Math 104 Online

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Contents

1 Applications of Integration

We study some important application of integrations: computing volumes of a variety of complicated three-dimensional objects, computing arc length and surface area, and finding centers of mass.

Integration is the tool to use whenever a quantity can be conceived as an *accumulation of infinitesimal parts*. Volume is one of the most basic and important of such quantities. In the activities that follow, we regard volume as the accumulated size of infinitely thin slices and use this perspective to derive and apply a number of formulas for computing volume.

1.1 Volume By General Cross Sections

We use cross-sectional area to compute volume.

Example 1. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 2. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`volumes/01genslicepractice.tex`

1.2 Exercises: General Slicing

Exercises computing volume by cross-sectional area.

Exercise 1 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length $L = \boxed{x - x^2}$ and area $A = \boxed{(x - x^2)^2}$.*
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of $x = \boxed{0}$ up to a maximum of $x = \boxed{1}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 2 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length $\boxed{1}$ in the x-direction and width $\boxed{z^2}$ in the y-direction.*
- *The area of a z cross section is $A(z) = \boxed{z^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

Exercise 3 *A right circular cylinder of radius 1 and height 3 is twisted along its axis so that the disk at height z is centered on the axis $x = \cos(2\pi z/3), y = \sin(2\pi z/3))$, which corresponds to one full twist along the axis. Compute the volume of this twisted cylinder.*

$$V = \boxed{3\pi}.$$

Exercise 4 *A certain three-dimensional region has a base in the xy-plane which is bounded above by the graph $y = 1 - x^2$ and below by $y = 0$. Slices perpendicular to the y-axis are equilateral triangles whose base lies in the xy-plane as well. Compute the volume of the region.*

$$V = \boxed{\frac{\sqrt{3}}{2}}.$$

Sample Exam Questions

Question 5 *(2018 Midterm 1) Compute the volume of the region in 3-dimensional space which satisfies the inequalities*

$$0 \leq x \leq (1 - z^2) \qquad \text{and} \qquad 0 \leq y \leq (1 + z^2) \qquad \text{and} \qquad 0 \leq z \leq 1.$$

Multiple Choice:

(a) $\frac{2}{3}$

(b) $\frac{3}{4}$

(c) $\frac{4}{5}$ ✓

(d) $\frac{5}{6}$

(e) $\frac{6}{7}$

(f) *none of these*

Question 6 *(2019 Midterm 1) The inequality*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

defines an ellipse in the xy-plane whose area is πab for any positive values of the constants a and b. Compute the three dimensional volume of the region defined by

$$4x^2 + z^2y^2 \leq z^2 \text{ for } 0 \leq z \leq 1.$$

(Hints won't reveal until after you choose a response.)

Multiple Choice:

(a) $4\pi z^2$

(b) 4π

(c) $\frac{\pi}{4z}$

(d) $\frac{\pi}{4}$ ✓

(e) πz

(f) π

Feedback(`attempt`): *Dividing the first inequality by z^2 on both sides gives*

$$\frac{x^2}{\frac{z^2}{4}} + \frac{y^2}{1} \leq 1,$$

Hint: *which means that slices in the z-direction are ellipses with area $\pi \frac{z}{2} \cdot 1 = \frac{\pi z}{2}$.*

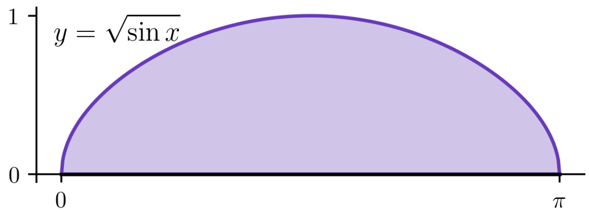
Hint: *Volume is obtained by integration:*

$$V = \int_0^1 \frac{\pi z}{2} dz = \left. \frac{\pi z^2}{4} \right|_0^1 = \frac{\pi}{4}.$$

1.3 The Disk and Washer Methods

We practice setting up calculations related to the disk and washer methods.

Example 3. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*



- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

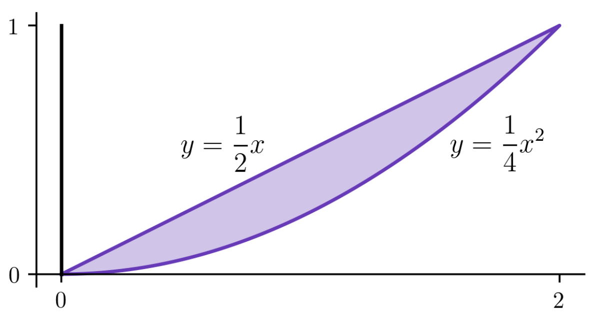
  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓

  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}} \right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 4. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*



- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*

- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***
  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$
  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓
- *We conclude that*

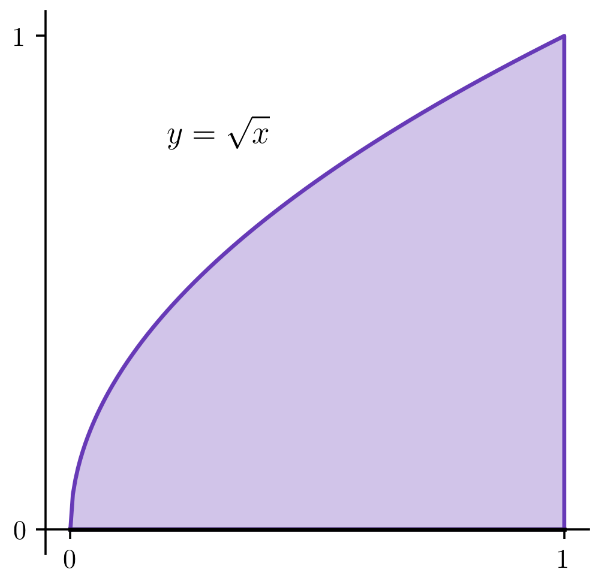
$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}}\right)^2 - \left(\boxed{2y}\right)^2 \right] d\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

volumes/02washerpractice.tex

1.4 Exercises: Disks and Washers

Exercises for the disk and washer methods.

Exercise 7 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the x-axis. Use the disk method to find the volume of the solid of revolution.*

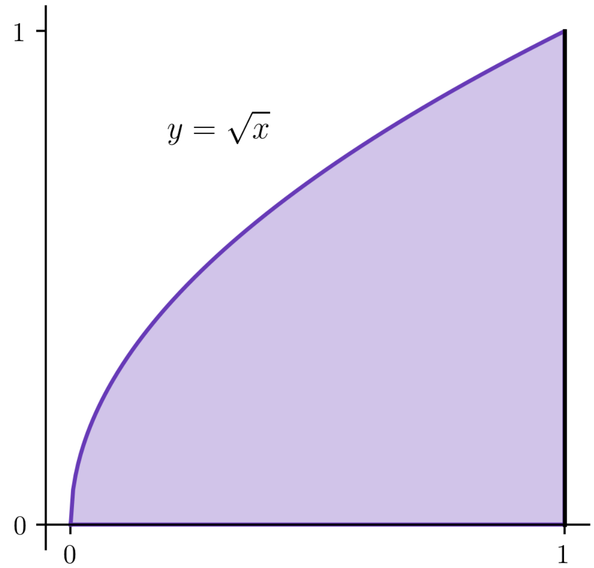


Hint: *The radius $R(x)$ will be a difference of y-values because slices are indexed by the variable x. Each slice will extend from $y = 0$ to $y = \sqrt{x}$, and so $R(x)$ must be the larger of these y-values minus the smaller of these y-values.*

$$R(x) = \boxed{\sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(x))^2 dx = \boxed{\frac{\pi}{2}}$$

Exercise 8 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the axis $x = 1$. Use the disk method to find the volume of the solid of revolution.*

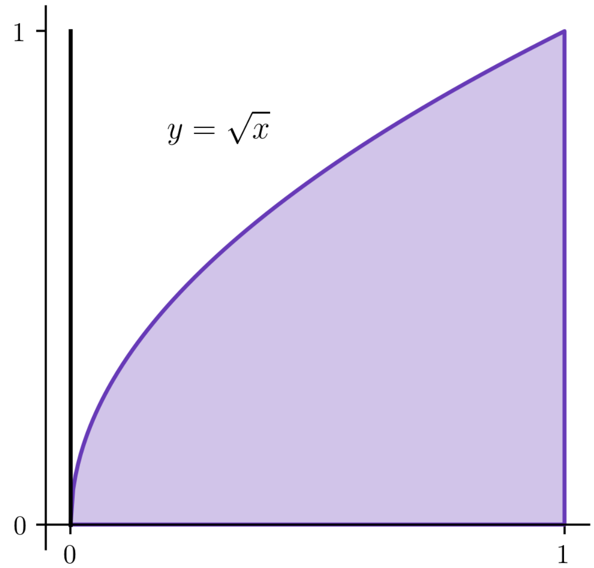


Hint: *The radius $R(y)$ will be a difference of x-values because slices are indexed by the variable y. Each slice will extend from $x = y^2$ to $x = 1$, and so $R(y)$ must be the larger of these x-values minus the smaller of these x-values*

$$R(y) = \boxed{1 - y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(y))^2 dy = \boxed{\frac{8\pi}{15}}$$

Exercise 9 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the axis $x = 0$. Use the washer method to find the volume of the solid of revolution.*

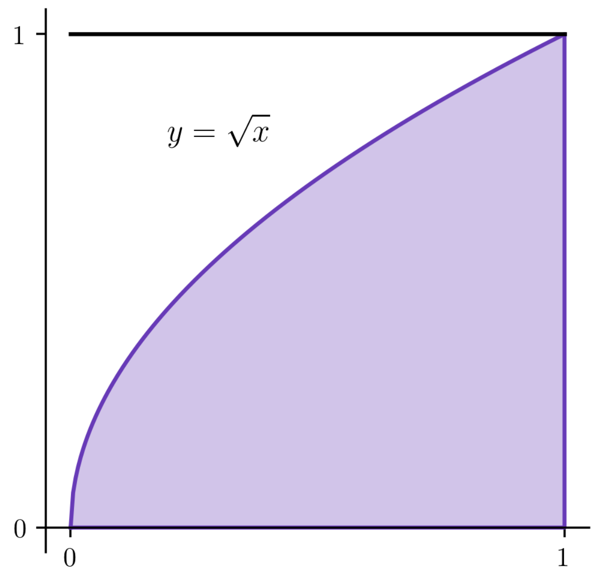


Hint: *Each radius will be a difference of x-values because slices are indexed by the variable y. The distance from the axis $x = 0$ to the line $x = 1$ is 1, and the distance from the axis $x = 0$ to $x = y^2$ is y^2.*

$$R_{\text{outer}}(y) = \boxed{1} \text{ and } r_{\text{inner}}(y) = \boxed{y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(y))^2 - (r_{\text{inner}}(y))^2\right] dy = \boxed{\frac{4\pi}{5}}$$

Exercise 10 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $y = 1$. Use the washer method to find the volume of the solid of revolution.*



Hint: *Each radius will be a difference of y-values because slices are indexed by the variable x. The distance from the axis $y = 1$ to the line $y = 0$ is 1, and the distance from the axis $y = 1$ to $y = \sqrt{x}$ is $1 - \sqrt{x}$.*

$$R_{\text{outer}}(x) = \boxed{1} \text{ and } r_{\text{inner}}(x) = \boxed{1 - \sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(x))^2 - (r_{\text{inner}}(x))^2 \right] dx = \boxed{\frac{5\pi}{6}}$$

Sample Quiz Questions

Question 11 *The region in the plane bounded on the left by the curve $x = -y^2$, on the right by the curve $x = y^2+2y+2$, above by the line $y = 0$, and below by the line $y = -2$ is revolved around the axis $x = 2$. Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)*

Multiple Choice:

(a) 16π

(b) 20π

(c) 24π ✓

(d) 28π

(e) 32π

(f) 36π

Feedback(*`attempt`*): *The axis $x = 2$ is perpendicular to the direction of slices using the integration variable y, which indicates the washer method. The region lies to the left of the axis. One way to see this is to evaluate $x = -y^2$ at $y = -2$, giving $x = -4$, which is to the left of the axis $x = 2$.*

Hint: *The integral to compute equals*

$$\begin{aligned} V &= \int_{-2}^{0} \pi \left((2 - (-y^2))^2 - (2 - (y^2 + 2y + 2))^2 \right) \, dy \\ &= \pi \int_{-2}^{0} (-4y^3 + 4) \, dy \\ &= \pi \left(-y^4 + 4y \right) \Big|_{-2}^{0} = 24\pi. \end{aligned}$$

Question 12 *The region in the plane bounded below by the curve $y = -2x^2 + 5x + 2$, above by the curve $y = -2x^2 + 2x + 2$, on the right by the line $x = 0$, and on the left by the line $x = -1$ is revolved around the axis $y = 2$. Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)*

Multiple Choice:

(a) 10π ✓

(b) 14π

(c) 18π

(d) 22π

(e) 26π

(f) 30π

Feedback(*`attempt`*): *The axis $y = 2$ is perpendicular to the direction of slices using the integration variable x, which indicates the washer method. The region lies below the axis. One way to see this is to evaluate $y = -2x^2 + 5x + 2$ at $x = -1$, giving $y = -5$, which is below the axis $y = 2$.*

Hint: *The integral to compute equals*

$$\begin{aligned} V &= \int_{-1}^{0} \pi \left((2-(-2x^2+5x+2))^2 - (2-(-2x^2+2x+2))^2 \right) \, dx \\ &= \pi \int_{-1}^{0} \left(-12x^3 + 21x^2 \right) \, dx \\ &= \pi \left(-3x^4 + 7x^3 \right) \Big|_{-1}^{0} = 10\pi. \end{aligned}$$

Question 13 *The region in the plane given by* $\left| -\frac{x}{2} + \frac{1}{2}\sqrt{9-6x^2} \right| \leq y \leq \frac{x}{2} + \frac{1}{2}\sqrt{9-6x^2}$ *and* $0 \leq x \leq \frac{2}{3}\sqrt{3}$ *is revolved around the* x*-axis. Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)*

Multiple Choice:

(a) $\frac{13}{9}\pi$ ✓

(b) $\frac{19}{9}\pi$

(c) $\frac{26}{9}\pi$

(d) $\frac{28}{9}\pi$

(e) $\frac{37}{9}\pi$

(f) $\frac{49}{9}\pi$

Feedback(`attempt`): *If the variable* x *is used for slicing, then slices are perpendicular to the axis of rotation, which indicates the washer method should be used.*

Hint: *The inequalities for* y *give the outer and inner radii, and*

$$\left(\frac{x}{2} + \frac{1}{2}\sqrt{9-6x^2} \right)^2 - \left(\left| -\frac{x}{2} + \frac{1}{2}\sqrt{9-6x^2} \right| \right)^2 = x\sqrt{9-6x^2}.$$

(Note that the absolute values go away when the radius is squared.)

Hint: *To compute the integral*

$$\int_{0}^{\frac{2}{3}\sqrt{3}} \pi x \sqrt{9-6x^2} \, dx$$

we can use the substitution $u = 9 - 6x^2$ which implies the equality $du = (-12x)\,dx$ for the differentials. This gives the equality

$$\begin{aligned}\int \pi x\sqrt{9-6x^2}\,dx &= \int \left(-\frac{\pi}{12}\sqrt{u}\right)\,du \\ &= -\frac{\pi}{18}u^{\frac{3}{2}}.\end{aligned}$$

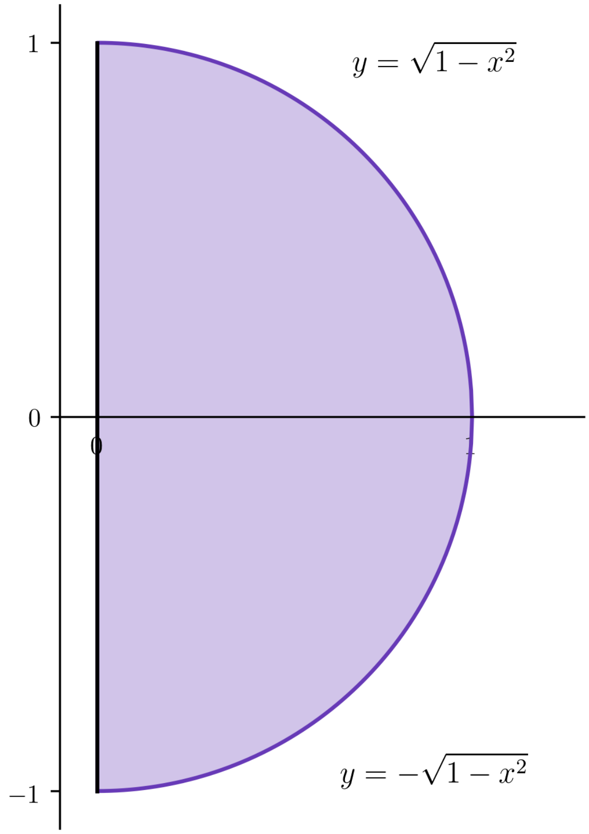
Reversing the substitution gives

$$\begin{aligned}\int_0^{\frac{2}{3}\sqrt{3}} \pi x\sqrt{9-6x^2}\,dx &= \left[-\frac{\pi}{18}\left(9-6x^2\right)^{\frac{3}{2}}\right]\Bigg|_0^{\frac{2}{3}\sqrt{3}} \\ &= \left(-\frac{\pi}{18}\right) - \left(-\frac{3}{2}\pi\right) = \frac{13}{9}\pi.\end{aligned}$$

1.5 The Shell Method

We practice setting up setting up volume calculations using the shell method.

Example 5. *The region defined by the inequalities $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ and $x \geq 0$ (shown below) is revolved around the y-ais. Compute the volume using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

Multiple Choice:

(a) $h(x) = \sqrt{1-x^2}$

(b) $h(x) = -\sqrt{1-x^2}$

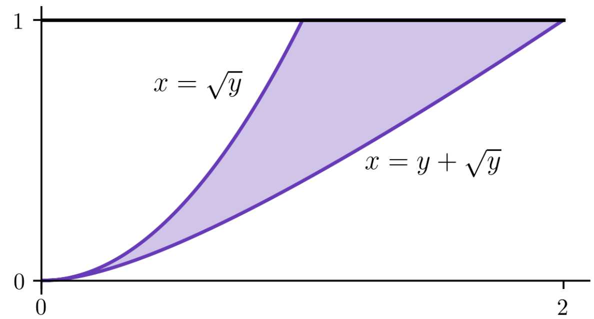
(c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\, dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$.)

Example 6. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*



- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$

  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$

  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

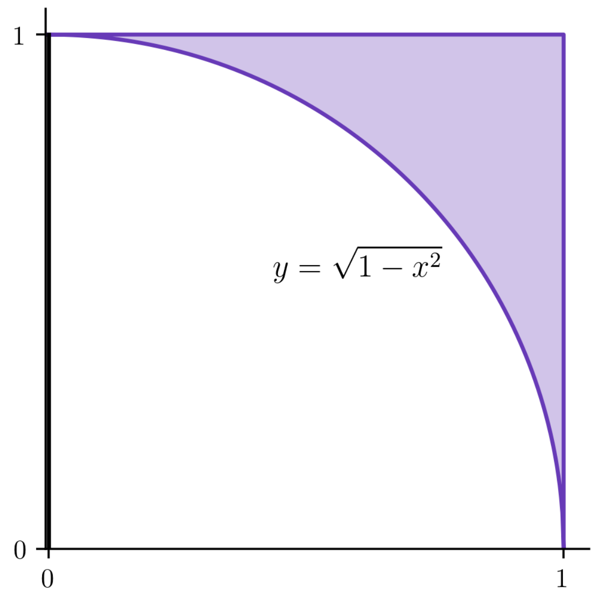
$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\, dy = \boxed{\frac{\pi}{3}}.$$

volumes/03shellpractice.tex

1.6 Exercises: Shell Method

Exercises for using the shell method.

Exercise 14 *The region defined by the inequalities* $\sqrt{1-x^2} \leq y \leq 1$ *for* $0 \leq y \leq 1$ *is revolved around the* y*-axis. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is* x*, the radius of a shell is the (horizontal ✓/ vertical) distance from an* x*-slice to the axis of rotation. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an* x*-slice is equal to*

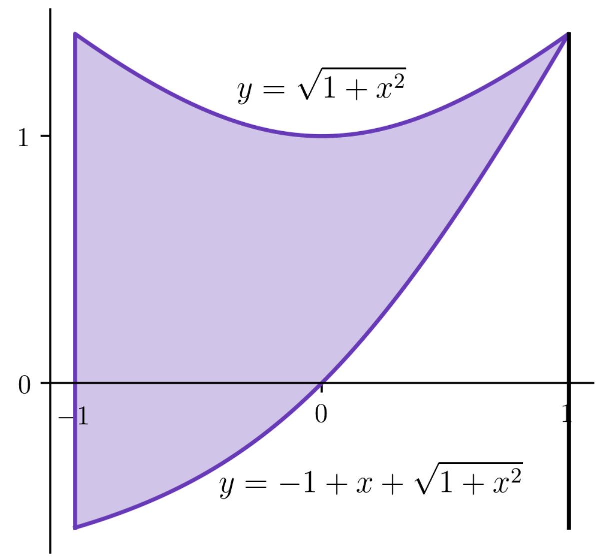
$$h(x) = \boxed{1 - \sqrt{1-x^2}}.$$

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi x(1 - \sqrt{1-x^2})}\, dx = \boxed{\frac{\pi}{3}}.$$

(Note: to compute the integral, split it into two parts and make the substitution $u = 1 - x^2$ *for one of them.)*

Exercise 15 *The region in the plane bounded above by the graph $y = \sqrt{1+x^2}$, below by $y = -1+x+\sqrt{1+x^2}$, and on the left by $x = 0$ is revolved around the axis $x = 1$. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{1} - \boxed{x}.$$

- *The height of an x-slice is equal to*

$$h(x) = \boxed{-1+x}.$$

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{-1}}^{\boxed{1}} \boxed{2\pi(1-x)^2}\, dx = \boxed{\frac{16\pi}{3}}.$$

Exercise 16 *The region in the plane $y = \sqrt{x}$, $y = 0$, and $x = 1$ is revolved around the y-axis. Use the shell method to compute the volume.*

$$V = \boxed{\frac{4\pi}{5}}.$$

Exercise 17 *The same region as above (bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$) is revolved around the axis $x = 1$. Use the shell method to compute the volume.*

$$V = \boxed{\frac{8\pi}{15}}.$$

Exercise 18 *The same region as above (bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$) is revolved around the x-axis. Use the shell method to compute the volume.*

Hint: *The "height" of a shell is $1 - y^2$ in this case.*

$$V = \boxed{\frac{\pi}{2}}.$$

Exercise 19 *For the same region as above (bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$), use the shell method to compute the volume when revolved around the axis $y = 1$.*

$$V = \boxed{\frac{5\pi}{6}}.$$

Sample Quiz Questions

Question 20 *The region in the plane bounded below by the curve $y = -x^2$, above by the curve $y = x^2+2x+2$, on the right by the line $x = 0$, and on the left by the line $x = -2$ is revolved around the axis $x = -2$. Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)*

Multiple Choice:

(a) 4π

(b) 6π

(c) 8π ✓

(d) 10π

(e) 12π

(f) 14π

Feedback(*`attempt`*): *The axis $x = -2$ is parallel to the direction of slices using the integration variable x, which indicates the shell method. The region lies to the right of the axis, which must be the case because the interval $-2 \leq x \leq 0$ lies to the right of the axis $x = -2$.*

Hint: *The integral to compute equals*

$$\begin{aligned} V &= \int_{-2}^{0} 2\pi(x - (-2))((x^2 + 2x + 2) - (-x^2))\ dx \\ &= \pi \int_{-2}^{0} (4x^3 + 12x^2 + 12x + 8)\ dx \\ &= \pi \left(x^4 + 4x^3 + 6x^2 + 8x\right)\Big|_{-2}^{0} = 8\pi. \end{aligned}$$

Question 21 *The region in the plane bounded on the left by the curve $x = -y^2 + 4y + 1$, on the right by the curve $x = y^2 + 2y + 1$, and below by the line $y = -1$ is revolved around the axis $y = -1$. Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)*

Multiple Choice:

(a) π ✓

(b) 5π

(c) 9π

(d) 13π

(e) 17π

(f) 21π

Feedback(*`attempt`*): *The axis $y = -1$ is parallel to the direction of slices using the integration variable y, which indicates the shell method. The lower endpoint of integration will be $y = -1$; the upper endpoint can be determined by setting $-y^2 + 4y + 1 = y^2 + 2y + 1$ and choosing the solution which is greater than -1. This gives the range $-1 \leq y \leq 0$. The region lies above the axis, which must be the case because the interval $-1 \leq y \leq 0$ lies above the axis $y = -1$.*

Hint: *The integral to compute equals*

$$\begin{aligned} V &= \int_{-1}^{0} 2\pi(y - (-1))((y^2 + 2y + 1) - (-y^2 + 4y + 1))\ dy \\ &= \pi \int_{-1}^{0} (4y^3 - 4y)\ dy \\ &= \pi \left(y^4 - 2y^2\right)\Big|_{-1}^{0} = 1\pi. \end{aligned}$$

Question 22 *The region in the plane between the x-axis and the graph*

$$y = \frac{1}{2\sqrt{\frac{x^2}{3}+1}}$$

in the range $0 \le x \le 3$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid. (Hints won't reveal until after you choose a response.)

