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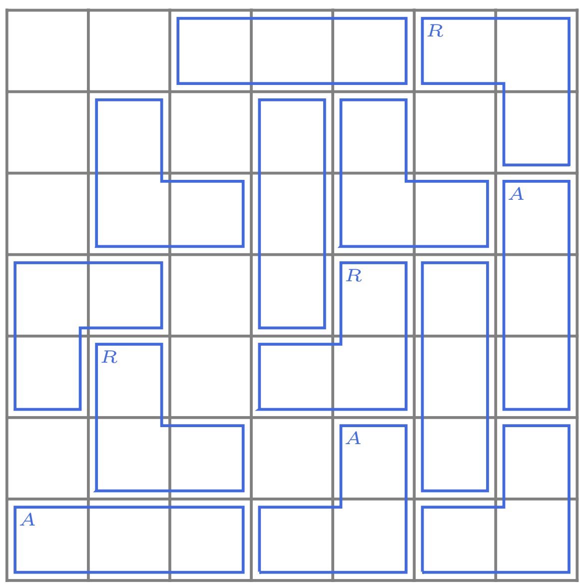
## Problem 1

***Question***

> Fill each cell with an integer from $1-7$ so each number appears exactly once in each row and column. In each "cage" of three cells, the three numbers must be valid lengths for the sides of a non-degenerate triangle. Additionally, if a cage has an "A", the triangle must be acute, and if the cage has an "R", the triangle must be right.
>
> 
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>
> There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

**Solution:**

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | 1 | 7 | 6 | 2 | *R* 4 | 5 |
| 4 | 7 | 2 | 5 | 6 | 1 | 3 |
| 7 | 3 | 5 | 2 | 1 | 6 | *A* 4 |
| 1 | 2 | 6 | 4 | *R* 5 | 3 | 7 |
| 2 | *R* 5 | 1 | 3 | 4 | 7 | 6 |
| 6 | 4 | 3 | 1 | *A* 7 | 5 | 2 |
| *A* 5 | 6 | 4 | 7 | 3 | 2 | 1 |

The problem was solved by trial and error. The squares marked $R$ are filled with the Pythagorean triple $(3, 4, 5)$. The squares marked $A$ are filled with the acute triangles $(3, 7, 7)$, $(4, 5, 6)$, and $(4, 6, 7)$. The other cages include acute and obtuse triangles. There are three iscosceles triangles, $(1, 2, 2)$ (which appears twice), $(1, 6, 6)$, and $(3, 7, 7)$.

## Problem 2

### *Question*

> In how many ways can a $3 \times 3$ grid be filled with integers from 1 to 12 such that all three of the following conditions are satisfied: (a) both 1 and 2 appear in the grid, (b) the grid contains at most 8 distinct values, and (c) the sums of the numbers in each row, each column, and both main diagonals are all the same? Rotations and reflections are considered the same.

### Aside On The Luo Shu Magic Square

It is well known that the "Luo Shu" magic square is the only $3 \times 3$ magic square that uses all integers 1 to 9 and that its magic sum is 15. The Luo Shu magic square uses 9 distinct values and, therefore, violates condition (b).

| | | |
|---|---|---|
| 8 | 1 | 6 |
| 3 | 5 | 7 |
| 4 | 9 | 2 |

The Luo-Shu magic square violates condition (b).

**Solution:** $\boxed{\text{THREE magic square satisfy all of the stated conditions.}}$

The following magic squares use 1 and 2 and contains at most 8 values from 1 to 12. They satisfy conditions (a), (b) and (c).

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 1 | 2 | 3 |
| 2 | 3 | 1 |

**(a)** $M = 6 \mid 3$ values.

| | | |
|---|---|---|
| 2 | 3 | 4 |
| 5 | 3 | 1 |
| 2 | 3 | 4 |

**(b)** $M = 9 \mid 5$ values.

| | | |
|---|---|---|
| 2 | 5 | 5 |
| 7 | 4 | 1 |
| 3 | 3 | 6 |

**(c)** $M = 12 \mid 7$ values.

**Solution:** These magic squares satisfy conditions (a), (b) and (c).

**Proof:**

A generic $3 \times 3$ magic square will be denoted with letters from $a$ to $i$ as follows:

| | | |
|---|---|---|
| $a$ | $b$ | $c$ |
| $d$ | $e$ | $f$ |
| $g$ | $h$ | $i$ |

generic notation for the values (not necessarily distinct).

Let $M$ denote the magic constant. By construction, $M \in \mathbb{N}$. Summing across the three rows, the three columns, and the two diagonals yields:

$$\begin{aligned} \text{rows}: &\quad a+b+c=M, \quad d+e+f=M, \quad g+h+i=M, \\ \text{columns}: &\quad a+d+g=M, \quad b+e+h=M, \quad c+f+i=M, \\ \text{diagonals}: &\quad a+e+i=M, \quad c+e+g=M. \end{aligned}$$

This yields 8 independent equations with 10 unknowns.

Let $S$ denote the sum of all the integers used in the square. $S$ must be a multiple of 3, because

$$S = a+b+c+d+g+h+i = 3M$$

Consider now the other constraints of the problem.

The integers $\{a, b, \ldots, i\}$ are in $\{1, 2, \ldots, 12\}$, with at most 8 distinct value, and with values 1 and 2 occurring at least once. It follows that a candidate for the largest possible sum $S$ is

$$1 + 2 + 7 \times 12 = 87$$

This candidate is a multiple of 3, which gives $M \leq 29$.

A candidate for the smallest possible sum $S$ is

$$1 + 2 + 7 \times 1 = 10$$

The nearest multiple of 3 is 12, which gives $M \geq 4$.

Taking both conditions together implies that candidate values for the magic constant are in $M \in \{4, 5, \ldots, 29\}$.

Further considerations imply that $M$ must be a multiple of 3.

