Russian School of Math: Lesson 6

James & Patrick

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Abstract

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

1

Convert 100_{b+1} to base b, where $b \geq 3$.

Solution

Convert 100_{b+1} to base b, where $b \geq 3$.

$$100_{b+1} = 1 \times (b+1)^2 + 0 \times (b+1)^1 + 0 \times (b+1)^0 = b^2 + 2b + 1 = 1 \times b^2 + 2 \times b^1 + 1 \times b^0 \to 121_b$$

If $b < 3$, the factor 2 in front of b^2 would unravel.
Solution: $\boxed{121_b.}$

2

The repeating decimals of $0.\overline{ab}$ and $0.\overline{abc}$ satisfy $0.\overline{ab} + 0.\overline{abc} = \frac{33}{37}$, where a, b, c are (not necessarily distinct) digits. Find the three-digit number $\overline{abc}$.

Solution

First, note that $\dfrac{33}{37} = 0.\overline{891}$. Next, note that adding $0.\overline{ab}$ and $0.\overline{abc}$ gives:

Column No. $\rightarrow$		1	2	3	4	5	6	
	0.	a	b	a	b	a	b	...
+	0.	a	b	c	a	b	c	...
=	0.	2a	2b	a+c	b+a	a+b	b+c	...
=	0.	8	9	1	8	9	1	...

We attempt to match $2a$, $2b$, $(a+c)$, etc. with the digits 8, 9, and 1.
The first attempt fails. First, since the sum is less than 1, this suggests that $2a = 8$, or $a = 4$. Next, since $2b$ is not a multiple of 9, this suggests that there was a carry, so we take 10 away from the previous position and add 1 to the current position. Putting it together,

$$\begin{aligned} 2a &= 8 \\ 2b + 1 &= 9 \\ a + c - 10 &= 1 \end{aligned}$$

Solving the system gives $a = 8$, $b = 2$ $c = 7$. But clearly that is not right! The second attempt succeeds. Note that for the adjacent $(b+a) \to 8$ and $(a+b) \to 9$ in columns 4 and 5 to match, there must have been a carry that stopped there, that is:

$$\begin{aligned} a + b + 1 &= 9 \\ b + a &= 8 \end{aligned}$$

Next, column 3 suggests $a + c = 1$, but that clearly is not consistent with $a + b = 8$, so there must be a carry, and so $a + c - 10 = 1$. Next, column 2 suggests $2b = 9$, which does not divide evenly, so there must be a carry: $2b + 1 = 9$. Putting it together,

$$\begin{aligned} a + b &= 8 \\ a + c - 10 &= 1 \\ 2b + 1 &= 9 \end{aligned}$$

which yields $a = 4$, $b = 4$, $c = 7$.
Solution: $\boxed{\overline{abc} = \overline{447}.}$

3

Find the number of ending zeros of 2018! in base 9. Give your answer in base 9.

Solution

Related question:

Find the number of ending zeros of 21! in base 9. Count the number of times 3 appears as a factor in 21!

1 from 3
1 from 6
2 from 9
1 from 12
1 from 15
2 from 18
1 from 21

There are a total of 9 factors, so 21! is divisible by $3^9 = 3 \times 9^4$. It is divisible by 9 at most 4 times, so it ends with four zeroes in base 9.

We count the number of times 3 appears as a factor in 2018!.

1 from 3
1 from 6
2 from 9
1 from 12
1 from 15
2 from 18
1 from 21
...

The total count is incremented by 1 for the number of times 3 goes into 2018; by 1 for the number of times 6 goes into 2018, by 2 for the number of times 9 goes into 2018, and the cycle repeats.

1×224 from 3, 12, ..., 2001, 2010.
2×224 from 6, 18, ..., 2004, 2013.
1×224 from 9, 21, ..., 2007, 2016.

There are a total of $4 \times 224 = 896$ factors, so 218! is divisible by 3^{896}. Since $896 = 3 \times 298 + 2$, we have $3^{896} = 3^{3 \times 298 + 2} = 9^{298} \times 3^2 = 9^{299}$. It is divisible by 9 exactly 299 times, so it ends in 299 zeros in base 9.
Solution: $\boxed{299.}$

4

How many natural decimal numbers are 3-digit numbers when written in base 12 and 4-digit numbers when written in base 8.

Solution

We first find the ranges of numbers that correspond to these digit requirements in each base.

Step 1: Find the range for 3-digit numbers in base 12

A natural number n requires 3 digits in base 12 if it satisfies the following inequality:

$$122 \leq n < 123$$

The powers of 12 are:

$$12^2 = 144$$
$$12^3 = 1728$$

so the range for n in base 12 is:

$$144 \leq n < 1728$$

Step 2: Find the range for 4-digit numbers in base 9

A natural number n requires 4 digits in base 9 if it satisfies the following inequality:

$$93 \leq n < 94$$

The powers of 9 are:

$$9^3 = 729$$
$$9^4 = 6561$$

so the range for n in base 9 is:

$$729 \leq n < 6561$$

Step 3: Find the intersection of the two ranges

$$\{144 \leq n < 1728\} \cap \{729 \leq n < 6561\}$$

The overlap of these two ranges is:

$$729 \leq n < 1728$$

Step 4: Calculate the number of natural numbers in the intersection

The smallest integer in the range is 729. The largest integer in the range is 1727, so the count is $1727729 + 1 = 999$. Solution: $\boxed{999.}$

5

A number N has three digits when expressed in base 7. When N is expressed in base 9 the digits are reversed. Find the middle digit in either representation of N.