Multiple Choice:

(a) $\frac{3}{2}\pi$

(b) $\frac{8}{5}\pi$

(c) $\frac{5}{3}\pi$

(d) 2π

(e) 3π ✓

(f) 5π

Feedback(`attempt`): *If the variable x is used for slicing, then slices are parallel to the axis of rotation, which indicates the shell method should be used. The radius of a shell is x. The height of a shell is exactly $\frac{1}{2\sqrt{\frac{x^2}{3}+1}}$.*

Hint: *The volume of the region is therefore given by*

$$\int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2+3}}\,dx.$$

Hint: *To compute the integral we can use the substitution $u = x^2 + 3$ which implies the equality $du = (2x)\,dx$ for the differentials. This gives the equality*

$$\int \frac{\sqrt{3}\pi x}{\sqrt{x^2+3}}\,dx = \int \frac{\sqrt{3}\pi}{2\sqrt{u}}\,du$$
$$= \sqrt{3}\pi\sqrt{u}.$$

Reversing the substitution gives

$$\int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2+3}}\,dx = \left[\sqrt{3}\pi\sqrt{x^2+3}\right]\Big|_0^3$$
$$= (6\pi) - (3\pi) = 3\pi.$$

1.7 Synthesis: Choose Your Method

We practice choosing a method for computing volume when none is specified.

Example 7. *(Spring 2016 Final Exam) Find the volume of the solid generated by revolving the region in the first quadrant bounded by $y = 1 - x^2$, $y = 0$, and $x = 0$ about the line $x = 2$.*

- *Using x as the slicing variable, slices are (parallel ✓/ perpendicular) to the axis $x = 2$. This alignment corresponds to the (disk or washer/ shell ✓) method. (Expand to continue.)*

- *The height of a shell is*

  Multiple Choice:

  (a) 2
  (b) $2 - x$
  (c) 0
  (d) $x - 2$
  (e) $1 - x^2$ ✓
  (f) $1 - x^2 - 2$

  and the radius is

  Multiple Choice:

  (a) 2
  (b) $2 - x$ ✓
  (c) 0
  (d) $x - 2$
  (e) $1 - x^2$
  (f) $1 - x^2 - 2$

Therefore

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi(2-x)(1-x^2)}\, dx = \boxed{\frac{13\pi}{6}}.$$

Example 8. *(Fall 2012 Final Exam) The region in the xy-plane bounded by $y = (x-1)^{1/4}$ and the x-axis for $1 \le x \le 2$ is rotated about the x-axis. What is the volume of the resulting solid of revolution?*

- *Using x as the slicing variable, slices are (parallel/ perpendicular ✓) to the axis $x = 2$. This alignment corresponds to the (disk or washer ✓/ shell) method. (Expand to continue.)*

- *The outer radius of a disk is*

  Multiple Choice:

  (a) 2
  (b) $y^4 + 1$
  (c) 0
  (d) $(x - 1)^{1/4}$ ✓

  and the inner radius is

  Multiple Choice:

  (a) 2
  (b) $y^4 + 1$
  (c) 0 ✓
  (d) $(x - 1)^{1/4}$

Therefore

$$V = \int_{\boxed{1}}^{\boxed{2}} \boxed{\pi(x-1)^{1/2}}\, dx = \boxed{\frac{2\pi}{3}}.$$

Example 9. *Find the volume of the solid generated by revolving the region in the first quadrant bounded on the left by $y = \sqrt{x}$, on the right by $y = x - 2$, and below by $y = 0$ about the axis $y = 0$.*

- *To carry out the calculation as efficiently as possible, the (x/ y ✓)-variable should be used as the variable for slicing. (Expand to continue.)*

- *Using y as the slicing variable, slices are (parallel ✓/ perpendicular) to the axis $y = 0$. This alignment corresponds to the (disk or washer/ shell ✓) method. (Expand to continue.)*

- *The height of a shell is*

  Multiple Choice:

  (a) $y + 2$
  (b) y^2

(c) $x - 2$
(d) $\sqrt{x}$
(e) $y^2 - y - 2$
(f) $y + 2 - y^2$ ✓

and the radius is

Multiple Choice:

(a) y ✓
(b) y^2
(c) $x - 2$
(d) $\sqrt{x}$
(e) $y^2 - y - 2$
(f) $y + 2 - y^2$

Therefore

$$V = \int_{\boxed{0}}^{\boxed{2}} \boxed{2\pi y(y + 2 - y^2)}\, dy = \boxed{\frac{16\pi}{3}}.$$

`volumes/04choosepractice.tex`

1.8 Exercises: Choose Your Method

Exercises choosing a method for computing volume.

Exercise 23 *The region in the plane bounded by $y = e^{-x/2}$ and the x-axis for $0 \leq x \leq \ln 2$ is rotated about the x-axis. The volume of the resulting solid of revolution is*

$$V = \boxed{\frac{\pi}{2}}.$$

(Hints won't be revealed until after you choose a response.)

Feedback(`attempt`): *If x is used as the slicing variable, then slices are vertical and consequently perpendicular to the axis of rotation.*

Hint: *Furthermore one side of the region lies along the axis, so the disk method is appropriate in this case.*

Hint: *The distance from the axis to the upper edge of the region is $e^{-x/2}$, so*

$$\begin{aligned} V &= \int_0^{\ln 2} \pi \left(e^{-x/2}\right)^2 dx = \pi \int_0^{\ln 2} e^{-x} dx \\ &= -\pi e^{-x}\Big|_{x=0}^{\ln 2} = \pi(-e^{-\ln 2} + e^0) = \pi\left(-\frac{1}{2} + 1\right) = \frac{\pi}{2}. \end{aligned}$$

Exercise 24 *The region in the plane bounded on the right by the curve $x = 2 - y^2$, on the left by the curve $x = y^2$, and on the bottom by $y = 0$ is revolved around the y-axis. Compute the volume of the resulting solid.*

$$V = \boxed{\frac{8\pi}{3}}.$$

Exercise 25 *Compute the volume of the solid of revolution obtained by rotating the region between $x = 0$, $y = 0$, and $x = \sqrt{2 + 3y^2 - 5y^4}$ around the y-axis.*

$$V = \boxed{2\pi}.$$

Exercise 26 *The region between the graph of $y = 1 - x^2$ and the x-axis is rotated around the line $y = 1$. What is the volume of the resulting solid?*

$$V = \boxed{\frac{8\pi}{5}}.$$

Exercise 27 *Find the volume obtained by rotating the region between the graph $x = \frac{1}{2}\sin(y^2)$ and the y-axis for $0 \le y \le \sqrt{\pi}$ about the x-axis.*

$$V = \boxed{\pi}.$$

Sample Exam Questions

Question 28 *Calculate the volume of the solid obtained by rotating the area between the graphs of $y = \dfrac{1}{\sqrt{x^2 - 1}}$ and the x-axis for $1 < x < \sqrt{5}$ around the y-axis.*

Multiple Choice:

(a) π

(b) 4π ✓

(c) 6π

(d) 8π

(e) 3π

(f) 2π

Question 29 *Let $f(x)$ be a continuous function that satisfies $f(0) = 0$ and $f(x) > 0$ for $x > 0$. For every $b > 0$, when the region between the graph of $y = f(x)$, the x-axis, and the line $x = b$ is rotated around the x-axis, the volume of the resulting solid is $18\pi b^2$. What is $f(x)$? (Hints will not be revealed until after you choose a response.)*

Multiple Choice:

(a) $9x$

(b) $3x^2$

(c) $6\sqrt{x}$ ✓

(d) $27x^{3/2}$

(e) $9x^2$

(f) $\sqrt{3x}$

Feedback(`attempt`): *By the disk method, we have that*

$$\int_0^b \pi(f(x))^2 dx = 18\pi b^2$$

for each $b > 0$. Solve this equation for b.

Hint: *Differentiate both sides with respect to b; use the Fundamental Theorem of Calculus to differentiate the left-hand side.*

Question 30 *Find the volume of the solid generated by revolving the region bounded above by $y = \sec x$ and bounded below by $y = 0$ for $0 \leq x \leq \pi/3$ about the x-axis.*

Multiple Choice:

(a) π

(b) 2π

(c) $\pi\sqrt{3}$ ✓

(d) 3π

(e) 4π

(f) *none of these*

1.9 Arc Length

We practice setting up and executing arc length calculations.

Example 10. *Compute the arc length of the curve*

$$x = -\frac{1}{8}y^{-2} - \frac{1}{4}y^4 - 2$$

between the endpoints $y = 1/\sqrt{2}$ *and* $y = 1$.

- *First we compute the derivative:*

$$\frac{dx}{dy} = \frac{1}{4}y^{-3} - y^3.$$

- *Next, we write the arc length element:*

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \left(\frac{1}{4}y^{-3} - y^3\right)^2}\, dy.$$

- *The key to integrating is to first fully simplify:*

$$\begin{aligned} 1 + \left(\frac{1}{4}y^{-3} - y^3\right)^2 &= 1 + \frac{1}{16}y^{-6} + 2\left(\frac{1}{4}y^{-3}\right)\left(-y^3\right) + y^6 \\ &= 1 + \frac{1}{16}y^{-6} - \frac{1}{2} + y^6 \end{aligned}$$

Observation 1. *When the middle term in a binomial (i.e., FOIL) expansion is* $-1/2$ *adding one* ***always*** *gives a perfect square. In this case, the algebraic fact is*

$$1 + \frac{1}{16}y^{-6} - \frac{1}{2} + y^6 = \frac{1}{16}y^{-6} + \frac{1}{2} + y^6 = \left(\frac{1}{4}y^{-3} + y^3\right)^2.$$

The general phenomenon is that what was formerly the square of a difference— in this case, $\left(\frac{1}{4}y^{-3} - y^3\right)^2$ *—becomes the square of the corresponding sum.*

Therefore

$$\begin{aligned} L &= \int_{1/\sqrt{2}}^{1} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_{1/\sqrt{2}}^{1} \sqrt{1 + \left(\frac{1}{4}y^{-3} - y^3\right)^2}\, dy \\ &= \int_{1/\sqrt{2}}^{1} \sqrt{1 + \frac{1}{16}y^{-6} - \frac{1}{2} + y^6}\, dy = \int_{1/\sqrt{2}}^{1} \sqrt{\left(\boxed{\frac{1}{4}y^{-3} + y^3}\right)^2}\, dy \\ &= \int_{1/\sqrt{2}}^{1} \boxed{\frac{1}{4}y^{-3} + y^3}\, dy = \left(\boxed{-\frac{1}{8}y^{-2} + \frac{1}{4}y^4}\right)\Bigg|_{1/\sqrt{2}}^{1} \\ &= \left(-\frac{1}{8} + \frac{1}{4}\right) - \left(-\frac{1}{4} + \frac{1}{16}\right) = \frac{5}{16}. \end{aligned}$$

arclengths/05arclengthpractice.tex

1.10 Exercises: Arc Length

We practice computing arc length.

Exercise 31 *Find the arc length of the function on the given interval:* $f(x) = \sqrt{8}x$ *on* $[-1, 1]$

$$L = \boxed{6}.$$

Exercise 32 *Find the arc length of the function on the given interval:* $f(x) = \ln(\cos x)$ *on* $[0, \pi/4]$. *(You may use the fact that* $\int \sec x\, dx = \ln|\sec x + \tan x| + C$.*)*

$$L = \boxed{\ln(1+\sqrt{2})}.$$

Exercise 33 *Set up the integral to compute the arc length of the function on the given interval:* $f(x) = x^2$ *on* $[0, 1]$.

$$L = \int_{\boxed{0}}^{\boxed{1}} \boxed{\sqrt{1+4x^2}}\, dx$$

Question 34 *Let* $y = \frac{x^4}{16} + \frac{1}{2x^2}$. *Find the arc length for* $1 \le x \le \sqrt{2}$.

$$L = \boxed{\frac{7}{16}}.$$

Sample Quiz Questions

Question 35 *Compute the arc length of the curve*

$$y = \frac{3}{4}x^{-2} + \frac{1}{24}x^4 - 1$$

between the endpoints $x = \sqrt{3}$ *and* $x = \sqrt{6}$. *(Hints won't be revealed until after you choose a response.)*

Multiple Choice:

(a) $\frac{5}{12}$

(b) $\frac{7}{12}$

(c) $\frac{3}{4}$

(d) $\frac{11}{12}$

(e) $\frac{13}{12}$

(f) $\frac{5}{4}$ ✓

Feedback(`attempt`): *Applying the formula for arc length gives that*

$$L = \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{1 + \left(-\frac{3}{2}x^{-3} + \frac{1}{6}x^3\right)^2}\, dx$$

Hint:

$$\begin{aligned}
&= \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{1 + \frac{9}{4}x^{-6} - \frac{1}{2} + \frac{1}{36}x^6}\, dx = \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{\left(\frac{3}{2}x^{-3} + \frac{1}{6}x^3\right)^2}\, dx \\
&= \int_{\sqrt{3}}^{\sqrt{6}} \left(\frac{3}{2}x^{-3} + \frac{1}{6}x^3\right) dx = \left(-\frac{3}{4}x^{-2} + \frac{1}{24}x^4\right)\Big|_{\sqrt{3}}^{\sqrt{6}} \\
&= \left(-\frac{1}{8} + \frac{3}{2}\right) - \left(-\frac{1}{4} + \frac{3}{8}\right) = \frac{5}{4}.
\end{aligned}$$

Note that you must always take the positive square root in going from line two to line three. In particular, if you get a negative answer, you have likely taken the negative square root.

Sample Exam Questions

Question 36 *(2017 Midterm 1) Compute the length of the curve* $x = \frac{1}{8}(y^2 + 2y) - \ln(y+1)$ *between* $y = 0$ *and* $y = 2$*.*

Multiple Choice:

(a) $1 + \ln 3$ ✓

(b) $2 + \ln 6$

(c) $3 + \ln 9$

(d) $4 + \ln 12$

(e) $5 + \ln 15$

(f) *none of the above*

Question 37 *Find the arc length of the following curve between* $x = -1$ *and* $x = 1$*:*

$$y = 3\cosh\frac{x}{3}.$$

(Note: $\cosh x = (e^x + e^{-x})/2$*.)*

Multiple Choice:

(a) $\frac{e}{3} - \frac{1}{3e}$

(b) $\frac{e}{2} - \frac{1}{2e}$

(c) $e - \frac{1}{e}$

(d) $2e - \frac{2}{e}$

(e) $3e - \frac{3}{e}$ ✓

(f) *none of the above*

Question 38 *A certain curve $y = f(x)$ in the plane has the property that its length between the endpoints $x = 0$ and $x = a$ is equal to*

$$\int_0^a \sqrt{1 + \sin^2 t}\, dt$$

for every value of $a > 0$. Assuming the curve passes through the points $(0, 0)$ and $\left(\frac{\pi}{2}, 1\right)$, what is $f\left(\frac{\pi}{4}\right)$?

Multiple Choice:

(a) $\frac{1}{2}$

(b) $\frac{1}{\sqrt{2}}$

(c) $1 - \frac{1}{\sqrt{2}}$ ✓

(d) 0

(e) $-\frac{1}{\sqrt{2}}$

(f) *none of these*

Question 39 *Find the length of the part of the curve $y = \frac{3}{16}e^{2x} + \frac{1}{3}e^{-2x}$ for $0 \le x \le \ln 2$.*

Multiple Choice:

(a) $\frac{13}{16}$ ✓

(b) $\frac{11}{16}$

(c) $\frac{3}{8}$

(d) $\frac{9}{8}$

(e) $\frac{29}{64}$

(f) $\frac{3}{4}$

Question 40 *Find the length of the part of the curve* $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ *for* $1 \le x \le 2$.

Multiple Choice:

(a) $\dfrac{13}{16}$ ✓

(b) $\dfrac{11}{16}$

(c) $\dfrac{7}{8}$

(d) $\dfrac{13\sqrt{2}}{16}$

(e) $\dfrac{11\sqrt{2}}{16}$

(f) $\dfrac{7\sqrt{2}}{8}$

1.11 Surface Area

We practice setting up integrals for the surface area of surfaces of revolution.

Example 11. *Suppose the graph* $y = 2\sqrt{x}$ *between* $x = 0$ *and* $x = 1$ *is revolved around the* x*-axis. Compute the area of the resulting surface.*

- *The distance from the* x*-axis to the point* $(x, 2\sqrt{x})$ *is* $\boxed{2\sqrt{x}}$.
- *For this curve, the arc length element satisfies* $ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\,dx = \boxed{\sqrt{1 + \dfrac{1}{x}}}\,dx$

$$\begin{aligned} A &= \int_{\boxed{0}}^{\boxed{1}} \boxed{2\sqrt{x}}\,ds \\ &= \int_{\boxed{0}}^{\boxed{1}} \boxed{2\sqrt{x+1}}\,dx \\ &= \boxed{\frac{4}{3}\left(2\sqrt{2} - 1\right)}. \end{aligned}$$

`arclengths/06surfacepractice.tex`

1.12 Exercises: Surface Area

Various exercises related to the computation of areas of surfaces of revolution.

Exercise 41 *Find the surface area of the solid formed by revolving $y = 2x$ on $[0, 1]$ about the x-axis.*

$$A = \boxed{2\pi\sqrt{5}}.$$

Exercise 42 *Find the surface area of the solid formed by revolving $y = x^2$ on $[0, 1]$ about the y-axis.*

Hint: *To compute the integral, you will need to make a substitution like $u = 1+4x^2$ or something similar.*

$$A = \boxed{\frac{(5\sqrt{5}-1)\pi}{6}}.$$

Exercise 43 *Find the surface area of the solid formed by revolving $y = x^3$ on $[0, 1]$ about the x-axis.*

Hint: *To compute the integral, you will need to make a substitution like $u = 1+9x^4$ or something similar.*

$$L = \boxed{\frac{(10\sqrt{10}-1)\pi}{27}}.$$

Sample Exam Questions

Question 44 *Give an integral formula for the area of the surface generated by revolving the curve $y = \ln x$ between $x = 1$ and $x = 2$ about the y-axis. Explain your answer. You do not need to evaluate the integral.*

Multiple Choice:

(a) $\displaystyle\int_1^2 2\pi\sqrt{x^2+1}\,dx$ ✓

(b) $\displaystyle\int_1^2 2\pi(\ln x)\frac{\sqrt{x^2+1}}{x}\,dx$

(c) $\displaystyle\int_1^2 \frac{2\pi}{x}\sqrt{1+(\ln x)^2}\,dx$

(d) $\displaystyle\int_1^2 \frac{1}{2\pi\sqrt{x^2+1}}\,dx$

(e) $\displaystyle\int_1^2 2\pi(\ln x)\frac{x}{\sqrt{x^2+1}}\,dx$

(f) $\displaystyle\int_1^2 \frac{2\pi x}{\sqrt{1+(\ln x)^2}}\,dx$

Question 45 *The curve* $y=\dfrac{x^2}{8}$ *between* $x=0$ *and* $x=3$ *is revolved around the* y*-axis. Compute the surface area of the resulting surface.*

Multiple Choice:

(a) $\dfrac{31\pi}{6}$

(b) $\dfrac{41\pi}{6}$

(c) $\dfrac{61\pi}{6}$ ✓

(d) $\dfrac{71\pi}{6}$

(e) $\dfrac{91\pi}{6}$

(f) *none of the above*

1.13 Centers of Mass and Centroids

We practice setting up calculations for centers of mass and centroids.

Example 12. *Compute the centroid of the region bounded by the inequalities*

$$0 \le x \le 2 \qquad \text{and} \qquad -\frac{9}{2}x^2 + x \le y \le \frac{9}{2}x^2 + x.$$

- *The term "centroid" refers to the geometric center of a region. Practically speaking, this means we may assume constant density (e.g., density* 1*).*
- *First we compute the "mass" of the region, which in this case is simply the area between curves:*

$$M = \int_{\boxed{0}}^{\boxed{2}} \left[\left(\boxed{\frac{9}{2}x^2 + x} \right) - \left(\boxed{-\frac{9}{2}x^2 + x} \right) \right] dx = \boxed{24}.$$

- *Next we compute the moments about the y and x axes. This always involves multiplying the integrand above by $\tilde{x}$ and $\tilde{y}$, respectively (note the reversal), where $(\tilde{x}, \tilde{y})$ are the coordinates of the geometric center of a typical slice.*
- *Using x as the slicing variable, slices are (horizontal / vertical ✓) and consequently the x-coordinate of the geometric center of a slice is just x (but note that this would be different if y were the slicing variable). Thus*

$$M_y = \int_{\boxed{0}}^{\boxed{2}} x \left[\boxed{9x^2} \right] dx = \boxed{36}.$$

- *The y-coordinate of the geometric center of a slice will be the average of y-coordinates at the top and bottom of a slice. Therefore*

$$\tilde{y} = \boxed{x}.$$

Thus

$$M_x = \int_{\boxed{0}}^{\boxed{2}} \boxed{9x^3} dx = \boxed{36}.$$

- *Thus*

$$\overline{x} = \frac{M_y}{M} = \boxed{\frac{3}{2}},$$

$$\overline{y} = \frac{M_x}{M} = \boxed{\frac{3}{2}}.$$

Example 13. *Compute the centroid of the region given by* $-\frac{3}{2}y^2 + 2y \le x \le \frac{3}{2}y^2 + 2y$ *between* $y = 0$ *and* $y = 2$*.*

- *In this example, we should use y as the slicing variable, so the roles of x and y are largely switched in comparison to the previous example.*
- *First compute the mass:*

$$M = \int_0^2 \left[\left(\boxed{\frac{3}{2}y^2 + 2y} \right) - \left(\boxed{-\frac{3}{2}y^2 + 2y} \right) \right] dy = \boxed{8}.$$

- *In this case, $\tilde{y} = y$ since slices are (horizontal ✓/ vertical), so*

$$M_x = \int_0^2 y \left[\boxed{3y^2} \right] dy = \boxed{12}.$$

- *Likewise, $\tilde{x}$ is the average of x-coordinates of endpoints of a slice. Thus*

$$\tilde{x} = \boxed{2y}.$$

Therefore

$$M_y = \int_{\boxed{0}}^{\boxed{2}} \left[\boxed{6y^3} \right] dy = \boxed{24}$$

- *To conclude,*

$$\overline{x} = \frac{M_y}{M} = \boxed{3},$$
$$\overline{y} = \frac{M_x}{M} = \boxed{\frac{3}{2}}.$$

centroids/07centroidpractice.tex

1.14 Exercises: Centers of Mass and Centroids

Various questions relating to centers of mass and centroids.

Exercise 46 *Find the centroid of the region bounded above by $y = x$ and below by $y = x^2$.*

$$\overline{x} = \boxed{\frac{1}{2}} \text{ and } \overline{y} = \boxed{\frac{2}{5}}.$$

Exercise 47 *Find the centroid of the region bounded above by $y = 4 - x^2$ and below by $y = 0$.*

$$\overline{x} = \boxed{0} \text{ and } \overline{y} = \boxed{\frac{8}{5}}.$$

Exercise 48 *A thin plate in the plane defined by $x^2 \le y \le 1$ and $x \ge 0$ has density y at the point (x, y). Compute the center of mass.*

Hint: *Use y as the variable of slicing. The center of mass of a single slice $(\tilde{x}, \tilde{y})$ is then $(\sqrt{y}/2, y)$.*

$$\overline{x} = \boxed{\frac{1}{2}} \text{ and } \overline{y} = \boxed{\frac{5}{7}}.$$

Exercise 49 *A thin plate in the plane defined by $x^2 \le y \le 2x^2$ and $0 \le x \le 1$ has density x at the point (x, y). Compute the center of mass.*

Hint: *Use x as the variable of slicing. The center of mass of a single slice $(\tilde{x}, \tilde{y})$ is then $(x, 3x^2/2)$.*

$$\overline{x} = \boxed{\frac{4}{5}} \text{ and } \overline{y} = \boxed{1}.$$

Exercise 50 *The same thin plate as above ($x^2 \leq y \leq 2x^2$ and $0 \leq x \leq 1$) now has density x^{-2} at the point (x, y). Because the density of the plate is now higher near the origin than in the previous problem, this suggests that the center of mass will shift (away from/ towards ✓) the origin relative to the previous exercise.*

Compute the center of mass.

$$\overline{x} = \boxed{\frac{1}{2}} \text{ and } \overline{y} = \boxed{\frac{1}{2}}.$$

Exercise 51 *Compute the centroid of a thin wire along the graph $y = \sqrt{1-x^2}$ between $x = 0$ and $x = 1$.*

Hint: *Recall that*

$$M = \int ds$$
$$\overline{x} = \frac{1}{M} \int x ds$$
$$\overline{y} = \frac{1}{M} \int y ds$$

where ds is the arc length element. We also know that

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C.$$

$$\overline{x} = \boxed{\frac{2}{\pi}} \text{ and } \overline{y} = \boxed{\frac{2}{\pi}}.$$

Sample Quiz Questions

Question 52 *Compute the centroid of the region bounded by the inequalities*

$$-2 \leq x \leq 0 \qquad \textit{and} \qquad 3x^2 - \frac{13}{2} \leq y \leq 3x^2 - \frac{11}{2}.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $\left(-\frac{3}{2}, -3\right)$

(b) $(-1, -3)$

(c) $\left(-\frac{1}{2}, -3\right)$

(d) $\left(-\frac{3}{2}, -2\right)$

(e) $(-1, -2)$ ✓

(f) $\left(-\frac{1}{2}, -2\right)$

Feedback(*`attempt`*): *The key calculations are as follows:*

$$\begin{aligned}
M &= \int_{-2}^{0} \left[\left(3x^2 - \frac{11}{2}\right) - \left(3x^2 - \frac{13}{2}\right)\right] dx, \\
M_y &= \int_{-2}^{0} x \left[\left(3x^2 - \frac{11}{2}\right) - \left(3x^2 - \frac{13}{2}\right)\right] dx, \\
M_x &= \frac{1}{2}\int_{-2}^{0} \left[\left(3x^2 - \frac{11}{2}\right)^2 - \left(3x^2 - \frac{13}{2}\right)^2\right] dx.
\end{aligned}$$

Hint:

$$\begin{aligned}
M &= \int_{-2}^{0} \left[\left(3x^2 - \frac{11}{2}\right) - \left(3x^2 - \frac{13}{2}\right)\right] dx = \int_{-2}^{0} [1]\, dx = 2, \\
M_y &= \int_{-2}^{0} x \left[\left(3x^2 - \frac{11}{2}\right) - \left(3x^2 - \frac{13}{2}\right)\right] dx = \int_{-2}^{0} x\, [1]\, dx = -2, \\
M_x &= \frac{1}{2}\int_{-2}^{0} \left[\left(3x^2 - \frac{11}{2}\right)^2 - \left(3x^2 - \frac{13}{2}\right)^2\right] dx = \int_{-2}^{0} \left[3x^2 - 6\right] = -4, \\
\overline{x} &= \frac{M_y}{M} = -1, \\
\overline{y} &= \frac{M_x}{M} = -2.
\end{aligned}$$

Sample Exam Questions

Question 53 *Find the y-coordinate of the centroid of the region bounded by the x-axis, the y-axis, and the graph of $y = \cos x$ for $0 \le x \le \pi/2$ if the density is constant.*

Hint: *Use the identity*

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

to calculate the integral of $\cos^2 x$.

