Math 104 Online

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Contents

1 Applications of Integration

We study some important application of integrations: computing volumes of a variety of complicated three-dimensional objects, computing arc length and surface area, and finding centers of mass.

Integration is the tool to use whenever a quantity can be conceived as an *accumulation of infinitesimal parts.* Volume is one of the most basic and important of such quantities. In the activities that follow, we regard volume as the accumulated size of infinitely thin slices and use this perspective to derive and apply a number of formulas for computing volume.

1.1 Volume By General Cross Sections

We use cross-sectional area to compute volume.

Example 1. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 2. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

volumes/01genslicepractice.tex

1.2 Exercises: General Slicing

Exercises computing volume by cross-sectional area.

Exercise 1 *The base of a solid region is bounded by the curves* $x = 0$, $y = x^2$ *and* $y = x$. *The cross sections perpendicular to the* x*-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the* x*-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 2 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

- *Cross sections perpendicular to the* z*-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the* x*-direction and width* $\boxed{z^2}$ *in the* y*-direction.*
- *The area of a* z *cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

Sample Exam Questions

Question 3 *(2018 Midterm 1) Compute the volume of the region in 3-dimensional space which satisfies the inequalities*

$$0 \leq x \leq (1-z^2) \qquad \text{and} \qquad 0 \leq y \leq (1+z^2) \qquad \text{and} \qquad 0 \leq z \leq 1.$$

Multiple Choice:

(a) $\frac{2}{3}$

(b) $\frac{3}{4}$

(c) $\frac{4}{5}$ ✓

(d) $\frac{5}{6}$

(e) $\frac{6}{7}$

(f) *none of these*

Question 4 *(2019 Midterm 1) The inequality*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$$

defines an ellipse in the xy-plane whose area is πab for any positive values of the constants a and b. Compute the three dimensional volume of the region defined by

$$4x^2 + z^2y^2 \leq z^2 \text{ for } 0 \leq z \leq 1.$$

Multiple Choice:

(a) $4\pi z^2$

(b) 4π

(c) $\frac{\pi}{4z}$

(d) $\frac{\pi}{4}$ ✓

(e) πz

(f) π

Feedback*(attempt)*: *Dividing the first inequality by z^2 on both sides gives*

$$\frac{x^2}{\frac{z^2}{4}} + \frac{y^2}{1} \leq 1,$$

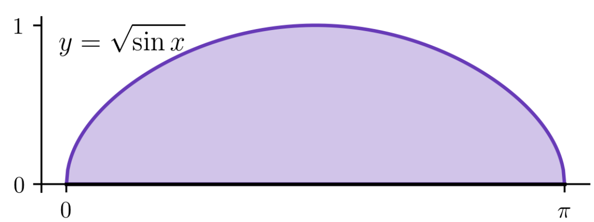
which means that slices in the z-direction are ellipses with area $\pi\frac{z}{2} \cdot 1 = \frac{\pi z}{2}$. Volume is obtained by integration:

$$V = \int_0^1 \frac{\pi z}{2} dz = \left.\frac{\pi z^2}{4}\right|_0^1 = \frac{\pi}{4}.$$

1.3 The Disk and Washer Methods

We practice setting up calculations related to the disk and washer methods.

Example 3. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*



- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

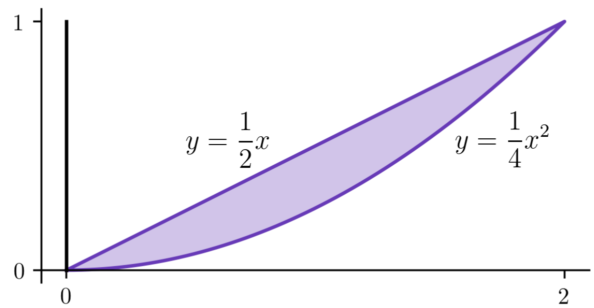
  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓

  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}} \right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 4. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*



- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***
  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$
  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓
- *We conclude that*

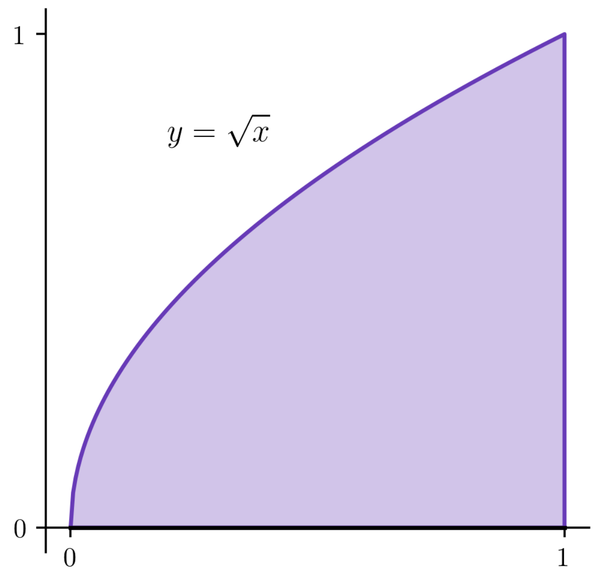
$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

`volumes/02washerpractice.tex`

1.4 Exercises: Disks and Washers

Exercises for the disk and washer methods.

Exercise 5 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the x-axis. Use the disk method to find the volume of the solid of revolution.*

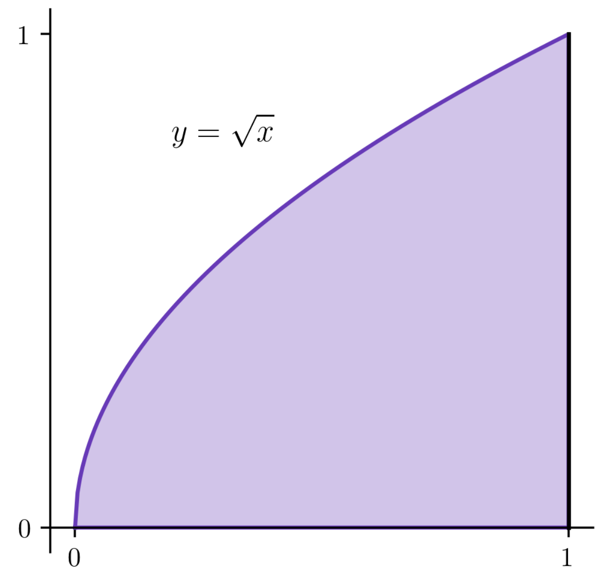


Hint: *The radius $R(x)$ will be a difference of y-values because slices are indexed by the variable x. Each slice will extend from $y = 0$ to $y = \sqrt{x}$, and so $R(x)$ must be the larger of these y-values minus the smaller of these y-values.*

$$R(x) = \boxed{\sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(x))^2 dx = \boxed{\frac{\pi}{2}}$$

Exercise 6 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the axis $x = 1$. Use the disk method to find the volume of the solid of revolution.*

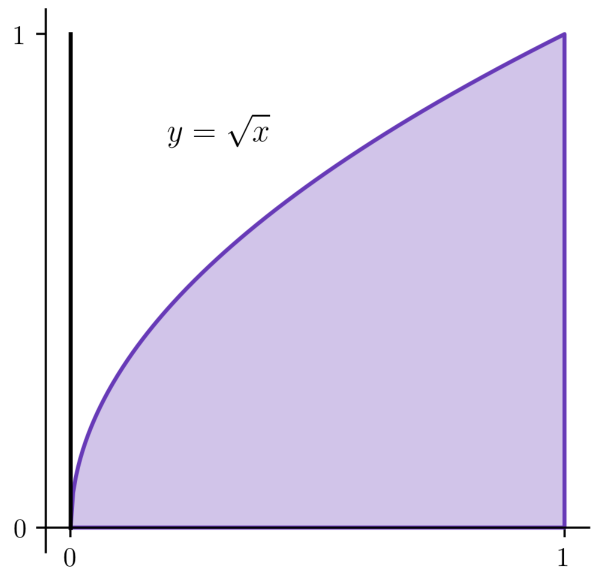


Hint: *The radius $R(y)$ will be a difference of x-values because slices are indexed by the variable y. Each slice will extend from $x = y^2$ to $x = 1$, and so $R(y)$ must be the larger of these x-values minus the smaller of these x-values*

$$R(y) = \boxed{1 - y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi (R(y))^2 dy = \boxed{\frac{8\pi}{15}}$$

Exercise 7 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 0$. Use the washer method to find the volume of the solid of revolution.*

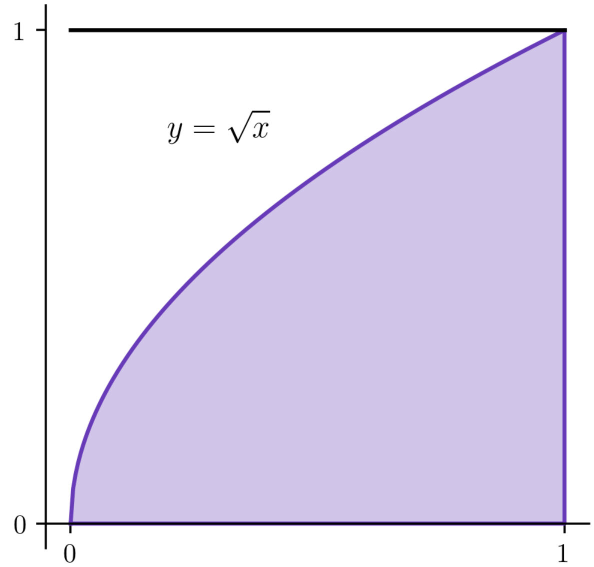


Hint: *Each radius will be a difference of x-values because slices are indexed by the variable y. The distance from the axis $x = 0$ to the line $x = 1$ is 1, and the distance from the axis $x = 0$ to $x = y^2$ is y^2.*

$$R_{\text{outer}}(y) = \boxed{1} \text{ and } r_{\text{inner}}(y) = \boxed{y^2}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(y))^2 - (r_{\text{inner}}(y))^2\right] dy = \boxed{\frac{4\pi}{5}}$$

Exercise 8 *The region $0 \le y \le \sqrt{x}$ with $x \le 1$, shown below, is revolved around the axis $y = 1$. Use the washer method to find the volume of the solid of revolution.*



Hint: *Each radius will be a difference of y-values because slices are indexed by the variable x. The distance from the axis $y = 1$ to the line $y = 0$ is 1, and the distance from the axis $y = 1$ to $y = \sqrt{x}$ is $1 - \sqrt{x}$.*

$$R_{\text{outer}}(x) = \boxed{1} \text{ and } r_{\text{inner}}(x) = \boxed{1 - \sqrt{x}}$$

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(x))^2 - (r_{\text{inner}}(x))^2 \right] dx = \boxed{\frac{5\pi}{6}}$$

Sample Quiz Questions

Question 9 *The region in the plane bounded on the left by the curve $x = -y^2$, on the right by the curve $x = y^2 + 2y + 2$, above by the line $y = 0$, and below by the line $y = -2$ is revolved around the axis $x = 2$. Compute the volume of the resulting solid.*

Multiple Choice:

(a) 16π

(b) 20π

(c) 24π ✓

(d) 28π

(e) 32π

(f) 36π

Feedback(*`attempt`*): *The axis $x = 2$ is perpendicular to the direction of slices using the integration variable y, which indicates the washer method. The region lies to the left of the axis. One way to see this is to evaluate $x = -y^2$ at $y = -2$, giving $x = -4$, which is to the left of the axis $x = 2$. The integral to compute equals*

$$\begin{aligned} V &= \int_{-2}^{0} \pi \left((2-(-y^2))^2 - (2-(y^2+2y+2))^2 \right) \, dy \\ &= \pi \int_{-2}^{0} (-4y^3 + 4) \, dy \\ &= \pi \left(-y^4 + 4y \right) \Big|_{-2}^{0} = 24\pi. \end{aligned}$$

Question 10 *The region in the plane bounded below by the curve $y = -2x^2 + 5x + 2$, above by the curve $y = -2x^2 + 2x + 2$, on the right by the line $x = 0$, and on the left by the line $x = -1$ is revolved around the axis $y = 2$. Compute the volume of the resulting solid.*

Multiple Choice:

(a) 10π ✓

(b) 14π

(c) 18π

(d) 22π

(e) 26π

(f) 30π

Feedback(*`attempt`*): *The axis $y = 2$ is perpendicular to the direction of slices using the integration variable x, which indicates the washer method. The region lies below the axis. One way to see this is to evaluate $y = -2x^2 + 5x + 2$ at $x = -1$, giving $y = -5$, which is below the axis $y = 2$. The integral to compute equals*

$$\begin{aligned} V &= \int_{-1}^{0} \pi \left((2-(-2x^2+5x+2))^2 - (2-(-2x^2+2x+2))^2 \right) \, dx \\ &= \pi \int_{-1}^{0} (-12x^3 + 21x^2) \, dx \\ &= \pi \left(-3x^4 + 7x^3 \right) \Big|_{-1}^{0} = 10\pi. \end{aligned}$$

Question 11 *The region in the plane given by* $\left|-\frac{x}{2}+\frac{1}{2}\sqrt{9-6x^2}\right| \leq y \leq \frac{x}{2}+\frac{1}{2}\sqrt{9-6x^2}$ *and* $0 \leq x \leq \frac{2}{3}\sqrt{3}$ *is revolved around the* x*-axis. Compute the volume of the resulting solid.*

Multiple Choice:

(a) $\frac{13}{9}\pi$ ✓

(b) $\frac{19}{9}\pi$

(c) $\frac{26}{9}\pi$

(d) $\frac{28}{9}\pi$

(e) $\frac{37}{9}\pi$

(f) $\frac{49}{9}\pi$

Feedback(`attempt`): *If the variable* x *is used for slicing, then slices are perpendicular to the axis of rotation, which indicates the washer method should be used. The inequalities for* y *give the outer and inner radii, and*

$$\left(\frac{x}{2}+\frac{1}{2}\sqrt{9-6x^2}\right)^2 - \left(\left|-\frac{x}{2}+\frac{1}{2}\sqrt{9-6x^2}\right|\right)^2 = x\sqrt{9-6x^2}.$$

(Note that the absolute values go away when the radius is squared.)To compute the integral

$$\int_0^{\frac{2}{3}\sqrt{3}} \pi x\sqrt{9-6x^2}\,dx$$

we can use the substitution $u = 9-6x^2$ *which implies the equality* $du = (-12x)\,dx$ *for the differentials. This gives the equality*

$$\begin{aligned}\int \pi x\sqrt{9-6x^2}\,dx &= \int \left(-\frac{\pi}{12}\sqrt{u}\right)\,du \\ &= -\frac{\pi}{18}u^{\frac{3}{2}}.\end{aligned}$$

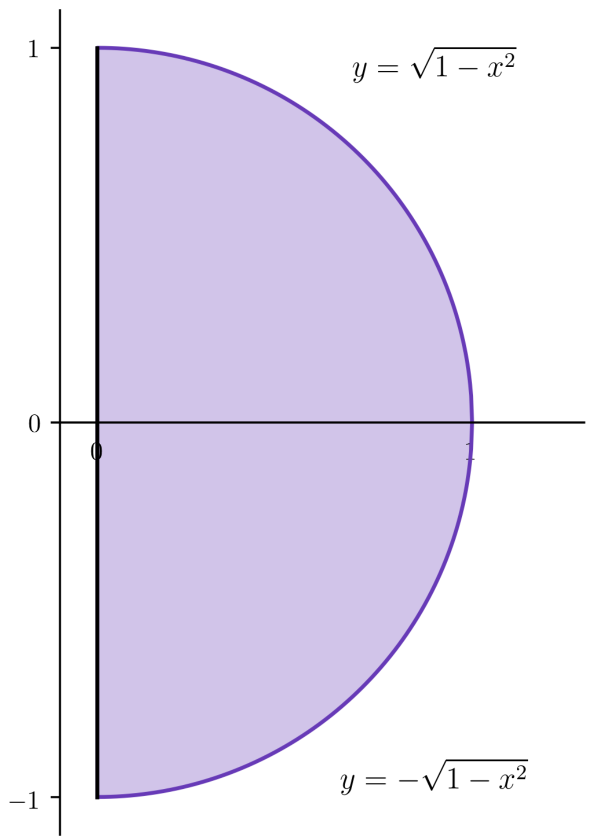
Reversing the substitution gives

$$\begin{aligned}\int_0^{\frac{2}{3}\sqrt{3}} \pi x\sqrt{9-6x^2}\,dx &= \left[-\frac{\pi}{18}\left(9-6x^2\right)^{\frac{3}{2}}\right]\Bigg|_0^{\frac{2}{3}\sqrt{3}} \\ &= \left(-\frac{\pi}{18}\right) - \left(-\frac{3}{2}\pi\right) = \frac{13}{9}\pi.\end{aligned}$$

1.5 The Shell Method

We practice setting up setting up volume calculations using the shell method.

