Review of Probability: Distribution Function

Dr. Patrick Toche

Textbook:

> **James H. Stock and Mark W. Watson**, *Introduction to Econometrics*, 4th Edition, Pearson.

Other references:

> **Joshua D. Angrist and Jörn-Steffen Pischke**, *Mostly Harmless Econometrics: An Empiricist's Companion*, 1st Edition, Princeton University Press.
>
> **Jeffrey M. Wooldridge**, *Introductory Econometrics: A Modern Approach*, 7th Edition, Cengage Learning.

The textbook comes with online resources and study guides. Other references will be given from time to time.

Probability Distribution

Let Y denote the number of "heads" that occur when two coins are tossed.

1. Derive the probability distribution of Y.

Let the sample space be $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$, with associated probabilities

$$\Pr[(H, H)] = \Pr[(H, T)] = \Pr[(T, H)] = \Pr[(H, H)] = \frac{1}{4}$$

We have $Y = 0$ if (T, T); $Y = 1$ if (H, T) or (T, H); and $Y = 2$ if (H, H). The sample space is $\Omega_Y = \{0, 1, 2\}$ with probability distribution:

$$\Pr[Y = 0] = \frac{1}{4}$$
$$\Pr[Y = 1] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$\Pr[Y = 2] = \frac{1}{4}$$

Importantly, check that $\Pr[Y = 0] + \Pr[Y = 1] + Pr[Y = 2] = 1$.

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Probability Distribution

Let Y denote the number of "heads" that occur when two coins are tossed.

2. Derive the cumulative probability distribution of Y.

The cumulative distribution adds up the probabilities for each outcome from $Y = 0$ to $Y = 2$:

$$\Pr[Y = 0] = \frac{1}{4}$$

$$\Pr[Y = 1 \text{ or } Y = 0] = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

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By an abuse of notation, we sometimes write $\Pr[Y < 0] = 0$ and $\Pr[Y < \infty] = 1$.

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Probability Distribution

Let Y denote the number of "heads" that occur when two coins are tossed.

3. Derive the mean and variance of Y.

Intuitively (but beware of intuitions!) we guess $E[Y] = 1$, which is easy to check:

$$E[Y] = \frac{1}{4} \cdot 0 + \frac{1}{2} \cdot 1 + \frac{1}{4} \cdot 2 = 1$$

The variance may be calculated directly:

$$var[Y] = \frac{1}{4} \cdot (0 - 1)^2 + \frac{1}{2} \cdot (1 - 1)^2 + \frac{1}{4} \cdot (2 - 1)^2 = \frac{1}{2}$$

Or indirectly:

$$var[Y] = E[Y^2] - (E[Y])^2$$
$$= \frac{1}{4} \cdot 0^2 + \frac{1}{2} \cdot 1^2 + \frac{1}{4} \cdot 2^2 - 1^2 = \frac{3}{2} - 1 = \frac{1}{2} \checkmark$$

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