Slopes of Perpendicular Lines

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Abstract

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In mathematics, perpendicular lines are more commonly called orthogonal lines. On a plane, the two concepts are equivalent: orthogonality is an extension of perpendicularity to spaces of higher dimension than the plane. The situation is depicted in Figure 1. Note that $a < 0$ and $b > 0$ in this example. Are the slopes always of opposite signs?

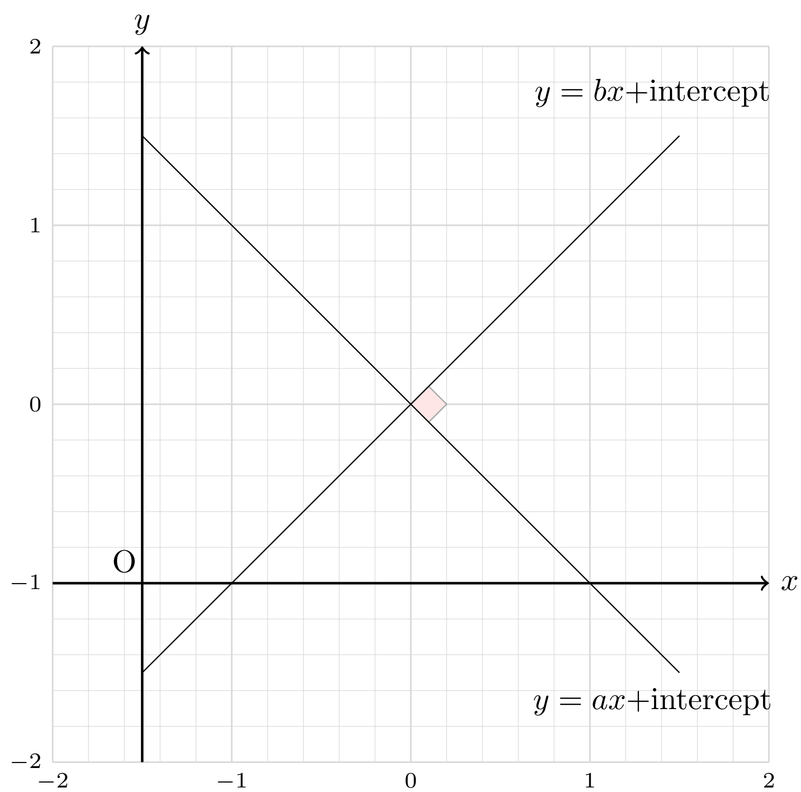

Figure 1: **Two Orthogonal Lines in a Cartesian Coordinate System**.

We have very little information to go by. Can we infer the slope b from a? Consider a special case. Take the 45° line and consider the line orthogonal to it that goes through the origin. The orthogonal line is clearly the $-45°$ line — or equivalently the $315°$ line $(360-45)$. In radians, the $45°$ line has angle $\frac{\pi}{4}$, while the $-45°$ line has angle $\frac{-\pi}{4} = \frac{7\pi}{8} \mod 2\pi$. By inspection and considerations of symmetry, the slope of the $-45°$ line is clearly -1. So is the answer simply $b = -a$? No! Read on.