Example 5. *The region defined by the inequalities* $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ *and* $x \geq 0$ *(shown below) is revolved around the* y*-ais. Compute the volume using the shell method.*



- *When the slicing variable is* x*, the radius of a shell is the (horizontal ✓/ vertical) distance from an* x*-slice to the axis* $x = 0$*. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an* x*-slice is equal to*

  Multiple Choice:

(a) $h(x) = \sqrt{1-x^2}$

(b) $h(x) = -\sqrt{1-x^2}$

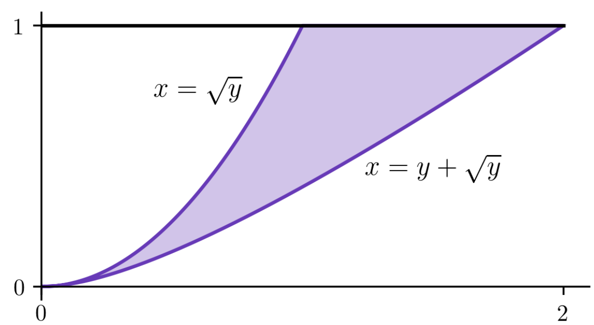
(c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\,dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$.)

Example 6. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*



- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

Multiple Choice:

(a) $h(y) = \sqrt{y}$

(b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$

(c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

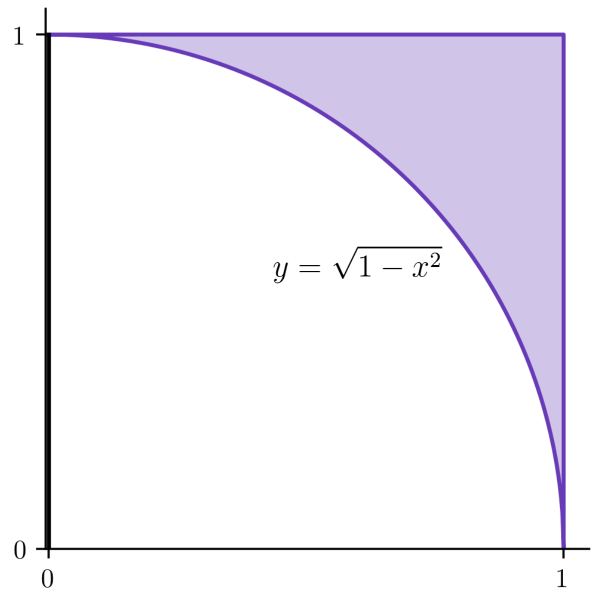
$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\,dy = \boxed{\frac{\pi}{3}}.$$

`volumes/03shellpractice.tex`

1.6 Exercises: Shell Method

Exercises for using the shell method.

Exercise 12 *The region defined by the inequalities* $\sqrt{1-x^2} \leq y \leq 1$ *for* $0 \leq y \leq 1$ *is revolved around the* y*-axis. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is* x*, the radius of a shell is the (horizontal ✓/ vertical) distance from an* x*-slice to the axis of rotation. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an* x*-slice is equal to*

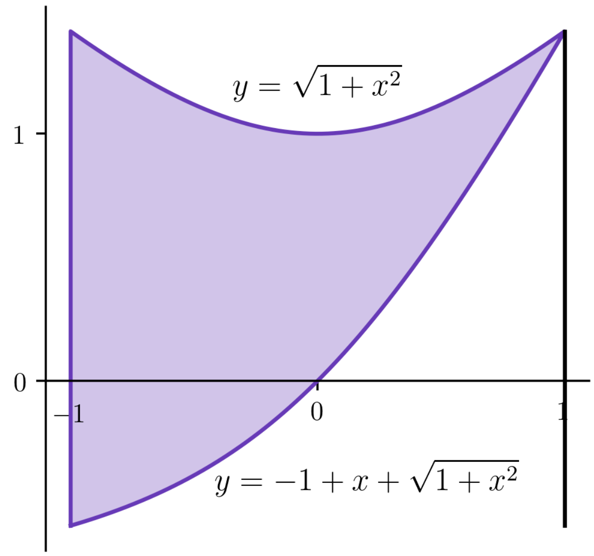
$$h(x) = \boxed{1 - \sqrt{1-x^2}}.$$

- *The volume is equal to the integral of* $2\pi r h$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi x(1 - \sqrt{1-x^2})}\, dx = \boxed{\frac{\pi}{3}}.$$

(Note: to compute the integral, split it into two parts and make the substitution $u = 1 - x^2$ *for one of them.)*

Exercise 13 *The region in the plane bounded above by the graph $y = \sqrt{1+x^2}$, below by $y = -1+x+\sqrt{1+x^2}$, and on the left by $x = 0$ is revolved around the axis $x = 1$. Compute the volume of the resulting solid using the shell method.*



- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{1} - \boxed{x}.$$

- *The height of an x-slice is equal to*

$$h(x) = \boxed{-1+x}.$$

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{-1}}^{\boxed{1}} \boxed{2\pi(1-x)^2}\, dx = \boxed{\frac{16\pi}{3}}.$$

Sample Quiz Questions

Question 14 *The region in the plane bounded below by the curve $y = -x^2$, above by the curve $y = x^2 + 2x + 2$, on the right by the line $x = 0$, and on the left by the line $x = -2$ is revolved around the axis $x = -2$. Compute the volume of the resulting solid.*

Multiple Choice:

(a) 4π

(b) 6π

(c) 8π ✓

(d) 10π

(e) 12π

(f) 14π

Feedback(*`attempt`*): *The axis $x = -2$ is parallel to the direction of slices using the integration variable x, which indicates the shell method. The region lies to the right of the axis, which must be the case because the interval $-2 \leq x \leq 0$ lies to the right of the axis $x = -2$. The integral to compute equals*

$$\begin{aligned} V &= \int_{-2}^{0} 2\pi(x - (-2))((x^2 + 2x + 2) - (-x^2))\ dx \\ &= \pi \int_{-2}^{0} (4x^3 + 12x^2 + 12x + 8)\ dx \\ &= \pi \left(x^4 + 4x^3 + 6x^2 + 8x\right)\Big|_{-2}^{0} = 8\pi. \end{aligned}$$

Question 15 *The region in the plane bounded on the left by the curve $x = -y^2 + 4y + 1$, on the right by the curve $x = y^2 + 2y + 1$, and below by the line $y = -1$ is revolved around the axis $y = -1$. Compute the volume of the resulting solid.*

Multiple Choice:

(a) π ✓

(b) 5π

(c) 9π

(d) 13π

(e) 17π

(f) 21π

Feedback(*`attempt`*): *The axis $y = -1$ is parallel to the direction of slices using the integration variable y, which indicates the shell method. The lower endpoint of integration will be $y = -1$; the upper endpoint can be determined by setting*

$-y^2+4y+1 = y^2+2y+1$ and choosing the solution which is greater than -1. This gives the range $-1 \le y \le 0$. The region lies above the axis, which must be the case because the interval $-1 \le y \le 0$ lies above the axis $y = -1$. The integral to compute equals

$$\begin{aligned} V &= \int_{-1}^{0} 2\pi(y-(-1))((y^2+2y+1)-(-y^2+4y+1))\ dy \\ &= \pi \int_{-1}^{0} (4y^3-4y)\ dy \\ &= \pi \left(y^4 - 2y^2\right)\Big|_{-1}^{0} = 1\pi. \end{aligned}$$

Question 16 *The region in the plane between the x-axis and the graph*

$$y = \frac{1}{2\sqrt{\frac{x^2}{3}+1}}$$

in the range $0 \le x \le 3$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.

Multiple Choice:

(a) $\frac{3}{2}\pi$

(b) $\frac{8}{5}\pi$

(c) $\frac{5}{3}\pi$

(d) 2π

(e) 3π ✓

(f) 5π

Feedback(*`attempt`*): *If the variable x is used for slicing, then slices are parallel to the axis of rotation, which indicates the shell method should be used. The radius of a shell is x. The height of a shell is exactly $\frac{1}{2\sqrt{\frac{x^2}{3}+1}}$. The volume of the region is therefore given by*

$$\int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2+3}}\, dx.$$

To compute the integral we can use the substitution $u = x^2+3$ which implies the equality $du = (2x)\, dx$ for the differentials. This gives the equality

$$\begin{aligned} \int \frac{\sqrt{3}\pi x}{\sqrt{x^2+3}}\, dx &= \int \frac{\sqrt{3}\pi}{2\sqrt{u}}\, du \\ &= \sqrt{3}\pi\sqrt{u}. \end{aligned}$$

Reversing the substitution gives

$$\begin{aligned}\int_0^3 \frac{\sqrt{3}\pi x}{\sqrt{x^2+3}}\,dx &= \left[\sqrt{3}\pi\sqrt{x^2+3}\right]\Big|_0^3\\ &= (6\pi)-(3\pi)=3\pi.\end{aligned}$$

1.7 Synthesis: Choose Your Method

We practice choosing a method for computing volume when none is specified.

Problem 17 *Type* 2. $\boxed{2}$, $\underset{\text{given}}{\boxed{2}}$, $\boxed{\dfrac{1}{2}}$, $\displaystyle\int_a^b f(x)\ dx$,

$$-\frac{1}{\frac{12}{1}}\sum_{n=1}^{\boxed{\infty}} n$$

`volumes/04choosepractice.tex`

1.8 Exercises: Choose Your Method

Exercises choosing a method for computing volume.

Sample Exam Questions

Question 18 *The region in the plane bounded by $y = e^{-x/2}$ and the x-axis for $0 \le x \le \ln 2$ is rotated about the x-axis. The volume of the resulting solid of revolution is*

Multiple Choice:

(a) 4π

(b) 4π

(c) 4π

(d) 4π

(e) 4π

(f) 4π ✓

Feedback(`attempt`): *If x is used as the slicing variable, then slices are vertical and consequently perpendicular to the axis of rotation. Furthermore one side of the region lies along the axis, so the disk method is appropriate in this case. The distance from the axis to the upper edge of the region is $e^{-x/2}$, so*

$$\begin{aligned} V &= \int_0^{\ln 2} \pi \left(e^{-x/2}\right)^2 dx = \pi \int_0^{\ln 2} e^{-x} dx \\ &= -\pi e^{-x}\Big|_{x=0}^{\ln 2} = \pi(-e^{-\ln 2} + e^0) = \pi\left(-\frac{1}{2} + 1\right) = \frac{\pi}{2}. \end{aligned}$$

Question 19 *Compute the volume of the solid of revolution obtained by rotating the region between $x = 0$, $y = 0$, and $y = \sqrt{2 + 3x^2 - 5x^4}$ around the x-axis.*

Multiple Choice:

(a) 4π

(b) 4π

(c) 4π

(d) 4π ✓

(e) 4π

(f) 4π

Question 20 *The region in the plane bounded on the right by the curve $x = 2 - y^2$, on the left by the curve $x = y^2$, and on the bottom by $y = 0$ is revolved around the y-axis. Compute the volume of the resulting solid.*

Multiple Choice:

(a) $\dfrac{8\pi}{3}$ ✓

(b) $\dfrac{9\pi}{4}$

(c) $\dfrac{10\pi}{7}$

(d) $\dfrac{11\pi}{5}$

(e) $\dfrac{12\pi}{11}$

(f) *none of these*

Question 21 *The region between the graph of $y = 1 - x^2$ and the x-axis is rotated around the line $y = 1$. What is the volume of the resulting solid?*

Multiple Choice:

(a) $\dfrac{2\pi}{5}$

(b) $\dfrac{4\pi}{5}$

(c) $\dfrac{6\pi}{5}$

(d) $\dfrac{8\pi}{5}$ ✓

(e) 2π

(f) $\dfrac{12\pi}{5}$

Question 22 *Calculate the volume of the solid obtained by rotating the area between the graphs of* $y = \dfrac{1}{\sqrt{x^2-1}}$ *and the* x*-axis for* $1 < x < \sqrt{5}$ *around the* y*-axis.*

Multiple Choice:

(a) π

(b) 4π ✓

(c) 6π

(d) 8π

(e) 3π

(f) 2π

Question 23 *Let* $f(x)$ *be a continuous function that satisfies* $f(0) = 0$ *and* $f(x) > 0$ *for* $x > 0$*. For every* $b > 0$*, when the region between the graph of* $y = f(x)$*, the* x*-axis, and the line* $x = b$ *is rotated around the* x*-axis, the volume of the resulting solid is* $18\pi b^2$*. What is* $f(x)$*?*

Multiple Choice:

(a) $9x$

(b) $3x^2$

(c) $6\sqrt{x}$ ✓

(d) $27x^{3/2}$

(e) $9x^2$

(f) $\sqrt{3x}$

Question 24 *Find the volume of the solid generated by revolving the region bounded above by $y = \sec x$ and bounded below by $y = 0$ for $0 \leq x \leq \pi/3$ about the x-axis.*

Multiple Choice:

(a) π

(b) 2π

(c) $\pi\sqrt{3}$ ✓

(d) 3π

(e) 4π

(f) *none of these*

1.9 Arc Length

We practice setting up and executing arc length calculations.

Example 7. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 8. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph* $x = \boxed{\sqrt{y}}$ *to the graph* $x = \boxed{1}$. *The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is* $\boxed{1-\sqrt{y}}$, *giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ *between* $y = 0$ *and* $y = 1$, *since these are the most extreme values of y found in our region (note that we can find the upper value* $y = 1$ *by solving for the intersection of the curves* $x = \sqrt{y}$ *and* $x = 1$*). Therefore we integrate* $A(y)dy$ *to conclude*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`arclengths/05arclengthpractice.tex`

1.10 Exercises: Arc Length

We practice computing arc length.

Sample Quiz Questions

Question 25 *Compute the arc length of the curve*

$$y = \frac{3}{4}x^{-2} + \frac{1}{24}x^4 - 1$$

between the endpoints $x = \sqrt{3}$ *and* $x = \sqrt{6}$.

Multiple Choice:

(a) $\frac{5}{12}$

(b) $\frac{7}{12}$

(c) $\frac{3}{4}$

(d) $\frac{11}{12}$

(e) $\frac{13}{12}$

(f) $\frac{5}{4}$ ✓

Feedback(`attempt`): *Applying the formula for arc length gives that*

$$\begin{aligned} L &= \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{1 + \left(-\frac{3}{2}x^{-3} + \frac{1}{6}x^3\right)^2}\, dx \\ &= \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{1 + \frac{9}{4}x^{-6} - \frac{1}{2} + \frac{1}{36}x^6}\, dx = \int_{\sqrt{3}}^{\sqrt{6}} \sqrt{\left(\frac{3}{2}x^{-3} + \frac{1}{6}x^3\right)^2}\, dx \\ &= \int_{\sqrt{3}}^{\sqrt{6}} \left(\frac{3}{2}x^{-3} + \frac{1}{6}x^3\right)\, dx = \left(-\frac{3}{4}x^{-2} + \frac{1}{24}x^4\right)\Bigg|_{\sqrt{3}}^{\sqrt{6}} \\ &= \left(-\frac{1}{8} + \frac{3}{2}\right) - \left(-\frac{1}{4} + \frac{3}{8}\right) = \frac{5}{4}. \end{aligned}$$

Note that you must always take the positive square root in going from line two to line three. In particular, if you get a negative answer, you have likely taken the negative square root.

Sample Exam Questions

Question 26 *(2017 Midterm 1) Compute the length of the curve* $x = \frac{1}{8}(y^2 + 2y) - \ln(y+1)$ *between* $y = 0$ *and* $y = 2$.

Multiple Choice:

(a) $1 + \ln 3$ ✓

(b) $2 + \ln 6$

(c) $3 + \ln 9$

(d) $4 + \ln 12$

(e) $5 + \ln 15$

(f) *none of the above*

Question 27 *Find the arc length of the following curve between* $x = -1$ *and* $x = 1$:

$$y = 3\cosh\frac{x}{3}.$$

Multiple Choice:

(a) $3e - \frac{3}{e}$

(b) $3e - \frac{3}{e}$

(c) $3e - \frac{3}{e}$

(d) $3e - \frac{3}{e}$

(e) $3e - \frac{3}{e}$ ✓

(f) *none of the above*

Question 28 *A certain curve $y = f(x)$ in the plane has the property that its length between the endpoints $x = 0$ and $x = a$ is equal to*

$$\int_0^a \sqrt{1+\sin^2 t}\; dt$$

for every value of $a > 0$. Assuming the curve passes through the points $(0,0)$ and $(\frac{\pi}{2}, 1)$, what is $f(\frac{\pi}{4})$?

