Math 104 Practice

Philip T. Gressman

November 16, 2019

Contents

1 Computing Volume

A sample application of integration: computing volumes of a variety of complicated three-dimensional objects.

Integration is the tool to use whenever a quantity can be conceived as an *accumulation of infinitesimal parts*. Volume is one of the most basic and important of such quantities. In the activities that follow, we regard volume as the accumulated size of infinitely thin slices and use this perspective to derive and apply a number of formulas for computing volume.

1.1 Volume By General Cross Sections

We use cross-sectional area to compute volume.

Example 1. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 2. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

Exercise 1 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length $L = \boxed{x - x^2}$ and area $A = \boxed{(x - x^2)^2}$.*
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of $x = \boxed{0}$ up to a maximum of $x = \boxed{1}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 2 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$0 \le x \le 1,$$
$$0 \le y \le z^2,$$
$$0 \le z \le 3.$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length $\boxed{1}$ in the x-direction and width $\boxed{z^2}$ in the y-direction.*
- *The area of a z cross section is $A(z) = \boxed{z^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

volumes/genslicepractice.tex

1.2 Exercises: General Slicing

Exercises computing volume by cross-sectional area.

Exercise 3 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 4 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

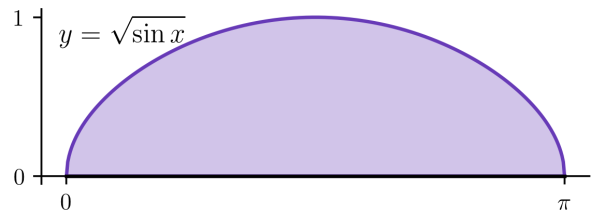
- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

1.3 The Disk and Washer Methods

We practice setting up calculations related to the disk and washer methods.

Example 3. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*



- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

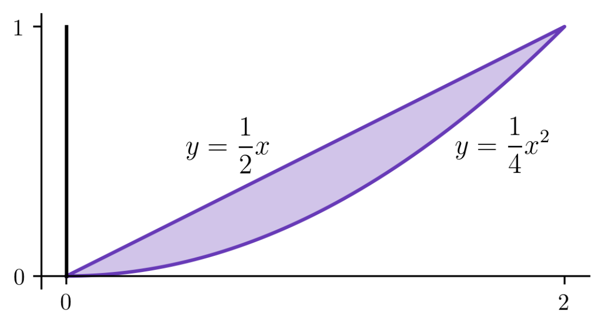
  Multiple Choice:

  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓
  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}} \right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 4. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*



- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

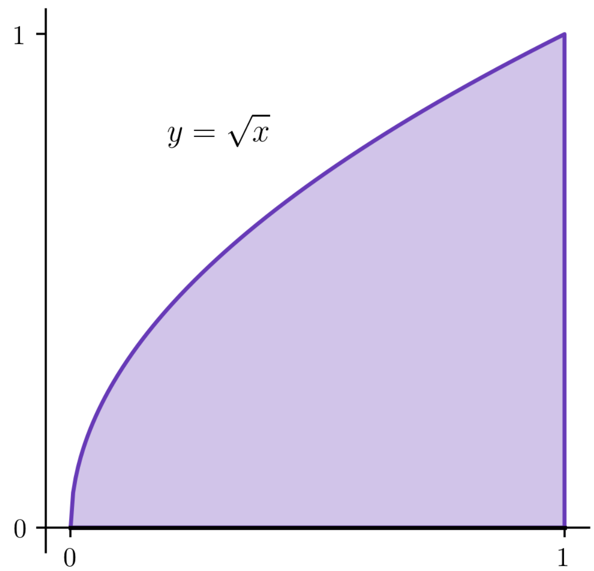
$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\,\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

volumes/washerpractice.tex

1.4 Exercises: Disks and Washers

Exercises for the disk and washer methods.

