AMC 10 Problem Series (2804)

Jon Joseph

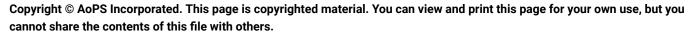**Friday**
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Overview

Week 5 (Jul 2) Class Transcript - Tools of Algebra

💬 ✏️

⊙ Go back to the class overview page

Display all student messages • Show few student messages • Hide student messages

jonjoseph　2021-07-02 19:30:55
Nice @ ypang9804. Doesn't even look like you were trying!!

jonjoseph　2021-07-02 19:31:36
Well, lots of claps anyway.

jonjoseph　2021-07-02 19:31:52
AMC 10 Problem Series
Week 5: Tools of Algebra

jonjoseph　2021-07-02 19:31:59
In today's class, we will look at a number of algebraic topics. We will start with percent, ratio, and proportion. We will also cover some basic statistics, and end with some problems that can be solved using factorization.

jonjoseph　2021-07-02 19:32:13
PERCENT, RATIO, and PROPORTION

jonjoseph　2021-07-02 19:32:18
Let's do a quick computation, to make sure everyone understands how to work with percentages.

jonjoseph　2021-07-02 19:32:22
Suppose I initially have $\$80$ in my bank account. If the amount of money in my account increases by 15% in one year, then how much will I have in my bank account after a year?

jonjoseph　2021-07-02 19:33:26
15% **of** 80 **is** $\dfrac{15}{100} \cdot 80 = 12$**, so I will have**

$$80 + \frac{15}{100} \cdot 80 = 80 + 12 = \$92$$

in my bank account after a year.

jonjoseph　2021-07-02 19:33:34
It is convenient to factor the 80 **out of the two terms on the far left hand side above, so we can also write that computation as**

$$80 + \frac{15}{100} \cdot 80 = \left(1 + \frac{15}{100}\right) \cdot 80 = \frac{115}{100} \cdot 80 = \$92.$$

jonjoseph　2021-07-02 19:34:05
Or, as many of you said, $80 \cdot 1.15.$

jonjoseph　2021-07-02 19:34:16
More generally, by the same reasoning, if I have some amount c **and it increases by** $x\%$**, then its end value is**

$$\left(1 + \frac{x}{100}\right)c.$$

jonjoseph   2021-07-02 19:34:36
Last year Mr. John Q. Public received an inheritance. He paid 20% in federal taxes on the inheritance, and paid 10% of what he had left in state taxes. He paid a total of $\$10,500$ for both taxes. How many dollars was the inheritance?

(A) 30,000   (B) 32,500   (C) 35,000   (D) 37,500   (E) 40,000

jonjoseph   2021-07-02 19:34:43
First, what's an inheritance?

jonjoseph   2021-07-02 19:35:54
Some good answers.

jonjoseph   2021-07-02 19:35:58
An inheritance is money you receive when a relative or someone close to you dies; they leave some of their money for you. You don't get to keep all the money, however, since it is taxed by both the federal and state governments.

jonjoseph   2021-07-02 19:36:11
We want to find the inheritance, so let x be the inheritance.

jonjoseph   2021-07-02 19:36:16
What we know is the total amount he paid in taxes. To get an equation we can use to solve for x, let's figure out how much he paid in taxes in terms of x.

jonjoseph   2021-07-02 19:36:25
Mr. Public paid 20% in federal taxes on the inheritance. What is this in terms of x?

jonjoseph   2021-07-02 19:37:30
The amount of federal taxes is $\frac{20}{100} \cdot x = \frac{x}{5}$. So how much did he have left after paying the federal taxes?

jonjoseph   2021-07-02 19:38:16
Mr. Public had $x - \frac{x}{5} = \frac{4x}{5}$ left.

jonjoseph   2021-07-02 19:38:22
Mr. Public then paid another 10% on the remainder (how much was left over after federal taxes were paid) in state taxes. How much is this in terms of x?

jonjoseph   2021-07-02 19:39:49
The amount of state taxes is $\frac{10}{100} \cdot \frac{4x}{5} = \frac{4x}{50} = \frac{2x}{25}$. So what equation can we write down?

