to fundamental properties of series.***Exercise 169** Compute the sum of the series:**Hint:** Split the series into two separate geometric series and evaluate each separately.

$$\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n} = \boxed{2}.$$

**Exercise 170** Compute the sum of the series:**Hint:** One way to proceed is to reindex the series so that it starts at  $n = 0$ .

$$\sum_{n=3}^{\infty} \frac{(-1)^{n+1} 2^n}{3^n} = \boxed{\frac{8}{45}}$$

**Exercise 171** Compute the sum of the series:

$$\sum_{n=1}^{\infty} \frac{3^{2n+1}}{2^{3n} \cdot 9^n} = \boxed{\frac{3}{7}}$$

**Hint:** Simplify as much as possible and write in standard form for a geometric series.**Exercise 172** • Give an example of a geometric series whose first term is 3 and which sums to 6. If no such series exists, enter N/A in both spaces below:

$$6 = \sum_{n=0}^{\infty} \boxed{3} \left( \boxed{\frac{1}{2}} \right)^n$$

• Give an example of a geometric series whose first term is 3 and which sums to 2. If no such series exists, enter N/A in both spaces below:

$$2 = \sum_{n=0}^{\infty} \boxed{3} \left( \boxed{-\frac{1}{2}} \right)^n$$

- Give an example of a geometric series whose first term is 3 and which sums to 1. If no such series exists, enter N/A in both spaces below:

$$1 = \sum_{n=0}^{\infty} \boxed{N/A} \left( \boxed{N/A} \right)^n$$

**Exercise 173** Find the correct value of the constant in the blank below which makes the sum of the series equal to zero:

$$0 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \boxed{8} - 2^n}{3^n}$$

**Hint:** One reasonable strategy is to separate the series into two geometric series and reindex to apply the standard formula.

**Exercise 174** Compute the sum of the infinite series below:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \boxed{\frac{1}{4}}.$$

**Hint:** There is a closely-related telescoping series:

$$\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} = \frac{2}{n(n+1)(n+2)}$$

**Exercise 175** Compute the sum of the infinite series below.

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \boxed{\frac{3}{2}}$$

**Hint:** Expand the terms using partial fractions and compute several partial sums by hand. What you get is something like a telescoping series, but cancellations occur in a slightly different way than usual.

**Hint:**

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

**Hint:** The formula for a general partial sum is

$$\frac{2}{1 \cdot 3} + \cdots + \frac{2}{N \cdot (N+2)} = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

**Exercise 176** • Use your answer to the previous exercise to compute the sum of the series:

$$\sum_{n=3}^{\infty} \frac{1}{n(n-2)} = \boxed{\frac{3}{4}}$$

• Use the answer you just found to compute the sum of the series:

$$\sum_{n=4}^{\infty} \frac{1}{n(n-2)} = \boxed{\frac{5}{12}}$$

**Hint:** To drop the  $n = 3$  term, you can just calculate it and then subtract it from the previous series.

## Sample Quiz Questions

**Question 177** Compute the exact value of the infinite series

$$\sum_{n=1}^{\infty} \ln \left( \frac{1 + n^{-1}}{1 + (n+1)^{-1}} \right).$$

**Multiple Choice:**

- (a)  $\ln 2$  ✓
- (b)  $\ln 3$
- (c)  $\ln 4$
- (d)  $\ln 5$
- (e)  $\ln 6$
- (f)  $\ln 7$

**Feedback(attempt):** The series is not a geometric series or Taylor series, we com-

pute the first few partial sums:

$$\begin{aligned} S_1 &= \ln \left( \frac{1+1}{1+2^{-1}} \right) = \ln \left( \frac{2}{\frac{3}{2}} \right) \\ S_2 &= \ln \left( \frac{1+1}{1+2^{-1}} \right) + \ln \left( \frac{1+2^{-1}}{1+3^{-1}} \right) = \ln \left( \frac{1+1}{1+3^{-1}} \right) = \ln \left( \frac{2}{\frac{4}{3}} \right) \\ S_3 &= \ln \left( \frac{2}{1+3^{-1}} \right) + \ln \left( \frac{1+3^{-1}}{1+4^{-1}} \right) = \ln \left( \frac{2}{1+4^{-1}} \right) = \ln \left( \frac{2}{\frac{5}{4}} \right) \\ &\vdots \\ S_n &= \ln \left( \frac{2}{1+(n+1)^{-1}} \right). \end{aligned}$$

In particular, writing the sum of logarithms as a logarithm of a product leads to substantial cancellation. By letting  $n \rightarrow \infty$ , we get  $S_n \rightarrow \ln 2$ .

## 5.5 Series Comparison Tests

We study the direct and limit comparison theorems for infinite series and practice their application.

### Online Texts

- OpenStax II 5.4: Comparison Tests<sup>56</sup>
- Ximera OSU: Comparison Tests<sup>57</sup>
- Community Calculus 11.5: Comparison Tests<sup>58</sup>

### Examples

**Example 45.**

**Example 46.**

<sup>56</sup>See OpenStax II 5.4: Comparison Tests at <https://openstax.org/books/calculus-volume-2/pages/5-4-comparison-tests>

<sup>57</sup>See Ximera OSU: Comparison Tests at <https://ximera.osu.edu/mooculus/calculus2/comparisonTests/titlePage>

<sup>58</sup>See Community Calculus 11.5: Comparison Tests at [https://www.whitman.edu/mathematics/calculus\\_online/section11.05.html](https://www.whitman.edu/mathematics/calculus_online/section11.05.html)

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## 5.6 Exercises: Series Comparison Tests

Exercises relating to the direct and limit comparison tests for series.

**Exercise 178** Suppose you wished to use the Direct Comparison Test to establish convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + n}.$$

Which of the following options below would be valid series for comparison? Select all that apply.

Select All Correct Answers:

(a)  $\sum_{n=2}^{\infty} \frac{1}{n^5}$

(b)  $\sum_{n=2}^{\infty} \frac{1}{n^4}$  ✓

(c)  $\sum_{n=2}^{\infty} \frac{1}{n^3}$  ✓

(d)  $\sum_{n=2}^{\infty} \frac{1}{n^2}$  ✓

(e)  $\sum_{n=2}^{\infty} \frac{1}{n}$

**Feedback(attempt):** Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^5}$ : This comparison wouldn't be valid because

$$\frac{1}{n^5} \leq \frac{1}{n^4 + n} \quad \text{for all } n \geq 2,$$

which is the wrong direction for the inequality when showing convergence.

Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^4}$ : This comparison is valid because

$$\frac{1}{n^4 + n} \leq \frac{1}{n^4} \quad \text{for all } n \geq 2,$$



Exercises: Series Comparison Tests

which is the proper direction for the comparison inequality when showing convergence. Also key is that the  $p$ -series for  $p = 4$  is convergent.

Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^3}$ : This comparison is valid because

$$\frac{1}{n^4 + n} \leq \frac{1}{n^3} \quad \text{for all } n \geq 2,$$

which is the proper direction for the comparison inequality when showing convergence. Also key is that the  $p$ -series for  $p = 3$  is convergent.

Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^2}$ : This comparison is valid because

$$\frac{1}{n^4 + n} \leq \frac{1}{n^2} \quad \text{for all } n \geq 2,$$

which is the proper direction for the comparison inequality when showing convergence. Also key is that the  $p$ -series for  $p = 2$  is convergent.

Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n}$ : This comparison is not valid because the series (known as the harmonic series) is not convergent, so no comparisons to it can establish convergence.

**Exercise 179** Suppose you wished to use the Direct Comparison Test to establish convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n^4 - n}.$$

Which of the following options below would be valid series for comparison? Select all that apply.

**Select All Correct Answers:**

(a)  $\sum_{n=2}^{\infty} \frac{1}{n^4}$

(b)  $\sum_{n=2}^{\infty} \frac{1}{n^3}$  ✓

(c)  $\sum_{n=2}^{\infty} \frac{1}{n^2}$  ✓

**Feedback(attempt):** Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^4}$ : This comparison is not valid because

$$\frac{1}{n^4} \leq \frac{1}{n^4 - n^2} \quad \text{for all } n \geq 2,$$

which is the wrong direction for the comparison inequality when showing convergence.

Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^3}$ : This comparison is valid because

$$\frac{1}{n^4 - n} \leq \frac{1}{n^3} \quad \text{for all } n \geq 2$$

(because  $n^4 - n = n(n^3 - 1)$  and  $n^3 - 1$  is always greater than  $n^2$  whenever  $n \geq 2$ .) which is the proper direction for the comparison inequality when showing convergence. Also key is that the  $p$ -series for  $p = 3$  is convergent.

Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^2}$ : This comparison is valid because

$$\frac{1}{n^4 - n} \leq \frac{1}{n^2} \quad \text{for all } n \geq 2,$$

(because  $n^4 - n = n(n^3 - 1)$  and  $n^3 - 1$  is always greater than  $n$  whenever  $n \geq 2$ .) which is the proper direction for the comparison inequality when showing convergence. Also key is that the  $p$ -series for  $p = 2$  is convergent.

**Exercise 180** Using only the comparison functions listed here:

$$\sum_{n=1}^{\infty} \frac{1}{1 + \sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{n+1}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} e^{-n},$$

which of the following series can be proved to either diverge or converge using the Direct Comparison Test? (Select all that apply.)

**Select All Correct Answers:**

(a)  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} + n^{1/2}}$

(b)  $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^{1/2}}$  ✓

(c)  $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$  ✓

(d)  $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$

(e)  $\sum_{n=1}^{\infty} \frac{1}{e^n + 1}$  ✓

(f)  $\sum_{n=3}^{\infty} \frac{1}{2n+3}$

**Feedback(attempt):** For the series  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} + n^{1/2}}$ : This series is convergent, but none of the allowed comparisons will work: Only  $1/(n+1)$  and  $1/(1+\sqrt{n})$  are larger than  $1/(n^{3/2} + n^{1/2})$ , but neither of those series converges.

For the series  $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^{1/2}}$ : Comparison to  $1/n^2$  works fine in this case.

For the series  $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ :  $\ln(n+1) \leq n+1$  so  $1/\ln(n+1) \geq 1/(n+1)$ , so actually diverges by direct comparison.

For the series  $\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}$ : This series is convergent, but none of the convergent comparison options  $1/n^2$  and  $e^{-n}$  is greater than  $1/n^2$ .

For the series  $\sum_{n=1}^{\infty} \frac{1}{e^n + 1}$ : The direct comparison  $1/(e^n + 1) \leq e^{-n}$  works fine to show convergence of the series.

For the series  $\sum_{n=3}^{\infty} \frac{1}{2n+3}$ : This is a divergent series, but direct comparison with the allowed options doesn't work because  $1/(2n+3)$  is smaller than the two divergent options  $1/(n+1)$  and  $1/(1+\sqrt{n})$ .

**Exercise 181** Suppose you wished to use the Limit Comparison Test to establish convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^4 + n}.$$

Which of the following options below would be valid series for comparison? Select all that apply.

**Select All Correct Answers:**

(a)  $\sum_{n=2}^{\infty} \frac{1}{n^5}$

(b)  $\sum_{n=2}^{\infty} \frac{1}{n^4}$  ✓

(c)  $\sum_{n=2}^{\infty} \frac{1}{n^3}$  ✓

(d)  $\sum_{n=2}^{\infty} \frac{1}{n^2}$  ✓

(e)  $\sum_{n=2}^{\infty} \frac{1}{n}$

**Feedback(attempt):** Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^5}$ : This comparison wouldn't be valid because  $1/n^5$  is too small: the limit of  $1/(n^4 + n)$  divided by  $1/n^5$  is infinite, which is an inconclusive case.

Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^4}$ : This comparison is valid because the limit  $1/(n^4 - n)$  divided by  $1/n^4$  is 1, which means that they both converge (because a  $p$ -series for  $p = 4$  is convergent).

Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^3}$ : This comparison is valid because  $1/n^3$  is a convergent  $p$ -series which is much larger than  $1/(n^4 + n)$ , i.e.,

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n^4 + n}}{\frac{1}{n^3}} = \lim_{n \rightarrow \infty} \frac{n^3}{n^4 + n} = 0.$$

Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n^2}$ : Also works for the same reason as  $1/n^3$ : it's a convergent  $p$ -series and the original series divided by the comparison series goes to zero.

Expand for feedback on  $\sum_{n=2}^{\infty} \frac{1}{n}$ : This comparison is not valid because the series (known as the *harmonic series*) is not convergent, so no comparisons to it can establish convergence.

**Exercise 182** Suppose you wished to use the Limit Comparison Test to establish convergence of the series

$$\sum_{n=2}^{\infty} \frac{1}{n^4 - n}.$$

Which of the following options below would be valid series for comparison?  
Select all that apply.

**Select All Correct Answers:**

(a)  $\sum_{n=2}^{\infty} \frac{1}{n^4}$  ✓

(b)  $\sum_{n=2}^{\infty} \frac{1}{n^3}$  ✓

(c)  $\sum_{n=2}^{\infty} \frac{1}{n^2}$  ✓

**Feedback(attempt):** For the Limit Comparison Test, non-dominant terms don't matter, so the results will be the same here as they were for the series  $1/(n^4 + n)$ .

**Exercise 183** Using only the comparison functions listed here:

$$\sum_{n=1}^{\infty} \frac{1}{1 + \sqrt{n}}, \quad \sum_{n=1}^{\infty} \frac{1}{n + 1}, \quad \sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} e^{-n}$$

, which of the following series can be proved to either diverge or converge using the Limit Comparison Test? (Select all that apply.)

**Select All Correct Answers:**

(a)  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} + n^{1/2}}$

(b)  $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^{1/2}}$  ✓

(c)  $\sum_{n=1}^{\infty} \frac{1}{\ln(n + 1)}$  ✓

(d)  $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$  ✓

(e)  $\sum_{n=1}^{\infty} \frac{1}{e^n + 1}$  ✓

$$(f) \sum_{n=3}^{\infty} \frac{1}{2n+3} \quad \checkmark$$

**Feedback(attempt):** For the series  $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} + n^{1/2}}$ : This series is convergent, but none of the allowed comparisons will work: Only  $1/(n+1)$  and  $1/(1+\sqrt{n})$  are larger than  $1/(n^{3/2} + n^{1/2})$ , but neither of those series converges.

For the series  $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^{1/2}}$ : Comparison to  $1/n^2$  works fine in this case.

For the series  $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ :  $\ln(n+1) \leq n+1$  so  $1/\ln(n+1) \geq 1/(n+1)$ , so the limit

$$\frac{n+1}{\ln(n+1)} \rightarrow \infty \text{ as } n \rightarrow \infty$$

so we get divergence by the divergence of the comparison series.

For the series  $\sum_{n=1}^{\infty} \frac{1}{n^2 - 1}$ : Here the non-dominant term  $-1$  in the denominator is no longer relevant, so limit comparison to  $1/n^2$  works fine.

For the series  $\sum_{n=1}^{\infty} \frac{1}{e^n + 1}$ : The direct comparison  $1/(e^n + 1)/e^{-n} \rightarrow 1$  works fine to show convergence of the series.

For the series  $\sum_{n=3}^{\infty} \frac{1}{2n+3}$ : Limit comparison with  $1/(n+1)$  works great:  $1/(2n+3)$  divided by  $1/(n+1)$  equals

$$\frac{n+1}{2n+3} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty.$$

## 5.7 The Ratio and Root Tests

We study the ratio and root tests for infinite series and practice their application.

