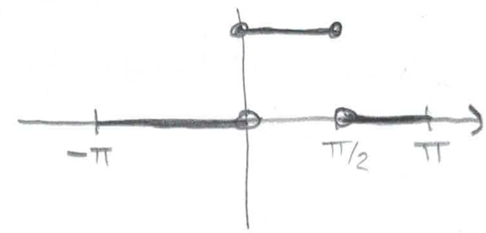
Phys 364 Tutorial #2   Sept 22, 2020

Section 7

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx; \quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx}$$

7.2

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ 1 & 0 < x < \pi/2 \\ 0 & \pi/2 < x < \pi \end{cases}$$



Steps $f(x) =$

1) solve c_0 separately, it is usually a special case!
(why?)

2) solve c_n generally

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi/2} dx = \frac{1}{4}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{\pi/2} e^{-inx}$$

$$= \frac{-1}{in2\pi} e^{-inx}\Big|_0^{\pi/2}$$

$$= \frac{i}{n2\pi}\left[e^{-in\pi/2} - 1\right]$$

$$= \frac{i}{2\pi n}\left[\cos(n\pi/2) - i\sin(n\pi/2) - 1\right] = \frac{1}{2\pi n}\left[\sin(n\pi/2) + i(\cos(n\pi/2) -\right.$$

Now

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-inx} = \sum_{n=-\infty}^{\infty} c_n[\cos(nx) - i\sin(nx)]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n[\cos(nx) - i\sin(nx)] + \sum_{n=-\infty}^{-1} c_n[\cos(nx) - i\sin(nx)]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n[\cos(nx) - i\sin(nx)] + \sum_{n=1}^{\infty} c_{-n}[\cos(nx) + i\sin(nx)]$$

$$= c_0 + \sum_{n=1}^{\infty}(c_n + c_{-n})\cos(nx) + i\sum_{n=1}^{\infty}(c_n - c_{-n})\sin(nx)$$

$$\therefore \quad c_n + c_{-n} = \frac{1}{\pi n}\sin(n\pi/2)$$

$$c_n - c_{-n} = \frac{i}{n\pi}(\cos(n\pi/2) - 1)$$

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin(n\pi/2)\cos(nx) + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}[1-\cos(n\pi/2)]\sin(nx)$$

first few term ($\sin(n\pi/2) = 0$ if n even)

$1-\cos(n\pi/2) = 0$ ~~if n odd~~ if

$$f(x) = \frac{1}{4} + \frac{1}{\pi}\left[\cos(x) - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} + \cdots\right]$$

$$+ \frac{1}{\pi}\left[\sin(x) \frac{2\sin(2x)}{2} + \frac{\sin(3x)}{3} + \cdots\right]$$

which is identical to the old method

Section 8

☆ Important! Read the textbook

- Fourier series over $[-\pi,\pi] \neq [0,2\pi]$ for few reasons

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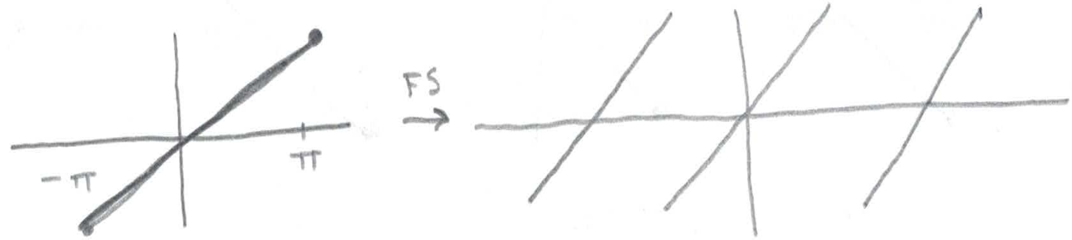
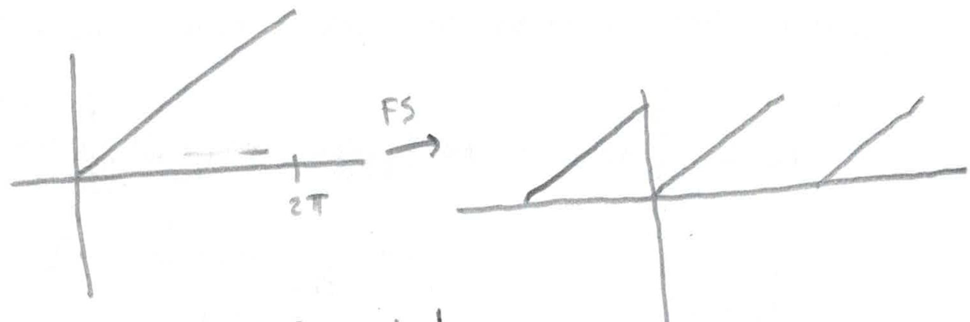




So Fourier extension looks different!