Multiple Choice:

(a) $\frac{1}{2}$

(b) $\frac{1}{\sqrt{2}}$

(c) $1 - \frac{1}{\sqrt{2}}$ ✓

(d) 0

(e) $-\frac{1}{\sqrt{2}}$

(f) *none of these*

Question 29 *Find the length of the part of the curve $y = \frac{3}{16}e^{2x} + \frac{1}{3}e^{-2x}$ for $0 \leq x \leq \ln 2$.*

Multiple Choice:

(a) $\frac{13}{16}$ ✓

(b) $\frac{11}{16}$

(c) $\frac{3}{8}$

(d) $\frac{9}{8}$

(e) $\frac{29}{64}$

(f) $\frac{3}{4}$

Question 30 *Find the length of the part of the curve* $y = \dfrac{x^4}{4} + \dfrac{1}{8x^2}$ *for* $1 \leq x \leq 2$.

Multiple Choice:

(a) $\dfrac{13}{16}$ ✓

(b) $\dfrac{11}{16}$

(c) $\dfrac{7}{8}$

(d) $\dfrac{13\sqrt{2}}{16}$

(e) $\dfrac{11\sqrt{2}}{16}$

(f) $\dfrac{7\sqrt{2}}{8}$

Question 31 *Let* $y = \dfrac{x^4}{16} + \dfrac{1}{2x^2}$. *Find the arc length for* $1 \leq x \leq \sqrt{2}$.

Multiple Choice:

(a) $\dfrac{5}{7}$

(b) $\dfrac{6}{7}$

(c) $\dfrac{5}{6}$

(d) $\dfrac{7}{4}$

(e) $\dfrac{7}{16}$ ✓

(f) *none of these*

1.11 Surface Area

We practice setting up integrals for the surface area of surfaces of revolution.

Example 9. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 10. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`arclengths/06surfacepractice.tex`

1.12 Exercises: Surface Area

Various exercises related to the computation of areas of surfaces of revolution.

Sample Exam Questions

Question 32 *Give an integral formula for the area of the surface generated by revolving the curve $y = \ln x$ between $x = 1$ and $x = 2$ about the y-axis. Explain your answer. You do not need to evaluate the integral.*

Multiple Choice:

(a) $\displaystyle\int_1^2 \frac{2\pi x}{\sqrt{1+(\ln x)^2}}\,dx$ ✓

(b) $\displaystyle\int_1^2 \frac{2\pi x}{\sqrt{1+(\ln x)^2}}\,dx$

(c) $\displaystyle\int_1^2 \frac{2\pi x}{\sqrt{1+(\ln x)^2}}\,dx$

(d) $\displaystyle\int_1^2 \frac{2\pi x}{\sqrt{1+(\ln x)^2}}\,dx$

(e) $\displaystyle\int_1^2 \frac{2\pi x}{\sqrt{1+(\ln x)^2}}\,dx$

(f) $\displaystyle\int_1^2 \frac{2\pi x}{\sqrt{1+(\ln x)^2}}\,dx$

Question 33 *The curve $y = \dfrac{x^2}{8}$ between $x = 0$ and $x = 3$ is revolved around the y-axis. Compute the surface area of the resulting surface.*

Multiple Choice:

(a) $\dfrac{31\pi}{6}$

(b) $\dfrac{41\pi}{6}$

(c) $\frac{61\pi}{6}$ ✓

(d) $\frac{71\pi}{6}$

(e) $\frac{91\pi}{6}$

(f) *none of the above*

1.13 Centers of Mass and Centroids

We practice setting up calculations for centers of mass and centroids.

Example 11. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 12. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

centroids/07centroidpractice.tex

1.14 Exercises: Centers of Mass and Centroids

Various questions relating to centers of mass and centroids.

Sample Quiz Questions

Question 34 *Compute the centroid of the region bounded by the inequalities*

$$-2 \le x \le 0 \qquad \text{and} \qquad 3x^2 - \frac{13}{2} \le y \le 3x^2 - \frac{11}{2}.$$

Multiple Choice:

(a) $\left(-\frac{3}{2}, -3\right)$

(b) $(-1, -3)$

(c) $\left(-\frac{1}{2}, -3\right)$

(d) $\left(-\frac{3}{2}, -2\right)$

(e) $(-1, -2)$ ✓

(f) $\left(-\frac{1}{2}, -2\right)$

Feedback(*`attempt`*): *The key calculations are as follows:*

$$\begin{aligned}
M &= \int_{-2}^{0} \left[\left(3x^2 - \frac{11}{2}\right) - \left(3x^2 - \frac{13}{2}\right)\right] dx = \int_{-2}^{0} [1]\, dx = 2, \\
M_y &= \int_{-2}^{0} x \left[\left(3x^2 - \frac{11}{2}\right) - \left(3x^2 - \frac{13}{2}\right)\right] dx = \int_{-2}^{0} x\, [1]\, dx = -2, \\
M_x &= \frac{1}{2}\int_{-2}^{0} \left[\left(3x^2 - \frac{11}{2}\right)^2 - \left(3x^2 - \frac{13}{2}\right)^2\right] dx = \int_{-2}^{0} \left[3x^2 - 6\right] = -4, \\
\overline{x} &= \frac{M_y}{M} = -1, \\
\overline{y} &= \frac{M_x}{M} = -2.
\end{aligned}$$

Sample Exam Questions

Question 35 *Which of the following is the centroid of the region given by* $x^2 + y^2 \leq 1$, $x \geq 0$, *and* $y \geq 0$? *Justify your response.* **Without doing the calculation,** *is* $\overline{x}$ *greater than or less than* $\frac{1}{2}$? *Give a brief geometric explanation.*

Multiple Choice:

(a) $\left(\frac{5}{3\pi}, \frac{4}{3\pi}\right)$ ✓

(b) $\left(\frac{5}{3\pi}, \frac{4}{3\pi}\right)$

(c) $\left(\frac{5}{3\pi}, \frac{4}{3\pi}\right)$

(d) $\left(\frac{5}{3\pi}, \frac{4}{3\pi}\right)$

(e) $\left(\frac{5}{3\pi}, \frac{4}{3\pi}\right)$

(f) $\left(\frac{5}{3\pi}, \frac{4}{3\pi}\right)$

Feedback(`attempt`):

$$M = \frac{\pi}{4}$$

$$\int_0^1 x\sqrt{1-x^2}dx = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}\Big|_0^1$$

The value of $\overline{x}$ *should be less than* $\frac{1}{2}$ *since, when compared symmetrically about the axis* $x = \frac{1}{2}$, *the quarter circle has longer slices to the left of the axis than it does to the right.*

Question 36 *Find the* y*-coordinate of the centroid of the region bounded by the* x*-axis, the* y*-axis, and the graph of* $y = \cos x$ *for* $0 \leq x \leq \pi/2$ *if the density is constant.*

Multiple Choice:

(a) $\frac{\pi}{18}$

(b) $\dfrac{\pi}{12}$

(c) $\dfrac{\pi}{8}$ ✓

(d) $\dfrac{\pi}{6}$

(e) $\dfrac{\pi}{4}$

(f) $\dfrac{\pi}{2}$

Feedback(*`attempt`*): *The area of the region is given by*

$$M = \int_0^{\frac{\pi}{2}} \cos x \; dx = 1$$

and

$$M_x = \int_0^{\frac{\pi}{2}} \frac{0 + \cos x}{2} \cos x \; dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^2 x \; dx$$
$$\frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2x}{2} dx = \frac{\pi}{8}.$$

Therefore $\overline{y} = M_x/M = \pi/8$.

Question 37 *Find the y-coordinate of the centroid of the region in the upper half-plane (i.e., for $y > 0$) bounded by the semicircle $y = \sqrt{1 - x^2}$. (It is easiest to use a geometric formula to find the area of the region.)*

Multiple Choice:

(a) $\dfrac{4\pi}{3}$

(b) $\dfrac{4}{3\pi}$ ✓

(c) $\dfrac{7\pi}{3}$

(d) $\dfrac{7}{3\pi}$

(e) $\dfrac{28\pi}{9}$

(f) $\dfrac{28}{9\pi}$

2 Integration Techniques

We begin a study of techniques for computing integrals.

Example 13. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 14. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

2.1 Substitution and Tables

We review substitution and the use of integral tables.

Example 15. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 16. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

techniques/08substitutionpractice.tex

2.2 Exercises: Substitution and Tables

Various exercises relating to substitution and the use of integral tables.

Example 17. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 18. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

2.3 Integration by Parts

We study the integration technique of integration by parts.

Example 19. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 20. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

`techniques/09ibppractice.tex`

2.4 Exercises: Integration by Parts

Various exercises relating to integration by parts.

Sample Quiz Questions

Question 38 *Compute the definite integral*

$$\int_1^4 e^{3x}(x+1)\ dx.$$

Multiple Choice:

(a) $\frac{14}{9}e^{12} - \frac{5}{9}e^3$ ✓

(b) $\frac{17}{9}e^{12} - \frac{5}{9}e^3$

(c) $\frac{17}{9}e^{12} - \frac{8}{9}e^3$

(d) $\frac{20}{9}e^{12} - \frac{8}{9}e^3$

(e) $\frac{20}{9}e^{12} - \frac{11}{9}e^3$

(f) $\frac{23}{9}e^{12} - \frac{11}{9}e^3$

Feedback(`attempt`): *Integrate by parts, integrating the exponential and differentiating polynomials.*

$$\begin{aligned} &\int_1^4 e^{3x}(x+1)\ dx \\ &\qquad = \frac{e^{3x}}{3}(x+1)\bigg|_1^4 - \int_1^4 \frac{e^{3x}}{3}dx \\ &\qquad = \frac{5}{3}e^{12} - \frac{2}{3}e^3 - \frac{e^{3x}}{9}\bigg|_1^4 \\ &\qquad = \frac{14}{9}e^{12} - \frac{5}{9}e^3 \end{aligned}$$

Question 39 *Compute the definite integral*

$$\int_{\pi}^{2\pi} x \sin 4x \; dx.$$

Multiple Choice:

(a) 0

(b) $\dfrac{\pi}{4}$

(c) $-\dfrac{\pi}{4}$ ✓

(d) $\dfrac{3\pi}{4}$

(e) $-\dfrac{3\pi}{4}$

(f) $\dfrac{7\pi}{4}$

Feedback(`attempt`): *Integrate by parts, integrating the trig functions and differentiating polynomials.*

$$\begin{aligned} &\int_{\pi}^{2\pi} x \sin 4x \; dx \\ &\qquad = -\frac{\cos 4x}{4} x \bigg|_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \frac{\cos 4x}{4} dx \\ &\qquad = -\frac{2\pi}{4} + \frac{(-1)^4 \pi}{4} - \frac{\sin 4x}{16} \bigg|_{\pi}^{2\pi} = -\frac{\pi}{4} \end{aligned}$$

Question 40 *Compute the indefinite integral*

$$\int \arctan 5x \; dx.$$

Multiple Choice:

(a) $x \arctan 5x - \dfrac{1}{10} \ln|1 + 25x^2| + C$ ✓

(b) $x \arctan 5x - \dfrac{1}{12} \ln|1 + 25x^2| + C$

(c) $x \arctan 5x - \dfrac{1}{14} \ln|1 + 25x^2| + C$

(d) $x \arctan 5x + \frac{1}{10} \ln|1+25x^2| + C$

(e) $x \arctan 5x + \frac{1}{12} \ln|1+25x^2| + C$

(f) $x \arctan 5x + \frac{1}{14} \ln|1+25x^2| + C$

Feedback(`attempt`): *Integrate by parts, integrating the coefficient 1 and differentiating arctangent.*

$$\begin{aligned}
&\int \arctan 5x \ dx \\
&\quad = x \arctan 5x - \int \frac{5x}{1+25x^2} \ dx \\
&\quad = x \arctan 5x - \frac{1}{10} \int \frac{50x}{1+25x^2} \ dx \\
&\quad = x \arctan 5x - \frac{1}{10} \ln|1+25x^2| + C
\end{aligned}$$

Question 41 *Compute the indefinite integral*

$$\int e^{3x} \cos 5x \ dx.$$

Multiple Choice:

(a) $\dfrac{e^{3x}(2\cos 5x + 5\sin 5x)}{29} + C$

(b) $\dfrac{e^{3x}(3\cos 5x + 5\sin 5x)}{34} + C$ ✓

(c) $\dfrac{e^{3x}(\cos 5x + 3\sin 5x)}{20} + C$

(d) $\dfrac{e^{3x}(\cos 5x + 2\sin 5x)}{15} + C$

(e) $\dfrac{e^{3x}(2\cos 5x + 7\sin 5x)}{53} + C$

(f) $\dfrac{e^{3x}(3\cos 5x + 7\sin 5x)}{58} + C$

Feedback(`attempt`): *Integrate by parts, integrating the exponential and differentiating cosine.*

$$\begin{aligned}
&\int e^{3x}\cos 5x\ dx \\
&\qquad = \frac{e^{3x}}{3}\cos 5x - \int \frac{e^{3x}}{3}(-5\sin 5x)\ dx \\
&\qquad = \frac{e^{3x}\cos 5x}{3} + \frac{5}{3}\int e^{3x}\sin 5x\ dx \\
&\qquad = \frac{e^{3x}\cos 5x}{3} + \frac{5}{3}\frac{e^{3x}}{3}\sin 5x - \frac{5}{3}\int \frac{e^{3x}}{3}(5\cos 5x)\ dx \\
&\qquad = \frac{e^{3x}(3\cos 5x + 5\sin 5x)}{9} - \frac{25}{9}\int e^{3x}\cos 5x\ dx \\
&\Rightarrow \quad \frac{34}{9}\int e^{3x}\cos 5x\ dx = \frac{e^{3x}(3\cos 5x + 5\sin 5x)}{9} \\
&\Rightarrow \quad \int e^{3x}\cos 5x\ dx = \frac{e^{3x}(3\cos 5x + 5\sin 5x)}{34} + C
\end{aligned}$$

Sample Exam Questions

Question 42 *Compute the integral below.*

$$\int_{\frac{1}{2}}^{\infty} \frac{\ln(2x)}{x^2} dx$$

Multiple Choice:

(a) $1 - \ln 2$

(b) 2 ✓

(c) $\ln 2 - \frac{1}{2}$

(d) $\frac{1}{2}$

(e) $2 - 2\ln 2$

(f) *the integral diverges*

Question 43 *Compute the indefinite integral indicated below. [Hint: Write* $\frac{1}{\cos^2\theta} = \sec^2\theta$ *and integrate by parts.]*

$$\int \left(1 + \frac{\ln|\sin\theta|}{\cos^2\theta}\right) d\theta$$

Multiple Choice:

(a) $(\sin\theta)\ln|\sin\theta| + C$

(b) $(\cos\theta)\ln|\sin\theta| + C$

(c) $(\tan\theta)\ln|\sin\theta| + C$ ✓

(d) $(\csc\theta)\ln|\sin\theta| + C$

(e) $(\sec\theta)\ln|\sin\theta| + C$

(f) $(\cot\theta)\ln|\sin\theta| + C$

2.5 Trigonometric Integrals

We learn various techniques for integrating certain combinations of trigonometric functions.

Example 21. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 22. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference* *of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is* $\boxed{1-\sqrt{y}}$*, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

`techniques/10trigonometricpractice.tex`

2.6 Exercises: Trigonometric Integral

Various exercises relating to the integration of trigonometric functions.

Sample Quiz Questions

Question 44 *Compute the value of the integral*

$$\int_0^{\frac{\pi}{4}} \sin^3 2x \; dx.$$

Multiple Choice:

(a) $\frac{1}{5}$

(b) $\frac{1}{3}$ ✓

(c) $\frac{1}{2}$

(d) 1

(e) 2

(f) 3

Feedback(`attempt`): *To simplify the calculation, begin with a substitution which replaces x with $x/2$. The question reduces to computing*

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^3 x \; dx.$$

This integral is compatible with the substitution $u = \cos x$. By the substitution formula, this means $dx = -du/\sin x$, and one must also replace $\sin^2 x$ by $1 - u^2$. Furthermore, by virtue of the special angle formulas $\cos 0 = 1$ and $\cos\frac{\pi}{2} = 0$, the problem is reduced to computing the integral

$$-\frac{1}{2}\int_1^0 (1 - u^2) \; du.$$

Carrying out this calculation in the usual way gives a final answer of $\frac{1}{3}$.