Add all lines that go through the center:

$$\begin{aligned} \text{middle row} \mid \text{column}: \quad & d + e + f = M, \quad b + e + h = M, \\ \text{both diagonals}: \quad & a + e + i = M, \quad c + e + g = M. \end{aligned}$$

Adding these sums, rearranging, and using the previous calculations gives

$$\begin{aligned} 4M &= (d + e + f) + (b + e + h) + (a + e + i) + (c + e + g) \\ &= (a + b + c) + (d + e + f) + (g + h + i) + 3e \\ &= 3M + 3e \\ \implies M &= 3e \end{aligned}$$

Thus, $M$ must be a multiple of 3 and, furthermore, the central value $e$ must equal one-third of the magic constant $e = M/3$.

Further considerations imply $M \leq 18$. The central value associated with $M = 21$ is $e = 7$. The other two cells in a row/column/diagonal alignment must therefore add up to 14. One of these cells must be at least 1, since 1 and 2 are required values in the grid, which would require a 13 to be placed in the alignment. But 13 is not an admissible value in the grid. Therefore no magic square with $M = 21$ can satisfy the stated conditions. A similar argument applies to larger values of $M$. And therefore $M \leq 18$.

We have shown that $M \in \{6, 9, 12, 15, 18\}$, with corresponding central value $e \in \{2, 3, 4, 5, 6\}$. As we have reduced the number of candidate magic sums to a manageable level, we conduct an exhaustive search for each case.

**Case** $M = 6$

*There is exactly one magic squares with magic sum* $M = 6$ *that satisfies the stated conditions.*

We have already established that the central value must be $e = 2$. Since 1 and 2 must be used, there are only two possible starting grids: 1 at a corner and 1 at an edge. Consider first the case where 1 is at a corner. The magic square can be filled in terms of the parameter $b$. To ensure the values are admissible, we must have $b - 2 > 0$ and $4 - b > 0$ and therefore $b = 3$. This is a magic square that satisfies conditions (a), (b) and (c).

| | | |
|---|---|---|
| $1$ | $b$ | $5-b$ |
| $6-b$ | $2$ | $b-2$ |
| $b-1$ | $4-b$ | $3$ |

**(a)** Generic pattern

$\xrightarrow{b=3}$

| | | |
|---|---|---|
| 1 | 3 | 2 |
| 3 | 2 | 1 |
| 2 | 1 | 3 |

**(b)** Unique solution

$M = 6$: One magic squares satisfies the stated conditions.

Consider next the case where 1 is at an edge. The magic square can be filled in terms of $a$ and $d$ as follows. Only the value $(a, d) = (2, 3)$ and $(a, d) = (3, 1)$ are consistent with the admissible values. Each of these yields a magic square that satisfies conditions (a), (b) and (c).

| | | |
|---|---|---|
| $a$ | 1 | $5-a$ |
| $d$ | 2 | $4-d$ |
| $6-a-d$ | 3 | $a+d-3$ |

**(a)** Generic pattern

$\xrightarrow[d=3]{a=2}$

| | | |
|---|---|---|
| 2 | 1 | 3 |
| 3 | 2 | 1 |
| 1 | 3 | 2 |

**(b)** One solution

$\xrightarrow[d=1]{a=3}$

| | | |
|---|---|---|
| 3 | 1 | 2 |
| 1 | 2 | 3 |
| 2 | 3 | 1 |

**(c)** Only other solution

$M=6$: Two magic squares that satisfy the stated conditions.

These squares can be obtained from each other by reflection/rotation, so count as one.

**Case** $M = 9$

*There is exactly one magic squares with magic sum* $M = 9$ *that satisfies the stated conditions.*

We have already established that the central value must be $e = 3$. Since 1 and 2 must be used, the only possible starting grids would be:

| | | |
|---|---|---|
| 1 | 2 | $c$ |
| $d$ | 3 | $f$ |
| $g$ | $h$ | $i$ |

**(a)** First, $c \to 6$ and $i \to 5$, which makes the sum along column $cfi$ too large.□

| | | |
|---|---|---|
| 1 | $b$ | $c$ |
| $d$ | 3 | 2 |
| $g$ | $h$ | $i$ |

**(b)** First, $d \to 4$ and $i \to 5$. Next, $g \to 4$, which makes the sum in row $ghi$ too large.□

| | | |
|---|---|---|
| 1 | $b$ | 2 |
| $d$ | 3 | $f$ |
| $g$ | $h$ | $i$ |

**(c)** First, $g \to 4$ and $i \to 5$, which makes the sum in row $ghi$ too large. □

| | | |
|---|---|---|
| 1 | $b$ | $c$ |
| $d$ | 3 | $f$ |
| $g$ | $h$ | 2 |

**(d)** This position is a non-starter since the sum along the diagonal is not 9. □

| | | |
|---|---|---|
| 2 | 1 | $c$ |
| $d$ | 3 | $f$ |
| $g$ | $h$ | $i$ |

**(e)** First, $i \to 4$ and $c \to 6$, which makes the sum in the third column too large. □

| | | |
|---|---|---|
| 2 | $b$ | $c$ |
| $d$ | 3 | 1 |
| $g$ | $h$ | $i$ |

**(f)** First, $d \to 5$ and $i \to 4$. Next, $g \to 2$ and $c \to 4$. Last, $h \to 3$ and $b \to 3$. Eureka! □

$M = 9$: Ruling out all but one magic square for the stated conditions.

**Case** $M = 12$

*There is exactly one magic squares with magic sum* $M = 12$ *that satisfies the stated conditions.*

We have already established that the central value must be $e = 4$. Since 1 and 2 must be used, the only possible starting grids would be:

| | | |
|---|---|---|
| 1 | 2 | $c$ |
| $d$ | 4 | $f$ |
| $g$ | $h$ | $i$ |

**(a)** First, $c \to 9$ and $i \to 7$, which makes the sum along column $cfi$ too large. □

| | | |
|---|---|---|
| 1 | $b$ | $c$ |
| $d$ | 4 | 2 |
| $g$ | $h$ | $i$ |

**(b)** First, $d \to 6$ and $i \to 7$. Next, $g \to 5$, which makes the sum along row $ghi$ too large. □

| | | |
|---|---|---|
| 1 | $b$ | 2 |
| $d$ | 4 | $f$ |
| $g$ | $h$ | $i$ |

**(c)** First, $b \to 9$, which makes the sum along the middle column too large. □

| | | |
|---|---|---|
| 1 | $b$ | $c$ |
| $d$ | 4 | $f$ |
| $g$ | $h$ | 2 |

**(d)** This position is a non-starter since the sum along the diagonal is not equal to 12. □

| | | |
|---|---|---|
| 2 | 1 | $c$ |
| $d$ | 4 | $f$ |
| $g$ | $h$ | $i$ |

**(e)** First, $c \to 9$ and $i \to 6$, which makes the sum along column $cfi$ too large. □

| | | |
|---|---|---|
| 2 | $b$ | $c$ |
| $d$ | 4 | 1 |
| $g$ | $h$ | $i$ |

**(f)** First, $d \to 7$ and $i \to 6$. Next, $g \to 3$ and $c \to 5$. Last, $b \to 5$ and $h \to 3$. Eureka! □

$M = 12$: Ruling out all but one magic square for the stated conditions.

