Math 104 Ximera Content

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Contents

1 Computing Volume

A sample application of integration: computing volumes of a variety of complicated three-dimensional objects.

Integration is the tool to use whenever a quantity can be conceived as an *accumulation of infinitesimal parts*. Volume is one of the most basic and important of such quantities. In the activities that follow, we regard volume as the accumulated size of infinitely thin slices and use this perspective to derive and apply a number of formulas for computing volume.

1.1 Volume By General Cross Sections

We use cross-sectional area to compute volume.

Example 1. *The base of a solid region is bounded by the curves $x = 0$, $y = 0$, and $y = \sqrt{1 - x^2}$. The cross sections perpendicular to the x axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the x-axis are vertical, so the base of a typical x cross section will extend from $y = 0$ to $y = \sqrt{1 - x^2}$. Since each cross section will have area*

$$A(x) = \left(\sqrt{1 - x^2} - 0\right)^2 = 1 - x^2.$$

To compute volume, we integrate $dV = A(x)dx$ between $x = 0$ and $x = 1$, since these are the most extreme values of x found in our region. Therefore

$$V = \int_0^1 (1 - x^2)dx = x - \frac{x^3}{3}\Big|_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$

Example 2. *The base of a solid region is bounded by the curves $y = 0$, $x = \sqrt{y}$, and $x = 1$. The cross sections perpendicular to the y-axis are squares. Compute the volume of the region.*

Solution: *Lines in the xy-plane which are perpendicular to the y-axis are (horizontal ✓/ vertical), so the base of a typical y cross section will extend from the graph $x = \boxed{\sqrt{y}}$ to the graph $x = \boxed{1}$. The length of the base is the difference of x-coordinates (since all points on a slice have the same y-coordinate), so the length of the base is $\boxed{1 - \sqrt{y}}$, giving the square an area of*

$$A(y) = \boxed{(1 - \sqrt{y})^2}$$

(note that the answer is a function of y because different y cross sections will generally have different areas). To compute volume, we integrate $dV = A(y)dy$ between $y = 0$ and $y = 1$, since these are the most extreme values of y found in our region (note that we can find the upper value $y = 1$ by solving for the intersection of the curves $x = \sqrt{y}$ and $x = 1$). Therefore we integrate $A(y)dy$ to conclude

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(1 - \sqrt{y})^2}\, dy = \boxed{\frac{1}{6}}.$$

Exercise 1 *The base of a solid region is bounded by the curves $x = 0$, $y = x^2$ and $y = x$. The cross sections perpendicular to the x-axis are squares. Compute the volume of the region.*

- *Possible x-coordinates of points in the base range from a minimum value of $x = \boxed{0}$ up to a maximum of $x = \boxed{1}$.*
- *A typical square cross section has side length $L = \boxed{x - x^2}$ and area $A = \boxed{(x - x^2)^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{1}} \boxed{(x - x^2)^2}\, d\boxed{x} = \boxed{\frac{1}{30}}.$$

Exercise 2 *Find the volume of the region in three-dimensional space defined by the inequalities*

$$0 \le x \le 1,$$
$$0 \le y \le z^2,$$
$$0 \le z \le 3.$$

- *Cross sections perpendicular to the z-axis are (square / rectangular ✓ / triangluar) with length $\boxed{1}$ in the x-direction and width $\boxed{z^2}$ in the y-direction.*
- *The area of a z cross section is $A(z) = \boxed{z^2}$.*
- *To compute volume, integrate:*

$$V = \int_{\boxed{0}}^{\boxed{3}} \boxed{z^2}\, dz = \boxed{9}.$$

1.2 Testing answer

A regression test of `answer`*.*

Problem 3 *Type* 2. $\boxed{2}$, $\underset{\text{given}}{\boxed{2}}$, $\boxed{\frac{1}{2}}$, $\int_a^b f(x)\,dx$,

$$-\frac{1}{\frac{12}{1}}\sum_{n=1}^{\boxed{\infty}} n$$

1.3 Testing answer

A regression test of `answer`.

