A PERTURBED LINEAR MIXING MODEL ACCOUNTING FOR SPECTRAL VARIABILITY

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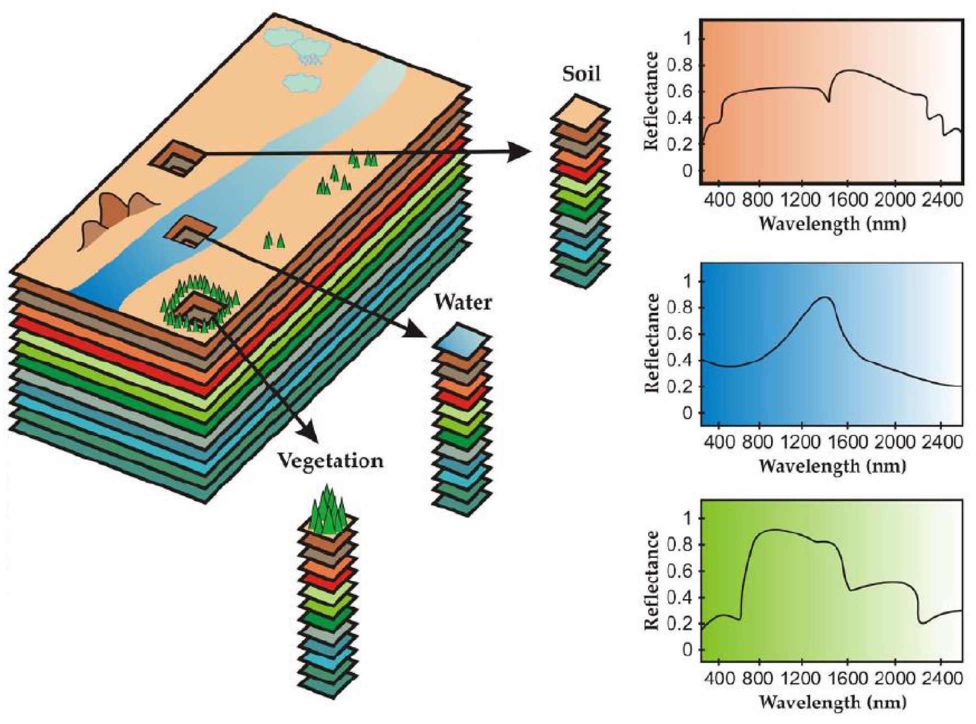
1. Introduction

Hyperspectral imagery

- high spectral resolution, low spatial resolution ⇒ hyperspectral unmixing
- hyperspectral unmixing
  - ▷ identifying the reference spectral signatures in the data (*endmembers*)
  - ▷ estimating the endmember relative fraction in each pixel (*abundances*).

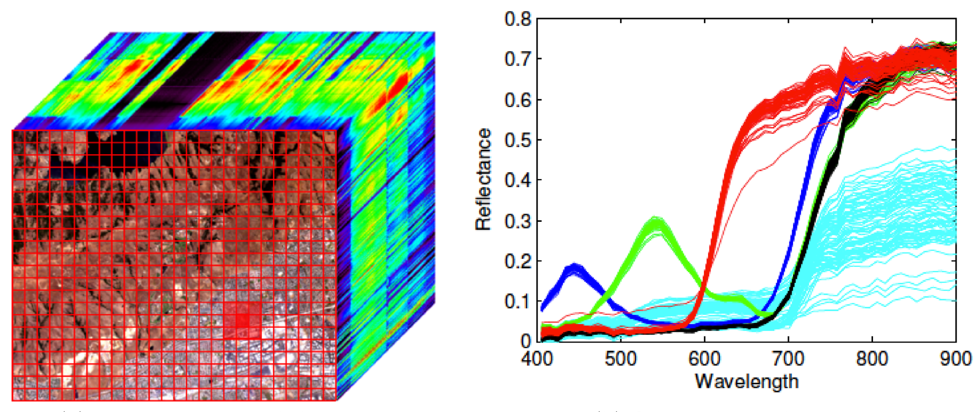


Linear mixing model (LMM)

$$\mathbf{y}_n = \sum_{k=1}^{K} a_{kn}\mathbf{m}_k + \mathbf{b}_n, \text{ for } n = 1,\dots,N. \quad (1)$$

Limitation of the LMM

- spatially varying acquisition conditions + inherent variability of the imaged scene (natural evolution)
  - ▷ spectral variability
- estimation errors may be propagated into the unmixing process (unsupervised procedures)
  - ▷ new models need to be studied to account for spectral variability.