**Case** $M = 15$

*No magic squares with magic sum* $M = 15$ *satisfies the stated conditions.*

We have already established that the central value must be $e = 5$. Since 1 and 2 must be used, the only possible starting grids would be:

| | | |
|---|---|---|
| 1 | 2 | $c$ |
| $d$ | 5 | $f$ |
| $g$ | $h$ | $i$ |

**(a)** $c \to 12$, which makes the sum along the diagonal too large. $\square$

| | | |
|---|---|---|
| 1 | $b$ | $c$ |
| $d$ | 5 | 2 |
| $g$ | $h$ | $i$ |

**(b)** First, $d \to 8$ and $i \to 9$. Next $g \to 6$, which makes the sum along row $ghi$ too large. $\square$

| | | |
|---|---|---|
| 1 | $b$ | 2 |
| $d$ | 5 | $f$ |
| $g$ | $h$ | $i$ |

**(c)** $b \to 12$, which makes the sum in the middle column too large. $\square$

| | | |
|---|---|---|
| 1 | $b$ | $c$ |
| $d$ | 5 | $f$ |
| $g$ | $h$ | 2 |

**(d)** This position is a non-starter since the sum along the diagonal is not equal to 15. $\square$

| | | |
|---|---|---|
| 2 | 1 | $c$ |
| $d$ | 5 | $f$ |
| $g$ | $h$ | $i$ |

**(e)** $c \to 12$, which makes the sum along the diagonal too large. $\square$

| | | |
|---|---|---|
| 2 | $b$ | $c$ |
| $d$ | 5 | 1 |
| $g$ | $h$ | $i$ |

**(f)** This yields the Luo Shu magic square, which uses more than 8 distinct values. $\square$

$M = 15$: Ruling out all magic square for the stated conditions.

**Case** $M = 18$

*No magic squares with magic sum* $M = 18$ *satisfies the stated conditions.*

We have already established that the central value must be $e = 6$. Since 1 and 2 must be used, the only possible starting grids would be:

| | | |
|---|---|---|
| 1 | 2 | $c$ |
| $d$ | 6 | $f$ |
| $g$ | $h$ | $i$ |

**(a)** $c \to 15$, but 15 is not admissible. □

| | | |
|---|---|---|
| 1 | $b$ | $c$ |
| $d$ | 6 | 2 |
| $g$ | $h$ | $i$ |

**(b)** First, $d \to 10$ and $i \to 11$. Next, $g \to 7$, which makes the sum along the row $ghi$ too large. □

| | | |
|---|---|---|
| 1 | $b$ | 2 |
| $d$ | 6 | $f$ |
| $g$ | $h$ | $i$ |

**(c)** $b \to 15$, but 15 is not admissible. □

| | | |
|---|---|---|
| 1 | $b$ | $c$ |
| $d$ | 6 | $f$ |
| $g$ | $h$ | 2 |

**(d)** This position is a non-starter since the sum along the diagonal is not equal to 18. □

| | | |
|---|---|---|
| 2 | 1 | $c$ |
| $d$ | 6 | $f$ |
| $g$ | $h$ | $i$ |

**(e)** $c \to 15$, but 15 is not admissible. □

| | | |
|---|---|---|
| 2 | $b$ | $c$ |
| $d$ | 6 | 1 |
| $g$ | $h$ | $i$ |

**(f)** A magic square exists (see below), but it uses more than 8 distinct values. □

$M = 18$: Ruling out all magic square for the stated conditions.

A strong candidate emerges from panel (f): a magic square that uses both 1 and 2; but it uses 9 distinct values. Thus conditions (a) and (c) are satisfied, but condition (b) is violated.

| | | |
|---|---|---|
| 2 | 9 | 7 |
| 11 | 6 | 1 |
| 5 | 3 | 10 |

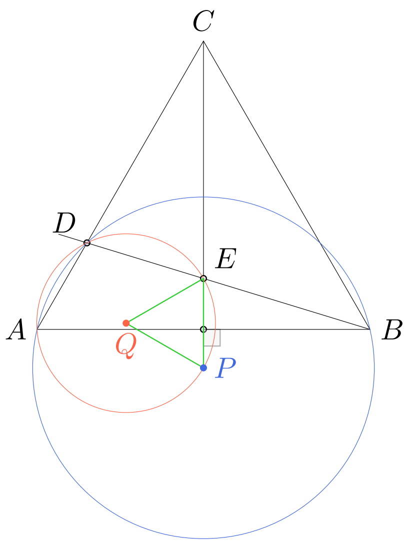
This magic square with magic sum $M = 18$ satisfies conditions (a) and (c), but violates condition (b).

