MathCounts Competition Practice III, December 2020 Sprint Round

Clairbourn School Grade 7/8

Revised: February 11, 2021

Abstract

This note reviews the Sprint Round of MathCounts Competition Practice III, January 2021.

Sprint Round

1.

How many integers between 500 and 1000 contain both the digits 3 and 4?

$\square$ integers

Only two integers, 534 and 543, satisfy the criterion between 500 and 599. Likewise 634, 643 and 734, 743 and 834, 843 and 934, 943. So there are $5 \times 2 = 10$ integers.

10 integers

Note that if the range had been, say, between 300 and 400, the calculation would have been different. In this case, valid integers are $34n$, for any $n \in [0, 9]$ (ten integers) and $3n4$, for any $n \in [0, 9]$ (ten integers).

2.

Tom rides the bus part of the way to school and then he walks the rest of the way. He walks five minutes longer than he rides. The whole trip takes 27 minutes. For how many minutes does he walk?

$\square$ minutes

Let t_1 denote the time spent riding the bus and t_2 the time spent walking. The text suggests the following system (of 2 equations and 2 unknown):

$$t_2 - t_1 = 5$$
$$t_1 + t_2 = 27$$

Adding both equations yields:

$$2t_2 = 5 + 27$$
$$\Rightarrow t_2 = 16$$

16 minutes

3.

Liberty Middle School's enrollment increased to 660 students. This is an increase of 10% over last year's enrollment. What was last year's enrollment?

[ ] students

Let e denote last year's enrollment. An increase of 10% is $1.1e$. Thus,

$$1.1e = 660 \Rightarrow e = \frac{6,600}{11} = \frac{6 \times 11 \times 100}{11} = 600$$

Last year's enrollment was:

600 students

4.

The distance from the earth to the sun is $93,000,000$ miles, and light travels at $186,000$ miles per second. How many seconds does light from the sun take to reach the earth?

[ ] seconds

The time it takes is given by $t = d/v$, where t is time (seconds), d is distance (miles) and v is velocity (miles per second). To see this quickly, note that velocity v is measured in units of distance per time (miles per second), so that its inverse is in units of time per distance (seconds per mile): keeping track of units shows that d/v is measured in units of time (seconds).

$$\frac{d}{v} = \frac{93,000,000}{186,000} = \frac{93}{186}10^3 = \frac{3 \times 31}{3 \times 62}10^3 = \frac{1000}{2} = 500$$

Light takes:

500 seconds

5.

Each of four test scores in Connie's class is to be weighted equally. On the first three tests Connie scored 80%, 90% and 95%. What percent must she score on her fourth test to have an overall average of exactly 90%?

[ ] %

Let x denote the unknown grade. The average is:

$$\frac{80+90+95+x}{4} = 90$$

Solving for x yields

$$x = 4 \times 90 - 265 = 95$$

The score on her fourth test is:

95%

6.

Haley has enlarged a 3-inch by 5-inch picture so that both the length and width are tripled. The area of the enlarged photo is how many times the area of the original photo?

[ ] times

The original area is 3×5. The enlarged area is $(3 \times 3) \times (3 \times 5)$. The ratio is therefore

$$\frac{(3 \times 3) \times (3 \times 5)}{3 \times 5} = 3^2$$

9 times

This is a general result: scaling up the lengths of a 2-d figure by a factor m yields to an increase in area of m^2. For a 3-d figure, the increase in volume would be proportional to m^3.

7.

What is the least positive integer divisible by each of 1, 3, 5, and 7?

[ ]

The least common multiple is

$$3 \times 5 \times 7 = 105$$

105

8.

If $m \diamondsuit n = (m^2 - n) \div n$ for all real numbers m and n, where $n \neq 0$, what is the value of $6 \diamondsuit 3$?

Applying the rule yields

$$6 \diamondsuit 3 = (6^2 - 3) \div 3 = 33 \div 3 = 11$$

11

9.

New York and Denver are in different time zones. When it is noon in New York, it is $10:00 a.m.$ in Denver. At $2:00 p.m.$ in New York, a train departs, and it arrives in Denver 45 hours later. What time is it in Denver when the train arrives?

a.m.