Exercise 5 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the x-axis. Use the disk method to find the volume of the solid of revolution.*

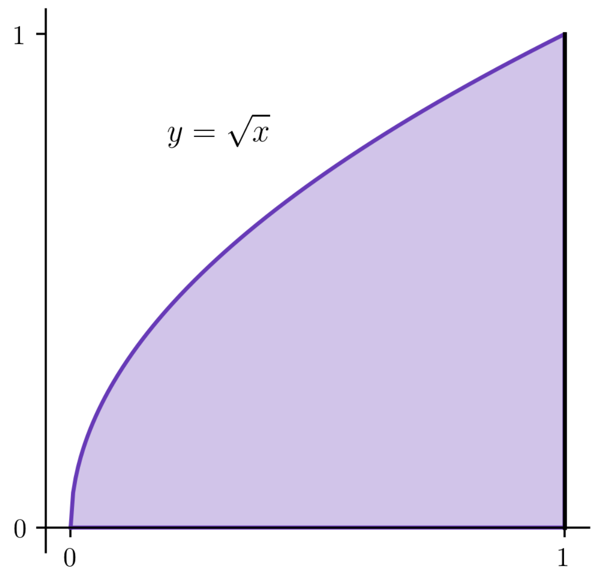


Hint: *The radius $R(x)$ will be a difference of y-values because slices are indexed by the variable x. Each slice will extend from $y = 0$ to $y = \sqrt{x}$, and so $R(x)$ must be the larger of these y-values minus the smaller of these y-values.*

$$R(x) = \boxed{\sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(x))^2 dx = \boxed{\frac{\pi}{2}}$$

Exercise 6 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 1$. Use the disk method to find the volume of the solid of revolution.*

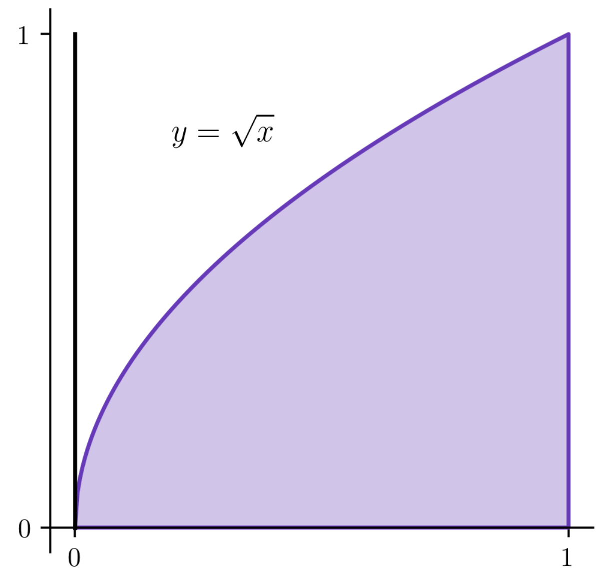


Hint: *The radius $R(y)$ will be a difference of x-values because slices are indexed by the variable y. Each slice will extend from $x = y^2$ to $x = 1$, and so $R(y)$ must be the larger of these x-values minus the smaller of these x-values*

$$R(y) = \boxed{1 - y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(y))^2 dy = \boxed{\frac{8\pi}{15}}$$

Exercise 7 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 0$. Use the washer method to find the volume of the solid of revolution.*

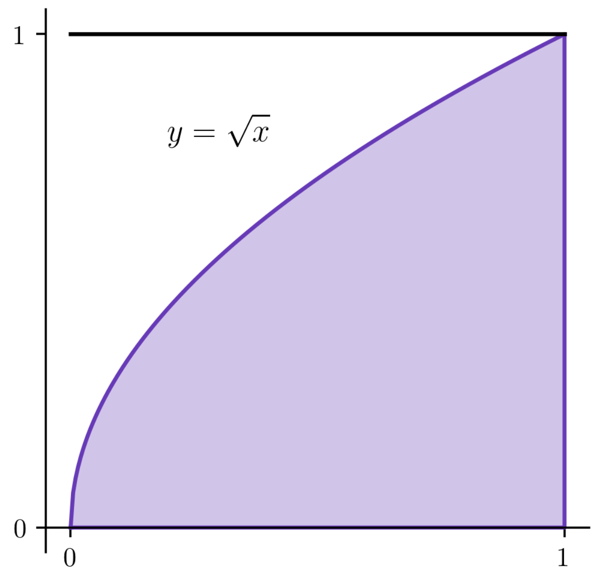


Hint: *Each radius will be a difference of x-values because slices are indexed by the variable y. The distance from the axis $x = 0$ to the line $x = 1$ is 1, and the distance from the axis $x = 0$ to $x = y^2$ is y^2.*