### (Video) Calculus: Single Variable

YouTube link: [https://www.youtube.com/watch?v=D-E4lQY-w\\_c](https://www.youtube.com/watch?v=D-E4lQY-w_c)

## Online Texts

- OpenStax II 5.6: Ratio and Root Tests<sup>59</sup>
- Ximera OSU: Ratio and Root Tests<sup>60</sup>
- Community Calculus 11.7: Ratio and Root Tests<sup>61</sup>

## Examples

**Example 47.**

**Example 48.**

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<sup>59</sup>See OpenStax II 5.6: Ratio and Root Tests at <https://openstax.org/books/calculus-volume-2/pages/5-6-ratio-and-root-tests>

<sup>60</sup>See Ximera OSU: Ratio and Root Tests at <https://ximera.osu.edu/mooculus/calculus2/ratioAndRootTest/titlePage>

<sup>61</sup>See Community Calculus 11.7: Ratio and Root Tests at [https://www.whitman.edu/mathematics/calculus\\_online/section11.07.html](https://www.whitman.edu/mathematics/calculus_online/section11.07.html)

series/21ratirootpractice.tex

## 5.8 Exercises: Ratio and Root Tests

*Exercises relating to the Ratio and Root Tests.*

Note: As always, you can type the word “infinity” or “infty” (without quotes) in any entry box to indicate that the answer is infinite.

**Exercise 184** Consider the infinite series

$$\sum_{n=1}^{\infty} n2^{-n}.$$

Apply the Ratio Test:

$$\lim_{n \rightarrow \infty} \frac{(n+1)2^{-n-1}}{n2^{-n}} = \frac{1}{2}.$$

Apply the Root Test:

$$\lim_{n \rightarrow \infty} \left| n2^{-n} \right|^{\frac{1}{n}} = \frac{1}{2}.$$

Both tests indicate that the series (converges ✓/ diverges).

**Exercise 185** Oftentimes the Ratio Test is easier to apply than the Root Test when dealing with factorials. Use the Ratio Test to determine convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}.$$

In the spaces below, record the eponymous “ratio” in the first blanks, simplify it in the second blanks, and then record the limit.

$$\lim_{n \rightarrow \infty} \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4}.$$

The test (indicates convergence ✓/ indicates divergence/ is inconclusive).



**Exercise 186** Apply the Ratio Test to the series given below.

$$\sum_{n=0}^{\infty} \frac{5^n - 3n}{4^n}$$

$$\lim_{n \rightarrow \infty} \frac{\frac{5^{n+1} - 3(n+1)}{4^{n+1}}}{\frac{5^n - 3n}{4^n}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{4} \frac{5^{n+1} - 3(n+1)}{5^n - 3n}}{1} = \frac{5}{4}.$$

The test (indicates convergence/ indicates divergence ✓/ is inconclusive).

**Exercise 187** Apply the Ratio Test to the series given below.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

$$\lim_{n \rightarrow \infty} \frac{((n+1)^2 + 1)^{-1}}{(n^2 + 1)^{-1}} = \lim_{n \rightarrow \infty} \frac{n^2 + 1}{(n+1)^2 + 1} = 1.$$

The test (indicates convergence/ indicates divergence ✓/ is inconclusive ✓).

**Exercise 188** Apply the Ratio Test to the series given below.

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^n}$$

$$\lim_{n \rightarrow \infty} \frac{3^{n+1}/((n+1)^2 2^{n+1})}{3^n/(n^2 2^n)} = \lim_{n \rightarrow \infty} \frac{\frac{3}{2} \frac{n^2}{(n+1)^2}}{1} = \frac{3}{2}.$$

The test (indicates convergence/ indicates divergence ✓/ is inconclusive).

**Exercise 189** Apply the Ratio Test to the series given below.

$$\sum_{n=0}^{\infty} \frac{4^n}{4^n + 1}$$

$$\lim_{n \rightarrow \infty} \frac{\frac{4^{n+1}}{4^{n+1} + 1}}{\frac{4^n}{4^n + 1}} = \lim_{n \rightarrow \infty} 4 \frac{4^n + 1}{4^{n+1} + 1} = \boxed{1}.$$

The test (indicates convergence/ indicates divergence/ is inconclusive ✓).

**Exercise 190** Oftentimes the Root Test is easier to apply when terms have large exponents (growing faster than a constant times  $n$ ). Use the Root Test to determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} 4^n \left(1 - \frac{1}{n}\right)^{n^2}.$$

First say what quantity should have its  $n$ -th root taken, then simplify, and lastly record the value of the limit.

$$\lim_{n \rightarrow \infty} \left[ 4^n \left(1 - \frac{1}{n}\right)^{n^2} \right]^{\frac{1}{n}} = \lim_{n \rightarrow \infty} 4 \left(1 - \frac{1}{n}\right)^n = \boxed{\frac{4}{e}}.$$

The test (indicates convergence/ indicates divergence ✓/ is inconclusive).

**Exercise 191** Apply the Root Test to the series given below.

$$\sum_{n=1}^{\infty} 2^{-\ln n}$$

$$\lim_{n \rightarrow \infty} \left[ 2^{-\ln n} \right]^{\frac{1}{n}} = 2^{-\lim_{n \rightarrow \infty} \frac{\ln n}{n}} = \boxed{1}.$$

The test (indicates convergence/ indicates divergence/ is inconclusive ✓).

**Exercise 192** Apply the Root Test to the series given below.

$$\sum_{n=1}^{\infty} 2^{-n^2}$$

$$\lim_{n \rightarrow \infty} \left[ 2^{-n^2} \right]^{\frac{1}{n}} = \lim_{n \rightarrow \infty} 2^{-n} = \boxed{0}.$$

The test (indicates convergence ✓/ indicates divergence/ is inconclusive).

**Exercise 193** Apply the Root Test to the series given below.

$$\sum_{n=1}^{\infty} e^{\ln n} \left(1 - \frac{1}{n}\right)^{n^2}$$

$$\lim_{n \rightarrow \infty} \left| e^{\ln n} \left(1 - \frac{1}{n}\right)^{n^2} \right|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \boxed{e^{(\ln n)/n} \left(1 - \frac{1}{n}\right)^n} = \boxed{1/e}.$$

The test (indicates convergence ✓/ indicates divergence / is inconclusive).

**Exercise 194** Apply the Root Test to the series given below.

$$\sum_{n=1}^{\infty} \frac{e^{n^2}}{n^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{e^{n^2}}{n^n} \right|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \boxed{e^n/n} = \boxed{\infty}$$

The test (indicates convergence ✓/ indicates divergence / is inconclusive).

## Sample Quiz Questions

**Question 195** Determine which of the following three infinite series will lead to inconclusive results for the Ratio Test and then determine whether that series is convergent or divergent.

$$I: \sum_{k=1}^{\infty} \frac{1}{k - e^{-k}} \quad II: \sum_{m=1}^{\infty} \frac{1}{m^2 - e^m} \quad III: \sum_{l=1}^{\infty} \frac{e^{-l}}{l^2 + 1}$$

**Multiple Choice:**

- (a) I inconclusive, converges
- (b) I inconclusive, diverges ✓
- (c) II inconclusive, converges
- (d) II inconclusive, diverges
- (e) III inconclusive, converges

(f) III inconclusive, diverges

**Feedback(attempt):** The first series will give an inconclusive result for the Ratio Test because

$$\lim_{k \rightarrow \infty} \frac{k - e^{-k}}{k + 1 - e^{-k-1}} = \lim_{k \rightarrow \infty} \frac{1 - k^{-1}e^{-k}}{\frac{k+1}{k} - k^{-1}e^{-k-1}} = \frac{1-0}{1-0} = 1.$$

However, we know that the harmonic series diverges and that

$$\frac{1}{k - e^{-k}} > \frac{1}{k},$$

so by direct comparison to the harmonic series, series I must diverge.

## 5.9 The Integral Test

We study the integral test for infinite series and related concepts.

### (Video) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=TNcr4MAKR1s>

### Online Texts

- OpenStax II 5.3: The Integral Test<sup>62</sup>
- Ximera OSU: The Integral Test<sup>63</sup> and Remainders for the Integral Test<sup>64</sup>
- Community Calculus 11.3: The Integral Test<sup>65</sup>

### Examples

**Example 49.**

**Example 50.**

<sup>62</sup>See OpenStax II 5.3: The Integral Test at <https://openstax.org/books/calculus-volume-2/pages/5-3-the-divergence-and-integral-tests>

<sup>63</sup>See Ximera OSU: The Integral Test at <https://ximera.osu.edu/mooculus/calculus2/integralTest/titlePage>

<sup>64</sup>See Remainders for the Integral Test at <https://ximera.osu.edu/mooculus/calculus2/remainers/digInRemainersIntegralTest>

<sup>65</sup>See Community Calculus 11.3: The Integral Test at [https://www.whitman.edu/mathematics/calculus\\_online/section11.03.html](https://www.whitman.edu/mathematics/calculus_online/section11.03.html)

series/22integralpractice.tex

**5.10 Exercises: The Integral Test***Exercises relating to the integral test.***Exercise 196** The function

$$\frac{\cos \pi x}{x}$$

is decreasing for  $x \geq \boxed{N/A}$  (if the function is not ultimately decreasing, enter  $N/A$ ). The integral test (does/ does not ✓) apply to the series

$$\sum_{n=1}^{\infty} \frac{\cos \pi n}{n}.$$

The series (converges/ diverges/ can't be determined using this test ✓).

**Exercise 197** The function

$$\frac{\ln x}{x}$$

is decreasing for  $x \geq \boxed{e}$  (if the function is not ultimately decreasing, enter  $N/A$ ). The integral test (does ✓/ does not) apply to the series

$$\sum_{n=3}^{\infty} \frac{\ln n}{n}.$$

We have

$$\int_3^{\infty} \frac{\ln x}{x} dx = \boxed{\infty},$$

so the series (converges/ diverges ✓/ can't be determined using this test).

**Exercise 198** The function

$$\frac{1}{x^2 + 1}$$

is decreasing for  $x \geq \boxed{0}$  (if the function is not ultimately decreasing, enter  $N/A$ ). The integral test (does ✓/ does not) apply to the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}.$$

We have that

$$\int_1^{\infty} \frac{dx}{x^2 + 1} = \boxed{\frac{\pi}{4}},$$

so the series (converges ✓/ diverges/ can't be determined using this test).

---

**Exercise 199** The integral test (does ✓/ does not) apply to the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

The series (converges ✓/ diverges/ can't be determined using this test).

**Hint:** Make a substitution  $u = \ln x$ .

---

**Exercise 200** The integral test (does ✓/ does not) apply to the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}.$$

The series (converges/ diverges ✓/ can't be determined using this test).

**Hint:** Make a substitution  $u = \ln x$ .

---

**Exercise 201** The integral test (does/ does not ✓) apply to the series

$$\sum_{n=2}^{\infty} e^{-n^2} \cos n\pi.$$

The series (converges/ diverges/ can't be determined using this test ✓).

---

**Exercise 202** The integral test (does ✓/ does not) apply to the series

$$\sum_{n=2}^{\infty} n^3 e^{-n^2}.$$

The series (converges ✓/ diverges/ can't be determined using this test).

**Hint:** Make a substitution  $u = x^2$  and then integrate by parts.

**Exercise 203** The function

$$\frac{x}{(x^2 + 3)^2}$$

is decreasing for  $x \geq \boxed{1}$ . By the integral test,

$$\int_{\boxed{8}}^{\infty} \frac{x}{(x^2 + 3)^2} dx \leq \sum_{n=8}^{\infty} \frac{n}{(n^2 + 3)^2} \leq \int_{\boxed{7}}^{\infty} \frac{x}{(x^2 + 3)^2} dx.$$

We can approximate the infinite series by the sum of the first seven terms with what bounds on the error?

$$\frac{1}{\boxed{134}} + \sum_{n=1}^7 \frac{n}{(n^2 + 3)^2} \leq \sum_{n=1}^{\infty} \frac{n}{(n^2 + 3)^2} \leq \frac{1}{\boxed{104}} + \sum_{n=1}^7 \frac{n}{(n^2 + 3)^2}.$$

**Hint:** Write

$$\sum_{n=1}^{\infty} \frac{n}{(n^2 + 3)^2} = \sum_{n=1}^7 + \sum_{n=8}^{\infty} \frac{n}{(n^2 + 3)^2}$$

and then use the bounds we know for the “tail” (i.e., the sum over  $n \geq 8$ ).

**Exercise 204** Using the integral test, we can determine that the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is (equal to/ greater than/ less than  $\sqrt{2}$ ).

**Hint:** Compare the series to the partial sum of the first three terms. Don’t forget to include an estimate of the remainder.

**Hint:**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leq \sum_{n=1}^3 \frac{1}{n^2} + \int_3^{\infty} \frac{dx}{x^2}.$$

## Sample Quiz Questions

**Question 205** When approximating the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

by the sum of the first  $N$  terms, how large must  $N$  be to ensure that the approximation error is less than  $1/200$ ? Choose the smallest correct bound among those listed.

**Multiple Choice:**

- (a)  $N > 5$
- (b)  $N > 10$  ✓
- (c)  $N > 20$
- (d)  $N > 400$
- (e)  $N > 8000$
- (f)  $N > 160000$

**Feedback(attempt):** Because the terms  $n^{-3}$  are positive and decreasing, we know that the partial sums are always less than or equal to the sum of the series. By the Integral Test, we can further say that

$$\sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^N \frac{1}{n^3} \leq \int_N^{\infty} \frac{1}{x^3} dx = \frac{1}{2N^2}.$$

To be certain that the error is less than  $1/200$ , we set  $(2N^2)^{-1} < 1/200$ , which gives  $N > 10$ .

## 5.11 Alternating Series

We study the notion of alternating series and related concepts.

### (Video) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=E122N0GM2p4>

### Online Texts

- OpenStax II 5.5: Alternating Series<sup>66</sup>

<sup>66</sup>See OpenStax II 5.5: Alternating Series at <https://openstax.org/books/calculus-volume-2/pages/5-5-alternating-series>



- Ximera OSU: Alternating Series<sup>67</sup> and Remainders for Alternating Series<sup>68</sup> and Absolute and Conditional Convergence<sup>69</sup>
- Community Calculus 11.4: Alternating Series<sup>70</sup> and 11.6: Absolute Convergence<sup>71</sup>

## Examples

**Example 51.**

**Example 52.**

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<sup>67</sup>See Ximera OSU: Alternating Series at <https://ximera.osu.edu/mooculus/calculus2/alternatingSeries/titlePage>

<sup>68</sup>See Remainders for Alternating Series at <https://ximera.osu.edu/mooculus/calculus2/remainers/digInAlternatingSeriesTestRemainers>

<sup>69</sup>See Absolute and Conditional Convergence at <https://ximera.osu.edu/mooculus/calculus2/absoluteAndConditionalConvergence/titlePage>

<sup>70</sup>See Community Calculus 11.4: Alternating Series at [https://www.whitman.edu/mathematics/calculus\\_online/section11.04.html](https://www.whitman.edu/mathematics/calculus_online/section11.04.html)

<sup>71</sup>See 11.6: Absolute Convergence at [https://www.whitman.edu/mathematics/calculus\\_online/section11.06.html](https://www.whitman.edu/mathematics/calculus_online/section11.06.html)

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## 5.12 Exercises: Alternating Series

*Exercises relating to alternating series and absolute or conditional convergence.*

**Exercise 206** For the infinite series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n},$$

the function  $1/\ln n$  (is  $\checkmark$ / is not) positive and (is  $\checkmark$ / is not) decreasing when  $n \geq 2$ . Furthermore it (does  $\checkmark$ / does not) tend to zero as  $n \rightarrow \infty$ . The alternating series test (does  $\checkmark$ / does not) apply and (implies convergence  $\checkmark$ / implies divergence/ says nothing about the series).

