2016 AIME I Problems/Problem 15

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Problem

Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

Solution

Using the radical axis theorem, the lines $\overline{AD}, \overline{BC}, \overline{XY}$ are all concurrent at one point, call it F. Now recall by Miquel's theorem in $\triangle FDC$ the fact that quadrilaterals $DAXY$ and $CBXY$ are cyclic implies $FAXB$ is cyclic as well. Denote $\omega_3 \equiv (FAXB)$ and $Z \equiv \ell \cap \overline{FXY}$.

Since point Z lies on the radical axis of ω_1, ω_2, it has equal power with respect to both circles, thus

$$AZ^2 = \mathrm{Pow}_{\omega_1}(Z) = ZX \cdot ZY = \mathrm{Pow}_{\omega_2}(Z) = ZB^2 \implies AZ = ZB.$$

Also, notice that

$$AZ \cdot ZB = \mathrm{Pow}_{\omega_3}(Z) = ZX \cdot ZF \implies ZY = ZF.$$

The diagonals of quadrilateral $FAYB$ bisect each other at Z, so we conclude that $FAYB$ is a parallelogram. Let $u := ZX$, so that $ZY = ZF = u + 47$.

Because $FAYB$ is a parallelogram and quadrilaterals $DAXY, CBXY$ are cyclic,

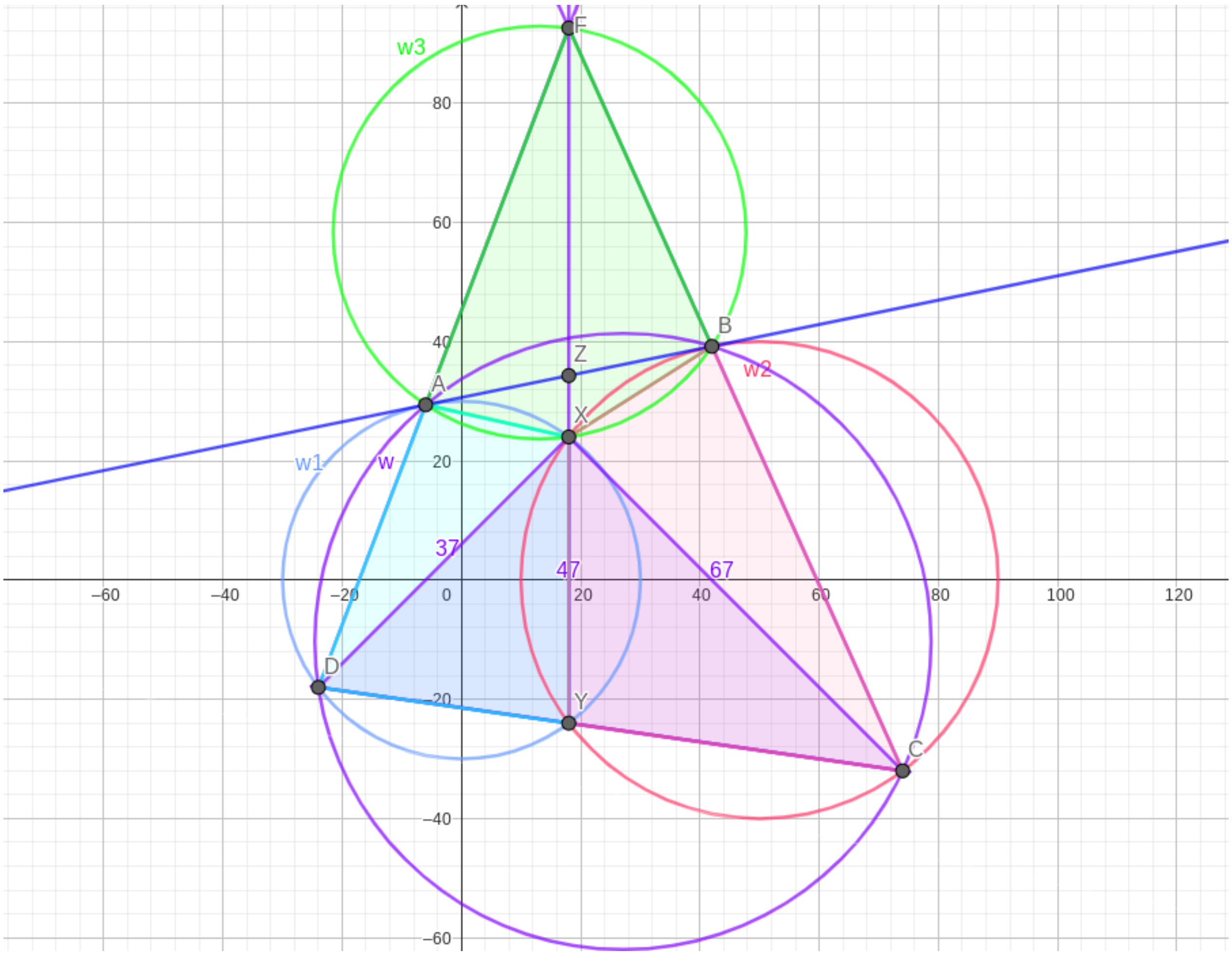
$$\angle DFX = \angle AFX = \angle BYX = \angle BCX = \angle FCX \text{ and } \angle XDF = \angle XDA = \angle XYA = \angle XFB = \angle XFC$$

