CSCI-4113
Assignment 2

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Contents

<u>Note:</u> To avoid confusing myself, i redefined y as w so the resulting LP P is:

$$Mazimize(3x + 4w + 2z)$$

Such That:

$$4x + w - 2z \leq 20 \tag{1}$$
$$3x - w + 2 \geq 1 \tag{2}$$
$$x - 2w + 2z = -8 \tag{3}$$
$$4x - w + 3z \leq 21 \tag{4}$$

and

$$x \geq 0$$
$$w \leq 0$$

1 Question 1

Given the LP P that is clearly not in Canonical Form as it has a Minimization Objective Function, Lower Bound and equality Constriants, and negative/unbounded variables.
We define a sequence of equivalent LPs $P^{(i)}, 0 \leq i \leq 3$ such that $P^{(3)} = P'$ is an LP equivalent to P and in Canonical form:

<u>First</u>

We need to define an LP $P^{(0)}$ that is equivalent to P but with the a negated Objective Function for $P^{(0)}$ to be a maximization LP.

1. Since P and $P^{(0)}$ have the same variables and constraints, this means that they have the same set of feasible solutions.
2. Minimizing $f(x)$ is the same as maximizing $f'(x)$, thus P and P' have the same Objective Function value.

$P^{(0)}$ will have an objective Function:

$$Maximize(-3x - 4w - 2z)$$

<u>Second</u>

We now construct a new LP $P^{(1)}$ that is equivalent to $P^{(0)}$ but with no equality constraints.
That is, we replace all the equalities with inequalities.

1. Since $a = b$ is clearly equivalent to $a \leq b$ && $a \geq b$
2. Thus $P^{(0)}$ and $P^{(1)}$ are equivalent since they have the same set of variables, the same objective function and equivalent constraints.

Thus we replace constraint (3) in $P^{(0)}$ with $\downarrow$ in $P^{(1)}$

$$x - 2w + 2z \leq -8 \tag{3.1}$$

$$x - 2w + 2z \geq -8 \tag{3.2}$$

Third

The next step is to replace all the lower bound inequality constraints in $P^{(1)}$ to upper bound constraints, thus constructing a new LP $P^{(2)}$.

1. An inequality $a \leq b$ is equivalent to its negation $-a \geq -b$
2. Just like in the prev step, we are not changing the Objective function and replacing some constraints with equivalent one's thus both $P^{(2)}$ and $P^{(1)}$ have the same Objective Function values and set of feasible solutions and are thus equivalent.

So we replace constraint (1) and (3.2) respectivley with:

$$-3x + w - z \leq -1 \tag{1.1}$$

$$-x + 2w - 2z \leq 8 \tag{3.2.1}$$

This gives us an LP $P^{(2)}$

Fourth

We now need to add non-negativity constarints to $P^{(2)}$.

1. x already has the constraint so we dont need to modify it.
2. w has a non-positivitey constraint, which is equivalent to:

$$-w' \geq 0$$

   Thus we replace every instance of w with $-w'$ (note that $w = w'$) and the constarint $w \leq 0$ with $w' \geq 0$.
3. The variable z is unbounded. We can add a non-negativitey constarint on it by representing it as the differnce of two non-negative numbers. It is clear to see that this holds for any number. So:

$$z = z' - z''$$

   And add the bellow constarints to the LP

$$z' \geq 0, z'' \geq 0$$

   We then replace all instances of z with $z' - z''$ in $P^{(2)}$ Thus giving us a new LP $P^{(3)}$.

Notice that all the LP‘s in the sequence 0,1,2,3 are equivalent and since P is equivalent to $P^{(0)}$. This means that it is also equivalent to $P^{(3)}$. And since $P^{(3)}$ is a maximization LP whom constraints all define upper bounds, and has non-negativitey constarints on all its variables this means that it is in Canonical form, so $P = P^{(3)} = P'$)(The equality here denotes equivalence).

Then P' is our Canonical LP such that (im reusing equation numbers, a hard reset):

$$Maximize(-3x + 4w - 2z' + 2z'')$$

Such that:

$$4x - w - 2z' + 2z'' \leq 20 \tag{1}$$

$$-3x - w - z' + z'' \leq -1 \tag{2}$$

$$x + 2w + 2z' - 2z'' \leq -8 \tag{3}$$

$$-x - 2w - 2z' + 2z'' \leq 8 \tag{4}$$

$$4x + w + 3z' - 3z'' \leq 21 \tag{5}$$

$$x \geq 0, w \geq 0, z' \geq 0, z'' \geq 0$$

2 Question 2

An LP is in standard form if it follows all the restrictions of the Canonical form with the changed restriction of having constarints be equality instead of inequalities. We do this by adding a slack variable y_i for the i^{th} constraint.
So LP equivalent to P and P' but in standard form is P'' such that:

