Problems and Applications: Bernoulli Distribution

Let Y be a Bernoulli random variable with success probability $\Pr(Y = 1) = p$, and let $Y_1, \ldots, Y_n$ be i.i.d. draws from this distribution. Let $\hat{p}$ be the fraction of successes (1s) in this sample.

1. Show that $\hat{p} = \overline{Y}$.

   $\hat{p}$ is the fraction of successes in this sample, so

$$\hat{p} = \frac{1}{n} \cdot (Y_1 + \ldots + Y_n)$$

   The mean $\overline{Y}$ is the expected value of Y,

$$\overline{Y} = \mathrm{E}[Y] = \frac{1}{n} \cdot (\mathrm{E}[Y_1] + \ldots \mathrm{E}[Y_n])$$

   And since $\hat{p}$ is a fixed sample value,

$$\hat{p} = \mathrm{E}[\hat{p}] = \frac{1}{n} \cdot (\mathrm{E}[Y_1] + \ldots \mathrm{E}[Y_n]) = \overline{Y}$$

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2. Show that $\hat{p}$ is an unbiased estimator of p.

   The random variable Y_i takes value 1 with probability p and 0 with probability $1 - p$,

$$\mathrm{E}[Y_i] = p \cdot 1 + (1 - p) \cdot 0 = p$$

   By definition of the fraction of successes $\hat{p}$,

$$\begin{aligned} \hat{p} &= \frac{1}{n} \cdot (Y_1 + \ldots + Y_n) \\ \Longrightarrow \quad \mathrm{E}[\hat{p}] &= \frac{1}{n} \cdot (\mathrm{E}[Y_1] + \ldots + \mathrm{E}[Y_n]) \\ &= \frac{1}{n} \cdot (p + \ldots + p) \\ &= \frac{1}{n} \cdot n \cdot p = p \quad \Longrightarrow \quad \mathrm{E}[\hat{p} - p] = 0 \end{aligned}$$

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3. Show that $\mathrm{var}(\hat{p}) = p(1 - p)/n$.

$$\begin{aligned} \mathrm{var}(\hat{p}) = \mathrm{var}(\overline{Y}) = \mathrm{var}\left(\frac{1}{n}(Y_1 + \ldots + Y_n)\right) \\ &= \frac{1}{n^2} \cdot \mathrm{var}(Y_1 + \ldots + Y_n) && \text{because } n \text{ is constant} \\ &= \frac{1}{n^2} \cdot (\mathrm{var}(Y_1) + \ldots + \mathrm{var}(Y_n)) && \text{because the } Y_i\text{s are i.i.d.} \\ &= \frac{1}{n^2} \cdot (\mathrm{var}(Y) + \ldots + \mathrm{var}(Y)) && \text{because the } Y_i\text{s are i.i.d.} \\ &= \frac{1}{n^2} \cdot n \cdot \mathrm{var}(Y) \\ &= \frac{1}{n} \cdot \mathrm{var}(Y) \end{aligned}$$

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4. Show that $\mathrm{var}(\hat{p}) = p(1 - p)/n$.

   Now compute $\mathrm{var}(Y)$

$$\begin{aligned} \mathrm{E}[Y] &= p \cdot 1 \ + (1 - p) \cdot 0 \ = p \\ \mathrm{E}[Y^2] &= p \cdot 1^2 + (1 - p) \cdot 0^2 = p \\ \mathrm{var}(Y) &= \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2 = p - p^2 = p(1 - p) \\ \Longrightarrow \quad \mathrm{var}(\hat{p}) = \mathrm{var}(\overline{Y}) &= \frac{\mathrm{var}(Y)}{n} = \frac{p(1 - p)}{n} \end{aligned}$$