Russian School of Math Test 2

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Abstract

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

1

Given a regular 27-gon $A_1A_2\dots A_{27}$. Find the number of non-isosceles triangles with vertices at points A_1, A_2, $\dots A_{27}$. Triangles that differ in vertex order (for example, $A_1A_2A_4$ and $A_2A_4A_1$) are counted as one triangle.

$$\frac{n(n-4)(n-5)}{6} = \frac{27 \cdot 23 \cdot 22}{6} = 9 \cdot 23 \cdot 11 = 2277.$$

Solution: $\boxed{2277.}$

Information below borrowed from `stackexchange.com`

Number of triangles formed using the vertices of a regular n-gon

Number of triangles formed using the vertices of an n-gon

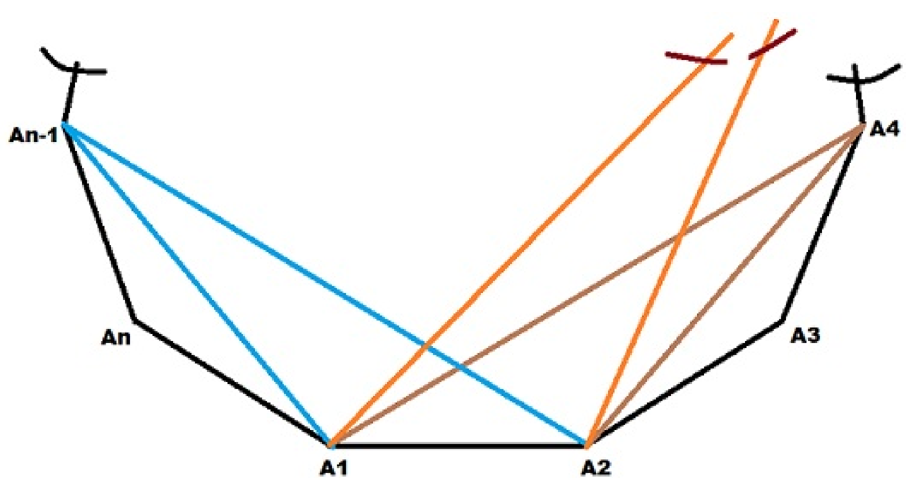
Consider a regular polygon with n number of vertices $A_1, A_2, A_3, A_3, \dots, A_{n-1}, A_n$.

The total number of triangles formed by joining the vertices of an n-sided regular polygon is:

$$\begin{aligned} N &= \text{number of ways of selecting 3 vertices from } n \\ &= \binom{n}{3} = \frac{n(n-1)(n-2)}{6} \quad \forall\ n \geq 3 \end{aligned}$$

Consider a side A_1A_2 of a regular n-polygon. Join the vertices A_1A_2 to any of $(n-4)$ vertices i.e. $A_4, A_5, A_6, \dots, A_{n-1}$ to get triangles with only one side in common. Thus there are $(n-4)$ different triangles with only one side A_1A_2 common. Similarly, there are $(n-4)$ different triangles with only one side A_2A_3 in common, and so on. There are $(n-4)$ different triangles with each of n sides common. Therefore, the number of triangles N_1 that have only one side common with that of the polygon

$$N_1 = (\text{No. of triangles corresponding to one side})(\text{No. of sides}) = (n-4)n$$



Now, join the alternate vertices A_1 and A_3 by a straight (blue) line to get a triangle $A_1A_2A_3$ with two sides A_1A_2 and A_2A_3 common. Similarly, join alternate vertices A_2 and A_4 to get another triangle

$A_2A_3A_4$ with two sides A_2A_3 and A_3A_4 common, and so on. There are n pairs of alternate and consecutive vertices to get n different triangles with two sides common. Therefore, the number of triangles N_2 that have two sides common with that of the polygon $N_2 = n$.

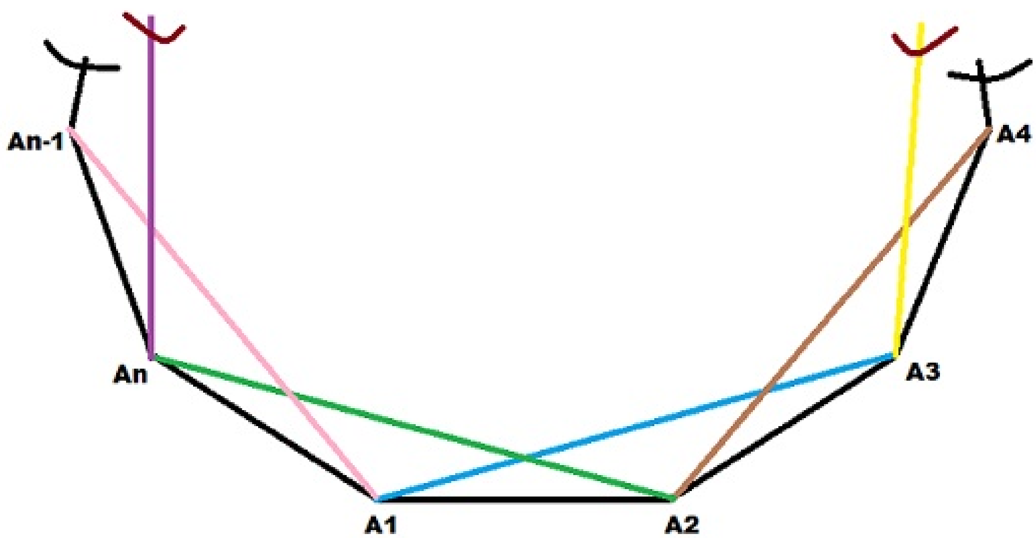
If N_0 is the number of triangles having no side common with that of the polygon then we have