Problem 4 *Type* 2. $\boxed{2}$, $\underset{\text{given}}{\boxed{2}}$, $\boxed{\frac{1}{2}}$, $\int_a^b f(x)\,dx$,

$$-\frac{1}{\frac{12}{1}}\sum_{n=1}^{\boxed{\infty}} n$$

1.4 Testing answer

A regression test of `answer`*.*

Problem 5 *Type* 2. $\boxed{2}$, $\underbrace{\boxed{2}}_{\text{given}}$, $\boxed{\frac{1}{2}}$, $\int_a^b f(x)\,dx$,

$$-\frac{1}{\frac{12}{1}}\sum_{n=1}^{\boxed{\infty}} n$$

1.5 Testing answer

A regression test of `answer`*.*

Problem 6 *Type* 2. $\boxed{2}$, $\underset{\text{given}}{\boxed{2}}$, $\boxed{\dfrac{1}{2}}$, $\displaystyle\int_a^b f(x)\;dx$,

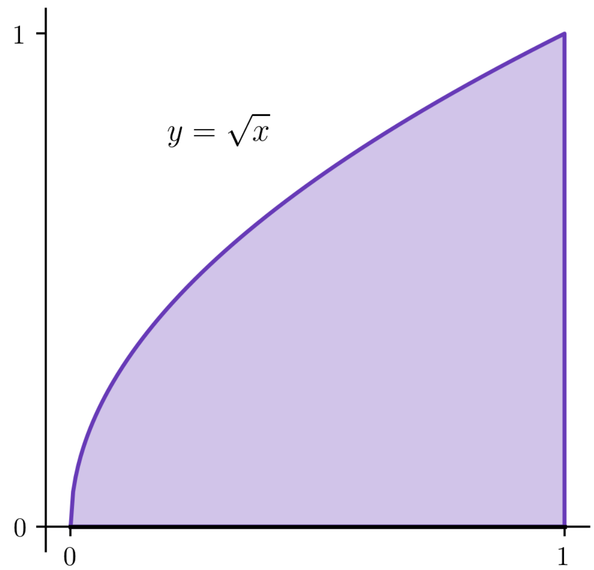
$$-\frac{1}{\frac{12}{1}}\sum_{n=1}^{\boxed{\infty}} n$$

volumes/washerpractice.tex

1.6 Exercises: Disks and Washers

Exercises for the disk and washer methods.

Problem 7 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the x-axis. Use the disk method to find the volume of the solid of revolution.*



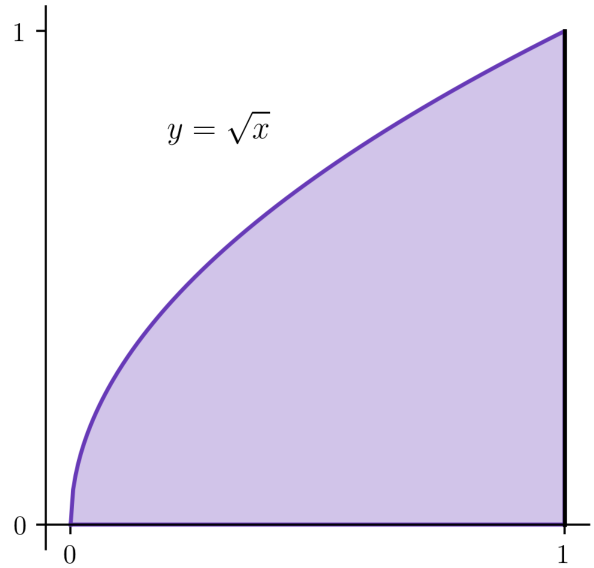
$$R(x) = \boxed{\sqrt{x}}$$

Hint: *The radius $R(x)$ will be a difference of y-values because slices are indexed by the variable x.*

Hint: *Each slice will extend from $y = 0$ to $y = \sqrt{x}$, and so $R(x)$ must be the larger of these y-values minus the smaller of these y-values*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi(R(x))^2 dx = \boxed{\frac{\pi}{2}}$$

Problem 8 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 1$. Use the disk method to find the volume of the solid of revolution.*