The clock resets every 24 hours. The journey lasts 45 hours, where $45 = (2 \times 24) - 3$, which we can use for a quick calculation. The train leaves NY at $2 p.m.$, so it arrives at $(2 - 3) p.m.$, which is $11 a.m.$ NY time, or $9 a.m.$ Denver time.

$9:00\ a.m.$

10.

A month ago, the ratio of nurses to doctors on a hospital staff was 3:5. Since that time, two additional nurses joined the staff, no nurses left and the number of doctors remained the same. The ratio of nurses to doctors on the hospital staff is now 4:5. How many nurses are **now** on the staff?

nurses

Let n denote the number of nurses and d the number of doctors one month ago. The verbal statement yields a linear system in n and d:

$$\begin{aligned} \frac{n}{d} &= \frac{3}{5} \\ \frac{n+2}{d} &= \frac{4}{5} \end{aligned}$$

Or equivalently:

$$\begin{aligned} 3d - 5n &= 0 \\ 4d - 5n &= 10 \end{aligned}$$

Subtracting the first equation from the second gives $d = 10$, from which $n = 6$ follows. And so $n + 2 = 8$.

8 nurses

11.

A sequence is formed by multiplying each term by 3 and then adding 3 to get the next term. If the third term is 39, what is the value of the first term?

To go backwards from the third to the first term, we apply the reverse steps twice. We first subtract 3, then divide by 3, starting from 39.

$$\begin{aligned} (39 - 3)/3 &= 12 \\ (12 - 3)/3 &= 3 \end{aligned}$$

The first term is:

3

12.

Bill and Jill both exercise on Monday, January 1. Bill exercises every 5th day, and Jill exercises every 4th day. What is the next date on which Bill and Jill both exercise?

The least common multiple is $5 \times 4 = 20$. They exercise on the same day every 20 days.

$01(January), 21st$

13.

Consider the square with vertices at $(3, 3)$, $(-3, 3)$, $(-3, -3)$ and $(3, -3)$. How many points with integer coordinates lie strictly in the interior of this square?

[ ] points

Because the figure is a square, we can count the points along one edge and square that number. The horizontal length between -3 and 3 contains points at -2, -1, 0, $+1$, $+2$, or five points:

$5^2 = 25$

25 points

14.

A bracelet is made by stringing together four beads. Each bead is either red or green. How many different color patterns are possible for the bracelet, where the patterns are considered the same if rotating one will produce the other, as shown here?

[ ] patterns

We are not told how many beads of each color to string. "Each bead is either red or green" means that the beads must be either red or green, including possibly all red or all green. There are therefore 4 possibilities:

1. Colors: $\{GGGG\}$. Patterns: $[GGGG]$. Count: 1
2. Colors: $\{RGGG\}$. Patterns: $[RGGG]$. Count: 1
3. Colors: $\{RRGG\}$. Patterns: $[RRGG, RGRG]$. Count: 2
4. Colors: $\{RRRG\}$. Patterns: $[RRRG]$. Count: 1
5. Colors: $\{RRRR\}$. Patterns: $[RRRR]$. Count: 1

Note: $RGGR$ is a repeat of $RRGG$ by rotation. Total: $1 + 1 + 2 + 1 + 1 = 6$

6 patterns

15.

To create a unique house paint color, Melton mixes together a sample that is 12 gallons of red, 2.5 gallons of yellow and 0.5 gallons of blue paint. He then mixes a main batch of paint using 30 gallons of yellow paint and enough red and blue paint to maintain the original ratio. How many total gallons of paint did he use to make the main batch of paint?

[ ] gallons

The sample mix of red/yellow/blue weighs $(12 + 2.5 + 0.5)$ gallons. The main batch is increased in proportion by the same factor that 2.5 is increased to 30, a factor of 12:

$$\frac{30}{2.5} = 12$$

The main batch therefore weighs:

$$12 \times (12 + 2.5 + 0.5) = 12 \times 12 + 12 \times 2.5 + 12 \times 0.5 = 144 + 30 + 6 = 180$$

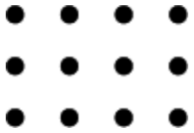
The total weight is:

180 gallons

16.