so we have the pair of similar triangles $\triangle DFX \sim \triangle FCX$. Thus

$$\frac{37}{2u+47} = \frac{2u+47}{67} \implies 2u{+}47 = \sqrt{37 \cdot 67} \implies u = \frac{1}{2}\left(\sqrt{37 \cdot 67} - 47\right).$$

Now compute

$$AB^2 = 4AZ^2 = 4 \cdot ZX \cdot ZY = 4u(u+47) = 37 \cdot 67 - 47^2 = \mathbf{270}.$$



Solution 1

Let $Z = XY \cap AB$. By the radical axis theorem AD, XY, BC are concurrent, say at P. Moreover, $\triangle DXP \sim \triangle PXC$ by simple angle chasing. Let $y = PX, x = XZ$. Then

$$\frac{y}{37} = \frac{67}{y} \qquad \Longrightarrow \qquad y^2 = 37 \cdot 67.$$

Now, $AZ^2 = \frac{1}{4}AB^2$, and by power of a point,

$$x(y-x) = \tfrac{1}{4}AB^2, \quad \text{and}$$
$$x(47+x) = \tfrac{1}{4}AB^2$$

Solving, we get

$$\tfrac{1}{4}AB^2 = \tfrac{1}{2}(y-47) \cdot \tfrac{1}{2}(y+47) \qquad \Longrightarrow$$

$$AB^2 = 37 \cdot 67 - 47^2 = \boxed{270}$$

Solution 2

By the Radical Axis Theorem AD, XY, BC concur at point E.

Let AB and EY intersect at S. Note that because $AXDY$ and $CYXB$ are cyclic, by Miquel's Theorem $AXBE$ is cyclic as well. Thus

$$\angle AEX = \angle ABX = \angle XCB = \angle XYB$$

and

$$\angle XEB = \angle XAB = \angle XDA = \angle XYA.$$

Thus $AY \parallel EB$ and $YB \parallel EA$, so $AEBY$ is a parallelogram. Hence $AS = SB$ and $SE = SY$. But notice that DXE and EXC are similar by AA Similarity, so $XE^2 = XD \cdot XC = 37 \cdot 67$. But

$$XE^2 - XY^2 = (XE+XY)(XE-XY) = EY \cdot 2XS = 2SY \cdot 2SX = 4SA^2 = AB^2.$$

Hence $AB^2 = 37 \cdot 67 - 47^2 = \boxed{270}$.

