ME561/EECS 561

Winter 2021 - Homework 5

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Problem 1: [12 point(s)]

Find the modified Z transform of the following functions by hand.

1.1 [6 point(s)]

$$E(s) = \frac{20}{(s+2)(s+5)}$$

1.2 [6 point(s)]

$$E(s) = \frac{5}{s(s+1)}$$

1.1

E(s) = $\frac{20}{(s+2)(s+5)}$

Because $Zm(f,m) = z^{-1}\sum_{at\ every\ pole\ of\ F(s)} Residue\ of\ \frac{F(s)e^{mTs}}{1-z^{-1}e^{Ts}}, pole$

poles are

s= -2,-5

At pole = -2 residue is:

$$= \frac{20e^{mT\lambda}}{(\lambda+5)(1-z^{-1}e^{T\lambda})}|_{\lambda=-2}$$

$$= \frac{20e^{-2mT}}{(3)(1-z^{-1}e^{-2T})}$$

At pole = -5 residue is:

$$= \frac{20e^{mT\lambda}}{(\lambda+2)(1-z^{-1}e^{T\lambda})}|_{\lambda=-5}$$

$$= \frac{20e^{-5mT}}{(-3)(1-z^{-1}e^{-5T})}$$

$$E(z,m) = z^{-1}\left(\frac{20e^{-2mT}}{(3)(1-z^{-1}e^{-2T})} + \frac{20e^{-5mT}}{(-3)(1-z^{-1}e^{-5T})}\right) = \frac{20}{3}\left(\frac{z^{-1}e^{-2mT}}{(1-z^{-1}e^{-2T})} - \frac{z^{-1}e^{-5mT}}{(1-z^{-1}e^{-5T})}\right)$$

$$= \frac{20}{3}\left(\frac{e^{-2mT}}{(z-e^{-2T})} - \frac{e^{-5mT}}{(z-e^{-5T})}\right)$$

1.2

E(s) = $\frac{5}{s(s+1)}$

Because $Zm(f,m) = z^{-1}\sum_{at\ every\ pole\ of\ F(s)} Residue\ of\ \frac{F(s)e^{mTs}}{1-z^{-1}e^{Ts}}, pole$

poles are

s= 0,-1

At pole = 0 residue is:

$$= \frac{5e^{mT\lambda}}{(\lambda+1)(1-z^{-1}e^{T\lambda})}|_{\lambda=0}$$

$$= \frac{5e^{0}}{(1)(1-z^{-1}e^{0})} = \frac{5}{(1-z^{-1})}$$

At pole = -1 residue is:

$$= \frac{5e^{mT\lambda}}{(\lambda)(1-z^{-1}e^{T\lambda})}|_{\lambda=-1}$$

$$= \frac{5e^{-mT}}{(-1)(1-z^{-1}e^{-T})} = \frac{-5e^{-mT}}{(1-z^{-1}e^{-T})}$$

$$E(z,m) = z^{-1}\left(\frac{5}{(1-z^{-1})} + \frac{-5e^{-mT}}{(1-z^{-1}e^{-T})}\right) = 5z^{-1}(\frac{1}{(1-z^{-1})} - \frac{e^{-mT}}{(1-z^{-1}e^{-T})})$$

$$= 5(\frac{z^{-1}}{(1-z^{-1})} - \frac{z^{-1}e^{-mT}}{(1-z^{-1}e^{-T})}) = 5(\frac{1}{(z-1)} - \frac{e^{-mT}}{(z-e^{-T})})$$

Problem 2: [12 point(s)]

Find the Z transform of the following functions by hand. The results from 1 may be helpful.