jonjoseph   2021-07-02 19:41:33
We can write $\frac{x}{5} + \frac{2x}{25} = 10500$. What is x?

jonjoseph   2021-07-02 19:42:48
We see that $\frac{x}{5} + \frac{2x}{25} = \frac{5x}{25} + \frac{2x}{25} = \frac{7x}{25}$, so $\frac{7x}{25} = 10500$. Then $x = \frac{25}{7} \cdot 10500 = 37500$. The answer is (D).

jonjoseph   2021-07-02 19:43:49
True although they can be in the millions of dollars along with the beach house in Tahiti.

jonjoseph   2021-07-02 19:44:11
In a certain year the price of gasoline rose by 20% during January, fell by 20% during February, rose by 25% during March, and fell by $x\%$ during April. The price of gasoline at the end of April was the same as it had been at the beginning of January. To the nearest integer, what is x?

(A) 12   (B) 17   (C) 20   (D) 25   (E) 35

jonjoseph   2021-07-02 19:44:38
How can we get started?

jonjoseph   2021-07-02 19:45:41
We can get started by setting the initial price of gasoline as a variable. Let's call the initial price p.

jonjoseph   2021-07-02 19:45:49
We eventually want an equation involving x, but before we do that, we need to figure out how the price changes from month to month up to the end.

jonjoseph   2021-07-02 19:45:57
The price rose 20% in January. What is the price after this raise, in terms of p?

jonjoseph   2021-07-02 19:46:52
After the price rises 20%, the price is

$$p + \frac{20}{100}p = \frac{120}{100} \cdot p = \frac{6}{5} \cdot p.$$

jonjoseph   2021-07-02 19:46:57
The price then fell 20% in February. What is the price after this fall?

jonjoseph   2021-07-02 19:48:04
After the price falls 20%, only 80% is left. So, the new price is

$$\frac{80}{100} \cdot \frac{6}{5} \cdot p = \frac{4}{5} \cdot \frac{6}{5} \cdot p = \frac{24}{25} \cdot p.$$

jonjoseph   2021-07-02 19:48:11
The price then rose 25% in March. What is the price after this raise?

jonjoseph   2021-07-02 19:49:25
After the price rises 25% in March, the new price is

$$\frac{24}{25} \cdot p + \frac{25}{100} \cdot \frac{24}{25} \cdot p = \frac{125}{100} \cdot \frac{24}{25} \cdot p$$
$$= \frac{5}{4} \cdot \frac{24}{25} \cdot p$$
$$= \frac{6}{5} \cdot p.$$

jonjoseph   2021-07-02 19:49:42
Maybe you have some happy realization here that going down by 20% is multiplying by $\frac{4}{5}$ and going up by 25% is multiplying by $\frac{5}{4}$ so these undo each other!

jonjoseph   2021-07-02 19:49:57
Finally, the price then fell by $x\%$ in April, back to the same price p.

jonjoseph   2021-07-02 19:50:12
So what equation can we write?

jonjoseph   2021-07-02 19:52:13

We can write $\left(1 - \dfrac{x}{100}\right) \cdot \dfrac{6}{5} p = p$. **How can we solve for x?**

jonjoseph   2021-07-02 19:52:23

Since $p \neq 0$, we can divide both sides by p. Doing so gives $\left(1 - \dfrac{x}{100}\right) \cdot \dfrac{6}{5} = 1$. What is x?

jonjoseph   2021-07-02 19:53:35

Solving for x, we find:

$$\left(1 - \frac{x}{100}\right) \cdot \frac{6}{5} = 1,$$
$$1 - \frac{x}{100} = \frac{5}{6},$$
$$\frac{1}{6} = \frac{x}{100},$$
$$x = \frac{100}{6}.$$

jonjoseph   2021-07-02 19:53:49

To the nearest integer, this is 17. The answer is (B).

jonjoseph   2021-07-02 19:53:57

Note that the final answer does not depend on the initial price p. It canceled in the end. Let's think about this in general.

jonjoseph   2021-07-02 19:54:16

(Some of you have already noticed this.)