- What about if I also shift by π?

$$\int_{-\pi}^{\pi} f(x) e^{-inx}\,dx \quad \text{v.s.} \quad \int_{0}^{2\pi} f(x-\pi) e^{-inx}\,dx$$

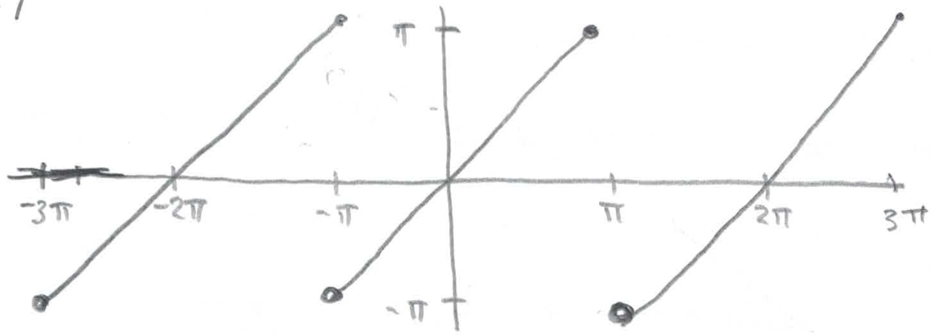
$$\int_{0}^{2\pi} f(x-\pi)e^{-inx}\,dx \underset{y=x+\pi}{=} \int_{0}^{2\pi} f(y-2\pi)\, e^{-iny}\, e^{-in\pi}\,dx$$

$$= \int_{0}^{2\pi} \underbrace{f(y-2\pi)}_{f(y)}\, e^{-iny}\, e^{-in\pi}\,dx$$

So there is a shift! (Also see this for Fourier transforms)

☆ FS are not invariant under translations ☆

10a)



First odd function so we know $a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{1}{\pi}\left(\frac{-x}{n}\cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos(nx)\,dx\right)$$

$u = x$ $\quad dv = \sin(nx)$

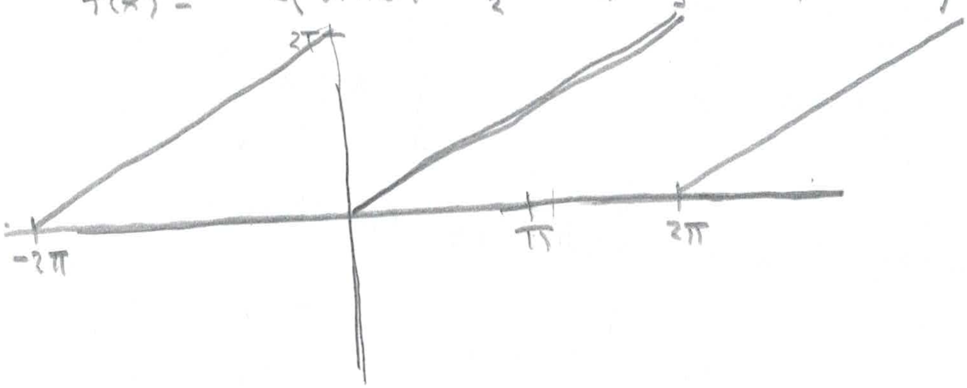
$du = dx$ $\quad v = \frac{-1}{n}\cos(nx)$

$$= -\frac{2}{n}\cos(n\pi) + 0$$

$$= \frac{2}{n}(-1)^{n+1}$$

$$\therefore \quad f(x) = 2\left(\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \cdots\right)$$

b)



Wait this is the previous function shifted to the left by π & up by π so let's use this

