2021 Fall AMC 10B Problems/Problem 15

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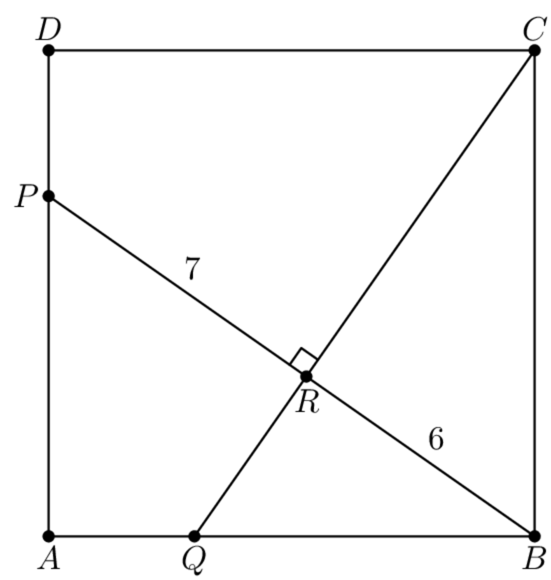
Problem

In square $ABCD$, points P and Q lie on $\overline{AD}$ and $\overline{AB}$, respectively. Segments $\overline{BP}$ and $\overline{CQ}$ intersect at right angles at R, with $BR = 6$ and $PR = 7$. What is the area of the square?



$\textbf{(A) }85\qquad\textbf{(B) }93\qquad\textbf{(C) }100\qquad\textbf{(D) }117\qquad\textbf{(E) }125$

Solution 1

Note that $\triangle APB \cong \triangle BQC.$ Then, it follows that $\overline{PB} \cong \overline{QC}.$ Thus, $QC = PB = PR + RB = 7 + 6 = 13.$ Define x to be the length of side $CR,$ then $RQ = 13 - x.$ Because $\overline{BR}$ is the altitude of the triangle, we can use the property that $QR \cdot RC = BR^2.$ Substituting the given lengths, we have

$$(13 - x) \cdot x = 36.$$

Solving, gives $x = 4$ and $x = 9.$ We eliminate the possibilty of $x = 4$ because $RC > QR.$ Thus, the side length of the square, by Pythagorean Theorem, is

$$\sqrt{9^2 + 6^2} = \sqrt{81 + 36} = \sqrt{117}.$$

Thus, the area of the sqaure is $(\sqrt{117})^2 = 117$. Thus, the answer is $\boxed{\textbf{(D) } 117}$.

~NH14

Solution 2 (Similarity, Pythagorean Theorem, and Systems of Equations

As above, note that $\triangle BPA \cong \triangle CQB$, which means that $QC = 13$. In addition, note that BR is the altitude of a right triangle to its hypotenuse, so $\triangle BQR \sim \triangle CBR \sim \triangle CQB$. Let the side length of the square be x; using similarity side ratios of $\triangle BQR$ to $\triangle CQB$, we get

$$\frac{6}{x} = \frac{QB}{13} \implies QB \cdot x = 78$$

Note that $QB^2 + x^2 = 13^2 = 169$ by the Pythagorean theorem, so we can use the expansion $(a+b)^2 = a^2 + 2ab + b^2$ to produce two equations and two variables;

$$(QB+x)^2 = QB^2 + 2QB \cdot x + x^2 \implies (QB+x)^2 = 169 + 2 \cdot 78 \implies QB + x = \sqrt{13(13) + 13(12)} = \sqrt{13 \cdot 25} = 5\sqrt{13}$$

$$(QB-x)^2 = QB^2 - 2QB \cdot x + x^2 \implies (QB-x)^2 = 169 - 2 \cdot 78 \implies QB - x = \sqrt{13(13) - 13(12} = \sqrt{13 \cdot 1} = \sqrt{13}$$