We have exhausted all cases up to reflection/rotation. $\square$

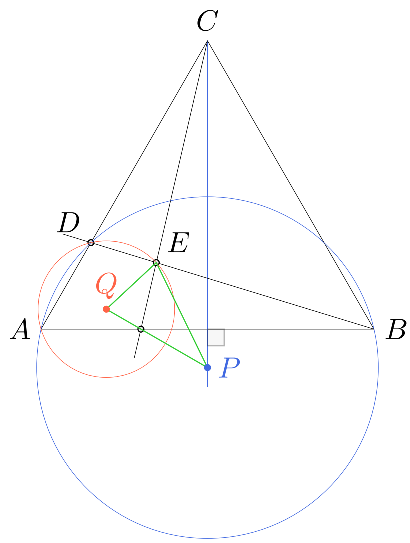
## Problem 3

### *Question*

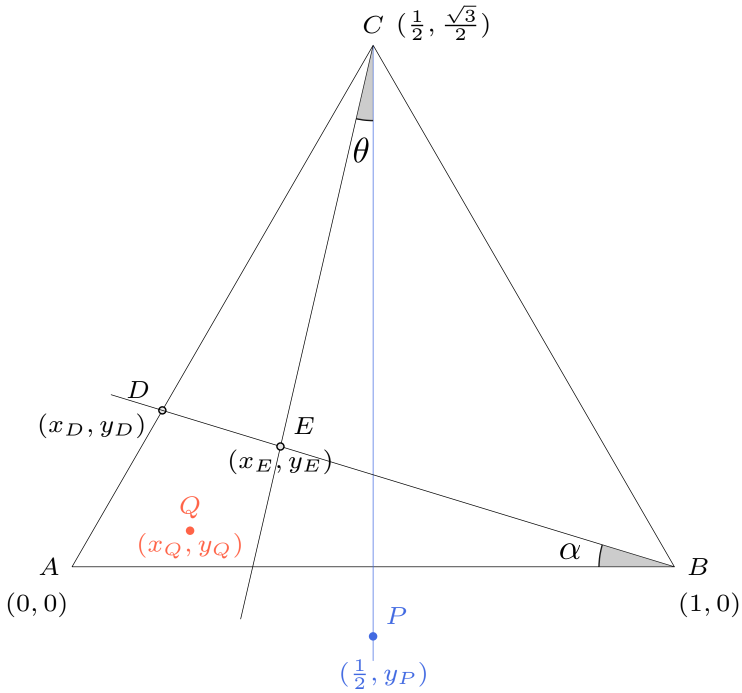
> $\triangle ABC$ is an equilateral triangle. $D$ is a point on $A$C, and $E$ is a point on $B$D. Let $P$ and $Q$ be the circumcenters of $\triangle ABD$ and $\triangle AED$, respectively. Prove that $\triangle EPQ$ is an equilateral triangle if and only if $AB \perp CE$.




**(a)** $AB \perp CE \implies \overline{QE} = \overline{QP} = \overline{EP}$.




**(b)** $CE \not\perp AB \implies \overline{QE} \neq \overline{QP}$.




We assume $D \neq A$ to ensure the circumcenters $P$ and $Q$ are unique. We associate Cartesian coordinates to the points $A$, $B$, $C$, $D$, $E$, $P$, and $Q$. We associate equations to the lines $AC$, $AE$, and $BE$. We then calculate the distances $EP$, $EQ$, and $PQ$ in this coordinate system and use them to prove the theorem.

**Theorem:** $\overline{QE} = \overline{QP} = \overline{EP} \iff AB \perp CE.$

We split the proof into three parts:

**Theorem** (Part I)**:** $AB \perp CE \implies \overline{QE} = \overline{QP}.$

**Theorem** (Part II)**:** $AB \perp CE \implies d(Q, EP) = \frac{\sqrt{3}}{2}\,\overline{EP}.$

Taken together, Part I and Part II are equivalent to $AB \perp CE \implies \overline{QE} = \overline{QP} = \overline{EP}$. The equivalence follows from the property that the height of an equilateral triangle (here $d(Q, EP)$) is $\frac{\sqrt{3}}{2}$ times the base (here $\overline{EP}$).

**Theorem** (Part III)**:** $CE \not\perp AB \implies \overline{QE} \neq \overline{EP}.$

Useful coordinates and equations summarized in the table are proved below.

| Point | $x$ | $y$ |
|---|---|---|
| $A$ | $0$ | $0$ |
| $B$ | $1$ | $0$ |
| $C$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $D$ | $\frac{\tan(\alpha)}{\sqrt{3}+\tan(\alpha)}$ | $\frac{\sqrt{3}\tan(\alpha)}{\sqrt{3}+\tan(\alpha)}$ |
| $E$ | $\frac{1}{2}\cdot\frac{2\tan(\alpha)+\cot(\theta)-\sqrt{3}}{\cot(\theta)+\tan(\alpha)}$ | $\frac{1}{2}\cdot\frac{\tan(\alpha)(\cot(\theta)+\sqrt{3})}{\cot(\theta)+\tan(\alpha)}$ |
| $P$ | $\frac{1}{2}$ | $\frac{1}{2}\cdot\frac{\sqrt{3}\tan(\alpha)-1}{\sqrt{3}+\tan(\alpha)}$ |
| $Q$ | $x_Q$ | $y_Q$ |

| Line | Equation |
|---|---|
| $AD$ | $y=\sqrt{3}x$ |
| $BE$ | $y=\tan(\alpha)(1-x)$ |
| $CE$ | $y=\cot(\theta)(x-1/2)+\sqrt{3}/2$ |

Setting the coordinates of point $B$ as $(1,0)$ is without loss of generality: The units can be redefined to represent any length $AB$, while the $AB$ line can be rotated to represent any triangle $ABC$.

**Equations of $AD$, $BE$ and $CE$**

The coordinate system gives $A:(0,0)$, $B:(1,0)$, and $C:(1/2,\sqrt{3}/2)$. The coordinates of $C$ follow from the well-known result that the height of triangle $ABC$ has length $\sqrt{3}/2$, which is an application of the Pythagoras triangle equality.

- The equation of line $AD = AC$ is $y = \sqrt{3}\,x$. This follows from $\tan 60° = \sqrt{3}$. One can verify that $A : (0, 0)$ and $C : (1/2, \sqrt{3}/2)$ satisfy the equation.

- The equation of line $BE = BD$ is $y = \tan(\alpha)(1 - x)$. This follows from a generic $y - y_B = \tan(\pi - \alpha)(x - x_B)$ by substituting the $B$ coordinates $(1, 0)$ and noting that $\tan(\pi - \alpha) = -\tan(\alpha)$.

- The equation of line $CE$ is $y = \cot(\theta)(x - 1/2) + \sqrt{3}/2$. This follows from a generic $y - y_C = \tan(\pi/2 - \theta)(x - x_C)$ by substituting the $C$ coordinates $(1/2, \sqrt{3}/2)$ and noting that $\tan(\pi/2 - \theta) = \cot(\theta)$.