2.1 [6 point(s)]

$$E(s) = \frac{20e^{-0.3Ts}}{(s+2)(s+5)}$$

2.2 [6 point(s)]

$$E(s) = \frac{5e^{-0.6Ts}}{s(s+1)}$$

2.1

Because $e^{-0.3Ts}$ is shifting the Z transform by (1-m)T where m = 0.7. Therefore,

$$E(z) = E(z, 0.7) = \frac{20}{3}\left(\frac{e^{-2*0.7T}}{(z - e^{-2T})} - \frac{e^{-5*0.7T}}{(z - e^{-5T})}\right) = \frac{20}{3}\left(\frac{e^{-1.4T}}{(z - e^{-2T})} - \frac{e^{-3.5T}}{(z - e^{-5T})}\right)$$

2.2

Because $e^{-0.6Ts}$ is shifting the Z transform by (1-m)T where m = 0.4. Therefore,

$$E(z) = E(z,m) = z^{-1}\left(\frac{5}{(1-z^{-1})} + \frac{-5e^{-0.4*T}}{(1-z^{-1}e^{-T})}\right) = 5z^{-1}(\frac{1}{(1-z^{-1})} - \frac{e^{-0.4T}}{(1-z^{-1}e^{-T})})$$

$$= 5(\frac{z^{-1}}{(1-z^{-1})} - \frac{z^{-1}e^{-0.4T}}{(1-z^{-1}e^{-T})}) = 5(\frac{1}{(z-1)} - \frac{e^{-0.4T}}{(z-e^{-T})})$$

Problem 3: [21 point(s)]

Consider the following system in Fig. 1. (Note that epsilon in the diagram is equivalent to e. $\mathcal{E}^{-Ts} = e^{-Ts}$)

3.1 [9 point(s)] Find the output $c(kT)$ by hand for the system for $e(t)$ equal to a unit-step function.

3.2 [6 point(s)] Explain the effect of the sampler and data hold on $c(kT)$ in the upper path.

3.3 [6 point(s)] Sketch the unit-response $c(t)$ of the system. This sketch can be made without mathematically solving for $C(s)$.

3.1

Because e(t) is equal to a unit-step function:

e(t) = u(t) = $\begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$

Therefore:

E(s) $= \frac{1}{s}$

Upper path:

X(s) $= E(s) * \frac{2}{s} = \frac{2}{s^2}$

pole is s = 0:

At pole=0 residue is:

$$\frac{d}{ds}\left(\frac{2}{(1-e^{-T(s-\lambda)})}\right)|_{\lambda=0}$$

$$= \frac{2e^{-T(s-\lambda)}T}{(1-e^{-T(s-\lambda)})^2}|_{\lambda=0}$$

$$= \frac{2e^{-Ts}T}{(1-e^{-Ts})^2}$$

Therefore,

$$X^*(s) = \frac{2e^{-Ts}T}{(1-e^{-Ts})^2}$$

$$C(s)_{upper} = X^*(s) * ZOH(s) = \frac{2e^{-Ts}T}{(1-e^{-Ts})^2} * \frac{1-e^{-Ts}}{s} = \frac{2e^{-Ts}T}{1-e^{-Ts}} * \frac{1}{s}$$

$We\ set$ $\frac{2e^{-Ts}T}{1-e^{-Ts}}$ as $F^*(s)$ and $\frac{1}{s}$ $as\ B(s)$

$$C(s)_{upper} = F^*(s)B(s)$$

$F^*(s) = \frac{2e^{-Ts}T}{1-e^{-Ts}} \rightarrow F(z) = \frac{2z^{-1}T}{1-z^{-1}}$, B(s) $= \frac{1}{s}$

pole is s = 0:

At pole=0 residue is:

$\frac{1}{(1-e^{-T(s-\lambda)})}|_{\lambda=0}$

$$= \frac{1}{(1-e^{-T(s)})}$$

$$B^*(s) = \frac{1}{(1-e^{-T(s)})}$$

Hence, B(z) $= \frac{1}{1-z^{-1}}$

Therefore,

$$C(z)_{upper} = F(z)B(z) = \frac{2z^{-1}T}{1-z^{-1}} * \frac{1}{1-z^{-1}} = \frac{2z^{-1}T}{(1-z^{-1})^2}$$

Lower path:

Because e(t) is equal to a unit-step function:

e(t) = u(t) = $\begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$

Therefore:

E(s) $= \frac{1}{s}$

pole is s = 0:

At pole=0 residue is:

$\frac{1}{(1-e^{-T(s-\lambda)})}|_{\lambda=0}$

$$= \frac{1}{1-e^{-Ts}}$$

Therefore,

$$E^*(s) = \frac{1}{1-e^{-Ts}}$$

$$C(s)_{lower} = E^*(s) * \frac{1-e^{-Ts}}{s} * \frac{2}{s} = \frac{1}{1-e^{-Ts}} * \frac{1-e^{-Ts}}{s} * \frac{2}{s} = \frac{2}{s^2}$$

$C(s)_{lower} = \frac{2}{s^2}$

pole is s = 0:

At pole=0 residue is:

$\frac{d}{ds}\left(\frac{2}{(1-e^{-T(s-\lambda)})}\right)|_{\lambda=0}$

$= \frac{2e^{-T(s-\lambda)}T}{(1-e^{-T(s-\lambda)})^2}|_{\lambda=0}$

$= \frac{2e^{-Ts}T}{(1-e^{-Ts})^2}$

Hence,

$C^*(s)_{lower} = \frac{2e^{-Ts}T}{\left(1-e^{-Ts}\right)^2} = C(z)_{lower}|_{z=e^{sT}} \rightarrow C(z)_{lower} = \frac{2z^{-1}T}{(1-z^{-1})^2}$

Therefore,

C(z) = $C(z)_{lower} - C(z)_{upper} = \frac{2z^{-1}T}{(1-z^{-1})^2} - \frac{2z^{-1}T}{(1-z^{-1})^2} = 0$

Hence, c(kT) = 0

3.2

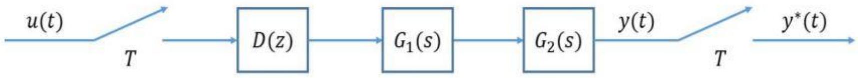
There is no effect on c(kT) of the sampler and on the data hold in the upper path because the upper path will cancel out the lower path. As the sampling period in the upper path of the system and lower path of the system is equal to that making the output c(kT) equal to zero. And also it sampling the output of E(s).

3.3

A straight line only as $c(t) = 0$

Problem 4: [30 point(s)]

Consider the following system with sampling period $T = 1$s.



4.1 [2 point(s)] List those transfer functions in the block diagram that contain the transfer function of a data hold.

4.2 [8 point(s)] Suppose

$$D(z) = \frac{z-1}{z}, \quad G_1(s)G_2(s) = \frac{s(s+1)}{s^2+2s+2} \text{ (not including data hold)}, \quad u(t) = t$$

and we use a zero-order hold. Solve for $y(t)$ by hand.

4.3 [6 point(s)] Find the pulse transfer function of the whole system by hand.

4.4 [6 point(s)] Using the pulse transfer function, find $y^*(t)$ by hand.

4.5 [6 point(s)] Sketch $y(t)$ and $y^*(t)$.

4.6 [2 point(s)] The dc gain is defined as the steady-state output when the input is the Heaviside function. What is the dc gain of the system?

4.1

$G_1(s)$ *contain the transfer function of a data hold.*

4.2

$$y(s) = \left(G_2(s)G_1(s)\left(\frac{1-e^{-Ts}}{s}\right)\right)D^*(s)\,U^*(s)$$

$$= \left(G_2(s)G_1(s)\left(\frac{1-e^{-Ts}}{s}\right)\right)D^*(s)\,U^*(s)$$

$$D(z) = \frac{z-1}{z} \to D^*(s) = \frac{e^{sT}-1}{e^{sT}}$$

$$u(t) = t \to U(s) = \frac{1}{s^2}$$

$$U^*(s) = \frac{e^{-Ts}T}{(1-e^{-Ts})^2}$$

$$y(s) = \left(\frac{s(s+1)}{s^2+2s+2}\left(\frac{1-e^{-Ts}}{s}\right)\right)D^*(s)\,U^*(s) = \left(\frac{s(s+1)}{s^2+2s+2}\left(\frac{1-e^{-Ts}}{s}\right)\right)\frac{e^{sT}-1}{e^{sT}} * \frac{e^{-Ts}T}{(1-e^{-Ts})^2}$$

$$y(s) = \left(\frac{s(s+1)}{s^2+2s+2}\left(\frac{1-e^{-Ts}}{s}\right)\right)\frac{e^{sT}-1}{e^{sT}} * \frac{e^{-Ts}T}{(1-e^{-Ts})^2}$$