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**Exercise 207** For the infinite series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n + (-1)^n \sqrt{\ln n}},$$

the function  $1/(\ln n + (-1)^n \sqrt{\ln n})$  (is  $\checkmark$ / is not) positive and (is  $\checkmark$ / is not  $\checkmark$ ) decreasing when  $n \geq 2$ . Furthermore it (does  $\checkmark$ / does not) tend to zero as  $n \rightarrow \infty$ . The alternating series test (does  $\checkmark$ / does not  $\checkmark$ ) apply and (implies convergence/ implies divergence/ says nothing about the series  $\checkmark$ ).

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**Exercise 208** For the infinite series

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln n - (\ln n)^{-1/2}},$$

the function  $1/(\ln n - (\ln n)^{-1/2})$  (is  $\checkmark$ / is not) positive and (is  $\checkmark$ / is not) decreasing when  $n \geq 3$ . Furthermore it (does  $\checkmark$ / does not) tend to zero as  $n \rightarrow \infty$ . The alternating series test (does  $\checkmark$ / does not) apply and (implies convergence  $\checkmark$ / implies divergence/ says nothing about the series).

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**Exercise 209** For the infinite series

$$\sum_{n=2}^{\infty} \frac{(-1)^n (n-1)^2}{n^2},$$

the function  $1/(\ln n + (-1)^n \sqrt{\ln n})$  (is ✓/ is not) positive and (is ✓/ is not) decreasing when  $n \geq 2$ . Furthermore it (does/ does not ✓) tend to zero as  $n \rightarrow \infty$ . The alternating series test (does/ does not ✓) apply and (implies convergence/ implies divergence/ says nothing about the series ✓).

**Exercise 210** For the infinite series

$$\sum_{n=1}^{\infty} \frac{\cos \ln n}{e^n},$$

the function  $1/(e^n)$  (is ✓/ is not) positive and (is ✓/ is not) decreasing when  $n \geq 1$ . Furthermore it (does ✓/ does not) tend to zero as  $n \rightarrow \infty$ . The alternating series test (does/ does not ✓) apply and (implies convergence/ implies divergence/ says nothing about the series ✓).

**Hint:** First determine whether the series is alternating.

**Exercise 211** Does the alternating series test apply to the series?

$$\sum_{n=5}^{\infty} \frac{\cos n\pi}{\sqrt{n}}$$

**Multiple Choice:**

- (a) Yes ✓
- (b) No, it's not alternating
- (c) No, the terms are not decreasing
- (d) No, the terms do not go to zero

If yes, for what minimum value of  $N$  can you be certain that

$$\sum_{n=5}^{N-1} \frac{\cos n\pi}{\sqrt{n}}$$

differs from the sum of the series by at most  $10^{-3}$ ? If no such  $N$  exists, write N/A.

$$N \geq \boxed{10^6}.$$

**Hint:** If the alternating series test applies, we would need the magnitude (i.e., absolute value) of the first term not included in the partial sum to be no greater than  $10^{-3}$ .

**Exercise 212** Find an interval of length  $\frac{1}{4}$  which contains the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

$$\text{Interval} = \left[ \frac{3}{4}, 1 \right]$$

**Hint:** Partial sums of an alternating series also alternate above and below the sum of the series itself.

## Sample Quiz Questions

**Question 213** For each series below, determine whether it converges absolutely (A), converges conditionally (C), or diverges (D). Show how you used convergence tests to arrive at your answer.

$$\text{I: } \sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt[3]{n+3}} \quad \text{II: } \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+3} \quad \text{III: } \sum_{n=2}^{\infty} \frac{\cos n\pi}{\ln(n^2+1)}$$

**Multiple Choice:**

- (a) I: C, II: D, III: D
- (b) I: C, II: A, III: C ✓
- (c) I: A, II: C, III: A
- (d) I: D, II: C, III: D
- (e) I: C, II: D, III: C
- (f) I: C, II: A, III: A

**Feedback(attempt):** I: converges conditionally. The value of  $\cos n\pi$  alternates  $\pm 1$ . The terms  $(n+3)^{-1/3}$  decrease to zero, so the series converges by the alternating series test. The series is not absolutely convergent because the  $p$ -series with  $p = -1/3$  is divergent.

II: converges absolutely. The series converges absolutely by direct comparison to a  $p$ -series with  $p = 2$ .

III: converges conditionally. The series converges by the alternating series test because  $1/\ln(n^2 + 1)$  decreases to 0 as  $n \rightarrow \infty$  and  $\cos n\pi$  alternates in value between +1 and -1. However,  $1/\ln(n^2 + 1) \geq 1/n$  for all large  $n$ , so by direct comparison to the harmonic series, the series is not absolutely convergent. Therefore the convergence is conditional.

**Question 214** For each series below, determine whether it converges absolutely (A), converges conditionally (C), or diverges (D). Show how you used convergence tests to arrive at your answer.

$$\text{I: } \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2n^2 + 1} \quad \text{II: } \sum_{n=1}^{\infty} \frac{1}{1 + n^3 e^{-n}} \quad \text{III: } \sum_{n=1}^{\infty} \frac{(-1)^n n + 2}{n^2}$$

**Multiple Choice:**

- (a) I: D, II: D, III: D
- (b) I: D, II: A, III: C
- (c) I: C, II: C, III: A
- (d) I: A, II: C, III: D
- (e) I: D, II: D, III: C ✓
- (f) I: D, II: A, III: A

**Feedback(attempt):** I: diverges. The series diverges because  $n^2/(n^2 + 1) \rightarrow 1$ , meaning that the terms do not go to zero. The  $n$ -th term divergence test implies divergence.

II: diverges. The series diverges because  $n/(n + n^3 e^{-n}) \rightarrow 1$  (because  $n^3 e^{-n} \rightarrow 0$ ). By the limit comparison theorem, this means the series has the same behavior as a  $p$ -series with  $p = 1$ , which means it diverges.

III: converges conditionally. The series converges because it is the sum of two convergent series: one with terms  $(-1)^n/n$  (which is a convergent series by the alternating series test because  $1/n$  decreases to zero) and a second with terms  $2/n^2$  (which is a convergent  $p$ -series). However, the series is not absolutely convergent, because

$$\left| \frac{(-1)^n n + 2}{n^2} \right| = \frac{n + (-1)^n 2}{n^2}$$

for  $n \geq 2$ , which is a sum of a divergent  $p$ -series with  $p = 1$  and an absolutely convergent alternating  $p$ -series with  $p = 2$ . Thus the series is conditionally convergent.

**Question 215** Which of the following intervals contains the value of the infinite series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}?$$

**Multiple Choice:**

(a)  $\left[\frac{1}{4}, \frac{1}{3}\right]$

(b)  $\left[\frac{1}{3}, \frac{1}{2}\right]$

(c)  $\left[\frac{1}{2}, \frac{7}{12}\right]$

(d)  $\left[\frac{7}{12}, \frac{5}{6}\right]$  ✓

(e)  $\left[\frac{5}{6}, \frac{11}{12}\right]$

(f)  $\left[\frac{11}{12}, \frac{7}{6}\right]$

**Feedback(attempt):** The function  $1/(n+1)$  is positive and decreases to zero, so by the Alternating Series Test, we know that partial sums alternate above and below the actual value of the sum. In particular, if we call the value of the sum  $L$ , then

$$\begin{aligned} 1 &\geq L \\ 1 - \frac{1}{2} &\leq L \\ 1 - \frac{1}{2} + \frac{1}{3} &\geq L \\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} &\leq L \end{aligned}$$

and so on. The last two inequalities together imply that  $L$  belongs to the interval  $\left[\frac{7}{12}, \frac{5}{6}\right]$ .

## Sample Exam Questions

**Question 216** Determine whether the following series converge absolutely (A), converge conditionally (C), or diverge (D). For full credit be sure to explain

your reasoning and specify which tests were used.

$$\sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n}}{3^n} \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

**Multiple Choice:**

- (a) both A
- (b) one A, the other C
- (c) one A, the other D
- (d) both C
- (e) one C, the other D ✓
- (f) both D

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## 5.13 Exercises: Cumulative

*Exercises relating to various topics we have studied.*

**Exercise 217** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n - e^{-n}}$$

**Multiple Choice:**

- (a) Absolute
- (b) Conditional ✓
- (c) Diverge

**Exercise 218** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n e^{-n}}{n - e^{-n}}$$

**Multiple Choice:**

- (a) Absolute ✓
- (b) Conditional
- (c) Diverge

**Hint:** Take absolute values and try direct comparison test.**Hint:** Try direct comparison to  $e^{-n}$ .**Exercise 219** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n - e^{-n}}$$

**Multiple Choice:**

- (a) Absolute
- (b) Conditional
- (c) Diverge ✓

**Hint:** What is the dominant term in the denominator as  $n \rightarrow \infty$ ?**Hint:** The dominant term in the denominator is  $n$ .**Hint:** This means that the terms do not go to zero.**Exercise 220** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + (-1)^n n}$$

**Multiple Choice:**

- (a) Absolute ✓
- (b) Conditional



(c) Diverge

**Hint:** What is the dominant term in the denominator as  $n \rightarrow \infty$ ?

**Hint:** The dominant term in the denominator is  $n^2$ .

**Exercise 221** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n - e^{-n}}$$

**Multiple Choice:**

- (a) Absolute
- (b) Conditional ✓
- (c) Diverge

**Exercise 222** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln(n+1)}$$

**Multiple Choice:**

- (a) Absolute
- (b) Conditional
- (c) Diverge ✓

**Hint:** Which term dominates as  $n \rightarrow \infty$ :  $\sqrt{n}$  or  $\ln(n+1)$ ?

**Hint:** Answer:  $\sqrt{n}$  dominates  $\ln(n+1)$  as  $n \rightarrow \infty$ .

**Exercise 223** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n} + \ln(n+1)}$$

**Multiple Choice:**

- (a) Absolute
- (b) Conditional ✓
- (c) Diverge

**Hint:** To show that convergence is not absolute, try limit comparison.

**Hint:** The comparison series can be taken to be  $1/\sqrt{n}$  in this case.

**Exercise 224** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} \ln \left( 2 + \frac{1}{n} \right)$$

**Multiple Choice:**

- (a) Absolute
- (b) Conditional
- (c) Diverge ✓

**Hint:** Try limit comparison with the harmonic series.

**Exercise 225** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{n \cos n\pi}{n^2 - 1}$$

**Multiple Choice:**

- (a) Absolute
- (b) Conditional ✓
- (c) Diverge

**Hint:** With absolute values, compare to a harmonic series.

**Hint:** How do we know that the alternating series test applies?

## Sample Exam Questions

**Question 226** Determine whether the following series converge or diverge.

$$I: \sum_{n=1}^{\infty} \frac{n^3}{n^4 + 4} \quad II: \sum_{n=1}^{\infty} \frac{3^n}{n!} \quad III: \sum_{n=2}^{\infty} \frac{\ln \ln n}{\ln n} \quad IV: \sum_{n=1}^{\infty} \frac{3n^2}{(n!)^2}$$

**Multiple Choice:**

- (a) I & II converge; III & IV diverge
- (b) I & III converge; II & IV diverge
- (c) I & IV converge; II & III diverge
- (d) II & III converge; I & IV diverge
- (e) II & IV converge; I & III diverge ✓
- (f) III & IV converge; I & II diverge

**Question 227** Determine whether the following series are convergent or divergent. Justify your answers.

$$I: \sum_{n=1}^{\infty} \frac{n^2 - 3n}{\sqrt[3]{n^{10} - 4n^2}} \quad II: \sum_{n=1}^{\infty} \frac{(-n)^n}{5^{2n+3}}$$

**Multiple Choice:**

- (a) I & II divergent
- (b) I convergent, II divergent ✓
- (c) I divergent, II convergent
- (d) I & II convergent

**Question 228** Determine whether the following series are convergent or divergent. Justify your answers.

$$I: \sum_{n=1}^{\infty} \frac{\arctan n}{n^4} \quad II: \sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{n^2}$$

**Multiple Choice:**

- (a) *I & II divergent*
- (b) *I convergent, II divergent*
- (c) *I divergent, II convergent*
- (d) *I & II convergent ✓*

**Question 229** Determine which of the following series are convergent. For full credit, be sure to explain your reasoning and specify which tests were used.

$$I: \sum_{n=2}^{\infty} 2ne^{-n^2} \quad II: \sum_{n=2}^{\infty} \frac{n + 2 \ln n}{2n^4} \quad III: \sum_{n=2}^{\infty} \frac{n^n}{n!}$$

**Multiple Choice:**

- (a) *only I*
- (b) *only I and II ✓*
- (c) *only I and III*
- (d) *only II*
- (e) *only II and III*
- (f) *only III*

## 6 Power Series

*We undertake a study of an important class of infinite series.*

**Power series** are a special and very important type of infinite series. These are series which involve a variable (usually  $x$  or  $t$ ) and are much like polynomials in the variable except that the degree of each term increases without limit. We will learn about the basic form of power series, determine when (for which values of the variable) the series is convergent, and use these series to approximate important functions (and sometimes use knowledge of special functions to evaluate the sum of the series).

## 6.1 Power Series

*We introduce the concept of a power series and some related fundamental properties.*

### (Video) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=S1aouJc4hoM>

### Online Texts

- OpenStax II 6.1: Power Series<sup>72</sup> and 6.2: Properties of Power Series<sup>73</sup>
- Ximera OSU: Power Series<sup>74</sup>
- Community Calculus 11.8: Power Series<sup>75</sup> and 11.9: Calculus with Power Series<sup>76</sup>

### Examples

**Example 53.**

**Example 54.**

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<sup>72</sup>See OpenStax II 6.1: Power Series at <https://openstax.org/books/calculus-volume-2/pages/6-1-power-series-and-functions>

<sup>73</sup>See 6.2: Properties of Power Series at <https://openstax.org/books/calculus-volume-2/pages/6-2-properties-of-power-series>

<sup>74</sup>See Ximera OSU: Power Series at <https://ximera.osu.edu/mooculus/calculus2/powerSeries/titlePage>

<sup>75</sup>See Community Calculus 11.8: Power Series at [https://www.whitman.edu/mathematics/calculus\\_online/section11.08.html](https://www.whitman.edu/mathematics/calculus_online/section11.08.html)

<sup>76</sup>See 11.9: Calculus with Power Series at [https://www.whitman.edu/mathematics/calculus\\_online/section11.09.html](https://www.whitman.edu/mathematics/calculus_online/section11.09.html)

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**6.2 Exercises: Power Series and Convergence***Exercises relating to power series and their convergence properties.***Exercise 230** Compute the radius of convergence of the power series below.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n + 1}$$

$$R = \boxed{2}$$

**Hint:** When computing a limit like

$$\lim_{n \rightarrow \infty} \frac{2^n + 1}{2^{n+1} + 1},$$

remember the effect of orders of growth:  $1 \ll 2^n$ , and so from the perspective of the limit, the +1's in both the numerator and denominator are both negligible.

