AMC 10 Problem Series (2804)

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Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Homework

Homework: Lesson 11

You have completed **10** of **10** challenge problems.
Due **Aug 21**.

Readings

Lesson 11 Transcript: Fri, Aug 13

Challenge Problems

Total Score: 60 / 60

Problem 1 – Correct! – Score: 6 / 6 (3019)

Problem: Report Error

Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Walter entered?

(A) 71 **(B)** 76 **(C)** 80 **(D)** 82 **(E)** 91

Solution:

After each score was entered, the average was always an integer, so for $1 \le n \le 5$, the sum of the first n scores must be a multiple of n.

In particular, the sum of the first three scores must be a multiple of 3. When 71, 76, 80, 82, and 91 are divided by 3, the remainders are 2, 1, 2, 1, and 1, respectively. This tells us that the only choice of three numbers that add up to a multiple of 3 are 76, 82, and 91.

Then the fourth score must be 71 or 80. The sum of the first four scores must be a multiple of 4, and the only score that satisfies this condition is 71 (since $76 + 82 + 91 + 71 = 320$). Therefore, the last score is $\boxed{80}$. The answer is (C).

Your Response(s):

C

Problem 2 – Correct! – Score: 6 / 6 (3020)

Problem: Report Error

The digits 1, 2, 3, 4, 5, 6, 7, and 9 are used to form four two-digit prime numbers, with each digit used exactly once. What is the sum of these four primes?

(A) 150 **(B)** 160 **(C)** 170 **(D)** 180 **(E)** 190

Solution:

The digits 2, 4, 6 cannot be units digits (otherwise, the two-digit number would be divisible by 2), and the digit 5 cannot be a units digit (otherwise, the two-digit number would be divisible by 5), so they must all be tens digits. The remaining digits 1, 3, 7, and 9 must be units digits.

Therefore, the sum of all four primes must be $10(2+4+5+6)+1+3+7+9=\boxed{190}$. The answer is (E). (The set $\{23, 47, 59, 61\}$ satisfies the conditions.)

Your Response(s):

E

Problem 3 – Correct! – Score: 6 / 6 (3021)

Problem:

Report Error

Suppose that m and n are positive integers such that $75m = n^3$. What is the minimum possible value of $m + n$?

(A) 15 **(B)** 30 **(C)** 50 **(D)** 60 **(E)** 5700

Solution:

In the equation $75m = n^3$, m is an increasing function of n (in other words, if n increases, so does m), so minimizing $m + n$ is equivalent to minimizing n.

The prime factorization of 75 is $3 \cdot 5^2$, so n must contain at least one factor of 3 and one factor of 5. Hence, n must be at least 15. If $n = 15$, then

$$m = \frac{n^3}{75} = \frac{15^3}{75} = 45.$$

Hence, the smallest possible value of $m + n$ is $45 + 15 = \boxed{60}$. The answer is (D).

Your Response(s):

D

Problem 4 – Correct! – Score: 6 / 6 (3022)

Problem:

Report Error

What is the largest integer that is a divisor of $(n+1)(n+3)(n+5)(n+7)(n+9)$ for all positive even integers n?

(A) 3 **(B)** 5 **(C)** 11 **(D)** 15 **(E)** 165

Solution:

Let $P_n = (n+1)(n+3)(n+5)(n+7)(n+9)$, and let d be the greatest common divisor of all P_n, where n is even. The fastest way to narrow down the possible values of d is to test certain values. For $n = 2$, 10, and 12, we get

$$\begin{aligned} P_2 &= 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 = 3^3 \cdot 5 \cdot 7 \cdot 11, \\ P_{10} &= 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19 = 3 \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 19, \\ P_{12} &= 13 \cdot 15 \cdot 17 \cdot 19 \cdot 21 = 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19. \end{aligned}$$

Hence, d must divide the greatest common divisor of P_2, P_{10}, and P_{12}, which is $3 \cdot 5 = 15$.

The numbers $n+1$, $n+3$, $n+5$, $n+7$, and $n+9$ form five consecutive odd numbers. At least one of these numbers is divisible 3, and at least one of these numbers is divisible by 5, so P_n is divisible by 15 for all even integers n.

Therefore, $d = \boxed{15}$. The answer is (D).

Your Response(s):

D

Problem 5 – Correct! – Score: 6 / 6 (3023)

Problem:

Report Error

Let S be the set of the 2005 smallest positive multiples of 4, and let T be the set of the 2005 smallest positive multiples of 6. How many elements are common to S and T?

(A) 166 **(B)** 333 **(C)** 500 **(D)** 668 **(E)** 1001

Solution:

A number is a multiple of both 4 and 6 if and only if it is a multiple of $\operatorname{lcm}(4, 6) = 12$.

Among the 2005 smallest positive multiples of 4, every third number is a multiple of 12. When we divide 2005 by 3, we get a quotient of 668 and a remainder of 1, so S and T have $\boxed{668}$ elements in common. The answer is (D).