Multiple Choice:

(a) $\frac{\pi}{18}$

(b) $\frac{\pi}{12}$

(c) $\frac{\pi}{8}$ ✓

(d) $\frac{\pi}{6}$

(e) $\frac{\pi}{4}$

(f) $\frac{\pi}{2}$

Feedback(`attempt`): *The area of the region is given by*

$$M = \int_0^{\frac{\pi}{2}} \cos x \; dx = 1$$

and

$$M_x = \int_0^{\frac{\pi}{2}} \frac{0 + \cos x}{2} \cos x \; dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^2 x \; dx$$
$$\frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} dx = \frac{\pi}{8}.$$

Therefore $\overline{y} = M_x/M = \pi/8$.

Question 54 *Find the y-coordinate of the centroid of the region in the upper half-plane (i.e., for $y > 0$) bounded by the semicircle $y = \sqrt{1 - x^2}$. (It is easiest to use a geometric formula to find the area of the region.)*

Multiple Choice:

(a) $\frac{4\pi}{3}$

(b) $\frac{4}{3\pi}$ ✓

(c) $\frac{7\pi}{3}$

(d) $\frac{7}{3\pi}$

(e) $\frac{28\pi}{9}$

(f) $\dfrac{28}{9\pi}$

2 Integration Techniques

We begin a study of techniques for computing integrals.

Introductory matter goes here...

2.1 Substitution and Tables

We review substitution and the use of integral tables.

Example 14. *The region in the plane given by*

$$\left|\frac{x}{2} - \frac{1}{2\sqrt{\frac{x^2}{3}+1}}\right| \leq y \leq \frac{x}{2} + \frac{1}{2\sqrt{\frac{x^2}{3}+1}}$$

and $0 \leq x \leq 3$ *is revolved around the* x*-axis. Compute the volume of the resulting solid.*

If the variable x *is used for slicing, then slices are perpendicular to the axis of rotation, which indicates the washer method should be used. The inequalities for* y *give the outer and inner radii, and*

$$\left(\frac{x}{2} + \frac{1}{2\sqrt{\frac{x^2}{3}+1}}\right)^2 - \left(\left|\frac{x}{2} - \frac{1}{2\sqrt{\frac{x^2}{3}+1}}\right|\right)^2 = \frac{\sqrt{3}x}{\sqrt{x^2+3}}.$$

(Note that the absolute values go away when the radius is squared.) This leads to the conclusion

$$V = \int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2+3}} dx.$$

Among the options below, the best choice for a potential substitution is

Multiple Choice:

(a) $u = \sqrt{3x}$

(b) $u = 1/\sqrt{x^2+3}$

(c) $u = \sqrt{x^2+3}$

(d) $u = x^2 + 3$ ✓

Using the substitution above, the differentials satisfy $du = \boxed{2x}dx$*. This gives the equality*

$$\int \frac{\sqrt{3}\pi x}{\sqrt{x^2+3}} dx = \int \boxed{\frac{\sqrt{3}\pi}{2\sqrt{u}}} du = \boxed{\sqrt{3}\pi\sqrt{u}}.$$

Reversing the substitution gives

$$\begin{aligned}\int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2+3}} dx &= \left[\boxed{\sqrt{3}\pi\sqrt{x^2+3}}\right]\Big|_0^3 \\ &= \boxed{3\pi}.\end{aligned}$$

Note: *We would not have to reverse the substitution if we also determined the new bounds. In this case, if $u = x^2 + 3$ and $x = 0$, then $u = \boxed{3}$. Likewise if $x = 3$, then $u = \boxed{12}$. Thus we could also have carried out the calculation by changing bounds:*

$$\int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2+3}}\,dx = \int_{\boxed{3}}^{\boxed{12}} \boxed{\frac{\sqrt{3}\pi}{2\sqrt{u}}}\,du.$$

Example 15. *Using the table of integrals below, compute the indefinite integral*

$$\int \frac{d\theta}{(1-\theta)\sqrt{\theta^2-2\theta-3}}.$$

- *When dealing with quadratic expressions such as the ones appearing in this integrand, it is often necessary to complete the square before appealing to a table. In this case,*

$$\theta^2 - 2\theta - 3 = \left(\theta - \boxed{1}\right)^2 - \boxed{4}.$$

- *Using the most appropriate entry of the table and plugging in the correct value of a gives*

$$\int \frac{dx}{x\sqrt{x^2 - \boxed{4}}} = \frac{1}{\boxed{2}}\operatorname{arcsec}\frac{x}{\boxed{2}} + C.$$

- *Based on the results of completing the square, we make a substitution $x = \theta - \boxed{1}$ and conclude*

$$\int \frac{d\theta}{(1-\theta)\sqrt{\theta^2-2\theta-3}} = \int \frac{dx}{\boxed{-x}\sqrt{x^2 - \boxed{4}}} = \boxed{-\frac{1}{2}}\operatorname{arcsec}\boxed{\frac{\theta-1}{2}} + C.$$

Example 16. *Use a table of integrals to compute the antiderivative below.*

$$\int \frac{x^2\,dx}{\sqrt{16+x^6}}$$

- *The key is to make the substitution $u = x^{\boxed{3}}$ so that the expression $16+x^6$ can be understood as a quadratic function of u.*

- *Specifically $du = \boxed{3x^2}dx$, so*

$$\int \frac{x^2\,dx}{\sqrt{16+x^6}} = \frac{1}{3}\int \frac{du}{\sqrt{16+u^2}}.$$

Aside from the factor of $1/3$, the u integral belongs to the table:

$$\int \frac{du}{\sqrt{16+u^2}} = \boxed{\ln|u + \sqrt{16+u^2}|} + C$$

- *We conclude that*

$$\int \frac{x^2\ dx}{\sqrt{16+x^6}} = \frac{1}{3}\int \frac{du}{\sqrt{16+u^2}} = \frac{\ln(x^3+\sqrt{16+x^6})}{3} + C.$$

Absolute values are not needed in the logarithm because $x^3+\sqrt{16+x^6}$ *can never be negative.*

A Basic Table of Integrals

(Note: To use these tables, k and a must represent *constants* in the integral you wish to compute and cannot depend on the variable of integration.)

$$\int x^k dx = \frac{x^{k+1}}{k+1} + C \qquad (k \neq -1)$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C \qquad (a > 0, a \neq 1)$$

$$\int e^x dx = e^x + C$$

$$\int x^k \ln|x| dx = \frac{x^{k+1}}{k+1} \ln|x| - \frac{x^{k+1}}{(k+1)^2} + C \qquad (k \neq -1)$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C \qquad (a \neq 0)$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \ln \left| x + \sqrt{a^2 + x^2} \right| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C \qquad (a > 0, |x| < a)$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| + C \qquad (|x| > |a|)$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{x}{a} + C \qquad (a > 0, x > a)$$

$$\int \sin x \; dx = -\cos x + C$$

$$\int \cos x \; dx = \sin x + C$$

$$\int \tan x \; dx = \ln|\sec x| + C$$

$$\int \csc x \; dx = -\ln|\csc x + \cot x| + C$$

$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x \; dx = \ln|\sin x| + C$$

$$\int \sec^2 x \; dx = \tan x + C$$

$$\int \csc^2 x \; dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C$$

$$\int \csc x \cot x dx = -\csc x + C$$

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2.2 Exercises: Substitution and Tables

Various exercises relating to substitution and the use of integral tables.

Compute the indefinite integrals below. Since there are many possible answers (which differ by constant values), use the given instructions if needed to choose which possible answer to use.

Exercise 55

$$\int (12x+14)\left(3x^2+7x-1\right)^5 dx = \boxed{\frac{1}{3}(3x^2+7x-1)^6} + C$$

(Add a constant to your answer if needed so that it equals 1/3 at $x=0$.)

Exercise 56

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \boxed{2e^{\sqrt{x}}} + C$$

(Add a constant to your answer if needed so that it equals 2 at $x=0$.)

Exercise 57

$$\int \frac{x}{\sqrt{x+3}} dx = \boxed{\frac{2}{3}(x-6)\sqrt{x+3}} + C$$

(Add a constant to your answer if needed so that it equals 0 at $x=6$.)

Exercise 58

$$\int \frac{\ln|x|}{x} dx = \boxed{\frac{1}{2}\ln^2|x|} + C$$

Remember absolute value in your logarithm. (Add a constant to your answer if needed so that it equals 0 at $x=1$.)

Exercise 59

$$\int \sin x\sqrt{\cos x} dx = \boxed{-\frac{2}{3}\cos^{\frac{3}{2}}(x)} + C$$

(Add a constant to your answer if needed so that it equals $-2/3$ at $x=0$.)

Exercise 60

$$\int \frac{9(2x+3)}{3x^2+9x+7} dx = \boxed{3\ln\left|3x^2+9x+7\right|} + C$$

Use absolute values as needed in logarithms. (Add a constant to your answer if needed so that it equals $3\ln 7$ at $x=0$.)

Exercise 61

$$\int \frac{x}{x^4+81} dx = \boxed{\frac{1}{18}\arctan\left(\frac{x^2}{9}\right)} + C$$

(Add a constant to your answer if needed so that it equals 0 at $x=0$.)

Hint: *Make a substitution $x^2 = 9u$.*

Exercise 62 *Evaluate the definite integral* $\displaystyle\int_{-2}^{-1} (x+1)e^{x^2+2x+1}\, dx$.

$$\text{Value} = \boxed{(1-e)/2}$$

Sample Exam Questions

Question 63 *Evaluate the integral* $\displaystyle\int_1^3 \left(x - \sqrt{4x^2-8x+13}\right) dx$ *using the fact that* $\displaystyle\int_0^4 \sqrt{x^2+9}\, dx = \frac{20+9\ln 3}{2}$. *(Hints will not be displayed until you have chosen a response.)*

Multiple Choice:

(a) $-\dfrac{4+9\ln 3}{6}$

(b) $-\dfrac{4+9\ln 3}{4}$ ✓

(c) $-\dfrac{4+9\ln 3}{2}$

(d) $-4-9\ln 3$

(e) $-8 - 18\ln 3$

(f) $-12 - 27\ln 3$

Feedback(*attempt*): *First complete the square:*

$$4x^2 - 8x + 13 = 4x^2 - 8x + 4 + (13 - 4) = (2x - 2)^2 + 9.$$

Hint: *Next substitute $u = 2x - 2$ (i.e., $x = \frac{u}{2} + 1$). Then $du = 2dx$ and $x = 1 \leftrightarrow u = 0$, $x = 3 \leftrightarrow u = 4$, so*

$$\int_1^3 \left(x - \sqrt{4x^2 - 8x + 13}\right) dx = \int_0^4 \left(\frac{u}{2} + 1 - \sqrt{u^2 + 9}\right) \frac{du}{2}.$$

Hint: *Now use linearity of the integral to finish:*

$$\int_0^4 \frac{u}{4}\, du + \int_0^4 \frac{1}{2}\, du - \int_0^4 \frac{1}{2}\sqrt{u^2+9}\, du = \left.\frac{u^2}{8}\right|_0^4 + \left.\frac{u}{2}\right|_0^4 - \frac{20 + 9\ln 3}{4} = 4 - \frac{20 + 9\ln 3}{4} = -\frac{4 + 9\ln 3}{4}.$$

2.3 Integration by Parts

We study the integration technique of integration by parts.

Example 17. *Compute the indefinite integral*

$$\int xe^{3x} dx.$$

- *Because integrating e^{3x} and differentiating e^{3x} are at a similar level of difficulty, we opt to differentiate x so that its degree as a polynomial will be decreasing.*
- *This gives*

$$\int xe^{3x} dx = \boxed{\frac{1}{3}xe^{3x}} - \int \boxed{\frac{1}{3}e^{3x}} dx = \boxed{\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}} + C.$$

Example 18. *Compute the indefinite integral*

$$\int x^2 \ln|x| dx.$$

- *Because integrating $\ln|x|$ is much harder than differentiating it, we choose to differentiate $\ln|x|$ and integrate x^2.*
- *This gives*

$$\int x^2 \ln|x| dx = \boxed{\frac{x^3}{3}} \ln|x| - \int \boxed{\frac{x^2}{3}} dx = \boxed{\frac{x^3}{3} \ln|x| - \frac{x^3}{9}} + C.$$

Example 19. *Compute the indefinite integral*

$$\int x^4 e^{2x} dx.$$

- *We'll do this using an organizational technique called "tabular integration" that many people find helpful when doing repeated integrations by parts.*
- *Make a table: one column for u and another for dv. In the first row, rewrite the functions that you will use for u and dv:*

u	dv
x^4	e^{2x}

- *Add new rows: differentiate items in the u column and integrate items in the dv column to determine what items go in the next row.*

u	dv
x^4	e^{2x}
$4x^3$	$e^{2x}/2$
$\boxed{12x^2}$	$\boxed{e^{2x}/4}$
$\boxed{24x}$	$\boxed{e^{2x}/8}$
$\boxed{24}$	$\boxed{e^{2x}/16}$

Now you combine terms by matching items in the first column with items one row down. For the last item in the u column, match it with the last item in the dv column (which you'll end up using twice):

$$x^4 \cdot \frac{e^{2x}}{2}, \qquad 4x^3 \cdot \boxed{\frac{e^{2x}}{4}}, \qquad \boxed{12x^2} \cdot \boxed{\frac{e^{2x}}{8}},$$
$$\boxed{24x} \cdot \boxed{\frac{e^{2x}}{16}}, \qquad \boxed{24} \cdot \boxed{\frac{e^{2x}}{16}}$$

- *To finish, you alternate addition and subtraction. Give a $+$ to the first term, a $-$ to the second, and so on. Last but not least, put an integral on the last term as well:*

$$+x^4 \frac{e^{2x}}{2} - \boxed{4x^3 \frac{e^{2x}}{4}} + \boxed{12x^2 \frac{e^{2x}}{8}} - \boxed{24x \frac{e^{2x}}{16}} + \int \boxed{24 \frac{e^{2x}}{16}} dx$$

In this way, we arrive at the formula

$$\int x^4 e^{2x} dx = \frac{1}{2} x^4 e^{2x} - x^3 e^{2x} + \frac{3}{2} x^2 e^{2x} - \frac{3}{2} x e^{2x} + \int \frac{3}{2} e^{2x} dx$$

- *Sometimes the integrand of the last term is zero. This would happen when u is a polynomial if you have enough lines in your table. In that case there would be nothing else to compute. (We stopped one line short of that point on purpose in this example to show how you would handle a case where there the last term isn't zero.)*

- *In this case, the final answer is*

$$\int x^4 e^{2x} dx = \frac{1}{2} x^4 e^{2x} - x^3 e^{2x} + \frac{3}{2} x^2 e^{2x} - \frac{3}{2} x e^{2x} + \frac{3}{4} e^{2x} + C$$

Example 20. *Compute the indefinite integral below using tabular integration:*

$$\int \cos 3x \cos 5x \; dx$$

- *Let us take $u = \cos 3x$ and $dv = \cos 5x$ (it would have been fine to choose them the other way around as well). The table is then*

u	dv
$\boxed{\cos 3x}$	$\boxed{\cos 5x}$
$\boxed{-3 \sin 3x}$	$\boxed{(\sin 5x)/5}$
$\boxed{-9 \cos 3x}$	$\boxed{-(\cos 5x)/25}$

Which gives

$$\int \cos 3x \cos 5x \; dx =$$

$$\boxed{\frac{1}{5} \cos 3x \sin 5x} - \boxed{\frac{3}{25} \sin 3x \cos 5x} + \int \boxed{\frac{9}{25} \cos 3x \cos 5x} \; dx.$$

- *This example has a twist: the integral on the right-hand side is just a constant times the integral on the left-hand side. This indicates that further integrations by parts would be unfruitful because you'd end up in something like a cycle. What you can do is solve for the answer by moving both integrals to the left side of the equation and then combining like terms:*

$$\boxed{\frac{16}{25}} \int \cos 3x \cos 5x \; dx = \boxed{\frac{1}{5} \cos 3x \sin 5x} - \boxed{\frac{3}{25} \sin 3x \cos 5x}$$

and therefore

$$\int \cos 3x \cos 5x \; dx = \boxed{\frac{5}{16} \cos 3x \sin 5x - \frac{3}{16} \sin 3x \cos 5x} + C.$$

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2.4 Exercises: Integration by Parts

Various exercises relating to integration by parts.

Compute the indefinite integrals below. Since there are many possible answers (which differ by constant values), use the given instructions if needed to choose which possible answer to use.

Exercise 64

$$\int x \sin x \; dx = \boxed{\sin x - x \cos x} + C$$

(Add a constant to your answer if needed so that it equals 0 *at* $x = 0$*.)*

Exercise 65

$$\int xe^{-x} \; dx = \boxed{-e^{-x} - xe^{-x}} + C$$

(Add a constant to your answer if needed so that it equals -1 *at* $x = 0$*.)*

Exercise 66

$$\int x^3 e^x \; dx \;\; = \boxed{x^3 e^x - 3x^2 e^x + 6xe^x - 6e^x} + C$$

(Write your answer so that it has no constant term.)

Exercise 67

$$\int x^2 \ln|x| \; dx = \boxed{\frac{1}{3}x^3 \ln|x| - \frac{x^3}{9}} + C$$

Don't forget absolute values in your logarithm. (Add a constant to your answer as necessary so that it equals $-1/9$ *at* $x = 1$*.)*

Exercise 68

$$\int e^x \sin x \; dx = \boxed{\frac{1}{2}e^x(\sin x - \cos x)} + C$$

(Add a constant to your answer if needed so that it equals $-1/2$ *at* $x = 0$*.)*

Hint: *This is a case in which you need to treat the integration by parts as an equation and solve for the answer.*

Exercise 69

$$\int \arcsin x \; dx = \boxed{\sqrt{1-x^2} + x\arcsin(x)} + C$$

(Add a constant to your answer if needed so that it equals 1 at $x = 0$.)

Hint: *Write* $\arcsin x = 1 \cdot \arcsin x$.

Sample Quiz Questions

Question 70 *Compute the definite integral*

$$\int_1^4 e^{3x}(x+1)\; dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $\frac{14}{9}e^{12} - \frac{5}{9}e^{3}$ ✓

(b) $\frac{17}{9}e^{12} - \frac{5}{9}e^{3}$

(c) $\frac{17}{9}e^{12} - \frac{8}{9}e^{3}$

(d) $\frac{20}{9}e^{12} - \frac{8}{9}e^{3}$

(e) $\frac{20}{9}e^{12} - \frac{11}{9}e^{3}$

(f) $\frac{23}{9}e^{12} - \frac{11}{9}e^{3}$

Feedback(`attempt`): *Integrate by parts, integrating the exponential and differentiating polynomials.*

Hint:

$$\begin{aligned}\int_1^4 e^{3x}(x+1)\ dx \\ &= \frac{e^{3x}}{3}(x+1)\bigg|_1^4 - \int_1^4 \frac{e^{3x}}{3}dx \\ &= \frac{5}{3}e^{12} - \frac{2}{3}e^3 - \frac{e^{3x}}{9}\bigg|_1^4 \\ &= \frac{14}{9}e^{12} - \frac{5}{9}e^3\end{aligned}$$

Question 71 *Compute the definite integral*

$$\int_\pi^{2\pi} x\sin 4x\ dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) 0

(b) $\frac{\pi}{4}$

(c) $-\frac{\pi}{4}$ ✓

(d) $\frac{3\pi}{4}$

(e) $-\frac{3\pi}{4}$

(f) $\frac{7\pi}{4}$

Feedback(`attempt`): *Integrate by parts, integrating the trig functions and differentiating polynomials.*

Hint:

$$\begin{aligned}\int_\pi^{2\pi} x\sin 4x\ dx \\ &= -\frac{\cos 4x}{4}x\bigg|_\pi^{2\pi} + \int_\pi^{2\pi} \frac{\cos 4x}{4}dx \\ &= -\frac{2\pi}{4} + \frac{(-1)^4\pi}{4} - \frac{\sin 4x}{16}\bigg|_\pi^{2\pi} = -\frac{\pi}{4}\end{aligned}$$

Question 72 *Compute the indefinite integral*

$$\int \arctan 5x \; dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $x \arctan 5x - \frac{1}{10} \ln|1 + 25x^2| + C$ ✓

(b) $x \arctan 5x - \frac{1}{12} \ln|1 + 25x^2| + C$

(c) $x \arctan 5x - \frac{1}{14} \ln|1 + 25x^2| + C$

(d) $x \arctan 5x + \frac{1}{10} \ln|1 + 25x^2| + C$

(e) $x \arctan 5x + \frac{1}{12} \ln|1 + 25x^2| + C$

(f) $x \arctan 5x + \frac{1}{14} \ln|1 + 25x^2| + C$

Feedback(`attempt`): *Integrate by parts, integrating the coefficient 1 and differentiating arctangent.*

Hint:

$$\begin{aligned}
&\int \arctan 5x \; dx \\
&\quad = x \arctan 5x - \int \frac{5x}{1 + 25x^2} \; dx \\
&\quad = x \arctan 5x - \frac{1}{10} \int \frac{50x}{1 + 25x^2} \; dx \\
&\quad = x \arctan 5x - \frac{1}{10} \ln|1 + 25x^2| + C
\end{aligned}$$

Question 73 *Compute the indefinite integral*

$$\int e^{3x} \cos 5x \; dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $\dfrac{e^{3x}(2\cos 5x + 5\sin 5x)}{29} + C$

(b) $\dfrac{e^{3x}(3\cos 5x + 5\sin 5x)}{34} + C$ ✓

(c) $\dfrac{e^{3x}(\cos 5x + 3\sin 5x)}{20} + C$

(d) $\dfrac{e^{3x}(\cos 5x + 2\sin 5x)}{15} + C$

(e) $\dfrac{e^{3x}(2\cos 5x + 7\sin 5x)}{53} + C$

(f) $\dfrac{e^{3x}(3\cos 5x + 7\sin 5x)}{58} + C$

Feedback(`attempt`): *Integrate by parts, integrating the exponential and differentiating cosine (or vice-versa), then solve for the antiderivative.*

Hint:

$$\begin{aligned}
&\int e^{3x}\cos 5x\ dx \\
&\quad = \frac{e^{3x}}{3}\cos 5x - \int \frac{e^{3x}}{3}(-5\sin 5x)\ dx \\
&\quad = \frac{e^{3x}\cos 5x}{3} + \frac{5}{3}\int e^{3x}\sin 5x\ dx \\
&\quad = \frac{e^{3x}\cos 5x}{3} + \frac{5}{3}\frac{e^{3x}}{3}\sin 5x - \frac{5}{3}\int \frac{e^{3x}}{3}(5\cos 5x)\ dx \\
&\quad = \frac{e^{3x}(3\cos 5x + 5\sin 5x)}{9} - \frac{25}{9}\int e^{3x}\cos 5x\ dx \\
\Rightarrow &\quad \frac{34}{9}\int e^{3x}\cos 5x\ dx = \frac{e^{3x}(3\cos 5x + 5\sin 5x)}{9} \\
\Rightarrow &\quad \int e^{3x}\cos 5x\ dx = \frac{e^{3x}(3\cos 5x + 5\sin 5x)}{34} + C
\end{aligned}$$

Sample Exam Questions

Question 74 *Compute the integral below.*

$$\int_{\frac{1}{2}}^{\infty} \frac{\ln(2x)}{x^2} dx$$

Multiple Choice:

(a) $1 - \ln 2$

(b) 2 ✓

(c) $\ln 2 - \dfrac{1}{2}$

(d) $\dfrac{1}{2}$

(e) $2 - 2\ln 2$

(f) *the integral diverges*

Question 75 *Compute the indefinite integral indicated below. [Hint: Write* $\dfrac{1}{\cos^2\theta} = \sec^2\theta$ *and integrate by parts.]*

$$\int \left(1 + \frac{\ln|\sin\theta|}{\cos^2\theta}\right) d\theta$$

Multiple Choice:

(a) $(\sin\theta)\ln|\sin\theta| + C$

(b) $(\cos\theta)\ln|\sin\theta| + C$

(c) $(\tan\theta)\ln|\sin\theta| + C$ ✓

(d) $(\csc\theta)\ln|\sin\theta| + C$

(e) $(\sec\theta)\ln|\sin\theta| + C$

(f) $(\cot\theta)\ln|\sin\theta| + C$

2.5 Trigonometric Integrals

We learn various techniques for integrating certain combinations of trigonometric functions.