$$Maximize(-3x + 4w' - 2z' + 2z'' + d)$$

Such that:

$$4x - w - 2z' + 2z'' + y_1 = 20 \tag{1}$$

$$-3x - w - z' + z'' + y_2 = -1 \tag{2}$$

$$x + 2w + 2z' - 2z'' + y_3 = -8 \tag{3}$$

$$-x - 2w - 2z' + 2z'' + y_4 = 8 \tag{4}$$

$$4x + w + 3z' - 3z'' + y_5 = 21 \tag{5}$$

$$x \geq 0, w \geq 0, z' \geq 0, z'' \geq 0$$

$$y_i \geq 0 \text{ where } 1 \leq i \leq 5 \tag{2.0.1}$$

3 Question 3

b	y_1	y_2	y_3	y_4	y_5	x	w	z'	z''
20	1					4	-1	-2	2
-1		1				-3	-1	-1	1
-8			1			1	2	2	-2
8				1		-1	-2	-2	2
21					1	4	1	3	-3
						-3	4	-2	2

The basic solutions $\{b_1, b_2, b_3, b_4, b_5\}$ are $\{20, -1, -8, 8, 21\}$ respectivley. This solution is not feasible since b_1 and b_2 both have negative values.

4 Question 4

We modify the Original LP by adding a new variable s to form an Auxiliary LP Q:

$$Maximize(-s)$$

Such that:

$$4x - w - 2z' + 2z'' + y_1 - s = 20 \tag{1}$$
$$-3x - w - z' + z'' + y_2 - s = -1 \tag{2}$$
$$x + 2w + 2z' - 2z'' + y_3 - s = -8 \tag{3}$$
$$-x - 2w - 2z' + 2z'' + y_4 - s = 8 \tag{4}$$
$$4x + w + 3z' - 3z'' + y_5 - s = 21 \tag{5}$$
$$x \geq 0, w \geq 0, z' \geq 0, z'' \geq 0, s \geq 0$$
$$y_i \geq 0 \text{ where } 1 \leq i \leq 5$$

Giving us the following tableau

b	y_1	y_2	y_3	y_4	y_5	x	w	z'	z''	s
20	1					4	-1	-2	2	-1
-1		1				-3	-1	-1	1	-1
-8			1			1	2	2	-2	-1
8				1		-1	-2	-2	2	-1
21					1	4	1	3	-3	-1
						-3	4	-2	2	-1

Steps:

1. Choose the column whom basic variable corresponds to the lowest basic solution value. In our case the lowest value in b is -8 which corresponds to the variable y_3. So we swap them, s enters the basis and y_3 leaves.

2. We now perform the bellow operations to restore the identity matrix:

   (a) (4) * -1

   (b) (1) + (4)

   (c) (2) + (4)

   (d) (3) + (4)

   (e) (5) + (4)

   (f) (6) + (4)

This gives the bellow tableau:

b	y_1	y_2	s	y_4	y_5	x	w	z'	z''	y_3
28	1					3	-3	-4	4	-1
7		1				-4	-3	-3	3	-1
8			1			-1	-2	-2	2	-1
16				1		-2	-4	-4	4	-1
29					1	2	-3	-1	1	-1
8						-4	2	-4	4	-1

Now we have that Q has a feasible solution, so we continue solving the LP until we find an optimal solution.

3. Choose an arbirtary non-basic variable whom objective function value is non-negative, I'm choosing z'', we then find the row with the $Min(\frac{b_i}{a_{z'',i}})$.
   In this case the min is $\frac{b_5}{a_{w,2}} = \frac{7}{3} = 2.333$

   So pivot z'' into the basis and y_2 out. Then perform the basic row operations to restore the identity matrix in the basis.

   (a) (2) / 3
   (b) (1) - 4*(2)
   (c) (3) - 2*(2)
   (d) (4) - 4(2)
   (e) (5) - (2)
   (f) (6) - 4(2)

   Resulting in:

b	y_1	z''	s	y_4	y_5	x	w	z'	y_2	y_3
$\frac{56}{3}$	1					$\frac{25}{3}$	1		$-\frac{4}{3}$	$\frac{1}{3}$
$\frac{7}{3}$		1				$-\frac{4}{3}$	-1	-1	$\frac{1}{3}$	$-\frac{1}{3}$
$\frac{10}{3}$			1			$\frac{5}{3}$			$-\frac{2}{3}$	$-\frac{1}{3}$
$\frac{20}{3}$				1		$\frac{10}{3}$			$-\frac{4}{3}$	$\frac{1}{3}$
$\frac{80}{3}$					1	$\frac{10}{3}$	-2		$-\frac{1}{3}$	$-\frac{2}{3}$
$\frac{10}{3}$						$\frac{4}{3}$	6		$-\frac{4}{3}$	$\frac{1}{3}$