Solution

Let $\overline{abc}$ denote the number in base 7. Breaking down the number gives:

$$49a + 7b + c = 81c + 9b + 1a$$

Simplifying gives $48a - 2b - 80c = 0$. Writing the middle digit b in terms of a and c:

$$b = 24a - 40c = 8(3a - 5c)$$

Since b is a multiple of 8, $b = 0$ in base 7.
Solution: $\boxed{0.}$

6

The number n can be written in base 14 as $\overline{abc}_{14}$; it can be written in base 15 as $\overline{acb}_{15}$; and in base 6 as $\overline{acac}_6$, where $a > 0$. Find the base 10 representation of n.

Solution

The number n can be written as follows:

$$\begin{aligned} n &= a \times 14^2 + b \times 14^1 + c \times 14^0 \\ &= a \times 15^2 + c \times 15^1 + b \times 15^0 \\ &= a \times 6^3 + c \times 6^2 + a \times 6^1 + c \times 6^0 \end{aligned}$$

This is a system of 3 equations in 4 unknowns to be solved for integers.

$$\begin{aligned} &(1) \quad n = 196a + 14b + c \\ &(2) \quad\;\; = 225a + 15c + b \\ &(3) \quad\;\; = 222a + 37c \end{aligned}$$

From (2)−(1):

$$(4) \quad 29a + 14c = 13b$$

From (2)−(3):

$$(5) \quad 3a + b = 22c$$

And thus from (4)−(5):

$$(6) \quad 26a + 36c = 14b \implies 13a + 18c = 7b$$

Eliminating b seems like a good approach, so we multiply (5) by 7 and combine it with (6):

$$\begin{aligned} &13a + 18c = 7b \\ &21a + 7b = 154c \\ &\implies 34a = 136c \implies 17a = 68c \implies a = 4c \end{aligned}$$

Substituting back into (5) yields $b = 10c$. We solve the system for a, b, c in integers:

$$a = 4c$$
$$b = 10c$$

The only solution with $a < 6$ and $c < 6$ is: $a = 4$, $b = 10$, $c = 1$. Substituting back into (3) to solve for n:

$$n = 222 \times 4 + 37 \times 1 = 925$$

Solution: $\boxed{n = 925.}$

7

What is the largest positive integer n less than $10,000$ such that in base 4, n and $3n$ have the same number of digits; in base 8, n and $7n$ have the same number of digits; and in base 16, n and $15n$ have the same number of digits? Express your answer in base 10.

Solution

Since $16^4 > 10,000$, the greatest n which satisfies the constraint for base 16 is

$$\begin{aligned}
4369 &= 1 \times 16^3 + 1 \times 16^2 + 1 \times 16^1 + 1 \times 16^0 &&= 1111_{16} \\
&= 1 \times 8^4 + 4 \times 8^2 + 2 \times 8^1 + 1 \times 8^0 &&= 10111_8 \\
&= 1 \times 4^6 + 1 \times 4^4 + 1 \times 4^2 + 1 \times 4^0 &&= 1010101_4 \\
&= 1 \times 2^{12} + 1 \times 2^8 + 1 \times 2^4 + 1 \times 2^0 &&= 1000100010001_2
\end{aligned}$$

8

Let $b(n)$ be the number of digits in the base-4 representation of n. Evaluate

$$\sum_{i=1}^{2013} b(i)$$

Solution

$$\begin{aligned}
3 \times 4^0 &= 3 \\
4^2 - 3 \times 4^1 &= 4 \\
3 \times 4^2 + 3 \times 4^0 &= 51 \\
4^4 - 3 \times 4^3 - 3 \times 4 &= 52 \\
3 \times 4^4 + 3 \times 4^2 + 3 \times 4^0 &= 819 \\
4^6 - 3 \times 4^5 - 3 \times 4^3 &= 820 \\
3 \times 4^6 + 3 \times 4^4 + 3 \times 4^2 + 3 \times 4^0 &= 13827
\end{aligned}$$

The number of digits in base-4 representation is summarized in the table:

$1 \leq n \leq 3$	$4 \leq n \leq 51$	$52 \leq n \leq 819$	$820 \leq n \leq 2013$
1	3	5	7

Adding up the 4 digits weighted by the number of cases gives:

$$\begin{aligned}\sum_{i=1}^{2013} b(i) &= 1 \times (3-1+1) + 3 \times (51-4+1) + 5 \times (819-52+1) + 7 \times (2013-820+1) \\ &= 3 + 48 + 768 + 1195 = 12345\end{aligned}$$

Solution: $\boxed{12345.}$