Example 21. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from* $y = 0$ *to* $y = \sqrt{1-x^2}$. *Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ *between* $x = 0$ *and* $x = 1$, *since these are the most extreme values of x found in our region. Therefore*

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 22. *The base of a solid region is bounded by the curves* $y = 0$, $x = \sqrt{y}$, *and* $x = 1$. *The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph* $x = \boxed{\sqrt{y}}$ *to the graph* $x = \boxed{1}$. *The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is* $\boxed{1-\sqrt{y}}$, *giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ *between* $y = 0$ *and* $y = 1$, *since these are the most extreme values of y found in our region (note that we can find the upper value* $y = 1$ *by solving for the intersection of the curves* $x = \sqrt{y}$ *and* $x = 1$*). Therefore we integrate* $A(y)dy$ *to conclude*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`techniques/10trigonometricpractice.tex`

2.6 Exercises: Trigonometric Integral

Various exercises relating to the integration of trigonometric functions.

Compute the indefinite integrals below. Since there are many possible answers (which differ by constant values), use the given instructions if needed to choose which possible answer to use.

Exercise 76

$$\int \sin x \cos^4 x \; dx = \boxed{-\frac{1}{5}\cos^5(x)} + C$$

(Your answer should not include any constant term.)

Exercise 77

$$\int \sin^3 x \cos^3 x \; dx = \boxed{\frac{1}{6}\cos^6 x - \frac{1}{4}\cos^4 x} + C$$

(Your answer should not include any constant terms.)

Exercise 78

$$\int \tan^4 x \sec^2 x \; dx = \boxed{\frac{\tan^5(x)}{5}} + C$$

(Add a constant to your answer if needed so that it equals 0 at $x = 0$.)

Exercise 79

$$\int \tan^3 x \sec^3 x \; dx = \boxed{\frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3}} + C$$

(Add a constant to your answer if needed so that it equals $-2/15$ at $x = 0$.)

Exercise 80

$$\int \sin^2 x \cos^7 x \; dx = \boxed{-\frac{1}{9}\sin^9(x) + \frac{3\sin^7(x)}{7} - \frac{3\sin^5(x)}{5} + \frac{\sin^3(x)}{3}} + C$$

(Your answer should not include any constant terms.)

Hint:

$$(1-u^2)^3 = 1 - 3u^2 + 3u^4 - u^6.$$

Exercise 81

$$\int \sin(5x)\cos(3x)\ dx = \boxed{\frac{1}{2}\left(-\frac{1}{8}\cos(8x) - \frac{1}{2}\cos(2x)\right)} + C$$

(Your answer should not include any constant terms.)

Exercise 82

$$\int \sin^2 x \cos^2 x\ dx = \boxed{\frac{x}{8} - \frac{1}{32}\sin(4x)} + C$$

(Your answer should not include any constant terms and should equal 0 at $x = 0$.)

Hint: *Use power reduction formulas.*

Sample Quiz Questions

Question 83 *Compute the value of the integral*

$$\int_0^{\frac{\pi}{4}} \sin^3 2x\ dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $\frac{1}{5}$

(b) $\frac{1}{3}$ ✓

(c) $\frac{1}{2}$

(d) 1

(e) 2

(f) 3

Feedback(`attempt`): *To simplify the calculation, begin with a substitution which replaces x with $x/2$. The question reduces to computing*

$$\frac{1}{2}\int_{0}^{\frac{\pi}{2}} \sin^3 x \; dx.$$

This integral is compatible with the substitution $u = \cos x$.

Hint: *By the substitution formula, this means $dx = -du/\sin x$, and one must also replace $\sin^2 x$ by $1-u^2$. Furthermore, by virtue of the special angle formulas $\cos 0 = 1$ and $\cos\frac{\pi}{2} = 0$, the problem is reduced to computing the integral*

$$-\frac{1}{2}\int_{1}^{0} (1-u^2) \; du.$$

Hint: *Carrying out this calculation in the usual way gives a final answer of $\frac{1}{3}$.*

Question 84 *Compute the value of the integral*

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \tan^{-6} x \sec^5 x \; dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $\frac{17}{5}$

(b) $\frac{19}{5}$

(c) $\frac{23}{5}$

(d) $\frac{29}{5}$

(e) $\frac{31}{5}$ ✓

(f) $\frac{37}{5}$

Feedback(`attempt`): *Since the power of secant is odd and the power of tangent is even, try rewriting the integral in terms of sine and cosine. This gives*

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^{-6} x \cos x \, dx.$$

This integral is compatible with the substitution $u = \sin x$.

Hint: *By the substitution formula, this means $dx = du/\cos x$. Furthermore, by virtue of the special angle formulas $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\sin\frac{\pi}{2} = 1$, the problem is reduced to computing the integral*

$$\int_{\frac{1}{2}}^{1} u^{-6} \, du.$$

Hint: *Carrying out this calculation in the usual way gives a final answer of $\frac{31}{5}$.*

Question 85 *Compute the value of the integral*

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^{-2} x \cos^3 x \, dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $\frac{1}{5}$

(b) $\frac{1}{3}$

(c) $\frac{1}{2}$ ✓

(d) 1

(e) 2

(f) 3

Feedback(`attempt`): *This integral is compatible with the substitution $u = \sin x$. By the substitution formula, this means $dx = du/\cos x$, and one must also replace $\cos^2 x$ by $1 - u^2$. Furthermore, by virtue of the special angle formulas $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\sin\frac{\pi}{2} = 1$, the problem is reduced to computing the integral*

Hint:

$$\int_{\frac{1}{2}}^{1} u^{-2}(1-u^2)\ du.$$

Hint: *Carrying out this calculation in the usual way gives a final answer of* $\frac{1}{2}$.

Sample Exam Questions

Question 86 *Compute the integral below.*

$$\int_0^{\frac{\pi}{8}} \tan^4 2x \sec^4 2x\ dx$$

Multiple Choice:

(a) $\frac{4}{9}$

(b) $\frac{7}{24}$

(c) $\frac{5}{14}$

(d) $\frac{9}{28}$

(e) $\frac{6}{35}$ ✓

(f) $\frac{1}{7}$

2.7 Trigonometric Substitions

We practice executing trigonometric substitutions.

Example 23. *Suppose the region below the graph* $y = \sqrt{\sin x}$ *and above the* x*-axis between* $x = 0$ *and* $x = \pi$ *is revolved around the* x*-axis. Compute the volume of the resulting solid.*

- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*

- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓

  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}} \right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 24. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*

- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

`techniques/11trigsubpractice.tex`

2.8 Exercises: Trigonometric Substitions

Various exercises relating to trigonometric substitutions.

Compute the indefinite integrals below. Since there are many possible answers (which differ by constant values), use the given instructions if needed to choose which possible answer to use. Do not forget absolute value signs inside logarithms when they are needed.

Exercise 87

$$\int x^2\sqrt{1-x^2}\ dx = \boxed{\frac{1}{8}\arcsin x - \frac{1}{8}x\sqrt{1-x^2}(1-2x^2)} + C$$

(Choose your answer to equal 0 *at* $x=0$*.)*

Exercise 88

$$\int \frac{1}{(x^2+1)^2}\ dx = \boxed{\frac{1}{2}\left(\arctan x + \frac{x}{x^2+1}\right)} + C$$

(Choose your answer to equal 0 *at* $x=0$*.)*

Exercise 89

$$\int \frac{x^2}{\sqrt{x^2+3}}\ dx = \boxed{\frac{1}{2}x\sqrt{x^2+3} - \frac{3}{2}\ln\left|\frac{\sqrt{x^2+3}}{\sqrt{3}} + \frac{x}{\sqrt{3}}\right|} + C$$

(Choose your answer to equal 0 *at* $x=0$*.)*

Sample Quiz Questions

Question 90 *Compute the integral*

$$\int_{-2}^{2} \frac{5}{(5-x^2)^{3/2}}\ dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) 2

(b) 3

(c) 4 ✓

(d) 5

(e) 6

(f) 7

Feedback(`attempt`): *Begin by making the trig substitution* $x = \sqrt{5}\sin\theta$.

Hint: *It follows that*

$$\begin{aligned}\int \frac{5}{(5-x^2)^{3/2}}\ dx &= \int \frac{5}{(5-(\sqrt{5}\sin\theta)^2)^{3/2}} \cdot (\sqrt{5}\cos\theta)\ d\theta \\ &= \int (\cos\theta)^{-2}\ d\theta \\ &= \int (\sec\theta)^2\ d\theta = (\tan\theta) + C.\end{aligned}$$

Hint: *To finish, use the inversion identity*

$$\tan\theta = \frac{x}{\sqrt{5-x^2}}.$$

Therefore

$$\int_{-2}^{2} \frac{5}{(5-x^2)^{3/2}}\ dx = \left.\frac{x}{\sqrt{5-x^2}}\right|_{-2}^{2} = (2) - (-2) = 4.$$

Question 91 *Compute the integral*

$$\int_{-1}^{1} \frac{3}{(3+x^2)^{3/2}}\ dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $\frac{1}{2}$

(b) 1 ✓

(c) $\frac{3}{2}$

(d) 2

(e) $\dfrac{5}{2}$

(f) 3

Feedback(*`attempt`*): *Begin by making the trig substitution* $x = \sqrt{3}\tan\theta$.

Hint: *It follows that*

$$\begin{aligned}\int \frac{3}{(3+x^2)^{3/2}}\ dx &= \int \frac{3}{(3+(\sqrt{3}\tan\theta)^2)^{3/2}} \cdot (\sqrt{3}\sec^2\theta)\ d\theta \\ &= \int (\sec\theta)^{-1}\ d\theta \\ &= \int (\cos\theta)\ d\theta = (\sin\theta) + C.\end{aligned}$$

Hint: *To finish, use the inversion identity*

$$\cos\theta = \frac{x}{\sqrt{3+x^2}}.$$

Therefore

$$\int_{-1}^{1} \frac{3}{(3+x^2)^{3/2}}\ dx = \left.\frac{x}{\sqrt{3+x^2}}\right|_{-1}^{1} = \left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right) = 1.$$

Question 92 *Compute the integral*

$$\int_4^5 \frac{16\sqrt{x^2-16}}{x^4}\ dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $\dfrac{4}{125}$

(b) $\dfrac{1}{25}$

(c) $\dfrac{6}{125}$

(d) $\dfrac{7}{125}$

(e) $\dfrac{8}{125}$

(f) $\dfrac{9}{125}$ ✓

Feedback(`attempt`): *Begin by making the trig substitution $x = 4\sec\theta$.*

Hint: *It follows that*

$$\begin{aligned}\int \frac{16\sqrt{x^2-16}}{x^4}\,dx &= \int \frac{16\sqrt{(4\sec\theta)^2-16}}{(4\sec\theta)^4}\cdot(4\sec\theta\tan\theta)\;d\theta\\ &= \int (\sec\theta)^{-3}(\tan\theta)^2\;d\theta\\ &= \int (\sin\theta)^2(\cos\theta)\;d\theta = \frac{1}{3}(\sin\theta)^3 + C.\end{aligned}$$

Hint: *To finish, use the inversion identity*

$$\sin\theta = \frac{\sqrt{x^2-16}}{x}.$$

Therefore

$$\int_4^5 \frac{16\sqrt{x^2-16}}{x^4}\,dx = \frac{1}{3}\,\frac{(x^2-16)^{3/2}}{x^3}\bigg|_4^5 = \left(\frac{9}{125}\right) - (0) = \frac{9}{125}.$$

Sample Exam Questions

Question 93 *Compute the value of the integral below.*

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$$

Multiple Choice:

(a) 0

(b) 1 ✓

(c) 2

(d) 3

(e) 4

(f) *none of these*

Question 94 *Evaluate* $\displaystyle\int_0^3 \frac{dx}{(25-x^2)^{3/2}}$.

Multiple Choice:

(a) 0

(b) $\frac{1}{100}$

(c) $\frac{3}{100}$ ✓

(d) $\frac{5}{100}$

(e) $\frac{7}{100}$

(f) *none of these*

2.9 Partial Fractions

We study the technique of partial fractions and its application to integration.

Example 25. *The region defined by the inequalities* $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ *and* $x \ge 0$ *(shown below) is revolved around the y-ais. Compute the volume using the shell method.*

- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis* $x = 0$*. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

  Multiple Choice:

  (a) $h(x) = \sqrt{1-x^2}$

  (b) $h(x) = -\sqrt{1-x^2}$

  (c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\, dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$*.)*

Example 26. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*

- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$
  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$
  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\, dy = \boxed{\frac{\pi}{3}}.$$

techniques/12partialfractionspractice.tex

2.10 Exercises: Partial Fractions

Various exercises relating to partial fractions and integration.

Compute the indefinite integrals below. Since there are many possible answers (which differ by constant values), use the given instructions if needed to choose which possible answer to use. Do not forget absolute value signs inside logarithms when they are needed.

Exercise 95

$$\frac{7x+7}{x^2+3x-10} = \frac{\boxed{3}}{x-2} + \frac{\boxed{4}}{x+5}$$

$$\int \frac{7x+7}{x^2+3x-10}\,dx = \boxed{3\ln|x-2| + 4\ln|x+5|} + C$$

(Do not include any constant terms in your antiderivative.)

Exercise 96

$$\int \frac{7x-2}{x^2+x}\,dx = \boxed{9\ln|x+1| - 2\ln|x|} + C$$

(Do not include any constant terms in your antiderivative.)

Exercise 97

$$\frac{x+7}{(x+5)^2} = \frac{\boxed{1}}{x+5} + \frac{\boxed{2}}{(x+5)^2}$$

$$\int \frac{x+7}{(x+5)^2}\,dx = \boxed{\ln|x+5| - \frac{2}{x+5}} + C$$

(Do not include any constant terms in your antiderivative.)

Exercise 98

$$\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx = \boxed{\frac{5}{x+1} + 7\ln|x| + 2\ln|x+1|} + C$$

(Do not include any constant terms in your answer.)

Exercise 99

$$\int \frac{x^2+x+1}{x^2+x-2}\,dx = \boxed{x+\ln|x-1|-\ln|x+2|}+C$$

(Do not include any constant terms in your answer.)

Hint: *Don't forget polynomial long division; it is needed in this case because the degree of the numerator is at least as large as the degree of the denominator.*

Exercise 100

$$\int \frac{x^2+x+5}{x^2+4x+10}\,dx = \boxed{-\frac{3}{2}\ln\left|x^2+4x+10\right|+x+\frac{\arctan\left(\frac{x+2}{\sqrt{6}}\right)}{\sqrt{6}}}+C$$

(Do not include any constant terms in your answer.)

Hint:

$$\frac{x^2+x+5}{x^2+4x+10} = \boxed{1}+\frac{\boxed{-3}x+\boxed{-5}}{x^2+4x+10}$$

Since the derivative of the denominator is $2x+4$, we should rewrite the numerator of the big fraction to have $x+2$'s if possible:

$$\frac{x^2+x+5}{x^2+4x+10} = \boxed{1}+\frac{\boxed{-3}(x+2)+\boxed{1}}{x^2+4x+10}.$$

For expressions like

$$\int \frac{x+2}{x^2+4x+10}dx$$

we should do a substitution. For terms like

$$\int \frac{1}{x^2+4x+10}dx$$

we should first complete the square: $x^2+4x+10=(x+2)^2+6$ and then make the substitution $x+2=u\sqrt{6}$.

Exercise 101

$$\int \frac{2x^2+x+1}{(x+1)(x^2+9)}\,dx = \boxed{\frac{9}{10}\ln\left|x^2+9\right|+\frac{1}{5}\ln|x+1|-\frac{4}{15}\arctan\left(\frac{x}{3}\right)}+C$$

(Do not include any constant terms in your answer.)

Sample Quiz Questions

Question 102 *Compute the integral*

$$\int_3^4 \frac{2x-3}{x^2-3x+2}\, dx.$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $\ln 2$

(b) $\ln 3$ ✓

(c) $\ln 4$

(d) $\ln 5$

(e) $\ln 6$

(f) $\ln 7$

Feedback(`attempt`): *First factor the denominator of the integrand:* $x^2 - 3x + 2 = (x-1)(x-2)$. *Since the roots are distinct, it is possible to use the Heaviside cover-up method.*

Hint: *The partial fractions expansion will take the form*

$$\frac{A}{x-1} + \frac{B}{x-2},$$

where the coefficient A *can be computed by cancelling the factor of* $x-1$ *in the denominator and evaluating the result at* $x = 1$, *i.e.,*

$$A = \frac{2(1)-3}{(1)-2} = 1.$$

Similarly,

$$B = \frac{2(2)-3}{(2)-1} = 1,$$

which gives that

$$\frac{2x-3}{(x-1)(x-2)} = \frac{1}{x-1} + \frac{1}{x-2}.$$

Hint: *Therefore*

$$\begin{aligned}\int_3^4 \frac{2x-3}{x^2-3x+2}\, dx &= \int_3^4 \left(\frac{1}{x-1} + \frac{1}{x-2}\right) dx \\ &= (\ln|4-1| + \ln|4-2|) - (\ln|3-1| + \ln|3-2|) \\ &= \ln 3 + \ln 2 + \ln\frac{1}{2} + 0 = \ln 3.\end{aligned}$$

Sample Exam Questions

Question 103 *Compute the volume of the solid of revolution obtained by revolving around the y-axis the region below the graph*

$$y = \frac{1}{(x-1)^2},$$

above $y = 0$, and between $x = 2$ and $x = 3$. (Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) π

(b) $\pi(\ln 2 + 3)$

(c) $\pi(2\ln 2 + 1)$ ✓

(d) $\pi(2\ln 3 + 1)$

(e) $\pi(3\ln 2 + 1)$

(f) $\pi(3\ln 3 + 1)$

Feedback(`attempt`): *Choosing x as the variable of integration, slices will be parallel to the y-axis, indicating that the shell method should be used. The radius of a shell is x (because the axis lies to the left of the region) and the height will be $(x-1)^{-2}$, so*

$$V = \int_2^3 \frac{2\pi x}{(x-1)^2}\,dx = 2\pi \int_2^3 \frac{x}{(x-1)^2}\,dx.$$

Hint: *The integral can be computed by partial fractions; the expansion has the form*

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

The coefficients A and B can be found by usual methods (but note that the Heaviside cover up method will not work in this case), but it is also possible to find them directly by carefully rewriting the numerator of the fraction in terms of $x-1$:

$$\frac{x}{(x-1)^2} = \frac{(x-1)+1}{(x-1)^2} = \frac{(x-1)}{(x-1)^2} + \frac{1}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2}.$$

Therefore

$$V = 2\pi \int_2^3 \left[\frac{1}{x-1} + \frac{1}{(x-1)^2}\right] dx = 2\pi \left[\ln|x-1| - \frac{1}{x-1}\right]\Bigg|_2^3$$
$$= 2\pi\left(\ln 2 - \frac{1}{2}\right) - 2\pi\,(0-1) = \pi(2\ln 2 + 1).$$

Question 104 *Compute the constants A and B in the partial fractions expansion indicated below. To receive full credit, it is not necessary to compute C, D, or E.*

$$\frac{x^4+16}{x^4-16} = A + \frac{B}{x-2} + \frac{C}{x+2} + \frac{Dx+E}{x^2+4}$$

(Hints won't be revealed until after you choose a response.)

Multiple Choice:

(a) $A=-1, B=1$

(b) $A=0, B=1$

(c) $A=1, B=1$ ✓

(d) $A=-1, B=-1$

(e) $A=0, B=-1$

(f) $A=1, B=-1$

Feedback(`attempt`): *You'll need to do polynomial long division first. To compute B, you can use Heaviside cover-up.*

Hint:

$$\frac{x^4+16}{x^4-16} = 1 + \frac{1}{x-2} - \frac{1}{x+2} - \frac{4}{x^2+4}$$

Question 105 *Evaluate* $\displaystyle\int_1^2 \frac{x^2+x+1}{x^2+x}dx$.

Multiple Choice:

(a) 0

(b) 1

(c) $1+\ln\left(\frac{4}{3}\right)$ ✓

(d) 2

(e) $2+\ln\left(\frac{8}{3}\right)$

(f) *none of these*

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techniques/13finalpractice.tex

2.11 Exercises: Cumulative

Exercises relating to various topics we have studied.

Sample Quiz Questions

Question 106 *The region in the plane between the x-axis and the graph*

$$y = \frac{1}{2}\ln x + 1$$

in the range $\frac{1}{5} \le x \le 1$ is revolved around the axis $x = \frac{1}{10}$. Compute the volume of the resulting solid.

Multiple Choice:

(a) $\frac{11}{25}\pi$

(b) $\frac{14}{25}\pi$

(c) $\frac{16}{25}\pi$ ✓

(d) $\frac{19}{25}\pi$

(e) $\frac{21}{25}\pi$

(f) $\frac{24}{25}\pi$

Feedback(*`attempt`*): *If the variable x is used for slicing, then slices are parallel to the axis of rotation, which indicates the shell method should be used. The radius of a shell is $x - \frac{1}{10}$. The height of a shell is exactly $\frac{1}{2}\ln x + 1$. The volume of the region is therefore given by*

$$\int_{\frac{1}{5}}^{1} \frac{\pi}{10}\left(10x - 1\right)\left(\ln x + 2\right)\, dx.$$

To compute the integralwe can use integration by parts. A reasonable strategy is to integrate $x - \frac{1}{10}$ and differentiate $\ln x + 2$. This gives the equality

$$\begin{aligned}\pi \int \left(x - \frac{1}{10}\right)\left(\ln x + 2\right)\, dx &= \pi\left(\left(\frac{x^2}{2} - \frac{x}{10}\right)\left(\ln x + 2\right) - \int \frac{1}{x}\left(\frac{x^2}{2} - \frac{x}{10}\right) dx\right) \\ &= -\pi\left(\frac{x^2}{4} - \frac{x}{10}\right) + \pi\left(\frac{x^2}{2} - \frac{x}{10}\right)\left(\ln x + 2\right).\end{aligned}$$

Therefore

$$\pi \int_{\frac{1}{5}}^{1} \left(x - \frac{1}{10}\right) (\ln x + 2)\ dx = \left[-\pi \left(\frac{x^2}{4} - \frac{x}{10} \right) + \pi \left(\frac{x^2}{2} - \frac{x}{10} \right) (\ln x + 2) \right] \Bigg|_{\frac{1}{5}}^{1}$$
$$= \left(\frac{13}{20}\pi \right) - \left(\frac{\pi}{100} \right) = \frac{16}{25}\pi.$$

Question 107 *Consider the region given by* $2\pi \leq x \leq \frac{5}{2}\pi$ *and* $0 \leq y \leq \sin x$. *Compute the* x*-coordinate of the centroid (i.e., assuming constant density).*

Multiple Choice:

(a) $-1 + \frac{5}{2}\pi$

(b) $1 + 2\pi$ ✓

(c) $\frac{5}{2}\pi$

(d) $-1 + 3\pi$

(e) 3π

(f) 4π

Feedback(`attempt`): *The mass* M *will be given by the integral*

$$\int_{2\pi}^{\frac{5}{2}\pi} \sin x\, dx$$

One can check that

$$\int_{2\pi}^{\frac{5}{2}\pi} \sin x\, dx = 1.$$

To compute the x*-coordinate of the centroid, we also need to compute the integral*

$$\int_{2\pi}^{\frac{5}{2}\pi} x \sin x\, dx$$

To compute the integralwe can use integration by parts. A reasonable strategy is to integrate $\sin x$ *and differentiate* x. *This gives the equality*

$$\int x \sin x\, dx = -x \cos x - \int (-\cos x)\ dx$$
$$= -x \cos x + \sin x.$$

Therefore

$$\int_{2\pi}^{\frac{5}{2}\pi} x \sin x\, dx = \left[-x \cos x + \sin x \right] \Big|_{2\pi}^{\frac{5}{2}\pi}$$
$$= 1 - (-2\pi) = 1 + 2\pi.$$

The corret answer is the ratio of the integrals, i.e.,

$$\overline{x} = \frac{1 + 2\pi}{1} = 1 + 2\pi.$$

Sample Exam Questions

Question 108 *An object moves in such a way that its acceleration at time t seconds is $(t^2 + 5t + 6)^{-1}$ meters per second squared. If the initial velocity of the object is $2/3$ meters per second, what is the limit of its velocity as $t \to \infty$?*

Multiple Choice:

(a) $\ln \frac{3}{2}$ *meters per second*

(b) $\ln 6$ *meters per second*

(c) 1 *meters per second*

(d) $\ln \frac{4}{9}$ *meters per second*

(e) $\ln \frac{9}{4}$ *meters per second*

(f) 0 *meters per second* ✓

Question 109 *Find the volume of the solid generated by revolving the region bounded above by $y = \sin x$ and bounded below by $y = 0$ for $0 \leq x \leq \pi$ about the line $x = \pi$.*

Multiple Choice:

(a) π^2

(b) $2\pi^2$ ✓

(c) $4\pi^2$

(d) $\frac{\pi^2}{2}$

(e) $\frac{\pi^2}{4}$

(f) *none of these*

Question 110 *Evaluate* $\int_1^2 x\ln(x^2+1)dx$.