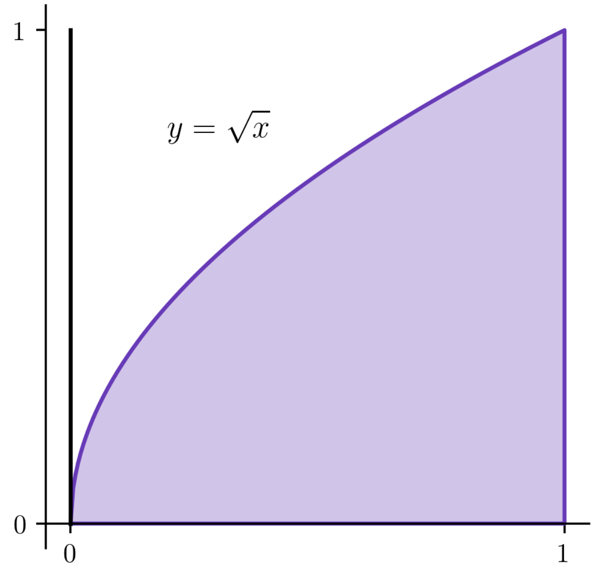
$$R(y) = \boxed{1 - y^2}$$

Hint: *The radius $R(y)$ will be a difference of x-values because slices are indexed by the variable y.*

Hint: *Each slice will extend from $x = y^2$ to $x = 1$, and so $R(y)$ must be the larger of these x-values minus the smaller of these x-values*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi(R(y))^2 dy = \boxed{\frac{8\pi}{15}}$$

Problem 9 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $x = 0$. Use the washer method to find the volume of the solid of revolution.*



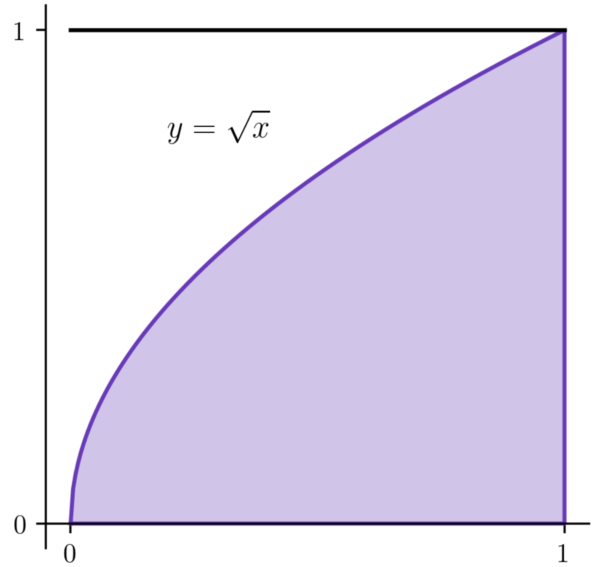
$$R_{\text{outer}}(y) = \boxed{1} \text{ and } r_{\text{inner}}(y) = \boxed{y^2}$$

Hint: *Each radius will be a difference of x-values because slices are indexed by the variable y.*

Hint: *The distance from the axis $x = 0$ to the line $x = 1$ is 1, and the distance from the axis $x = 0$ to $x = y^2$ is y^2.*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(y))^2 - (r_{\text{inner}}(y))^2 \right] dy = \boxed{\frac{4\pi}{5}}$$

Problem 10 *The region $0 \leq y \leq \sqrt{x}$ with $x \leq 1$, shown below, is revolved around the axis $y = 1$. Use the washer method to find the volume of the solid of revolution.*



$R_{\text{outer}}(x) = \boxed{1}$ *and* $r_{\text{inner}}(x) = \boxed{1 - \sqrt{x}}$

Hint: *Each radius will be a difference of y-values because slices are indexed by the variable x.*

Hint: *The distance from the axis $y = 1$ to the line $y = 0$ is 1, and the distance from the axis $y = 1$ to $y = \sqrt{x}$ is $1 - \sqrt{x}$.*

$$V = \int_{\boxed{0}}^{\boxed{1}} \pi \left[(R_{\text{outer}}(x))^2 - (r_{\text{inner}}(x))^2 \right] dx = \boxed{\frac{5\pi}{6}}$$