Correlations

a. Show that the regression R^2 in the regression of Y on X is the squared value of the sample correlation between X and Y. That is, show that $R^2 = r^2_{XY}$.

Pearson's sample correlation coefficient r_{XY} (population is denoted ρ_{XY}) is

$$r_{XY} = \frac{s_{XY}}{s_X s_Y}$$

The OLS estimator may be written:

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^n (X_i - \overline{X})^2} = \frac{s_{XY}}{s_X^2}$$

where s_{XY} denotes the sample covariance and s_X^2 denotes the variance, $s_{XY} = s_X^2$. The proof of this is standard and follows from minimizing the sum of squared residuals of the regression of Y on X.

The coefficient of determination of the regression of Y on X is

$$R^2 = 1 - \frac{\sum_{i=1}^n \hat{u}_i^2}{\sum_{i=1}^n (Y_i - \overline{Y})^2} = 1 - \frac{\sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2}{s_{YY}}$$

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$$\begin{aligned}\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1 \overline{X} \implies \sum_{i=1}^n (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)^2 &= \sum_{i=1}^n [(Y_i - \overline{Y}) - \hat{\beta}_1 (X_i - \overline{X})])^2 \\ &= \sum_{i=1}^n (Y_i - \overline{Y})^2 - 2\hat{\beta}_1 \sum_{i=1}^n (X_i - \overline{X})(Y_i - \overline{Y}) + \hat{\beta}_1^2 \sum_{i=1}^n (X_i - \overline{X})^2 \\ &= s_{YY} - 2\hat{\beta}_1 s_{XY} + \hat{\beta}_1^2 s_{XX}\end{aligned}$$

Plugging the above into the coefficient of determination:

$$\begin{aligned}R^2 &= 1 - \frac{s_{YY} - 2\hat{\beta}_1 s_{XY} + \hat{\beta}_1^2 s_{XX}}{s_{YY}} = 2\hat{\beta}_1 \frac{s_{XY}}{s_{YY}} - \hat{\beta}_1^2 \frac{s_{XX}}{s_{YY}} \\ &= 2\frac{s_{XY}}{s_X^2}\frac{s_{XY}}{s_{YY}} - \left(\frac{s_{XY}}{s_X^2}\right)^2 \frac{s_{XX}}{s_{YY}} = \frac{s_{XY}^2}{s_X^2 s_Y^2} = r^2_{XY}\end{aligned}$$

Correlations

b. Show that the R^2 from the regression of Y on X is the same as the R^2 from the regression of X on Y.

Since we have shown that $R^2 = r^2_{XY}$, it follows from $s_{XY} = s_{YX}$:

$$R^2 \text{ of } Y \text{ on } X = r^2_{XY} = \frac{s_{XY}}{s_X s_Y} = \frac{s_{YX}}{s_Y s_X} = r^2_{YX} = R^2 \text{ of } X \text{ on } Y$$

Correlations

c. Show that $\hat{\beta}_1 = r_{XY}(s_Y/s_X)$, where r_{XY} is the sample correlation between X and Y, and s_X and s_Y are the sample standard deviations of X and Y.

We've done most of the work already:

$$\hat{\beta}_1 = \frac{s_{XY}}{s_X^2} = \frac{r_{XY} s_X s_Y}{s_X^2} = \frac{r_{XY} s_Y}{s_X}$$