jonjoseph   2021-07-02 19:54:20

Suppose that for a certain initial price x, after a number of percent increases and decreases, the price eventually doubles to $2x$. If we started with another initial price y, and we experienced the same percent increases and decreases, then what would the eventual price be?

jonjoseph   2021-07-02 19:55:02

The eventual price would be $2y$ as well! This is because the factor by which the price of gas increases depends only on the sequence of percent increases and decreases (which scale different values proportionally!). If the price doubles for some initial price, then it doubles for any initial price.

jonjoseph   2021-07-02 19:55:31

Heather compares the price of a new computer at two different stores. Store A offers 15% off the sticker price followed by a $\$90$ rebate, and store B offers 25% off the same sticker price with no rebate. Heather saves $\$15$ by buying the computer at store A instead of store B. What is the sticker price of the computer, in dollars?

(A) 750   (B) 900   (C) 1000   (D) 1050   (E) 1500

jonjoseph   2021-07-02 19:55:47

First, a note for anyone who doesn't know: a rebate means the store gives you the money back. So a price of $\$200$ with a $\$20$ rebate means you pay $\$180$.

jonjoseph   2021-07-02 19:56:06

We can let x be the sticker price of the computer. We can then express the prices of the computer at stores A and B in terms of x.

jonjoseph   2021-07-02 19:56:14

Store A offers 15% off the sticker price, followed by a $\$90$ rebate. So what is store A's price in terms of x?

jonjoseph 2021-07-02 19:57:27

After 15% off, the price of the computer becomes $0.85x$. With a $\$90$ rebate, the price at Store A drops to $0.85x - 90$.

jonjoseph 2021-07-02 19:57:38

Store B offers 25% off the sticker price. So what is store B's price in terms of x?

jonjoseph 2021-07-02 19:58:18

After 25% off, the price of the computer becomes $0.75x$. So what equation can we write down?

jonjoseph 2021-07-02 20:00:06

Store A's price is lower than store B's price by $\$15$, so we can write

$$0.85x - 90 = 0.75x - 15.$$

jonjoseph 2021-07-02 20:00:15

So what is x?

jonjoseph 2021-07-02 20:01:33

The equation simplifies to $0.1x = 75$.

jonjoseph 2021-07-02 20:01:36

Solving, we have $x = \dfrac{75}{0.1} = 750$. The answer is (A).

jonjoseph 2021-07-02 20:01:49

The interior angles of quadrilateral $ABCD$ satisfy $\angle A = 2\angle B = 3\angle C = 4\angle D$. What is the degree measure of $\angle A$, rounded to the nearest whole number?

$(\text{A})\ 125 \quad (\text{B})\ 144 \quad (\text{C})\ 153 \quad (\text{D})\ 173 \quad (\text{E})\ 180$

jonjoseph 2021-07-02 20:02:09

Let's start by letting $\angle A = w, \angle B = x, \angle C = y$ and $\angle D = z$.

jonjoseph 2021-07-02 20:02:13

What relationship can we write down between w, x, y, z?

jonjoseph 2021-07-02 20:04:02

(I see many of you just restated what was given. I'll try to rephrase this next time.)

jonjoseph 2021-07-02 20:04:13

Since $ABCD$ is a quadrilateral then

$$w + x + y + z = 360°.$$

jonjoseph 2021-07-02 20:04:20

What is the relationship between angle w and angle x?

jonjoseph 2021-07-02 20:05:36

From the given information we have $2x = w$ or $x = \dfrac{w}{2}$.

jonjoseph 2021-07-02 20:05:43

Applying this same reasoning to each of the angles, what does our quadrilateral equation become in terms of w?

jonjoseph 2021-07-02 20:06:51

Correct:

$$w + \frac{w}{2} + \frac{w}{3} + \frac{w}{4} = 360°.$$

jonjoseph   2021-07-02 20:06:55
Fractions can be a pain - what's a quick way to get rid of all the fractions at once?

jonjoseph   2021-07-02 20:07:45
If we multiply through by $\operatorname{lcm}(2, 3, 4) = 12$ then

$$12w + 6w + 4w + 3w = 12 \cdot 360.$$

jonjoseph   2021-07-02 20:07:51
Can you finish? What's the answer?