**Coordinates of $A$, $B$, $C$, $D$, $E$, $P$, $Q$**

- The coordinates of $D$ can be found by solving the $AD \cap BE$ system:

$$\begin{aligned} AD: &\quad y_D = \sqrt{3}\,x_D \\ BE: &\quad y_D = \tan(\alpha)(1 - x_D) \end{aligned}$$

which yields

$$\begin{aligned} x_D &= \frac{\tan(\alpha)}{\sqrt{3} + \tan(\alpha)} \\ y_D &= \frac{\sqrt{3}\tan(\alpha)}{\sqrt{3} + \tan(\alpha)} \end{aligned}$$

- The coordinates of $E$ can be found by solving the $BE \cap CE$ system:

$$\begin{aligned} BE: &\quad y_E = \tan(\alpha)(1 - x_E) \\ CE: &\quad y_E = \cot(\theta)(x_E - 1/2) + \sqrt{3}/2 \end{aligned}$$

which yields

$$\begin{aligned} x_E &= \frac{1}{2} \cdot \frac{2\tan(\alpha) + \cot(\theta) - \sqrt{3}}{\cot(\theta) + \tan(\alpha)} \\ y_E &= \frac{1}{2} \cdot \frac{\tan(\alpha)(\cot(\theta) + \sqrt{3})}{\cot(\theta) + \tan(\alpha)} \end{aligned}$$

where $-\pi/2 < \theta < \pi/2$ implies $\cot(\theta) \neq 0$.

In the limit, where $CP \perp AB$, as $\theta \to 0$ implies $\cot(\theta) \to \infty$, and the coordinates simplify to

$$\theta \to 0: \quad x_E = \frac{1}{2}$$
$$\theta \to 0: \quad y_E = \frac{1}{2} \cdot \tan(\alpha)$$

In the limit where $\theta = 0$, $E$ lies on $CP$: See panel (a) of the figure.

- The coordinates of $P$ can be found by solving $\overline{PA} = \overline{PB} = \overline{PD}$. Coordinate $x_P$ can be found directly by a simple geometric argument: The point $P$ is the circumcenter of the circle going through $A$ and $B$, so it satisfies $\overline{PA} = \overline{PB}$. The point $C$ is the vertex of the equilateral triangle $ABC$, so it satisfies $\overline{CA} = \overline{CB}$. It follows that $CP \perp AB$ and therefore

  $$x_P = \frac{1}{2}$$

  Coordinate $y_P$ can be found from $\overline{PA} = \overline{PD}$:

  $$\begin{aligned}
  &(x_P - x_A)^2 + (y_P - y_A)^2 = (x_P - x_D)^2 + (y_P - y_D)^2 \\
  &\implies 2(x_D\, x_P + y_D\, y_P) = x_D^2 + y_D^2 \\
  &\implies y_P = \frac{x_D^2 + y_D^2 - x_D}{2y_D} = \frac{4x_D - 1}{2\sqrt{3}} = \frac{1}{2} \cdot \frac{\sqrt{3}\tan(\alpha) - 1}{\sqrt{3} + \tan(\alpha)}
  \end{aligned}$$

  where we have substituted $x_A = y_A = 0$, $y_D^2 = 3x_D^2$, $y_D = \sqrt{3}\, x_D$, and $x_D = \tan(\alpha)/(\sqrt{3} + \tan(\alpha))$, for $\alpha \neq 0$ (i.e. $D \neq A$, as assumed).

- The coordinates of $Q$ can be found by solving $\overline{QA} = \overline{QD} = \overline{QE}$. The first few steps are similar to the calculations above.

  $$\begin{cases} (x_Q - x_A)^2 + (y_Q - y_A)^2 = (x_Q - x_D)^2 + (y_Q - y_D)^2 \\ (x_Q - x_A)^2 + (y_Q - y_A)^2 = (x_Q - x_E)^2 + (y_Q - y_E)^2 \end{cases}$$
  $$\implies \begin{cases} x_D\, x_Q + y_D\, y_Q = \frac{1}{2}(x_D^2 + y_D^2) \\ x_E\, x_Q + y_E\, y_Q = \frac{1}{2}(x_E^2 + y_E^2) \end{cases}$$

  Solving the system in terms of the coordinates of $D$ and $E$ yields:

  $$x_Q = \frac{1}{2} \cdot \frac{(x_E^2 + y_E^2)\, y_D - (x_D^2 + y_D^2)\, y_E}{x_E\, y_D - y_E\, x_D}$$
  $$y_Q = \frac{1}{2} \cdot \frac{(x_D^2 + y_D^2)\, x_E - (x_E^2 + y_E^2)\, x_D}{x_E\, y_D - y_E\, x_D}$$

**Theorem (Part I):** To prove $\overline{QE} = \overline{QP}$, we show that:

$$(x_Q - x_E)^2 + (y_Q - y_E)^2 = (x_Q - x_P)^2 + (y_Q - y_P)^2$$

To show the equality, we use the condition $CP \perp AB$ and substitute the values of the coordinates:

- Simplify and rewrite the equality:

$$(x_P - x_E)(x_P + x_E - 2x_Q) + (y_P - y_E)(y_P + y_E - 2y_Q) = 0$$

- $CP \perp AB \implies \theta = 0 \implies x_P = x_E$, and $\alpha > 0 \implies y_P < y_E$.
- With $x_P = x_E$ and $y_P < y_E$, the equality simplifies to:

$$y_P + y_E - 2y_Q = 0 \implies y_Q = \frac{1}{2}(y_P + y_E)$$

  This condition on the $y$ coordinates of $Q$, $P$, and $E$ states that point $Q$ is the vertex of the isosceles triangle $\triangle EQP$. This is proved below in two steps. 1. Write $y_Q$ in terms of $\tan(\alpha)$. 2. Write $\frac{1}{2}(y_P + y_E)$ in terms of $\tan(\alpha)$. Apply Lemma 1 to prove equality between the two expressions.