Multiple Choice:

(a) 0

(b) 1

(c) $\ln 2$

(d) $\frac{1}{2}$

(e) $\ln(2)-\frac{1}{2}$

(f) *none of these* ✓

Feedback(*`attempt`*): *This integral can be computed via integration by parts. If we integrate x and differentiate $\ln(x^2+1)$, we get*

$$\begin{aligned}\int_1^2 x\ln(x^2+1)dx &= \frac{x^2}{2}\ln(x^2+1)\bigg|_1^2 - \int_1^2 \frac{x^2}{2}\frac{2x}{x^2+1}dx\\ &= 2\ln 5 - \frac{1}{2}\ln 2 - \int_1^2 \frac{x^3}{x^2+1}dx.\end{aligned}$$

The latter integral can be simplified using polynomial long division: $\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$. *Therefore*

$$\begin{aligned}\int_1^2 x\ln(x^2+1)dx &= 2\ln 5 - \frac{1}{2}\ln 2 - \int_1^2 xdx + \int_1^2 \frac{x}{x^2+1}dx\\ &= 2\ln 5 - \frac{1}{2}\ln 2 - \frac{x^2}{2}\bigg|_1^2 + \frac{1}{2}\ln(x^2+1)\bigg|_1^2\\ &= 2\ln 5 - \frac{1}{2}\ln 2 - 2 + \frac{1}{2} + \frac{1}{2}\ln 5 - \frac{1}{2}\ln 2\\ &= \frac{5}{2}\ln 5 - \frac{2}{2}\ln 4 - \frac{3}{2} - \frac{3}{2} = \ln\left(\frac{5^5}{4}\right) - \frac{3}{2}.\end{aligned}$$

3 Further Topics in Integration

We study additional topics relating to applications of integration.

Example 27. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 28. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

3.1 Numerical Integration

We study the problem of numerically approximating the value of an integral.

Example 29. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 30. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

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3.2 Exercises: Numerical Integration

Various exercises relating to numerical integration.

Exercise 111 *Consider the definite integral* $\int_{-1}^{1} x^2\,dx$.

- *The trapeziod rule with* $n = 4$ *gives the approximation*

$$\int_{-1}^{1} x^2\,dx \approx \boxed{\frac{3}{4}}.$$

- *Simpson's rule with* $n = 4$ *gives the approximation*

$$\int_{-1}^{1} x^2\,dx \approx \boxed{\frac{2}{3}}.$$

- *The exact value of the integral is*

$$\int_{-1}^{1} x^2\,dx = \boxed{\frac{2}{3}}.$$

Exercise 112 *When estimating the integral below using Simpson's rule, what is the minimum number of intervals that would be required to guarantee that the approximation error does not exceed* 2×10^{-5}*? (Enter the smallest value which you know is correct.)*

$$\int_{-1}^{0} e^{x\sqrt{6}} dx$$

$$n \geq \boxed{10}.$$

Exercise 113 *Find* n *such that the error in approximating the given definite integral is less than* 0.0001 *when using:*

- *The trapezoid rule:* $n \geq \boxed{\sqrt{\frac{10000\pi^3}{12}}} \approx 161$. *(Enter your answer as the exact result of your calculation; do not round or approximate.)*

(b) *Simpson's rule:* $n \geq \boxed{\left(\frac{10000\pi^5}{180}\right)^{1/4}} \approx 12$. *(Enter your answer as the exact result of your calculation; do not round or approximate.)*

Exercise 114 *How many equally spaced intervals N are sufficient for the trapezoidal rule to estimate the value of the following integral with an error less than or equal to 10^{-6}? (Enter the smallest value which you know is correct.)*

$$\int_{-1}^{1} e^{x^2-1}dx.$$

$$N \geq \boxed{2000}.$$

Hint: *The second derivative of the integrand is $(4x^2+2)e^{x^2-1}$. Since $-1 \leq x \leq 1$, it follows that*

$$(4x^2+2)e^{x^2-1} \leq (4 \cdot 1 + 2)e^0 = 6.$$

Hint: *By the trapezoid rule error formula, the error E satisfies*

$$|E| \leq \frac{6(1-(-1))^3}{12N^2} = \frac{4}{N^2},$$

where N is the number of intervals.

Hint: *To be sure the error is small enough, we need*

$$\frac{4}{N^2} \leq 10^{-6}$$

3.3 Orders of Growth

We study the use of orders of growth to compute limits, in preparation for improper integrals.

Example 31. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 32. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

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3.4 Exercises: Orders of Growth

Various exercises relating to orders of growth.

Sample Quiz Questions

Question 115 *Arrange the functions*

$$x^x \qquad \frac{e^x}{\ln x} \qquad \ln x$$

in order from least rate of growth to greatest rate of growth as $x \to \infty$. Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

Multiple Choice:

(a) $x^x \ll \dfrac{e^x}{\ln x} \ll \ln x$

(b) $\ln x \ll x^x \ll \dfrac{e^x}{\ln x}$

(c) $\dfrac{e^x}{\ln x} \ll \ln x \ll x^x$

(d) $x^x \ll \ln x \ll \dfrac{e^x}{\ln x}$

(e) $\dfrac{e^x}{\ln x} \ll x^x \ll \ln x$

(f) $\ln x \ll \dfrac{e^x}{\ln x} \ll x^x$ ✓

Feedback(*`attempt`*): *General Remarks:*

- *Higher powers of x grow faster at infinity than lower powers of x.*
- *As $x \to \infty$, $\ln x$ goes to infinity slower than x^p for any (presumably small) positive constant p.*
- *As $x \to \infty$, e^x goes to infinity faster than x^n for any (presumably large) positive constant n.*
- *As $x \to \infty$, x^x goes to infinity faster than any exponential of the form e^{cx} for any constant c.*

Question 116 *Arrange the functions*

$$\frac{e^{-x}}{\ln x} \qquad x^3 \ln x \qquad e^x$$

in order from least rate of growth to greatest rate of growth as $x \to \infty$. Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

Multiple Choice:

(a) $\dfrac{e^{-x}}{\ln x} \ll x^3 \ln x \ll e^x$ ✓

(b) $e^x \ll \dfrac{e^{-x}}{\ln x} \ll x^3 \ln x$

(c) $x^3 \ln x \ll e^x \ll \dfrac{e^{-x}}{\ln x}$

(d) $\dfrac{e^{-x}}{\ln x} \ll e^x \ll x^3 \ln x$

(e) $x^3 \ln x \ll \dfrac{e^{-x}}{\ln x} \ll e^x$

(f) $e^x \ll x^3 \ln x \ll \dfrac{e^{-x}}{\ln x}$

Feedback(*`attempt`*): *General Remarks:*

- *Higher powers of x grow faster at infinity than lower powers of x.*
- *As $x \to \infty$, $\ln x$ goes to infinity slower than x^p for any (presumably small) positive constant p.*
- *As $x \to \infty$, e^x goes to infinity faster than x^n for any (presumably large) positive constant n.*

Question 117 *Arrange the functions*

$$\left(\ln \frac{1}{x}\right)^2 \qquad \frac{e^{-x}}{x^3} \ln x \qquad x^3 e^x$$

in order from least rate of growth to greatest rate of growth as $x \to 0^+$. Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

Multiple Choice:

(a) $\left(\ln \frac{1}{x}\right)^2 \ll \frac{e^{-x}}{x^3} \ln x \ll x^3 e^x$

(b) $x^3 e^x \ll \left(\ln \frac{1}{x}\right)^2 \ll \frac{e^{-x}}{x^3} \ln x$ ✓

(c) $\frac{e^{-x}}{x^3} \ln x \ll x^3 e^x \ll \left(\ln \frac{1}{x}\right)^2$

(d) $\left(\ln \frac{1}{x}\right)^2 \ll x^3 e^x \ll \frac{e^{-x}}{x^3} \ln x$

(e) $\frac{e^{-x}}{x^3} \ln x \ll \left(\ln \frac{1}{x}\right)^2 \ll x^3 e^x$

(f) $x^3 e^x \ll \frac{e^{-x}}{x^3} \ln x \ll \left(\ln \frac{1}{x}\right)^2$

Feedback(`attempt`): *General Remarks:*

- *Lower powers of x grow faster as $x \to 0^+$ than higher powers of x.*
- *As $x \to 0^+$, $-\ln x = \ln x^{-1}$ goes to ∞ slower than x^{-p} for any (presumably small) positive p.*
- *As $x \to 0^+$, $e^x \to 1$ and so does not influence the growth rate.*
- *As $x \to 0^+$, $e^{-x} \to 1$ and so does not influence the growth rate.*

Question 118 *Arrange the functions*

$$\frac{x^3 e^x}{\ln x} \qquad \frac{e^{-x}}{x^3} \qquad \frac{e^{-x}}{x^3 \ln x}$$

in order from least rate of growth to greatest rate of growth as $x \to 0^+$. Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

Multiple Choice:

(a) $\frac{x^3 e^x}{\ln x} \ll \frac{e^{-x}}{x^3} \ll \frac{e^{-x}}{x^3 \ln x}$

(b) $\frac{e^{-x}}{x^3 \ln x} \ll \frac{x^3 e^x}{\ln x} \ll \frac{e^{-x}}{x^3}$

(c) $\frac{e^{-x}}{x^3} \ll \frac{e^{-x}}{x^3 \ln x} \ll \frac{x^3 e^x}{\ln x}$

(d) $\frac{x^3 e^x}{\ln x} \ll \frac{e^{-x}}{x^3 \ln x} \ll \frac{e^{-x}}{x^3}$ ✓

(e) $\frac{e^{-x}}{x^3} \ll \frac{x^3 e^x}{\ln x} \ll \frac{e^{-x}}{x^3 \ln x}$

(f) $\frac{e^{-x}}{x^3 \ln x} \ll \frac{e^{-x}}{x^3} \ll \frac{x^3 e^x}{\ln x}$

Feedback(`attempt`): *General Remarks:*

- *Lower powers of x grow faster as $x \to 0^+$ than higher powers of x.*
- *As $x \to 0^+$, $-\ln x = \ln x^{-1}$ goes to ∞ slower than x^{-p} for any (presumably small) positive p.*
- *As $x \to 0^+$, $e^x \to 1$ and so does not influence the growth rate.*
- *As $x \to 0^+$, $e^{-x} \to 1$ and so does not influence the growth rate.*

3.5 Improper Integrals

We study the concept of improper integrals.

Example 33. *The region defined by the inequalities $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ and $x \ge 0$ (shown below) is revolved around the y-ais. Compute the volume using the shell method.*

- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

  Multiple Choice:

  (a) $h(x) = \sqrt{1-x^2}$
  (b) $h(x) = -\sqrt{1-x^2}$
  (c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\,dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$.)

Example 34. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*

- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$
  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$
  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\,dy = \boxed{\frac{\pi}{3}}.$$

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3.6 Exercises: Improper Integrals

Various exercises relating to improper integrals.

Exercise 119 *Evaluate the improper integral:* $\displaystyle\int_0^{\infty} e^{5-2x}\ dx = \boxed{(e^5)/2}$.

Exercise 120 *Evaluate the given improper integral:* $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+9}\ dx = \boxed{\pi/3}$.

Exercise 121 *Evaluate the improper integral:* $\displaystyle\int_0^{1} \ln x\ dx = \boxed{-1}$.

Exercise 122 *Evaluate the given improper integral.* $\displaystyle\int_3^{\infty} \frac{1}{x^2-4}\ dx = \boxed{\frac{\ln 5}{4}}$.

Exercise 123 *Use the Direct Comparison Test or the Limit Comparison Test to determine whether the integral converges or diverges:* $\displaystyle\int_{10}^{\infty} \frac{3}{\sqrt{3x^2+2x-5}}\ dx$. *Answer: the integral (converges/ diverges ✓) by (direct/ limit ✓) comparison with the function* $\dfrac{1}{x^{\boxed{1}}}$.

Exercise 124 *Use the Direct Comparison Test or the Limit Comparison Test to determine whether the integral converges or diverges:* $\displaystyle\int_{2}^{\infty} \frac{2}{\sqrt{7x^3-x}}\ dx$. *Answer: the integral (converges ✓/ diverges) by (direct/ limit ✓) comparison with the function* $\dfrac{1}{x^{\boxed{3/2}}}$ *(select the largest exponent for the denominator which makes the statement true).*

Exercise 125 *Use the Direct Comparison Test or the Limit Comparison Test to determine whether the integral converges or diverges:* $\displaystyle\int_{1}^{\infty} e^{-x}\ln x\ dx$. *Answer: the integral (converges ✓/ diverges) by direct comparison with the function*

Multiple Choice:

(a) e^{-x}

(b) xe^{-x} ✓

(c) e^{-x}/x

Exercise 126 *Use the Direct Comparison Test or the Limit Comparison Test to determine whether the integral converges or diverges:* $\int_1^{\infty} e^{-x^2+3x+1}\, dx$.
Answer: the integral (converges ✓/ diverges) by direct comparison with the function

Multiple Choice:

(a) e^{-x^2}

(b) e^{-x} ✓

(c) e^{3x+1}

Exercise 127 *Use the Direct Comparison Test or the Limit Comparison Test to determine whether the integral converges or diverges:* $\int_1^{\infty} \frac{x}{x^2+\cos x}\, dx$.
Answer: the integral (converges/ diverges ✓) by (direct/ limit ✓) comparison with the function

Multiple Choice:

(a) $1/x$ ✓

(b) $x/\cos x$

(c) $1/(x^2+\cos x)$

Exercise 128 *Use the Direct or Limit Comparison Test to determine whether the integral converges or diverges:*

$$\int_0^{1/e} \frac{(\ln x)^2-1}{x^3+x^2+x} dx$$

Answer: The integral (converges/ diverges ✓) by (direct/ limit ✓) comparison with the function

Multiple Choice:

(a) $\dfrac{(\ln x)^2}{x^3}$

(b) $\dfrac{(\ln x)^2}{x^2}$

(c) $\dfrac{(\ln x)^2}{x}$ ✓

(d) $\dfrac{1}{x^3}$

(e) $\dfrac{1}{x^2}$

Exercise 129 *Use the Direct or Limit Comparison Test to determine whether the integral converges or diverges:*

$$\int_0^{1/e} \frac{(\ln x)^2 - 1}{x + \sqrt{x} + e^{-1/x}} dx$$

Answer: The integral (converges ✓/ diverges) by direct comparison with the function

Multiple Choice:

(a) $\dfrac{(\ln x)^2}{x}$

(b) $\dfrac{(\ln x)^2}{\sqrt{x}}$ ✓

(c) $\dfrac{(\ln x)^2}{e^{-1/x}}$

(d) $\dfrac{1}{x}$

(e) $\dfrac{1}{\sqrt{x}}$

(f) $\dfrac{1}{e^{-1/x}}$

Sample Quiz Questions

Question 130 *Which of the following improper integrals is convergent? Show how you used comparison tests to justify your answer.*

$$\mathrm{I}: \int_0^1 \frac{\sqrt{1+x^2e^{-x}}}{(\cos x)x^2}\,dx \qquad \mathrm{II}: \int_2^\infty \frac{e^x}{xe^x+x^2}\,dx \qquad \mathrm{III}: \int_2^\infty \frac{x^2}{x^4+1}\,dx$$

Multiple Choice:

(a) *only* I *converges*

(b) *only* II *converges*

(c) *only* III *converges* ✓

(d) I *and* II *converge*

(e) II *and* III *converge*

(f) I *and* III *converge*

Feedback(`attempt`): *Integral* I *is divergent by direct comparison to the function* $\frac{1}{x^2}$. *Integral* II *is divergent by limit comparison to the function* $\frac{1}{x}$. *Integral* III *is convergent by direct comparison to the function* $\frac{1}{x^2}$.

Question 131 *Which of the following improper integrals is convergent? Show how you used comparison tests to justify your answer.*

$$\mathrm{I}: \int_0^1 \frac{\sqrt{e^{2x}+x^3}}{x}\,dx \qquad \mathrm{II}: \int_0^1 \frac{x^2}{x^2\sqrt{x}+x^3}\,dx \qquad \mathrm{III}: \int_2^\infty \frac{x^2\ln x}{-x+x^4}\,dx$$

Multiple Choice:

(a) *only* I *converges*

(b) *only* II *converges*

(c) *only* III *converges*

(d) I *and* II *converge*

(e) II *and* III *converge* ✓

(f) I *and* III *converge*

Feedback*(attempt)*: *Integral* I *is divergent by direct comparison to the function* $\frac{1}{x}$*. Integral* II *is convergent by direct comparison to the function* $\frac{1}{\sqrt{x}}$*. Integral* III *is convergent by limit comparison to the function* $\frac{\ln x}{x^2}$*.*

Sample Exam Questions

Question 132 *Only one of the following four improper integrals diverges. Choose that improper integral and justify why it diverges. (You need NOT justify why the other integrals converge.)*

Multiple Choice:

(a) $\displaystyle\int_2^\infty \frac{\arctan x}{1+x^3}dx$

(b) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^4+x^2}}dx$

(c) $\displaystyle\int_2^\infty \frac{1+\sin x}{x^2}dx$

(d) $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x^2-1}}dx$ ✓

3.7 Probability

We study probability and its connections to integration.

Example 35. *Suppose the region below the graph* $y = \sqrt{\sin x}$ *and above the* x*-axis between* $x = 0$ *and* $x = \pi$ *is revolved around the* x*-axis. Compute the volume of the resulting solid.*

- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius* R *is the length of a (horizontal/ vertical ✓) extending from the axis to the graph* $y = \sqrt{\sin x}$*.*
- *Thus we know that the radius* R *must equal*

  Multiple Choice:

(a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓

(b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}} \right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 36. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*

- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

(a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

(b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

applications/17probabilitypractice.tex

3.8 Exercises: Probability

Various exercises relating to probability.

Sample Quiz Questions

Question 133 *Find the value of c which makes the function*

$$f(x) = \frac{1}{2}e^{-x} - ce^{-2x}$$

a probability density function on the interval $[0, \infty)$. What is the value of the mean μ of the corresponding random variable?

Multiple Choice:

(a) $c = 1,\ \mu = \frac{1}{2}$

(b) $c = -1,\ \mu = \frac{3}{4}$ ✓

(c) $c = 1,\ \mu = 1$

(d) $c = -1,\ \mu = \frac{5}{4}$

(e) $c = 1,\ \mu = \frac{3}{2}$

(f) $c = -1,\ \mu = \frac{7}{4}$

Feedback(`attempt`): *To compute the constant c, we use the fact that the integral of a probability density function must equal 1, so*

$$\mu = \int_0^\infty \left(\frac{1}{2}e^{-x} - ce^{-2x}\right)\ dx = 1.$$

This gives the equation

$$\frac{1}{2} - \frac{1}{2}c = 1,$$

which then implies that $c = -1$. To compute the mean μ, we use the formula

$$\mu = \int_0^\infty x\left(\frac{1}{2}e^{-x} + e^{-2x}\right)\ dx.$$

Calculating the integral gives $\mu = 3/4$.

Question 134 *A certain random variable X takes values in the interval $\left[2\pi, \frac{5}{2}\pi\right]$. If the probability density function is given by*

$$A \sin x$$

for some appropriate value of the constant A, compute the expected value μ of X.

Multiple Choice:

(a) $\mu = -1 + 2\pi$

(b) $\mu = 1 + \frac{3}{2}\pi$

(c) $\mu = 2\pi$

(d) $\mu = -1 + \frac{5}{2}\pi$

(e) $\mu = 1 + 2\pi$ ✓

(f) $\mu = \frac{5}{2}\pi$

Feedback(*`attempt`*): *The constant A will be the reciprocal of the integral*

$$\int_{2\pi}^{\frac{5}{2}\pi} \sin x \, dx$$

One can check that

$$\int_{2\pi}^{\frac{5}{2}\pi} \sin x \, dx = 1.$$

To compute the expected value μ we also need to compute the integral

$$\int_{2\pi}^{\frac{5}{2}\pi} x \sin x \, dx$$

To compute the integralwe can use integration by parts. A reasonable strategy is to integrate $\sin x$ and differentiate x. This gives the equality

$$\begin{aligned}\int x \sin x \, dx &= -x \cos x - \int (-\cos x) \; dx \\ &= -x \cos x + \sin x.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{2\pi}^{\frac{5}{2}\pi} x \sin x \, dx &= \left[-x \cos x + \sin x\right]\Big|_{2\pi}^{\frac{5}{2}\pi} \\ &= 1 - (-2\pi) = 1 + 2\pi.\end{aligned}$$

Therefore the expected value is the ratio of the integrals, i.e.,

$$\mu = \frac{1 + 2\pi}{1} = 1 + 2\pi.$$

Sample Exam Questions

Question 135 *A certain random variable X has values in $(1, \infty)$ and has the property that there is some constant C such that*

$$P(X > a) = C \ln \frac{a^3 + 1}{a^3}$$

for every $a > 1$. Compute the value of C and determine whether the expected value μ of X is finite or infinite. [Hint: There is enough information given to compute C without calculating any integrals.]

Multiple Choice:

(a) $C = \ln 2$ *and* $\mu < \infty$

(b) $C = 1$ *and* $\mu < \infty$

(c) $C = (\ln 2)^{-1}$ *and* $\mu < \infty$ ✓

(d) $C = \ln 2$ *and* $\mu = \infty$

(e) $C = 1$ *and* $\mu = \infty$

(f) $C = (\ln 2)^{-1}$ *and* $\mu = \infty$

Feedback(*`attempt`*): *We know that X is always greater than one, so*

$$1 = P(X > 1) = C \ln \frac{1 + 1}{1},$$

which gives $C = (\ln 2)^{-1}$. If we let $f(x)$ denote the probability density function of X, then

$$\frac{1}{\ln 2} \ln \frac{a^3 + 1}{a^3} = P(X > a) = \int_a^\infty f(x)dx.$$

Differentiating both sides with respect to a gives

$$\frac{1}{\ln 2} \left[\frac{3a^2}{a^3 + 1} - \frac{3}{a} \right] = -f(a)$$

so

$$f(a) = \frac{1}{\ln 2} \left[\frac{3}{a} - \frac{3a^2}{a^3 + 1} \right] = \frac{3}{a(a^3 + 1) \ln 2}.$$

The expected value of X must equal

$$\int_1^\infty \frac{3a}{a(a^3 + 1) \ln 2} da = \frac{3}{\ln 2} \int_1^\infty \frac{da}{a^3 + 1}.$$

This integral will be finite by direct comparison to the convergent integral $\int_1^\infty a^{-3} da$.

Question 136 *The function*

$$f(x) = \begin{cases} \dfrac{k}{x^3} & 1 \leq x < \infty \\ 0 & \textit{otherwise} \end{cases}$$

is a probability density function for a certain value of k. For that probability density function, find the probability that $x > 2$.

Multiple Choice:

(a) $\frac{1}{2}$

(b) $\frac{1}{3}$

(c) $\frac{1}{4}$ ✓

(d) $\frac{2}{3}$

(e) $\frac{1}{5}$

(f) $\frac{1}{6}$

Question 137 *For a certain real number k, the function*

$$f(X) = \begin{cases} \dfrac{k}{X^2+1} & \textit{if } X \geq 0 \\ 0 & \textit{otherwise} \end{cases}$$

is a probability density function for a continuous random variable X. For this value of k, find the probability that $X > 1$.

Multiple Choice:

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{2}{3}$

(d) 1

(e) $\frac{1}{2}$ ✓

(f) $\frac{1}{4}$

Question 138 *Let*

$$f(r) = \begin{cases} Cr^2 e^{-2r/b} & r \geq 0 \\ 0 & r < 0 \end{cases}.$$

Find C so that this is a probability density function (pdf) for the random variable r. Here b is a positive constant. This function is used to model the distance between the nucleus and the electron in a hydrogen atom. The constant b is called the Bohr length. Find the mean of the pdf.

Multiple Choice:

(a) $C = \frac{b^3}{4}$, mean $= b$

(b) $C = \frac{4}{b^2}$, mean $= b$

(c) $C = \frac{4}{b}$, mean $= b^2$

(d) $C = \frac{4}{b^3}$, mean $= \frac{3}{2}b$ ✓

(e) $C = \frac{4}{b^2}$, mean $= \frac{3}{2}b^2$

(f) $C = \frac{4}{b}$, mean $= \frac{3}{2}b^3$

---

4 Sequences and Series

We begin a study of sequences and series.