**Exercise 231** Compute the radius of convergence of the power series below.

$$\sum_{n=1}^{\infty} \frac{(-3)^n (\ln n) (x-3)^n}{4^n}$$

$$R = \boxed{\frac{4}{3}}$$

**Hint:** When computing a limit like

$$\lim_{n \rightarrow \infty} (\ln n)^{1/n},$$

remember that we already know  $n^{1/n} \rightarrow 1$  and that  $\ln n \ll n$  as well, so we expect  $(\ln n)^{1/n} \rightarrow 1$  as well.

**Exercise 232** Compute the radius of convergence of the power series below.

$$\sum_{n=1}^{\infty} \frac{(-2)^n n! (x-3)^n}{3^n + n^2}$$

$$R = \boxed{0}$$

Exercises: Power Series and Convergence

**Exercise 233** Compute the interval of convergence for the power series below.

$$\sum_{n=1}^{\infty} \frac{(-2)^n (x-3)^n}{\ln n}$$

The left endpoint of the interval is  $\boxed{5/2}$ ; it (is/ is not  $\checkmark$ ) included in the interval of convergence. The right endpoint of the interval is  $\boxed{7/2}$ ; it is (is  $\checkmark$ / is not) included in the interval of convergence.

**Exercise 234** Reindex the series below:

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n+2} = \sum_{n=0}^{\infty} x^{\boxed{2n+3}} \frac{\boxed{2n+3}}{\boxed{n+3}}$$

**Exercise 235** Differentiate the series below term-by-term:

$$\frac{d}{dx} \sum_{n=0}^{\infty} \frac{n}{n+1} x^n = \sum_{n=1}^{\infty} \left[ \frac{n^2}{n+1} \right] x^{\boxed{n-1}}.$$

(Note that the  $n = 0$  term goes away because the derivative of a constant is zero.)

**Exercise 236** Integrate the series below term-by-term:

$$\int_0^x \left[ \sum_{n=0}^{\infty} \frac{x^n}{(2n+3)^2} \right] = \sum_{n=0}^{\infty} \left[ \frac{1}{(n+1)(2n+3)^2} \right] x^{\boxed{n+1}}.$$

**Exercise 237** For each step below, apply a term-by-term operation, a multiplication by a monomial, or a substitution to derive a new series formula from a known formula.

- Use the formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Exercises: Power Series and Convergence

to derive a summation formula for  $1/(1+x^2)$ , i.e.,

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} \boxed{(-1)^n} x^{\boxed{2n}}.$$

- Use the formula you derived above to develop a power series expansion for arctangent:

$$\arctan x = \sum_{n=0}^{\infty} \boxed{\frac{(-1)^n}{(2n+1)}} x^{\boxed{2n+1}}.$$

- The radius of convergence of this last series is  $R = \boxed{1}$ .

**Exercise 238** For each step below, apply a term-by-term operation, a multiplication by a monomial, or a substitution to derive a new series formula from a known formula.

- Use the formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

to determine the sum of the series below for  $-1 < x < 1$ :

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \boxed{-\ln|1-x|}.$$

(Don't forget absolute values if you need them.)

- Use the formula you just derived to determine the sum of the series

$$\sum_{n=0}^{\infty} \frac{x^{2n+3}}{n+1} = \boxed{-x \ln|1-x^2|}.$$

- Use the formula you just derived to determine the sum of the series

$$\sum_{n=0}^{\infty} \frac{2^{2n} x^{2n+3}}{n+1} = \boxed{-\frac{1}{4} x \ln|1-4x^2|}.$$



## Sample Quiz Questions

**Question 239** Find the full interval of convergence for the power series

$$\sum_{m=2}^{\infty} \frac{(-3)^m m^2 (x-5)^m}{\ln m}.$$

(Hints will not be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\left(\frac{14}{3}, \frac{16}{3}\right)$  ✓
- (b)  $\left[\frac{14}{3}, \frac{16}{3}\right)$
- (c)  $(2, 8]$
- (d)  $[2, 8]$
- (e)  $(-\infty, \infty)$

**Feedback(attempt):** First observe that

$$\begin{aligned} \frac{1}{R} &= \lim_{m \rightarrow \infty} \left| \frac{\frac{(-3)^{(m+1)}(m+1)^2}{\ln(m+1)}}{\frac{(-3)^m m^2}{\ln m}} \right| \\ &= \lim_{m \rightarrow \infty} \left| \frac{-3(m+1)^2 \ln m}{m^2 \ln(m+1)} \right| \\ &= 3 \end{aligned}$$

because

$$\lim_{m \rightarrow \infty} \frac{(m+1)^2 \ln m}{m^2 \ln(m+1)} = \lim_{m \rightarrow \infty} \frac{(m+1)^2}{m^2} \lim_{m \rightarrow \infty} \frac{\ln m}{\ln(m+1)} = 1$$

by virtue of l'Hospital's rule applied to both limits on the right-hand side.

**Hint:** This means that the radius equals  $1/3$ . At the endpoint  $x = 14/3$ , the series equals

$$\sum_{m=2}^{\infty} \frac{m^2}{\ln m},$$

which diverges by the  $n$ -th term divergence test because  $\lim_{m \rightarrow \infty} m^2 / \ln m = \infty \neq 0$ . At the endpoint  $x = 16/3$ , the series equals

$$\sum_{m=2}^{\infty} (-1)^m \frac{m^2}{\ln m},$$

which diverges for the same reason as the other endpoint, i.e., the terms do not go to zero.

**Question 240** Find the full interval of convergence for the power series

$$\sum_{m=2}^{\infty} \frac{(-4)^m (\ln m) (x-1)^m}{(-2)^m m}.$$

(Hints will not be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\left(\frac{1}{2}, \frac{3}{2}\right)$
- (b)  $\left[\frac{1}{2}, \frac{3}{2}\right)$  ✓
- (c)  $(-1, 3]$
- (d)  $[-1, 3]$
- (e)  $(-\infty, \infty)$

**Feedback(attempt):** First observe that

$$\begin{aligned} \frac{1}{R} &= \lim_{m \rightarrow \infty} \left| \frac{\frac{(-4)^{(m+1)} (\ln(m+1))}{(-2)^{(m+1)} (m+1)}}{\frac{(-4)^m (\ln m)}{(-2)^m m}} \right| \\ &= \lim_{m \rightarrow \infty} \left| \frac{-4(\ln(m+1))m}{-2(\ln m)(m+1)} \right| \\ &= 2 \end{aligned}$$

because

$$\lim_{m \rightarrow \infty} \frac{m \ln(m+1)}{(m+1) \ln m} = \lim_{m \rightarrow \infty} \frac{m}{m+1} \lim_{m \rightarrow \infty} \frac{\ln(m+1)}{\ln m} = 1$$

by virtue of l'Hospital's rule applied to both limits on the right-hand side.

**Hint:** This means that the radius equals  $1/2$ . At the endpoint  $x = 3/2$ , the series equals

$$\sum_{m=2}^{\infty} \frac{(\ln m)}{m},$$

which diverges by direct comparison to the harmonic series, i.e., the  $p$ -series with  $p = 1$ . At the endpoint  $x = 1/2$ , the series equals

$$\sum_{m=2}^{\infty} (-1)^m \frac{(\ln m)}{m},$$

which converges by the alternating series test because the sign of the terms alternates and  $\ln m/m$  decreases to zero as  $m \rightarrow \infty$ .

**Question 241** Find the full interval of convergence for the power series

$$\sum_{m=2}^{\infty} \frac{(-2)^m (\ln m) (x+4)^m}{6^m m}.$$

(Hints will not be revealed until after you choose a response.)

**Multiple Choice:**

- (a)  $\left(-\frac{13}{3}, -\frac{11}{3}\right)$
- (b)  $\left[-\frac{13}{3}, -\frac{11}{3}\right)$
- (c)  $(-7, -1]$  ✓
- (d)  $[-7, -1]$
- (e)  $(-\infty, \infty)$

**Feedback(attempt):** First observe that

$$\begin{aligned} \frac{1}{R} &= \lim_{m \rightarrow \infty} \left| \frac{\frac{(-2)^{(m+1)} (\ln(m+1))}{6^{(m+1)} (m+1)}}{\frac{(-2)^m (\ln m)}{6^m m}} \right| \\ &= \lim_{m \rightarrow \infty} \left| \frac{-2(\ln(m+1))m}{6(\ln m)(m+1)} \right| \\ &= \frac{1}{3} \end{aligned}$$

because

$$\lim_{m \rightarrow \infty} \frac{m \ln(m+1)}{(m+1) \ln m} = \lim_{m \rightarrow \infty} \frac{m}{m+1} \lim_{m \rightarrow \infty} \frac{\ln(m+1)}{\ln m} = 1$$

by virtue of l'Hospital's rule applied to both limits on the right-hand side.

**Hint:** This means that the radius equals 3. At the endpoint  $x = -7$ , the series equals

$$\sum_{m=2}^{\infty} \frac{(\ln m)}{m},$$

which diverges by direct comparison to the harmonic series, i.e., the  $p$ -series with  $p = 1$ . At the endpoint  $x = -1$ , the series equals

$$\sum_{m=2}^{\infty} (-1)^m \frac{(\ln m)}{m},$$

which converges by the alternating series test because the sign of the terms alternates and  $\ln m/m$  decreases to zero as  $m \rightarrow \infty$ .

**Question 242** Find the full interval of convergence for the power series

$$\sum_{m=1}^{\infty} \frac{\sqrt[3]{m}(x-2)^m}{m!}.$$

**Multiple Choice:**

- (a)  $(1, 3)$
- (b)  $[1, 3)$
- (c)  $(1, 3]$
- (d)  $[1, 3]$
- (e)  $(-\infty, \infty)$  ✓

**Feedback(attempt):** First observe that

$$\begin{aligned} \frac{1}{R} &= \lim_{m \rightarrow \infty} \left| \frac{\frac{\sqrt[3]{(m+1)}}{(m+1)!}}{\frac{\sqrt[3]{m}}{m!}} \right| \\ &= \lim_{m \rightarrow \infty} \left| \frac{\sqrt[3]{(m+1)}}{(m+1)\sqrt[3]{m}} \right| \\ &= 0 \end{aligned}$$

because  $m+1$  in the denominator tends to  $\infty$  and

$$\lim_{m \rightarrow \infty} \frac{\sqrt[3]{m+1}}{\sqrt[3]{m}} = \lim_{m \rightarrow \infty} (1 + m^{-1})^{1/3} = \left(1 + \lim_{m \rightarrow \infty} m^{-1}\right)^{1/3} = 1.$$

This means that the radius is infinite and the interval of convergence is  $(-\infty, \infty)$ .

## Sample Exam Questions

**Question 243** For which values of  $x$  does the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n4^n}$  converge?

**Multiple Choice:**

- (a)  $-3 < x < 5$

Exercises: Power Series and Convergence

- (b)  $-3 \leq x < 5$
  - (c)  $-3 < x \leq 5$  ✓
  - (d)  $-5 < x \leq 3$
  - (e)  $-5 \leq x < 3$
  - (f)  $-5 \leq x \leq 3$
- 

**Question 244** Find the interval of convergence of the power series below.

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^{\frac{3}{4}}(n^2+2)}$$

**Multiple Choice:**

- (a)  $\left(0, \frac{1}{2}\right]$
  - (b)  $\left[0, \frac{1}{2}\right]$  ✓
  - (c)  $\left(0, \frac{1}{2}\right)$
  - (d)  $\left[0, \frac{1}{2}\right)$
  - (e)  $\left(-\frac{1}{2}, 0\right]$
  - (f)  $(-\infty, \infty)$
- 

**Question 245** Find the interval of convergence of the power series  $\sum_{n=2}^{\infty} \frac{2^n(x+5)^n}{\sqrt[3]{n}}$ .

**Multiple Choice:**

- (a)  $\left[-\frac{11}{2}, -\frac{9}{2}\right]$
- (b)  $\left[-\frac{11}{2}, -\frac{9}{2}\right)$  ✓

(c)  $\left(-\frac{11}{2}, -\frac{9}{2}\right)$

(d)  $\left[\frac{9}{2}, \frac{11}{2}\right)$

(e)  $\left(\frac{9}{2}, \frac{11}{2}\right)$

(f)  $\left[\frac{9}{2}, \frac{11}{2}\right]$

## 6.3 Taylor Series

We introduce the notion of a Taylor Series.

### (Video) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=CU1f2cn0fok>

### Online Texts

- OpenStax II 6.3: Taylor Series<sup>77</sup>
- Ximera OSU: Taylor Series<sup>78</sup>
- Community Calculus 11.10: Taylor Series<sup>79</sup>

### Examples

**Example 55.**

**Example 56.**

<sup>77</sup>See OpenStax II 6.3: Taylor Series at <https://openstax.org/books/calculus-volume-2/pages/6-3-taylor-and-maclaurin-series>

<sup>78</sup>See Ximera OSU: Taylor Series at <https://ximera.osu.edu/mooculus/calculus2/introductionToTaylorSeries/titlePage>

<sup>79</sup>See Community Calculus 11.10: Taylor Series at [https://www.whitman.edu/mathematics/calculus\\_online/section11.10.html](https://www.whitman.edu/mathematics/calculus_online/section11.10.html)

powerseries/26taylorseriespractice.tex

## 6.4 Exercises: Taylor Series

Exercises relating to Taylor series and their computation.