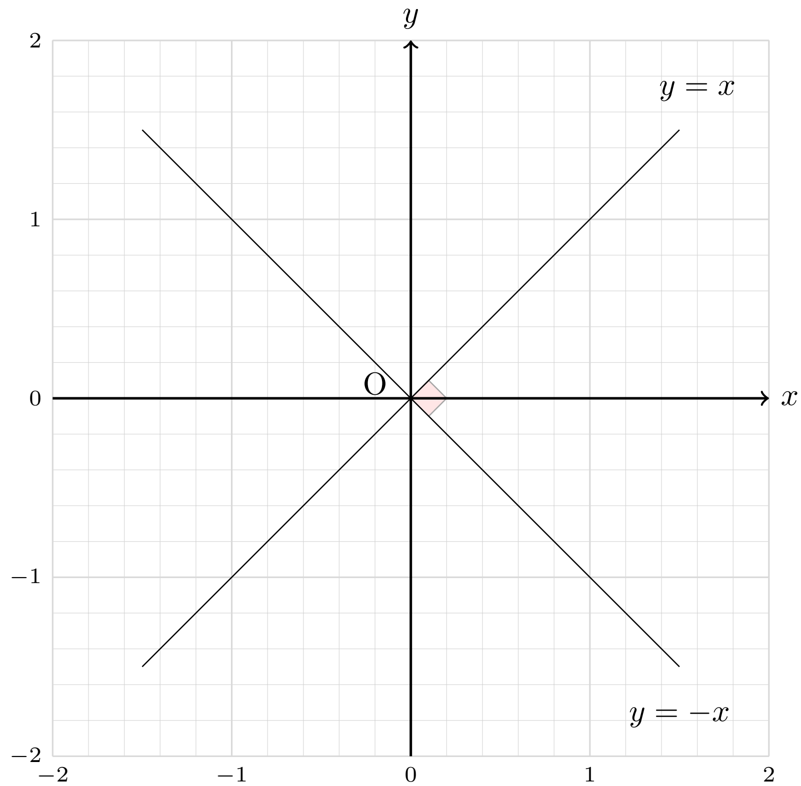

Figure 2: **Simple Example: The $45°$ line**.

Consider now the general case. Since only the slope matters in this problem, we can take two lines that intersect at the origin. Figure 3 shows that we can also represent the slopes graphically. As we are now considering two lines that intersect at the origin, their equations are simply $y = ax$ and $y = bx$. And so for $x = 1$, say, we have $y = a$ on line OA and $y = b$ on line OB. Can we find an expression for b in terms of a? Amazingly we can! Thanks to several right triangles and the Pythagoras theorem.

Triangle AOB yields:

$$(b-a)^2 = OA^2 + OB^2$$

Triangle OaA yields:

$$OA^2 = 1^2 + a^2$$

Triangle ObB yields:

$$OB^2 = 1^2 + b^2$$

Putting it all together gives:

$$\begin{aligned} (b-a)^2 &= \;\; OA^2 \quad + \quad OB^2 \\ &= 1^2 + a^2 \;\; + \;\; 1^2 + b^2 \\ b^2 - 2ab + a^2 &= a^2 + b^2 + 2 \\ -2ab &= 2 \\ ab &= -1 \end{aligned}$$

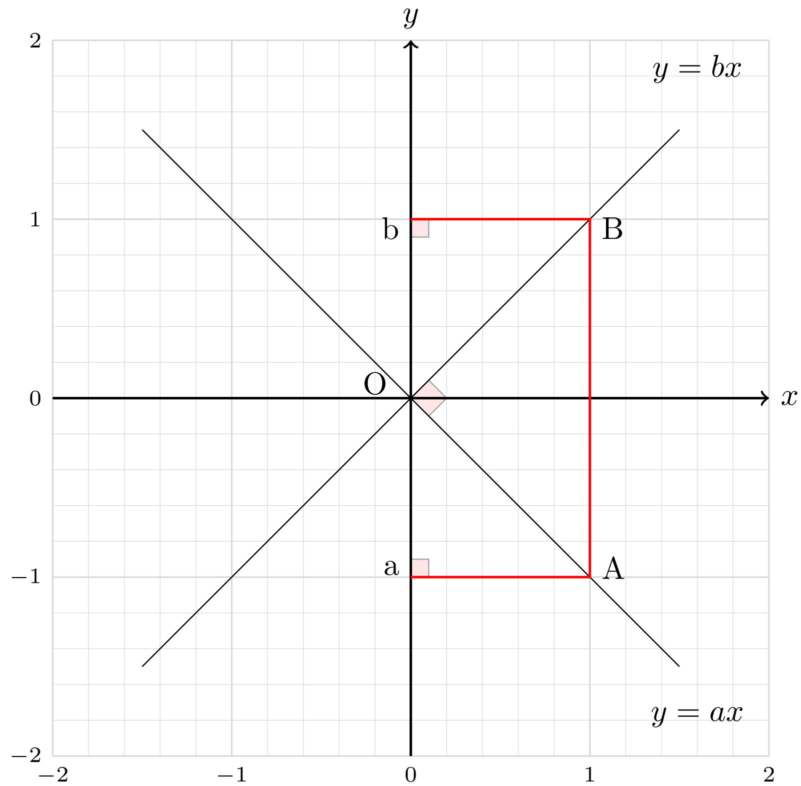

Figure 3: **Two Orthogonal Lines form Three Pythagorean Triangles**.

The slope of any perpendicular line is therefore equal to **minus the inverse** slope:

$$\boxed{b = -\frac{1}{a}}$$