Probability Distribution

Let Y denote the number of "heads" that occur when two coins are tossed.

1. Derive the probability distribution of Y.

Let the sample space be $\Omega = \{(H,H),(H,T),(T,H),(T,T)\}$, with associated probabilities

$$\Pr[(H,H)] = \Pr[(H,T)] = \Pr[(T,H)] = \Pr[(H,H)] = \frac{1}{4}$$

We have $Y = 0$ if (T,T); $Y = 1$ if (H,T) or (T,H); and $Y = 2$ if (H,H). The sample space is $\Omega_Y = \{0,1,2\}$ with probability distribution:

$$\Pr[Y=0] = \frac{1}{4}$$
$$\Pr[Y=1] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$\Pr[Y=2] = \frac{1}{4}$$

Importantly, check that $\Pr[Y=0] + \Pr[Y=1] + Pr[Y=2] = 1$.

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2. Derive the cumulative probability distribution of Y.

The cumulative distribution adds up the probabilities for each outcome from $Y = 0$ to $Y = 2$:

$$\Pr[Y=0] = \frac{1}{4}$$
$$\Pr[Y=1 \text{ or } Y=0] = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$
$$\Pr[Y=2 \text{ or } Y=1 \text{ or } Y=0] = \frac{1}{4} + \frac{3}{4} = 1$$

By an abuse of notation, we sometimes write $\Pr[Y<0] = 0$ and $\Pr[Y<\infty] = 1$.

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3. Derive the mean and variance of Y.

Intuitively (but beware of intuitions!) we guess $\mathrm{E}[Y] = 1$, which is easy to check:

$$\mathrm{E}[Y] = \frac{1}{4}\cdot 0 + \frac{1}{2}\cdot 1 + \frac{1}{4}\cdot 2 = 1$$

The variance may be calculated directly:

$$\mathrm{var}[Y] = \frac{1}{4}\cdot(0-1)^2 + \frac{1}{2}\cdot(1-1)^2 + \frac{1}{4}\cdot(2-1)^2 = \frac{1}{2}$$

Or indirectly:

$$\begin{aligned}\mathrm{var}[Y] &= \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2 \\ &= \frac{1}{4}\cdot 0^2 + \frac{1}{2}\cdot 1^2 + \frac{1}{4}\cdot 2^2 - 1^2 = \frac{3}{2} - 1 = \frac{1}{2} \quad ✔\end{aligned}$$