- $\theta = 0 \implies x_E = \frac{1}{2},\ y_E = \frac{1}{2}\tan(\alpha),\ x_E^2 + y_E^2 = \frac{1}{4}(1 + \tan^2(\alpha))$.
- Step 1. $x_D^2 + y_D^2 = 4x_D^2 = 4\tan^2(\alpha)/(\sqrt{3} + \tan(\alpha))^2$.
- Substitute $y_D = \sqrt{3}\,x_D$ and $x_D^2 + y_D^2 = 4x_D^2$ into $y_Q$, cancel out $x_D$ from numerator and denominator, substitute the values calculated above:

$$\begin{aligned} y_Q &= \frac{1}{2} \cdot \frac{(x_D^2 + y_D^2)\,x_E - (x_E^2 + y_E^2)\,x_D}{x_E\,y_D - y_E\,x_D} \\ &= \frac{1}{2} \cdot \frac{4x_D\,\frac{1}{2} - \frac{1}{4}(1 + \tan^2(\alpha))}{\frac{1}{2}\sqrt{3} - \frac{1}{2}\tan(\alpha)} \\ &= \frac{1}{4} \cdot \frac{8\tan(\alpha) - (1 + \tan^2(\alpha))(\sqrt{3} + \tan(\alpha))}{3 - \tan^2(\alpha)} \end{aligned}$$

- Step 2. Substitute the values of $y_P$ and $y_E$:

$$y_P + y_E = \frac{1}{2} \cdot \tan(\alpha) + \frac{1}{2} \cdot \frac{\sqrt{3}\tan(\alpha) - 1}{\sqrt{3} + \tan(\alpha)}$$

- The equality that remains to be proved reduces to

$$\frac{8\tan(\alpha) - (1+\tan^2(\alpha))(\sqrt{3}+\tan(\alpha))}{3-\tan^2(\alpha)} = \tan(\alpha) + \frac{\sqrt{3}\tan(\alpha)-1}{\sqrt{3}+\tan(\alpha)}$$

- Lemma 1 proves the above equality and therefore $y_Q = \frac{1}{2}(y_P + y_E)$.

□

**Lemma 1:** *The following equality holds for all $x \neq \pm\sqrt{3}$:*

$$\frac{8x - (1+x^2)(\sqrt{3}+x)}{3-x^2} = x + \frac{\sqrt{3}\,x-1}{\sqrt{3}+x}$$

**Proof:** Multiplying the right-hand side of the equality by $(\sqrt{3}-x)$, so that the denominator on both sides is equal to $3-x^2 = (\sqrt{3}\,x-1)(\sqrt{3}-x)$, gives:

$$x(\sqrt{3}-x)(\sqrt{3}+x) + (\sqrt{3}\,x-1)(\sqrt{3}-x) = 7x - \sqrt{3} - \sqrt{3}\,x^2 - x^3$$

Expanding the numerator on the left-hand side of the equality gives:

$$7x - \sqrt{3} - \sqrt{3}\,x^2 - x^3$$

□

**Theorem (Part II):** To prove $AB \perp CE \implies d(Q, EP) = \frac{\sqrt{3}}{2}\,\overline{EP}$, we show $\theta = 0 \implies x_P - x_Q = \frac{\sqrt{3}}{2}(y_E - y_P)$.

- Substitute the values of the coordinates $y_E$, $y_P$ into the right-hand side:

$$y_E - y_P = \frac{1}{2}\cdot\tan(\alpha) - \frac{1}{2}\cdot\frac{\sqrt{3}\tan(\alpha)-1}{\sqrt{3}+\tan(\alpha)} = \frac{1}{2}\cdot\frac{1+\tan^2(\alpha)}{\sqrt{3}+\tan(\alpha)}$$

- Substitute the values of the coordinates $x_P$, $x_Q$ into the left-hand side:

$$x_P - x_Q = \frac{1}{2} - \frac{1}{2}\cdot\frac{(x_E^2+y_E^2)\,y_D - (x_D^2+y_D^2)\,y_E}{x_E\,y_D - y_E\,x_D}$$

- Substitute $x_E^2 + y_E^2 = \frac{1}{4}(1 + \tan^2(\alpha))$, $y_D = \sqrt{3}\, x_D$, $x_D^2 + y_D^2 = 4x_D^2$, $x_E = \frac{1}{2}$, $y_E = \frac{1}{2}\tan(\alpha)$ into the above and simplify:

$$\begin{aligned} x_P - x_Q &= \frac{1}{2} - \frac{1}{2} \cdot \frac{\frac{\sqrt{3}}{4}\,(1+\tan^2(\alpha)) - 4\,\frac{\tan(\alpha)}{\sqrt{3}+\tan(\alpha)}\,\frac{1}{2}\tan(\alpha)}{\frac{1}{2}\,(\sqrt{3}-\tan(\alpha))} \\ &= \frac{1}{2} + \frac{1}{4} \cdot \frac{8\tan^2(\alpha) - \sqrt{3}(1+\tan^2(\alpha))(\sqrt{3}+\tan(\alpha))}{3-\tan^2(\alpha)} \\ &= \frac{1}{4} \cdot \frac{3 + 3\tan^2(\alpha) - \sqrt{3}\tan(\alpha) - \sqrt{3}\tan^3(\alpha)}{3-\tan^2(\alpha)} \end{aligned}$$

- The equality that remains to be proved reduces to

$$\frac{1}{4} \cdot \frac{3 + 3\tan^2(\alpha) - \sqrt{3}\tan(\alpha) - \sqrt{3}\tan^3(\alpha)}{3-\tan^2(\alpha)} = \frac{\sqrt{3}}{4} \cdot \frac{1+\tan^2(\alpha)}{\sqrt{3}+\tan(\alpha)}$$

  From $\alpha > 0 \implies \tan(\alpha) \neq \pm\sqrt{3}$. Multiply the right-hand side by $\sqrt{3} - \tan(\alpha)$ to get common denominators on both sides. The numerators on both sides must be equal:

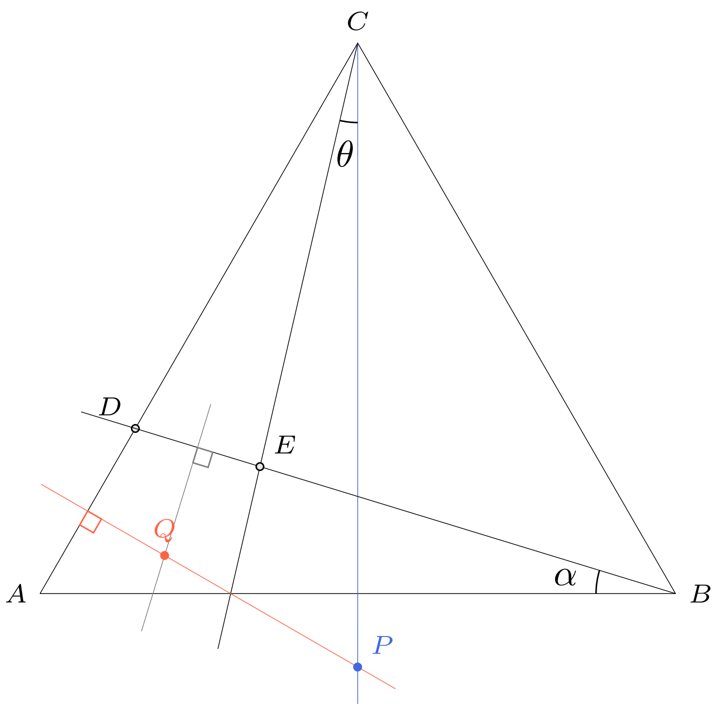
$$3 + 3\tan^2(\alpha) - \sqrt{3}\tan(\alpha) - \sqrt{3}\tan^3(\alpha) = \sqrt{3}(1+\tan^2(\alpha))(\sqrt{3}-\tan(\alpha))$$

  Expanding the right-hand side shows that the equality holds.

□

To recap, Part I of the Theorem proves that if $AB \perp CE$, then triangle $\triangle PQE$ is isosceles, $\overline{QE} = \overline{QP}$. Part II of the Theorem proves that if $AB \perp CE$, then the height of the triangle $\triangle PQE$ is equal to $\sqrt{3}/2$ times the length of the base $\overline{EP}$. Taken together, the Theorem proves that if $AB \perp CE$, then triangle $\triangle PQE$ is equilateral.

We now turn to the converse of the Theorem. Refer to the figure.




**Theorem (Part III):** $CE \not\perp AB \implies \overline{QE} \neq \overline{EP}$ We give a geometric proof. Since Point $P$ is the circumcenter of $ABD$, it is fixed for a given value of $\alpha$ and independent of $\theta$. Starting from $\theta = 0$ and $\triangle EPQ$ equilateral, let $\theta$ increase. While $P$ remains unchanged, distances $\overline{EP}$ and $\overline{QP}$ both increase: $E$ moves towards $D$ along line $BD$, while $Q$ moves towards the midpoint of $AD$ along the perpendicular to $AD$. For any angle other than $\pi/6 = 30°$, $E$ and $Q$ are moving along different directions and must therefore cover different distances. It follows that $\triangle EPQ$ is not equilateral for $\theta > 0$. In the special case where $\alpha = \pi/6$, the distance between $Q$ and $E$ remains unchanged as $\theta$ is increased, but since distances $EP$ and $QP$ are increased, the triangle cannot be equilateral in this special case. $\square$

The only configuration in which $\triangle EPQ$ is equilateral is when $AB \perp CE$.

## Problem 4

***Question***

> Let $x_1 < x_2 < \ldots < x_n$ (with $n \geq 2$) and let $S$ be the set of all the $x_i$. Let $T$ be a randomly chosen subset of $S$. What is the expected value of the indexed alternating sum of $T$? Express your answer in terms of the $x_i$.
> **Note:** We define the indexed alternating sum of $T$ as
>
> $$\sum_{i=1}^{|T|}(-1)^{i+1}(i)T[i],$$
>
> where $T[i]$is the $i$th element of $T$ when listed in increasing order. For example, if $T = \{1, 3, 5\}$, then the indexed alternating sum of $T$ is
>
> $$1 \cdot 1 - 2 \cdot 3 + 3 \cdot 5 = 10$$
>
> Alternating sums of empty sets are defined to be 0.

**Solution:** The expected value is $\sum_{T \subseteq S_n} \mathcal{I}(T) = 0$.

**Proof:**

$$\begin{aligned}
\sum_{T \subseteq S_n} \mathcal{I}(T) &= \sum_{T \subseteq S_{n-1}} \mathcal{I}(T) \ + \sum_{\substack{T \subseteq S_n \\ x_n \in T}} \mathcal{I}(T) \\
&= \sum_{T \subseteq S_{n-1}} \mathcal{I}(T) \ + \sum_{\substack{T \subseteq S_n \\ x_n \in T}} (-1)^{n+1}(n)T[n] - \mathcal{I}(T \setminus \{x_n\})) \\
&= \sum_{T \subseteq S_{n-1}} \mathcal{I}(T) \ + \sum_{\substack{T \subseteq S_n \\ x_n \in T}} (-1)^{n+1}(n)T[n] \ - \sum_{T \subseteq S_{n-1}} \mathcal{I}(T)) \\
&= \ \sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k} \\
&= \ 0
\end{aligned}$$

To prove the last step, set $a = 1, b = -1$ in the Binomial expansion formula

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

## Problem 5

***Question***

> Prove that there is no polynomial $P(x)$ with integer coefficients such that
>
> $$P(\sqrt[3]{5} + \sqrt[3]{25}) = 2\sqrt[3]{5} + 3\sqrt[3]{25}$$

The proof is adapted from [2]. The proof is by contradiction. We suppose a polynomial $P(x)$ exists and derive a contradiction.

Let $\alpha$ denote the irrational number $\alpha = \sqrt[3]{5} + \sqrt[3]{25}$, $\alpha \in \mathbb{R}/\mathbb{Q}$. Let $\beta$ denote the irrational number $\beta = 2\sqrt[3]{5} + 3\sqrt[3]{25}$, $\beta \in \mathbb{R}/\mathbb{Q}$. The problem is to prove that there is no polynomial $P(x)$ with integer coefficients such that $P(\alpha) = \beta$.