**Exercise 246** Use the Maclaurin series for  $\sin x$  to give a complete formula for the Maclaurin series for  $x \sin x^2$ :

$$x \sin(x^2) = \sum_{m=0}^{\infty} \left[ \frac{(-1)^m}{(2m+1)!} \right] x^{4m+3}$$

**Exercise 247** Use the Maclaurin series for  $e^x$  and  $e^{-x}$  to give a complete formula for the Maclaurin series for  $\cosh x$ :

$$\cosh x = \sum_{n=0}^{\infty} \left[ \frac{1}{(2n)!} \right] x^{2n}.$$

**Hint:** Your formula should account for the fact that there are only even powers of  $x$ .

**Exercise 248** Compute the Taylor series of the function  $f(x) = \ln x$  centered at the point  $x_0 = 2$ .

$$\ln 2 + \left[ \frac{1}{2} \right] (x-2) + \left[ -\frac{1}{8} \right] (x-2)^2 + \left[ \frac{1}{24} \right] (x-2)^3 + \cdots = \ln 2 + \sum_{n=1}^{\infty} \left[ \frac{(-1)^{n-1} 2^{-n}}{n} \right] (x-2)^n$$

**Exercise 249** Compute the Maclaurin series of the function identified below.

$$\int_0^x \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n+1)!(2n+1)} \right] x^{2n+1}$$

## Sample Quiz Questions

**Question 250** Compute the first 4 nonzero terms in the Taylor series at  $x = 0$  of the function

$$\frac{d}{dx} [xe^{x^2}].$$

**Multiple Choice:**

(a)  $1 + 3x^2 + \frac{5}{2}x^4 + \frac{7}{6}x^6$  ✓

(b)  $-1 - 3x^2 + \frac{5}{2}x^4 + \frac{7}{6}x^6$

(c)  $1 - 6x^2 - \frac{5}{4}x^4 + \frac{7}{6}x^6$

(d)  $-1 - 6x^2 + \frac{5}{4}x^4 + \frac{7}{6}x^6$

(e)  $2 + 3x^2 + \frac{5}{4}x^4 + \frac{7}{6}x^6$

(f)  $2 - 3x^2 - \frac{5}{4}x^4 + \frac{7}{6}x^6$

**Feedback(attempt):** Compute the series in stages beginning with substitution into known series:

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots$$

$$xe^{x^2} = x + x^3 + \frac{1}{2}x^5 + \frac{1}{6}x^7 + \cdots$$

$$\frac{d}{dx} [xe^{x^2}] = 1 + 3x^2 + \frac{5}{2}x^4 + \frac{7}{6}x^6 + \cdots$$

**Question 251** Compute the first 4 nonzero terms in the Taylor series at  $x = 0$  of the function

$$\int_0^x (x \ln(1-x)) \, dx.$$

**Multiple Choice:**

(a)  $-\frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{2}{15}x^5 - \frac{1}{24}x^6$

(b)  $\frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{2}{15}x^5 - \frac{1}{24}x^6$



$$(c) -\frac{1}{6}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 - \frac{1}{24}x^6$$

$$(d) \frac{1}{6}x^3 + \frac{1}{4}x^4 - \frac{1}{15}x^5 - \frac{1}{24}x^6$$

$$(e) -\frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{15}x^5 - \frac{1}{24}x^6 \quad \checkmark$$

$$(f) -\frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{15}x^5 - \frac{1}{24}x^6$$

**Feedback(attempt):** Compute the series in stages beginning with substitution into known series:

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$x \ln(1-x) = -x^2 - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{1}{4}x^5 + \cdots$$

$$\int_0^x (x \ln(1-x)) \, dx = -\frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{15}x^5 - \frac{1}{24}x^6 + \cdots$$

## Sample Exam Questions

**Question 252** The first few nonzero terms of the Maclaurin series for  $f(x) = \ln(1 + \sin x)$  are:

**Multiple Choice:**

$$(a) 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3 + \cdots$$

$$(b) 1 + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{48}x^3 + \cdots$$

$$(c) x - \frac{1}{2}x^2 + \frac{1}{8}x^3 - \frac{1}{24}x^4 + \cdots$$

$$(d) 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{6}x^4 \cdots$$

$$(e) x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots \quad \checkmark$$

$$(f) 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4 + \cdots$$

(Hints will not be revealed until you choose your response.)

**Feedback(attempt):** The first few derivatives of  $f(x)$  are:

$$\begin{aligned} f(x) &= \ln(1 + \sin(x)) \\ f'(x) &= \frac{\cos x}{1 + \sin x} \\ f''(x) &= -\frac{\sin x}{1 + \sin x} - \frac{\cos^2 x}{(1 + \sin x)^2} \\ f'''(x) &= -\frac{\cos x}{1 + \sin x} + \frac{\sin x \cos x}{(1 + \sin x)^2} - \frac{2 \sin x \cos x}{(1 + \sin x)^2} + 2 \frac{\cos^3 x}{(1 + \sin x)^3} \end{aligned}$$

**Hint:** Evaluating at  $x = 0$  gives  $f(0) = \ln 1 = 0$ ,  $f'(0) = 1$ ,  $f''(0) = -1$ , and  $f'''(0) = 1$ . Therefore the series starts with the terms  $x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots$ .

**Question 253** Find the Taylor polynomial of degree 2 for  $f(x) = \sqrt{x+16}$  centered at  $x = 9$ .

**Multiple Choice:**

- (a)  $5 + \frac{4}{5}x + \frac{9}{250}x^2$
- (b)  $5 - \frac{3}{5}(x-5) + \frac{1}{125}(x-5)^2$
- (c)  $5 + \frac{1}{10}(x-9) - \frac{1}{1000}(x-9)^2$  ✓
- (d)  $5 + \frac{3}{5}(x-5) + \frac{8}{125}(x-5)^2$
- (e)  $5 + \frac{1}{5}(x-9) + \frac{16}{125}(x-9)^2$
- (f) none of these

**Question 254** Use the Taylor polynomial of degree 3 for  $f(x) = \ln(1+x)$  centered at  $x_0 = 0$  to approximate the value of  $\ln\left(\frac{3}{2}\right)$ .

**Multiple Choice:**

- (a)  $\frac{2}{3}$
- (b)  $\frac{3}{2}$

(c)  $\frac{15}{4}$

(d)  $\frac{5}{12}$  ✓

(e)  $\frac{9}{24}$

(f)  $\frac{11}{24}$

**Question 255** Let  $F(x)$  be the unique function that satisfies  $F(0) = 0$ ,  $F'(0) = 0$ , and  $F'(x) = \frac{1}{x} \sin x^3$  for all  $x \neq 0$ . Find the Taylor Series of  $F(x)$  centered at  $x_0 = 0$ .

**Multiple Choice:**

(a)  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$

(b)  $\sum_{n=0}^{\infty} \frac{(-1)^n (6n+3)x^{6n+2}}{(2n+1)!}$

(c)  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(6n+3)(2n+1)!}$  ✓

(d)  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n+1)!}$

(e)  $\sum_{n=0}^{\infty} \frac{(-1)^n (6n+2)x^{6n+2}}{(2n+1)!}$

(f)  $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(6n+3)(2n+1)!}$

## 6.5 Taylor Series Applications

We study the use of Taylor series for evaluating infinite series and limits.

## (Video) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=fUbBKN70Ag8> YouTube link:  
[https://www.youtube.com/watch?v=td\\_QqFXq0u4](https://www.youtube.com/watch?v=td_QqFXq0u4)

## Online Texts

- OpenStax II 6.4: Working with Taylor Series<sup>80</sup>
- Ximera OSU: Numbers and Taylor Series<sup>81</sup> and Calculus and Taylor Series<sup>82</sup>
- Community Calculus 11.11: Taylor's Theorem<sup>83</sup>

## Examples

**Example 57.**

**Example 58.**

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<sup>80</sup>See OpenStax II 6.4: Working with Taylor Series at <https://openstax.org/books/calculus-volume-2/pages/6-4-working-with-taylor-series>

<sup>81</sup>See Ximera OSU: Numbers and Taylor Series at <https://ximera.osu.edu/mooculus/calculus2/numbersAndTaylorSeries/digInNumbersAndTaylorSeries>

<sup>82</sup>See Calculus and Taylor Series at <https://ximera.osu.edu/mooculus/calculus2/calculusAndTaylorSeries/titlePage>

<sup>83</sup>See Community Calculus 11.11: Taylor's Theorem at [https://www.whitman.edu/mathematics/calculus\\_online/section11.11.html](https://www.whitman.edu/mathematics/calculus_online/section11.11.html)

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## 6.6 Exercises: Taylor Series Applications

Various exercises relating to the application of Taylor Series to other problems of interest.

**Exercise 256** Compute the limit

$$\lim_{n \rightarrow \infty} n^3 \left[ \arctan \frac{1}{n} - \frac{1}{n} \right].$$

- The first few terms of the Maclaurin series for  $\arctan x$  are

$$\arctan x = \boxed{x} - \boxed{\frac{x^3}{3}} + \boxed{\frac{x^5}{5}} + \cdots$$

- Plug in  $x = \frac{1}{n}$  and subtract  $1/n$ :

$$\arctan \frac{1}{n} - \frac{1}{n} = -\frac{1}{3n^3} + \boxed{\frac{1}{5n^5}} + \cdots$$

- Multiply by  $n^3$ , neglect all but the dominant term, and conclude

$$\lim_{n \rightarrow \infty} n^3 \left[ \arctan \frac{1}{n} - \frac{1}{n} \right] = \boxed{-\frac{1}{3}}$$

**Exercise 257** Compute the limit

$$\lim_{n \rightarrow \infty} \frac{\ln(n+3) - \ln n}{\ln(n+1) - \ln n}.$$

- First combine logarithms to simplify a bit:

$$\ln(n+3) - \ln n = \ln \boxed{1 + \frac{3}{n}} \quad \text{and} \quad \ln(n+1) - \ln n = \ln \boxed{1 + \frac{1}{n}}.$$

- Write out the first few terms of the Maclaurin series for  $\ln(1+x)$  and substitute in the appropriate expression involving  $n$ :

$$\ln(1+x) = \boxed{x} - \boxed{\frac{x^2}{2}} + \boxed{\frac{x^3}{3}} - \cdots$$

$$\ln(n+3) - \ln n = \boxed{\frac{3}{n}} - \boxed{\frac{9}{2n^2}} + \boxed{\frac{9}{n^3}} - \cdots$$

$$\ln(n+1) - \ln n = \boxed{\frac{1}{n}} - \boxed{\frac{1}{2n^2}} + \boxed{\frac{1}{3n^3}} - \cdots$$

- Multiply by  $n$ , neglect all but the dominant term (as  $n \rightarrow \infty$ ) to conclude

$$\lim_{n \rightarrow \infty} \frac{\ln(n+3) - \ln n}{\ln(n+1) - \ln n} = \boxed{3}.$$

**Exercise 258** Compute the limit

$$\lim_{x \rightarrow 1} \frac{\ln x - x + 1}{\sin \pi x}.$$

- Compute the first few terms of the Taylor series of numerator and denominator centered at  $x = 1$ :

$$\ln x - x + 1 = \boxed{0} + \boxed{0}(x-1) + \boxed{-\frac{1}{2}}(x-1)^2 + \boxed{\frac{1}{3}}(x-1)^3 + \cdots$$

$$\sin \pi x = \boxed{0} + \boxed{-\pi}(x-1) + \boxed{0}(x-1)^2 + \boxed{\frac{\pi^3}{6}}(x-1)^3 + \cdots$$

- Neglect all but the dominant terms in numerator and denominator. Note that dominant here means as  $x \rightarrow 1$ . Then take the limit  $x \rightarrow 1$ :

$$\lim_{x \rightarrow 1} \frac{\ln x - x + 1}{\sin \pi x} = \boxed{0}.$$

**Exercise 259** Use the remainder formula for Taylor series to determine which partial sums of the series

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

differ from  $e$  by at most  $1/100$ ?

- The remainder formula says that

$$e^1 - \sum_{n=0}^N \frac{1}{n!} = \frac{e^{\xi}}{(N+1)!} (1-0)^{N+1}$$

for some point  $\xi$  which depends on  $N$  and is somewhere in the interval  $(\boxed{0}, \boxed{1})$ . The upper bound for the magnitude of  $e^{\xi}$  on this interval is  $M = \boxed{e}$ .

Exercises: Taylor Series Applications

- For which values of  $N$  do we have

$$\frac{M}{(N+1)!} \leq \frac{1}{50}?$$

(Check manually for small values of  $N$ ; use the fact that  $e$  is between 2 and 3.)

$$N \geq \boxed{4}.$$

**Exercise 260** In this exercise, we will investigate two different ways of numerically approximating the value of  $\ln 2$  using infinite series.

- The Maclaurin series of the function  $\ln(1+x)$  converges conditionally at  $x = 1$  to  $\ln 2$ . Compute the Maclaurin series:

$$\ln(1+x) = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{n+1} \right] x^{n+1}$$

(Reindex your answer to match the template above if your answer doesn't work as-is.)

- What degree Taylor polynomial would you need to use to approximate the value of  $\ln 2$  to an error strictly less than  $10^{-3}$ ? Answer: Taylor polynomial used should have degree  $N = \boxed{1000}$  or greater.
- A similar but distinct strategy would be to compute  $\ln(1/2)$  instead because we know  $-\ln(1/2) = \ln 2$ . Evaluating the Maclaurin series at  $x = -1/2$  and doing a little simplification, we see from the above series that

$$\ln 2 = \sum_{n=1}^{\infty} \left[ \frac{1}{n2^n} \right].$$

(Once again, reindex if your answer does not already start at  $n = 1$ .)

- The presence of a factor exponential in  $n$  suggests that this second series converges to  $\ln 2$  (much slower / slightly slower / slightly faster / much faster ✓) than the first series. Of the two series, then, the (first / second ✓) series presents a more efficient way to compute  $\ln 2$  numerically.

**Hint:** For the first estimation, use the series estimation features of the Alternating Series Test.

**Hint:** For an alternating series, the error of approximating the series by a partial sum is never greater than the magnitude of the first term omitted.

**Exercise 261** Use the remainder formula for Taylor series to determine which partial sums of the series

$$\sum_{n=1}^{\infty} \frac{1}{n3^n}$$

differ from  $\ln(3/2)$  by at most  $10^{-2}$ ?

- Guided by the example above, we will use the Maclaurin series

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

First, compute the derivatives of  $-\ln(1-x)$ . Find a pattern which holds for all  $N$ :

$$\frac{d^{N+1}}{dx^{N+1}} (-\ln(1-x)) = \boxed{\frac{N!}{(1-x)^{N+1}}}.$$

When  $x$  is between 0 and  $1/3$ , the largest value of the  $(N+1)$ -st derivative is what?

$$\left| \frac{d^{N+1}}{dx^{N+1}} (-\ln(1-x)) \right| \leq \boxed{N!3^{N+1}/2^{N+1}}.$$

- Using this upper bound for the  $(N+1)$ -st derivative, we know that

$$\left| \ln \frac{3}{2} - \sum_{n=0}^N \frac{1}{n3^n} \right| \leq \boxed{\frac{1}{(N+1)2^{N+1}}}$$

(use the upper bound and the remainder formula.)

- For which values of  $N$  is the expression you just found less than  $1/100$ ? Check by hand for smallish values of  $N$  to find the smallest one which works.

$$N \geq \boxed{4}.$$

## Sample Quiz Questions

**Question 262** Use Taylor series to estimate the value of

$$\sqrt[3]{\frac{11}{10}}$$

to within an error of at most  $1/900$ . (Hints will not be revealed until you choose a response.)



**Multiple Choice:**

- (a)  $\frac{31}{30}$  ✓
- (b)  $\frac{47}{45}$
- (c)  $\frac{19}{18}$
- (d)  $\frac{16}{15}$
- (e)  $\frac{97}{90}$
- (f)  $\frac{49}{45}$

**Feedback(attempt):** We may use the remainder formula for Taylor series to approach this problem. Suppose  $p_n(x)$  is the degree  $n$  Taylor polynomial of the function

$$f(x) = \sqrt[3]{1+x}$$

with center  $a = 0$ . Then the error  $E_n(x)$ , i.e., the difference between the polynomial and the function, does not exceed  $\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$ , where  $\xi$  is some unknown point in the range  $0 \leq \xi \leq x$ .

**Hint:** In this case one should take  $x = 1/10$  and determine how many derivatives are required to make this error estimate less than the given threshold.

**Hint:** This means checking by hand for small numbers of derivatives. For the specific problem at hand, if we approximate  $f(x)$  by the Taylor polynomial of degree  $n = 1$ , we have

$$\left| \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \right| = \left| \frac{x^{n+1}}{(n+1)!} \left( -\frac{2}{9}(\xi+1)^{-5/3} \right) \right| \leq \left| \frac{x^{n+1}}{(n+1)!} \left( \frac{2}{9} \right) \right| = \frac{1}{900}$$

when  $x = 1/10$ .

**Hint:** We conclude that the correct Taylor approximation is

$$p_n\left(\frac{1}{10}\right) = \left(\frac{1}{10}\right)^0 + \frac{1}{3}\left(\frac{1}{10}\right)^1 = 1 + \frac{1}{30} = \frac{31}{30}.$$

**Question 263** Use Taylor series to estimate the value of

$$e^{-\frac{1}{3}}$$

to within an error of at most  $1/162$ .

