

# **CSCI-4113**

## **Assignment 2**

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**Note:** To avoid confusing myself, i redefined  $y$  as  $w$  so the resulting LP  $P$  is:

$$\text{Minimize}(3x + 4w + 2z)$$

Such That:

$$4x + w - 2z \leq 20 \tag{1}$$

$$3x - w + z \geq 1 \tag{2}$$

$$x - 2w + 2z = -8 \tag{3}$$

$$4x - w + 3z \leq 21 \tag{4}$$

and

$$x \geq 0$$

$$w \leq 0$$

## 1 Question 1

Given the LP  $P$  that is clearly not in Canonical Form as it has a Minimization Objective Function, Lower Bound and equality Constraints, and negative/unbounded variables.

We define a sequence of equivalent LPs  $P^{(i)}, 0 \leq i \leq 3$  such that  $P^{(3)} = P'$  is an LP equivalent to  $P$  and in Canonical form:

### First

We need to define an LP  $P^{(0)}$  that is equivalent to  $P$  but with the a negated Objective Function for  $P^{(0)}$  to be a maximization LP.

1. Since  $P$  and  $P^{(0)}$  have the same variables and constraints, this means that they have the same set of feasible solutions.
2. Minimizing  $f(x)$  is the same as maximizing  $f'(x)$ , thus  $P$  and  $P'$  have the same Objective Function value.

$P^{(0)}$  will have an objective Function:

$$\text{Maximize}(-3x - 4w - 2z)$$

### Second

We now construct a new LP  $P^{(1)}$  that is equivalent to  $P^{(0)}$  but with no equality constraints. That is, we replace all the equalities with inequalities.

1. Since  $a = b$  is clearly equivalent to  $a \leq b \ \&\& \ a \geq b$
2. Thus  $P^{(0)}$  and  $P^{(1)}$  are equivalent since they have the same set of variables, the same objective function and equivalent constraints.

Thus we replace constraint (3) in  $P^{(0)}$  with  $\downarrow$  in  $P^{(1)}$

$$x - 2w + 2z \leq -8 \tag{3.1}$$

$$x - 2w + 2z \geq -8 \tag{3.2}$$

### **Third**

The next step is to replace all the lower bound inequality constraints in  $P^{(1)}$  to upper bound constraints, thus constructing a new LP  $P^{(2)}$ .

1. An inequality  $a \leq b$  is equivalent to its negation  $-a \geq -b$
2. Just like in the prev step, we are not changing the Objective function and replacing some constraints with equivalent one's thus both  $P^{(2)}$  and  $P^{(1)}$  have the same Objective Function values and set of feasible solutions and are thus equivalent.

So we replace constraint (1) and (3.2) respectively with:

$$-3x + w - z \leq -1 \quad (1.1)$$

$$-x + 2w - 2z \leq 8 \quad (3.2.1)$$

This gives us an LP  $P^{(2)}$

### **Fourth**

We now need to add non-negativity constraints to  $P^{(2)}$ .

1.  $x$  already has the constraint so we don't need to modify it.
2.  $w$  has a non-positivity constraint, which is equivalent to:

$$-w' \geq 0$$

Thus we replace every instance of  $w$  with  $-w'$  (note that  $w = w'$ ) and the constraint  $w \leq 0$  with  $w' \geq 0$ .

3. The variable  $z$  is unbounded. We can add a non-negativity constraint on it by representing it as the difference of two non-negative numbers. It is clear to see that this holds for any number. So:

$$z = z' - z''$$

And add the below constraints to the LP

$$z' \geq 0, z'' \geq 0$$

We then replace all instances of  $z$  with  $z' - z''$  in  $P^{(2)}$  Thus giving us a new LP  $P^{(3)}$ .

Notice that all the LP's in the sequence 0,1,2,3 are equivalent and since  $P$  is equivalent to  $P^{(0)}$ . This means that it is also equivalent to  $P^{(3)}$ . And since  $P^{(3)}$  is a maximization LP whose constraints all define upper bounds, and has non-negativity constraints on all its variables this means that it is in Canonical form, so  $P = P^{(3)} = P'$  (The equality here denotes equivalence).