$$= \left(\frac{(s+1)}{s^2+2s+2}\right)\frac{e^{sT}-1}{e^{sT}} * \frac{e^{-Ts}T}{1-e^{-Ts}}$$

$$= \left(\frac{(s+1)}{s^2+2s+2}\right)\frac{e^{sT}-1}{e^{sT}} * \frac{T}{e^{Ts}-1}$$

$$= \left(\frac{(s+1)}{s^2+2s+2}\right)\frac{1}{e^{sT}} * \frac{T}{1}$$

$$= \left(\frac{(s+1)}{s^2+2s+2}\right)\frac{T}{e^{sT}}$$

Because T = 1s.

Therefore,

$$y(s) = \left(\frac{(s+1)}{s^2+2s+2}\right)\frac{1}{e^s}$$

$$= \left(\frac{(s+1)}{s^2+2s+2}\right) e^{-s}$$

$$= \frac{(s+1)}{(s+1)^2+1}\ e^{-s}$$

$$y(t) = L^{-1}\{y(s)\}$$

Because $L^{-1}\left\{\frac{(s+1)}{(s+1)^2+1}\right\} = e^{-t}\cos(t)$

$$L\{f(t-a)S(t-a)\} = e^{-as}F(s)$$

Hence,

$$y(t) = e^{-(t-1)}\cos(t-1)\,S(t-1)$$

4.3

$$G(z) = \frac{y(z)}{U(z)}$$

$$u(t) = t \ \rightarrow U(s) = \frac{1}{s^2}$$

$$U^*(s) = \frac{e^{-Ts}T}{(1-e^{-Ts})^2}$$

$$U(z) = \ \frac{z^{-1}T}{(1-z^{-1})^2} = \frac{z^{-1}T}{1-2z^{-1}+z^{-2}} = \frac{zT}{z^2-2z+1} = \frac{zT}{(z-1)^2}$$

$$D(z) = \frac{z-1}{z}$$

$$y(z) = \ Z\{G_1(s)G_2(s) * ZOH\}D(z)\,U(z)$$

$$G_1(s)G_2(s) * ZOH = \frac{s(s+1)}{s^2+2s+2}\left(\frac{1-e^{-Ts}}{s}\right) = \frac{(s+1)}{s^2+2s+2} * (1-e^{-Ts})$$

Let B(s) $=\frac{(s+1)}{s^2+2s+2}$ $and\ F^*(s) = \ 1-e^{-Ts} \ \rightarrow F(z) = 1-z^{-1}$

$$L^{-1}(B(s)) = \ L^{-1}\left\{\frac{(s+1)}{s^2+2s+2}\right\} = e^{-t}\cos(t)$$

Therefore,

Z { $B(s)$} = B(z)

$$= \frac{z(z-e^{-T}\cos(T))}{z^2-2ze^{-T}\cos(T)+e^{-2T}}$$

Hence,

$$y(z) = \frac{z\left(z-e^{-T}\cos(T)\right)}{z^2-2ze^{-T}\cos(T)+e^{-2T}} * (1-z^{-1}) * \frac{z-1}{z} * \frac{zT}{(z-1)^2}$$

$$y(z) = \frac{z\left(z-e^{-T}\cos(T)\right)}{z^2-2ze^{-T}\cos(T)+e^{-2T}} * (1-z^{-1}) * \frac{1}{1} * \frac{T}{z-1}$$

$$= \frac{z\left(z-e^{-T}\cos(T)\right)}{z^2-2ze^{-T}\cos(T)+e^{-2T}} * (1-z^{-1}) * \frac{1}{1} * \frac{z^{-1}T}{1-z^{-1}}$$

$$= \frac{z\left(z-e^{-T}\cos(T)\right)}{z^2-2ze^{-T}\cos(T)+e^{-2T}} * \frac{T}{z} = \frac{T\left(z-e^{-T}\cos(T)\right)}{z^2-2ze^{-T}\cos(T)+e^{-2T}}$$