Your Response(s):

D

Problem 6 – Correct! – Score: 6 / 6 (3024)

Problem:

Report Error

For how many positive integers n less than or equal to 24 is $n!$ evenly divisible by $1 + 2 + \cdots + n$?

(A) 8 **(B)** 12 **(C)** 16 **(D)** 17 **(E)** 21

Solution:

We have that $1 + 2 + \cdots + n = n(n+1)/2$, so $n!$ is evenly divisible by $1 + 2 + \cdots + n$ if and only if

$$\frac{n!}{1 + 2 + \cdots + n} = \frac{n!}{n(n+1)/2} = \frac{2n!}{n(n+1)}$$

is an integer. We can write $n! = n \cdot (n-1)!$, so

$$\frac{2n!}{n(n+1)} = \frac{2(n-1)!}{n+1}.$$

To see when $n + 1$ divides $2(n-1)!$, we divide into the following cases.

Case 1: $n + 1$ is an even prime.

If $n + 1$ is an even prime, then $n = 1$. In this case, $2(n-1)!$ is divisible by $n + 1$.

Case 2: $n + 1$ is an odd prime.

If $n + 1$ is an odd prime, then $(n-1)! = 1 \cdot 2 \cdots (n-1)$ does not have any factors of $n + 1$, so $2(n-1)!$

is not divisible by $n+1$.

Case 3: $n+1$ is a composite number, and can be written in the form $n+1=ab$, where a and b are distinct positive integers, both greater than 1.

In this case, a and b appear as factors in $(n-1)!=1\cdot 2\cdots(n-1)$, so $2(n-1)!$ is divisible by $n+1$.

Case 4: $n+1$ is a composite number, and is of the form p^2, where p is a prime.

The only integers n in question for which $n+1$ is the square of a prime are 3, 8, and 24. It is easy to check that $2(n-1)!$ is divisible by $n+1$ for these values.

(Make sure you see why these cases cover all possibilities, and also why we have cases 3 and 4.)

Hence, $2(n-1)!$ is divisible by $n+1$ if and only if $n=1$ or $n+1$ is composite. It is easy to check that $n+1$ is composite for 15 values of n, $1\le n\le 24$, for a total of $15+1=\boxed{16}$ values of n. The answer is (C).

Your Response(s):

C

Problem 7 – Correct! – Score: 6 / 6 (3025)

Problem: Report Error

A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

(A) 3 **(B)** 7 **(C)** 13 **(D)** 37 **(E)** 43

Solution:

We claim that the largest prime number that always divides S is 37.

First, we look at an example. For the sequence 124, 241, 412, the sum S is equal to $124+241+412=777$. The prime factorization of 777 is $3\cdot 7\cdot 37$, so the largest prime number that always divides S must be 3, 7, or 37.

If a digit appears as a units digit, then it also appears as a tens digit in another number, and a hundreds digit in another number. If we let k be the sum of all units digits, then $S=(100+10+1)k=111k=37\cdot 3k$. Hence, S is always divisible by 37.

Therefore, the largest prime number that always divides S is $\boxed{37}$. The answer is (D).

Your Response(s):

D

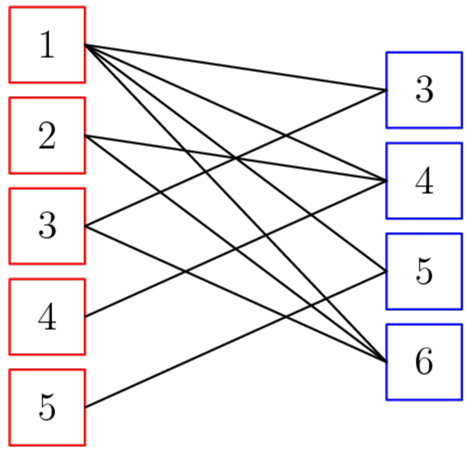
Problem 8 – Correct! – Score: 6 / 6 (3026)

Problem: Report Error

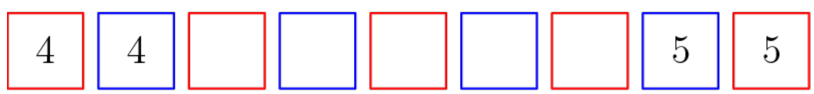
Sally has five red cards numbered 1 through 5 and four blue cards numbered 3 through 6. She stacks the cards so that the colors alternate and so that the number on each red card divides evenly into the number on each neighboring blue card. What is the sum of the numbers on the middle three cards?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

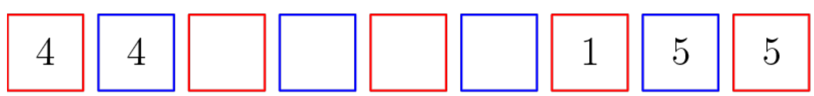
Solution:
First, we see which red cards can be placed next to which blue cards. We draw a line joining two cards if they can be placed next to each other.



The only blue card that can go next to the red 4 is the blue 4, and the only blue card that can go next to the red 5 is the blue 5, so the red 4 and the red 5 must be at the ends. The blue 4 and the blue 5 must then be next to these cards, respectively.