4. Repeate steps 4 and 5, this time we pivot x into the basis and s out.
    (a) (3) * 3/5
    (b) (1) - 25/3(3)
    (c) (2) + 4/3(3)
    (d) (4) - 10/3(3)
    (e) (5) - 10/3(3)
    (f) (6) - 4/3(3)

b	y_1	z''	x	y_4	y_5	s	w	z'	y_2	y_3
2	1						1		2	2
5		1					-1	-1	$-\frac{1}{5}$	$-\frac{3}{5}$
2			1			$\frac{3}{5}$			$-\frac{2}{5}$	$-\frac{1}{5}$
				1		-2				1
20					1	-2	-2		1	
						$-\frac{4}{5}$	6		$-\frac{4}{5}$	$\frac{3}{5}$

5. Again now with y_3 as the variable entering the basis, the corresponding variable to leave is y_4.
    (a) (1) - 2(4)
    (b) (2) + 3/5(4)
    (c) (3) + 1/5(4)
    (d) (6) - 3/5(4)

b	y_1	z''	x	y_3	y_5	s	w	z'	y_2	y_4
2	1					4	1		2	-2
5		1				$-\frac{6}{5}$	-1	-1	$-\frac{1}{5}$	$\frac{3}{5}$
2			1			$\frac{1}{5}$			$-\frac{2}{5}$	$\frac{1}{5}$
				1		-2				1
20					1	-2	-2		1	
						$\frac{2}{5}$	6		$-\frac{4}{5}$	$-\frac{3}{5}$

6. Again this time w enters the basis, and y_1 leaves.
   (a) (2) + (1)
   (b) (5) + 2(1)
   (c) (6) - 6(1)

b	w	z''	x	y_3	y_5	s	y_1	z'	y_2	y_4
2	1					4	1		2	-2
7		1				$\frac{14}{5}$	1	-1	$\frac{9}{5}$	$-\frac{7}{5}$
2			1			$\frac{1}{5}$			$-\frac{2}{5}$	$\frac{1}{5}$
				1		-2				1
24					1	6	2		5	-4
						$-\frac{118}{5}$	-6		$-\frac{64}{5}$	$\frac{57}{5}$

7. Again, now y_4 enters and y_3 leaves the basis
   (a) (1) + 2(4)
   (b) (2) + 7/5(4)
   (c) (3) - 1/5(4)
   (d) (5) + 4(4)
   (e) (6) -57/5(4)

b	w	z''	x	y_4	y_5	s	y_1	z'	y_2	y_3
2	1						1		2	2
7		1					1	-1	$\frac{9}{5}$	$\frac{7}{5}$
2			1			$\frac{3}{5}$		$-\frac{2}{5}$		$-\frac{1}{5}$
				1		-2				1
24					1	-2		2		4
						$-\frac{4}{5}$	-6		$-\frac{64}{5}$	$-\frac{57}{5}$

8. We now recover the original objective function values of x, w, z', z'' and perform the following row operations to fix the basis:
   (a) (6) - 4(1)
   (b) (6) - 2(2)
   (c) (6) + 3(3)
9. The final LP is: ¡insert table bellow¿

5 Question 5

Steps:

1. We only have one non-basic variable with possitive objective function coefficient , so we start by swapping z' and z''.
   (a) (2) * -1
   (b) (6) + 2(2)
2. repeat with z'' entering the basis and z' leaving
   (a) (2) * -1

6 Question 6

Brief Overview/Definitions:

1. At every run, Simplex attempts to increase the value of a non-basic variable by the tightest upper bound imposed on the variable by the constraints.
2. Recall that in the notes, we stated that the feasible region of every individual constarint forms a half space, and since a feasible solution must satisfy all constraints this means that it lies in the intersection of these halfspaces which is a convex polytope. (I focus on the fact that the feasible region is convex)

Proof

Given a tablue in which all the coefficients in the objective function are negative, this means that increasing the value of any of the non-basic variables will reduce the objective function value.

Further, since the feasible region of our tablue is convex this means that a local maxima on the objective function is also a global maxima. Thus the solution can not be "pushed up" any further without breaking any of the constraints. And the BFS at the point where the coefficients of the objective function are all negative is an optimal solution for that tablue.

Finally since Simplex always constructs tablues with the same objective function and equivalent constraints to that of the original LP, means that for a feasible solution of the original LP say z' to have an objective function value larger than $\hat{z}$ implies that the tablue constructed by simplex is not equivalent to the original LP, since we know that $\hat{z}$ is at a golbal maxima of it's feasibility region so z' must have a different feasibility region. And since we only use basic row operations when constructing subsequent tablues we know that this is not true.