**Multiple Choice:**

- (a)  $\frac{5}{9}$
- (b)  $\frac{13}{18}$  ✓
- (c)  $\frac{8}{9}$
- (d)  $\frac{19}{18}$
- (e)  $\frac{11}{9}$
- (f)  $\frac{25}{18}$

**Feedback(attempt):** We may use the remainder formula for Taylor series to approach this problem. Suppose  $p_n(x)$  is the degree  $n$  Taylor polynomial of the function

$$f(x) = e^{-x}$$

with center  $a = 0$ . Then the error  $E_n(x)$ , i.e., the difference between the polynomial and the function, does not exceed  $\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$ , where  $\xi$  is some unknown point in the range  $0 \leq \xi \leq x$ .

**Hint:** In this case one should take  $x = 1/3$  and determine how many derivatives are required to make this error estimate less than the given threshold.

**Hint:** This means checking by hand for small numbers of derivatives. For the specific problem at hand, if we approximate  $f(x)$  by the Taylor polynomial of degree  $n = 2$ , we have

$$\left| \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi) \right| = \left| \frac{x^{n+1}}{(n+1)!} (e^{-\xi}) \right| \leq \left| \frac{x^{n+1}}{(n+1)!} (1) \right| = \frac{1}{162}$$

when  $x = 1/3$ .

**Hint:** We conclude that the correct Taylor approximation is

$$p_n\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^0 - 1\left(\frac{1}{3}\right)^1 + \frac{1}{2}\left(\frac{1}{3}\right)^2 = 1 - \frac{1}{3} + \frac{1}{18} = \frac{13}{18}.$$

## 6.7 Exercises: Cumulative

Exercises relating to various topics we have studied.

**Exercise 264** The series

$$\sum_{n=1}^{\infty} \left( e^{\frac{1}{n}} - 1 \right)$$

is (convergent / divergent ✓) by limit comparison to the  $p$ -series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

with  $p = \boxed{1}$ . Likewise, the series

$$\sum_{n=1}^{\infty} \left( e^{\frac{1}{n}} - 1 - \frac{1}{n} \right)$$

is (convergent ✓ / divergent) by limit comparison to the  $p$ -series with  $p = \boxed{2}$ .

**Hint:** Use a Maclaurin series to determine the dominant behavior of

$$e^{\frac{1}{n}} - 1$$

as  $n \rightarrow \infty$ .

**Exercise 265** Fill in the blank below with an appropriate constant to make the series absolutely convergent:

$$\sum_{n=1}^{\infty} \left[ \ln \left( 1 + \frac{1}{n} \right) - \frac{\boxed{1}}{n} \right]$$

**Hint:** Use the Maclaurin series for  $\ln(1+x)$  and substitute  $x = 1/n$ .

**Exercise 266** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} n \sin \frac{1}{n}$$

**Multiple Choice:**

- (a) Absolute
- (b) Conditional
- (c) Diverge ✓

**Hint:** Show that the terms do not go to zero as  $n \rightarrow \infty$ .

---

**Exercise 267** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$$

**Multiple Choice:**

- (a) Absolute ✓
- (b) Conditional
- (c) Diverge

**Hint:** Use the Maclaurin series for  $\sin x$  and evaluate at  $x = 1/n$ . Find the dominant term.

---

**Exercise 268** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} \ln \left( 2 + \frac{1}{n} \right)$$

**Multiple Choice:**

- (a) Absolute
- (b) Conditional
- (c) Diverge ✓

**Hint:** Try limit comparison with the harmonic series.

---

**Exercise 269** Determine whether the series below converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} \ln \left( 1 + \frac{1}{n} \right)$$

**Multiple Choice:**

- (a) Absolute ✓
- (b) Conditional
- (c) Diverge

**Hint:** Do a Taylor expansion of  $\ln(1+x)$  and evaluate at  $x = 1/n$ .

**Exercise 270** Compute the sum of the series below.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{1}{3^n}$$

- We know the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \boxed{\arctan x}$$

for  $-1 < x < 1$ .

- This means

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2n+1} = \boxed{\frac{\arctan x}{x}}.$$

- We conclude

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{1}{3^n} = \boxed{\frac{\pi\sqrt{3}}{6}}.$$

**Exercise 271** Compute the sum of the series below.

$$\sum_{n=0}^{\infty} n 3^{-n} = \boxed{\frac{3}{4}}.$$

**Hint:**

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{when } -1 < x < 1.$$

**Hint:**

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1} \quad \text{when } -1 < x < 1.$$

**Exercise 272** Compute the sum of the series below.

$$\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = \boxed{3}.$$

**Hint:** You don't need Taylor series for this one.**Hint:** It's a telescoping series

## Sample Exam Questions

**Question 273** If it converges, find the sum of the series  $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!}$ . If the series diverges, explain why.

**Multiple Choice:**

- (a)  $\ln 2$
- (b)  $\ln 3 - \ln 2$
- (c)  $e^{-2}$
- (d)  $\frac{1}{2}$  ✓
- (e)  $\frac{2}{e}$
- (f) diverges

**Feedback(attempt):** We recognize the Taylor series for cosine:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

The series in question is exactly

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\pi/3)^{2n}}{(2n)!} = \cos \frac{\pi}{3} = \frac{1}{2}.$$

**Question 274** What is the limit of the sequence  $\left\{n^2 \left(1 - \cos \frac{1}{n}\right)\right\}$ ?

**Multiple Choice:**

- (a) 1
- (b)  $-1$
- (c)  $\frac{\sqrt{3}}{2}$
- (d)  $\frac{1}{2}$  ✓
- (e)  $-\frac{\sqrt{3}}{2}$
- (f) diverges

**Question 275** Find the limit of the sequence

$$a_n = \{n [\ln(n+3) - \ln n]\}.$$

**Multiple Choice:**

- (a) 0
- (b) 1
- (c)  $\ln 3$
- (d) 3 ✓
- (e)  $\infty$
- (f) the limit does not exist

## 7 Ordinary Differential Equations

We begin a study of first-order ordinary differential equations.

## Ordinary Differential Equations

In the following sections, we will study some basic *ordinary differential equations*, which are equations that involve an unknown function (rather than simply an unknown number) and its derivatives. In our case, we will restrict attention to *first-order* ODEs, which involve only the unknown function  $y(x)$ , its parameter  $x$ , and its derivative  $y'(x)$ . We will discuss how to numerically and graphically approximate solutions of such ODEs, study two important types of first-order ODEs, and see some applications of these ODEs to various geometric and physical problems.



## 7.1 ODEs: Foundations

*We study the fundamental concepts and properties associated with ODEs.*

### (Video) Calculus: Single Variable

**Note:** For now you can begin the video at around 1:45. We will discuss linear and separable ODEs shortly. The midpoint method and Runge-Kutta beginning around 12:15 are important things to understand, but you will not be expected to compute these yourself like you will for Euler's method. YouTube link: <https://www.youtube.com/watch?v=UISSWvg1pg0>

### Online Texts

- OpenStax II 4.1: ODEs<sup>84</sup> and Direction Fields and Numerical Methods<sup>85</sup>
- Ximera OSU: ODEs<sup>86</sup> and Numerical Methods<sup>87</sup>

### Examples

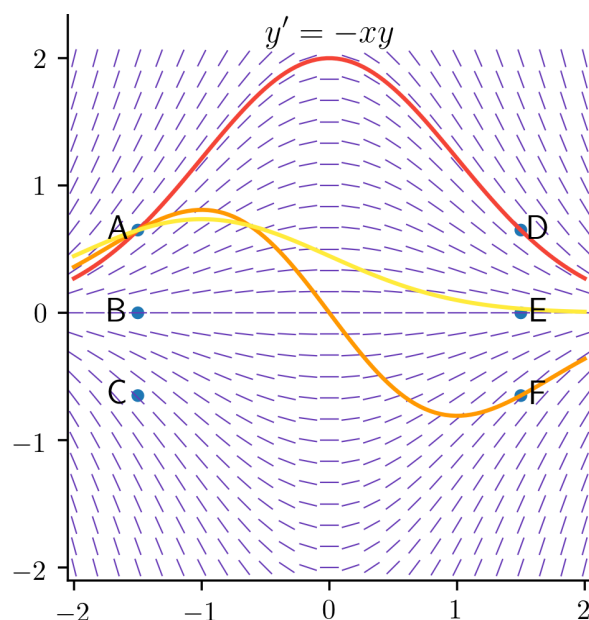
**Example 59.** *Below you will find a slope field for the ODE  $y' = -xy$ .*

<sup>84</sup>See OpenStax II 4.1: ODEs at <https://openstax.org/books/calculus-volume-2/pages/4-1-basics-of-differential-equations>

<sup>85</sup>See Direction Fields and Numerical Methods at <https://openstax.org/books/calculus-volume-2/pages/4-2-direction-fields-and-numerical-methods>

<sup>86</sup>See Ximera OSU: ODEs at <https://ximera.osu.edu/mooculus/calculus2/differentialEquations/titlePage>

<sup>87</sup>See Numerical Methods at <https://ximera.osu.edu/mooculus/calculus2/numericalMethods/titlePage>



There are three curves shown:

In Red:  $y = 2e^{-\frac{x^2}{2}},$

In Orange:  $y = -\frac{4x}{3}e^{-\frac{x^2}{2}},$

In Yellow:  $y = e^{-\frac{x^2}{2}-x-\frac{3}{2}}.$

Which of the three curves is a solution of the ODE  $y' = -xy$ ? In other words, which of the three curves is correctly aligned with the slope field?

**Multiple Choice:**

- (a) Red ✓
- (b) Orange
- (c) Yellow

There are also a number of points marked on the graph: point B at  $(-1.5, 0)$ , point C at  $(-1.5, -0.65)$ , point D at  $(1.5, 0.65)$ , point E at  $(1.5, 0)$ , and point F at  $(1.5, -0.65)$ . The solution of the ODE which begins at the point B will pass through the point (D/ E ✓/ F). Similarly, the solution which begins at C will pass through (D/ E/ F ✓).

**Example 60.** Can two different solutions of the same first-order ODE  $y' = f(x, y)$  have graphs which cross?

**Multiple Choice:**

- (a) Yes: There are infinitely many solutions, so they can always cross.
- (b) Yes: You can generally have many different initial conditions, so they can be arranged to cross.
- (c) No: There is only one solution passing through any horizontal line.
- (d) No: If there were a point of intersection, the tangent lines would be the same so the curves couldn't actually cross each other there. ✓

**Example 61.** The concept behind Euler's method is that we approximate a solution of an ODE of the form  $y' = f(x, y)$  with a polygonal graph. Usually each segment of the graph has the same width (referred to as the step size). We use the function  $f(x, y)$  to dictate the slope of each piece of the approximation. Let  $y(x)$  be the solution to the initial value problem  $y' = x + \frac{\ln y}{\ln 2} - \frac{y}{2}$  with  $y(0) = 1$ . Use Euler's Method with step size  $h = 1$  to approximate the value of  $y(3)$ .

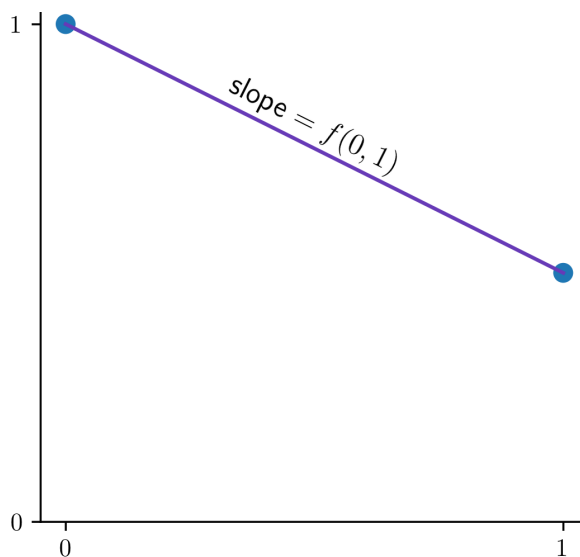
- For this particular ODE, we have

$$f(x, y) = \boxed{x + \frac{\ln y}{\ln 2} - \frac{y}{2}}.$$

- Supposing that we start at the point  $x = 0, y = 1$ , the slope will be

$$f(0, 1) = \boxed{-\frac{1}{2}}.$$

- Consider a line segment beginning at the point  $(0, 1)$  having slope  $f(0, 1)$  which you just calculated.

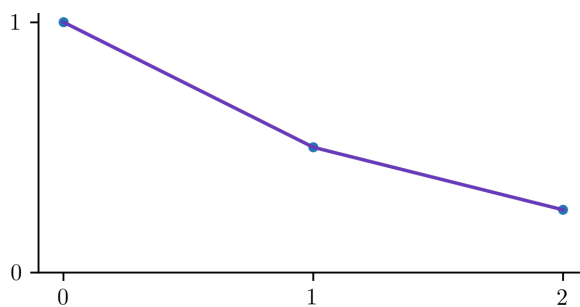


The line segment will pass through the point

$$\left(1, \boxed{\frac{1}{2}}\right)$$

(use your value of  $f(0, 1)$  to compute a numerical value of  $y$  corresponding to  $x = 1$ ).

- Now we repeat: Take a new line segment beginning at the point you just found.

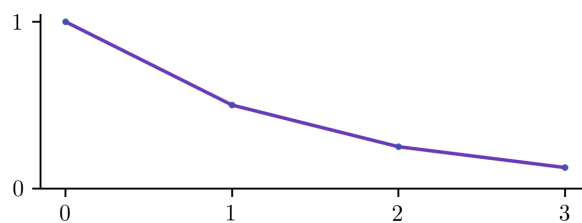


Its slope is given by

$$f\left(1, \boxed{\frac{1}{2}}\right) = \boxed{-\frac{1}{4}}.$$

This new line segment passes through the point  $(2, \boxed{1/4})$ .

- Repeat again:



the line segment beginning at the point you just found will have slope

$$f\left(2, \frac{1}{4}\right) = -\frac{1}{8}$$

and will pass through the point  $(3, \frac{1}{8})$ . Since we have arrived at an  $x$ -value of 3, we may stop. The  $y$ -value of this most recent point is our answer:

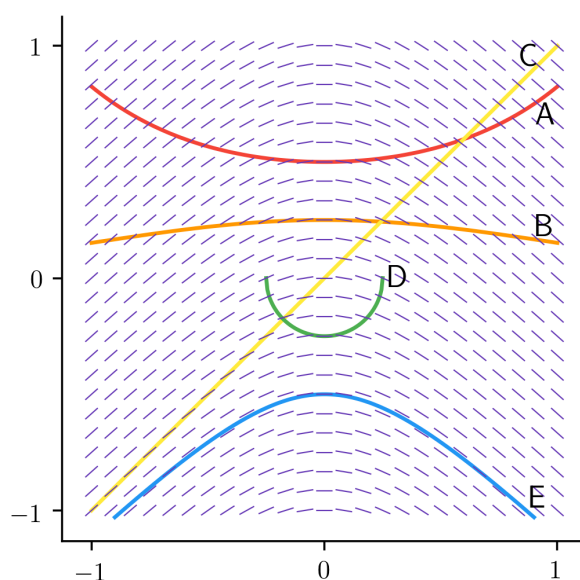
$$y(3) \approx \frac{1}{8}.$$

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## 7.2 Exercises: ODEs

*Exercises relating to fundamental properties of ODEs.*

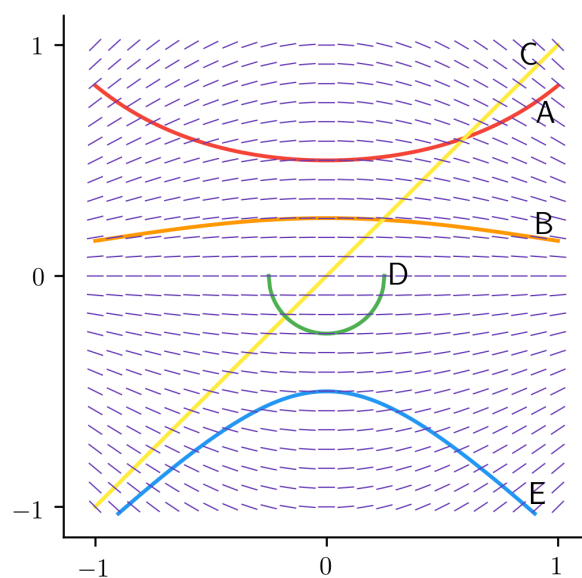
**Exercise 276** Which of the curves below is a solution of the ODE illustrated by the slope field?