Then the only card that go next to the blue 5 is the red 1.



Then only the red 2 can go next to the blue 4, so the red 3 must go in the remaining red slot.

Finally, the blue 6 must go between the red 2 and the red 3, so the blue 3 goes in the remaining blue slot.

Therefore, the sum of the numbers on the middle three cards is $6+3+3=\boxed{12}$. The answer is (E).

Your Response(s):

E

Problem 9 – Correct! – Score: 6 / 6 (3027)

Problem: Report Error

Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2-y^2=m^2$ for some positive integer m. What is $x+y+m$?

(A) 88 **(B)** 112 **(C)** 116 **(D)** 144 **(E)** 154

Solution:

Since x is a two-digit integer, we can write $x=10a+b$, where a and b are decimal digits. Then $y=10b+a$. We can factor x^2-y^2 as $(x+y)(x-y)$, and

$$(x+y)(x-y)=(11a+11b)(9a-9b)=99(a+b)(a-b).$$

We are told that $(x+y)(x-y)$ is a perfect square, so $99(a+b)(a-b)$ is a perfect square. Since 9 is a perfect square, $11(a+b)(a-b)$ must be a perfect square.

Since m is a positive integer, $a>b$. If $11(a+b)(a-b)$ is a perfect square, then at least one of the factors $(a+b)(a-b)$ must have a factor of 11, which means that $a+b$ or $a-b$ must have a factor of 11. Since a and b are digits, $a+b<18$ and $a-b<8$, so $a+b$ must be the factor that is divisible by 11. In fact, $a+b$ must be equal to 11.

Therefore, (a,b) must be one of the pairs $(6,5)$, $(7,4)$, $(8,3)$, or $(9,2)$. Since $11(a+b)(a-b)$ is a perfect square and $a+b=11$, $a-b$ must also be a perfect square. The only pair (a,b) for which $a-b$ is a perfect square is $(6,5)$.

Hence, $a=6$ and $b=5$, which means $x=65$ and $y=56$. Then

$$m^2=99(a+b)(a-b)=99\cdot 11\cdot 1=33^2,$$

which means $m=33$, so $x+y+m=65+56+33=\boxed{154}$. The answer is (E).

Your Response(s):

E

Problem 10 – Correct! – Score: 6 / 6 (3028)

Problem: Report Error

A high school basketball game between the Raiders and the Wildcats was tied at the end of the first quarter. The number of points scored by the Raiders in each of the four quarters formed an increasing geometric sequence, and the number of points scored by the Wildcats in each of the four quarters formed an increasing arithmetic sequence. At the end of the fourth quarter, the Raiders had won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

(A) 30 **(B)** 31 **(C)** 32 **(D)** 33 **(E)** 34

Solution:
Let a be the score (of either team) at the end of the first quarter. Let d be the common difference in the scores of the Wildcats. Then the scores of the Wildcats in each quarter are a, $a+d$, $a+2d$, and $a+3d$, so their total score is $4a+6d$.

Let r be the common ratio in the scores of the Raiders, so $r>1$. We know that r is rational, so let $r=m/n$, where m and n are relatively prime positive integers, with $m>n$. (Note that r is not necessarily an integer!) Then the Raiders scored

$$ar^3 = a \cdot \frac{m^3}{n^3}$$

in the last quarter, which means a is divisible by n^3. Let $a=An^3$. Then the scores of the Raiders in each quarter are An^3, Amn^2, Am^2n, and Am^3, so their total score is $A(n^3+mn^2+m^2n+m^3)$. This total score is at most 100, so the only possible pairs (m,n) are $(2,1)$, $(3,1)$, $(3,2)$, and $(4,1)$. We look at these cases separately.

Case 1: $(m,n)=(2,1)$.

In this case, the Raiders total score is $15A=15a$. The Raiders win by 1 point, so $15a=4a+6d+1$, which means $11a=6d+1$. The only solution in positive integers to this equation where $15a\le 100$ is $(a,d)=(5,9)$. Then, the total number of points scored by the two teams in the first half is $a+(a+d)+a+ar=5+14+5+10=34$.

Case 2: $(m,n)=(3,1)$.

In this case, the Raiders total score is $40A=40a$. The Raiders win by 1 point, so $40a=4a+6d+1$. But $40A$ is even and $4a+6d+1$ is odd, so there are no solutions in this case.

Case 3: $(m,n)=(3,2)$.

In this case, the Raiders total score is $65A$. The Raiders score at most 100 points, so $A=1$, which means $a=8$. Also, the Raiders win by 1 point, so $65=4a+6d+1$. Substituting $a=8$, we get $6d=32$, which has no solutions in integers.

Case 4: $(m,n)=(4,1)$.

In this case, the Raiders total score is $85A$. The Raiders score at most 100 points, so $A=1$, which means $a=1$. Also, the Raiders win by 1 point, so $85=4a+6d+1$. Substituting $a=1$, we get $6d=80$, which has no solutions in integers.

Therefore, the total number of points scored by the two teams in the first half is $\boxed{34}$. The answer is (E).

Your Response(s):
- E