Question 45 *Compute the value of the integral*

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \tan^{-6} x \sec^5 x \; dx.$$

Multiple Choice:

(a) $\dfrac{17}{5}$

(b) $\dfrac{19}{5}$

(c) $\dfrac{23}{5}$

(d) $\dfrac{29}{5}$

(e) $\dfrac{31}{5}$ ✓

(f) $\dfrac{37}{5}$

Feedback(*`attempt`*): *Since the power of secant is odd and the power of tangent is even, try rewriting the integral in terms of sine and cosine. This gives*

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^{-6} x \cos x \; dx.$$

This integral is compatible with the substitution $u = \sin x$*. By the substitution formula, this means* $dx = du/\cos x$*. Furthermore, by virtue of the special angle formulas* $\sin\frac{\pi}{6} = \frac{1}{2}$ *and* $\sin\frac{\pi}{2} = 1$*, the problem is reduced to computing the integral*

$$\int_{\frac{1}{2}}^{1} u^{-6} \; du.$$

Carrying out this calculation in the usual way gives a final answer of $\dfrac{31}{5}$.

Question 46 *Compute the value of the integral*

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^{-2} x \cos^3 x \; dx.$$

Multiple Choice:

(a) $\frac{1}{5}$

(b) $\frac{1}{3}$

(c) $\frac{1}{2}$ ✓

(d) 1

(e) 2

(f) 3

Feedback(*`attempt`*): *This integral is compatible with the substitution $u = \sin x$. By the substitution formula, this means $dx = du/\cos x$, and one must also replace $\cos^2 x$ by $1 - u^2$. Furthermore, by virtue of the special angle formulas $\sin \frac{\pi}{6} = \frac{1}{2}$ and $\sin \frac{\pi}{2} = 1$, the problem is reduced to computing the integral*

$$\int_{\frac{1}{2}}^{1} u^{-2}(1 - u^2)\ du.$$

Carrying out this calculation in the usual way gives a final answer of $\frac{1}{2}$.

Sample Exam Questions

Question 47 *Compute the integral below.*

$$\int_0^{\frac{\pi}{8}} \tan^4 2x \sec^4 2x\ dx$$

Multiple Choice:

(a) $\frac{4}{9}$

(b) $\frac{7}{24}$

(c) $\frac{5}{14}$

(d) $\frac{9}{28}$

(e) $\frac{6}{35}$ ✓

(f) $\frac{1}{7}$

2.7 Trigonometric Substitions

We practice executing trigonometric substitutions.

Example 23. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*

- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓
  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}} \right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 24. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*

- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$
  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

`techniques/11trigsubpractice.tex`

2.8 Exercises: Trigonometric Substitions

Various exercises relating to trigonometric substitutions.

Sample Quiz Questions

Question 48 *Compute the integral*

$$\int_{-2}^{2} \frac{5}{(5-x^2)^{3/2}}\ dx.$$

Multiple Choice:

(a) 2

(b) 3

(c) 4 ✓

(d) 5

(e) 6

(f) 7

Feedback(`attempt`): *Begin by making the trig substitution $x = \sqrt{5}\sin\theta$. It follows that*

$$\begin{aligned}\int \frac{5}{(5-x^2)^{3/2}}\ dx &= \int \frac{5}{(5-(\sqrt{5}\sin\theta)^2)^{3/2}} \cdot (\sqrt{5}\cos\theta)\ d\theta \\ &= \int (\cos\theta)^{-2}\ d\theta \\ &= \int (\sec\theta)^2\ d\theta = (\tan\theta) + C.\end{aligned}$$

To finish, use the inversion identity

$$\tan\theta = \frac{x}{\sqrt{5-x^2}}.$$

Therefore

$$\int_{-2}^{2} \frac{5}{(5-x^2)^{3/2}}\ dx = \left.\frac{x}{\sqrt{5-x^2}}\right|_{-2}^{2} = (2) - (-2) = 4.$$

Question 49 *Compute the integral*

$$\int_{-1}^{1} \frac{3}{(3+x^2)^{3/2}}\ dx.$$

Multiple Choice:

(a) $\frac{1}{2}$

(b) 1 ✓

(c) $\frac{3}{2}$

(d) 2

(e) $\frac{5}{2}$

(f) 3

Feedback(`attempt`): *Begin by making the trig substitution $x = \sqrt{3}\tan\theta$. It follows that*

$$\begin{aligned}\int \frac{3}{(3+x^2)^{3/2}}\ dx &= \int \frac{3}{(3+(\sqrt{3}\tan\theta)^2)^{3/2}} \cdot (\sqrt{3}\sec^2\theta)\ d\theta \\ &= \int (\sec\theta)^{-1}\ d\theta \\ &= \int (\cos\theta)\ d\theta = (\sin\theta) + C.\end{aligned}$$

To finish, use the inversion identity

$$\cos\theta = \frac{x}{\sqrt{3+x^2}}.$$

Therefore

$$\int_{-1}^{1} \frac{3}{(3+x^2)^{3/2}}\ dx = \left.\frac{x}{\sqrt{3+x^2}}\right|_{-1}^{1} = \left(\frac{1}{2}\right) - \left(-\frac{1}{2}\right) = 1.$$

Question 50 *Compute the integral*

$$\int_{4}^{5} \frac{16\sqrt{x^2-16}}{x^4}\ dx.$$

Multiple Choice:

(a) $\frac{4}{125}$

(b) $\dfrac{1}{25}$

(c) $\dfrac{6}{125}$

(d) $\dfrac{7}{125}$

(e) $\dfrac{8}{125}$

(f) $\dfrac{9}{125}$ ✓

Feedback(`attempt`): *Begin by making the trig substitution $x = 4\sec\theta$. It follows that*

$$\begin{aligned}\int \frac{16\sqrt{x^2-16}}{x^4}\,dx &= \int \frac{16\sqrt{(4\sec\theta)^2-16}}{(4\sec\theta)^4}\cdot(4\sec\theta\tan\theta)\;d\theta \\ &= \int (\sec\theta)^{-3}(\tan\theta)^2\;d\theta \\ &= \int (\sin\theta)^2(\cos\theta)\;d\theta = \frac{1}{3}(\sin\theta)^3 + C.\end{aligned}$$

To finish, use the inversion identity

$$\sin\theta = \frac{\sqrt{x^2-16}}{x}.$$

Therefore

$$\int_4^5 \frac{16\sqrt{x^2-16}}{x^4}\,dx = \frac{1}{3}\frac{(x^2-16)^{3/2}}{x^3}\bigg|_4^5 = \left(\frac{9}{125}\right) - (0) = \frac{9}{125}.$$

Samle Exam Questions

Question 51 *Compute the value of the integral below.*

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{(1-x^2)^{\frac{3}{2}}}\,dx$$

Multiple Choice:

(a) 0

(b) 1 ✓

(c) 2

(d) 3

(e) 4

(f) *none of these*

Question 52 *Evaluate* $\displaystyle\int_0^3 \frac{dx}{(25-x^2)^{3/2}}$.

Multiple Choice:

(a) 0

(b) $\frac{1}{100}$

(c) $\frac{3}{100}$ ✓

(d) $\frac{5}{100}$

(e) $\frac{7}{100}$

(f) *none of these*

2.9 Partial Fractions

We study the technique of partial fractions and its application to integration.

Example 25. *The region defined by the inequalities* $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ *and* $x \geq 0$ *(shown below) is revolved around the* y*-ais. Compute the volume using the shell method.*

- *When the slicing variable is* x*, the radius of a shell is the (horizontal ✓/ vertical) distance from an* x*-slice to the axis* $x = 0$*. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an* x*-slice is equal to*

  Multiple Choice:

(a) $h(x) = \sqrt{1-x^2}$

(b) $h(x) = -\sqrt{1-x^2}$

(c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\,dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$.)

Example 26. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*

- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

Multiple Choice:

(a) $h(y) = \sqrt{y}$

(b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$

(c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\,dy = \boxed{\frac{\pi}{3}}.$$

`techniques/12partialfractionspractice.tex`

2.10 Exercises: Partial Fractions

Various exercises relating to partial fractions and integration.

Sample Quiz Questions

Question 53 *Compute the integral*

$$\int_3^4 \frac{2x-3}{x^2-3x+2}\,dx.$$

Multiple Choice:

(a) $\ln 2$

(b) $\ln 3$ ✓

(c) $\ln 4$

(d) $\ln 5$

(e) $\ln 6$

(f) $\ln 7$

Feedback(`attempt`): *First factor the denominator of the integrand: $x^2-3x+2 = (x-1)(x-2)$. Since the roots are distinct, it is possible to use the Heaviside cover-up method. The partial fractions expansion will take the form*

$$\frac{A}{x-1}+\frac{B}{x-2},$$

where the coefficient A can be computed by cancelling the factor of $x-1$ in the denominator and evaluating the result at $x=1$, i.e.,

$$A = \frac{2(1)-3}{(1)-2} = 1.$$

Similarly,

$$B = \frac{2(2)-3}{(2)-1} = 1,$$

which gives that

$$\frac{2x-3}{(x-1)(x-2)} = \frac{1}{x-1}+\frac{1}{x-2}.$$

Therefore

$$\int_3^4 \frac{2x-3}{x^2-3x+2}\,dx = \int_3^4 \left(\frac{1}{x-1}+\frac{1}{x-2}\right)\,dx$$
$$= (\ln|4-1| + \ln|4-2|) - (\ln|3-1| + \ln|3-2|)$$
$$= \ln 3 + \ln 2 + \ln\frac{1}{2} + 0 = \ln 3.$$

Sample Exam Questions

Question 54 *Compute the volume of the solid of revolution obtained by revolving around the y-axis the region below the graph*

$$y = \frac{1}{(x-1)^2},$$

above $y=0$, and between $x=2$ and $x=3$.

Multiple Choice:

(a) $\pi(3\ln 3 + 1)$

(b) $\pi(3\ln 3 + 1)$

(c) $\pi(3\ln 3 + 1)$ ✓

(d) $\pi(3\ln 3 + 1)$

(e) $\pi(3\ln 3 + 1)$

(f) $\pi(3\ln 3 + 1)$

Feedback(*`attempt`*): *Choosing x as the variable of integration, slices will be parallel to the y-axis, indicating that the shell method should be used. The radius of a shell is x (because the axis lies to the left of the region) and the height will be $(x-1)^{-2}$, so*

$$V = \int_2^3 \frac{2\pi x}{(x-1)^2}dx = 2\pi\int_2^3 \frac{x}{(x-1)^2}dx.$$

The integral can be computed by partial fractions; the expansion has the form

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}.$$

The coefficients A and B can be found by usual methods (but note that the Heaviside cover up method will not work in this case), but it is also possible to find them directly by carefully rewriting the numerator of the fraction in terms of $x-1$:

$$\frac{x}{(x-1)^2} = \frac{(x-1)+1}{(x-1)^2} = \frac{(x-1)}{(x-1)^2} + \frac{1}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2}.$$

Therefore

$$V = 2\pi \int_2^3 \left[\frac{1}{x-1} + \frac{1}{(x-1)^2}\right] dx = 2\pi \left[\ln|x-1| - \frac{1}{x-1}\right]\Big|_2^3$$
$$= 2\pi \left(\ln 2 - \frac{1}{2}\right) - 2\pi\,(0-1) = \pi(2\ln 2 + 1).$$

Question 55 *Compute the constants A and B in the partial fractions expansion indicated below. To receive full credit, it is not necessary to compute C, D, or E.*

$$\frac{x^4+16}{x^4-16} = A + \frac{B}{x-2} + \frac{C}{x+2} + \frac{Dx+E}{x^2+4}$$

Multiple Choice:

(a) $A = -1, B = 1$

(b) $A = 0, B = 1$

(c) $A = 1, B = 1$ ✓

(d) $A = -1, B = -1$

(e) $A = 0, B = -1$

(f) $A = 1, B = -1$

Feedback(`attempt`):

$$\frac{x^4+16}{x^4-16} = 1 + \frac{1}{x-2} - \frac{1}{x+2} - \frac{4}{x^2+4}$$

Question 56 *Evaluate* $\displaystyle\int_1^2 \frac{x^2+x+1}{x^2+x}dx.$

Multiple Choice:

(a) 0

(b) 1

(c) $1 + \ln\left(\frac{4}{3}\right)$ ✓

(d) 2

(e) $2 + \ln\left(\dfrac{8}{3}\right)$

(f) *none of these*

techniques/13finalpractice.tex

2.11 Exercises: Cumulative

Exercises relating to various topics we have studied.

Sample Quiz Questions

Question 57 *The region in the plane between the x-axis and the graph*

$$y = \frac{1}{2}\ln x + 1$$

in the range $\frac{1}{5} \le x \le 1$ is revolved around the axis $x = \frac{1}{10}$. Compute the volume of the resulting solid.

Multiple Choice:

(a) $\dfrac{11}{25}\pi$

(b) $\dfrac{14}{25}\pi$

(c) $\dfrac{16}{25}\pi$ ✓

(d) $\dfrac{19}{25}\pi$

(e) $\dfrac{21}{25}\pi$

(f) $\dfrac{24}{25}\pi$

Feedback(*`attempt`*): *If the variable x is used for slicing, then slices are parallel to the axis of rotation, which indicates the shell method should be used. The radius of a* *shell is* $x - \frac{1}{10}$*. The height of a shell is exactly* $\frac{1}{2}\ln x + 1$*. The volume of the region is therefore given by*

$$\int_{\frac{1}{5}}^{1} \frac{\pi}{10}\,(10x-1)\,(\ln x + 2)\;dx.$$

To compute the integralwe can use integration by parts. A reasonable strategy is to integrate $x - \frac{1}{10}$ *and differentiate* $\ln x + 2$*. This gives the equality*

$$\begin{aligned}\pi\int\left(x-\frac{1}{10}\right)(\ln x+2)\;dx &= \pi\left(\left(\frac{x^2}{2}-\frac{x}{10}\right)(\ln x+2) - \int\frac{1}{x}\left(\frac{x^2}{2}-\frac{x}{10}\right)\,dx\right)\\ &= -\pi\left(\frac{x^2}{4}-\frac{x}{10}\right)+\pi\left(\frac{x^2}{2}-\frac{x}{10}\right)(\ln x+2)\,.\end{aligned}$$

Therefore

$$\begin{aligned}\pi\int_{\frac{1}{5}}^{1}\left(x-\frac{1}{10}\right)(\ln x+2)\;dx &= \left[-\pi\left(\frac{x^2}{4}-\frac{x}{10}\right)+\pi\left(\frac{x^2}{2}-\frac{x}{10}\right)(\ln x+2)\right]\Bigg|_{\frac{1}{5}}^{1}\\ &= \left(\frac{13}{20}\pi\right)-\left(\frac{\pi}{100}\right)=\frac{16}{25}\pi.\end{aligned}$$

Question 58 *Consider the region given by* $2\pi \le x \le \frac{5}{2}\pi$ *and* $0 \le y \le \sin x$*. Compute the* x*-coordinate of the centroid (i.e., assuming constant density).*

Multiple Choice:

(a) $-1+\frac{5}{2}\pi$

(b) $1+2\pi$ ✓

(c) $\frac{5}{2}\pi$

(d) $-1+3\pi$

(e) 3π

(f) 4π

Feedback(*`attempt`*): *The mass* M *will be given by the integral*

$$\int_{2\pi}^{\frac{5}{2}\pi}\sin x\,dx$$

One can check that

$$\int_{2\pi}^{\frac{5}{2}\pi}\sin x\,dx = 1.$$

To compute the x-coordinate of the centroid, we also need to compute the integral

$$\int_{2\pi}^{\frac{5}{2}\pi} x \sin x \, dx$$

To compute the integralwe can use integration by parts. A reasonable strategy is to integrate $\sin x$ and differentiate x. This gives the equality

$$\begin{aligned}\int x \sin x \, dx &= -x \cos x - \int (-\cos x) \, dx \\ &= -x \cos x + \sin x.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{2\pi}^{\frac{5}{2}\pi} x \sin x \, dx &= \left[-x \cos x + \sin x\right]\Big|_{2\pi}^{\frac{5}{2}\pi} \\ &= 1 - (-2\pi) = 1 + 2\pi.\end{aligned}$$

The corret answer is the ratio of the integrals, i.e.,

$$\overline{x} = \frac{1 + 2\pi}{1} = 1 + 2\pi.$$

Sample Exam Questions

Question 59 *An object moves in such a way that its acceleration at time t seconds is $(t^2 + 5t + 6)^{-1}$ meters per second squared. If the initial velocity of the object is $2/3$ meters per second, what is the limit of its velocity as $t \to \infty$?*

Multiple Choice:

(a) $\ln \frac{3}{2}$ *meters per second*

(b) $\ln 6$ *meters per second*

(c) 1 *meters per second*

(d) $\ln \frac{4}{9}$ *meters per second*

(e) $\ln \frac{9}{4}$ *meters per second*

(f) 0 *meters per second* ✓

Question 60 *Find the volume of the solid generated by revolving the region bounded above by $y = \sin x$ and bounded below by $y = 0$ for $0 \leq x \leq \pi$ about the line $x = \pi$.*

Multiple Choice:

(a) π^2

(b) $2\pi^2$ ✓

(c) $4\pi^2$

(d) $\dfrac{\pi^2}{2}$

(e) $\dfrac{\pi^2}{4}$

(f) *none of these*

Question 61 *Evaluate* $\displaystyle\int_1^2 x \ln(x^2+1)dx$.

