Phys 364 Tutorial #1 Sept 15 2020

Section II

Find amplitude, period, frequency, and velocity

1) $s(t) = 3\cos 5t$

$A = 3$

$\nu = 5/2\pi \qquad p = \frac{2\pi}{5}$

$\omega = 5$

$v(t) = \dot{s}(t) = -15\sin(5t)$

2) $s(t) = 2\sin(4t - 1)$ ↳ amp. ↳ ang freq ↙ phase

$A = 2$

$\omega = 4$

$p = \frac{2\pi}{\omega} = \frac{\pi}{2}$

$\phi = -1$

$v(t) = 8\cos(4t-1)$

Recall for SHM have

$s(t) = A\cos(\omega t + \phi)$

or $A\sin(\omega t + \phi)$

or $Ae^{i\omega t + i\phi}$

A = amplitude, or max displacement

ω = angular freq,

$p = \frac{2\pi}{\omega}$ = period $(f(x+p) = f(x) \ \forall x)$

$\nu = \frac{\omega}{2\pi}$ = freq.

ϕ = wave phase

$v = \dot{s}$ = velocity of displacement

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7) $z = 5e^{it} = 5\cos t + i5\sin t$

$Re(z) = 5\cos t$

$Im(z) = 5\sin t$

$A = 5$

$\omega = 1$

$p = 2\pi$

$\dot{z} = i5e^{it} = i5\cos t - 5\sin(t)$

$\dot{x} = Re(\dot{z}) = -5\sin(t)$

$\dot{y} = Im(\dot{z}) = 5\cos t$

$\vec{v} = -5\begin{pmatrix}\sin(t)\\ \cos(t)\end{pmatrix}$

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9) $z = 2e^{i\pi t}$

$x = 2\cos\pi t$

$y = 2\sin\pi t$

$A = 2$

$\omega = \pi$

$p = \frac{2\pi}{\omega} = 2$

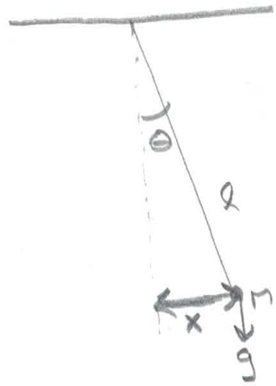
$\nu = 1/2$

Find A^2 through $zz' = 2e^{i\pi t}2e^{-i\pi t}$

$= 4e^0 = 4$

~~Sect~~

14)



$x_1(t) = 4\sin(\frac{\pi}{3}t)$

$x_2(t) = 3\sin(\frac{\pi}{4}t)$

Now $t=0$ $x_1(0) = 0 = x_2(0)$

when is $x_1 = x_2 = 0$ again?

$x_1(t) = 0 = \sin(\frac{\pi}{3}t)$

Now $\sin(x)=0$ when $x = \{n\pi,\ n\in\mathbb{Z}\}$

so $x = \frac{\pi}{3}t = n\pi$

so $t = 3n,\ n\in\mathbb{Z}$

so $x_1(t)=0$ when $t = 0, 3, 6, 9, 12, \ldots$

Similarly

$x_2(t) = 0 = \sin(\frac{\pi}{4}t)$

$t = 4n,\ n\in\mathbb{Z}$

$t = 0, 4, 8, 12$

Therefore when $t=12$ we get $x_1 = x_2 = 0$

Extra: what is the general relation for when

$x_1(t) = x_2(t)$

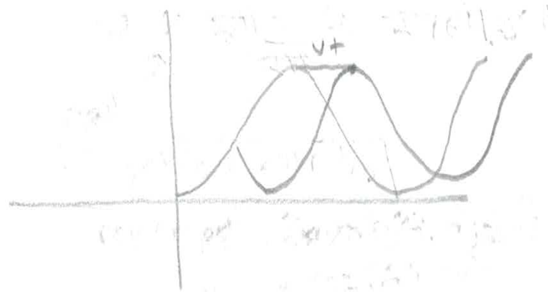
Section II Problem 23

Sound wave has solution

$$s(x,t) = A \sin\left[(x - vt)\frac{2\pi}{\lambda}\right]$$

Want amp = 1 $\nu = 440 s^{-1}$ (vel = 350 m/s)

So we also know $\lambda \nu = c_s$ since



$$\frac{2\pi}{\lambda}(x - vt)$$

suppose x = 0 @ t = 0 Now if wave has wavelength λ and period T we have

$$\cancel{x} - vT$$

must be same point so

$$v = \frac{\lambda}{T} = \lambda \nu$$

Therefore, if $\nu = 440 s^{-1}$ then

$$\frac{2\pi}{\lambda} = \frac{2\pi\nu}{v} = 7.9 m^{-1}$$

so $\boxed{s(x,t) = \sin[7.9(x - 350t)]}$

Section III

8) $Y = (A + B\sin[2\pi f t])\sin[2\pi f_c(t - x/v)]$

Goal 3 wave f_c, $f_c - f$, $f_c + f$

$$Y = A\sin[2\pi f_c(t - x/v)] + B\sin[2\pi f t]\sin[2\pi f_c(t - x/v)]$$

Use Trig iden. Recall

$$e^{i(A+B)} = e^{iA}e^{iB}$$
$$= (\cos(A) + i\sin(A))(\cos(B) + i\sin(B))$$
$$= \cos(A)\cos(B) - \sin(A)\sin(B) + i(\cos(A)\sin(B) + \cos(B)\sin(A))$$