$$G(z) = \frac{T\left(z-e^{-T}\cos(T)\right)}{z^2-2ze^{-T}\cos(T)+e^{-2T}} * \frac{(z-1)^2}{zT} = \frac{\left(z-e^{-T}\cos(T)\right)}{z^2-2ze^{-T}\cos(T)+e^{-2T}} * \frac{(z-1)^2}{z}$$

$$= \frac{(z-1)^2(1-e^{-T}\cos(T)\,z^{-1})}{z^2-2ze^{-T}\cos(T)+e^{-2T}}$$

Because T =1s,

$$G(z) = \frac{(z-1)^2(1-e^{-1}\cos(1)\,z^{-1})}{z^2-2ze^{-1}\cos(1)+e^{-2}}$$

4.4

$$G(z) = \frac{(z-1)^2(1-e^{-T}\cos(T)\,z^{-1})}{z^2 - 2ze^{-T}\cos(T) + e^{-2T}}$$

y(z) = $G(z) * U(z) = \frac{(z-1)^2\left(1-e^{-T}\cos(T)z^{-1}\right)}{z^2-2ze^{-T}\cos(T)+e^{-2T}} * \frac{zT}{(z-1)^2} = \frac{T\left(z-e^{-T}\cos(T)\right)}{z^2-2ze^{-T}\cos(T)+e^{-2T}}$

Therefore,

$$y^*(s) = \frac{T\left(e^{sT} - e^{-T}\cos(T)\right)}{e^{2sT} - 2e^{s}\cos(T) + e^{-2T}}$$

Because T=1s,

$$y^*(s) = \frac{\left(e^{s} - e^{-1}\cos(1)\right)}{e^{2s} - 2e^{s}\cos(1) + e^{-2}}$$

$y^*(t) \ = L^{-1}\{y^*(s)\}$

4.5

4.6

$$G(z) = \frac{(z-1)^2(1-e^{-T}\cos(T)\,z^{-1})}{z^2-2ze^{-T}\cos(T)+e^{-2T}}$$

In order to find DC gain:

$$G(z)_{z=1} = G(1) = \frac{(1-1)^2(1-e^{-T}\cos(T)\,z^{-1})}{z^2-2ze^{-T}\cos(T)+e^{-2T}} = 0$$

For this problem consider the values

$$m = 300[\mathrm{kg}] \quad k = 18000[\mathrm{N/m}] \quad b = 1200[\mathrm{N \cdot s/m}]$$

Suppose we want to know the response of the system given the input $u = \dot{r}(t)$. In the Fig. 3, $D(z)$ is the digital filter, and $G(s)$ is the transfer function of the system. For the sake of simplicity, let's assume $D(z) = 1$, and sampling period $T = 1\mathrm{s}$. Answer the following questions:

5.1 [6 point(s)] Find the (analog) transfer function $G_0(s) = G_{\mathrm{ZOH}} \cdot G(s)$.

5.2 [6 point(s)] Find the Z transform of $G_0(s)$.

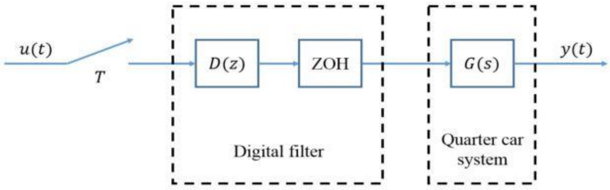


Figure 3: Signal flow of system

5.3 [7 point(s)] Find the discrete-time, state space representation of the system. Remember in homework 1, we proved:

$$A_d = e^{AT}, \quad B_d = \left(\int_0^T e^{Av} dv\right) B, \quad C_d = C$$

5.4 [6 point(s)] Find the transfer function of the discretized system.

