CSCI-4113
Assignment 2

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Note: To avoid confusing myself, i redefined y as w so the resulting LP P is:

$$Mazimize(3x + 4w + 2z)$$

Such That:

$$4x + w - 2z \leq 20 \tag{1}$$
$$3x - w + 2 \geq 1 \tag{2}$$
$$x - 2w + 2z = -8 \tag{3}$$
$$4x - w + 3z \leq 21 \tag{4}$$

and

$$x \geq 0$$
$$w \leq 0$$

1 Question 1

Given the LP P that is clearly not in Canonical Form as it has a Minimization Objective Function, Lower Bound and equality Constriants, and negative/unbounded variables.
We define a sequence of equivalent LPs $P^{(i)}, 0 \leq i \leq 3$ such that $P^{(3)} = P'$ is an LP equivalent to P and in Canonical form:

First

We need to define an LP $P^{(0)}$ that is equivalent to P but with the a negated Objective Function for $P^{(0)}$ to be a maximization LP.

1. Since P and $P^{(0)}$ have the same variables and constraints, this means that they have the same set of feasible solutions.
2. Minimizing $f(x)$ is the same as maximizing $f'(x)$, thus P and P' have the same Objective Function value.

$P^{(0)}$ will have an objective Function:

$$Maximize(-3x - 4w - 2z)$$

Second

We now construct a new LP $P^{(1)}$ that is equivalent to $P^{(0)}$ but with no equality constraints.
That is, we replace all the equalities with inequalities.

1. Since $a = b$ is clearly equivalent to $a \leq b$ && $a \geq b$
2. Thus $P^{(0)}$ and $P^{(1)}$ are equivalent since they have the same set of variables, the same objective function and equivalent constraints.

Thus we replace constraint (3) in $P^{(0)}$ with $\downarrow$ in $P^{(1)}$

$$x - 2w + 2z \leq -8 \tag{3.1}$$

$$x - 2w + 2z \geq -8 \tag{3.2}$$

Third

The next step is to replace all the lower bound inequality constraints in $P^{(1)}$ to upper bound constraints, thus constructing a new LP $P^{(2)}$.

1. An inequality $a \leq b$ is equivalent to its negation $-a \geq -b$
2. Just like in the prev step, we are not changing the Objective function and replacing some constraints with equivalent one's thus both $P^{(2)}$ and $P^{(1)}$ have the same Objective Function values and set of feasible solutions and are thus equivalent.

So we replace constraint (1) and (3.2) respectivley with:

$$-3x + w - z \leq -1 \tag{1.1}$$

$$-x + 2w - 2z \leq 8 \tag{3.2.1}$$

This gives us an LP $P^{(2)}$

Fourth

We now need to add non-negativity constarints to $P^{(2)}$.

1. x already has the constraint so we dont need to modify it.
2. w has a non-positivitey constraint, which is equivalent to:

   $$-w' \geq 0$$

   Thus we replace every instance of w with $-w'$ (note that $w = w'$) and the constarint $w \leq 0$ with $w' \geq 0$.
3. The variable z is unbounded. We can add a non-negativitey constarint on it by representing it as the differnce of two non-negative numbers. It is clear to see that this holds for any number. So:

   $$z = z' - z''$$

   And add the bellow constarints to the LP

   $$z' \geq 0, z'' \geq 0$$

   We then replace all instances of z with $z' - z''$ in $P^{(2)}$ Thus giving us a new LP $P^{(3)}$.

Notice that all the LP‘s in the sequence 0,1,2,3 are equivalent and since P is equivalent to $P^{(0)}$. This means that it is also equivalent to $P^{(3)}$. And since $P^{(3)}$ is a maximization LP whom constraints all define upper bounds, and has non-negativitey constarints on all its variables this means that it is in Canonical form, so $P = P^{(3)} = P'$)(The equality here denotes equivalence).

Then P' is our Canonical LP such that (im reusing equation numbers, a hard reset):

$$Maximize(-3x + 4w - 2z' + 2z'')$$

Such that:

$$4x - w - 2z' + 2z'' \leq 20 \tag{1}$$

$$-3x - w - z' + z'' \leq -1 \tag{2}$$

$$x + 2w + 2z' - 2z'' \leq -8 \tag{3}$$

$$-x - 2w - 2z' + 2z'' \leq 8 \tag{4}$$

$$4x + w + 3z' - 3z'' \leq 21 \tag{5}$$

$$x \geq 0, w \geq 0, z' \geq 0, z'' \geq 0$$

2 Question 2

An LP is in standard form if it follows all the restrictions of the Canonical form with the changed restriction of having constarints be equality instead of inequalities. We do this by adding a slack variable y_i for the i^{th} constraint.
So LP equivalent to P and P' but in standard form is P'' such that:

$$Maximize(-3x + 4w' - 2z' + 2z'' + d)$$

Such that:

$$4x - w - 2z' + 2z'' + y_1 = 20 \tag{1}$$

$$-3x - w - z' + z'' + y_2 = -1 \tag{2}$$

$$x + 2w + 2z' - 2z'' + y_3 = -8 \tag{3}$$

$$-x - 2w - 2z' + 2z'' + y_4 = 8 \tag{4}$$

$$4x + w + 3z' - 3z'' + y_5 = 21 \tag{5}$$

$$x \geq 0, w \geq 0, z' \geq 0, z'' \geq 0$$

$$y_i \geq 0 \text{ where } 1 \leq i \leq 5 \tag{2.0.1}$$

3 Question 3

b	y_1	y_2	y_3	y_4	y_5	x	w	z'	z''
20	1					4	-1	-2	2
-1		1				-3	-1	-1	1
-8			1			1	2	2	-2
8				1		-1	-2	-2	2
21					1	4	1	3	-3
						-3	4	-2	2

The basic solutions $\{b_1, b_2, b_3, b_4, b_5\}$ are $\{20, -1, -8, 8, 21\}$ respectivley. This solution is not feasible since b_1 and b_2 both have negative values.

4 Question 4

We modify the Original LP by adding a new variable s to form an Auxiliary LP Q:

$$Maximize(-s)$$

Such that:

$$4x - w - 2z' + 2z'' + y_1 - s = 20 \tag{1}$$
$$-3x - w - z' + z'' + y_2 - s = -1 \tag{2}$$
$$x + 2w + 2z' - 2z'' + y_3 - s = -8 \tag{3}$$
$$-x - 2w - 2z' + 2z'' + y_4 - s = 8 \tag{4}$$
$$4x + w + 3z' - 3z'' + y_5 - s = 21 \tag{5}$$
$$x \geq 0, w \geq 0, z' \geq 0, z'' \geq 0, s \geq 0$$
$$y_i \geq 0 \text{ where } 1 \leq i \leq 5$$

Steps:

1. Choose the column whom basic variable corresponds to the lowest basic solution value. In our case the lowest value in b is -8 which corresponds to the variable y_3
2. Swap column of s with y_3, so y_3 leaves the basis and s enters
3. Looking at row (4), multiple the entries in (4) by -1, Then:
   (a) (1) + (4)
   (b) (2) + (4)
   (c) (3) + (4)
   (d) (5) + (4)
   (e) (6) + (4)

   Now we have that Q has a feasible solution, so we continue solving the LP until we find an optimal solution.
4. We choose an arbirtary non-basic variable whom objective function value is non-negative, I'm choosing z'', we then find the row with the $Min(\frac{b_i}{a_{z'',i}})$.
   In this case the min is $\frac{b_5}{a_{w,2}} = \frac{7}{3} = 2.333$
5. Swap columns z'' and y_2 Then perform the basic row operations to restore the identity matrix in the basis.
   (a) (2) / 3
   (b) (1) - 4*(2)
   (c) (3) - 2*(2)
   (d) (4) - 4(2)

    (e) (5) - (2)
    (f) (6) - 4(2)

6. Repeate steps 4 and 5, this time we choose the column x so the $Min(\frac{b_i}{a_{x,i}})$ is $\frac{10}{3} \times \frac{3}{5}$ corresponding to column s
    (a) (3) * 3/5
    (b) (1) - 25/3(3)
    (c) (2) + 4/3(3)
    (d) (4) - 10/3(3)
    (e) (5) - 10/3(3)
    (f) (6) - 4/3(3)
7. Again now with y_3 as the variable entering the basis, the corresponding variable to leave is y_4.
    (a) (1) - 2(4)
    (b) (2) + 3/5(4)
    (c) (3) + 1/5(4)
    (d) (6) - 3/5(4)
8. Again this time w enters the basis, and y_1 leaves.
    (a) (2) + (1)
    (b) (5) + 2(1)
    (c) (6) - 6(1)
9. Again, now y_4 enters and y_3 leaves the basis
    (a) (1) + 2(4)
    (b) (2) + 7/5(4)
    (c) (3) - 1/5(4)
    (d) (5) + 4(4)
    (e) (6) -57/5(4)
10. We now recover the original objective function values of x, w, z', z'' and perform the following row operations to fix the basis:
    (a) (6) - 4(1)
    (b) (6) - 2(2)
    (c) (6) + 3(3)
11. The final LP is: ¡insert table bellow¿

5 Question 5