(a) An hyperspectral datacube (b) Spectral variability [a]

Figure 1: Spectral variability: an illustration

[a] P. Gader, A. Zare, R. Close, J. Aitken, G. Tuell, MUUFL Gulfport Hyperspectral and LiDAR Airborne Data Set, University of Florida, Gainesville, FL, Tech. Rep. REP-2013-570, Oct. 2013.

2. Perturbed LMM

Perturbed linear mixing model (PLMM)

- pixel represented by a linear combination of corrupted endmembers
- corrupted endmembers = endmembers affected by an additive spatially varying perturbation vector

$$\mathbf{y}_n = \sum_{k=1}^{K} a_{kn}\Big(\mathbf{m}_k + \mathbf{dm}_{n,k}\Big) + \mathbf{b}_n \text{ for } n = 1,\dots,N. \quad (2)$$

Matrix formulation

$$\mathbf{Y} = \mathbf{MA} + \underbrace{\Big[\mathbf{dM}_1\mathbf{a}_1\Big|\dots\Big|\mathbf{dM}_N\mathbf{a}_N\Big]}_{\boldsymbol{\Delta}} + \mathbf{B}. \quad (3)$$

- ▶ N : number of pixels
- ▶ L : number of spectral bands
- ▶ K : number of endmembers
- ▶ $\mathbf{Y} = [\mathbf{y}_1,\dots,\mathbf{y}_N] \in \mathbb{R}^{L\times N}$: matrix of hyperspectral pixels
- ▶ $\mathbf{M} = [\mathbf{m}_1,\dots,\mathbf{m}_K] \in \mathbb{R}^{L\times K}$: endmember matrix
- ▶ $\mathbf{A} = [\mathbf{a}_1,\dots,\mathbf{a}_N] \in \mathbb{R}^{K\times N}$: abundance matrix
- ▶ $\mathbf{dM}_n = [\mathbf{dm}_{n,1},\dots,\mathbf{dm}_{n,K}] \in \mathbb{R}^{L\times K}$: nth variability matrix
- ▶ $\boldsymbol{\Delta} = \Big[\mathbf{dM}_1\mathbf{a}_1\Big|\dots\Big|\mathbf{dM}_N\mathbf{a}_N\Big]$: deviation from the LMM

Constraints accounting for physical considerations

$$\begin{aligned} &\mathbf{A} \succeq \mathbf{O}_{K,N}, \quad \mathbf{A}^T\mathbf{1}_K = \mathbf{1}_N \\ &\mathbf{M} \succeq \mathbf{O}_{L,K}, \quad \mathbf{M} + \mathbf{dM}_n \succeq \mathbf{O}_{L,K}, \forall n = 1,\dots,N. \end{aligned} \quad (4)$$

3. Parameter estimation

$$(\mathbf{M}^*, \mathbf{dM}^*, \mathbf{A}^*) \in \underset{\mathbf{M},\mathbf{dM},\mathbf{A}}{\arg\min} \Big\{\mathcal{J}(\mathbf{M}, \mathbf{dM}, \mathbf{A}) \text{ s.t. } (4)\Big\} \quad (5)$$

with

$$\mathcal{J}(\mathbf{M}, \mathbf{dM}, \mathbf{A}) = \frac{1}{2}\|\mathbf{Y} - \mathbf{MA} - \boldsymbol{\Delta}\|_F^2 + \alpha\Phi(\mathbf{A}) + \beta\Psi(\mathbf{M}) + \gamma\Upsilon(\mathbf{dM}) \quad (6)$$

Trade-off between the data fitting term and the penalties $\Phi(\mathbf{A})$, $\Psi(\mathbf{M})$ and $\Upsilon(\mathbf{dM})$ controlled by (α, β, γ).

Abundance penalization

Spatially smooth abundances

$$\Phi(\mathbf{A}) = \frac{1}{2}\|\mathbf{AH}\|_F^2 \quad (7)$$

where $\mathbf{H} \in \mathbb{R}^{N\times 4N}$ computes the differences between the abundances of a given pixel and those of its 4 nearest neighbors.