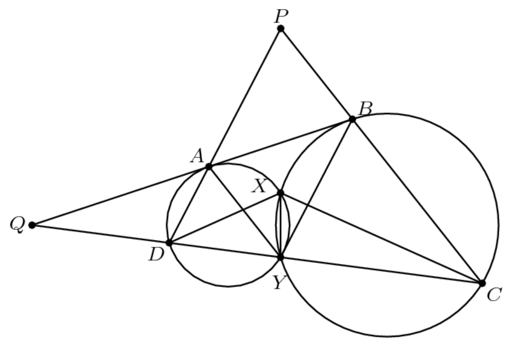
Solution 3

First, we note that as $\triangle XDY$ and $\triangle XYC$ have bases along the same line, $\frac{[\triangle XDY]}{[\triangle XYC]} = \frac{DY}{YC}$. We can also find the ratio of their areas using the circumradius area formula. If R_1 is the radius of ω_1 and if R_2 is the radius of ω_2, then

$$\frac{[\triangle XDY]}{[\triangle XYC]} = \frac{(37 \cdot 47 \cdot DY)/(4R_1)}{(47 \cdot 67 \cdot YC)/(4R_2)} = \frac{37 \cdot DY \cdot R_2}{67 \cdot YC \cdot R_1}.$$

Since we showed this to be $\frac{DY}{YC}$, we see that $\frac{R_2}{R_1} = \frac{67}{37}$.

We extend AD and BC to meet at point P, and we extend AB and CD to meet at point Q as shown below.



As $ABCD$ is cyclic, we know that $\angle BCD = 180 - \angle DAB = \angle BAP$. But then as AB is tangent to ω_2 at B, we see that $\angle BCD = \angle ABY$. Therefore, $\angle ABY = \angle BAP$, and $BY \parallel PD$. A similar argument shows $AY \parallel PC$. These parallel lines show $\triangle PDC \sim \triangle ADY \sim \triangle BYC$. Also, we showed that $\frac{R_2}{R_1} = \frac{67}{37}$, so the ratio of similarity between $\triangle ADY$ and $\triangle BYC$ is $\frac{37}{67}$, or rather

$$\frac{AD}{BY} = \frac{DY}{YC} = \frac{YA}{CB} = \frac{37}{67}.$$

We can now use the parallel lines to find more similar triangles. As $\triangle AQD \sim \triangle BQY$, we know that

$$\frac{QA}{QB} = \frac{QD}{QY} = \frac{AD}{BY} = \frac{37}{67}.$$

Setting $QA = 37x$, we see that $QB = 67x$, hence $AB = 30x$, and the problem simplifies to finding 30^2x^2. Setting $QD = 37^2y$, we also see that $QY = 37 \cdot 67y$, hence $DY = 37 \cdot 30y$. Also, as $\triangle AQY \sim \triangle BQC$, we find that

$$\frac{QY}{QC} = \frac{YA}{CB} = \frac{37}{67}.$$

As $QY = 37 \cdot 67y$, we see that $QC = 67^2y$, hence $YC = 67 \cdot 30y$.

Applying Power of a Point to point Q with respect to ω_2, we find

$$67^2x^2 = 37 \cdot 67^3y^2,$$

or $x^2 = 37 \cdot 67y^2$. We wish to find $AB^2 = 30^2x^2 = 30^2 \cdot 37 \cdot 67y^2$.