How many different squares can be formed by using four of the evenly spaced dots shown as vertices of the square?

[ ] squares



Let the distance between two adjacent dots be 1 unit. There are 6 squares formed by connecting four dots separated by 1 unit. There are 2 squares formed by connecting four dots separated by 2 units. In addition, there are 2 squares formed by connecting four dots separated by $\sqrt{2}$ units (along a diagonal).

$$6 + 2 + 2 = 10$$

10 squares

17.

Xinran walks 3 mi/h uphill, 4 mi/h on flat land and 5 mi/h downhill. If he walks one mile uphill, then one mile on flat land and then returns by the same route to his starting point, how many minutes does he walk?

[ ] minutes

At a speed of 3 mi/h uphill, Xinran walks 1 mile in 1/3 of an hour – or 20 minutes. At a speed of 4 mi/h on flat, Xinran walks 1 mile in 1/4 of an hour – or 15 minutes. At a speed of 5 mi/h downhill, Xinran walks 1 mile in 1/5 of an hour – or 12 minutes. Since Xinran backtracks after one mile, he walks a total of two miles on flat. Thus,

$$\left(\frac{1}{3} \times 60'\right) + 2 \times \left(\frac{1}{4} \times 60'\right) + \left(\frac{1}{5} \times 60'\right) = 20' + 30' + 12' = 62'$$

62 minutes

18.

Sassy Fashions buys dresses at wholesale and then marks them up for retail sale. They recently sold a dress at 40% discount off their marked-up price. What percent mark-up did they originally apply to the dress if they broke even on the sale? Express your answer to the nearest whole percent.

[ ] %

Let m denote the markup and c the cost of the dress at wholesale. The store breaks even if they sell the dress at cost. In other words, the discount exactly offsets the markup:

$$c = 0.6(1+m)c$$
$$\Rightarrow 1 + m = \frac{1}{0.6} \Rightarrow m = \frac{10}{6} - 1 = \frac{5-3}{3} = \frac{2}{3}$$

67 %

19.

For integers a, b and k, we know that $a > 12$, $b < 20$ and $a < b$. If $b = 7k$, what is the value of k?

b is a multiple of 7 between 12 and 20:

$$b = 14 \Rightarrow k = 2$$

2

20.

If one quart of paint is exactly enough for two coats of paint on a 9-foot by 10-foot wall, how many quarts of paint are needed to apply one coat of paint to a 10-foot by 12-foot wall? Express your answer as a common fraction.

$\frac{\quad}{\quad}$ quarts

Two coats cover 9×10 square feet, so one coat (0.5 quart) covers $45ft^2$. The amount of paint needed to cover $1ft^2$ is therefore $\frac{0.5}{45}$. The amount of paint needed to cover $120ft^2$ is:

$$120 \times \frac{0.5}{45} = \frac{120}{90} = \frac{4}{3}$$

$\frac{4}{3}$ quarts

21.

A mixture is made with 45 ounces of a 10% saline solution and x ounces of a 70% saline solution. The resulting mixture is a 25% saline solution. What is the value of x?

45 ounces of a 10% solution contains 0.1×45 ounces of salt, while x ounces of a 70% solution contains $0.7x$ ounces of salt. The resulting mix weighs $45 + x$ ounces. Therefore:

$$0.1 \times 45 + 0.7x = 0.25 \times (45 + x)$$

Instead of dealing with decimals and risking a silly mistake, multiply through by 100.

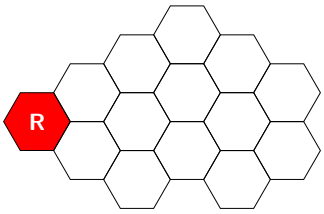
$$\begin{aligned} 10 \times 45 + 70x &= 25 \times 45 + 25x \\ \Rightarrow (70 - 25)x &= 25 \times 45 - 10 \times 45 \\ \Rightarrow 45x &= 45(25 - 10) \\ \Rightarrow x &= 25 - 10 = 15 \end{aligned}$$

15

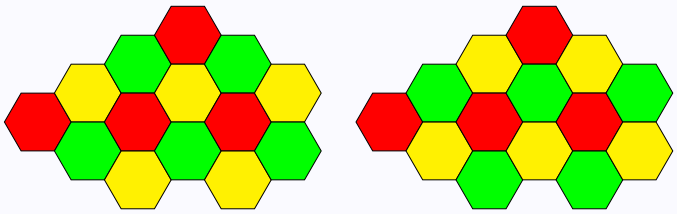
22.