Multiple Choice:

(a) 0

(b) 1

(c) $\ln 2$

(d) $\dfrac{1}{2}$

(e) $\ln(2) - \dfrac{1}{2}$

(f) *none of these* ✓

Feedback(*`attempt`*): *This integral can be computed via integration by parts. If we integrate x and differentiate $\ln(x^2+1)$, we get*

$$\begin{aligned}\int_1^2 x\ln(x^2+1)dx &= \frac{x^2}{2}\ln(x^2+1)\Big|_1^2 - \int_1^2 \frac{x^2}{2}\frac{2x}{x^2+1}dx \\ &= 2\ln 5 - \frac{1}{2}\ln 2 - \int_1^2 \frac{x^3}{x^2+1}dx.\end{aligned}$$

The latter integral can be simplified using polynomial long division: $\frac{x^3}{x^2+1} = x - \frac{x}{x^2+1}$. *Therefore*

$$\begin{aligned}\int_1^2 x\ln(x^2+1)dx &= 2\ln 5 - \frac{1}{2}\ln 2 - \int_1^2 xdx + \int_1^2 \frac{x}{x^2+1}dx \\ &= 2\ln 5 - \frac{1}{2}\ln 2 - \frac{x^2}{2}\Big|_1^2 + \frac{1}{2}\ln(x^2+1)\Big|_1^2 \\ &= 2\ln 5 - \frac{1}{2}\ln 2 - 2 + \frac{1}{2} + \frac{1}{2}\ln 5 - \frac{1}{2}\ln 2 \\ &= \frac{5}{2}\ln 5 - \frac{2}{2}\ln 4 - \frac{3}{2} - \frac{3}{2} = \ln\left(\frac{5^5}{4}\right) - \frac{3}{2}.\end{aligned}$$

3 Further Topics in Integration

We study additional topics relating to applications of integration.

Example 27. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 28. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference* *of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is* $\boxed{1-\sqrt{y}}$, *giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

3.1 Numerical Integration

We study the problem of numerically approximating the value of an integral.

Example 29. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 30. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

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3.2 Exercises: Numerical Integration

Various exercises relating to numerical integration.

Example 31. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 32. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

3.3 Orders of Growth

We study the use of orders of growth to compute limits, in preparation for improper integrals.

Example 33. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 34. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

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3.4 Exercises: Orders of Growth

Various exercises relating to orders of growth.

Sample Quiz Questions

Question 62 *Arrange the functions*

$$x^x \qquad \frac{e^x}{\ln x} \qquad \ln x$$

in order from least rate of growth to greatest rate of growth as $x \to \infty$. Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

Multiple Choice:

(a) $x^x \ll \dfrac{e^x}{\ln x} \ll \ln x$

(b) $\ln x \ll x^x \ll \dfrac{e^x}{\ln x}$

(c) $\dfrac{e^x}{\ln x} \ll \ln x \ll x^x$

(d) $x^x \ll \ln x \ll \dfrac{e^x}{\ln x}$

(e) $\dfrac{e^x}{\ln x} \ll x^x \ll \ln x$

(f) $\ln x \ll \dfrac{e^x}{\ln x} \ll x^x$ ✓

Feedback(*`attempt`*): *General Remarks:*

- *Higher powers of x grow faster at infinity than lower powers of x.*
- *As $x \to \infty$, $\ln x$ goes to infinity slower than x^p for any (presumably small) positive constant p.*
- *As $x \to \infty$, e^x goes to infinity faster than x^n for any (presumably large) positive constant n.*
- *As $x \to \infty$, x^x goes to infinity faster than any exponential of the form e^{cx} for any constant c.*

Question 63 *Arrange the functions*

$$\frac{e^{-x}}{\ln x} \qquad x^3 \ln x \qquad e^x$$

in order from least rate of growth to greatest rate of growth as $x \to \infty$. Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

Multiple Choice:

(a) $\dfrac{e^{-x}}{\ln x} \ll x^3 \ln x \ll e^x$ ✓

(b) $e^x \ll \dfrac{e^{-x}}{\ln x} \ll x^3 \ln x$

(c) $x^3 \ln x \ll e^x \ll \dfrac{e^{-x}}{\ln x}$

(d) $\dfrac{e^{-x}}{\ln x} \ll e^x \ll x^3 \ln x$

(e) $x^3 \ln x \ll \dfrac{e^{-x}}{\ln x} \ll e^x$

(f) $e^x \ll x^3 \ln x \ll \dfrac{e^{-x}}{\ln x}$

Feedback(`attempt`): *General Remarks:*

- *Higher powers of x grow faster at infinity than lower powers of x.*
- *As $x \to \infty$, $\ln x$ goes to infinity slower than x^p for any (presumably small) positive constant p.*
- *As $x \to \infty$, e^x goes to infinity faster than x^n for any (presumably large) positive constant n.*

Question 64 *Arrange the functions*

$$\left(\ln \frac{1}{x}\right)^2 \qquad \frac{e^{-x}}{x^3} \ln x \qquad x^3 e^x$$

in order from least rate of growth to greatest rate of growth as $x \to 0^+$. Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

Multiple Choice:

(a) $\left(\ln \frac{1}{x}\right)^2 \ll \frac{e^{-x}}{x^3} \ln x \ll x^3 e^x$

(b) $x^3 e^x \ll \left(\ln \frac{1}{x}\right)^2 \ll \frac{e^{-x}}{x^3} \ln x$ ✓

(c) $\frac{e^{-x}}{x^3} \ln x \ll x^3 e^x \ll \left(\ln \frac{1}{x}\right)^2$

(d) $\left(\ln \frac{1}{x}\right)^2 \ll x^3 e^x \ll \frac{e^{-x}}{x^3} \ln x$

(e) $\frac{e^{-x}}{x^3} \ln x \ll \left(\ln \frac{1}{x}\right)^2 \ll x^3 e^x$

(f) $x^3 e^x \ll \frac{e^{-x}}{x^3} \ln x \ll \left(\ln \frac{1}{x}\right)^2$

Feedback(`attempt`): *General Remarks:*

- *Lower powers of x grow faster as $x \to 0^+$ than higher powers of x.*
- *As $x \to 0^+$, $-\ln x = \ln x^{-1}$ goes to ∞ slower than x^{-p} for any (presumably small) positive p.*
- *As $x \to 0^+$, $e^x \to 1$ and so does not influence the growth rate.*
- *As $x \to 0^+$, $e^{-x} \to 1$ and so does not influence the growth rate.*

Question 65 *Arrange the functions*

$$\frac{x^3 e^x}{\ln x} \qquad \frac{e^{-x}}{x^3} \qquad \frac{e^{-x}}{x^3 \ln x}$$

in order from least rate of growth to greatest rate of growth as $x \to 0^+$. Compare on the basis of magnitude rather than sign, i.e., if a function is negative, take its absolute value first.

Multiple Choice:

(a) $\frac{x^3 e^x}{\ln x} \ll \frac{e^{-x}}{x^3} \ll \frac{e^{-x}}{x^3 \ln x}$

(b) $\frac{e^{-x}}{x^3 \ln x} \ll \frac{x^3 e^x}{\ln x} \ll \frac{e^{-x}}{x^3}$

(c) $\dfrac{e^{-x}}{x^3} \ll \dfrac{e^{-x}}{x^3 \ln x} \ll \dfrac{x^3 e^x}{\ln x}$

(d) $\dfrac{x^3 e^x}{\ln x} \ll \dfrac{e^{-x}}{x^3 \ln x} \ll \dfrac{e^{-x}}{x^3}$ ✓

(e) $\dfrac{e^{-x}}{x^3} \ll \dfrac{x^3 e^x}{\ln x} \ll \dfrac{e^{-x}}{x^3 \ln x}$

(f) $\dfrac{e^{-x}}{x^3 \ln x} \ll \dfrac{e^{-x}}{x^3} \ll \dfrac{x^3 e^x}{\ln x}$

Feedback(*`attempt`*): *General Remarks:*

- *Lower powers of x grow faster as $x \to 0^+$ than higher powers of x.*
- *As $x \to 0^+$, $-\ln x = \ln x^{-1}$ goes to ∞ slower than x^{-p} for any (presumably small) positive p.*
- *As $x \to 0^+$, $e^x \to 1$ and so does not influence the growth rate.*
- *As $x \to 0^+$, $e^{-x} \to 1$ and so does not influence the growth rate.*

3.5 Improper Integrals

We study the concept of improper integrals.

Example 35. *The region defined by the inequalities $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ and $x \geq 0$ (shown below) is revolved around the y-ais. Compute the volume using the shell method.*

- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

  Multiple Choice:

  (a) $h(x) = \sqrt{1-x^2}$

  (b) $h(x) = -\sqrt{1-x^2}$

  (c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\,dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$*.)*

Example 36. *The region between the curves* $x = \sqrt{y}$ *and* $x = y + \sqrt{y}$ *from* $y = 0$ *to* $y = 1$ *is revolved around the axis* $y = 1$*. Compute the volume of the resulting solid.*

- *When the slicing variable is* y*, the radius of a shell is the (horizontal/ vertical ✓) distance from a* y*-slice to the axis* $y = 1$*. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a* y*-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$

  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$

  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\,dy = \boxed{\frac{\pi}{3}}.$$

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3.6 Exercises: Improper Integrals

Various exercises relating to improper integrals.

Sample Quiz Questions

Question 66 *Which of the following improper integrals is convergent? Show how you used comparison tests to justify your answer.*

$$\text{I}: \int_0^1 \frac{\sqrt{1+x^2e^{-x}}}{(\cos x)x^2}\,dx \qquad \text{II}: \int_2^\infty \frac{e^x}{xe^x+x^2}\,dx \qquad \text{III}: \int_2^\infty \frac{x^2}{x^4+1}\,dx$$

Multiple Choice:

(a) *only* I *converges*

(b) *only* II *converges*

(c) *only* III *converges* ✓

(d) I *and* II *converge*

(e) II *and* III *converge*

(f) I *and* III *converge*

Feedback(*`attempt`*): *Integral* I *is divergent by direct comparison to the function* $\frac{1}{x^2}$*. Integral* II *is divergent by limit comparison to the function* $\frac{1}{x}$*. Integral* III *is convergent by direct comparison to the function* $\frac{1}{x^2}$*.*

Question 67 *Which of the following improper integrals is convergent? Show how you used comparison tests to justify your answer.*

$$\text{I}: \int_0^1 \frac{\sqrt{e^{2x}+x^3}}{x}\,dx \qquad \text{II}: \int_0^1 \frac{x^2}{x^2\sqrt{x}+x^3}\,dx \qquad \text{III}: \int_2^\infty \frac{x^2\ln x}{-x+x^4}\,dx$$

Multiple Choice:

(a) *only* I *converges*

(b) *only* II *converges*

(c) *only* III *converges*

(d) I *and* II *converge*

(e) II *and* III *converge* ✓

(f) I *and* III *converge*

Feedback(*`attempt`*): *Integral* I *is divergent by direct comparison to the function* $\frac{1}{x}$*. Integral* II *is convergent by direct comparison to the function* $\frac{1}{\sqrt{x}}$*. Integral* III *is convergent by limit comparison to the function* $\frac{\ln x}{x^2}$*.*

Sample Exam Questions

Question 68 *Only one of the following four improper integrals diverges. Choose that improper integral and justify why it diverges. (You need NOT justify why the other integrals converge.)*

Multiple Choice:

(a) $\displaystyle\int_2^\infty \frac{\arctan x}{1+x^3}\,dx$

(b) $\displaystyle\int_2^\infty \frac{1}{\sqrt{x^4+x^2}}\,dx$

(c) $\displaystyle\int_2^\infty \frac{1+\sin x}{x^2}\,dx$

(d) $\displaystyle\int_2^\infty \frac{1}{\sqrt[3]{x^2-1}}\,dx$ ✓

3.7 Probability

We study probability and its connections to integration.

Example 37. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*

- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓

  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}} \right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 38. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*

- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

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3.8 Exercises: Probability

Various exercises relating to probability.

Sample Quiz Questions

Question 69 *Find the value of c which makes the function*

$$f(x) = \frac{1}{2}e^{-x} - ce^{-2x}$$

a probability density function on the interval $[0, \infty)$. What is the value of the mean μ of the corresponding random variable?

Multiple Choice:

(a) $c = 1,\ \mu = \frac{1}{2}$

(b) $c = -1,\ \mu = \frac{3}{4}$ ✓

(c) $c = 1,\ \mu = 1$

(d) $c = -1,\ \mu = \frac{5}{4}$

(e) $c = 1,\ \mu = \frac{3}{2}$

(f) $c = -1,\ \mu = \frac{7}{4}$

Feedback(`attempt`): *To compute the constant c, we use the fact that the integral of a probability density function must equal 1, so*

$$\mu = \int_0^\infty \left(\frac{1}{2}e^{-x} - ce^{-2x}\right)\ dx = 1.$$

This gives the equation

$$\frac{1}{2} - \frac{1}{2}c = 1,$$

which then implies that $c = -1$. To compute the mean μ, we use the formula

$$\mu = \int_0^\infty x\left(\frac{1}{2}e^{-x} + e^{-2x}\right)\ dx.$$

Calculating the integral gives $\mu = 3/4$.

Question 70 *A certain random variable X takes values in the interval $\left[2\pi, \frac{5}{2}\pi\right]$. If the probability density function is given by*

$$A \sin x$$

for some appropriate value of the constant A, compute the expected value μ of X.

Multiple Choice:

(a) $\mu = -1 + 2\pi$

(b) $\mu = 1 + \frac{3}{2}\pi$

(c) $\mu = 2\pi$

(d) $\mu = -1 + \frac{5}{2}\pi$

(e) $\mu = 1 + 2\pi$ ✓

(f) $\mu = \frac{5}{2}\pi$

Feedback(*`attempt`*): *The constant A will be the reciprocal of the integral*

$$\int_{2\pi}^{\frac{5}{2}\pi} \sin x \, dx$$

One can check that

$$\int_{2\pi}^{\frac{5}{2}\pi} \sin x \, dx = 1.$$

To compute the expected value μ we also need to compute the integral

$$\int_{2\pi}^{\frac{5}{2}\pi} x \sin x \, dx$$

To compute the integralwe can use integration by parts. A reasonable strategy is to integrate $\sin x$ and differentiate x. This gives the equality

$$\begin{aligned} \int x \sin x \, dx &= -x \cos x - \int (-\cos x) \, dx \\ &= -x \cos x + \sin x. \end{aligned}$$

Therefore

$$\begin{aligned} \int_{2\pi}^{\frac{5}{2}\pi} x \sin x \, dx &= \left[-x \cos x + \sin x\right]\Big|_{2\pi}^{\frac{5}{2}\pi} \\ &= 1 - (-2\pi) = 1 + 2\pi. \end{aligned}$$

Therefore the expected value is the ratio of the integrals, i.e.,

$$\mu = \frac{1 + 2\pi}{1} = 1 + 2\pi.$$

Sample Exam Questions

Question 71 *A certain random variable X has values in $(1, \infty)$ and has the property that there is some constant C such that*

$$P(X > a) = C \ln \frac{a^3 + 1}{a^3}$$

for every $a > 1$. Compute the value of C and determine whether the expected value μ of X is finite or infinite. [Hint: There is enough information given to compute C without calculating any integrals.]