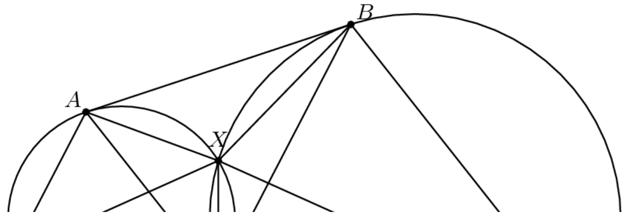
Applying Stewart's Theorem to $\triangle XDC$, we find

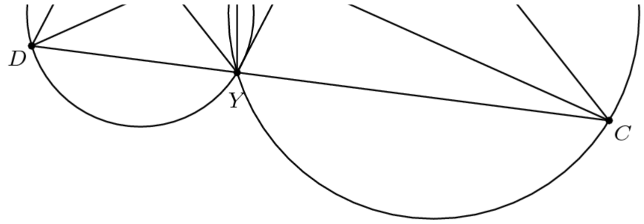
$$37^2 \cdot (67 \cdot 30y) + 67^2 \cdot (37 \cdot 30y) = (67 \cdot 30y) \cdot (37 \cdot 30y) \cdot (104 \cdot 30y) + 47^2 \cdot (104 \cdot 30y).$$

We can cancel $30 \cdot 104 \cdot y$ from both sides, finding $37 \cdot 67 = 30^2 \cdot 67 \cdot 37y^2 + 47^2$. Therefore,

$$AB^2 = 30^2 \cdot 37 \cdot 67y^2 = 37 \cdot 67 - 47^2 = \boxed{270}.$$

Solution 4





First of all, since quadrilaterals $ADYX$ and $XYCB$ are cyclic, we can let $\angle DAX = \angle XYC = \theta$, and $\angle XYD = \angle CBX = 180 - \theta$, due to the properties of cyclic quadrilaterals. In addition, let $\angle BAX = x$ and $\angle ABX = y$. Thus, $\angle ADX = \angle AYX = x$ and $\angle XYB = \angle XCB = y$. Then, since quadrilateral $ABCD$ is cyclic as well, we have the following sums:

$$\theta + x + \angle XCY + y = 180^\circ$$

$$180^\circ - \theta + y + \angle XDY + x = 180^\circ$$

Cancelling out 180° in the second equation and isolating θ yields $\theta = y + \angle XDY + x$. Substituting θ back into the first equation, we obtain

$$2x + 2y + \angle XCY + \angle XDY = 180^\circ$$

Since

$$x + y + \angle XAY + \angle XCY + \angle DAY = 180^\circ$$

$$x + y + \angle XDY + \angle XCY + \angle DAY = 180^\circ$$

we can then imply that $\angle DAY = x + y$. Similarly, $\angle YBC = x + y$. So then $\angle DXY = \angle YXC = x + y$, so since we know that XY bisects $\angle DXC$, we can solve for DY and YC with Stewart's Theorem. Let $DY = 37n$ and $YC = 67n$. Then

$$37n \cdot 67n \cdot 104n + 47^2 \cdot 104n = 37^2 \cdot 67n + 67^2 \cdot 37n$$

$$37n \cdot 67n + 47^2 = 37 \cdot 67$$

$$n^2 = \frac{270}{2479}$$

Now, since $\angle AYX = x$ and $\angle BYX = y$, $\angle AYB = x + y$. From there, let $\angle AYD = \alpha$ and $\angle BYC = \beta$. From angle chasing we can derive that $\angle YDX = \angle YAX = \beta - x$ and $\angle YCX = \angle YBX = \alpha - y$. From there, since $\angle ADX = x$, it is quite clear that $\angle ADY = \beta$, and $\angle YAB = \beta$ can be found similarly. From there, since $\angle ADY = \angle YAB = \angle BYC = \beta$ and $\angle DAY = \angle AYB = \angle YBC = x + y$, we have AA similarity between $\triangle DAY$, $\triangle AYB$, and $\triangle YBC$. Therefore the length of AY is the geometric mean of the lengths of DA and YB (from $\triangle DAY \sim \triangle AYB$). However, $\triangle DAY \sim \triangle AYB \sim \triangle YBC$ yields the proportion $\dfrac{AD}{DY} = \dfrac{YA}{AB} = \dfrac{BY}{YC}$; hence, the length of AB is the geometric mean of the lengths of DY and YC. We can now simply use arithmetic to calculate AB^2.

$$AB^2 = DY \cdot YC$$

$$AB^2 = 37 \cdot 67 \cdot \frac{270}{2479}$$

$$AB^2 = \boxed{270}$$

-Solution by TheBoomBox77

Solution 5 (not too different)

Let $E = DA \cap CB$. By Radical Axes, E lies on XY. Note that $EAXB$ is cyclic as X is the Miquel point of $\triangle EDC$ in this configuration.