$$N = N_0 + N_1 + N_2 \implies N_0 = N - N_1 - N_2$$

And therefore

$$N_0 = \binom{n}{3} - (n-4)n - n = \frac{n(n-1)(n-2)}{6} - n^2 + 3n = \frac{n(n-4)(n-5)}{6}$$

The above formula (N_0) is valid for polygon having n no. of the sides such that $n \geq 6$



Number of triangles formed using the vertices but not using any side of an n-gon

Apply the principle of inclusion-exclusion (PIE) to count the triangles that avoid the n properties that the triangle uses side $k \in \{1, \ldots, n\}$.

(a) Choose 0 properties and then all triangles: $\binom{n}{0}\binom{n}{3} = \binom{n}{3}$

(b) Choose 1 property and then all triangles that use (at least) that side: $\binom{n}{1}\binom{n-2}{1} = n(n-2)$

(c) Choose 2 properties (which must correspond to two adjacent sides) and then the unique triangle that uses both sides: $n \cdot 1 = n$

For $n \geq 4$, no triangle can satisfy more than 2 properties.

Applying PIE by adding and subtracting the three terms yields:

$$\binom{n}{3} - n(n-2) + n = \binom{n}{3} - n^2 + 3n$$

Another approach is to subtract the number of triangles that use exactly one side and the number that use exactly two sides. Exactly one side: $n(n-4)$. Exactly two sides: n. Now subtract both:

$$\binom{n}{3} - n(n-4) - n = \binom{n}{3} - n^2 + 3n$$

2

What is the probability that a randomly chosen positive factor of 1200 is not a multiple of 5? $1200 = 2^4 \cdot 3 \cdot 5^2$. So the probability is:

$$\frac{10}{30} = \frac{1}{3}$$

3

If a and b are positive integers such that $a \cdot b = 2400$, find the least possible value of $a + b$.

$$\begin{aligned} a \cdot b &= 2400 \\ 48 \cdot 50 &= 2400 \\ 48 + 50 &= 95 \end{aligned}$$

4

One week from Monday to Friday inclusive, Naina solved math problems. On Monday and Tuesday, she solved 8 problems each. On Wednesday, Naina solved 12 problems. On Thursday, she solved a whole number problem, and on Friday, Naina also solved a whole number of problems. After Friday, she calculated the median number of problems she solved over the past five days. What is the sum of all possible values of Naina's result?

5

A standard dice (a cube with 1 to 6 points its sides, the sum of the points on opposite sides is 7) is thrown. After the throw, we record the result, roll the cube randomly onto one of the adjacent faces, record a new result, then again randomly roll the cube onto one of the adjacent faces, and record the third result. How many different increasing sequences of points can we get?

6

The maximum value of the volume of a right circular cylinder inscribed in a sphere of radius $\sqrt{3}$ is equal to $a \cdot \pi$. Find a.

7

Christina wanted to draw a regular triangle ABC. But, since she drew inaccurately, she ended up with a triangle with $m\angle A = 59°$ and $m\angle B = 58.5°$. Then Christina drew the heights CE and BD, but since the triangular ruler was slightly skewed, she got $m\angle ADB = m\angle AEC = 91°$. Find the degree measure of $m\angle AED$.

8

A polynomial whose roots are all equal to each other is called a unicorn. Compute the number of distinct ordered triples (M, P, G), where M, P, G are complex numbers such that the polynomials

$$z^3 + Mz^2 + Pz + G \quad \text{and} \quad z^3 + Gz^2 + Pz + M$$

are both unicorns.

$$(z - r)^3 = z^3 - 3rz^2 + 3r^2z - r^3$$
$$(z - \rho)^3 = z^3 - 3\rho z^2 + 3\rho^2 z - \rho^3$$

$$-3r = M$$
$$3r^2 = P$$
$$-1 = G$$

Likewise

$$-3\rho = G$$
$$3\rho^2 = P$$
$$-1 = M$$

The number of distinct triples (M, P, G) is the number of triples $(-1, P, -1)$.

$$M = -1 \implies -3r = -1 \implies r = 1/3 \implies P = 1(1/3)^2 = 1/3$$
$$G = -1 \implies -3\rho = -1 \implies \rho = 1/3 \implies P = 1(1/3)^2 = 1/3$$