Multiple Choice:

(a) $C = \ln 2$ *and* $\mu < \infty$

(b) $C = 1$ *and* $\mu < \infty$

(c) $C = (\ln 2)^{-1}$ *and* $\mu < \infty$ ✓

(d) $C = \ln 2$ *and* $\mu = \infty$

(e) $C = 1$ *and* $\mu = \infty$

(f) $C = (\ln 2)^{-1}$ *and* $\mu = \infty$

Feedback(*`attempt`*): *We know that X is always greater than one, so*

$$1 = P(X > 1) = C \ln \frac{1+1}{1},$$

which gives $C = (\ln 2)^{-1}$. If we let $f(x)$ denote the probability density function of X, then

$$\frac{1}{\ln 2} \ln \frac{a^3+1}{a^3} = P(X > a) = \int_a^\infty f(x)dx.$$

Differentiating both sides with respect to a gives

$$\frac{1}{\ln 2} \left[\frac{3a^2}{a^3+1} - \frac{3}{a} \right] = -f(a)$$

so

$$f(a) = \frac{1}{\ln 2} \left[\frac{3}{a} - \frac{3a^2}{a^3+1} \right] = \frac{3}{a(a^3+1)\ln 2}.$$

The expected value of X must equal

$$\int_1^\infty \frac{3a}{a(a^3+1)\ln 2} da = \frac{3}{\ln 2} \int_1^\infty \frac{da}{a^3+1}.$$

This integral will be finite by direct comparison to the convergent integral $\displaystyle\int_1^\infty a^{-3} da$.

Question 72 *The function*

$$f(x) = \begin{cases} \dfrac{k}{x^3} & 1 \le x < \infty \\ 0 & \textit{otherwise} \end{cases}$$

is a probability density function for a certain value of k. For that probability density function, find the probability that $x > 2$.

Multiple Choice:

(a) $\frac{1}{2}$

(b) $\frac{1}{3}$

(c) $\frac{1}{4}$ ✓

(d) $\frac{2}{3}$

(e) $\frac{1}{5}$

(f) $\frac{1}{6}$

Question 73 *For a certain real number k, the function*

$$f(X) = \begin{cases} \dfrac{k}{X^2+1} & \textit{if } X \ge 0 \\ 0 & \textit{otherwise} \end{cases}$$

is a probability density function for a continuous random variable X. For this value of k, find the probability that $X > 1$.

Multiple Choice:

(a) 0

(b) $\frac{1}{3}$

(c) $\frac{2}{3}$

(d) 1

(e) $\frac{1}{2}$ ✓

(f) $\frac{1}{4}$

Question 74 *Let*

$$f(r) = \begin{cases} Cr^2 e^{-2r/b} & r \geq 0 \\ 0 & r < 0 \end{cases}.$$

Find C so that this is a probability density function (pdf) for the random variable r. Here b is a positive constant. This function is used to model the distance between the nucleus and the electron in a hydrogen atom. The constant b is called the Bohr length. Find the mean of the pdf.

Multiple Choice:

(a) $C = \frac{b^3}{4}$, mean $= b$

(b) $C = \frac{4}{b^2}$, mean $= b$

(c) $C = \frac{4}{b}$, mean $= b^2$

(d) $C = \frac{4}{b^3}$, mean $= \frac{3}{2}b$ ✓

(e) $C = \frac{4}{b^2}$, mean $= \frac{3}{2}b^2$

(f) $C = \frac{4}{b}$, mean $= \frac{3}{2}b^3$

4 Sequences and Series

We begin a study of sequences and series.

Example 39. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from* $y = 0$ *to* $y = \sqrt{1-x^2}$*. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ *between* $x = 0$ *and* $x = 1$*, since these are the most extreme values of x found in our region. Therefore*

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 40. *The base of a solid region is bounded by the curves* $y = 0$, $x = \sqrt{y}$, *and* $x = 1$*. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph* $x = \boxed{\sqrt{y}}$ *to the graph* $x = \boxed{1}$*. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is* $\boxed{1-\sqrt{y}}$*, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ *between* $y = 0$ *and* $y = 1$*, since these are the most extreme values of y found in our region (note that we can find the upper value* $y = 1$ *by solving for the intersection of the curves* $x = \sqrt{y}$ *and* $x = 1$*). Therefore we integrate* $A(y)dy$ *to conclude*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

4.1 Sequences

We study the mathematical concept of a sequence.

Example 41. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 42. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

series/18sequencepractice.tex

4.2 Exercises: Sequences

Exercises relating to sequences.

Sample Quiz Questions

Question 75 *Find the limit of the sequence*

$$\lim_{n\to\infty} \sqrt{\frac{2n-2}{2n^2-4n+3}}.$$

Justify your response.

Multiple Choice:

(a) 0 ✓

(b) $\frac{1}{3}$

(c) $\frac{1}{2}$

(d) 1

(e) 2

(f) 3

Feedback(*`attempt`*):

$$\begin{aligned}\lim_{n\to\infty} \sqrt{\frac{2n-2}{2n^2-4n+3}} &= \lim_{n\to\infty} \sqrt{\frac{2n^{-1}-2n^{-2}}{2-4n^{-1}+3n^{-2}}} \\ &= \sqrt{\frac{\lim\limits_{n\to\infty} 2n^{-1}-2n^{-2}}{\lim\limits_{n\to\infty} 2-4n^{-1}+3n^{-2}}} \\ &= \sqrt{\frac{0}{2}} = 0.\end{aligned}$$

Question 76 *Determine whether the limit below exists. If it exists, find its value.*

$$\lim_{n\to\infty} \frac{(-1)^{n+1}n^2-2^{-n-1}}{(-1)^n n^2+4^{-n}}.$$

Justify your response.

Multiple Choice:

(a) -1 ✓

(b) 0

(c) $\frac{1}{2}$

(d) 2

(e) 3

(f) *limit does not exist*

Feedback(*attempt*): *Comparing the orders of growth of the terms in the numerator, the first term dominates because* $|-1| > |1/2|$*. Likewise the first term dominates in the denominator because* $|-1| > |1/4|$*. Neglecting non-dominant terms leads to the limit*

$$\lim_{n\to\infty} \frac{(-1)^{n+1}n^2}{(-1)^n n^2}$$

which simply equals -1*.*

Question 77 *Determine whether the limit below exists. If it exists, find its value.*

$$\lim_{n\to\infty} \left(\frac{4n-3}{4n+1}\right)^n.$$

Justify your response.

Multiple Choice:

(a) 0

(b) 1

(c) e^{-1} ✓

(d) e

(e) e^2

(f) *limit does not exist*

Feedback(*attempt*): *First observe that*

$$\frac{4n-3}{4n+1} = 1 - \frac{4}{4n+1} \to 1$$

as $n \to \infty$. Next, in light of the known limit $(1 + x/k)^k \to e^x$ as $k \to \infty$, manipulate exponents to see that

$$\left(1 - \frac{4}{4n+1}\right)^n = \left(\left(1 - \frac{4}{4n+1}\right)^{4n+1}\right)^{1/4} \left(1 - \frac{4}{4n+1}\right)^{-1/4}.$$

As $n \to \infty$, the first term on the right-hand side tends to e^{-1} and the second term tends to 1. Thus the original sequence tends to e^{-1} as well.

Question 78 *Determine whether the limit below exists. If it exists, find its value.*

$$\lim_{n\to\infty} \left(\frac{n+3}{2n-2}\right)^{n^2}.$$

Justify your response.

Multiple Choice:

(a) 0 ✓

(b) 1

(c) e^{-1}

(d) e

(e) e^2

(f) *limit does not exist*

Feedback(*`attempt`*): *First observe that*

$$\frac{n+3}{2n-2} = \frac{1}{2} + \frac{2}{n-1} \to \frac{1}{2}$$

as $n \to \infty$. Since the limit is positive and less than one, raising this expression to increasingly large powers generates a sequence which converges rapidly to zero.

Sample Exam Questions

Question 79 *Determine whether the sequence $a_n = (-1)^{n-1}\dfrac{n^2}{1+n^2+n^3}$ converges or diverges. If it converges, find its limit.*

Multiple Choice:

(a) *divergent,* $\lim_{n\to\infty} a_n = 0$

(b) *convergent,* $\lim_{n\to\infty} a_n = 1$

(c) *convergent,* $\lim_{n\to\infty} a_n = 0$ ✓

(d) *convergent,* $\lim_{n\to\infty} a_n = -1$

(e) *divergent,* $\lim_{n\to\infty} a_n = \infty$

(f) *divergent, limit doesn't exist*

4.3 Series

We introduce the concept of a series and study some fundamental properties.

Example 43. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 44. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference* *of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is* $\boxed{1-\sqrt{y}}$*, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

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4.4 Exercises: Series

Exercises relating to fundamental properties of series.

Sample Quiz Questions

Question 80 *Compute the exact value of the infinite series*

$$\sum_{n=1}^{\infty} \ln\left(\frac{1+n^{-1}}{1+(n+1)^{-1}}\right).$$

Multiple Choice:

(a) $\ln 2$ ✓

(b) $\ln 3$

(c) $\ln 4$

(d) $\ln 5$

(e) $\ln 6$

(f) $\ln 7$

Feedback(*`attempt`*): *The series is not a geometric series or Taylor series, we compute the first few partial sums:*

$$\begin{aligned}
S_1 &= \ln\left(\frac{1+1}{1+2^{-1}}\right) = \ln\left(\frac{2}{\frac{3}{2}}\right)\\
S_2 &= \ln\left(\frac{1+1}{1+2^{-1}}\right) + \ln\left(\frac{1+2^{-1}}{1+3^{-1}}\right) = \ln\left(\frac{1+1}{1+3^{-1}}\right) = \ln\left(\frac{2}{\frac{4}{3}}\right)\\
S_3 &= \ln\left(\frac{2}{1+3^{-1}}\right) + \ln\left(\frac{1+3^{-1}}{1+4^{-1}}\right) = \ln\left(\frac{2}{1+4^{-1}}\right) = \ln\left(\frac{2}{\frac{5}{4}}\right)\\
&\vdots\\
S_n &= \ln\left(\frac{2}{1+(n+1)^{-1}}\right).
\end{aligned}$$

In particular, writing the sum of logarithms as a logarithm of a product leads to substantial cancellation. By letting $n \to \infty$, we get $S_n \to \ln 2$.

4.5 Series Comparison Tests

We study the direct and limit comparison theorems for infinite series and practice their application.

Example 45. *The region defined by the inequalities* $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$ *and* $x \geq 0$ *(shown below) is revolved around the* y*-ais. Compute the volume using the shell method.*

- *When the slicing variable is* x*, the radius of a shell is the (horizontal ✓/ vertical) distance from an* x*-slice to the axis* $x = 0$*. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an* x*-slice is equal to*

  Multiple Choice:

  (a) $h(x) = \sqrt{1-x^2}$
  (b) $h(x) = -\sqrt{1-x^2}$
  (c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of* $2\pi r h$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\, dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$*.)*

Example 46. *The region between the curves* $x = \sqrt{y}$ *and* $x = y + \sqrt{y}$ *from* $y = 0$ *to* $y = 1$ *is revolved around the axis* $y = 1$*. Compute the volume of the resulting solid.*

- *When the slicing variable is* y*, the radius of a shell is the (horizontal/ vertical ✓) distance from a* y*-slice to the axis* $y = 1$*. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a* y*-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$

(b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$

(c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of* $2\pi rh$*, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\, dy = \boxed{\frac{\pi}{3}}.$$

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4.6 Exercises: Series Comparison Tests

Exercises relating to the direct and limit comparison tests for series.

Exercise 81 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 82 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

4.7 The Ratio and Root Tests

We study the ratio and root tests for infinite series and practice their application.

Example 47. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*

- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓

  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}} \right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 48. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*

- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$

  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}} \right)^2 - \left(\boxed{2y} \right)^2 \right] d\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

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4.8 Exercises: Ratio and Root Tests

Exercises relating to the ratio and root tests.

Sample Quiz Questions

Question 83 *Determine which of the following three infinite series will lead to inconclusive results for the ratio test and then determine whether that series is convergent or divergent.*

$$\text{I: } \sum_{k=1}^{\infty} \frac{1}{k-e^{-k}} \quad \text{II: } \sum_{m=1}^{\infty} \frac{1}{m^2-e^{m}} \quad \text{III: } \sum_{l=1}^{\infty} \frac{e^{-l}}{l^2+1}$$

Multiple Choice:

(a) *I inconclusive, converges*

(b) *I inconclusive, diverges* ✓

(c) *II inconclusive, converges*

(d) *II inconclusive, diverges*

(e) *III inconclusive, converges*

(f) *III inconclusive, diverges*

Feedback(*`attempt`*): *The first series will give an inconclusive result for the ratio test because*

$$\lim_{k\to\infty} \frac{k-e^{-k}}{k+1-e^{-k-1}} = \lim_{k\to\infty} \frac{1-k^{-1}e^{-k}}{\frac{k+1}{k}-k^{-1}e^{-k-1}} = \frac{1-0}{1-0} = 1.$$

However, we know that the harmonic series diverges and that

$$\frac{1}{k-e^{-k}} > \frac{1}{k},$$

so by direct comparison to the harmonic series, series I must diverge.

4.9 The Integral Test

We study the integral test for infinite series and related concepts.

Example 49. *Suppose the region below the graph $y = \sqrt{\sin x}$ and above the x-axis between $x = 0$ and $x = \pi$ is revolved around the x-axis. Compute the volume of the resulting solid.*

- *Because the axis of rotation lies perfectly along the boundary of the region, the (disk ✓/ washer) method can be used.*
- *The radius R is the length of a (horizontal/ vertical ✓) extending from the axis to the graph $y = \sqrt{\sin x}$.*
- *Thus we know that the radius R must equal*

  Multiple Choice:

  (a) $R(x) = \sqrt{\sin x} - 0 = \sqrt{\sin x}$ ✓
  (b) $R(y) = \arcsin y^2 - 0 = \arcsin y^2$

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{\pi}} \pi \left(\boxed{\sqrt{\sin x}}\right)^2 d\boxed{x} = \boxed{2\pi}.$$

Example 50. *Suppose the region between the graphs $y = x/2$ and $y = x^2/4$ is revolved around the axis $x = 0$. Compute the volume of the resulting solid.*

- *Because the axis of rotation does not lie along the boundary of the region, the (disk/ washer ✓) method can be used.*
- *In this case, radius will equal the length of a (horizontal ✓/ vertical) extending from the axis to the graphs $y = x/2$ and $y = x^2/4$.*
- ***Multiple Choice:***

  (a) $R_{\text{outer}}(x) = x/2$ *and* $r_{\text{inner}}(x) = x^2/4$
  (b) $R_{\text{outer}}(y) = 2\sqrt{y}$ *and* $r_{\text{inner}}(y) = 2y$ ✓

- *We conclude that*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[\left(\boxed{2\sqrt{y}}\right)^2 - \left(\boxed{2y}\right)^2\right] d\boxed{y} = \boxed{\frac{2\pi}{3}}.$$

series/22integralpractice.tex

4.10 Exercises: The Integral Test

Exercises relating to the integral test.

Sample Quiz Questions

Question 84 *When approximating the sum of the infinite series*

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

by the sum of the first N terms, how large must N be to ensure that the approximation error is less than $1/200$? Choose the smallest correct bound among those listed.

Multiple Choice:

(a) $N > 5$

(b) $N > 10$ ✓

(c) $N > 20$

(d) $N > 400$

(e) $N > 8000$

(f) $N > 160000$

Feedback(*`attempt`*): *Because the terms n^{-3} are positive and decreasing, we know that the partial sums are always less than or equal to the sum of the series. By the Integral Test, we can further say that*

$$\sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{1}{n^3} \leq \int_{N}^{\infty} \frac{1}{x^3}\,dx = \frac{1}{2N^2}.$$

To be certain that the error is less than $1/200$, we set $(2N^2)^{-1} < 1/200$, which gives $N > 10$.

4.11 Alternating Series

We study the notion of alternating series and related concepts.

Example 51. *The region defined by the inequalities $-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$ and $x \ge 0$ (shown below) is revolved around the y-ais. Compute the volume using the shell method.*

- *When the slicing variable is x, the radius of a shell is the (horizontal ✓/ vertical) distance from an x-slice to the axis $x = 0$. Thus*

$$r(x) = \boxed{x} - \boxed{0}.$$

- *The height of an x-slice is equal to*

  Multiple Choice:

  (a) $h(x) = \sqrt{1-x^2}$
  (b) $h(x) = -\sqrt{1-x^2}$
  (c) $h(x) = \sqrt{1-x^2} - \left(-\sqrt{1-x^2}\right) = 2\sqrt{1-x^2}$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{4\pi x\sqrt{1-x^2}}\, dx = \boxed{\frac{4\pi}{3}}.$$

(Note: to compute the integral, we can make the substitution $u = 1 - x^2$.)