In the figure, the hexagon with the "R" is colored red, and each of the other hexagons will be colored red, yellow or green, so that no two hexagons with a common side are the same color. In how many different ways can the figure be colored?

ways



There are 2 choices at the first node. Beyond that, the choice of colors is unique. The two solutions are:



2 ways

23.

Five numbered balls, each with a different number from 1 through 5, are placed in a bowl. If Josh randomly chooses two balls, with replacement, what is the probability that the two numbers on the selected balls have a product that is even and greater than 10? Express your answer as a common fraction.

$\frac{\square}{\square}$

If Josh picks either a 1 or a 2, the product cannot exceed 10 no matter what the other ball is. There are five pairs that work: $(3,4)$, $(4,3)$, $(4,4)$, $(3,4)$, $(4,5)$, $(5,4)$. Since there are $5 \times 5 = 25$ possible pairs, the probability is

$$\frac{5}{25} = \frac{1}{5}$$

$\frac{1}{5}$

24.

On Claudia's birthday in 2019, her age was four times her brother's age on that day. On her birthday in 2020, her age was three times her brother's age on that day. In what year will Claudia's age, on her birthday, be twice her brother's age on that day?

Let c be Claudia's age, b her brother's age, and x the number of years that will pass. We have

$$\begin{aligned} c &= 4b \\ c+1 &= 3(b+1) \\ c+x &= 2(b+x) \end{aligned}$$

Solving the system by substitution yields $c = 8$, $b = 2$, and $x = 4$.

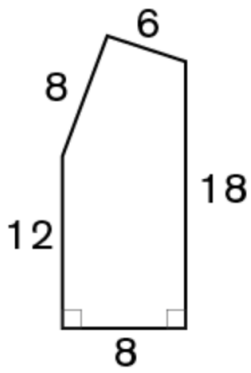
$$2019 + 4 = 2023$$

2023

25.

What is the area of the pentagon shown, with the indicated side lengths in inches?

in^2



Breaking down the areas into a rectangle and two triangles yields:

$$12 \cdot 8 + 2\left(\frac{1}{2} \cdot 8 \cdot 6\right) = 12 \cdot 8 + 8 \cdot 6 = 8(12+6) = 8 \cdot 18 = 144$$

144 in^2

26.

A bag contains exactly three red marbles, five yellow marbles and two blue marbles. If three marbles are drawn from the bag without replacement, what is the probability that all three will be the same color? Express your answer as a common fraction.

$\frac{\square}{\square}$

Three marbles of the same color could only be red or yellow. There are 10 marbles to begin with. The probability of drawing exactly three red marbles is:

$$\frac{3}{10} \times \frac{2}{9} \times \frac{1}{8}$$

The probability of drawing exactly three yellow marbles is:

$$\frac{5}{10} \times \frac{4}{9} \times \frac{3}{8}$$

Adding up the probabilities:

$$\frac{3}{10} \times \frac{2}{9} \times \frac{1}{8} + \frac{5}{10} \times \frac{4}{9} \times \frac{3}{8} = \frac{3 \times 2 + 5 \times 4 \times 3}{10 \times 9 \times 8} = \frac{66}{720} = \frac{33}{360} = \frac{11}{120}$$

$$\boxed{\frac{11}{120}}$$

27.

A peep increased by 25% is a pop. A pop decreased by 40% is a slug, and a slug increased by 100% is a slap. What percent of a peep is a slap?

$\square$ %

The relationships are:

$$\begin{aligned}\text{pop} &= (1 + 0.25)\ \text{peep}\\ \text{slug} &= (1 - 0.4)\ \text{pop}\\ \text{slap} &= (1 + 1)\ \text{slug}\end{aligned}$$

The relationship between a peep is a slap is:

$$\begin{aligned}\text{slap} &= (1 + 1)(1 - 0.4)(1 + 0.25)\ \text{peep}\\ &= 2 \cdot 0.6 \cdot 1.25\ \text{peep}\\ &= 1.5\ \text{peep}\end{aligned}$$

150 %

28.