jonjoseph   2021-07-02 20:09:17
We see that $\angle A = w = 12 \cdot \dfrac{72}{5} = \dfrac{864}{5} = 172.8$, which is 173 when rounded to the nearest integer. The answer is (D).

jonjoseph   2021-07-02 20:09:38
A bag initially contains red marbles and blue marbles only, with more blue than red. Red marbles are added to the bag until only $1/3$ of the marbles in the bag are blue. Then yellow marbles are added to the bag until only $1/5$ of the marbles in the bag are blue. Finally, the number of blue marbles in the bag is doubled. What fraction of the marbles now in the bag are blue?

(A) $\dfrac{1}{5}$ (B) $\dfrac{1}{4}$ (C) $\dfrac{1}{3}$ (D) $\dfrac{2}{5}$ (E) $\dfrac{1}{2}$

jonjoseph   2021-07-02 20:10:12
This problem is difficult because there are so many unknowns to work with: the original amount of blue marbles, the original amount of red marbles, the number of red marbles added, and the number of yellow marbles added.

jonjoseph   2021-07-02 20:10:24
We could create a variable for each of these quantities, but things get complex quickly. To keep things simple, we should try to use as few variables as possible.

jonjoseph   2021-07-02 20:10:44
Let's solve this "hard way" first. Then rethink it

jonjoseph   2021-07-02 20:10:53
The bag initially contains red and blue marbles. We then add red marbles until 1/3 of the marbles are blue. What can we say about the red marbles at this point?

jonjoseph   2021-07-02 20:11:44
We can say that $2/3$ of the marbles are red. So how can we represent the number of blue and red marbles by variables?

jonjoseph   2021-07-02 20:13:07
If we let b be the number of blue marbles, then there are twice as many red marbles. So, the number of red marbles is $2b$. We can double check this by seeing that there's a total of $b + 2b = 3b$ marbles, of which $\frac{1}{3} \cdot 3b = b$ are blue and $\frac{2}{3} \cdot 3b = 2b$ are red.

jonjoseph   2021-07-02 20:13:16
We then add yellow marbles, until $1/5$ of the marbles are blue.

jonjoseph   2021-07-02 20:13:21
We don't immediately have a way to write the number of yellow marbles in terms of b, so let's call this quantity y. What equation can we write down?

jonjoseph   2021-07-02 20:13:54
Hint: The total is now $3b + y$.

jonjoseph   2021-07-02 20:15:11

We previously had $3b$ total marbles, and now we've added y marbles, so we have $3b + y$ total marbles. Since there is a ratio of 1 blue marble (b) to 5 total marbles $(3b + y)$, we can write a proportion:

$$\frac{b}{3b + y} = \frac{1}{5}.$$

jonjoseph 2021-07-02 20:15:29

If we solve for y in terms of b, we'll know how many of each color of marble we have in terms of b. What is y in terms of b?

jonjoseph 2021-07-02 20:16:14

Cross-multiplying, we get $5b = 3b + y$, so $y = 2b$.

jonjoseph 2021-07-02 20:16:20

So now, there are b blue marbles, $2b$ red marbles, and $2b$ yellow marbles.

jonjoseph 2021-07-02 20:16:24

Finally, we double the number of blue marbles. What fraction of the marbles are now blue?

jonjoseph 2021-07-02 20:17:46

Doubling only the number of blue marbles (not red or yellow marbles), we get $2b$ blue marbles, $2b$ red marbles, and $2b$ yellow marbles. Since there are $6b$ total marbles, we have

$$\frac{2b}{6b} = \frac{1}{3}.$$

jonjoseph 2021-07-02 20:17:48

Therefore, the fraction of marbles that are now blue is $1/3$. The answer is (C).

jonjoseph 2021-07-02 20:18:15

Now, a couple of you saw the shortcut (shoutout to wendywxy and Sipdip!!)

jonjoseph 2021-07-02 20:18:23

Let's look at that now.

jonjoseph 2021-07-02 20:18:36

Is there one step of it we can look at that is enough to solve the problem?