Claim. $\triangle DXE \sim \triangle EXC$ Proof. We angle chase.

$$\measuredangle XEC = \measuredangle XEB = \measuredangle XAB = \measuredangle XDA = \measuredangle XDE$$

and

$$\measuredangle XCE = \measuredangle XCB = \measuredangle XBA = \measuredangle XEA = \measuredangle XED. \square$$

Let $F = EX \cap AB$. Note

$$FA^2 = FX \cdot FY = FB^2$$

and

$$EF \cdot FX = AF \cdot FB = FA^2 = FX \cdot FY \implies EF = FY$$

By our claim,

$$\frac{DX}{XE} = \frac{EX}{XC} \implies EX^2 = DX \cdot XC = 67 \cdot 37 \implies FY = \frac{EY}{2} = \frac{EX + XY}{2} = \frac{\sqrt{67 \cdot 37} + 47}{2}$$

and

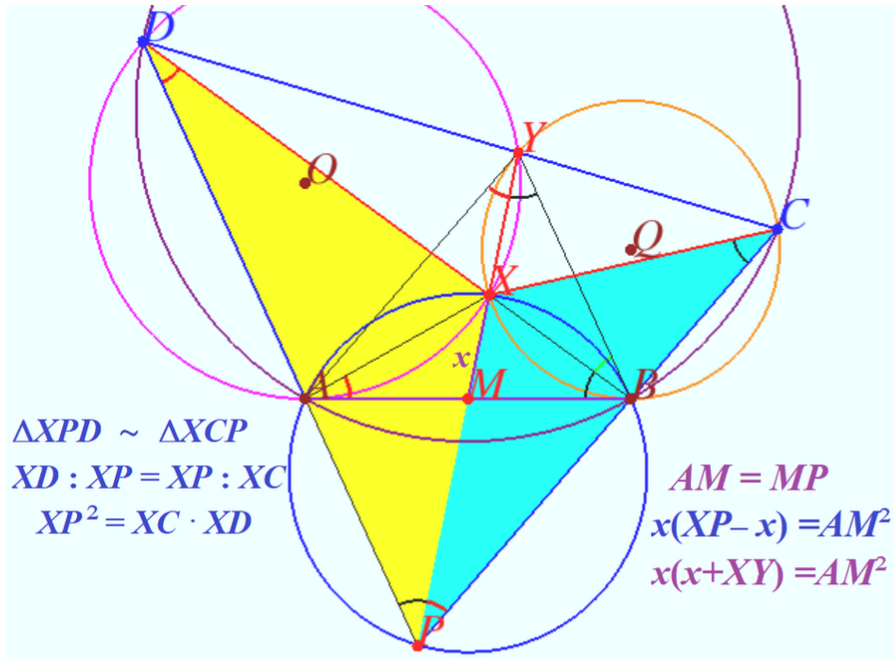
$$FX = FY - 47 = \frac{\sqrt{67 \cdot 37} - 47}{2}$$