Then  $P'$  is our Canonical LP such that (im reusing equation numbers, a hard reset):

$$\text{Maximize}(-3x + 4w - 2z' + 2z'')$$

Such that:

$$4x - w - 2z' + 2z'' \leq 20 \tag{1}$$

$$-3x - w - z' + z'' \leq -1 \tag{2}$$

$$x + 2w + 2z' - 2z'' \leq -8 \tag{3}$$

$$-x - 2w - 2z' + 2z'' \leq 8 \tag{4}$$

$$4x + w + 3z' - 3z'' \leq 21 \tag{5}$$

$$x \geq 0, w \geq 0, z' \geq 0, z'' \geq 0$$

## 2 Question 2

An LP is in standard form if it follows all the restrictions of the Canonical form with the changed restriction of having constraints be equality instead of inequalities. We do this by adding a slack variable  $y_i$  for the  $i^{th}$  constraint.

So LP equivalent to  $P$  and  $P'$  but in standard form is  $P''$  such that:

$$\text{Maximize}(-3x + 4w' - 2z' + 2z'' + d)$$

Such that:

$$4x - w - 2z' + 2z'' + y_1 = 20 \quad (1)$$

$$-3x - w - z' + z'' + y_2 = -1 \quad (2)$$

$$x + 2w + 2z' - 2z'' + y_3 = -8 \quad (3)$$

$$-x - 2w - 2z' + 2z'' + y_4 = 8 \quad (4)$$

$$4x + w + 3z' - 3z'' + y_5 = 21 \quad (5)$$

$$x \geq 0, w \geq 0, z' \geq 0, z'' \geq 0$$

$$y_i \geq 0 \text{ where } 1 \leq i \leq 5 \quad (2.0.1)$$

### 3 Question 3

$b$	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$x$	$w$	$z'$	$z''$
20	1					4	-1	-2	2
-1		1				-3	-1	-1	1
-8			1			1	2	2	-2
8				1		-1	-2	-2	2
21					1	4	1	3	-3
						-3	4	-2	2

The basic solutions  $\{b_1, b_2, b_3, b_4, b_5\}$  are  $\{20, -1, -8, 8, 21\}$  respectively. This solution is not feasible since  $b_1$  and  $b_2$  both have negative values.

## 4 Question 4

We modify the Original LP by adding a new variable  $s$  to form an Auxiliary LP  $Q$ :

$$\text{Maximize}(-s)$$

Such that:

$$4x - w - 2z' + 2z'' + y_1 - s = 20 \quad (1)$$

$$-3x - w - z' + z'' + y_2 - s = -1 \quad (2)$$

$$x + 2w + 2z' - 2z'' + y_3 - s = -8 \quad (3)$$

$$-x - 2w - 2z' + 2z'' + y_4 - s = 8 \quad (4)$$

$$4x + w + 3z' - 3z'' + y_5 - s = 21 \quad (5)$$

$$x \geq 0, w \geq 0, z' \geq 0, z'' \geq 0, s \geq 0$$

$$y_i \geq 0 \text{ where } 1 \leq i \leq 5$$

Giving us the following tableau

$b$	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$x$	$w$	$z'$	$z''$	$s$
20	1					4	-1	-2	2	-1
-1		1				-3	-1	-1	1	-1
-8			1			1	2	2	-2	-1
8				1		-1	-2	-2	2	-1
21					1	4	1	3	-3	-1
						-3	4	-2	2	-1

### Steps:

1. Choose the column whom basic variable corresponds to the lowest basic solution value. In our case the lowest value in  $b$  is  $-8$  which corresponds to the variable  $y_3$ . So we swap them,  $s$  enters the basis and  $y_3$  leaves.
2. We now perform the bellow operations to restore the identity matrix:
  - (a) (4) \* -1
  - (b) (1) + (4)
  - (c) (2) + (4)
  - (d) (3) + (4)
  - (e) (5) + (4)
  - (f) (6) + (4)



This gives the bellow tableau:

$b$	$y_1$	$y_2$	$s$	$y_4$	$y_5$	$x$	$w$	$z'$	$z''$	$y_3$
28	1					3	-3	-4	4	-1
7		1				-4	-3	-3	3	-1
8			1			-1	-2	-2	2	-1
16				1		-2	-4	-4	4	-1
29					1	2	-3	-1	1	-1
8						-4	2	-4	4	-1

Now we have that  $Q$  has a feasible solution, so we continue solving the LP until we find an optimal solution.