**Proof:** Suppose a polynomial $P(x)$ exists with integer coefficients and such that $P(\alpha) = \beta$. There exist polynomials $Q(x)$, $R(x)$ and $w(x)$ with integer coefficients such that

$$P(x) = Q(x)R(x) + w(x)$$

where $Q(x)$ satisfies $Q(\alpha) = 0$ (Lemma 4) and where $w(x)$ is a polynomial of degree either 1 or 2 such that $w(\alpha) = \beta$ (Lemma 3), implying

$$P(\alpha) = Q(\alpha)R(\alpha) + w(\alpha) = w(\alpha) = \beta$$

Thus, if $P(x)$ exists, $P(\alpha) = \beta$ implies $w(\alpha) = \beta$, a contradiction with lemma 2 and lemma 3, since the coefficients of $w(x)$ are in $\mathbb{Q}$ but not in $\mathbb{N}$. Conclusion: $P(x)$ does not exist. □

**Lemma 2:** *There exists no polynomial $w(x)$ of degree 1, with integer coefficients, such that $w(\sqrt[3]{5} + \sqrt[3]{25}) = 2\sqrt[3]{5} + 3\sqrt[3]{25}$.*

**Lemma 3:** *There exists exactly one polynomial $w(x)$ of degree 2, with rational coefficients, such that $w(\sqrt[3]{5} + \sqrt[3]{25}) = 2\sqrt[3]{5} + 3\sqrt[3]{25}$.*

**Lemma 4:** *There exists one polynomial $Q(x)$ of degree 3 with integer coefficients such that $Q(\alpha) = 0$.*

**Proof of Lemma 2:** Proof by contradiction. Suppose a polynomial $w(x) = ax + b$ exists, $a, b \in \mathbb{N}$. Since $\alpha \in \mathbb{R}/\mathbb{Q}$, it must be that $a \neq 0$ and $a \neq 1$. Moreover $w(\alpha) = \beta$ implies

$$
\begin{aligned}
&a(\sqrt[3]{5} + \sqrt[3]{25}) + b = 2\sqrt[3]{5} + 3\sqrt[3]{25} \\
&\implies (a-2)\sqrt[3]{5} + (a-3)\sqrt[3]{25} \in \mathbb{N} \subset \mathbb{Q} \\
&\implies \big((a-2)\sqrt[3]{5} + (a-3)\sqrt[3]{25}\big)^2 \in \mathbb{Q} \\
&\implies (a-2)^2\sqrt[3]{25} + (a-3)^2\sqrt[3]{25^2} + 2(a-2)(a-3)\sqrt[3]{5^3} \in \mathbb{Q} \\
&\implies (a-2)^2\sqrt[3]{25} + 5(a-3)^2\sqrt[3]{5} \in \mathbb{Q}
\end{aligned}
$$

A weighted average of the second and last expressions, with rational weights, is also in $\mathbb{Q}$. Select weights so that the $\sqrt[3]{5}$ term is eliminated:

$$
\begin{aligned}
&5(a-3)^2 \times \big((a-2)\sqrt[3]{5} + (a-3)\sqrt[3]{25}\big) - (a-2) \times \big((a-2)^2\sqrt[3]{25} + 5(a-3)^2\sqrt[3]{5}\big) \in \mathbb{Q} \\
&\implies \big(5(a-3)^2(a-3) - (a-2)^3\big)\sqrt[3]{25} \in \mathbb{Q} \\
&\implies \sqrt[3]{25} \in \mathbb{Q}
\end{aligned}
$$

where we have used the result that products, sums, and powers of expressions like $(a-2)$ and $(a-3)$ are in $\mathbb{Q}$ if $a \in \mathbb{Q}$, under the assumption that $w$ exists. Since $\sqrt[3]{25} \notin \mathbb{Q}$, we have reached a contradiction. $\square$

**Proof of Lemma 3:** Construct the polynomial $w(x) = ax^2 + bx + c$ by equating the coefficients derived by developing $w(\alpha) = \beta$:

$$
\begin{aligned}
a(\sqrt[3]{5} + \sqrt[3]{25})^2 + b(\sqrt[3]{5} + \sqrt[3]{25}) + c &= 2\sqrt[3]{5} + 3\sqrt[3]{25} \\
a\big(\sqrt[3]{25} + 10 + 5\sqrt[3]{5}\big) + b(\sqrt[3]{5} + \sqrt[3]{25}) + c &= 2\sqrt[3]{5} + 3\sqrt[3]{25}
\end{aligned}
$$

$$
\implies \begin{cases} a + b &= 3 \\ 5a + b &= 2 \\ 10a + c &= 0 \end{cases} \implies a = -\frac{1}{4},\ b = \frac{13}{4},\ c = \frac{5}{2}.
$$

The desired degree-2 polynomial $w(x)$ is: $w(x) = -(1/4)x^2 + (13/4)x + 5/24$. No other coefficients satisfy $w(\alpha) = \beta$. $\square$

**Proof of Lemma 4:** The polynomial $Q(x)$ is constructed from its root $\alpha$:

$$
\begin{aligned}
\alpha^3 = \big(\sqrt[3]{5} + \sqrt[3]{25}\big)^3 &= 5 + 3(\sqrt[3]{5})^2\sqrt[3]{25} + 3\sqrt[3]{5}(\sqrt[3]{25})^2 + 25 \\
&= 30 + 15(\sqrt[3]{5} + \sqrt[3]{25}) = 30 + 15\alpha
\end{aligned}
$$

Thus, $Q(x) = x^3 - 15x - 30$ satisfies $Q(\alpha) = 0$. $\square$

### Acknowledgments

#### Problem 1

I used trial and error to find the solution.

#### Problem 2

I used ideas described on the Wikipedia page
`https://en.wikipedia.org/wiki/Magic_square`.

#### Problem 3

I did not find similar geometry problems to guide me and followed an analytical approach based on a Cartesian coordinate system. The resulting proofs were more complicated than expected.

#### Problem 4

Without [1], I would not have known where to start.

#### Problem 5

Without [2], I would not have known where to start.

### References

## References

[1] The Art Of Problem Solving. *1983 AIME Problems / Problem 13*. Answer Key. 2024.

[2] *The 23rd Annual Vojtěch Jarník International Mathematical Competition*. Category II. Ostrava, Apr. 2013.