$$R_{\text{outer}}(y) = \boxed{1} \text{ and } r_{\text{inner}}(y) = \boxed{y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(y))^2 - (r_{\text{inner}}(y))^2 \right] dy = \boxed{\frac{4\pi}{5}}$$

Exercise 8 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the axis $y = 1$. Use the washer method to find the volume of the solid of revolution.*



Hint: *Each radius will be a difference of y-values because slices are indexed by the variable x. The distance from the axis $y = 1$ to the line $y = 0$ is 1, and the distance from the axis $y = 1$ to $y = \sqrt{x}$ is $1 - \sqrt{x}$.*

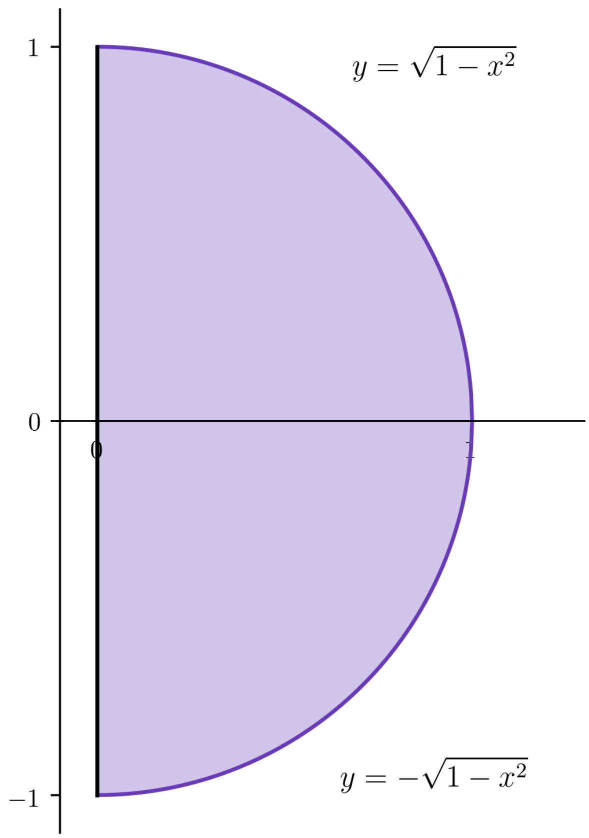
$$R_{\text{outer}}(x) = \boxed{1} \text{ and } r_{\text{inner}}(x) = \boxed{1 - \sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(x))^2 - (r_{\text{inner}}(x))^2 \right] dx = \boxed{\frac{5\pi}{6}}$$

1.5 The Shell Method

We practice setting up setting up volume calculations using the shell method.

Example 5. *The region defined by the inequalities $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ and $x \geq 0$ (shown below) is revolved around the y-ais. Compute the volume using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

Multiple Choice:

(a) $h(x) = \sqrt{1-x^2}$

(b) $h(x) = -\sqrt{1-x^2}$

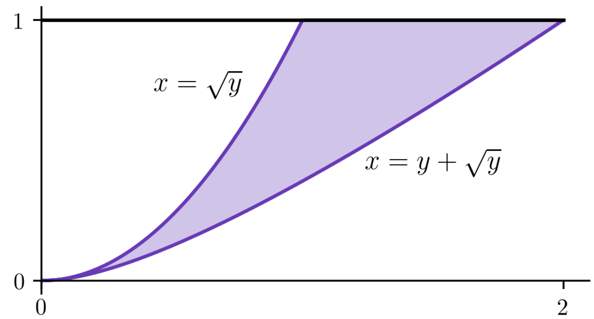
(c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\, dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$.)

