2021 AMC 10A Problems/Problem 22

Contents

Problem

Hiram's algebra notes are 50 pages long and are printed on 25 sheets of paper; the first sheet contains pages 1 and 2, the second sheet contains pages 3 and 4, and so on. One day he leaves his notes on the table before leaving for lunch, and his roommate decides to borrow some pages from the middle of the notes. When Hiram comes back, he discovers that his roommate has taken a consecutive set of sheets from the notes and that the average (mean) of the page numbers on all remaining sheets is exactly 19. How many sheets were borrowed?

$\textbf{(A)}\ 10\qquad\textbf{(B)}\ 13\qquad\textbf{(C)}\ 15\qquad\textbf{(D)}\ 17\qquad\textbf{(E)}\ 20$

Solution 1

Suppose the roommate took sheets a through b, or equivalently, page numbers $2a-1$ through $2b$. Because there are $(2b-2a+2)$ numbers taken,

$$\frac{(2a-1+2b)(2b-2a+2)}{2}+19(50-(2b-2a+2))=\frac{50\cdot 51}{2} \implies (2a+2b-39)(b-a+1)=\frac{50\cdot 13}{2}=25\cdot 13.$$

The first possible solution that comes to mind is if $2a+2b-39=25, b-a+1=13 \implies a+b=32, b-a=12$, which indeed works, giving $b=22$ and $a=10$. The answer is $22-10+1=\boxed{\textbf{(B)}\ 13}$.

~Lcz

Solution 2

Suppose the smallest page number borrowed is $k,$ and n pages are borrowed. It follows that the largest page number borrowed is $k+n-1.$

We have the following preconditions:

1. n pages are borrowed means that $\frac{n}{2}$ sheets are borrowed, from which n must be even.
2. k must be odd, as the smallest page number borrowed is on the right side (odd-numbered).
3. $1+2+3+\cdots+50=\frac{51(50)}{2}=1275.$
4. The sum of the page numbers borrowed is $\frac{(2k+n-1)n}{2}.$

Together, we have

$$\begin{aligned}
\frac{1275-\frac{(2k+n-1)n}{2}}{50-n} &= 19 \\
1275-\frac{(2k+n-1)n}{2} &= 19(50-n) \\
2550-(2k+n-1)n &= 38(50-n) \\
2550-(2k+n-1)n &= 1900-38n \\
650 &= (2k+n-39)n.
\end{aligned}$$

The factors of 650 are

$$1,2,5,10,13,25,26,50,65,130,325,650.$$

Since n is even, we only have a few cases to consider:

$\boldsymbol{n}$	$\boldsymbol{2k+n-39}$	$\boldsymbol{k}$
2	325	181
10	65	47
26	25	19
50	13	1
130	5	-43
650	1	-305

Since $1\leq k\leq 49,$ only $k=47,19,1$ are possible:

- If $k=47,$ then there will not be sufficient pages when we take 10 pages out starting from page $47.$
- If $k=1,$ then the average page number of all remaining sheets will be undefined, as there will be no sheets remaining after we take 50 pages (25 sheets) out starting from page $1.$

Therefore, the only possibility is $k=19.$ We conclude that $n=26$ pages, or $\frac{n}{2}=\boxed{\textbf{(B) } 13}$ sheets, are borrowed.

~MRENTHUSIASM

Solution 3

Let n be the number of sheets borrowed, with an average page number $k+25.5$. The remaining $25-n$ sheets have an average page number of 19 which is less than 25.5, the average page number of all 50 pages, therefore $k>0$. Since the borrowed sheets start with an odd page number and end with an even page number we have $k\in\mathbb{N}$. We notice that $n<25$ and $k\leq(49+50)/2-25.5=24<25$.

The weighted increase of average page number from 25.5 to $k+25.5$ should be equal to the weighted decrease of average page number from 25.5 to 19, where the weights are the page number in each group (borrowed vs. remained), therefore

$$2nk=2(25-n)(25.5-19)=13(25-n)\implies 13|n \text{ or } 13|k$$

Since $n,k<25$ we have either $n=13$ or $k=13$. If $n=13$ then $k=6$. If $k=13$ then $2n=25-n$ which is impossible. Therefore the answer should be $n=\boxed{\textbf{(B) } 13}$.

~asops

Solution 4

Let $(2k-1)-2n$ be pages be borrowed, the sum of the page numbers on those pages is $(2n+2k+1)(n-k)$

while the sum of the rest pages is $1275-(2n+2k+1)(n-k)$ and we know the average of the rest is $\frac{1275-(2n+2k+1)}{50-2n+2k}$ which equals to 19; multiply this out we got $950-38(n-k)=1275-(2n+2k+1)(n-k)$ and we got $(2n+2k-37)(n-k)=325$. As $325=25\cdot 13$, we can see $n-k=13$ and that is desired $\boxed{\textbf{(B)}\ 13}$.

~bluesoul

Video Solution by OmegaLearn (Arithmetic Sequences and System of Equations)

https://youtu.be/dWOLIdTxwa4

~ pi_is_3.14

Video Solution by MRENTHUSIASM (English & Chinese)

https://www.youtube.com/watch?v=28te8OUiVxE

~MRENTHUSIASM

See also

2021 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
Preceded by **Problem 21**	Followed by **Problem 23**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	