Endmember penalization

Constrains the volume of the simplex whose vertices are the endmember signatures

$$\Psi(\mathbf{M}) = \frac{1}{2}\sum_{i\neq j}\|\mathbf{m}_i - \mathbf{m}_j\|_2^2. \quad (8)$$

Variability penalization

Limits the norm of the spectral variability

$$\Upsilon(\mathbf{dM}_n) = \frac{1}{2}\sum_{n=1}^{N}\|\mathbf{dM}_n\|_F^2. \quad (9)$$

4. An ADMM-based algorithm

- Global algorithm : *block coordinate descent* (BCD), convergence to a critical point of $\mathcal{J}$ if each sub-problem is exactly minimized
  - ▷ each parameter estimated by the Alternating Direction Method of Multipliers (ADMM).

Algorithm 1: PLMM-unmixing: global algorithm.

Data: $\mathbf{Y}, \mathbf{A}^{(0)}, \mathbf{M}^{(0)}, \mathbf{dM}^{(0)}$
begin
  $k \leftarrow 1$;
  while *stopping criterion not satisfied* **do**
(a) $\mathbf{A}^{(k)} \leftarrow \underset{\mathbf{A}}{\arg\min}\ \mathcal{J}\Big(\mathbf{M}^{(k-1)}, \mathbf{dM}^{(k-1)}, \mathbf{A}\Big)$;
(b) $\mathbf{M}^{(k)} \leftarrow \underset{\mathbf{M}}{\arg\min}\ \mathcal{J}\Big(\mathbf{M}, \mathbf{dM}^{(k-1)}, \mathbf{A}^{(k)}\Big)$;
(c) $\mathbf{dM}^{(k)} \leftarrow \underset{\mathbf{dM}}{\arg\min}\ \mathcal{J}\Big(\mathbf{M}^{(k)}, \mathbf{dM}, \mathbf{A}^{(k)}\Big)$;
    $k \leftarrow k + 1$;
  $\mathbf{A} \leftarrow \mathbf{A}^{(k)}$;
  $\mathbf{M} \leftarrow \mathbf{M}^{(k)}$;
  $\mathbf{dM} \leftarrow \mathbf{dM}^{(k)}$;
Result: $\mathbf{A}, \mathbf{M}, \mathbf{dM}$

ADMM: general principle

Given $f : \mathbb{R}^p \to \mathbb{R}^+$, $g \in \mathbb{R}^m \to \mathbb{R}^+$, $\mathbf{A} \in \mathbb{R}^{n\times p}$ and $\mathbf{B} \in \mathbb{R}^{n\times m}$, solve

$$\min_{\mathbf{x},\mathbf{z}}\Big\{f(\mathbf{x}) + g(\mathbf{z})\Big|\mathbf{Ax} + \mathbf{Bz} = \mathbf{c}\Big\}. \quad (10)$$

Associated scaled augmented Lagrangian

$$\mathcal{L}_\mu(\mathbf{x}, \mathbf{z}, \mathbf{u}) = f(\mathbf{x}) + g(\mathbf{z}) + \frac{\mu}{2}\|\mathbf{Ax} + \mathbf{Bz} - \mathbf{c} + \mathbf{u}\|_2^2, \quad \mu > 0.$$

ADMM

- ▶ successively minimize the augmented Lagrangian $\mathcal{L}_\mu$ with respect to each variable

$$\begin{aligned} \mathbf{x}^{(k+1)} &\in \underset{\mathbf{x}}{\arg\min}\ \mathcal{L}_\mu\Big(\mathbf{x}, \mathbf{z}^{(k)}, \mathbf{u}^{(k)}\Big) \\ \mathbf{z}^{(k+1)} &\in \underset{\mathbf{z}}{\arg\min}\ \mathcal{L}_\mu\Big(\mathbf{x}^{(k+1)}, \mathbf{z}, \mathbf{u}^{(k)}\Big) \\ \mathbf{u}^{(k+1)} &= \mathbf{u}^{(k)} + \mathbf{Ax}^{(k+1)} + \mathbf{Bz}^{(k+1)} - \mathbf{c}. \end{aligned}$$