**Multiple Choice:**

- (a) A
- (b) B
- (c) C
- (d) D
- (e) E ✓

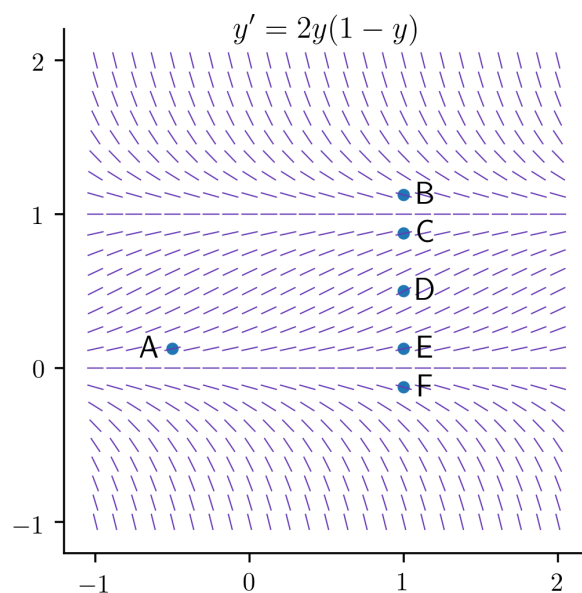
**Exercise 277** Which of the curves below is a solution of the ODE illustrated by the slope field?



**Multiple Choice:**

- (a) A
- (b) B ✓
- (c) C
- (d) D
- (e) E

**Exercise 278** Trace the solution of the given ODE which begins at the point A. Which of the other labelled points will the solution pass through?



**Multiple Choice:**

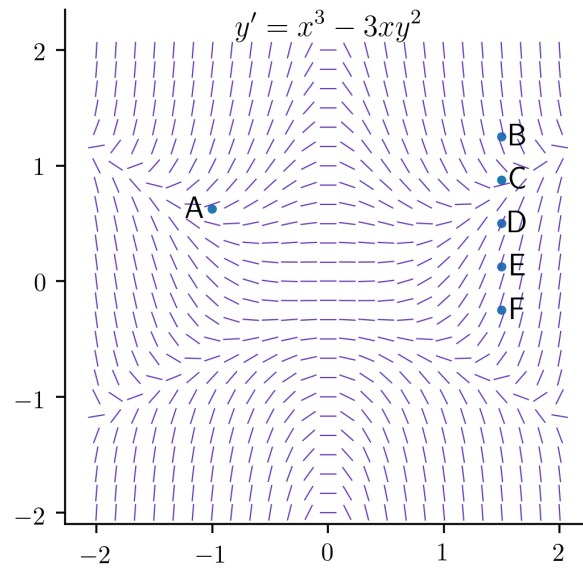
- (a)  $B$
- (b)  $C$  ✓
- (c)  $D$
- (d)  $E$

The ODE  $y' = 2y(1 - y)$  has one asymptotically stable constant solution. It is

$$y = \boxed{1}.$$

**Exercise 279** Trace the solution of the given ODE which begins at the point  $A$ . Which of the other labelled points will the solution pass through?





**Multiple Choice:**

- (a) B
- (b) C ✓
- (c) D
- (d) E

The ODE  $y' = x^3 - 3xy^2$  has (no ✓/ one/ two/ three) constant solutions.

**Exercise 280** Let  $y(x)$  be the solution to the initial value problem  $y' = y$  with  $y(0) = 1$ . Use Euler's Method with step size  $h = 1$  to approximate the value of  $y(4)$ . Fill in your work in the table below.

$x$	$y$	$y + yh$
0	1	2
1	<input type="text" value="2"/>	<input type="text" value="4"/>
<input type="text" value="2"/>	<input type="text" value="4"/>	<input type="text" value="8"/>
<input type="text" value="3"/>	<input type="text" value="8"/>	<input type="text" value="16"/>
<input type="text" value="4"/>	<input type="text" value="16"/>	

$$y(4) \approx \boxed{16}.$$

**Exercise 281** Let  $y(x)$  be the solution to the initial value problem  $y' = x - y$  with  $y(0) = 1$ . Use Euler's Method with step size  $h = 1$  to approximate the value of  $y(5)$ . Fill in your work in the table below.

$x$	$y$	$(x - y)$	$y + (x - y)h$
0	1	-1	0
1	0	1	1
2	1	1	2
3	2	1	3
4	3	1	4
5	4		

$$y(5) \approx \boxed{4}.$$

Is the function  $y = x - 1$  a solution of the ODE  $y' = x - y$ ?

**Multiple Choice:**

- (a) Yes ✓
- (b) No

If yes, is it a solution of the IVP  $y' = x - y$  and  $y(0) = 1$ ?

**Multiple Choice:**

- (a) Yes
- (b) No ✓
- (c) Not applicable

Is the function  $y = x - 1 + 2e^{-x}$  a solution of the IVP  $y' = x - y$  and  $y(0) = 1$ ?

**Multiple Choice:**

- (a) Yes ✓
- (b) No

Given this, what is the difference between the approximated value of  $y(5)$  that you calculated and the exact value. If you get a negative answer, take its absolute value:

$$|y_{\text{approx}}(5) - y_{\text{exact}}(5)| = \boxed{2e^{-5}} \approx 0.0135.$$

## 7.3 Separable and Linear ODEs

We learn techniques to solve first-order linear and separable ODEs.

### Online Texts

- OpenStax II 4.3: Separable Equations<sup>88</sup> and Linear Equations<sup>89</sup>
- Ximera OSU: Separable Equations<sup>90</sup>

### Examples

**Example 62.** *The ODE*

$$y' = x^2 e^y$$

*is separable.*

- To separate, we move all factors involving  $y$  to the left-hand side and all factors involving  $x$  to the right-hand side. We also formally write  $y'$  as  $dy/dx$  and verify that we can end up with  $dy$  on the left-hand side and  $dx$  on the right hand side:

$$\boxed{e^{-y}} dy = \boxed{x^2} dx.$$

- Apply an integral on both sides:

$$\int \boxed{e^{-y}} dy = \int \boxed{x^2} dx \Rightarrow \boxed{-e^{-y}} = \boxed{\frac{x^3}{3}} + C.$$

*That gives our general solution.*

- If we had initial conditions  $y(0) = 1$ , we would plug in  $x = \boxed{0}$ ,  $y = \boxed{1}$  to our general solution to determine that  $C = \boxed{-e^{-1}}$ . We can plug in this value for  $C$  and solve for  $y$  to determine that

$$y = \boxed{-\ln\left(e^{-1} - \frac{x^3}{3}\right)}.$$

<sup>88</sup>See OpenStax II 4.3: Separable Equations at <https://openstax.org/books/calculus-volume-2/pages/4-3-separable-equations>

<sup>89</sup>See Linear Equations at <https://openstax.org/books/calculus-volume-2/pages/4-5-first-order-linear-equations>

<sup>90</sup>See Ximera OSU: Separable Equations at <https://ximera.osu.edu/mooculus/calculus2/separableDifferentialEquations/titlePage>

**Example 63.** *The ODE*

$$xy' = -x^2y + e^{-x^2/2}$$

*is linear.*

- *To begin, we must write the ODE in standard form:*

$$y' + \boxed{x}y = \boxed{e^{-x^2/2}/x}.$$

- *We give the name  $P(x)$  to the coefficient of  $y$  (note that it doesn't have to depend on  $x$ . Writing  $P(x)$  is only meant to indicate that it cannot depend on  $y$ ). We give the name  $Q(x)$  to the right-hand side.*
- *The integrating factor is the exponential of the integral of  $P$ . In this case, we can take  $I(x) = e^{x^2/2}$ . Multiply both sides of the standard form equation by the integrating factor:*

$$\boxed{e^{x^2/2}}y' + \boxed{xe^{x^2/2}}y = \boxed{\frac{1}{x}}.$$

- *The integrating factor gets its name because it is now possible to integrate the left-hand side. By the product rule, we know that*

$$\left(e^{x^2/2}y\right)' = \boxed{e^{x^2/2}}y' + \boxed{xe^{x^2/2}}y,$$

*which is exactly the right-hand side. Thus we have*

$$\left(e^{x^2/2}y\right)' = \boxed{\frac{1}{x}}.$$

- *Integrating the left-hand side gives  $e^{x^2/2}y$ ; integrating both sides gives*

$$e^{x^2/2}y = \boxed{\ln|x|} + C$$

*(don't forget absolute values in the logarithm). Solving for  $y$  gives*

$$y = \boxed{e^{-x^2/2} \ln|x|} + Ce^{-x^2/2}$$

*as the general solution.*

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## 7.4 Exercises: Linear and Separable ODEs

*Exercises related to solving linear and separable ODEs.*

Remember: several of these exercises involve logarithms. Use absolute value signs inside the logarithm when they're needed.

**Exercise 282** Find the general solution of the ODE below.

$$y' = y - 2$$

General Solution:  $y = C \boxed{e^x} + \boxed{2}$

(For definiteness, the expression you enter in the second blank should equal 2 when  $x = 0$ .)

**Hint:** This ODE is both linear and separable, so either approach will work.

**Exercise 283** Find the general solution of the ODE below.

$$x^2 y' + xy = 1$$

General Solution:  $y = \boxed{\frac{\ln |x|}{x}} + \frac{C}{x}$

(For definiteness, the function you enter for your answer should equal 0 at  $x = 1$ . If that's not the case, you might need to rewrite your solution and redefine the constant  $C$ .)

**Hint:** This is a linear ODE.

**Exercise 284** Find the general solution of the ODE below.

$$yy' = 4x$$

General Solution:  $y^2 = \boxed{4x^2} + C$

(For definiteness, the function you enter for your answer should equal 0 at  $x = 0$ . If that's not the case, you might need to rewrite your solution and redefine the constant  $C$ .)

Exercises: Linear and Separable ODEs

**Hint:** This is a separable ODE. In the form written, it's already separated.

---

**Exercise 285**

$$y' - 3y = xe^{2x}$$

General Solution:  $y = C \boxed{e^{3x}} - (x + 1) \boxed{e^{2x}}$

**Hint:** This is a linear ODE.

---

**Exercise 286**

$$(x^2 + 1)y' = \frac{x}{y - 1}$$

General Solution:  $(y - 1)^2 = \boxed{\ln(x^2 + 1)} + C$

(For definiteness, the function you enter for your answer should equal 0 at  $x = 0$ . If that's not the case, you might need to rewrite your solution and redefine the constant  $C$ .)

**Hint:** This is a separable ODE.

---

**Exercise 287** Solve the initial value problem

$$y' = \cos^2 x \cos^2 2y \text{ with } y(0) = 0.$$

Solution:  $y = \boxed{\frac{1}{2} \arctan \left( x + \frac{1}{2} \sin 2x \right)}$

**Hint:** This is a separable ODE.

---

**Exercise 288** Solve the initial value problem

$$y' + (\tan x)y = \sec x \text{ with } y(0) = -3.$$

Solution:  $y = \boxed{\sin x - 3 \cos x}$

**Hint:** This is a linear ODE. The integral of  $\tan x$  is  $-\ln |\cos x|$ .

---

## Sample Quiz Questions

**Question 289** Let  $y(x)$  be the solution to the initial value problem

$$\frac{dy}{dx} = -(1 + 3x^2)y^2$$

and  $y(0) = 1/2$ . What is the value of  $y(1)$ ?

**Multiple Choice:**

- (a)  $\frac{1}{6}$
- (b)  $\frac{1}{4}$  ✓
- (c)  $\frac{1}{3}$
- (d)  $\frac{1}{2}$
- (e)  $\frac{\pi}{4}$
- (f) 1

**Feedback(attempt):** This is a separable ODE. Moving all functions of  $y$  to the left-hand side and all functions of  $x$  to the right-hand side and integrating gives

$$\int \frac{-1}{y^2} dy = \int (1 + 3x^2) dx,$$

which yields

$$\frac{1}{y} = x^3 + x + C.$$

Evaluating at  $x = 0$  and  $y = 1/2$  gives  $2 = 0 + C$ , so

$$\frac{1}{y} = x^3 + x + 2,$$

i.e.,

$$y = \frac{1}{x^3 + x + 2}.$$

Plugging in  $x = 1$  gives  $y = 1/4$ .

## Sample Exam Questions

**Question 290** The solution of the initial value problem  $x \frac{dy}{dx} + 3y = 7x^4$ ,  $y(1) = 1$ , satisfies  $y(2) =$

**Multiple Choice:**

- (a) 0
- (b) 1
- (c) 2
- (d) 4
- (e) 8
- (f) 16 ✓

**Question 291** The solution of the initial value problem  $\frac{dy}{dx} - 20x^4 e^{-y} = 0$ ,  $y(0) = 0$ , satisfies  $y(1) =$

**Multiple Choice:**

- (a)  $\ln 5$  ✓
- (b)  $\ln 4$
- (c)  $\ln 3$
- (d)  $\ln 2$
- (e) 1
- (f) 0

**Question 292** Let  $y(x)$  be the solution of the initial value problem

$$x \frac{dy}{dx} = e^x - y \quad \text{with} \quad y(\ln 2) = 0.$$

Find  $y(1)$ .

**Multiple Choice:**



- (a)  $\frac{e^2}{2}$
- (b)  $2e^2$
- (c)  $\frac{e}{2}$
- (d) 0
- (e)  $e - 2$  ✓
- (f) 1

**Question 293** Let  $y(x)$  be the solution of the initial value problem

$$x \frac{dy}{dx} = y + x^2 \sin x \quad \text{with} \quad y(\pi) = 0.$$

What is  $y(2\pi)$ ?

**Multiple Choice:**

- (a)  $-\pi$
- (b)  $-2\pi$
- (c)  $-4\pi$  ✓
- (d) 0
- (e)  $2\pi$
- (f)  $4\pi$

**Question 294** Consider the initial value problem

$$(1 + x^2) \frac{dy}{dx} = 2y \quad \text{with} \quad y(0) = 2.$$

What is  $\lim_{x \rightarrow \infty} y(x)$ ?

