Confidence Intervals

A researcher, using data on class size (CS) and average test scores from 100 third-grade classes, estimates the OLS regression:

$$\widehat{TestScore} = \underset{(20.4)}{520.4} - \underset{(2.21)}{5.82} \times CS, \quad R^2 = 0.08, \quad SER = 11.5$$

a. Construct a 95% confidence interval for β_1, the regression slope coefficient.

b. Calculate the p-value for the two-sided test of the null hypothesis H_0: $\beta_1 = 0$. Do you reject the null hypothesis at the 5% level? At the 1% level?

c. Calculate the p-value for the two-sided test of the null hypothesis $H_0: \beta_1 = -5.6$. Without doing any additional calculations, determine whether -5.6 is contained in the 95% confidence interval for β_1.

d. Construct a 99% confidence interval for β_0.

Confidence Intervals

$$\widehat{TestScore} = \underset{(20.4)}{520.4} - \underset{(2.21)}{5.82} \times CS, \quad R^2 = 0.08, \quad SER = 11.5$$

a. Construct a 95% confidence interval for β_1, the regression slope coefficient.

A two-tailed $\alpha\%$ confidence interval for β_1 is:

$$\hat{\beta}_1 \pm t_{\alpha/2} \cdot \text{SE}(\hat{\beta}_1)$$
$$= -5.82 \pm t_{\alpha/2} \cdot 2.21$$

As the sample size is $n = 100$, the Student-t distribution with $n - 2 = 98$ degrees of freedom is reasonably well approximated by the standard normal distribution. The critical t- or z-value may be read from a probability table or computed with, say, the R command `qt(1-0.05/2,df=98)` or `qnorm(1-0.05/2)`.

A two-tailed 95% confidence interval for β_1 is:

$-10.15 < \beta_1 < -1.49$: based on the standard normal distribution $z_{0.05/2} \approx 1.96$

$-10.20 < \beta_1 < -1.44$: based on the Student-t distribution $t_{0.05/2}(98) \approx 1.98$

Confidence Intervals

$$\widehat{TestScore} = \underset{(20.4)}{520.4} - \underset{(2.21)}{5.82} \times CS, \quad R^2 = 0.08, \quad SER = 11.5$$

b. Calculate the p-value for the two-sided test of the null hypothesis H_0: $\beta_1 = 0$. Do you reject the null hypothesis at the 5% level? At the 1% level?

The test statistic associated with H_0 is:

$$t_0 = \frac{\hat{\beta}_1 - \beta_{1,0}}{\text{SE}(\hat{\beta}_1)} = \frac{-5.82 - 0}{2.21} \approx -2.63$$

The two-sided p-value is therefore:

$$p\text{-value} = 2\Phi(t_0) \approx 0.0085$$

The p-value may be computed with `2*pnorm(-2.63)`.

Since the p-value is smaller than one percent, we reject the null hypothesis at the 5% significance level and at the 1% level.

Confidence Intervals

$$\widehat{TestScore} = \underset{(20.4)}{520.4} - \underset{(2.21)}{5.82} \times CS, \quad R^2 = 0.08, \quad SER = 11.5$$

c. Calculate the p-value for the two-sided test of the null hypothesis $H_0: \beta_1 = -5.6$. Without doing any additional calculations, determine whether -5.6 is contained in the 95% confidence interval for β_1.

The test statistic associated with H_0 is:

$$t_0 = \frac{\hat{\beta}_1 - \beta_{1,0}}{\text{SE}(\hat{\beta}_1)} = \frac{-5.82 + 5.6}{2.21} \approx -0.10$$

The two-sided p-value is therefore:

$$p\text{-value} = 2\Phi(t_0) \approx 0.92$$

The p-value may be computed with `2*pnorm(-0.10)`. The p-value is large and we cannot reject the null hypothesis at the usual significance levels. Since we cannot reject the null at the 5% significance level, -5.6 is contained in the 95% confidence interval for β_1.

Confidence Intervals

$$\widehat{TestScore} = \underset{(20.4)}{520.4} - \underset{(2.21)}{5.82} \times CS, \quad R^2 = 0.08, \quad SER = 11.5$$

d. Construct a 99% confidence interval for β_0.

A two-tailed $\alpha\%$ confidence interval for β_0 is:

$$\begin{aligned} &\hat{\beta}_0 \pm t_{\alpha/2} \cdot \text{SE}(\hat{\beta}_0) \\ &= 520.4 \pm t_{\alpha/2} \cdot 20.4 \end{aligned}$$

As the sample size is $n = 100$, the Student-t distribution with $n - 2 = 98$ degrees of freedom is reasonably well approximated by the standard normal distribution. The critical t-value, $t_{\alpha/2} \approx 2.5758$, may be read from a probability table or computed with the R command `qt(1-0.01/2,df=98)` or `qnorm(1-0.01/2)`.

A two-tailed 99% confidence interval for β_0 is:

$467.85 < \beta_0 < 572.95$: based on the standard normal distribution $z_{0.01/2} \approx 2.58$

$466.81 < \beta_0 < 573.99$: based on the Student-t distribution $t_{0.01/2}(98) \approx 2.63$