- ▶ convergence guaranteed when the problem is (strictly) convex
- ▶ adjustment rule for μ

$$\mu^{(k+1)} = \begin{cases} \tau^{\text{incr}}\mu^{(k)} & \text{if } \left\|\mathbf{r}^{(k)}\right\|_2 > \rho\left\|\mathbf{s}^{(k)}\right\|_2 \\ \mu^{(k)}/\tau^{\text{decr}} & \text{if } \left\|\mathbf{s}^{(k)}\right\|_2 > \rho\left\|\mathbf{r}^{(k)}\right\|_2 \\ \mu^{(k)} & \text{otherwise} \end{cases} \quad (11)$$

where the primal and dual residuals $\mathbf{r}^{(k+1)}$ and $\mathbf{s}^{(k+1)}$ at iteration $k+1$ are given by

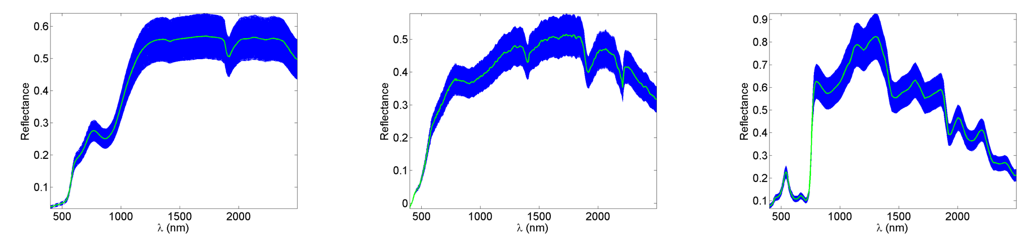
$$\mathbf{r}^{(k+1)} = \mathbf{Ax}^{(k+1)} + \mathbf{Bz}^{(k+1)} - \mathbf{c} \quad (12)$$

$$\mathbf{s}^{(k+1)} = \mu\mathbf{A}^T\mathbf{B}\Big(\mathbf{z}^{(k+1)} - \mathbf{z}^{(k)}\Big). \quad (13)$$

5. Results

5.1. Experiment with synthetic data

- Method evaluated on a 128 × 64-pixel image
- Linear mixtures of 3 endmembers with $L = 160$ spectral bands
- No pure pixel, mixture corrupted by an additive white Gaussian noise to ensure a SNR of 30dB
- Abundance and endmembers initialized with VCA/FCLS
- Simulation scenario: $\mu_n^{(\mathbf{A})(0)} = \mu_n^{(\mathbf{dM})(0)} = 10^{-4}$, $\mu_\ell^{(\mathbf{M})(0)} = 10^{-8}$, $\tau^{\text{incr}} = \tau^{\text{decr}} = 1.1$, $\rho = 10$, $\varepsilon^{\text{abs}} = 10^{-1}$ and $\varepsilon^{\text{rel}} = 10^{-4}$.



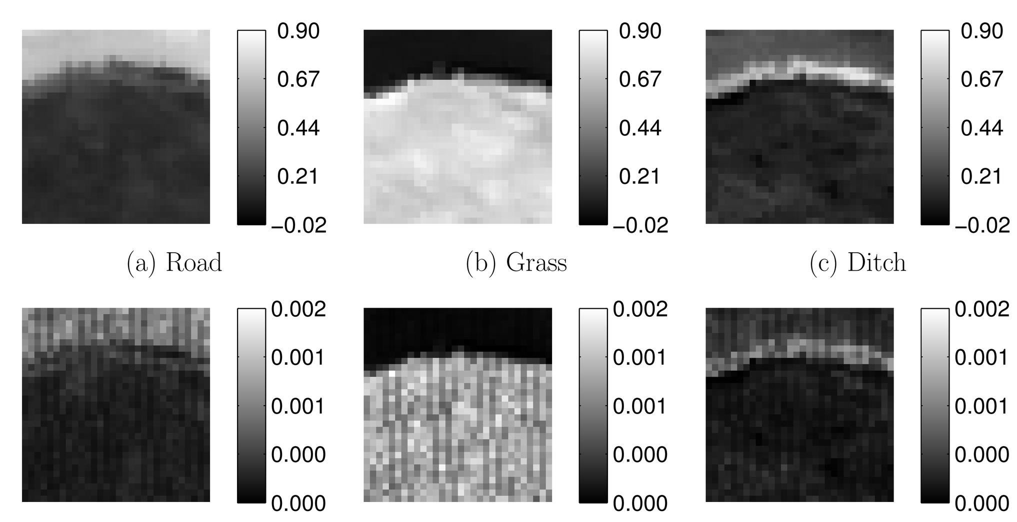
(a) (b) (c)

Figure 2: True endmembers (green) and variability (blue) used in the synthetic mixture.