Example 37. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from* $y = 0$ *to* $y = \sqrt{1-x^2}$*. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ *between* $x = 0$ *and* $x = 1$*, since these are the most extreme values of* x *found in our region. Therefore*

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 38. *The base of a solid region is bounded by the curves* $y = 0$, $x = \sqrt{y}$*, and* $x = 1$*. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph* $x = \boxed{\sqrt{y}}$ *to the graph* $x = \boxed{1}$*. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is* $\boxed{1 - \sqrt{y}}$*, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y *because different* y *cross sections will generally have different areas). To compute volume, we integrate* $dV = A(y)dy$ *between* $y = 0$ *and* $y = 1$*, since these are the most extreme values of* y *found in our region (note that we can find the upper value* $y = 1$ *by solving for the intersection of the curves* $x = \sqrt{y}$ *and* $x = 1$*). Therefore we integrate* $A(y)dy$ *to conclude*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

4.1 Sequences

We study the mathematical concept of a sequence.

Example 39. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 40. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

series/18sequencepractice.tex

4.2 Exercises: Sequences

Exercises relating to sequences.

Exercise 139 *The sequence $a_n = 1/n^3$ has limit $L = 0$. Suppose $\epsilon = 1/64$; find a threshold N such that*

$$|a_n - L| < \epsilon$$

is guaranteed to hold for all $n > N$ (take your value of N as small as possible).

$$N = \boxed{4}.$$

Exercise 140 *The sequence $a_n = (2n^2 + (-1)^n)/n^2$ has limit $L = 2$. Suppose $\epsilon = 1/100$; find a threshold N such that*

$$|a_n - L| < \epsilon$$

is guaranteed to hold for all $n > N$ (take your value of N as small as possible).

$$N = \boxed{10}.$$

Exercise 141 *Determine the n^{th} term of the given sequence. $a_1 = 4$, $a_2 = 7$, $a_3 = 10$, $a_4 = 13$, $a_5 = 16$, ...*

$$a_n = \boxed{3n + 1}.$$

Exercise 142 *Determine the n^{th} term of the given sequence. $a_1 = 3$, $a_2 = -5/2$, $a_3 = 7/4$, $a_4 = -9/8$, $a_5 = 11/16$, ...*

$$a_n = \boxed{\frac{(-1)^{n-1}(2n+1)}{2^{n-1}}}.$$

Exercise 143 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_n = \left\{(-1)^n \frac{n}{n+1}\right\}$$

The sequence (converges/ diverges ✓) to $\boxed{N/A}$ *(enter N/A if the sequence does not converge to a finite answer).*

Exercise 144 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_n = \frac{4n^2 - n + 5}{3n^2 + 1}$$

The sequence (converges ✓/ diverges) to $\boxed{4/3}$ *(enter N/A if the sequence does not converge to a finite answer).*

Exercise 145 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_n = \frac{4^n}{5^n}$$

The sequence (converges ✓/ diverges) to $\boxed{0}$ *(enter N/A if the sequence does not converge to a finite answer).*

Exercise 146 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_n = \left(1 - \frac{3}{n}\right)^{-n}$$

The sequence (converges ✓/ diverges) to $\boxed{e^3}$ *(enter N/A if the sequence does not converge to a finite answer).*

Hint: *Take the reciprocal and compare to your reference list of commonly-occurring limits.*

Exercise 147 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_n = \left(1 - \frac{3}{n}\right)^{-n^2}$$

The sequence (converges/ diverges ✓) to $\boxed{N/A}$ *(enter N/A if the sequence does not converge to a finite answer).*

Hint: *If a sequence b_n tends to e^3, what will the sequence $(b_n)^n$ do?*

Exercise 148 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_n = \frac{(1.1)^n}{n}$$

The sequence (converges/ diverges ✓) to $\boxed{N/A}$ *(enter N/A if the sequence does not converge to a finite answer).*

Hint: *What are the relative orders of growth of numerator versus denominator?*

Exercise 149 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_n = \frac{(0.9)^n}{n}$$

The sequence (converges ✓/ diverges) to $\boxed{0}$ *(enter N/A if the sequence does not converge to a finite answer).*

Exercise 150 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_n = n^{1000000}(0.9)^n$$

The sequence (converges ✓/ diverges) to $\boxed{0}$ *(enter N/A if the sequence does not converge to a finite answer).*

Exercise 151 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_n = \frac{\ln n}{n}$$

The sequence (converges ✓/ diverges) to $\boxed{0}$ *(enter N/A if the sequence does not converge to a finite answer).*

Exercise 152 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_n = \frac{\ln n}{n^{0.00001}}$$

The sequence (converges ✓/ diverges) to $\boxed{0}$ *(enter N/A if the sequence does not converge to a finite answer).*

Exercise 153 *Let b_n be the sequence given by*

$$b_1 = 0 \quad \text{and} \quad b_{n+1} = \frac{2 + b_n}{3} \quad \text{for } n \geq 1$$

converges. Compute its limit.

$$\lim_{n \to \infty} b_n = \boxed{1}.$$

Exercise 154 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{a_n} \quad \text{for } n \geq 1$$

The sequence (converges/ diverges ✓) to $\boxed{N/A}$ *(enter N/A if the sequence does not converge to a finite answer).*

Exercise 155 *Determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.*

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = a_n + \frac{1}{4}(4 - a_n^2) \quad \text{for } n \geq 1$$

The sequence (converges ✓/ diverges) to $\boxed{2}$ *(enter N/A if the sequence does not converge to a finite answer).*

Exercise 156 *Let a_n be the sequence given by*

$$a_1 = \frac{1}{4}, \quad \text{and} \quad a_{n+1} = 2a_n(1 - a_n) \quad \text{for } n \geq 1$$

converges. Compute its limit.

$$\lim_{n \to \infty} a_n = \boxed{\frac{1}{2}}.$$

Hint: *If a_n happens to be positive and less than $1/2$, then $2(1 - a_n) > 1$, so this forces $2a_n(1 - a_n) > a_n$ (meaning that the term after a_n will be larger than a_n.*

Hint: *The function $2x(1-x)$ is nonnegative on the interval $[0, 1]$ and has a maximum value of $1/2$ attained at $x = 1/2$. This means that if a_n is anything between 0 and 1, the next term of the sequence will always be between 0 and $1/2$.*

Sample Quiz Questions

Question 157 *Find the limit of the sequence*

$$\lim_{n \to \infty} \sqrt{\frac{2n - 2}{2n^2 - 4n + 3}}.$$

Justify your response. (Hints will not be revealed until after you choose a response.)

Multiple Choice:

(a) 0 ✓

(b) $\frac{1}{3}$

(c) $\frac{1}{2}$

(d) 1

(e) 2

(f) 3

Feedback(*`attempt`*): *Because the square root function is continuous, you can pass the limit through it and compute*

$$\sqrt{\lim_{n\to\infty} \frac{2n-2}{2n^2-4n+3}}.$$

Hint: *Reduce numerator and denominator to the dominant terms (in the regime $n \to \infty$).*

Hint:

$$\begin{aligned}\lim_{n\to\infty} \sqrt{\frac{2n-2}{2n^2-4n+3}} &= \lim_{n\to\infty} \sqrt{\frac{2n^{-1}-2n^{-2}}{2-4n^{-1}+3n^{-2}}} \\ &= \sqrt{\frac{\lim_{n\to\infty} 2n^{-1}-2n^{-2}}{\lim_{n\to\infty} 2-4n^{-1}+3n^{-2}}} \\ &= \sqrt{\frac{0}{2}} = 0.\end{aligned}$$

Question 158 *Determine whether the limit below exists. If it exists, find its value.*

$$\lim_{n\to\infty} \frac{(-1)^{n+1}n^2 - 2^{-n-1}}{(-1)^n n^2 + 4^{-n}}.$$

Justify your response. (Hints will not be revealed until after you choose a response.)

Multiple Choice:

(a) -1 ✓

(b) 0

(c) $\dfrac{1}{2}$

(d) 2

(e) 3

(f) *limit does not exist*

Feedback(*`attempt`*): *Comparing the orders of growth of the terms in the numerator, the first term dominates because $|-1| > |1/2|$.*

Hint: *Likewise the first term dominates in the denominator because $|-1| > |1/4|$.*

Hint: *Neglecting non-dominant terms leads to the limit*

$$\lim_{n\to\infty} \frac{(-1)^{n+1}n^2}{(-1)^n n^2}$$

which simply equals -1*.*

Question 159 *Determine whether the limit below exists. If it exists, find its value.*

$$\lim_{n\to\infty} \left(\frac{4n-3}{4n+1}\right)^n .$$

Justify your response. (Hints will not be revealed until after you choose a response.)

Multiple Choice:

(a) 0

(b) 1

(c) e^{-1} ✓

(d) e

(e) e^2

(f) *limit does not exist*

Feedback(*`attempt`*): *First observe that*

$$\frac{4n-3}{4n+1} = 1 - \frac{4}{4n+1} \to 1$$

as $n \to \infty$*.*

Hint: *Next, in light of the known limit* $(1 + x/k)^k \to e^x$ *as* $k \to \infty$*, manipulate exponents to see that*

$$\left(1 - \frac{4}{4n+1}\right)^n = \left(\left(1 - \frac{4}{4n+1}\right)^{4n+1}\right)^{1/4} \left(1 - \frac{4}{4n+1}\right)^{-1/4} .$$

Hint: *As* $n \to \infty$*, the first term on the right-hand side tends to* e^{-1} *and the second term tends to 1. Thus the original sequence tends to* e^{-1} *as well.*

Question 160 *Determine whether the limit below exists. If it exists, find its value.*

$$\lim_{n\to\infty}\left(\frac{n+3}{2n-2}\right)^{n^2}.$$

Justify your response. (Hints will not be revealed until after you choose a response.)

Multiple Choice:

(a) 0 ✓

(b) 1

(c) e^{-1}

(d) e

(e) e^2

(f) *limit does not exist*

Feedback(`attempt`): *First observe that*

$$\frac{n+3}{2n-2} = \frac{1}{2} + \frac{2}{n-1} \to \frac{1}{2}$$

as $n \to \infty$.

Hint: *Since the limit is positive and less than one, raising this expression to increasingly large powers generates a sequence which converges rapidly to zero.*

Sample Exam Questions

Question 161 *Determine whether the sequence* $a_n = (-1)^{n-1}\dfrac{n^2}{1+n^2+n^3}$ *converges or diverges. If it converges, find its limit.*

Multiple Choice:

(a) *divergent,* $\lim\limits_{n\to\infty} a_n = 0$

(b) *convergent,* $\lim\limits_{n\to\infty} a_n = 1$

(c) *convergent,* $\lim\limits_{n\to\infty} a_n = 0$ ✓

(d) *convergent*, $\lim_{n\to\infty} a_n = -1$

(e) *divergent*, $\lim_{n\to\infty} a_n = \infty$

(f) *divergent, limit doesn't exist*

4.3 Series

We introduce the concept of a series and study some fundamental properties.

Example 41. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 42. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$

between $y = 0$ *and* $y = 1$*, since these are the most extreme values of* y *found in our region (note that we can find the upper value* $y = 1$ *by solving for the intersection of the curves* $x = \sqrt{y}$ *and* $x = 1$*). Therefore we integrate* $A(y)dy$ *to conclude*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`series/19seriespractice.tex`

4.4 Exercises: Series

Exercises relating to fundamental properties of series.

Exercise 162 *Compute the sum of the infinite series below.*

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \boxed{\frac{3}{2}}$$

Hint: *Expand the terms using partial fractions and compute several partial sums by hand. What you get is something like a telescoping series, but cancellations occur in a slightly different way than usual.*

Hint:

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

Hint: *The formula for a general partial sum is*

$$\frac{2}{1\cdot 3} + \cdots + \frac{2}{N\cdot(N+2)} = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

Sample Quiz Questions

Question 163 *Compute the exact value of the infinite series*

$$\sum_{n=1}^{\infty} \ln\left(\frac{1+n^{-1}}{1+(n+1)^{-1}}\right).$$

Multiple Choice:

(a) $\ln 2$ ✓

(b) $\ln 3$

(c) $\ln 4$

(d) $\ln 5$

(e) $\ln 6$

(f) $\ln 7$

Feedback(*`attempt`*): *The series is not a geometric series or Taylor series, we compute the first few partial sums:*

$$\begin{aligned}
S_1 &= \ln\left(\frac{1+1}{1+2^{-1}}\right) = \ln\left(\frac{2}{\frac{3}{2}}\right)\\
S_2 &= \ln\left(\frac{1+1}{1+2^{-1}}\right) + \ln\left(\frac{1+2^{-1}}{1+3^{-1}}\right) = \ln\left(\frac{1+1}{1+3^{-1}}\right) = \ln\left(\frac{2}{\frac{4}{3}}\right)\\
S_3 &= \ln\left(\frac{2}{1+3^{-1}}\right) + \ln\left(\frac{1+3^{-1}}{1+4^{-1}}\right) = \ln\left(\frac{2}{1+4^{-1}}\right) = \ln\left(\frac{2}{\frac{5}{4}}\right)\\
&\vdots\\
S_n &= \ln\left(\frac{2}{1+(n+1)^{-1}}\right).
\end{aligned}$$

In particular, writing the sum of logarithms as a logarithm of a product leads to substantial cancellation. By letting $n \to \infty$, we get $S_n \to \ln 2$.

4.5 Series Comparison Tests

We study the direct and limit comparison theorems for infinite series and practice their application.

Example 43. *The region defined by the inequalities $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ and $x \ge 0$ (shown below) is revolved around the y-ais. Compute the volume using the shell method.*

- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

  Multiple Choice:

  (a) $h(x) = \sqrt{1-x^2}$
  (b) $h(x) = -\sqrt{1-x^2}$
  (c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\,dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$.)

Example 44. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*

- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$

  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$

  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\, dy = \boxed{\frac{\pi}{3}}.$$

`series/20comparisonpractice.tex`

4.6 Exercises: Series Comparison Tests

Exercises relating to the direct and limit comparison tests for series.

Exercise 164 *Suppose you wished to use the Direct Comparison Test to establish convergence of the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^4+n}.$$

Which of the following options below would be valid series for comparison? Select all that apply.

Select All Correct Answers:

(a) $\sum_{n=2}^{\infty} \frac{1}{n^5}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n^4}$ ✓

(c) $\sum_{n=2}^{\infty} \frac{1}{n^3}$ ✓

(d) $\sum_{n=2}^{\infty} \frac{1}{n^2}$ ✓

(e) $\sum_{n=2}^{\infty} \frac{1}{n}$

Feedback(`attempt`): *Expand for feedback on* $\sum_{n=2}^{\infty} \frac{1}{n^5}$*: This comparison wouldn't be valid because*

$$\frac{1}{n^5} \leq \frac{1}{n^4+n} \quad \text{for all } n \geq 2,$$

which is the wrong direction for the inequality when showing convergence.

Expand for feedback on $\sum_{n=2}^{\infty} \frac{1}{n^4}$*: This comparison is valid because*

$$\frac{1}{n^4+n} \leq \frac{1}{n^4} \quad \text{for all } n \geq 2,$$

which is the proper direction for the comparison inequality when showing convergence. Also key is that the p-series for $p = 4$ is convergent.

Expand for feedback on $\sum_{n=2}^{\infty} \frac{1}{n^3}$*: This comparison is valid because*

$$\frac{1}{n^4+n} \leq \frac{1}{n^3} \quad \text{for all } n \geq 2,$$

which is the proper direction for the comparison inequality when showing convergence. Also key is that the p-series for $p = 3$ is convergent.

Expand for feedback on $\sum_{n=2}^{\infty} \frac{1}{n^2}$*: This comparison is valid because*

$$\frac{1}{n^4+n} \leq \frac{1}{n^2} \quad \text{for all } n \geq 2,$$

which is the proper direction for the comparison inequality when showing convergence. Also key is that the p-series for $p = 2$ is convergent.

Expand for feedback on $\sum_{n=2}^{\infty} \frac{1}{n}$*: This comparison is not valid because the series (known as the* harmonic series*) is not convergent, so no comparisons to it can establish convergence.*

Exercise 165 *Suppose you wished to use the Direct Comparison Test to establish convergence of the series*

$$\sum_{n=2}^{\infty} \frac{1}{n^4 - n}.$$

Which of the following options below would be valid series for comparison? Select all that apply.

Select All Correct Answers:

(a) $\sum_{n=2}^{\infty} \frac{1}{n^4}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n^3}$ ✓

(c) $\sum_{n=2}^{\infty} \frac{1}{n^2}$ ✓

Feedback*(`attempt`)*: *Expand for feedback on* $\sum_{n=2}^{\infty} \frac{1}{n^4}$*: This comparison is not valid because*

$$\frac{1}{n^4} \leq \frac{1}{n^4 - n^2} \quad \text{for all } n \geq 2,$$

which is the wrong direction for the comparison inequality when showing convergence.

Expand for feedback on $\sum_{n=2}^{\infty} \frac{1}{n^3}$*: This comparison is valid because*

$$\frac{1}{n^4 - n} \leq \frac{1}{n^3} \quad \text{for all } n \geq 2$$

(because $n^4 - n = n(n^3 - 1)$ *and* $n^3 - 1$ *is always greater than* n^2 *whenever* $n \geq 2$*.) which is the proper direction for the comparison inequality when showing convergence. Also key is that the* p*-series for* $p = 3$ *is convergent.*

Expand for feedback on $\sum_{n=2}^{\infty} \frac{1}{n^2}$*: This comparison is valid because*

$$\frac{1}{n^4 - n} \leq \frac{1}{n^2} \quad \text{for all } n \geq 2,$$

(because $n^4 - n = n(n^3 - 1)$ *and* $n^3 - 1$ *is always greater than* n *whenever* $n \geq 2$*.) which is the proper direction for the comparison inequality when showing convergence. Also key is that the* p*-series for* $p = 2$ *is convergent.*

Exercise 166 *Using only the comparison functions listed here:*

$$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n+1}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} e^{-n},$$

which of the following series can be proved to either diverge or converge using the Direct Comparison Test? (Select all that apply.)

Select All Correct Answers:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} + n^{1/2}}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^{1/2}}$ ✓

(c) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ ✓

(d) $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$

(e) $\sum_{n=1}^{\infty} \frac{1}{e^n+1}$ ✓

(f) $\sum_{n=3} \infty \frac{1}{2n+3}$

Feedback(*`attempt`*): *For the series* $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n^{1/2}}$: *This series is convergent, but none of the allowed comparisons will work: Only* $1/(n+1)$ *and* $1/(1+\sqrt{n})$ *are larger than* $1/(n^{3/2}+n^{1/2})$, *but neither of those series converges.*

For the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n^{1/2}}$: *Comparison to* $1/n^2$ *works fine in this case.*

For the series $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$: $\ln(n+1) \le n+1$ *so* $1/\ln(n+1) \ge 1/(n+1)$, *so actually diverges by direct comparison.*

For the series $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$: *This series textitis convergent, but none of the convergent comparison options* $1/n^2$ *and* e^{-n} *is greater than* $1/n^2$.

For the series $\sum_{n=1}^{\infty} \frac{1}{e^n+1}$: *The direct comparison* $1/(e^n+1) \le e^{-n}$ *works fine to show convergence of the series.*

For the series $\sum_{n=3} \infty \frac{1}{2n+3}$: *This is a divergent series, but direct comparison with the allowed options doesn't work because* $1/(2n+3)$ *is smaller than the two divergent options* $1/(n+1)$ *and* $1/(1+\sqrt{n})$.

Exercise 167 *Suppose you wished to use the Limit Comparison Test to establish convergence of the series*

$$\sum_{n=1}^{\infty} \frac{1}{n^4+n}.$$

Which of the following options below would be valid series for comparison? Select all that apply.

Select All Correct Answers:

(a) $\sum_{n=2}^{\infty} \frac{1}{n^5}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n^4}$ ✓

(c) $\sum_{n=2}^{\infty} \frac{1}{n^3}$ ✓

(d) $\sum_{n=2}^{\infty} \frac{1}{n^2}$ ✓

(e) $\sum_{n=2}^{\infty} \frac{1}{n}$

Feedback(*attempt*): *Expand for feedback on* $\sum_{n=2}^{\infty} \frac{1}{n^5}$*: This comparison wouldn't be valid because* $1/n^5$ *is too small: the limit of* $1/(n^4+n)$ *divided by* $1/n^5$ *is infinite, which is an inconclusive case.*

Expand for feedback on $\sum_{n=2}^{\infty} \frac{1}{n^4}$*: This comparison is valid because the limit* $1/(n^4-n)$ *divided by* $1/n^4$ *is* 1*, which means that they both converge (because a* p*-series for* $p = 4$ *is convergent).*

Expand for feedback on $\sum_{n=2}^{\infty} \frac{1}{n^3}$*: This comparison is valid because* $1/n^3$ *is a convergent* p*-series which is much larger than* $1/(n^4+n)$*, i.e.,*

$$\lim_{n\to\infty} \frac{\frac{1}{n^4+n}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^4+n} = 0.$$

Expand for feedback on $\sum_{n=2}^{\infty} \frac{1}{n^2}$*: Also works for the same reason as* $1/n^3$*: it's a convergent* p*-series and the original series divided by the comparison series goes to zero.*

Expand for feedback on $\sum_{n=2}^{\infty} \frac{1}{n}$*: This comparison is not valid because the series (known as the harmonic series) is not convergent, so no comparisons to it can establish convergence.*

Exercise 168 *Suppose you wished to use the Limit Comparison Test to establish convergence of the series*

$$\sum_{n=2}^{\infty} \frac{1}{n^4-n}.$$

Which of the following options below would be valid series for comparison? Select all that apply.

Select All Correct Answers:

(a) $\sum_{n=2}^{\infty} \frac{1}{n^4}$ ✓

(b) $\sum_{n=2}^{\infty} \frac{1}{n^3}$ ✓

(c) $\sum_{n=2}^{\infty} \frac{1}{n^2}$ ✓

Feedback(`attempt`): *For the Limit Comparison Test, non-dominant terms don't matter, so the results will be the same here as they were for the series $1/(n^4 + n)$.*

Exercise 169 *Using only the comparison functions listed here:*

$$\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n+1}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} e^{-n}$$

, which of the following series can be proved to either diverge or converge using the Limit Comparison Test? (Select all that apply.)

Select All Correct Answers:

(a) $\sum_{n=1}^{\infty} \frac{1}{n^{3/2} + n^{1/2}}$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2 + n^{1/2}}$ ✓

(c) $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$ ✓

(d) $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ ✓

(e) $\sum_{n=1}^{\infty} \frac{1}{e^n + 1}$ ✓

(f) $\sum_{n=3} \infty \frac{1}{2n+3}$ ✓

Feedback(*attempt*): *For the series* $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+n^{1/2}}$*: This series is convergent, but none of the allowed comparisons will work: Only* $1/(n+1)$ *and* $1/(1+\sqrt{n})$ *are larger than* $1/(n^{3/2}+n^{1/2})$*, but neither of those series converges.*

For the series $\sum_{n=1}^{\infty} \frac{1}{n^2+n^{1/2}}$*: Comparison to* $1/n^2$ *works fine in this case.*

For the series $\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$*:* $\ln(n+1) \leq n+1$ *so* $1/\ln(n+1) \geq 1/(n+1)$*, so the limit*

$$\frac{n+1}{\ln(n+1)} \to \infty \text{ as } n \to \infty$$

so we get divergence by the divergence of the comparison series.

For the series $\sum_{n=1}^{\infty} \frac{1}{n^2-1}$*: Here the non-dominant term* -1 *in the denominator is no longer relevant, so limit comparison to* $1/n^2$ *works fine.*

For the series $\sum_{n=1}^{\infty} \frac{1}{e^n+1}$*: The direct comparison* $1/(e^n+1)/e^{-n} \to 1$ *works fine to show convergence of the series.*

For the series $\sum_{n=3} \infty \frac{1}{2n+3}$*: Limit comparison with* $1/(n+1)$ *works great:* $1/(2n+3)$ *divided by* $1/(n+1)$ *equals*

$$\frac{n+1}{2n+3} \to \frac{1}{2} \text{ as } n \to \infty.$$

4.7 The Ratio and Root Tests

We study the ratio and root tests for infinite series and practice their application.

Example 45. *Suppose the region below the graph* $y = \sqrt{\sin x}$ *and above the* x*-axis between* $x = 0$ *and* $x = \pi$ *is revolved around the* x*-axis. Compute the volume of the resulting solid.*

- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius* R *is the length of a (horizontal/ vertical ✓) extending from the axis to the graph* $y = \sqrt{\sin x}$*.*

- *Thus we know that the radius R must equal*

  Multiple Choice:

  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓
  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}}\right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 46. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*

- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$
  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}}\right)^2 - \left(\boxed{2y}\right)^2\right] d\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

`series/21ratiorootpractice.tex`

4.8 Exercises: Ratio and Root Tests

Exercises relating to the Ratio and Root Tests.

Exercise 170 *Consider the infinite series*

$$\sum_{n=1}^{\infty} n2^{-n}.$$

Apply the Ratio Test:

$$\lim_{n\to\infty} \frac{\boxed{(n+1)2^{-n-1}}}{\boxed{n2^{-n}}} = \boxed{\frac{1}{2}}.$$

Apply the Root Test:

$$\lim_{n\to\infty} \left|\boxed{n2^{-n}}\right|^{\frac{1}{n}} = \boxed{\frac{1}{2}}.$$

Both tests indicate that the series (converges ✓/ diverges).