jonjoseph 2021-07-02 20:19:18

The only step that matters is the last step, when we double the blue marbles!

jonjoseph 2021-07-02 20:19:21

Let b be the number of blue marbles in the bag before we double the number. How many total marbles are in the bag at that point?

jonjoseph 2021-07-02 20:20:13

Since the blue marbles make up $1/5$ of the bag, there are $5b$ total marbles. So how many non-blue marbles are in the bag?

jonjoseph 2021-07-02 20:20:49

There are $4b$ non-blue marbles. So how many blue and non-blue marbles are in the bag after we double the number of blue marbles?

jonjoseph 2021-07-02 20:21:53

Nice

jonjoseph 2021-07-02 20:21:55

There are $2b$ blue marbles and $4b$ non-blue marbles. What fraction of the marbles are blue at the end?

jonjoseph 2021-07-02 20:22:41

$2b/6b = 1/3$ of the marbles are blue at the end. So the answer is (C).

jonjoseph   2021-07-02 20:22:48
A tad easier.

jonjoseph   2021-07-02 20:23:24
I'm not sure. Let me know if you find it and I'll tell the group.

jonjoseph   2021-07-02 20:24:05
Watch out for those problems that seems to overwhelm you with information.

jonjoseph   2021-07-02 20:24:32
Yes. But they can go way, way back.

jonjoseph   2021-07-02 20:25:33
1980's?

jonjoseph   2021-07-02 20:25:42
STATISTICS

jonjoseph   2021-07-02 20:25:53
Given a list of numbers, what is the average?

jonjoseph   2021-07-02 20:27:37
The average is the sum of the numbers divided by the number of items in the list. For example, the average of the numbers 9, $6, 2$, and 3 is $\dfrac{9+6+2+3}{4} = \dfrac{20}{4} = 5.$

jonjoseph   2021-07-02 20:27:48
The average of a list is also known as the mean.

jonjoseph   2021-07-02 20:27:59
Uhm...

jonjoseph   2021-07-02 20:28:01
My statistics teacher once called me an average student. I thought that was mean.

jonjoseph   2021-07-02 20:28:32
The average value of all the pennies, nickels, dimes, and quarters in Paula's purse is 20 cents. If she had one more quarter, the average value would be 21 cents. How many dimes does she have in her purse?

$$\text{(A) } 0 \quad \text{(B) } 1 \quad \text{(C) } 2 \quad \text{(D) } 3 \quad \text{(E) } 4$$

jonjoseph   2021-07-02 20:29:00
We may be tempted to create four variables for the unknown amounts of pennies, nickels, dimes, and quarters, but let's see if we can keep things simple.

jonjoseph   2021-07-02 20:29:06
We are told that the average value of all of Paula's coins is 20 cents.

jonjoseph   2021-07-02 20:29:10
Averages and totals naturally relate to each other. What do we need to know how much money Paula has in total?

jonjoseph   2021-07-02 20:30:09
We need to know how many coins Paula has in total. We don't know this, so let n be the number of coins Paula has. Then how much money does Paula have in total, in terms of n?

jonjoseph   2021-07-02 20:31:26
In total, Paula has $20n$ cents.

jonjoseph   2021-07-02 20:31:29
If Paula had one more quarter, then the average value of her coins would be 21 cents. What equation does this give us?

jonjoseph   2021-07-02 20:33:16

If Paula had one more quarter, she would have $20n + 25$ cents, among $n + 1$ coins.

jonjoseph   2021-07-02 20:33:18

Therefore,

$$\frac{20n + 25}{n + 1} = 21.$$

jonjoseph   2021-07-02 20:33:20

So what is the value of $n?$

jonjoseph   2021-07-02 20:34:29

Multiplying both sides by $n + 1$, we get $20n + 25 = 21n + 21$.

jonjoseph   2021-07-02 20:34:36

Solving for n, we find $n = 4$. At this point, we might look at the answers and choose (E), and later, we would be sad.

jonjoseph   2021-07-02 20:34:39

What should we do instead?

jonjoseph   2021-07-02 20:35:23

Paula has $20 \cdot 4 = 80$ cents, among 4 coins.