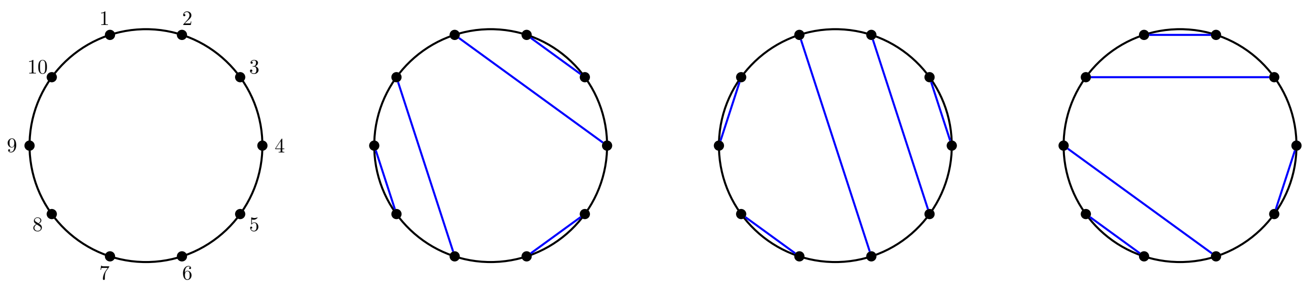
Solution: $(-1, 1/3, -1)$.

9

The centers of the three circles A, B, and C are collinear with the center of circle B lying between the centers of circles A and C. Circles A and C are both externally tangent to circle B, and the three circles share a common tangent line. Given that circle A has radius 12 and circle B has radius 42, find the radius of circle C.

10

Ten distinct points are placed in a circle. All ten of the points are paired so that the line segments connecting the pairs do not intersect. In how many different ways can this pairing be done?



11

The function f satisfies the equality $f(xy) = y \cdot f(x) + x \cdot f(y)$ for any $x, y > 0$. Find $f(\frac{1}{8})$ if $f(2) = 1$.

$$f(2) = 1 \implies x + 2f(x) = f(2x)$$

$$1/2 + 2f(1/2) = f(1) = 0 \implies f(1/2) = -1/4$$

$$1/4 + 2f(1/4) = f(1/2) = -1/4 \implies f(1/4) = (1/2)(-1/4 - 1/4) = -1/4$$

$$1/8 + 2f(1/8) = f(1/4) = -1/4 \implies f(1/8) = (1/2)(-1/4 - 1/8) = (1/2)(-3/8) = -3/16$$

Solution: $\boxed{f(1/8) = -3/16.}$

12

The are 3 red and 7 blue balls in the first box. There are 5 red and 4 blue balls in the second box. John randomly chooses two balls from the first box and two balls from the second box. Find the probability that the combination of chosen balls from the first box is the same as the combination of chosen balls from the second box, given that at least one of the chosen balls is red.

$$Prob(r,r)|_{1=r} = \frac{2}{9} \cdot (\frac{5}{9} \cdot \frac{4}{8})$$

$$Prob(r,b)|_{1=r} = \frac{7}{9} \cdot (\frac{5}{9} \cdot \frac{4}{8})$$

$$Prob(r,r)|_{2=r} = (\frac{3}{10} \cdot \frac{2}{9}) \cdot \frac{4}{8}$$

$$Prob(r,b)|_{2=r} = (\frac{3}{10} \cdot \frac{7}{9}) \cdot \frac{4}{8}$$

Solution:

$$\frac{2}{9} \cdot \frac{5}{9} \cdot \frac{4}{8} + \frac{7}{9} \cdot \frac{5}{9} \cdot \frac{4}{8} + \frac{3}{10} \cdot \frac{2}{9} \cdot \frac{4}{8} + \frac{3}{10} \cdot \frac{7}{9} \cdot \frac{4}{8}$$
$$= \frac{400 + 1400 + 216 + 756}{10 \cdot 9 \cdot 9 \cdot 8} = \frac{2772}{6480} = \frac{1386}{3240} = \frac{693}{1620} = \frac{77}{180} \approx 0.43$$

13

Let S be the set of all 10-term arithmetic progressions that include the numbers 4 and 10. For example, $(-2, 1, 4, 7, 10, 13, 16, 19, 22, 25)$ and $(10, 8.5, 7, 5.5, 4, 2.5, 1, -0.5, -2, -3.5)$ are both members of S. Find the sum of values of a_{10} for all such progressions belonging to S, that is find

$$\sum_{(a_1, a_2, \ldots, a_{10}) \in S} a_{10}.$$

14

A function f satisfies the equation

$$f(x) + f\left(1 - \frac{1}{x}\right) = 1 + x$$

for every real number x ($x \neq 0, x \neq 1$). Find $f(10)$.

15

Let c be a complex number. Suppose there exist distinct complex numbers r, s, and t such that for every complex number z, we have

$$(z - r)(z - s)(z - t) = (z - cr)(z - cs)(z - ct)$$

Compute the number of distinct possible values of c.