Example 6. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*



- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$
  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$
  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi r h$, so*

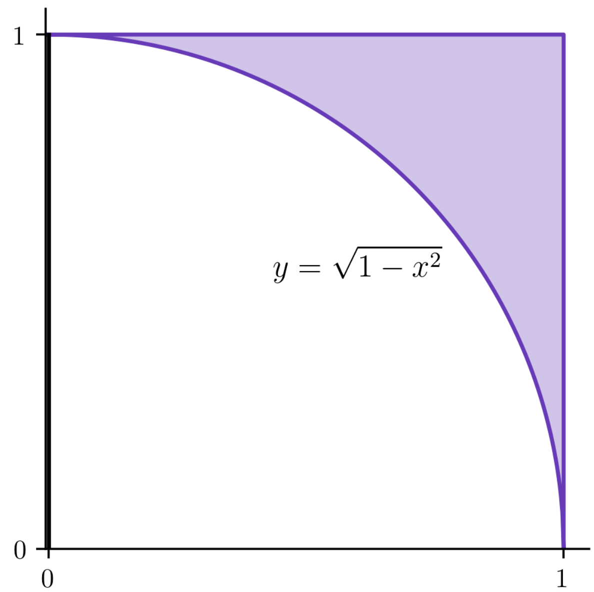
$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\, dy = \boxed{\frac{\pi}{3}}.$$

volumes/shellpractice.tex

1.6 Exercises: Shell Method

Exercises for using the shell method.

Exercise 9 *The region defined by the inequalities $\sqrt{1-x^2} \leq y \leq 1$ for $0 \leq y \leq 1$ is revolved around the y-axis. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis of rotation. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

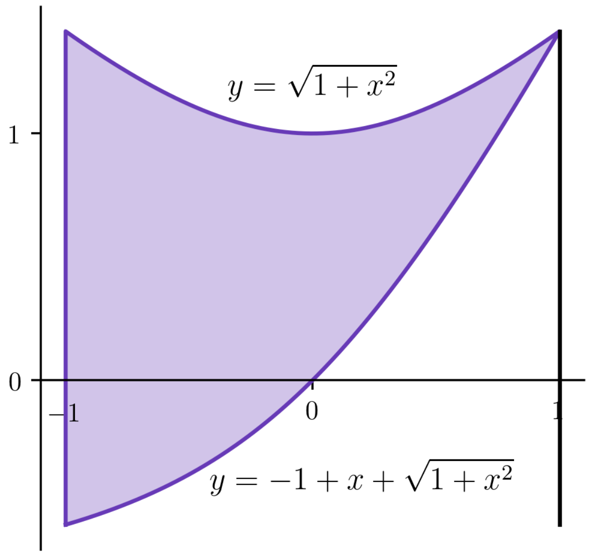
$$h(x) = \boxed{1 - \sqrt{1-x^2}}.$$

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi x(1 - \sqrt{1-x^2})}\,dx = \boxed{\frac{\pi}{3}}.$$

(Note: to compute the integral, split it into two parts and make the substitution $u = 1 - x^2$ for one of them.)

Exercise 10 *The region in the plane bounded above by the graph $y = \sqrt{1+x^2}$, below by $y = -1+x+\sqrt{1+x^2}$, and on the left by $x = 0$ is revolved around the axis $x = 1$. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{1} - \boxed{x}.$$

- *The height of an x-slice is equal to*

$$h(x) = \boxed{-1+x}.$$

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{-1}}^{\boxed{1}} \boxed{2\pi(1-x)^2}\,dx = \boxed{\frac{16\pi}{3}}.$$

1.7 Testing answer

A regression test of `answer`.

Problem 11 *Type* 2. $\boxed{2}$, $\underset{\text{given}}{\boxed{2}}$, $\boxed{\frac{1}{2}}$, $\int_a^b f(x)\ dx$,

$$-\frac{1}{\frac{12}{1}}\sum_{n=1}^{\boxed{\infty}} n$$