jonjoseph   2021-07-02 20:35:26

Can we now figure out how many dimes Paula has?

jonjoseph   2021-07-02 20:35:31

How many ways can you make 80 cents from 4 coins?

jonjoseph   2021-07-02 20:35:35

Hint: Take cases on the number of quarters.

jonjoseph   2021-07-02 20:36:44

Aha! The only way to make 80 cents from 4 coins is to use 3 quarters and 1 nickel. If you have fewer than 3 quarters, then you can't get up to 80 cents with the rest of the coins.

jonjoseph   2021-07-02 20:36:48

Hence, Paula has no dimes, so the answer is (A).

jonjoseph   2021-07-02 20:37:10

Ugh. I wanted at least one dime.

jonjoseph   2021-07-02 20:37:30

Me too.

jonjoseph   2021-07-02 20:37:42

Now for another idea. Given a list of numbers, the median is the middle element when the numbers are in increasing order. What is the median of the list $8, 1, 10, 2, 5?$

jonjoseph   2021-07-02 20:38:50

First, we must sort the list in increasing order, which gives us $1, 2, 5, 8, 10$.

jonjoseph   2021-07-02 20:38:53

The median of $1, 2, 5, 8, 10$ is $\boxed{5}$, the middle element. (The idea behind the median is that half the elements are greater than the median, and the other half are less than the median.)

jonjoseph   2021-07-02 20:39:01

So if a list contains an odd number of terms, then the median is always the middle term.

jonjoseph   2021-07-02 20:39:06

If a list contains an even number of terms, then the median is the average of the middle two terms.

jonjoseph   2021-07-02 20:39:19

What is the median of the list $8, 4, 7, 1, 12, 6$?

jonjoseph   2021-07-02 20:40:08

First, we sort the list in increasing order, to get $1, 4, 6, 7, 8, 12$.

jonjoseph   2021-07-02 20:40:12

The median of $1, 4, 6, 7, 8, 12$ **is** $\dfrac{6+7}{2} = \dfrac{13}{2}$.

jonjoseph   2021-07-02 20:40:21

The mean of three numbers is 10 **more than the least of the numbers and** 15 **less than the greatest. The median of the three numbers is** 5**. What is their sum?**

$(A)\ 5$ $(B)\ 20$ $(C)\ 25$ $(D)\ 30$ $(E)\ 36$

jonjoseph   2021-07-02 20:40:43

As usual, we want to start by defining variables. What variables should we define?

jonjoseph   2021-07-02 20:41:19

We can start by letting the three numbers be variables, but we may be able to make things easier by looking at the information that is provided in the problem.

jonjoseph   2021-07-02 20:41:23

For example, we are told that the median of the three numbers is 5**. What does that tell us?**

jonjoseph   2021-07-02 20:42:11

This tells us that the middle of the three numbers is 5**. So already we know one of the numbers.**

jonjoseph   2021-07-02 20:42:14

We are also told that the mean of the three numbers is 10 **more than the smallest number, and** 15 **less than the largest number. What does that suggest we make a variable for?**

jonjoseph   2021-07-02 20:42:56

When you think about it, that means we don't need separate variables for those two numbers. We can instead just make a single variable for the mean.

jonjoseph   2021-07-02 20:43:04

So let x **be the mean of the three numbers. Then the smallest number is** $x - 10$**, and the largest number is** $x + 15$**.**

jonjoseph   2021-07-02 20:43:08

So the three numbers are $x - 10, 5$**, and** $x + 15$**. Now what?**

jonjoseph   2021-07-02 20:44:54

We know that x **is the mean of the three numbers, so**

$$\frac{(x-10) + 5 + (x+15)}{3} = x.$$

jonjoseph   2021-07-02 20:45:00

Is that clear?

jonjoseph   2021-07-02 20:45:14

So what is the value of x?

jonjoseph   2021-07-02 20:46:36

Solving for x**, we find** $x = 10$**. So what is the sum of the three numbers?**

jonjoseph   2021-07-02 20:47:19

Our three numbers are $x - 10 = 10 - 10 = 0, 5$, and $10 + 15 = 25$. So, the sum is $0 + 5 + 25 = 30$. The answer is (D).

jonjoseph   2021-07-02 20:47:31

On a certain math exam, 10% of the students got 70 points, 25% got 80 points, 20% got 85 points, 15% got 90 points, and the rest got 95 points. What is the difference between the mean and the median score on this exam?