**Multiple Choice:**

- (a)  $2e^\pi$  ✓
- (b)  $2e^{\pi/2}$

(c)  $2e^{\pi/4}$

(d) 1

(e) 0

(f)  $e^{\pi}$

## 7.5 Applications of ODEs

We study some sample applications of ODEs.

### (Videos) Calculus: Single Variable

YouTube link: <https://www.youtube.com/watch?v=54HrJmeON24> YouTube link: <https://www.youtube.com/watch?v=xZ3CxLxljLA>

### Examples

**Example 64.** Newton's law of cooling states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the ambient temperature. For example, if the ambient temperature is 20 degrees Celsius, then

$$\frac{dT}{dt} = -k \left( T(t) - \boxed{20} \right).$$

(Here  $k$  is a positive number is called the heat transfer coefficient; it depends on the physical properties of the object and the surrounding environment).

- The equation is linear:

$$\frac{dT}{dt} + kT = \boxed{20}k.$$

- In this case,  $P(t) = \boxed{k}$  and the integrating factor is  $\boxed{e^{kt}}$  (since  $k$  is a constant with respect to  $t$ ). The general solution is

$$T(t) = \boxed{20} + C \boxed{e^{-kt}}.$$

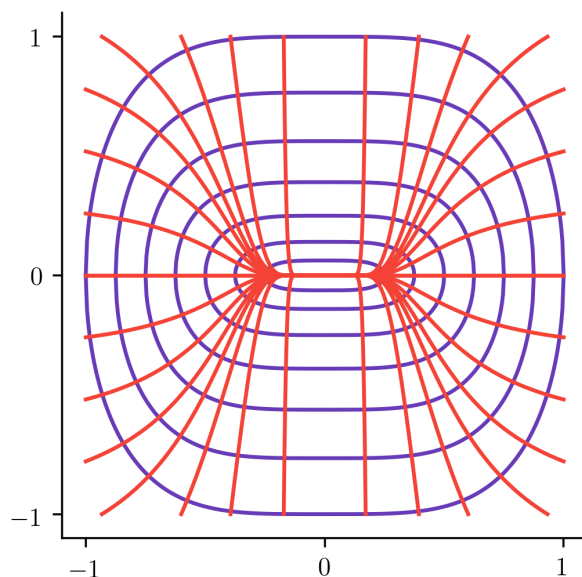
- If we subtract ambient temperature from both sides, this tells us that the temperature difference decays exponentially at some rate determined by the unknown constant  $k$ .

- We can generally solve for  $k$  if we have enough information. For example, if  $T(0) = 100$ , then  $C = \boxed{80}$ . If we also know that  $T(10) = 60$ , then  $Ce^{-10k} = 40$  (because 40 is the difference of 60 and 20), which means  $k = \boxed{(\ln 2)/10}$ . Another way of understanding this exponential decay is to observe that the temperature difference for this particular object will be cut in half every ten minutes.
- Now that we know  $k$ , we can compute  $T(t)$  for all time. For example,  $T(20) = \boxed{40}$ , which we could see by plugging in directly to the solution or by merely observing that the temperature difference will be cut in half  $\boxed{2}$  times in the course of 20 minutes.

**Example 65.** For the family of curves in the plane given by

$$x^4 + y^2 = C$$

(illustrated by the purple oval-shaped curves in the image below), find a formula for the orthogonal trajectories (shown in red).



- The first step is to find a first-order ODE that is satisfied by the family. This involves differentiating with respect to  $x$ . We get

$$4x^3 + 2yy' = 0, \quad \text{meaning } y' = \boxed{-\frac{2x^3}{y}}.$$

- If  $m$  and  $m'$  are slopes of orthogonal lines, then  $mm' = -1$ . So the orthogonal trajectories satisfy an ODE similar to the one above except

that the formula for  $y'$  is replaced by its negative reciprocal, i.e.,

$$y' = \boxed{\frac{y}{2x^3}}.$$

- This is a separable ODE. It separates to

$$\boxed{\frac{dy}{y}} = \frac{1}{2} \boxed{\frac{dx}{x^3}}.$$

Integrating both sides gives

$$\ln |y| = -\frac{1}{4}x^{-2} + C.$$

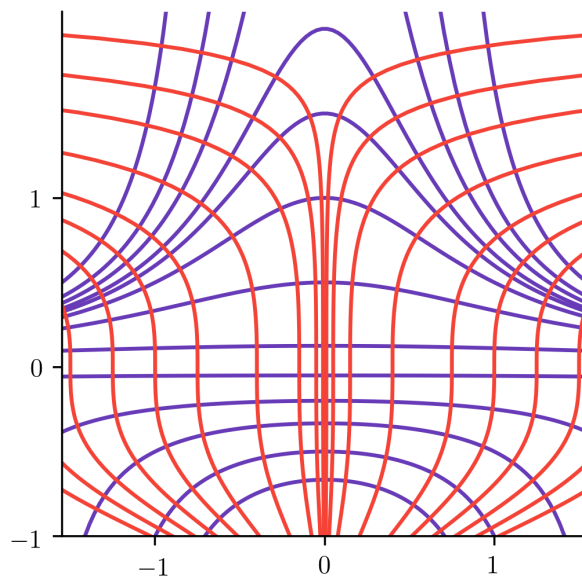
If we exponentiate both sides and reexpress the arbitrary constant, we get

$$y = C \boxed{e^{-\frac{1}{4x^2}}}$$

**Example 66.** For the family of curves in the plane given by

$$y = \frac{1}{C + x^2}$$

(shown in purple) find a formula for the orthogonal trajectories (shown in red).



- We first differentiate with respect to  $x$ :

$$y' = \boxed{-\frac{2x}{(x^2 + C)^2}}.$$

- Next we must eliminate  $C$  from the equation for  $y'$  by using the formula  $y = 1/(x^2 + C)$ . (For example, solve for  $C$  in terms of  $x$  and  $y$  and then substitute this in to your expression for  $y'$ .)

$$y' = \boxed{-2xy^2}$$

(your answer here will depend on the variables  $x$  and  $y$  but should not explicitly depend on  $C$ ).

- Then we write a new ODE whose slope is the negative reciprocal of the expression we just found:

$$y' = \boxed{\frac{1}{2xy^2}}.$$

This is a separable ODE:

$$\boxed{y^2} dy = \frac{1}{2} \frac{dx}{\boxed{x}}.$$

- Integrate both sides:

$$\boxed{\frac{y^3}{3}} = \frac{1}{2} \boxed{\ln |x|} + C.$$

(Don't forget absolute values on logarithms.)

**Example 67.** A large 100 liter tank is initially filled with fresh water. At time  $t = 0$ , a technician begins pouring in 2 liters per minute of sugar water which is at a concentration of 4 grams per liter. At the same time, another technician opens a valve at the bottom of the tank and begins to drain the tank at a rate of 1 liter per minute. Assuming the whole tank is kept well-mixed at all times, what is the concentration of the liquid in the tank after 100 minutes?

- It is always easiest to write an ODE for the total amount  $A$  of sugar in the tank (in grams). The concentration can be deduced later by dividing amount by volume.
- The first step is to find the function  $V(t)$  for the total volume of solution in the tank. The rate of fresh solution coming in is  $\boxed{2}$  liters per minute and the rate going out is  $\boxed{1}$  liter(s) per minute. So the net flow in of all liquid is  $\boxed{1}$  liter(s) per minute. This means that at time  $t$ ,  $V(t) = \boxed{100 + t}$ .
- Fill in the blanks:  $R_{\text{in}}$  is the rate of sugar coming in (measured in grams per minute),  $F_{\text{out}}$  is the volume rate of flow out from the bottom of the tank. Then

$$\frac{dA}{dt} = R_{\text{in}} - F_{\text{out}} \frac{A(t)}{V(t)} = \boxed{8} - \boxed{1} \frac{A(t)}{\boxed{t + 100}}.$$

- The equation is linear:

$$\frac{dA}{dt} + \frac{1}{100+t} A(t) = \boxed{8}.$$

- We compute the integrating factor:

$$I(t) = e^{\int P(t)dt} = e^{\int \boxed{(t+100)^{-1}} dt} = \boxed{t+100}.$$

(Note: logarithms don't need absolute values here because  $t > 0$ .)

- After multiplying both sides by  $I(t)$ , we have

$$\left( \boxed{(t+100)} A(t) \right)' = \boxed{8(t+100)}.$$

- Integrating both sides gives

$$(t+100)A(t) = 4(t+100)^2 + C.$$

At  $t = 0$ ,  $A(0)$  is given to be zero, so  $C = \boxed{-40000}$ . The full solution of the initial value problem is then

$$A(t) = \boxed{4(t+100) - \frac{40000}{t+100}}.$$

- $A(100) = \boxed{600}$  grams. Furthermore  $V(t) = \boxed{200}$  liters, so the concentration at time  $t = 100$  minutes is  $\boxed{3}$  grams per liter.

odes/31applypractice.tex

## 7.6 Exercises: ODE Applications

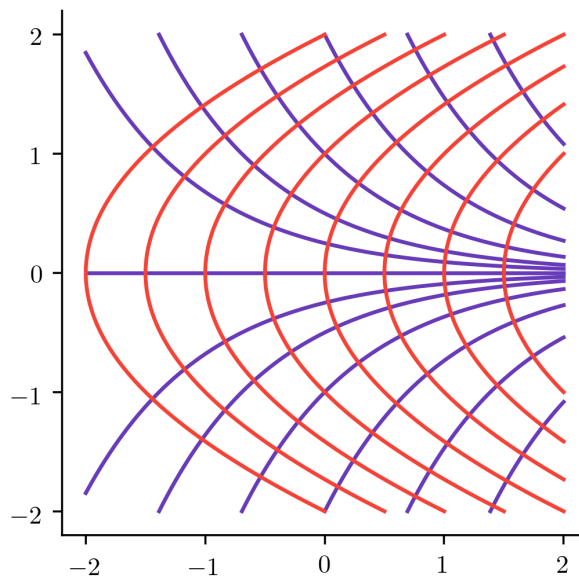
Exercises relating to the application of ODEs to solve problems.

**Exercise 295** Find the orthogonal trajectories of the given family (represented by the purple curves in the plot below):

$$y = Ce^{-x}.$$

Write your answer as  $f(y) = g(x) + C$ , where both  $f(y)$  and  $g(x)$  vanish at the origin. Moreover, you should multiply by constants if necessary so that  $f(1) = 1/2$  and  $g(1) = 1$ .

$$\boxed{\frac{1}{2}y^2} = \boxed{x} + C.$$

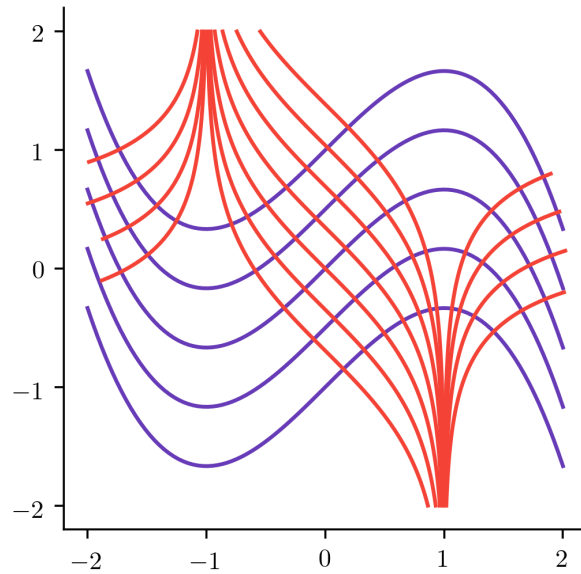


**Exercise 296** Find the orthogonal trajectories of the given family (represented by the purple curves in the plot below):

$$y = x - \frac{x^3}{3} + C.$$

Write your answer as  $y = f(x) + C$ , where  $f(0) = 0$

$$y = \boxed{\frac{1}{2} \ln \left| \frac{1-x}{1+x} \right|} + C.$$



**Exercise 297** Find the orthogonal trajectories of the given family:

$$y = \tan\left(C + x + \frac{x^3}{3}\right)$$

Write your answer as  $f(y) = g(x) + C$ , where both  $f(y)$  and  $g(x)$  vanish at the origin. Moreover, you should multiply by constants if necessary so that  $f(1) = 4/3$  and  $g(1) = -\pi/4$ .

$$\boxed{y + \frac{y^3}{3}} = \boxed{-\arctan x} + C$$

**Exercise 298** It takes about three days to defrost a 10 pound imitation frozen turkey in a home refrigerator. Suppose that the initial temperature is  $-16^\circ\text{C}$  and the ambient temperature is  $2^\circ\text{C}$  inside the refrigerator. If after exactly two days, the temperature of the frozen turkey is  $0^\circ\text{C}$ , find a formula for the temperature  $y(t)$  (in degrees Celsius) as a function of time for all  $t > 0$ , where  $t$  is measured in days

$$y = \boxed{-18e^{-t \ln 3} + 2}.$$



**Exercise 299** At time  $t = 0$ , a cup of tea has temperature  $46^\circ\text{C}$ . Ten minutes later, its temperature is  $34^\circ\text{C}$ . Ten minutes after that, the temperature is then  $28^\circ\text{C}$ . Assuming the temperature obeys Newton's Law of cooling, what is the temperature (in degrees Celsius) as a function of time  $y(t)$  for all  $t > 0$ , where  $t$  is measured in minutes?

$$y(t) = \boxed{24e^{-t \ln(2)/10} + 22}$$

**Hint:** Suppose  $A$  is the ambient temperature. We know that

$$\frac{48 - A}{34 - A} = \frac{34 - A}{28 - A}$$

because every ten minutes, the *difference* between tea temperature and room temperature decreases by the same factor.

**Exercise 300** A 1000 liter tank is filled with a sugar solution: 200 kilograms of sugar dissolved in 1000 liters of pure water. The solution in the tank is kept thoroughly mixed at all times. At time  $t = 0$ , the attendants begin adding 1 liter per minute of dissolved sugar at a concentration of 0.1 kilograms per liter. At the same time, One liter per minute is drained from the tank to keep the overall volume of solution in the tank at 1000 liters. Find a function  $S(t)$  for the total amount of sugar in the tank (in kilograms) a time  $t > 0$ , where  $t$  is measured in minutes.

$$S(t) = \boxed{100 + 100e^{-t/1000}}.$$

**Exercise 301** A small flask contains 10mL of solvent in which 1 gram of substance  $X$  is initially dissolved. The solution is slowly drained at a rate of 3mL per hour while 4mL per hour of pure solvent is added (with the whole solution being kept thoroughly mixed). Find a formula  $X(t)$  for the amount of substance  $X$  (in grams) still in the beaker at time  $t$ , where  $t$  is measured in hours.

$$X(t) = \boxed{\frac{1000}{(t + 10)^3}}.$$

## Sample Exam Questions

**Question 302** A tank contains 100 gallons of water in which 300 pounds of salt are dissolved. At some initial time, workers begin pumping in fresh water,

*i.e., containing no salt, at a rate of 10 gallons per minute. During the process, the tank is kept well-mixed and 20 gallons per minute of the resulting saltwater are pumped out of the tank (in particular, note that the tank will be empty after 10 minutes). Find the total amount of salt in the tank (measured in pounds) which remains 9 minutes after the process starts.*

**Multiple Choice:**

- (a) 1
  - (b) 2
  - (c) 3 ✓
  - (d) 4
  - (e) 5
  - (f) 6
-