Exercise 171 *Oftentimes the Ratio Test is easier to apply than the Root Test when dealing with factorials. Use the Ratio Test to determine convergence or divergence of the series*

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}.$$

In the spaces below, record the eponymous "ratio" in the first blanks, simplify it in the second blanks, and then record the limit.

$$\lim_{n\to\infty} \frac{\boxed{\frac{((n+1)!)^2}{(2n+2)!}}}{\boxed{\frac{(n!)^2}{(2n)!}}} = \lim_{n\to\infty} \boxed{\frac{(n+1)^2}{(2n+1)(2n+2)}} = \boxed{\frac{1}{4}}.$$

The test (indicates convergence ✓/ indicates divergence/ is inconclusive).

Exercise 172 *Apply the Ratio Test to the series given below.*

$$\sum_{n=0}^{\infty} \frac{5^n - 3n}{4^n}$$

$$\lim_{n\to\infty} \frac{\boxed{\dfrac{5^{n+1}-3(n+1)}{4^{n+1}}}}{\boxed{\dfrac{5^n-3n}{4^n}}} = \lim_{n\to\infty} \boxed{\frac{1}{4}\frac{5^{n+1}-3(n+1)}{5^n-3n}} = \boxed{\frac{5}{4}}.$$

The test (indicates convergence/ indicates divergence ✓/ is inconclusive).

Exercise 173 *Apply the Ratio Test to the series given below.*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

$$\lim_{n\to\infty} \frac{\boxed{((n+1)^2+1)^{-1}}}{\boxed{(n^2+1)^{-1}}} = \lim_{n\to\infty} \boxed{\frac{n^2+1}{(n+1)^2+1}} = \boxed{1}.$$

The test (indicates convergence/ indicates divergence/ is inconclusive ✓).

Exercise 174 *Apply the Ratio Test to the series given below.*

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^n}$$

$$\lim_{n\to\infty} \frac{\boxed{3^{n+1}/((n+1)^2 2^{n+1})}}{\boxed{3^n/(n^2 2^n)}} = \lim_{n\to\infty} \boxed{\frac{3}{2}\frac{n^2}{(n+1)^2}} = \boxed{\frac{3}{2}}.$$

The test (indicates convergence/ indicates divergence ✓/ is inconclusive).

Exercise 175 *Apply the Ratio Test to the series given below.*

$$\sum_{n=0}^{\infty} \frac{4^n}{4^n+1}$$

$$\lim_{n\to\infty} \frac{\boxed{\dfrac{4^{n+1}}{4^{n+1}+1}}}{\boxed{\dfrac{4^n}{4^n+1}}} = \lim_{n\to\infty} \boxed{4\frac{4^n+1}{4^{n+1}+1}} = \boxed{1}.$$

The test (indicates convergence/ indicates divergence/ is inconclusive ✓).

Exercise 176 *Oftentimes the Root Test is easier to apply when terms have large exponents (growing faster than a constant times n). Use the Root Test to determine the convergence or divergence of the series*

$$\sum_{n=1}^{\infty} 4^n \left(1 - \frac{1}{n}\right)^{n^2}.$$

First say what should quantity should have its n-th root taken, then simplify, and lastly record the value of the limit.

$$\lim_{n\to\infty} = \left|\boxed{4^n \left(1 - \frac{1}{n}\right)^{n^2}}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \boxed{4\left(1 - \frac{1}{n}\right)^{n}} = \boxed{\frac{4}{e}}.$$

The test (indicates convergence/ indicates divergence ✓/ is inconclusive).

Exercise 177 *Apply the Root Test to the series given below.*

$$\sum_{n=1}^{\infty} 2^{-\ln n}$$

$$\lim_{n\to\infty} \left|\boxed{2^{-\ln n}}\right|^{\frac{1}{n}} = 2^{-\lim_{n\to\infty}\boxed{\frac{\ln n}{n}}} = \boxed{1}.$$

The test (indicates convergence/ indicates divergence/ is inconclusive ✓).

Exercise 178 *Apply the Root Test to the series given below.*

$$\sum_{n=1}^{\infty} 2^{-n^2}$$

$$\lim_{n\to\infty} \left|\boxed{2^{-n^2}}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \boxed{2^{-n}} = \boxed{0}.$$

The test (indicates convergence ✓/ indicates divergence/ is inconclusive).

Exercise 179 *Apply the Root Test to the series given below.*

$$\sum_{n=1}^{\infty} e^{\ln n}\left(1 - \frac{1}{n}\right)^{n^2}$$

$$\lim_{n\to\infty}\left|\boxed{e^{\ln n}\left(1-\frac{1}{n}\right)^{n^2}}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\boxed{e^{(\ln n)/n}\left(1-\frac{1}{n}\right)^{n}} = \boxed{1/e}.$$

The test (indicates convergence ✓/ indicates divergence/ is inconclusive).

Exercise 180 *Apply the Root Test to the series given below.*

$$\sum_{n=1}^{\infty}\frac{e^{n^2}}{n^n}$$

$$\lim_{n\to\infty}\left|\frac{e^{n^2}}{n^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty}\boxed{e^n/n} = \boxed{\infty}$$

The test (indicates convergence/ indicates divergence ✓/ is inconclusive).

Sample Quiz Questions

Question 181 *Determine which of the following three infinite series will lead to inconclusive results for the Ratio Test and then determine whether that series is convergent or divergent.*

$$\text{I: }\sum_{k=1}^{\infty}\frac{1}{k-e^{-k}}\quad \text{II: }\sum_{m=1}^{\infty}\frac{1}{m^2-e^{m}}\quad \text{III: }\sum_{l=1}^{\infty}\frac{e^{-l}}{l^2+1}$$

Multiple Choice:

(a) *I inconclusive, converges*

(b) *I inconclusive, diverges* ✓

(c) *II inconclusive, converges*

(d) *II inconclusive, diverges*

(e) *III inconclusive, converges*

(f) *III inconclusive, diverges*

Feedback(`attempt`): *The first series will give an inconclusive result for the Ratio Test because*

$$\lim_{k\to\infty}\frac{k-e^{-k}}{k+1-e^{-k-1}} = \lim_{k\to\infty}\frac{1-k^{-1}e^{-k}}{\frac{k+1}{k}-k^{-1}e^{-k-1}} = \frac{1-0}{1-0} = 1.$$

However, we know that the harmonic series diverges and that

$$\frac{1}{k-e^{-k}} > \frac{1}{k},$$

so by direct comparison to the harmonic series, series I must diverge.

4.9 The Integral Test

We study the integral test for infinite series and related concepts.

Example 47. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*

- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓
  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}}\right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 48. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*

- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

(b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\,\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

`series/22integralpractice.tex`

4.10 Exercises: The Integral Test

Exercises relating to the integral test.

Sample Quiz Questions

Question 182 *When approximating the sum of the infinite series*

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

by the sum of the first N terms, how large must N be to ensure that the approximation error is less than $1/200$? Choose the smallest correct bound among those listed.

Multiple Choice:

(a) $N > 5$

(b) $N > 10$ ✓

(c) $N > 20$

(d) $N > 400$

(e) $N > 8000$

(f) $N > 160000$

Feedback(`attempt`): *Because the terms n^{-3} are positive and decreasing, we know that the partial sums are always less than or equal to the sum of the series. By the Integral Test, we can further say that*

$$\sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{1}{n^3} \leq \int_N^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2N^2}.$$

To be certain that the error is less than $1/200$, we set $(2N^2)^{-1} < 1/200$, which gives $N > 10$.

4.11 Alternating Series

We study the notion of alternating series and related concepts.

Example 49. *The region defined by the inequalities* $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ *and* $x \ge 0$ *(shown below) is revolved around the y-ais. Compute the volume using the shell method.*

- *When the slicing variable is* x*, the radius of a shell is the (horizontal ✓/ vertical) distance from an* x*-slice to the axis* $x = 0$*. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an* x*-slice is equal to*

  Multiple Choice:

  (a) $h(x) = \sqrt{1-x^2}$

  (b) $h(x) = -\sqrt{1-x^2}$

  (c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\,dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$*.)*

Example 50. *The region between the curves* $x = \sqrt{y}$ *and* $x = y + \sqrt{y}$ *from* $y = 0$ *to* $y = 1$ *is revolved around the axis* $y = 1$*. Compute the volume of the resulting solid.*

- *When the slicing variable is* y*, the radius of a shell is the (horizontal/ vertical ✓) distance from a* y*-slice to the axis* $y = 1$*. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a* y*-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$

  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$

  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\,dy = \boxed{\frac{\pi}{3}}.$$

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4.12 Exercises: Alternating Series

Exercises relating to alternating series and absolute or conditional convergence.

Sample Quiz Questions

Question 183 *For each series below, determine whether it converges absolutely (A), converges conditionally (C), or diverges (D). Show how you used convergence tests to arrive at your answer.*

$$\text{I: } \sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt[3]{n+3}} \qquad \text{II: } \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+3} \qquad \text{III: } \sum_{n=2}^{\infty} \frac{\cos n\pi}{\ln(n^2+1)}$$

Multiple Choice:

(a) *I: C, II: D, III: D*

(b) *I: C, II: A, III: C* ✓

(c) *I: A, II: C, III: A*

(d) *I: D, II: C, III: D*

(e) *I: C, II: D, III: C*

(f) *I: C, II: A, III: A*

Feedback(*`attempt`*): *I: converges conditionally. The value of $\cos n\pi$ alternates ± 1. The terms $(n+3)^{-1/3}$ decrease to zero, so the series converges by the alternating series test. The series is not absolutely convergent because the p-series with $p = -1/3$ is divergent.*

II: converges absolutely. The series converges absolutely by direct comparison to a p-series with $p = 2$.

III: converges conditionally. The series converges by the alternating series test because $1/\ln(n^2+1)$ decreases to 0 as $n \to \infty$ and $\cos n\pi$ alternates in value between $+1$ and -1. However, $1/\ln(n^2+1) \geq 1/n$ for all large n, so by direct comparison to the harmonic series, the series is not absolutely convergent. Therefore the convergence is conditional.

Question 184 *For each series below, determine whether it converges absolutely (A), converges conditionally (C), or diverges (D). Show how you used convergence tests to arrive at your answer.*

$$I: \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2n^2+1} \qquad II: \sum_{n=1}^{\infty} \frac{1}{1+n^3 e^{-n}} \qquad III: \sum_{n=1}^{\infty} \frac{(-1)^n n+2}{n^2}$$

Multiple Choice:

(a) *I: D, II: D, III: D*

(b) *I: D, II: A, III: C*

(c) *I: C, II: C, III: A*

(d) *I: A, II: C, III: D*

(e) *I: D, II: D, III: C* ✓

(f) *I: D, II: A, III: A*

Feedback(`attempt`): *I: diverges. The series diverges because $n^2/(n^2+1) \to 1$, meaning that the terms do not go to zero. The n-th term divergence test implies divergence.*

II: diverges. The series diverges because $n/(n+n^3e^{-n}) \to 1$ (because $n^3e^{-n} \to 0$). By the limit comparison theorem, this means the series has the same behavior as a p-series with $p=1$, which means it diverges.

III: converges conditionally. The series converges because it is the sum of two convergent series: one with terms $(-1)^n/n$ (which is a convergent series by the alternating series test because $1/n$ decreases to zero) and a second with terms $2/n^2$ (which is a convergent p-series). However, the series is not absolutely convergent, because

$$\left|\frac{(-1)^n n+2}{n^2}\right| = \frac{n+(-1)^n 2}{n^2}$$

for $n \geq 2$, which is a sum of a divergent *p-series with $p=1$ and an absolutely convergent alternating p-series with $p=2$. Thus the series is conditionally convergent.*

Question 185 *Which of the following intervals contains the value of the infinite series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}?$$

Multiple Choice:

(a) $\left[\frac{1}{4}, \frac{1}{3}\right]$

(b) $\left[\frac{1}{3}, \frac{1}{2}\right]$

(c) $\left[\frac{1}{2}, \frac{7}{12}\right]$

(d) $\left[\frac{7}{12}, \frac{5}{6}\right]$ ✓

(e) $\left[\frac{5}{6}, \frac{11}{12}\right]$

(f) $\left[\frac{11}{12}, \frac{7}{6}\right]$

Feedback(*`attempt`*): *The function $1/(n+1)$ is positive and decreases to zero, so by the Alternating Series Test, we know that partial sums alternate above and below the actual value of the sum. In particular, if we call the value of the sum L, then*

$$\begin{aligned} 1 &\geq L \\ 1 - \frac{1}{2} &\leq L \\ 1 - \frac{1}{2} + \frac{1}{3} &\geq L \\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} &\leq L \end{aligned}$$

and so on. The last two inequalities together imply that L belongs to the interval $[\frac{7}{12}, \frac{5}{6}]$.

Sample Exam Questions

Question 186 *Determine whether the following series converge absolutely (A), converge conditionally (C), or diverge (D). For full credit be sure to explain your reasoning and specify which tests were used.*

$$\sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n}}{3^n} \qquad \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Multiple Choice:

(a) *both A*

(b) *one A, the other C*

(c) *one A, the other D*

(d) *both C*

(e) *one C, the other D* ✓

(f) *both D*

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4.13 Exercises: Cumulative

Exercises relating to various topics we have studied.

Sample Exam Questions

Question 187 *Determine whether the following series converge or diverge.*

$$\textit{I: } \sum_{n=1}^{\infty} \frac{n^3}{n^4+4} \quad \textit{II: } \sum_{n=1}^{\infty} \frac{3^n}{n!} \quad \textit{III: } \sum_{n=2}^{\infty} \frac{\ln\ln n}{\ln n} \quad \textit{IV: } \sum_{n=1}^{\infty} \frac{3n^2}{(n!)^2}$$

Multiple Choice:

(a) *I & II converge; III & IV diverge*

(b) *I & III converge; II & IV diverge*

(c) *I & IV converge; II & III diverge*

(d) *II & III converge; I & IV diverge*

(e) *II & IV converge; I & III diverge* ✓

(f) *III & IV converge; I & II diverge*

Question 188 *Determine whether the following series are convergent or divergent. Justify your answers.*

$$\textit{I: } \sum_{n=1}^{\infty} \frac{n^2-3n}{\sqrt[3]{n^{10}-4n^2}} \quad \textit{II: } \sum_{n=1}^{\infty} \frac{(-n)^n}{5^{2n+3}}$$

Multiple Choice:

(a) *I & II divergent*

(b) *I convergent, II divergent* ✓

(c) *I divergent, II convergent*

(d) *I & II convergent*

Question 189 *Determine whether the following series are convergent or divergent. Justify your answers.*

$$\text{I: } \sum_{n=1}^{\infty} \frac{\arctan n}{n^4} \qquad \text{II: } \sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{n^2}$$

Multiple Choice:

(a) *I & II divergent*

(b) *I convergent, II divergent*

(c) *I divergent, II convergent*

(d) *I & II convergent* ✓

Question 190 *Determine which of the following series are convergent. For full credit, be sure to explain your reasoning and specify which tests were used.*

$$\text{I: } \sum_{n=2}^{\infty} 2ne^{-n^2} \qquad \text{II: } \sum_{n=2}^{\infty} \frac{n + 2\ln n}{2n^4} \qquad \text{III: } \sum_{n=2}^{\infty} \frac{n^n}{n!}$$

Multiple Choice:

(a) *only I*

(b) *only I and II* ✓

(c) *only I and III*

(d) *only II*

(e) *only II and III*

(f) *only III*

---

5 Power Series

We undertake a study of an important class of infinite series.

Example 51. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 52. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

5.1 Power Series

We introduce the concept of a power series and some related fundamental properties.

Example 53. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 54. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

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5.2 Exercises: Power Series and Convergence

Exercises relating to power series and their convergence properties.

Exercise 191 *Compute the radius of convergence of the power series below.*

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{2^n+1}$$

$$R = \boxed{2}$$

Hint: *When computing a limit like*

$$\lim_{n\to\infty} \frac{2^n+1}{2^{n+1}+1},$$

remember the effect of orders of growth: $1 \ll 2^n$, and so from the perspective of the limit, the $+1$'s in both the numerator and denominator are both negligible.

Exercise 192 *Compute the radius of convergence of the power series below.*

$$\sum_{n=1}^{\infty} \frac{(-3)^n(\ln n)(x-3)^n}{4^n}$$

$$R = \boxed{\frac{4}{3}}$$

Hint: *When computing a limit like*

$$\lim_{n\to\infty} (\ln n)^{1/n},$$

remember that we already know $n^{1/n} \to 1$ and that $\ln n \ll n$ as well, so we expect $(\ln n)^{1/n} \to 1$ as well.

Exercise 193 *Compute the radius of convergence of the power series below.*

$$\sum_{n=1}^{\infty} \frac{(-2)^n n!(x-3)^n}{3^n+n^2}$$

$$R = \boxed{0}$$

Exercise 194 *Compute the interval of convergence for the power series below.*

$$\sum_{n=1}^{\infty} \frac{(-2)^n (x-3)^n}{\ln n}$$

The left endpoint of the interval is $\boxed{5/2}$; *it (is/ is not ✓) included in the interval of convergence. The right endpoint of the interval is* $\boxed{7/2}$; *it is (is ✓/ is not) included in the interval of convergence.*

Exercise 195 *Reindex the series below:*

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n+2} = \sum_{n=0} \frac{x^{\boxed{2n+3}}}{\boxed{n+3}}$$

Exercise 196 *Differentiate the series below term-by-term:*

$$\frac{d}{dx} \sum_{n=0}^{\infty} \frac{n}{n+1} x^n = \sum_{n=1} \boxed{\frac{n^2}{n+1}} x^{\boxed{n-1}}.$$

(Note that the $n = 0$ term goes away because the derivative of a constant is zero.)

Exercise 197 *Integrate the series below term-by-term:*

$$\int_0^x \left[\sum_{n=0}^{\infty} \frac{x^n}{(2n+3)^2} \right] = \sum_{n=0} \boxed{\frac{1}{(n+1)(2n+3)^2}} x^{\boxed{n+1}}.$$

Exercise 198 *For each step below, apply a term-by-term operation, a multiplication by a monomial, or a substitution to derive a new series formula from a known formula.*

- *Use the formula*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

to derive a summation formula for $1/(1+x^2)$, *i.e.,*

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} \boxed{(-1)^n} x^{\boxed{2n}}.$$

- *Use the formula you derived above to develop a power series expansion for arctangent:*

$$\arctan x = \sum_{n=0}^{\infty} \boxed{\frac{(-1)^n}{(2n+1)}} x^{\boxed{2n+1}}.$$

- *The radius of convergence of this last series is* $R = \boxed{1}$.

Exercise 199 *For each step below, apply a term-by-term operation, a multiplication by a monomial, or a substitution to derive a new series formula from a known formula.*

- *Use the formula*

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

to determine the sum of the series below for $-1 < x < 1$*:*

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \boxed{-\ln|1-x|}.$$

(Don't forget absolute values if you need them.)

- *Use the formula you just derived to determine the sum of the series*

$$\sum_{n=0}^{\infty} \frac{x^{2n+3}}{n+1} = \boxed{-x\ln|1-x^2|.}$$

- *Use the formula you just derived to determine the sum of the series*

$$\sum_{n=0}^{\infty} \frac{2^{2n}x^{2n+3}}{n+1} = \boxed{-\frac{1}{4}x\ln|1-4x^2|.}$$

Sample Quiz Questions

Question 200 *Find the full interval of convergence for the power series*

$$\sum_{m=2}^{\infty} \frac{(-3)^m m^2 (x-5)^m}{\ln m}.$$

(Hints will not be revealed until after you choose a response.)

Multiple Choice:

(a) $\left(\frac{14}{3}, \frac{16}{3}\right)$ ✓

(b) $\left[\frac{14}{3}, \frac{16}{3}\right)$

(c) $(2, 8]$

(d) $[2, 8]$

(e) $(-\infty, \infty)$

Feedback(`attempt`): *First observe that*

$$\begin{aligned} \frac{1}{R} &= \lim_{m\to\infty} \left| \frac{\frac{(-3)^{(m+1)}(m+1)^2}{\ln(m+1)}}{\frac{(-3)^m m^2}{\ln m}} \right| \\ &= \lim_{m\to\infty} \left| \frac{-3(m+1)^2 \ln m}{m^2 \ln(m+1)} \right| \\ &= 3 \end{aligned}$$

because

$$\lim_{m\to\infty} \frac{(m+1)^2 \ln m}{m^2 \ln(m+1)} = \lim_{m\to\infty} \frac{(m+1)^2}{m^2} \lim_{m\to\infty} \frac{\ln m}{\ln(m+1)} = 1$$

by virtue of l'Hospital's rule applied to both limits on the right-hand side.

Hint: *This means that the radius equals* $1/3$. *At the endpoint* $x = 14/3$, *the series equals*

$$\sum_{m=2}^{\infty} \frac{m^2}{\ln m},$$

which diverges by the n*-th term divergence test because* $\lim\limits_{m\to\infty} m^2/\ln m = \infty \neq 0$. *At the endpoint* $x = 16/3$, *the series equals*

$$\sum_{m=2}^{\infty} (-1)^m \frac{m^2}{\ln m},$$

which diverges for the same reason as the other endpoint, i.e., the terms do not go to zero.

Question 201 *Find the full interval of convergence for the power series*

$$\sum_{m=2}^{\infty} \frac{(-4)^m (\ln m)(x-1)^m}{(-2)^m m}.$$

(Hints will not be revealed until after you choose a response.)

Multiple Choice:

(a) $\left(\frac{1}{2}, \frac{3}{2}\right)$

(b) $\left[\frac{1}{2}, \frac{3}{2}\right)$ ✓

(c) $(-1, 3]$

(d) $[-1, 3]$

(e) $(-\infty, \infty)$

Feedback(`attempt`): *First observe that*

$$\begin{aligned} \frac{1}{R} &= \lim_{m\to\infty} \left| \frac{\frac{(-4)^{(m+1)}(\ln(m+1))}{(-2)^{(m+1)}(m+1)}}{\frac{(-4)^m(\ln m)}{(-2)^m m}} \right| \\ &= \lim_{m\to\infty} \left| \frac{-4(\ln(m+1))m}{-2(\ln m)(m+1)} \right| \\ &= 2 \end{aligned}$$

because

$$\lim_{m\to\infty} \frac{m\ln(m+1)}{(m+1)\ln m} = \lim_{m\to\infty} \frac{m}{m+1} \lim_{m\to\infty} \frac{\ln(m+1)}{\ln m} = 1$$

by virtue of l'Hospital's rule applied to both limits on the right-hand side.

Hint: *This means that the radius equals $1/2$. At the endpoint $x = 3/2$, the series equals*

$$\sum_{m=2}^{\infty} \frac{(\ln m)}{m},$$

which diverges by direct comparison to the harmonic series, i.e., the p-series with $p = 1$. At the endpoint $x = 1/2$, the series equals

$$\sum_{m=2}^{\infty} (-1)^m \frac{(\ln m)}{m},$$

which converges by the alternating series test because the sign of the terms alternates and $\ln m/m$ decreases to zero as $m \to \infty$.

Question 202 *Find the full interval of convergence for the power series*

$$\sum_{m=2}^{\infty} \frac{(-2)^m (\ln m)(x+4)^m}{6^m m}.$$

(Hints will not be revealed until after you choose a response.)

Multiple Choice:

(a) $\left(-\frac{13}{3}, -\frac{11}{3}\right)$

(b) $\left[-\frac{13}{3}, -\frac{11}{3}\right)$

(c) $(-7, -1]$ ✓

(d) $[-7, -1]$

(e) $(-\infty, \infty)$

Feedback(`attempt`): *First observe that*

$$\begin{aligned} \frac{1}{R} &= \lim_{m\to\infty} \left| \frac{\frac{(-2)^{(m+1)}(\ln(m+1))}{6^{(m+1)}(m+1)}}{\frac{(-2)^m(\ln m)}{6^m m}} \right| \\ &= \lim_{m\to\infty} \left| \frac{-2(\ln(m+1))m}{6(\ln m)(m+1)} \right| \\ &= \frac{1}{3} \end{aligned}$$

because

$$\lim_{m\to\infty} \frac{m \ln(m+1)}{(m+1)\ln m} = \lim_{m\to\infty} \frac{m}{m+1} \lim_{m\to\infty} \frac{\ln(m+1)}{\ln m} = 1$$

by virtue of l'Hospital's rule applied to both limits on the right-hand side.

Hint: *This means that the radius equals 3. At the endpoint $x = -7$, the series equals*

$$\sum_{m=2}^{\infty} \frac{(\ln m)}{m},$$

which diverges by direct comparison to the harmonic series, i.e., the p-series with $p = 1$. At the endpoint $x = -1$, the series equals

$$\sum_{m=2}^{\infty} (-1)^m \frac{(\ln m)}{m},$$

which converges by the alternating series test because the sign of the terms alternates and $\ln m/m$ decreases to zero as $m \to \infty$.