$(A)\ 0$ $(B)\ 1$ $(C)\ 2$ $(D)\ 4$ $(E)\ 5$

jonjoseph   2021-07-02 20:48:05

We are told that 10% got 70 points, 25% got 80 points, 20% got 85 points, 15% got 90 points, and the rest got 95 points. As a percentage, how many got 95 points?

jonjoseph   2021-07-02 20:48:49

The first four percentages add up to $10\% + 25\% + 20\% + 15\% = 70\%$, so the remaining 30% got 95 points.

jonjoseph   2021-07-02 20:48:55

We want to find the difference between the mean and the median, so let's compute both.

jonjoseph   2021-07-02 20:48:57

To find the mean, do we take the average of $70, 80, 85, 90$ and 95?

jonjoseph   2021-07-02 20:49:24

Taking the average of $70, 80, 85, 90$, and 95 would give us the wrong answer.

jonjoseph   2021-07-02 20:49:30

If the number of students who got these points was equally distributed, then this would work. However, the percent of students who got these scores are not all the same, so we cannot simply take the average of these five numbers.

jonjoseph   2021-07-02 20:50:04

Is there anything about the problem that fixes the total number of students or can we choose a convenient number of students to work with?

jonjoseph   2021-07-02 20:51:02

It does not matter how many students are in the class, because we are dealing with percentages. All our quantities of students are proportional to the total number of students, which will cancel out when we take the mean.

jonjoseph   2021-07-02 20:51:49

Because percentages total 100% it is often very convenient to pick the total number at 100 (students in this case).

jonjoseph   2021-07-02 20:51:59

We know that 10% got 70 points, which would be 10 students. What is the sum of the scores for these 10 students?

jonjoseph   2021-07-02 20:52:34

The sum of the scores for these 10 students is $10 \cdot 70 = 700$.

jonjoseph   2021-07-02 20:52:37

We know that 25% got 80 points, which would be 25 students. What is the sum of the scores for these 25 students?

jonjoseph   2021-07-02 20:53:11

The sum of the scores for these 25 students is $25 \cdot 80 = 2000$.

jonjoseph   2021-07-02 20:53:12

Let's jump to the end. What is the sum of the scores for all 100 students?

jonjoseph   2021-07-02 20:54:31

The sum of the scores for all 100 students is $10 \cdot 70 + 25 \cdot 80 + 20 \cdot 85 + 15 \cdot 90 + 30 \cdot 95 = 8600$. So what is the average score?

jonjoseph   2021-07-02 20:55:25

The average score is $\dfrac{8600}{100} = 86.$

jonjoseph 2021-07-02 20:55:31

We could have also taken what is called a *weighted average* of the scores to get the same result. For example, 10% of students got 70 points, so that contributes $0.1(70) = 7$ points to the mean.

jonjoseph 2021-07-02 20:55:41

Altogether, the weighted average gives us a mean of $0.1(70) + 0.25(80) + 0.2(85) + 0.15(90) + 0.3(95)$, which simplifies to $7 + 20 + 17 + 13.5 + 28.5 = 86.$

jonjoseph 2021-07-02 20:55:56

Now we compute the median.

jonjoseph 2021-07-02 20:55:58

Here are the numbers again.

jonjoseph 2021-07-02 20:56:00

Number of Students	Score
10	70
25	80
20	85
15	90
30	95

jonjoseph 2021-07-02 20:56:03

If we list all 100 scores in increasing order, then the median is the average of the 50th score and the 51st score. So what is the median?

jonjoseph 2021-07-02 20:57:01

The first ten scores would be 70, and the next 25 scores would be 80. Then the next 20 scores are all 85. These include the 50th score and the 51st score.