5.1

A=$\begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ b/m \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$,where m =300[kg] ,k = 18000 [N/m], b=1200[N$\cdot$ s/m]

Hence,

A=$\begin{bmatrix} 0 & 1 \\ -60 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

G(s) $= C(sI - A)^{-1}B = \frac{4s}{s^2+4s+60} + \frac{60}{s^2+4s+60} = \frac{4s+60}{s^2+4s+60}$:from MATLAB

D(z) = 1 → $D^*(s) = 1$

$$G_{ZOH} = \left(\frac{1 - e^{-Ts}}{s}\right)$$

$$G_0(s) = \left(\frac{1 - e^{-Ts}}{s}\right) \frac{4s + 60}{s^2 + 4s + 60}$$

5.2

$$G_0(s) = \left(\frac{1-e^{-Ts}}{s}\right)\frac{4s+60}{s^2+4s+60} = (1-e^{-Ts})\frac{4s+60}{s(s^2+4s+60)}$$

We set $F^*(s) = 1-e^{-Ts} \rightarrow F(z) = 1-z^{-1}$

We set B(s) $= \frac{4s+60}{s(s^2+4s+60)}$,

$G_0(z) = F(z)B(z)$

By using MATLAB:

The z-transform of $B(s)$ *is* :

$$B(z) = \frac{z}{z-1} + \frac{z * e^2 * (\cos(2\sqrt{14}) - ze^2)}{z^2 * e^4 - 2 * z * \cos(2\sqrt{14})\, e^2 + 1} + \frac{z * \sqrt{14}\sin(2\sqrt{14}) * e^2}{14 * (z^2 * e^4 - 2 * z * \cos(2\sqrt{14})\, e^2 + 1)}$$

Hence,

$$\begin{aligned}
G_0(z) &= (1-z^{-1})\Big(\frac{z}{z-1} + \frac{z * e^2 * (\cos(2\sqrt{14}) - ze^2)}{z^2 * e^4 - 2 * z * \cos(2\sqrt{14})\, e^2 + 1} \\
&\qquad + \frac{z * \sqrt{14}\sin(2\sqrt{14}) * e^2}{14 * (z^2 * e^4 - 2 * z * \cos(2\sqrt{14})\, e^2 + 1)}\Big) \\
&= (1-z^{-1})\Big(\frac{1}{1-z^{-1}} + \frac{z * e^2 * (\cos(2\sqrt{14}) - ze^2)}{z^2 * e^4 - 2 * z * \cos(2\sqrt{14})\, e^2 + 1} \\
&\qquad + \frac{z * \sqrt{14}\sin(2\sqrt{14}) * e^2}{14 * (z^2 * e^4 - 2 * z * \cos(2\sqrt{14})\, e^2 + 1)}\Big) \\
&= 1 + (1-z^{-1})\Big(\frac{z * e^2 * (\cos(2\sqrt{14}) - ze^2)}{z^2 * e^4 - 2 * z * \cos(2\sqrt{14})\, e^2 + 1} + \frac{z * \sqrt{14}\sin(2\sqrt{14}) * e^2}{14 * (z^2 * e^4 - 2 * z * \cos(2\sqrt{14})\, e^2 + 1)}\Big)
\end{aligned}$$

5.3

A=$\begin{bmatrix} 0 & 1 \\ -60 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$

$A_d = e^{AT}$

T =1s

From MATLAB:

$$A_d = e^{A} = \begin{bmatrix} 0.0827 & 0.0169 \\ -1.0114 & 0.0153 \end{bmatrix}$$

$$C_d = C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$B_d = \left(\int_0^T e^{Av} dv \right) B = \begin{bmatrix} -0.0169 \\ 0.9847 \end{bmatrix}$$

Discrete-time state space representation of the system:

$$\begin{bmatrix} \dot{p} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} 0.0827 & 0.0169 \\ -1.0114 & 0.0153 \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix} + \begin{bmatrix} -0.0169 \\ 0.9847 \end{bmatrix} \dot{r}(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ w \end{bmatrix}$$

5.4

$$A_d = e^A = \begin{bmatrix} 0.0827 & 0.0169 \\ -1.0114 & 0.0153 \end{bmatrix}$$

$$C_d = C = [0 \quad 1]$$

$$B_d = \left(\int_0^T e^{Av} dv\right) B = \begin{bmatrix} -0.0169 \\ 0.9847 \end{bmatrix}$$

$$H(z) = C_d (zI - A_d)^{-1} B_d$$

From MATLAB:

$$H(z) = \frac{0.9847z - 0.0644}{z^2 - 0.0980z + 0.0183}$$