Table 1: Simulation results for synthetic data (GMSE($\mathbf{A}$)$\times 10^{-2}$, GMSE($\mathbf{dM}$)$\times 10^{-4}$, RE $\times 10^{-4}$).

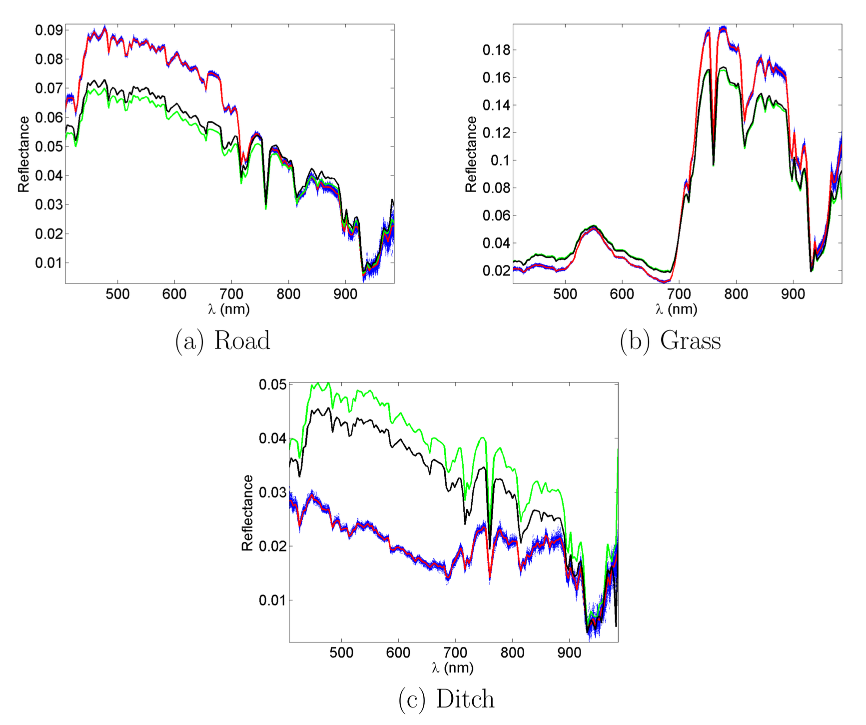
	VCA/FCLS	AEB	Proposed method
aSAM($\mathbf{M}$)	5.0639	5.1104	**4.1543**
GMSE($\mathbf{A}$)	2.07	2.11	**1.44**
GMSE($\mathbf{dM}$)	/	/	**4.36**
RE	2.66	2.66	**0.38**
time (s)	**1**	33	1990

5.2. Experiment with real data



(a) Road (b) Grass (c) Ditch (d) Variability (road) (e) Variability (grass) (f) Variability (ditch))

Figure 3: Abundance and variability distribution (real data).



(a) Road (b) Grass (c) Ditch

Figure 4: Endmember estimation on real data (PLMM in red, VCA in green, AEB in black and variability in blue dotted lines).

Table 2: Results on real data (Madonna) (RE $\times 10^{-6}$).

	VCA/FCLS	AEB	Proposed method
RE	8.64	5.25	**9.55**$\times 10^{-2}$
time (s)	**0.41**	1.77	24.5

6. Conclusion and future work

- ▶ Introduction of an explicit variability model
- ▶ Proposition of an unsupervised unmixing strategy
- ▷ Finding automatic rules to set the penalty parameters
- ▷ Modeling temporal variability in hyperspectral image time series

This work is supported by the Direction Générale de l'Armement (DGA), French Ministry of Defence, and by the Hypanema ANR Project n°ANR-12-BS03-003.