jonjoseph 2021-07-02 20:57:07

Therefore, the median score is $\dfrac{85 + 85}{2} = 85.$

jonjoseph 2021-07-02 20:57:15

Answer?

jonjoseph 2021-07-02 20:57:56

Hence, the difference between the mean and the median score is $86 - 85 = 1$. The answer is (B).

jonjoseph 2021-07-02 20:58:20

Here is our last problem. We may not finish but let's start it.

jonjoseph 2021-07-02 20:58:23

Every high school in the city of Euclid sent a team of 3 students to a math contest. Each participant in the contest received a different score. Andrea's score was the median among all students, and hers was the highest score on her team. Andrea's teammates Beth and Carla placed 37th and 64th, respectively. How many schools are in the city?

(A) 22   (B) 23   (C) 24   (D) 25   (E) 26

jonjoseph 2021-07-02 20:58:56

At first glance, it might not seem like there is enough information in the problem to determine the answer. So let's read the problem carefully to make sure we catch every possible clue.

jonjoseph   2021-07-02 20:59:10

We are looking for the number of schools in the city, so let n be this number.

jonjoseph   2021-07-02 20:59:13

Then how many students participated in the contest?

jonjoseph   2021-07-02 20:59:57

Each high school sent a team of 3 students, so $3n$ students participated in the contest.

jonjoseph   2021-07-02 21:00:00

The formula for the median depends on whether there are an even or odd number of students, so let's decide which is the right formula first.

jonjoseph   2021-07-02 21:00:05

Suppose there is an even number of students. Then what is the median score?

jonjoseph   2021-07-02 21:00:36

If there is an even number of students, then the median score is the average of the two middle scores.

jonjoseph   2021-07-02 21:00:40

We are told that Andrea obtained the median score. (Her score is equal to the median score.) What can we conclude from this?

jonjoseph   2021-07-02 21:01:10

Since Andrea obtained the median score, another student must have obtained the same score, so that the average of the two scores is also Andrea's score.

jonjoseph   2021-07-02 21:01:12

But we are told that every student got a different score! So this is impossible.

jonjoseph   2021-07-02 21:01:15

Therefore, there must be an odd number of students, and Andrea got the unique middle score.

jonjoseph   2021-07-02 21:01:26

There are $3n$ students, so what is Andrea's ranking?

jonjoseph   2021-07-02 21:02:05

Andrea's ranking is $\dfrac{3n+1}{2}$.

jonjoseph   2021-07-02 21:02:12

Now remember that we are told that Andrea got the highest score on her team.

jonjoseph   2021-07-02 21:02:14

One of her teammates Beth placed 37th. So what does that say about Andrea's ranking?

jonjoseph   2021-07-02 21:03:15

It says that Andrea's ranking must be 36th or better. So what can we write?

jonjoseph   2021-07-02 21:03:20

We can write the inequality

$$\frac{3n+1}{2} \le 36.$$

jonjoseph   2021-07-02 21:03:22

What does this inequality say about $n?$

jonjoseph   2021-07-02 21:04:06

This inequality says that $n \le \frac{71}{3}$, but since n is an integer, $n \le 23$.

jonjoseph   2021-07-02 21:04:10

At this point, we can look at our multiple choice options! The only options that are less than or equal to 23 are (A) and (B). Can

we say which one it is?

jonjoseph   2021-07-02 21:05:00

We proved that the total number of students $3n$ is odd, which means that n is odd. So the answer must be (B), 23.

jonjoseph   2021-07-02 21:05:05

Well done!!!

jonjoseph   2021-07-02 21:05:24

SUMMARY

jonjoseph   2021-07-02 21:05:26

In today's class, we have seen how to work with percents, ratios, and proportions effectively. We also learned how to work with the mean and median of numbers, and how to use factorization in problems.

jonjoseph   2021-07-02 21:05:35

These may look like different topics, but they all have the same underlying theme. The key step in all of these problems is to take the given information and express it algebraically. This step is usually not difficult, but some ways of using variables are better than others, so look for ways that will keep your equations simple. You should also read the problem carefully, to make sure your equations are the correct ones.

jonjoseph   2021-07-02 21:05:53

Have a great and safe weekend. See you next week!!