$\therefore$ equating Re & Im

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$
$$\sin(A+B) = \cos(A)\sin(B) - \sin(A)\cos(B)$$

Now want $\sin(A)\sin(B)$ so use cos iden

$$\cos(A+B) - \cos(A-B) = -2\sin(A)\sin(B)$$

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

let $A = 2\pi f_c(t - x/v)$
$B = 2\pi f t$

$A - B = 2\pi(f_c - f)t - 2\pi f_c x/v$
$A + B = 2\pi(f_c + f)t - 2\pi f_c x/v$

$$\therefore \quad Y(t) = A\sin[2\pi f_c(t - x/v)] + \frac{B}{2}\{\cos[(2\pi(f_c - f)t - 2\pi f_c x/v] - \cos[2\pi(f_c + f)t - 2\pi f_c x/v]\}$$

can also further manip to get

$$Y(t) = A\sin[2\pi f_c(t - x/v)] + \frac{B}{2}\{\sin[2\pi(f - f_c)t + 2\pi f_c x/v + \pi/2] + \sin[2\pi(f_c + f)t - 2\pi f_c x/v - \pi/2]\}$$

(exercise)

Section IV

5) $$\langle \cos^2(\tfrac{x}{2}) \rangle_{x\in[0,\pi/2]} = \frac{2}{\pi}\int_0^{\pi/2} \cos^2(\tfrac{x}{2})\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi/2} [1+\cos(\cancel{x})]\,dx$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + \sin(x)\Big|_0^{\pi/2}\right]$$

$$= \frac{1}{2} + \frac{1}{\pi}$$

recall $\left[\cos^2(x) - \sin^2(x) = \cos(2x)\right]$

$\Rightarrow 2\cos^2(x) - 1 = \cos(2x)$

$\Rightarrow \cos^2(x) = \frac{1}{2}(1+\cos(2x))$

$$\boxed{\langle \cos^2(\tfrac{x}{2}) \rangle_{x\in[0,\pi/2]} = \frac{1}{2} + \frac{1}{\pi}}$$

Section V/VI

FS review

- How I think of it

Have functions, f, that obey Dirichlet cond.

- $|f|$ is bounded on $[-\pi, \pi]$ and all subintervals
- f has finitely many discontinuities
- $\int |f| dx < \infty$ (although this also follows from 1.)

Then f can decomposed as F.S.

- Now recall Linear algebra any vector in a vector space V can be rep as linear comb of basis vectors $\hat{e}_n$

$$v = \sum_{n=1}^{N} v_n \hat{e}_n$$

if we also have inner product $\langle v, w \rangle$ (like dot product) then can form orthogonal & orthonormal basis

$$\text{orthog} \quad \langle e_n, e_m \rangle = 0 \text{ if } n \neq m$$

$$\text{orthon} \quad \langle e_n, e_m \rangle = \delta_{nm}$$

Now recall that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\cos(mx)dx = 0 \quad \forall n, m$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \neq 0 \\ 0 & m = n = 0 \end{cases}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx = \begin{cases} 0 & m \neq n \\ \frac{1}{2} & m = n \neq 0 \\ 1 & m = n = 0 \end{cases}$$

so $\cos(mx)$ & $\sin(nx)$ ~~form~~ look like orthogonal

if $\langle f, g \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) g(x) dx$

& in fact you can show they do form a basis of infinite dimensional vector space (Hilbert space) call $L^2([-\pi, \pi])$

So now let

$$e_n^{(cos)}(x) = \cos(nx) \qquad n > 1$$
$$e_n^{(s)}(x) = \sin(nx) \qquad n > 1$$
$$e_n(x) = 1$$

then we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n e_n^{(c)} + \sum_{n=1}^{\infty} b_n e_n^{(s)}$$

So how do we find a_n & b_n, well use your knowlegde from Linear algebra! Take an inner product of f with $e_n^{(c)}$ or $e_n^{(s)}$!

$m > 0$

$$\langle e_m^{(c)}, f\rangle = \langle e_m^{(c)}, a_0 e_0(x) + \sum_{n=1}^{\infty} a_n e_n^{(c)} + \sum_{n=1}^{\infty} b_n e_n^{(s)}\rangle$$

$\langle \cdot, \cdot \rangle$ is bilinear $$= a_0 \cancel{\langle e_m^{(c)}, e_0\rangle}^{0} + \underbrace{\sum_{n=1}^{\infty} a_n \langle e_m^{(c)}, e_n^{(c)}\rangle}_{\frac{a_n}{2}\delta_{mn}} + \sum_{n=1}^{\infty} b_n \cancel{\langle e_m^{(c)}, e_m^{(s)}\rangle}^{0}$$

$$= \frac{a_m}{2}$$

$$\therefore \quad a_m = 2\langle e_m^{(c)}, f\rangle$$
$$= \frac{2}{2\pi}\int_{-\pi}^{\pi} \cos(mx) f(x)\,dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(mx) f(x)\,dx$$

Aside

You've seen this before! Think of particle in a infinite well. Then $e_n(x)$ are actually the "eigenfunctions" that solve

$$-\frac{d^2}{dx^2} f = \lambda f$$

or

$$Mf = \lambda f$$

where $$M = \frac{-d^2}{dx}$$

Almost The entire course is just finding different basis functions for different eigenvalue problems! For example Legendre, Bessel. are all examples.

$$\boxed{L^{M} f = \lambda f}$$ remember