We want x^2, so we want to find x. Subtracting the first equation from the second, we get

$$2x = 6\sqrt{13} \implies x = 3\sqrt{13}$$

Then x^2 = $(3\sqrt{13}^2) = 9 \cdot 13 = 117 = \boxed{D}$

~KingRavi

Solution 3

We have that $\triangle CRB \sim \triangle BAP$. Thus, $\frac{\overline{CB}}{\overline{CR}} = \frac{\overline{PB}}{\overline{AB}}$. Now, let the side length of the square be s. Then, by the Pythagorean theorem, $CR = \sqrt{x^2 - 36}$. Plugging all of this information in, we get

$$\frac{s}{\sqrt{s^2 - 36}} = \frac{13}{s}.$$

Simplifying gives

$$s^2 = 13\sqrt{s^2 - 36},$$

Squaring both sides gives

$$s^4 = 169s^2 - 169 \cdot 36 \implies s^4 - 169s^2 + 169 \cdot 36 = 0.$$

We now set $s^2 = t$, and get the equation $t^2 - 169t + 169 \cdot 36 = 0$. From here, notice we want to solve for t, as it is precisely s^2, or the area of the square. So we use the Quadratic formula, and though it may seem bashy, we hope for a nice cancellation of terms.

$$t = \frac{169 \pm \sqrt{169^2 - 4 \cdot 36 \cdot 169}}{2}.$$

It seems scary, but factoring 169 from the square root gives us

$$\frac{169 \pm \sqrt{169 \cdot (169 - 144)}}{2} = \frac{169 \pm \sqrt{169 \cdot 25}}{2} = \frac{169 \pm 13 \cdot 5}{2} = \frac{169 \pm 65}{2}$$

$$t = \frac{169 \pm \sqrt{169 \cdot (169-144)}}{2} = \frac{169 \pm \sqrt{169 \cdot 25}}{2} = \frac{169 \pm 13 \cdot 5}{2} = \frac{169 \pm 65}{2},$$

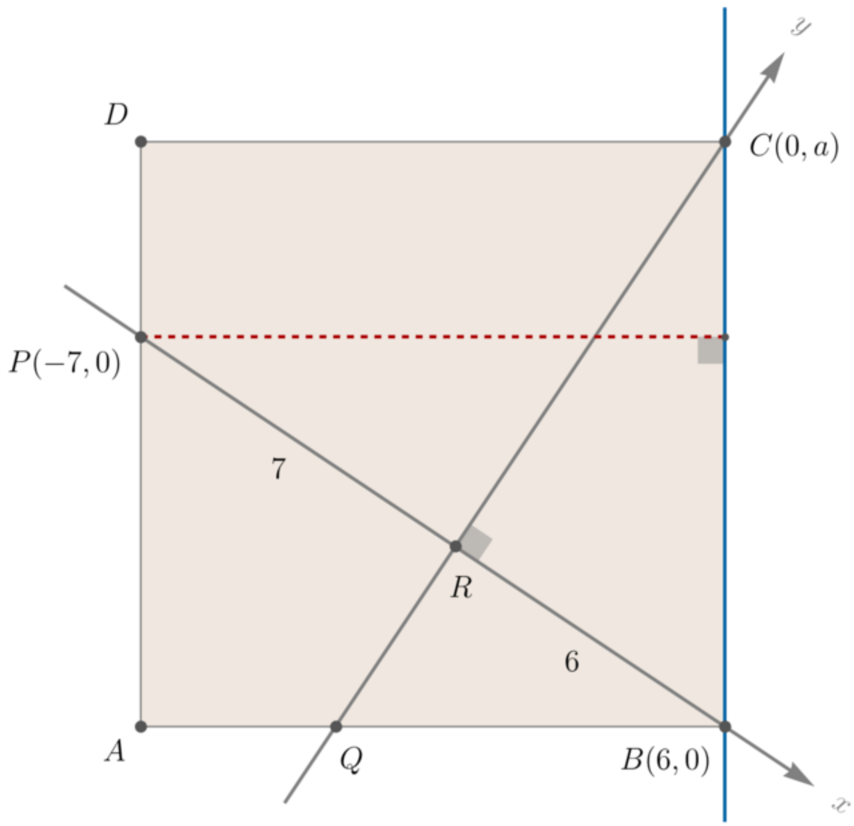
giving us the solutions

$$t = 52, 117.$$

We instantly see that $t = 52$ is way too small to be an area of this square (52 isn't even an answer choice, so you can skip this step if out of time) because then the side length would be $2\sqrt{13}$ and then, even the largest line you can draw inside the square (the diagonal) is $2\sqrt{26},$ which is less than 13 (line PB) And thus, t must be 117, and our answer is $\boxed{\textbf{(D)}}$. $\blacksquare$

