Review of Probability: Card Games

Dr. Patrick Toche

Textbook:

James H. Stock and Mark W. Watson, *Introduction to Econometrics*, 4th Edition, Pearson.

Other references:

Joshua D. Angrist and Jörn-Steffen Pischke, *Mostly Harmless Econometrics: An Empiricist's Companion*, 1st Edition, Princeton University Press.
Jeffrey M. Wooldridge, *Introductory Econometrics: A Modern Approach*, 7th Edition, Cengage Learning.

The textbook comes with online resources and study guides. Other references will be given from time to time.

Probability of a Hand

Five cards are randomly selected from a standard deck. What is the probability of drawing the Four kings and the queen of hearts (in any order)?

- A standard problem of drawing without replacement! Which of these solutions is correct?

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- **Solution 1**
  - The probability of drawing the 4 kings is:

$$\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49}$$

  - There is only one queen of hearts and 48 cards left after the kings have been drawn. The probability of drawing 4 kings and the queen of hearts is therefore:

$$\frac{4}{52} \times \frac{3}{51} \times \frac{2}{50} \times \frac{1}{49} \times \frac{1}{48} = \frac{24}{311875200} \approx 0.000000077$$

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- **Solution 2**
  - There are $\binom{52}{5}$ ways of selecting 5 cards from the deck.
  - There are exactly $\binom{4}{4}$ ways of selecting the 4 kings.
  - Of the 48 remaining cards, there are $\binom{48}{1}$ ways to select the fifth card.
  - The probability is therefore:

$$\frac{\binom{4}{4}\binom{48}{1}}{\binom{52}{5}} = \frac{1}{54145} \approx 0.000018$$

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- **Solution 3**

  - The probability of drawing 4 kings and the queen of hearts is equal to the probability of selecting 5 cards:

$$\frac{5}{52} \times \frac{4}{51} \times \frac{3}{50} \times \frac{2}{49} \times \frac{1}{48} = \frac{1}{2598960} \approx 0.00000038$$

Probability of a Hand

- Recall that the number of order-free combinations is:
$$\binom{n}{k} = {}^{n}C_{k} = \frac{n!}{k!(n-k)!}$$
- Solution 1 gives the probability of drawing four kings first, in any order, and the queen of hearts in one given position. As there are 5 positions the queen of hearts could be in, Solution 1 is too small by a factor of 5.
- Solution 2 gives the probability of drawing four kings and any other card (not specifically the queen of hearts). Solution 2 is too large by a factor of 48.
- Solution 3 gives the correct probability. There are $\binom{52}{5}$ ways to choose 5 cards from the deck, and only one of those ways is the event of interest. The probability is therefore:
$$\frac{1}{\binom{52}{5}} = \frac{5! \times 47!}{52!} \approx 0.00000038$$

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- Equivalently, the correct answer can be broken down as follows:

$$\frac{\binom{4}{4}\binom{1}{1}}{\binom{52}{5}} = \frac{1}{\binom{52}{5}}$$

  The numerator is the number of ways to choose 4 kings from the 4 available kings, $\binom{4}{4}$, multiplied by the number of ways to choose 1 queen of heart from the only available queen of heart $\binom{1}{1}$. Thus, there is only 1 such 5-card hand.
- The probability of drawing the four kings and the queen of hearts is therefore equal to the probability of drawing any *specific* hand, e.g. ace of heart, three of diamonds, seven of clubs, nine of diamonds, jack of spades.
- The probability of drawing the four kings and any queen (whether hearts, diamonds, spades or clubs) is 4 times as large (replace $\binom{1}{1}$ by $\binom{4}{1}$ in the formula above).
- The answers are related as follows:

$$\begin{aligned}\frac{1}{2598960} &= 5 \times \frac{24}{311875200} \\ &= \frac{1}{48} \times \frac{1}{54145}\end{aligned}$$

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