If Jonah reverses the two digits of his age, divides the resulting number by three, and then adds 20, the result is Jonah's age. How old is Jonah?

____ years old

Let x and y denote the digits. Jonah's age may be written $10x + y$. The verbal statement yields the equation:

$$\frac{10y + x}{3} + 20 = 10x + y$$

Multiplying through by 3 and combining terms yields:

$$29x = 7y + 60$$

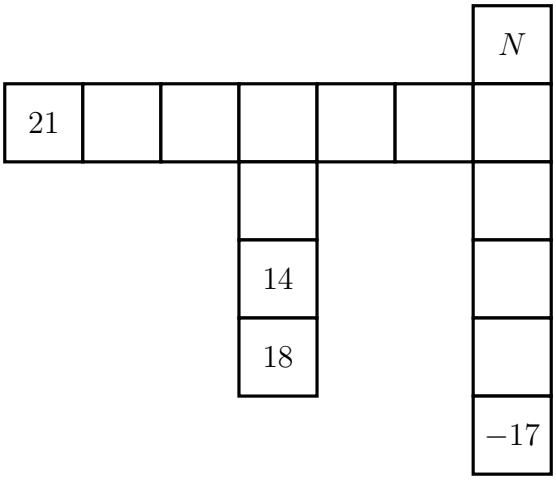
Based on trial and error (starting with low values of x and matching the last digit of multiples of 7), we find $x = 4$ and $y = 8$.

48 years old

29.

In the figure shown, the sequence of integers in the row of squares and in each of the two columns of squares form three distinct arithmetic sequences. What is the value of N?

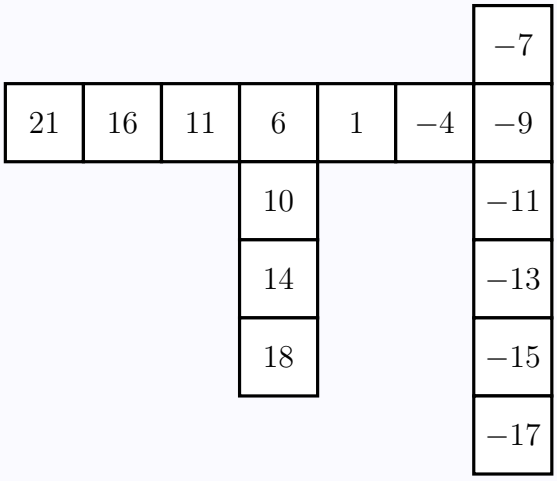
____



Refer to the figure. Since these are arithmetic sequences, we subtract 4 from 18 to get 14, 10, 6. This means we go from 21 down to 6 in three steps, i.e. by subtracting $15/3 = 5$ at each step. Thus going rightwards from 6 yields 1, -4, and -9 (immediately below N). This means we go from -17 to -9 in four steps, i.e. by adding $8/4 = 2$ at each step. This implies

$$N = -9 + 2 = -7$$

Complete solution:



-7

30.

If $\frac{a}{b} = \frac{3}{4}$, $\frac{b}{c} = \frac{8}{9}$, and $\frac{c}{d} = \frac{2}{3}$, what is the value of $\frac{ad}{b^2}$? Express your answer as a common fraction.

$$\frac{a}{b} = \frac{3}{4}$$
$$\frac{b}{c} = \frac{8}{9}$$
$$\frac{c}{d} = \frac{2}{3}$$

By dividing the first equation by the second and third equations, we get

$$\begin{aligned}\frac{a}{b} \times \frac{c}{b} \times \frac{d}{c} &= \frac{3}{4} \times \frac{9}{8} \times \frac{3}{2} \\ \frac{ad}{b^2} &= \frac{3 \times 9 \times 3}{4 \times 8 \times 2} \\ &= \frac{81}{64}\end{aligned}$$

$$\boxed{\frac{81}{64}}$$