~wamofan

Solution 4 (Point-line distance formula)



Denote $a = RC$. Now tilt your head to the right and view $R, \overrightarrow{RB}$ and $\overrightarrow{RC}$ as the origin, x-axis and y-axis, respectively. In particular, we have points $B(6,0), C(0,a), P(-7,0)$. Note that side length of the square $ABCD$ is $BC = \sqrt{a^2+36}$. Also equation of line BC is

$$\underbrace{\frac{x}{6} + \frac{y}{a} = 1}_{\text{intercepts form}} \qquad \Longrightarrow \qquad ax + 6y - 6a = 0.$$

Because the distance from $P(-7,0)$ to line $BC : ax + 6y - 6a = 0$ is also the side length $\sqrt{a^2+36}$, we can apply the point-line distance formula to get

$$\frac{|a \cdot (-7) + 6 \cdot 0 - 6a|}{\sqrt{a^2+36}} = \sqrt{a^2+36}$$

which reduces to $|13a| = a^2 + 36$. Since a is positive, the last equations factors as $a^2 - 13a + 36 = (a-4)(a-9) = 0$. Now judging from the figure, we learn that $a > RB = 6$. So $a = 9$. Therefore, the area of the square $ABCD$ is $BC^2 = RC^2 + RB^2 = a^2 + 6^2 = \mathbf{117}$. Choose $\boxed{\textbf{(D) } 117}$. $\blacksquare$

~VensL.

Solution 5

Denote $\angle PBA = \alpha$. Because $\angle QRB = \angle QBC = 90^\circ$, $\angle BCQ = \alpha$.

Hence, $AB = BP \cos \angle PBA = 13 \cos \alpha$, $BC = \dfrac{BR}{\sin \angle BCQ} = \dfrac{6}{\sin \alpha}$.

Because $ABCD$ is a square, $AB = BC$. Hence, $13 \cos \alpha = \dfrac{6}{\sin \alpha}$.

Therefore,

$$\begin{aligned} \sin 2\alpha &= 2 \sin \alpha \cos \alpha \\ &= \frac{12}{13}. \end{aligned}$$

Thus, $\cos 2\alpha = \pm \dfrac{5}{13}$.

$\textbf{Case 1}$: $\cos 2\alpha = \dfrac{5}{13}$.

Thus, $\cos \alpha = \sqrt{\dfrac{1 + \cos 2\alpha}{2}} = \dfrac{3}{\sqrt{13}}$.

Hence, $AB = 13 \cos \alpha = 3\sqrt{13}$.

Therefore, $\text{Area } ABCD = AB^2 = 117$.

$\textbf{Case 2}$: $\cos 2\alpha = -\dfrac{5}{13}$.

Thus, $\cos \alpha = \sqrt{\dfrac{1 + \cos 2\alpha}{2}} = \dfrac{2}{\sqrt{13}}$.

Hence, $AB = 13 \cos \alpha = 2\sqrt{13}$.

However, we observe $BQ = \dfrac{BR}{\cos \alpha} = 3\sqrt{13} > AB$. Therefore, in this case, point Q is not on the segment AB.

Therefore, this case is infeasible.

Putting all cases together, the answer is $\boxed{\textbf{(D) } 117}$.

~Steven Chen (www.professorchenedu.com)

Solution 6 (Answer choices and areas)

Note that if we connect points P and C, we get a triangle with height RC and length 13. This triangle has an area of $\frac{1}{2}$ the square. We can now use answer choices to our advantage!

Answer choice A: If BC was $\sqrt{85}$, RC would be 7. The triangle would therefore have an area of $\frac{91}{2}$ which is not half of the area of the square. Therefore, A is wrong.

Answer choice B: If BC was $\sqrt{93}$, RC would be $\sqrt{57}$. This is obviously wrong.

Answer choice C: If BC was 10, we would have that RC is 8. The area of the triangle would be 52, which is not half the area of the square. Therefore, C is wrong.

Answer choice D: If BC was $\sqrt{117}$, that would mean that RC is 9. The area of the triangle would therefore be $\frac{117}{2}$ which IS half the area of the square. Therefore, our answer is $\boxed{\textbf{(D)}\ 117}$.

~Arcticturn

Video Solution by Interstigation

https://www.youtube.com/watch?v=sKC0Yt6sPi0

See Also

2021 Fall AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
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