3. Choose an arbitrary non-basic variable whom objective function value is non-negative, I'm choosing  $z''$ , we then find the row with the  $Min(\frac{b_i}{a_{z'',i}})$ .

In this case the min is  $\frac{b_5}{a_{w,2}} = \frac{7}{3} = 2.333$

So pivot  $z''$  into the basis and  $y_2$  out. Then perform the basic row operations to restore the identity matrix in the basis.

- (a) (2) / 3
- (b) (1) - 4\*(2)
- (c) (3) - 2\*(2)
- (d) (4) - 4(2)
- (e) (5) - (2)
- (f) (6) - 4(2)

Resulting in:

$b$	$y_1$	$z''$	$s$	$y_4$	$y_5$	$x$	$w$	$z'$	$y_2$	$y_3$
$\frac{56}{3}$	1					$\frac{25}{3}$	1		$-\frac{4}{3}$	$\frac{1}{3}$
$\frac{7}{3}$		1				$-\frac{4}{3}$	-1	-1	$\frac{1}{3}$	$-\frac{1}{3}$
$\frac{10}{3}$			1			$\frac{5}{3}$			$-\frac{2}{3}$	$-\frac{1}{3}$
$\frac{20}{3}$				1		$\frac{10}{3}$			$-\frac{4}{3}$	$\frac{1}{3}$
$\frac{80}{3}$					1	$\frac{10}{3}$	-2		$-\frac{1}{3}$	$-\frac{2}{3}$
$\frac{10}{3}$						$\frac{4}{3}$	6		$-\frac{4}{3}$	$\frac{1}{3}$

4. Repeat steps 4 and 5, this time we pivot  $x$  into the basis and  $s$  out.

- (a) (3) \*  $3/5$
- (b) (1) -  $25/3(3)$
- (c) (2) +  $4/3(3)$
- (d) (4) -  $10/3(3)$
- (e) (5) -  $10/3(3)$
- (f) (6) -  $4/3(3)$

$b$	$y_1$	$z''$	$x$	$y_4$	$y_5$	$s$	$w$	$z'$	$y_2$	$y_3$
2	1						1		2	2
5		1					-1	-1	$-\frac{1}{5}$	$-\frac{3}{5}$
2			1			$\frac{3}{5}$			$-\frac{2}{5}$	$-\frac{1}{5}$
				1		-2				1
20					1	-2	-2		1	
						$-\frac{4}{5}$	6		$-\frac{4}{5}$	$\frac{3}{5}$

5. Again now with  $y_3$  as the variable entering the basis, the corresponding variable to leave is  $y_4$ .

- (a) (1) -  $2(4)$
- (b) (2) +  $3/5(4)$
- (c) (3) +  $1/5(4)$
- (d) (6) -  $3/5(4)$

$b$	$y_1$	$z''$	$x$	$y_3$	$y_5$	$s$	$w$	$z'$	$y_2$	$y_4$
2	1					4	1		2	-2
5		1				$-\frac{6}{5}$	-1	-1	$-\frac{1}{5}$	$\frac{3}{5}$
2			1			$\frac{1}{5}$			$-\frac{2}{5}$	$\frac{1}{5}$
				1		-2				1
20					1	-2	-2		1	
						$\frac{2}{5}$	6		$-\frac{4}{5}$	$-\frac{3}{5}$