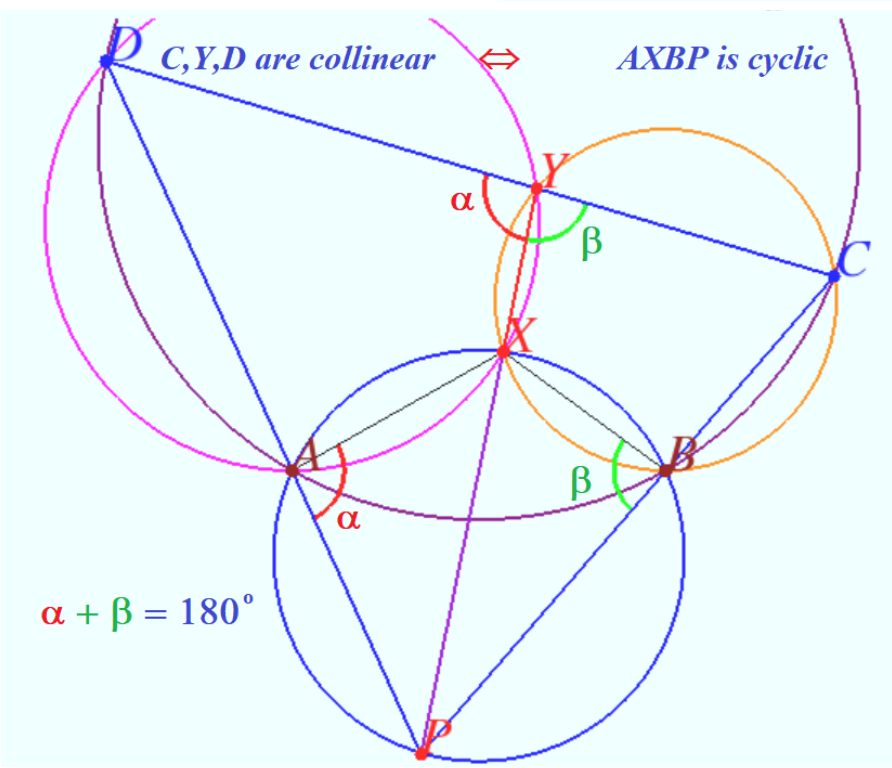
Finally,

$$AB^2 = (2 \cdot FA)^2 = 4 \cdot FX \cdot FY = 4 \cdot \frac{(67 \cdot 37) - 47^2}{4} = \boxed{270}. \blacksquare$$

~Mathscienceclass

Solution 6 (No words)





$AB^2 = 4AM^2 = 2x(2x + 2XY) = (XP - XY)(XP + XY) = XP^2 - XY^2 = XC \cdot XD - XY^2 = 67 \cdot 37 - 47^2 = \boxed{270}.$

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Solution 7 (Linearity of Power of a Point)

Extend $\overline{AD}$ and $\overline{BC}$ to meet at point P. Let M be the midpoint of segment AB. Then by radical axis on (ADY), (BCY) and $(ABCD)$, P lies on XY. By the bisector lemma, M lies on XY. It is well-known that P, A, X, and B are concyclic. By Power of a point on M with respect to $(PAXB)$ and (ADY),

$$|\mathrm{Pow}(M,(PAXB))| = MX{\cdot}MP = MA^2 = |\mathrm{Pow}(M,(ADY))| = MX{\cdot}MY,$$

so $MP = MY$. Thus AB and PY bisect each other, so $PAYB$ is a parallelogram. This implies that

$$\angle DAY = \angle YBC,$$

so by the inscribed angle theorem $\overline{XY}$ bisects $\angle DXC$.

Claim: $AB^2 = DY \cdot YC$.

Proof. Define the linear function $f(\bullet) := \mathrm{Pow}(\bullet, (ADY)) - \mathrm{Pow}(\bullet, (ABCD))$. Since $\overline{BY}$ is parallel to the radical axis $\overline{AD}$ of (ADY) and $(ABCD)$ by our previous parallelism, $f(B) = f(Y)$. Note that $f(B) = AB^2$ while $f(Y) = DY \cdot YC$, so we conclude. $\square$

By Stewart's theorem on $\triangle DXC$, $DY \cdot YC = 37 \cdot 67 - 47^2 = 270$, so $AB^2 = \boxed{270}$.

~ Leo.Euler

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