Example 52. *The region between the curves $x = \sqrt{y}$ and $x = y + \sqrt{y}$ from $y = 0$ to $y = 1$ is revolved around the axis $y = 1$. Compute the volume of the resulting solid.*

- *When the slicing variable is y, the radius of a shell is the (horizontal/ vertical ✓) distance from a y-slice to the axis $y = 1$. Thus*

$$r(y) = \boxed{1} - \boxed{y}.$$

- *The "height" of a y-slice is equal to*

  Multiple Choice:

  (a) $h(y) = \sqrt{y}$
  (b) $h(y) = \sqrt{y} - (y + \sqrt{y}) = -y$
  (c) $h(y) = (y + \sqrt{y}) - \sqrt{y} = y$ ✓

- *The volume is equal to the integral of $2\pi rh$, so*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{2\pi y(1-y)}\, dy = \boxed{\frac{\pi}{3}}.$$

series/23alternatepractice.tex

4.12 Exercises: Alternating Series

Exercises relating to alternating series and absolute or conditional convergence.

Sample Quiz Questions

Question 85 *For each series below, determine whether it converges absolutely (A), converges conditionally (C), or diverges (D). Show how you used convergence tests to arrive at your answer.*

$$\text{I: } \sum_{n=1}^{\infty} \frac{\cos n\pi}{\sqrt[3]{n+3}} \qquad \text{II: } \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+3} \qquad \text{III: } \sum_{n=2}^{\infty} \frac{\cos n\pi}{\ln(n^2+1)}$$

Multiple Choice:

(a) *I: C, II: D, III: D*

(b) *I: C, II: A, III: C* ✓

(c) *I: A, II: C, III: A*

(d) *I: D, II: C, III: D*

(e) *I: C, II: D, III: C*

(f) *I: C, II: A, III: A*

Feedback(`attempt`): *I: converges conditionally. The value of $\cos n\pi$ alternates ± 1. The terms $(n+3)^{-1/3}$ decrease to zero, so the series converges by the alternating series test. The series is not absolutely convergent because the p-series with $p = -1/3$ is divergent.*

II: converges absolutely. The series converges absolutely by direct comparison to a p-series with $p = 2$.

III: converges conditionally. The series converges by the alternating series test because $1/\ln(n^2+1)$ decreases to 0 as $n \to \infty$ and $\cos n\pi$ alternates in value between $+1$ and -1. However, $1/\ln(n^2+1) \geq 1/n$ for all large n, so by direct comparison to the harmonic series, the series is not absolutely convergent. Therefore the convergence is conditional.

Question 86 *For each series below, determine whether it converges absolutely (A), converges conditionally (C), or diverges (D). Show how you used convergence tests to arrive at your answer.*

$$\text{I: } \sum_{n=1}^{\infty} \frac{(-1)^n n^2}{2n^2+1} \qquad \text{II: } \sum_{n=1}^{\infty} \frac{1}{1+n^3 e^{-n}} \qquad \text{III: } \sum_{n=1}^{\infty} \frac{(-1)^n n + 2}{n^2}$$

Multiple Choice:

(a) *I: D, II: D, III: D*

(b) *I: D, II: A, III: C*

(c) *I: C, II: C, III: A*

(d) *I: A, II: C, III: D*

(e) *I: D, II: D, III: C* ✓

(f) *I: D, II: A, III: A*

Feedback(*`attempt`*): *I: diverges. The series diverges because $n^2/(n^2+1) \to 1$, meaning that the terms do not go to zero. The n-th term divergence test implies divergence.*

II: diverges. The series diverges because $n/(n+n^3e^{-n}) \to 1$ (because $n^3e^{-n} \to 0$). By the limit comparison theorem, this means the series has the same behavior as a p-series with $p = 1$, which means it diverges.

III: converges conditionally. The series converges because it is the sum of two convergent series: one with terms $(-1)^n/n$ (which is a convergent series by the alternating series test because $1/n$ decreases to zero) and a second with terms $2/n^2$ (which is a convergent p-series). However, the series is not absolutely convergent, because

$$\left| \frac{(-1)^n n + 2}{n^2} \right| = \frac{n + (-1)^n 2}{n^2}$$

for $n \geq 2$, which is a sum of a divergent p-series with $p = 1$ and an absolutely convergent alternating p-series with $p = 2$. Thus the series is conditionally convergent.

Question 87 *Which of the following intervals contains the value of the infinite series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}?$$

Multiple Choice:

(a) $\left[\frac{1}{4}, \frac{1}{3}\right]$

(b) $\left[\dfrac{1}{3}, \dfrac{1}{2}\right]$

(c) $\left[\dfrac{1}{2}, \dfrac{7}{12}\right]$

(d) $\left[\dfrac{7}{12}, \dfrac{5}{6}\right]$ ✓

(e) $\left[\dfrac{5}{6}, \dfrac{11}{12}\right]$

(f) $\left[\dfrac{11}{12}, \dfrac{7}{6}\right]$

Feedback(`attempt`): *The function $1/(n+1)$ is positive and decreases to zero, so by the Alternating Series Test, we know that partial sums alternate above and below the actual value of the sum. In particular, if we call the value of the sum L, then*

$$\begin{aligned} 1 &\geq L \\ 1 - \frac{1}{2} &\leq L \\ 1 - \frac{1}{2} + \frac{1}{3} &\geq L \\ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} &\leq L \end{aligned}$$

and so on. The last two inequalities together imply that L belongs to the interval $[\frac{7}{12}, \frac{5}{6}]$.

Sample Exam Questions

Question 88 *Determine whether the following series converge absolutely (A), converge conditionally (C), or diverge (D). For full credit be sure to explain your reasoning and specify which tests were used.*

$$\sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n}}{3^n} \qquad \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

Multiple Choice:

(a) *both A*

(b) *one A, the other C*

(c) *one A, the other D*

(d) *both C*

(e) *one C, the other D* ✓

(f) *both D*

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4.13 Exercises: Cumulative

Exercises relating to various topics we have studied.

Sample Exam Questions

Question 89 *Determine whether the following series converge or diverge.*

$$\text{I: } \sum_{n=1}^{\infty} \frac{n^3}{n^4+4} \quad \text{II: } \sum_{n=1}^{\infty} \frac{3^n}{n!} \quad \text{III: } \sum_{n=2}^{\infty} \frac{\ln\ln n}{\ln n} \quad \text{IV: } \sum_{n=1}^{\infty} \frac{3n^2}{(n!)^2}$$

Multiple Choice:

(a) *I & II converge; III & IV diverge*

(b) *I & III converge; II & IV diverge*

(c) *I & IV converge; II & III diverge*

(d) *II & III converge; I & IV diverge*

(e) *II & IV converge; I & III diverge* ✓

(f) *III & IV converge; I & II diverge*

Question 90 *Determine whether the following series are convergent or divergent. Justify your answers.*

$$\text{I: } \sum_{n=1}^{\infty} \frac{n^2-3n}{\sqrt[3]{n^{10}-4n^2}} \qquad \text{II: } \sum_{n=1}^{\infty} \frac{(-n)^n}{5^{2n+3}}$$

Multiple Choice:

(a) *I & II divergent*

(b) *I convergent, II divergent* ✓

(c) *I divergent, II convergent*

(d) *I & II convergent*

Question 91 *Determine whether the following series are convergent or divergent. Justify your answers.*

$$\text{I: } \sum_{n=1}^{\infty} \frac{\arctan n}{n^4} \qquad \text{II: } \sum_{n=1}^{\infty} \frac{\sin \frac{1}{n}}{n^2}$$

Multiple Choice:

(a) *I & II divergent*

(b) *I convergent, II divergent*

(c) *I divergent, II convergent*

(d) *I & II convergent* ✓

Question 92 *Determine which of the following series are convergent. For full credit, be sure to explain your reasoning and specify which tests were used.*

$$\text{I: } \sum_{n=2}^{\infty} 2ne^{-n^2} \qquad \text{II: } \sum_{n=2}^{\infty} \frac{n + 2\ln n}{2n^4} \qquad \text{III: } \sum_{n=2}^{\infty} \frac{n^n}{n!}$$

Multiple Choice:

(a) *only I*

(b) *only I and II* ✓

(c) *only I and III*

(d) *only II*

(e) *only II and III*

(f) *only III*

5 Power Series

We undertake a study of an important class of infinite series.

Example 53. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 54. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

5.1 Power Series

We introduce the concept of a power series and some related fundamental properties.

Example 55. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 56. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

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5.2 Exercises: Power Series and Convergence

Exercises relating to power series and their convergence properties.

Sample Quiz Questions

Question 93 *Find the full interval of convergence for the power series*

$$\sum_{m=2}^{\infty} \frac{(-3)^m m^2 (x-5)^m}{\ln m}.$$

Multiple Choice:

(a) $\left(\frac{14}{3}, \frac{16}{3}\right)$ ✓

(b) $\left[\frac{14}{3}, \frac{16}{3}\right)$

(c) $(2, 8]$

(d) $[2, 8]$

(e) $(-\infty, \infty)$

Feedback(`attempt`): *First observe that*

$$\begin{aligned}\frac{1}{R} &= \lim_{m\to\infty} \left| \frac{\frac{(-3)^{(m+1)}(m+1)^2}{\ln (m+1)}}{\frac{(-3)^m m^2}{\ln m}} \right| \\ &= \lim_{m\to\infty} \left| \frac{-3(m+1)^2 \ln m}{m^2 \ln (m+1)} \right| \\ &= 3\end{aligned}$$

because

$$\lim_{m\to\infty} \frac{(m+1)^2 \ln m}{m^2 \ln(m+1)} = \lim_{m\to\infty} \frac{(m+1)^2}{m^2} \lim_{m\to\infty} \frac{\ln m}{\ln(m+1)} = 1$$

by virtue of l'Hospital's rule applied to both limits on the right-hand side. This means that the radius equals $1/3$. At the endpoint $x = 14/3$, the series equals

$$\sum_{m=2}^{\infty} \frac{m^2}{\ln m},$$

which diverges by the n-th term divergence test because $\lim_{m\to\infty} m^2/\ln m = \infty \neq 0$. *At the endpoint* $x = 16/3$, *the series equals*

$$\sum_{m=2}^{\infty} (-1)^m \frac{m^2}{\ln m},$$

which diverges for the same reason as the other endpoint, i.e., the terms do not go to zero.

Question 94 *Find the full interval of convergence for the power series*

$$\sum_{m=2}^{\infty} \frac{(-4)^m (\ln m)(x-1)^m}{(-2)^m m}.$$

Multiple Choice:

(a) $\left(\frac{1}{2}, \frac{3}{2}\right)$

(b) $\left[\frac{1}{2}, \frac{3}{2}\right)$ ✓

(c) $(-1, 3]$

(d) $[-1, 3]$

(e) $(-\infty, \infty)$

Feedback(`attempt`): *First observe that*

$$\begin{aligned}
\frac{1}{R} &= \lim_{m\to\infty} \left| \frac{\frac{(-4)^{(m+1)}(\ln(m+1))}{(-2)^{(m+1)}(m+1)}}{\frac{(-4)^m(\ln m)}{(-2)^m m}} \right| \\
&= \lim_{m\to\infty} \left| \frac{-4(\ln(m+1))m}{-2(\ln m)(m+1)} \right| \\
&= 2
\end{aligned}$$

because

$$\lim_{m\to\infty} \frac{m\ln(m+1)}{(m+1)\ln m} = \lim_{m\to\infty} \frac{m}{m+1} \lim_{m\to\infty} \frac{\ln(m+1)}{\ln m} = 1$$

by virtue of l'Hospital's rule applied to both limits on the right-hand side. This means that the radius equals $1/2$. *At the endpoint* $x = 3/2$, *the series equals*

$$\sum_{m=2}^{\infty} \frac{(\ln m)}{m},$$

which diverges by direct comparison to the harmonic series, i.e., the p-series with $p = 1$*. At the endpoint* $x = 1/2$*, the series equals*

$$\sum_{m=2}^{\infty} (-1)^m \frac{(\ln m)}{m},$$

which converges by the alternating series test because the sign of the terms alternates and $\ln m/m$ *decreases to zero as* $m \to \infty$*.*

Question 95 *Find the full interval of convergence for the power series*

$$\sum_{m=2}^{\infty} \frac{(-2)^m (\ln m)(x+4)^m}{6^m m}.$$

Multiple Choice:

(a) $\left(-\dfrac{13}{3}, -\dfrac{11}{3}\right)$

(b) $\left[-\dfrac{13}{3}, -\dfrac{11}{3}\right)$

(c) $(-7, -1]$ ✓

(d) $[-7, -1]$

(e) $(-\infty, \infty)$

Feedback(`attempt`): *First observe that*

$$\begin{aligned} \frac{1}{R} &= \lim_{m\to\infty} \left| \frac{\frac{(-2)^{(m+1)}(\ln(m+1))}{6^{(m+1)}(m+1)}}{\frac{(-2)^m(\ln m)}{6^m m}} \right| \\ &= \lim_{m\to\infty} \left| \frac{-2(\ln(m+1))m}{6(\ln m)(m+1)} \right| \\ &= \frac{1}{3} \end{aligned}$$

because

$$\lim_{m\to\infty} \frac{m\ln(m+1)}{(m+1)\ln m} = \lim_{m\to\infty} \frac{m}{m+1} \lim_{m\to\infty} \frac{\ln(m+1)}{\ln m} = 1$$

by virtue of l'Hospital's rule applied to both limits on the right-hand side. This means that the radius equals 3*. At the endpoint* $x = -7$*, the series equals*

$$\sum_{m=2}^{\infty} \frac{(\ln m)}{m},$$

which diverges by direct comparison to the harmonic series, i.e., the p-series with $p = 1$. At the endpoint $x = -1$, the series equals

$$\sum_{m=2}^{\infty} (-1)^m \frac{(\ln m)}{m},$$

which converges by the alternating series test because the sign of the terms alternates and $\ln m/m$ decreases to zero as $m \to \infty$.

Question 96 *Find the full interval of convergence for the power series*

$$\sum_{m=1}^{\infty} \frac{\sqrt[3]{m}(x-2)^m}{m!}.$$

Multiple Choice:

(a) $(1, 3)$

(b) $[1, 3)$

(c) $(1, 3]$

(d) $[1, 3]$

(e) $(-\infty, \infty)$ ✓

Feedback(`attempt`): *First observe that*

$$\begin{aligned} \frac{1}{R} &= \lim_{m\to\infty} \left| \frac{\frac{\sqrt[3]{(m+1)}}{(m+1)!}}{\frac{\sqrt[3]{m}}{m!}} \right| \\ &= \lim_{m\to\infty} \left| \frac{\sqrt[3]{(m+1)}}{(m+1)\sqrt[3]{m}} \right| \\ &= 0 \end{aligned}$$

because $m + 1$ in the denominator tends to ∞ and

$$\lim_{m\to\infty} \frac{\sqrt[3]{m+1}}{\sqrt[3]{m}} = \lim_{m\to\infty} \left(1 + m^{-1}\right)^{1/3} = \left(1 + \lim_{m\to\infty} m^{-1}\right)^{1/3} = 1.$$

This means that the radius is infinite and the interval of convergence is $(-\infty, \infty)$.

Sample Exam Questions

Question 97 *For which values of x does the series* $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n4^n}$ *converge?*

Multiple Choice:

(a) $-3 < x < 5$

(b) $-3 \leq x < 5$

(c) $-3 < x \leq 5$ ✓

(d) $-5 < x \leq 3$

(e) $-5 \leq x < 3$

(f) $-5 \leq x \leq 3$

Question 98 *Find the interval of convergence of the power series below.*

$$\sum_{n=1}^{\infty} \frac{(4x-1)^n}{n^{\frac{3}{4}}(n^2+2)}$$

Multiple Choice:

(a) $\left(0, \frac{1}{2}\right]$

(b) $\left[0, \frac{1}{2}\right]$ ✓

(c) $\left(0, \frac{1}{2}\right)$

(d) $\left[0, \frac{1}{2}\right)$

(e) $\left(-\frac{1}{2}, 0\right]$

(f) $(-\infty, \infty)$

Question 99 *Find the interval of convergence of the power series* $\sum_{n=2}^{\infty} \frac{2^n (x+5)^n}{\sqrt[3]{n}}$.

Multiple Choice:

(a) $\left[-\frac{11}{2}, -\frac{9}{2}\right]$

(b) $\left[-\frac{11}{2}, -\frac{9}{2}\right)$ ✓

(c) $\left(-\frac{11}{2}, -\frac{9}{2}\right)$

(d) $\left[\frac{9}{2}, \frac{11}{2}\right)$

(e) $\left(\frac{9}{2}, \frac{11}{2}\right)$

(f) $\left[\frac{9}{2}, \frac{11}{2}\right]$

5.3 Taylor Series

We introduce the notion of a Taylor Series.