The shift up by π implies $a_0 = 1$ but $a_n = 0$ $n > 1$

$$f(x+\pi) = \pi + \sum_{n=1}^{\infty} b_n \sin(n(x+\pi))$$

(plugging in (a))

$$= \pi + \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n}\sin(nx + n\pi)$$

$\sin(nx + n\pi) = \cos(n\pi)\sin(nx)$

$= (-1)^n \sin(nx)$

$$= \pi + \sum_{n=1}^{\infty} \frac{2}{n}\underbrace{(-1)^{n+1}(-1)^n}_{(-1)}\sin(nx)$$

$$= \pi - 2\sum \frac{\sin(nx)}{n}$$

$\square$

Section 9

9.13

$$\int_{-\ell}^{\ell} f(x)dx = \begin{cases} 0 & -f(x) = f(-x) \\ 2\int_0^{\ell} f(x)dx & f(x) = f(-x) \end{cases}$$

Strategy for a proof

1) Start with assumptions (even & odd defn)
2) Use assumptions in equations & start to manipulate

Assump. f odd

$f(x) = -f(-x)$ & $\int_{-\ell}^{\ell} f(x)dx$

$$\int_{-\ell}^{\ell} f(x)dx = \int_0^{\ell} f(x)dx + \int_{-\ell}^{0} f(x)dx$$

$$(y=-x) \quad = \int_0^{\ell} f(x)dx - \int_{\ell}^{0} f(-y)dy$$

$$= \int_0^{\ell} f(x)dx + \int_0^{\ell} f(-y)dy$$

$$(\text{use odd}) \quad = \int_0^{\ell} f(x)dx - \int_0^{\ell} f(y)dy = 0$$

Use same strategy for even!

9.6

$$f(x) = \begin{cases} -1 & -\ell < x < 0 \\ 1 & 0 < x < \ell \end{cases}$$

Already from domain being symmetric across 0 & $f(-x) = f(x)$ on $[-\ell, \ell]$

we know this is a sine series! $a_n = 0$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{\ell}\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right)dx = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{\ell}\right)\Big|_0^{\ell}$$

$$= \frac{-2}{n\pi}\left[\cos(n\pi) - 1\right]$$

$$b_n = \frac{2}{n\pi}\left[1-\cos(n\pi)\right]$$

$$= \frac{2}{n\pi}\begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\sin\left(\frac{1\pi}{2}x\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}x\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}x\right) + \cdots\right]$$

Section 10

$\int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi} = 1+1-2$

10.4

$$f(x)=A\left|\sin(2\pi\nu t)\right| \qquad \nu = 60\text{ Hz},\ A=100$$

$$\Rightarrow\ p=\frac{1}{60}s \ \ \&\ \ \ell=\frac{p}{2}=\frac{1}{120}$$

Now since $f(x)$ is even we can use cosine expansion & $b_n=0$

$n=0$
$$a_0=\frac{2A}{\ell}\int_0^{\ell}\sin\left(\frac{\pi}{\ell}t\right)dt$$
$$=\frac{4A}{\pi}$$

$n>0$
$$a_n=\frac{2A}{\ell}\int_0^{\ell}\sin\left(\frac{\pi}{\ell}t\right)\cos\left(\frac{n\pi}{\ell}t\right)dt$$

$\sin x\cos y=\frac{1}{2}[\sin(x+y)-\sin(x-y)]$

$$=\frac{1A}{\ell}\int_0^{\ell}\left[\sin\left(\frac{\pi(n+1)}{\ell}t\right)-\sin\left(\frac{\pi(n-1)}{\ell}t\right)\right]dt$$
$$=-\frac{A}{\pi}\left[\frac{1}{n+1}\cos\left[\frac{\pi(n+1)}{\ell}t\right]\Big|_0^{\ell}-\frac{1}{n-1}\cos\left[\frac{\pi(n-1)}{\ell}t\right]\Big|_0^{\ell}\right]$$
$$=-\frac{A}{\pi}\left[\frac{\cos[\pi(n+1)]-1}{n+1}-\frac{\cos[\pi(n-1)]-1}{n-1}\right]$$

Now $\cos[\pi(n+1)]=-\cos(\pi n)=\cos[\pi(n-1)]$

so
$$a_n=-\frac{[-\cos(\pi n)-1]}{\pi}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)A$$
$$=+\frac{[-\cos(\pi n)-1]}{\pi}\,\frac{2}{n^2-1}=\begin{cases}0 & n\text{ odd}\\ \frac{-4}{n^2-1}\frac{1}{\pi}A\end{cases}$$