6. Again this time  $w$  enters the basis, and  $y_1$  leaves.

- (a)  $(2) + (1)$
- (b)  $(5) + 2(1)$
- (c)  $(6) - 6(1)$

$b$	$w$	$z''$	$x$	$y_3$	$y_5$	$s$	$y_1$	$z'$	$y_2$	$y_4$
2	1					4	1		2	-2
7		1				$\frac{14}{5}$	1	-1	$\frac{9}{5}$	$-\frac{7}{5}$
2			1			$\frac{1}{5}$			$-\frac{2}{5}$	$\frac{1}{5}$
				1		-2				1
24					1	6	2		5	-4
						$-\frac{118}{5}$	-6		$-\frac{64}{5}$	$\frac{57}{5}$

7. Again, now  $y_4$  enters and  $y_3$  leaves the basis

- (a)  $(1) + 2(4)$
- (b)  $(2) + 7/5(4)$
- (c)  $(3) - 1/5(4)$
- (d)  $(5) + 4(4)$
- (e)  $(6) - 57/5(4)$

$b$	$w$	$z''$	$x$	$y_4$	$y_5$	$s$	$y_1$	$z'$	$y_2$	$y_3$
2	1						1		2	2
7		1					1	-1	$\frac{9}{5}$	$\frac{7}{5}$
2			1			$\frac{3}{5}$			$-\frac{2}{5}$	$-\frac{1}{5}$
				1		-2				1
24					1	-2		2		4
						$-\frac{4}{5}$	-6		$-\frac{64}{5}$	$-\frac{57}{5}$

8. We now remove  $s$  and recover the original objective function values of  $x, w, z', z''$  and perform the following row operations to fix the basis:

$b$	$w$	$z''$	$x$	$y_4$	$y_5$	$y_1$	$z'$	$y_2$	$y_3$
2	1					1		2	2
7		1				1	-1	$\frac{9}{5}$	$\frac{7}{5}$
2			1					$-\frac{2}{5}$	$-\frac{1}{5}$
				1					1
24					1		2		4
	4	2	-3			-6	-2	$-\frac{64}{5}$	$-\frac{57}{5}$

9. Finally perform last set of row operations:

(a) (6) - 4(1)

(b) (6) - 2(2)

(c) (6) + 3(3)

The final LP is:

$b$	$w$	$z''$	$x$	$y_4$	$y_5$	$y_1$	$z'$	$y_2$	$y_3$
2	1					1		2	2
7		1				1	-1	$\frac{9}{5}$	$\frac{7}{5}$
2			1					$-\frac{2}{5}$	$-\frac{1}{5}$
				1					1
24					1		2		4
-16						-12		$-\frac{128}{5}$	$-\frac{114}{5}$

## 5 Question 5

### Steps:

- The tableau we get from Q4) can not be optimized any further as we have that all the coefficients in the objective function as negative values.
- The BFS values we get are:  $x = 2, w = 2, z' = 0, z'' = 7$  with the Objective function value being 16.
- The solution of the original LP is:  $3(2) + 4(-2) + 2(0 - 7) = -16$

## 6 Question 6

### Brief Overview/Definitions:

1. At every run, Simplex attempts to increase the value of a non-basic variable by the tightest upper bound imposed on the variable by the constraints.
2. Recall that in the notes, we stated that the feasible region of every individual constraint forms a half space, and since a feasible solution must satisfy all constraints this means that it lies in the intersection of these halfspaces which is a convex polytope. (I focus on the fact that the feasible region is convex)

### Proof

Given a tableau in which all the coefficients in the objective function are negative, this means that increasing the value of any of the non-basic variables will reduce the objective function value.

Further, since the feasible region of our tableau is convex this means that a local maxima on the objective function is also a global maxima. Thus the solution can not be “pushed up” any further without breaking any of the constraints. And the BFS at the point where the coefficients of the objective function are all negative is an optimal solution for that tableau.

Finally since Simplex always constructs tableaux with the same objective function and equivalent constraints to that of the original LP, means that for a feasible solution of the original LP say  $z'$  to have an objective function value larger than  $\hat{z}$  implies that the tableau constructed by simplex is not equivalent to the original LP, since we know that  $\hat{z}$  is at a global maxima of its feasibility region so  $z'$  must have a different feasibility region. And since we only use basic row operations when constructing subsequent tableaux we know that this is not true.