Question 203 *Find the full interval of convergence for the power series*

$$\sum_{m=1}^{\infty} \frac{\sqrt[3]{m}(x-2)^m}{m!}.$$

Multiple Choice:

(a) $(1,3)$

(b) $[1,3)$

(c) $(1,3]$

(d) $[1,3]$

(e) $(-\infty,\infty)$ ✓

Feedback(`attempt`): *First observe that*

$$\begin{aligned}\frac{1}{R} &= \lim_{m\to\infty}\left|\frac{\frac{\sqrt[3]{(m+1)}}{(m+1)!}}{\frac{\sqrt[3]{m}}{m!}}\right| \\ &= \lim_{m\to\infty}\left|\frac{\sqrt[3]{(m+1)}}{(m+1)\sqrt[3]{m}}\right| \\ &= 0\end{aligned}$$

because $m+1$ in the denominator tends to ∞ and

$$\lim_{m\to\infty}\frac{\sqrt[3]{m+1}}{\sqrt[3]{m}} = \lim_{m\to\infty}\left(1+m^{-1}\right)^{1/3} = \left(1+\lim_{m\to\infty} m^{-1}\right)^{1/3} = 1.$$

This means that the radius is infinite and the interval of convergence is $(-\infty,\infty)$.

Sample Exam Questions

Question 204 *For which values of x does the series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-1)^n}{n4^n}$ converge?*

Multiple Choice:

(a) $-3 < x < 5$

(b) $-3 \leq x < 5$

(c) $-3 < x \leq 5$ ✓

(d) $-5 < x \leq 3$

(e) $-5 \leq x < 3$

(f) $-5 \leq x \leq 3$

Question 205 *Find the interval of convergence of the power series below.*

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^{\frac{3}{4}}(n^2+2)}$$

Multiple Choice:

(a) $\left(0, \frac{1}{2}\right]$

(b) $\left[0, \frac{1}{2}\right]$ ✓

(c) $\left(0, \frac{1}{2}\right)$

(d) $\left[0, \frac{1}{2}\right)$

(e) $\left(-\frac{1}{2}, 0\right]$

(f) $(-\infty, \infty)$

Question 206 *Find the interval of convergence of the power series* $\sum_{n=2}^{\infty} \frac{2^n(x+5)^n}{\sqrt[3]{n}}$.

Multiple Choice:

(a) $\left[-\frac{11}{2}, -\frac{9}{2}\right]$

(b) $\left[-\frac{11}{2}, -\frac{9}{2}\right)$ ✓

(c) $\left(-\frac{11}{2}, -\frac{9}{2}\right)$

(d) $\left[\frac{9}{2}, \frac{11}{2}\right)$

(e) $\left(\frac{9}{2}, \frac{11}{2}\right)$

(f) $\left[\frac{9}{2}, \frac{11}{2}\right]$

5.3 Taylor Series

We introduce the notion of a Taylor Series.

Example 55. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 56. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y *because different* y *cross sections will generally have different areas). To compute volume, we integrate* $dV = A(y)dy$ *between* $y = 0$ *and* $y = 1$*, since these are the most extreme values of* y *found in our region (note that we can find the upper value* $y = 1$ *by solving for the intersection of the curves* $x = \sqrt{y}$ *and* $x = 1$*). Therefore we integrate* $A(y)dy$ *to conclude*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

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5.4 Exercises: Taylor Series

Exercises relating to Taylor series and their computation.

Exercise 207 *Use the Maclaurin series for* $\sin x$ *to give a complete formula for the Maclaurin series for* $x \sin x^2$*:*

$$x\sin(x^2) = \sum_{m=0}^{\infty} \boxed{\frac{(-1)^m}{(2m+1)!}} x^{\boxed{4m+3}}$$

Exercise 208 *Use the Maclaurin series for* e^x *and* e^{-x} *to give a complete formula for the Maclaurin series for* $\cosh x$*:*

$$\cosh x = \sum_{n=0}^{\infty} \boxed{\frac{1}{(2n)!}} x^{\boxed{2n}}.$$

Hint: *Your formula should account for the fact that there are only even powers of* x*.*

Exercise 209 *Compute the Taylor series of the function* $f(x) = \ln x$ *centered at the point* $x_0 = 2$*.*

$$\boxed{\ln 2} + \boxed{\frac{1}{2}}(x-2) + \boxed{-\frac{1}{8}}(x-2)^2 + \boxed{\frac{1}{24}}(x-2)^3 + \cdots = \ln 2 + \sum_{n=1}^{\infty} \boxed{\frac{(-1)^{n-1}2^{-n}}{n}}(x-2)^n$$

Exercise 210 *Compute the Maclaurin series of the function identified below.*

$$\int_0^x \frac{\sin t}{t} dt = \sum_{n=0}^{\infty} \boxed{\frac{(-1)^n}{(2n+1)!(2n+1)}} x^{\boxed{2n+1}}$$

Sample Quiz Questions

Question 211 *Compute the first 4 nonzero terms in the Taylor series at $x = 0$ of the function*

$$\frac{d}{dx}\left[xe^{x^2}\right].$$

Multiple Choice:

(a) $1 + 3x^2 + \frac{5}{2}x^4 + \frac{7}{6}x^6$ ✓

(b) $-1 - 3x^2 + \frac{5}{2}x^4 + \frac{7}{6}x^6$

(c) $1 - 6x^2 - \frac{5}{4}x^4 + \frac{7}{6}x^6$

(d) $-1 - 6x^2 + \frac{5}{4}x^4 + \frac{7}{6}x^6$

(e) $2 + 3x^2 + \frac{5}{4}x^4 + \frac{7}{6}x^6$

(f) $2 - 3x^2 - \frac{5}{4}x^4 + \frac{7}{6}x^6$

Feedback(`attempt`): *Compute the series in stages beginning with substitution into known series:*

$$\begin{aligned} e^{x^2} &= 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots \\ xe^{x^2} &= x + x^3 + \frac{1}{2}x^5 + \frac{1}{6}x^7 + \cdots \\ \frac{d}{dx}\left[xe^{x^2}\right] &= 1 + 3x^2 + \frac{5}{2}x^4 + \frac{7}{6}x^6 + \cdots \end{aligned}$$

Question 212 *Compute the first 4 nonzero terms in the Taylor series at $x = 0$ of the function*

$$\int_0^x (x\ln(1-x))\ dx.$$

Multiple Choice:

(a) $-\frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{2}{15}x^5 - \frac{1}{24}x^6$

(b) $\frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{2}{15}x^5 - \frac{1}{24}x^6$

(c) $-\frac{1}{6}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 - \frac{1}{24}x^6$

(d) $\frac{1}{6}x^3 + \frac{1}{4}x^4 - \frac{1}{15}x^5 - \frac{1}{24}x^6$

(e) $-\frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{15}x^5 - \frac{1}{24}x^6$ ✓

(f) $-\frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{15}x^5 - \frac{1}{24}x^6$

Feedback(*`attempt`*): *Compute the series in stages beginning with substitution into known series:*

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

$$x\ln(1-x) = -x^2 - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{1}{4}x^5 + \cdots$$

$$\int_0^x (x\ln(1-x))\ dx = -\frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{15}x^5 - \frac{1}{24}x^6 + \cdots$$

Sample Exam Questions

Question 213 *The first few nonzero terms of the Maclaurin series for* $f(x) = \ln(1+\sin x)$ *are:*

Multiple Choice:

(a) $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{24}x^3 + \cdots$

(b) $1 + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{48}x^3 + \cdots$

(c) $x - \frac{1}{2}x^2 + \frac{1}{8}x^3 - \frac{1}{24}x^4 + \cdots$

(d) $1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{6}x^4 \cdots$

(e) $x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \cdots$ ✓

(f) $1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4 + \cdots$

(Hints will not be revealed until you choose your response.)

Feedback(`attempt`): *The first few derivatives of $f(x)$ are:*

$$
\begin{aligned}
f(x) &= \ln(1+\sin(x)) \\
f'(x) &= \frac{\cos x}{1+\sin x} \\
f''(x) &= -\frac{\sin x}{1+\sin x} - \frac{\cos^2 x}{(1+\sin x)^2} \\
f'''(x) &= -\frac{\cos x}{1+\sin x} + \frac{\sin x\cos x}{(1+\sin x)^2} - \frac{2\sin x\cos x}{(1+\sin x)^2} + 2\frac{\cos^3 x}{(1+\sin x)^3}
\end{aligned}
$$

Hint: *Evaluating at $x=0$ gives $f(0)=\ln 1=0$, $f'(0)=1$, $f''(0)=-1$, and $f'''(0)=1$. Therefore the series starts with the terms $x-\frac{1}{2}x^2+\frac{1}{6}x^3+\cdots$.*

Question 214 *Find the Taylor polynomial of degree 2 for $f(x)=\sqrt{x+16}$ centered at $x=9$.*

Multiple Choice:

(a) $5+\frac{4}{5}x+\frac{9}{250}x^2$

(b) $5-\frac{3}{5}(x-5)+\frac{1}{125}(x-5)^2$

(c) $5+\frac{1}{10}(x-9)-\frac{1}{1000}(x-9)^2$ ✓

(d) $5+\frac{3}{5}(x-5)+\frac{8}{125}(x-5)^2$

(e) $5+\frac{1}{5}(x-9)+\frac{16}{125}(x-9)^2$

(f) *none of these*

Question 215 *Use the Taylor polynomial of degree 3 for $f(x)=\ln(1+x)$ centered at $x_0=0$ to approximate the value of $\ln\left(\frac{3}{2}\right)$.*

Multiple Choice:

(a) $\frac{2}{3}$

(b) $\frac{3}{2}$

(c) $\dfrac{15}{4}$

(d) $\dfrac{5}{12}$ ✓

(e) $\dfrac{9}{24}$

(f) $\dfrac{11}{24}$

Question 216 *Let $F(x)$ be the unique function that satisfies $F(0) = 0$, $F'(0) = 0$, and $F'(x) = \dfrac{1}{x}\sin x^3$ for all $x \neq 0$. Find the Taylor Series of $F(x)$ centered at $x_0 = 0$.*

Multiple Choice:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (6n+3) x^{6n+2}}{(2n+1)!}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(6n+3)(2n+1)!}$ ✓

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n+1)!}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (6n+2) x^{6n+2}}{(2n+1)!}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(6n+3)(2n+1)!}$

5.5 Taylor Series Applications

We study the use of Taylor series for evaluating infinite series and limits.

Example 57. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from* $y = 0$ *to* $y = \sqrt{1-x^2}$. *Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ *between* $x = 0$ *and* $x = 1$, *since these are the most extreme values of x found in our region. Therefore*

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 58. *The base of a solid region is bounded by the curves* $y = 0$, $x = \sqrt{y}$, *and* $x = 1$. *The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph* $x = \boxed{\sqrt{y}}$ *to the graph* $x = \boxed{1}$. *The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is* $\boxed{1-\sqrt{y}}$, *giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ *between* $y = 0$ *and* $y = 1$, *since these are the most extreme values of y found in our region (note that we can find the upper value* $y = 1$ *by solving for the intersection of the curves* $x = \sqrt{y}$ *and* $x = 1$*). Therefore we integrate* $A(y)dy$ *to conclude*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

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5.6 Exercises: Taylor Series Applications

Various exercises relating to the application of Taylor Series to other problems of interest.

Exercise 217 *In this exercise, we will investigate two different ways of numerically approximating the value of* $\ln 2$ *using infinite series.*

- *The Maclaurin series of the function* $\ln 1 + x$ *converges conditionally at* $x = 1$ *to* $\ln 2$*. Compute the Maclaurin series:*

$$\ln(1+x) = \sum_{n=0}^{\infty} \boxed{\frac{(-1)^n}{n+1}} x^{n+1}$$

  (Reindex your answer to match the template above if your answer doesn't work as-is.)

- *What degree Taylor polynomial would you need to use to approximate the value of* $\ln 2$ *to an error strictly less than* 10^{-3}*? Answer: Taylor polynomial used should have degree* $N = \boxed{1000}$ *or greater.*

- *A similar but distinct strategy would be to compute* $\ln(1/2)$ *instead because we know* $-\ln(1/2) = \ln 2$*. Evaluating the Maclaurin series at* $x = -1/2$ *and doing a little simplification, we see from the above series that*

$$\ln 2 = \sum_{n=1}^{\infty} \boxed{\frac{1}{n2^n}}.$$

  (Once again, reindex if your answer does not already start at $n = 1$*.)*

- *The presence of a factor exponential in* n *suggests that this second series converges to* $\ln 2$ *(much slower / slightly slower / slightly faster / much faster ✓) than the first series. Of the two series, then, the (first / second ✓) series presents a more efficient way to compute* $\ln 2$ *numerically.*

Hint: *For the first estimation, use the series estimation features of the Alternating Series Test.*

Hint: *For an alternating series, the error of approximating the series by a partial sum is never greater than the magnitude of the first term omitted.*

Sample Quiz Questions

Question 218 *Use Taylor series to estimate the value of*

$$\sqrt[3]{\frac{11}{10}}$$

to within an error of at most $1/900$. *(Hints will not be revealed until you choose a response.)*

Multiple Choice:

(a) $\frac{31}{30}$ ✓

(b) $\frac{47}{45}$

(c) $\frac{19}{18}$

(d) $\frac{16}{15}$

(e) $\frac{97}{90}$

(f) $\frac{49}{45}$

Feedback(`attempt`): *We may use the remainder formula for Taylor series to approach this problem. Suppose* $p_n(x)$ *is the degree* n *Taylor polynomial of the function*

$$f(x) = \sqrt[3]{1+x}$$

with center $a = 0$. *Then the error* $E_n(x)$, *i.e., the difference between the polynomial and the function, does not exceed* $\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$, *where* ξ *is some unknown point in the range* $0 \leq \xi \leq x$.

Hint: *In this case one should take* $x = 1/10$ *and determine how many derivatives are required to make this error estimate less than the given threshold.*

Hint: *This means checking by hand for small numbers of derivatives. For the specific problem at hand, if we approximate* $f(x)$ *by the Taylor polynomial of degree* $n = 1$, *we have*

$$\left|\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi)\right| = \left|\frac{x^{n+1}}{(n+1)!}\left(-\frac{2}{9}(\xi+1)^{-5/3}\right)\right| \leq \left|\frac{x^{n+1}}{(n+1)!}\left(\frac{2}{9}\right)\right| = \frac{1}{900}$$

when $x = 1/10$.

Hint: *We conclude that the correct Taylor approximation is*

$$p_n\left(\frac{1}{10}\right) = \left(\frac{1}{10}\right)^0 + \frac{1}{3}\left(\frac{1}{10}\right)^1 = 1 + \frac{1}{30} = \frac{31}{30}.$$

Question 219 *Use Taylor series to estimate the value of*

$$e^{-\frac{1}{3}}$$

to within an error of at most $1/162$.

Multiple Choice:

(a) $\frac{5}{9}$

(b) $\frac{13}{18}$ ✓

(c) $\frac{8}{9}$

(d) $\frac{19}{18}$

(e) $\frac{11}{9}$

(f) $\frac{25}{18}$

Feedback(*`attempt`*): *We may use the remainder formula for Taylor series to approach this problem. Suppose* $p_n(x)$ *is the degree* n *Taylor polynomial of the function*

$$f(x) = e^{-x}$$

with center $a = 0$. *Then the error* $E_n(x)$, *i.e., the difference between the polynomial and the function, does not exceed* $\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$, *where* ξ *is some unknown point in the range* $0 \le \xi \le x$.

Hint: *In this case one should take* $x = 1/3$ *and determine how many derivatives are required to make this error estimate less than the given threshold.*

Hint: *This means checking by hand for small numbers of derivatives. For the specific problem at hand, if we approximate* $f(x)$ *by the Taylor polynomial of degree* $n = 2$, *we have*

$$\left|\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi)\right| = \left|\frac{x^{n+1}}{(n+1)!}\left(e^{-\xi}\right)\right| \le \left|\frac{x^{n+1}}{(n+1)!}(1)\right| = \frac{1}{162}$$

when $x = 1/3$.

Hint: *We conclude that the correct Taylor approximation is*

$$p_n\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^0 - 1\left(\frac{1}{3}\right)^1 + \frac{1}{2}\left(\frac{1}{3}\right)^2 = 1 - \frac{1}{3} + \frac{1}{18} = \frac{13}{18}.$$

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5.7 Exercises: Cumulative

Exercises relating to various topics we have studied.

Sample Exam Questions

Question 220 *If it converges, find the sum of the series* $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!}$. *If the series diverges, explain why.*

Multiple Choice:

(a) $\ln 2$

(b) $\ln 3 - \ln 2$

(c) e^{-2}

(d) $\frac{1}{2}$ ✓

(e) $\frac{2}{e}$

(f) *diverges*

Feedback(*`attempt`*): *We recognize the Taylor series for cosine:*

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!}$$

The series in question is exactly

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\pi/3)^{2n}}{(2n)!} = \cos\frac{\pi}{3} = \frac{1}{2}.$$

Question 221 *What is the limit of the sequence* $\left\{n^2\left(1-\cos\frac{1}{n}\right)\right\}$?

Multiple Choice:

(a) 1

(b) -1

(c) $\frac{\sqrt{3}}{2}$

(d) $\frac{1}{2}$ ✓

(e) $-\frac{\sqrt{3}}{2}$

(f) *diverges*

Question 222 *Find the limit of the sequence*

$$a_n = \{n\,[\ln(n+3)-\ln n]\}\,.$$

Multiple Choice:

(a) 0

(b) 1

(c) $\ln 3$

(d) 3 ✓

(e) ∞

(f) *the limit does not exist*

6 Ordinary Differential Equations

We begin a study of first-order ordinary differential equations.

Example 59. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from* $y = 0$ *to* $y = \sqrt{1-x^2}$. *Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ *between* $x = 0$ *and* $x = 1$, *since these are the most extreme values of x found in our region. Therefore*

$$V = \int_0^1 (1-x^2)dx = x - \left.\frac{x^3}{3}\right|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 60. *The base of a solid region is bounded by the curves* $y = 0$, $x = \sqrt{y}$, *and* $x = 1$. *The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph* $x = \boxed{\sqrt{y}}$ *to the graph* $x = \boxed{1}$. *The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is* $\boxed{1 - \sqrt{y}}$, *giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ *between* $y = 0$ *and* $y = 1$, *since these are the most extreme values of y found in our region (note that we can find the upper value* $y = 1$ *by solving for the intersection of the curves* $x = \sqrt{y}$ *and* $x = 1$*). Therefore we integrate* $A(y)dy$ *to conclude*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

6.1 ODEs: Foundations

We study the fundamental concepts and properties associated with ODEs.

Example 61. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 62. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

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6.2 Exercises: ODEs

Exercises relating to fundamental properties of ODEs.

Exercise 223 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length $L = \boxed{x - x^2}$ and area $A = \boxed{(x - x^2)^2}$.*
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of $x = \boxed{0}$ up to a maximum of $x = \boxed{1}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 224 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length $\boxed{1}$ in the x-direction and width $\boxed{z^2}$ in the y-direction.*
- *The area of a z cross section is $A(z) = \boxed{z^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

6.3 Separable and Linear ODEs

We learn techniques to solve first-order linear and separable ODEs.

Example 63. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 64. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

odes/30solvepractice.tex

6.4 Exercises: Linear and Separable ODEs

Exercises related to solving linear and separable ODEs.

Remember: several of these exercises involve logarithms. Use absolute value signs inside the logarithm when they're needed.

Exercise 225 *Find the general solution of the ODE below. Write your answer in the form $f(y) = g(x) + C$ for appropriate functions $f(y)$ and $g(x)$.*

$$y' = y - 2$$

$$\textit{General Solution:} \quad \boxed{\ln|y-2|} = \boxed{x} + C$$

Hint: *This ODE is both linear and separable, so either approach will work.*

Exercise 226 *Find the general solution of the ODE below.*

$$x^2y' + xy = 1$$

$$\textit{General Solution:} \quad y = \boxed{\frac{\ln|x|}{x}} + \frac{C}{x}$$

(For definiteness, the function you enter for your answer should equal 0 at $x = 1$. If that's not the case, you might need to rewrite your solution and redefine the constant C.)

Hint: *This is a linear ODE.*

Exercise 227 *Find the general solution of the ODE below.*

$$yy' = 4x$$

$$\textit{General Solution:} \quad y^2 = \boxed{4x^2} + C$$

(For definiteness, the function you enter for your answer should equal 0 at $x = 0$. If that's not the case, you might need to rewrite your solution and redefine the constant C.)

Hint: *This is a separable ODE. In the form written, it's already separated.*

Exercise 228

$$y' - 3y = xe^{2x}$$

$$\textit{General Solution:} \quad y = C\boxed{e^{3x}} - (x+1)\boxed{e^{2x}}$$

Hint: *This is a linear ODE.*

Exercise 229

$$(x^2+1)y' = \frac{x}{y-1}$$

$$\textit{General Solution:} \quad (y-1)^2 = \boxed{\ln(x^2+1)} + C$$

(For definiteness, the function you enter for your answer should equal 0 *at* $x = 0$*. If that's not the case, you might need to rewrite your solution and redefine the constant* C*.)*

Hint: *This is a separable ODE.*

Exercise 230 *Solve the initial value problem*

$$y' = \cos^2 x \cos^2 2y \text{ with } y(0) = 0.$$

$$\textit{Solution:} \quad y = \boxed{\frac{1}{2}\arctan\left(x + \frac{1}{2}\sin 2x\right)}$$

Hint: *This is a separable ODE.*

Exercise 231 *Solve the initial value problem*

$$y' + (\tan x)y = \sec x \text{ with } y(0) = -3.$$

$$\textit{Solution:} \quad y = \boxed{\sin x - 3\cos x}$$

Hint: *This is a linear ODE. The integral of* $\tan x$ *is* $-\ln|\cos x|$*.*

Sample Quiz Questions

Question 232 *Let $y(x)$ be the solution to the initial value problem*

$$\frac{dy}{dx} = -(1+3x^2)y^2$$

and $y(0) = 1/2$. What is the value of $y(1)$?

Multiple Choice:

(a) $\frac{1}{6}$

(b) $\frac{1}{4}$ ✓

(c) $\frac{1}{3}$

(d) $\frac{1}{2}$

(e) $\frac{\pi}{4}$

(f) 1

Feedback(`attempt`): *This is a separable ODE. Moving all functions of y to the left-hand side and all functions of x to the right-hand side and integrating gives*

$$\int \frac{-1}{y^2}\, dy = \int (1+3x^2)\, dx,$$

which yields

$$\frac{1}{y} = x^3 + x + C.$$

Evaluating at $x = 0$ and $y = 1/2$ gives $2 = 0 + C$, so

$$\frac{1}{y} = x^3 + x + 2,$$

i.e.,

$$y = \frac{1}{x^3 + x + 2}.$$

Plugging in $x = 1$ gives $y = 1/4$.

Sample Exam Questions

Question 233 *The solution of the initial value problem* $x\dfrac{dy}{dx} + 3y = 7x^4$, $y(1) = 1$, *satisfies* $y(2) =$

Multiple Choice:

(a) 0

(b) 1

(c) 2

(d) 4

(e) 8

(f) 16 ✓

Question 234 *The solution of the initial value problem* $\dfrac{dy}{dx} - 20x^4e^{-y} = 0$, $y(0) = 0$, *satisfies* $y(1) =$

Multiple Choice:

(a) $\ln 5$ ✓

(b) $\ln 4$

(c) $\ln 3$

(d) $\ln 2$

(e) 1

(f) 0

Question 235 *Let* $y(x)$ *be the solution of the initial value problem*

$$x\frac{dy}{dx} = e^x - y \quad \textit{with} \quad y(\ln 2) = 0.$$

Find $y(1)$.

Multiple Choice:

(a) $\dfrac{e^2}{2}$

(b) $2e^2$

(c) $\dfrac{e}{2}$

(d) 0

(e) $e - 2$ ✓

(f) 1

Question 236 *Let $y(x)$ be the solution of the initial value problem*

$$x\frac{dy}{dx} = y + x^2 \sin x \quad \textit{with} \quad y(\pi) = 0.$$

What is $y(2\pi)$?

Multiple Choice:

(a) $-\pi$

(b) -2π

(c) -4π ✓

(d) 0

(e) 2π

(f) 4π

Question 237 *Consider the initial value problem*

$$(1 + x^2)\frac{dy}{dx} = 2y \quad \textit{with} \quad y(0) = 2.$$

What is $\lim_{x\to\infty} y(x)$?

Multiple Choice:

(a) $2e^{\pi}$ ✓

(b) $2e^{\pi/2}$

(c) $2e^{\pi/4}$

(d) 1

(e) 0

(f) e^{π}

6.5 Applications of ODEs

We study some sample applications of ODEs.

Example 65. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 66. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$

between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

odes/31applypractice.tex

6.6 Exercises: ODE Applications

Exercises relating to the application of ODEs to solve problems.

Sample Exam Questions

Question 238 *A tank contains 100 gallons of water in which 300 pounds of salt are dissolved. At some initial time, workers begin pumping in fresh water, i.e., containing no salt, at a rate of 10 gallons per minute. During the process, the tank is kept well-mixed and 20 gallons per minute of the resulting saltwater are pumped out of the tank (in particular, note that the tank will be empty after 10 minutes). Find the total amount of salt in the tank (measured in pounds) which remains 9 minutes after the process starts.*

Multiple Choice:

(a) 1

(b) 2

(c) 3 ✓

(d) 4

(e) 5

(f) 6