$\longrightarrow n=1$

special case $n=1$
$$a_1=\frac{2}{\ell}\int_0^{\ell}\sin\left(\frac{\pi}{\ell}t\right)\cos\left(\frac{\pi}{\ell}t\right)=\frac{1}{\ell}\int_0^{\ell}\sin\left(\frac{2\pi}{\ell}t\right)dt$$
$$=-\frac{1}{2\pi}\cos\left(\frac{2\pi}{\ell}t\right)\Big|_0^{\ell}=0$$

so

$$\boxed{f(x)=\frac{2A}{\pi}+\frac{4A}{\pi}\sum_{n\,\text{even}}\frac{\cos(2\pi\nu x)}{1-n^2}}$$

$$=\frac{200}{\pi}+\frac{400}{\pi}\sum_{n\,\text{even}}\frac{\cos(120\pi x)}{1-n^2}$$

11.4

Prove Parseval's theorem for $f(x)$ with period 2ℓ

Let

$$\langle f | g \rangle = \frac{1}{2\ell}\int_{-\ell}^{\ell} \bar{f}(x) g(x)\, dx$$

now f is usually real so $\bar{f} = f$

Now

$$\langle \cos(\tfrac{\pi}{\ell}nx) | \sin(\tfrac{\pi}{\ell}nx) \rangle = 0$$

$$\langle \sin(\tfrac{\pi}{\ell}nx) | \sin(\tfrac{\pi}{\ell}mx) \rangle = \tfrac{1}{2}\delta_{mn}$$

$$\langle \cos(\tfrac{\pi}{\ell}nx) | \cos(\tfrac{\pi}{\ell}mx) \rangle = \tfrac{1}{2}\delta_{mn}$$

or 1 if $m=n$

since $\frac{1}{2\ell}\int_{-\ell}^{\ell} f(\tfrac{\pi}{\ell}x) g(\tfrac{\pi}{\ell}x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x')g(x')\,dx'$ $\quad x' = \frac{\pi}{\ell}x$

So now let

$$f(x) = \frac{a_0}{2} + \sum a_n \cos(\tfrac{n\pi}{\ell}x) + \sum b_n \sin(\tfrac{n\pi}{\ell}x)$$

$$\langle f | f \rangle = \left\langle \frac{a_0}{2}\cdot 1 + \sum a_m \cos(\tfrac{m\pi}{\ell}x) + \sum b_m \sin(\tfrac{m\pi}{\ell}x) \,\middle|\, \frac{a_0}{2}\cdot 1 + \sum a_n \cos(\tfrac{n\pi}{\ell}x) + \sum b_n \sin(\tfrac{n\pi}{\ell}x) \right\rangle$$

$$= \left(\frac{a_0}{2}\right)^2 + \frac{a_0}{2}\sum_m \cancel{\langle 1, \cos(\tfrac{m\pi}{\ell}x)\rangle}^{\to 0} + \frac{a_0}{2}\sum_m b_m \cancel{\langle 1, \sin(\tfrac{m\pi}{\ell}x)\rangle}^{\to 0}$$

$$+ (\text{same with } n)^{\to 0}$$

$$+ \sum \cancel{\langle \cos(\tfrac{m\pi}{\ell}x) | \sin(\tfrac{n\pi}{\ell}x)\rangle}^{\to 0} \text{ terms}$$

$$+ \sum_{m,n} a_m a_n \langle \cos(\tfrac{m\pi}{\ell}x) | \cos(\tfrac{n\pi}{\ell}x)\rangle$$

$$+ \sum_{m,n} b_m b_n \langle \sin(\tfrac{m\pi}{\ell}x) | \sin(\tfrac{n\pi}{\ell}x)\rangle$$

$$= \left(\frac{a_0}{2}\right)^2 + \sum_{m,n} a_m a_n \tfrac{1}{2}\delta_{mn} + \sum_{m,n} b_m b_n \tfrac{1}{2}\delta_{mn}$$

$$= \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_n a_n^2 + \frac{1}{2}\sum_b b_n^2$$

$\square$