Example 57. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 58. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

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5.4 Exercises: Taylor Series

Exercises relating to Taylor series and their computation.

Sample Exam Questions

Question 100 *The first few nonzero terms of the Maclaurin series for* $f(x) = \ln(1+\sin x)$ *are:*

Multiple Choice:

(a) $1+\dfrac{1}{2}x-\dfrac{1}{8}x^2+\dfrac{1}{24}x^3+\cdots$

(b) $1+\dfrac{1}{2}x-\dfrac{1}{8}x^2-\dfrac{1}{48}x^3+\cdots$

(c) $x-\dfrac{1}{2}x^2+\dfrac{1}{8}x^3-\dfrac{1}{24}x^4+\cdots$

(d) $1+x+\dfrac{1}{2}x^2+\dfrac{1}{3}x^3+\dfrac{1}{6}x^4\cdots$

(e) $x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3-\dfrac{1}{12}x^4+\cdots$ ✓

(f) $1+x+\dfrac{1}{2}x^2+\dfrac{1}{3}x^3-\dfrac{1}{12}x^4+\cdots$

Feedback(*`attempt`*): *The first few derivatives of* $f(x)$ *are:*

$$\begin{aligned}
f(x) &= \ln(1+\sin(x))\\
f'(x) &= \frac{\cos x}{1+\sin x}\\
f''(x) &= -\frac{\sin x}{1+\sin x}-\frac{\cos^2 x}{(1+\sin x)^2}\\
f'''(x) &= -\frac{\cos x}{1+\sin x}+\frac{\sin x\cos x}{(1+\sin x)^2}-\frac{2\sin x\cos x}{(1+\sin x)^2}+2\frac{\cos^3 x}{(1+\sin x)^3}
\end{aligned}$$

Evaluating at $x=0$ *gives* $f(0)=\ln 1=0$, $f'(0)=1$, $f''(0)=-1$, *and* $f'''(0)=1$. *Therefore the series starts with the terms* $x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\cdots$.

Question 101 *Find the Taylor polynomial of degree 2 for* $f(x) = \sqrt{x+16}$ *centered at* $x = 9$.

Multiple Choice:

(a) $5 + \frac{4}{5}x + \frac{9}{250}x^2$

(b) $5 - \frac{3}{5}(x-5) + \frac{1}{125}(x-5)^2$

(c) $5 + \frac{1}{10}(x-9) - \frac{1}{1000}(x-9)^2$ ✓

(d) $5 + \frac{3}{5}(x-5) + \frac{8}{125}(x-5)^2$

(e) $5 + \frac{1}{5}(x-9) + \frac{16}{125}(x-9)^2$

(f) *none of these*

Question 102 *Use the Taylor polynomial of degree* 3 *for* $f(x) = \ln(1+x)$ *centered at* $x_0 = 0$ *to approximate the value of* $\ln\left(\frac{3}{2}\right)$.

Multiple Choice:

(a) $\frac{2}{3}$

(b) $\frac{3}{2}$

(c) $\frac{15}{4}$

(d) $\frac{5}{12}$ ✓

(e) $\frac{9}{24}$

(f) $\frac{11}{24}$

Question 103 *Let* $F(x)$ *be the unique function that satisfies* $F(0) = 0$, $F'(0) = 0$, *and* $F'(x) = \frac{1}{x}\sin x^3$ *for all* $x \neq 0$. *Find the Taylor Series of* $F(x)$ *centered at* $x_0 = 0$.

Multiple Choice:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (6n+3) x^{6n+2}}{(2n+1)!}$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(6n+3)(2n+1)!}$ ✓

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+2}}{(2n+1)!}$

(e) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (6n+2) x^{6n+2}}{(2n+1)!}$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{(6n+3)(2n+1)!}$

5.5 Taylor Series Applications

We study the use of Taylor series for evaluating infinite series and limits.

Example 59. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 60. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

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5.6 Exercises: Taylor Series Applications

Various exercises relating to the application of Taylor Series to other problems of interest.

Sample Quiz Questions

Question 104 *Compute the first 4 nonzero terms in the Taylor series at $x = 0$ of the function*

$$\frac{d}{dx}\left[xe^{x^2}\right].$$

Multiple Choice:

(a) $1 + 3x^2 + \frac{5}{2}x^4 + \frac{7}{6}x^6$ ✓

(b) $-1 - 3x^2 + \frac{5}{2}x^4 + \frac{7}{6}x^6$

(c) $1 - 6x^2 - \frac{5}{4}x^4 + \frac{7}{6}x^6$

(d) $-1 - 6x^2 + \frac{5}{4}x^4 + \frac{7}{6}x^6$

(e) $2 + 3x^2 + \frac{5}{4}x^4 + \frac{7}{6}x^6$

(f) $2 - 3x^2 - \frac{5}{4}x^4 + \frac{7}{6}x^6$

Feedback(*`attempt`*): *Compute the series in stages beginning with substitution into known series:*

$$e^{x^2} = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots$$

$$xe^{x^2} = x + x^3 + \frac{1}{2}x^5 + \frac{1}{6}x^7 + \cdots$$

$$\frac{d}{dx}\left[xe^{x^2}\right] = 1 + 3x^2 + \frac{5}{2}x^4 + \frac{7}{6}x^6 + \cdots$$

Question 105 *Compute the first 4 nonzero terms in the Taylor series at $x = 0$ of the function*

$$\int_0^x (x\ln(1-x))\ dx.$$

Multiple Choice:

(a) $-\frac{1}{6}x^3 - \frac{1}{8}x^4 - \frac{2}{15}x^5 - \frac{1}{24}x^6$

(b) $\frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{2}{15}x^5 - \frac{1}{24}x^6$

(c) $-\frac{1}{6}x^3 + \frac{1}{4}x^4 + \frac{1}{15}x^5 - \frac{1}{24}x^6$

(d) $\frac{1}{6}x^3 + \frac{1}{4}x^4 - \frac{1}{15}x^5 - \frac{1}{24}x^6$

(e) $-\frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{15}x^5 - \frac{1}{24}x^6$ ✓

(f) $-\frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{15}x^5 - \frac{1}{24}x^6$

Feedback(`attempt`): *Compute the series in stages beginning with substitution into known series:*

$$\begin{aligned} \ln(1-x) &= -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \\ x\ln(1-x) &= -x^2 - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{1}{4}x^5 + \cdots \\ \int_0^x (x\ln(1-x))\ dx &= -\frac{1}{3}x^3 - \frac{1}{8}x^4 - \frac{1}{15}x^5 - \frac{1}{24}x^6 + \cdots \end{aligned}$$

Question 106 *Use Taylor series to estimate the value of*

$$\sqrt[3]{\frac{11}{10}}$$

to within an error of at most $1/900$*.*

Multiple Choice:

(a) $\frac{31}{30}$ ✓

(b) $\frac{47}{45}$

(c) $\frac{19}{18}$

(d) $\frac{16}{15}$

(e) $\dfrac{97}{90}$

(f) $\dfrac{49}{45}$

Feedback(`attempt`): *We may use the remainder formula for Taylor series to approach this problem. Suppose $p_n(x)$ is the degree n Taylor polynomial of the function*

$$f(x) = \sqrt[3]{1+x}$$

with center $a = 0$. Then the error $E_n(x)$, i.e., the difference between the polynomial and the function, does not exceed $\dfrac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$, where ξ is some unknown point in the range $0 \le \xi \le x$. In this case one should take $x = 1/10$ and determine how many derivatives are required to make this error estimate less than the given threshold. This means checking by hand for small numbers of derivatives. For the specific problem at hand, if we approximate $f(x)$ by the Taylor polynomial of degree $n = 1$, we have

$$\left|\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi)\right| = \left|\frac{x^{n+1}}{(n+1)!}\left(-\frac{2}{9}(\xi+1)^{-5/3}\right)\right| \le \left|\frac{x^{n+1}}{(n+1)!}\left(\frac{2}{9}\right)\right| = \frac{1}{900}$$

when $x = 1/10$. We conclude that the correct Taylor approximation is

$$p_n\left(\frac{1}{10}\right) = \left(\frac{1}{10}\right)^0 + \frac{1}{3}\left(\frac{1}{10}\right)^1 = 1 + \frac{1}{30} = \frac{31}{30}.$$

Question 107 *Use Taylor series to estimate the value of*

$$e^{-\frac{1}{3}}$$

to within an error of at most $1/162$.

Multiple Choice:

(a) $\dfrac{5}{9}$

(b) $\dfrac{13}{18}$ ✓

(c) $\dfrac{8}{9}$

(d) $\dfrac{19}{18}$

(e) $\dfrac{11}{9}$

(f) $\dfrac{25}{18}$

Feedback(`attempt`): *We may use the remainder formula for Taylor series to approach this problem. Suppose $p_n(x)$ is the degree n Taylor polynomial of the function*

$$f(x) = e^{-x}$$

with center $a = 0$. Then the error $E_n(x)$, i.e., the difference between the polynomial and the function, does not exceed $\frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$, where ξ is some unknown point in the range $0 \le \xi \le x$. In this case one should take $x = 1/3$ and determine how many derivatives are required to make this error estimate less than the given threshold. This means checking by hand for small numbers of derivatives. For the specific problem at hand, if we approximate $f(x)$ by the Taylor polynomial of degree $n = 2$, we have

$$\left|\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi)\right| = \left|\frac{x^{n+1}}{(n+1)!}\left(e^{-\xi}\right)\right| \le \left|\frac{x^{n+1}}{(n+1)!}(1)\right| = \frac{1}{162}$$

when $x = 1/3$. We conclude that the correct Taylor approximation is

$$p_n\left(\frac{1}{3}\right) = \left(\frac{1}{3}\right)^0 - 1\left(\frac{1}{3}\right)^1 + \frac{1}{2}\left(\frac{1}{3}\right)^2 = 1 - \frac{1}{3} + \frac{1}{18} = \frac{13}{18}.$$

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5.7 Exercises: Cumulative

Exercises relating to various topics we have studied.

Sample Exam Questions

Question 108 *If it converges, find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!}$. If the series diverges, explain why.*

Multiple Choice:

(a) $\ln 2$

(b) $\ln 3 - \ln 2$

(c) e^{-2}

(d) $\frac{1}{2}$ ✓

(e) $\frac{2}{e}$

(f) *diverges*

Feedback(`attempt`): *We recognize the Taylor series for cosine:*

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{3^{2n}(2n)!}$$

The series in question is exactly

$$\sum_{n=0}^{\infty} \frac{(-1)^n (\pi/3)^{2n}}{(2n)!} = \cos\frac{\pi}{3} = \frac{1}{2}.$$

Question 109 *What is the limit of the sequence* $\left\{n^2\left(1-\cos\frac{1}{n}\right)\right\}$*?*

Multiple Choice:

(a) 1

(b) -1

(c) $\frac{\sqrt{3}}{2}$

(d) $\frac{1}{2}$ ✓

(e) $-\frac{\sqrt{3}}{2}$

(f) *diverges*

Question 110 *Find the limit of the sequence*

$$a_n = \{n\left[\ln(n+3) - \ln n\right]\}.$$

Multiple Choice:

(a) 0

(b) 1

(c) $\ln 3$

(d) 3 ✓

(e) ∞

(f) *the limit does not exist*

6 Ordinary Differential Equations

We begin a study of first-order ordinary differential equations.

Example 61. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 62. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the

intersection of the curves $x = \sqrt{y}$ *and* $x = 1$*). Therefore we integrate* $A(y)dy$ *to conclude*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

6.1 ODEs: Foundations

We study the fundamental concepts and properties associated with ODEs.

Example 63. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 64. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

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6.2 Exercises: ODEs

Exercises relating to fundamental properties of ODEs.

Exercise 111 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *A typical square cross section has side length* $L = \boxed{x - x^2}$ *and area* $A = \boxed{(x - x^2)^2}$.
- *Possible numerical values of the x-coordinates of points in the base range from a minimum value of* $x = \boxed{0}$ *up to a maximum of* $x = \boxed{1}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 112 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$\begin{aligned} 0 &\le x \le 1, \\ 0 &\le y \le z^2, \\ 0 &\le z \le 3. \end{aligned}$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length* $\boxed{1}$ *in the x-direction and width* $\boxed{z^2}$ *in the y-direction.*
- *The area of a z cross section is* $A(z) = \boxed{z^2}$.
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

6.3 Separable and Linear ODEs

We learn techniques to solve first-order linear and separable ODEs.

Example 65. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1-x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1-x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1-x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1-x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 66. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1-\sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1-\sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1-\sqrt{y})^2}\,dy = \boxed{\frac{1}{6}}.$$

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6.4 Exercises: Linear and Separable ODEs

Exercises related to solving linear and separable ODEs.

Sample Quiz Questions

Question 113 *Let $y(x)$ be the solution to the initial value problem*

$$\frac{dy}{dx} = -(1+3x^2)y^2$$

and $y(0) = 1/2$. What is the value of $y(1)$?

Multiple Choice:

(a) $\frac{1}{6}$

(b) $\frac{1}{4}$ ✓

(c) $\frac{1}{3}$

(d) $\frac{1}{2}$

(e) $\frac{\pi}{4}$

(f) 1

Feedback(*`attempt`*): *This is a separable ODE. Moving all functions of y to the left-hand side and all functions of x to the right-hand side and integrating gives*

$$\int \frac{-1}{y^2}\, dy = \int (1+3x^2)\, dx,$$

which yields

$$\frac{1}{y} = x^3 + x + C.$$

Evaluating at $x = 0$ and $y = 1/2$ gives $2 = 0 + C$, so

$$\frac{1}{y} = x^3 + x + 2,$$

i.e.,

$$y = \frac{1}{x^3 + x + 2}.$$

Plugging in $x = 1$ gives $y = 1/4$.

Sample Exam Questions

Question 114 *The solution of the initial value problem* $x\dfrac{dy}{dx} + 3y = 7x^4$, $y(1) = 1$, *satisfies* $y(2) =$

Multiple Choice:

(a) 0

(b) 1

(c) 2

(d) 4

(e) 8

(f) 16 ✓

Question 115 *The solution of the initial value problem* $\dfrac{dy}{dx} - 20x^4e^{-y} = 0$, $y(0) = 0$, *satisfies* $y(1) =$

Multiple Choice:

(a) $\ln 5$ ✓

(b) $\ln 4$

(c) $\ln 3$

(d) $\ln 2$

(e) 1

(f) 0

Question 116 *Let* $y(x)$ *be the solution of the initial value problem*

$$x\frac{dy}{dx} = e^x - y \quad \text{with} \quad y(\ln 2) = 0.$$

Find $y(1)$.

Multiple Choice:

(a) $\dfrac{e^2}{2}$

(b) $2e^2$

(c) $\dfrac{e}{2}$

(d) 0

(e) $e - 2$ ✓

(f) 1

Question 117 *Let $y(x)$ be the solution of the initial value problem*

$$x\frac{dy}{dx} = y + x^2 \sin x \quad \textit{with} \quad y(\pi) = 0.$$

What is $y(2\pi)$?

Multiple Choice:

(a) $-\pi$

(b) -2π

(c) -4π ✓

(d) 0

(e) 2π

(f) 4π

Question 118 *Consider the initial value problem*

$$(1 + x^2)\frac{dy}{dx} = 2y \quad \textit{with} \quad y(0) = 2.$$

What is $\lim_{x\to\infty} y(x)$?

Multiple Choice:

(a) $2e^{\pi}$ ✓

(b) $2e^{\pi/2}$

(c) $2e^{\pi/4}$

(d) 1

(e) 0

(f) e^{π}

6.5 Applications of ODEs

We study some sample applications of ODEs.

Example 67. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\bigg|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 68. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$

between $y = 0$ *and* $y = 1$*, since these are the most extreme values of* y *found in our region (note that we can find the upper value* $y = 1$ *by solving for the intersection of the curves* $x = \sqrt{y}$ *and* $x = 1$*). Therefore we integrate* $A(y)dy$ *to conclude*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

odes/31applypractice.tex

6.6 Exercises: ODE Applications

Exercises relating to the application of ODEs to solve problems.

Sample Exam Questions

Question 119 *A tank contains 100 gallons of water in which 300 pounds of salt are dissolved. At some initial time, workers begin pumping in fresh water, i.e., containing no salt, at a rate of 10 gallons per minute. During the process, the tank is kept well-mixed and 20 gallons per minute of the resulting saltwater are pumped out of the tank (in particular, note that the tank will be empty after 10 minutes). Find the total amount of salt in the tank (measured in pounds) which remains 9 minutes after the process starts.*

Multiple Choice:

(a) 1

(b) 2

(c) 3 ✓

(d) 4

(e) 5

(f) 6