AMC 10 Problem Series (2804)

Jon Joseph

Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Homework

Lesson: 1 2 3 **4** 5 6 7 8 9 10 11 12

Homework: Lesson 4

You have completed **10** of **10** challenge problems.
Past Due **Jul 3**.

Readings

Lesson 4 Transcript: Fri, Jun 25

Challenge Problems

Total Score: 60 / 60

Problem 1 – Correct! – Score: 6 / 6 (2865)

Problem:

Report Error

For how many integers n between 1 and 100 does $x^2 + x - n$ factor into the product of two linear factors with integer coefficients?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 9 **(E)** 10

Solution:

If $x^2 + x - n$ can be factored into two linear factors with integer coefficients, then we can write

$$x^2 + x - n = (x + a)(x + b),$$

where a and b are integers. Expanding the right-hand side, we get

$$x^2 + x - n = x^2 + (a + b)x + ab.$$

Equating the coefficients, we get $a + b = 1$ and $ab = -n$. Then $b = 1 - a$, so

$$n = -ab = -a(1 - a) = a^2 - a.$$

Completing the square, we get

$$a^2 - a = \left(a - \frac{1}{2}\right)^2 - \frac{1}{4}.$$

This expression tells us two important things. First, the function $a^2 - a$ is symmetric around $a = 1/2$. (For example, substituting $a = 4$ and $a = -3$ give the same value of $a^2 - a$, namely 12.) Second, the function $a^2 - a$ is increasing for $a \ge 1/2$. Hence, to find all the values of $n = a^2 - a$ that are between 1 and 100, it suffices to check the cases where $a \ge 1$.

Let $f(a) = a^2 - a$. Then $f(1) = 0$, $f(2) = 2$, $f(3) = 6$, and so on, up to $f(10) = 90$, and $f(11) = 110$. Hence, $n = a^2 - a$ is between 1 and 100 for $a = 2$, 3, . . ., 10, for a total of $\boxed{\text{nine}}$ values. The answer is (D).

Your Response(s):

D

Problem 2 – Correct! – Score: 6 / 6 (2866)

Problem: Report Error

Suppose that a and b are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. Then the pair (a, b) is

(A) $(-2, 1)$ **(B)** $(-1, 2)$ **(C)** $(1, -2)$ **(D)** $(2, -1)$ **(E)** $(4, 4)$

Solution:

By Vieta's formulas, $a + b = -a$ and $ab = b$. Since b is nonzero, we can divide both sides of the equation $ab = b$ by b to get $a = 1$. Then from the equation $a + b = -a$, $b = -2a = -2$, so $(a, b) = \boxed{(1, -2)}$. The answer is (C).

Your Response(s):

C

Problem 3 – Correct! – Score: 6 / 6 (2867)

Problem: Report Error

Let f be the function defined by $f(x) = ax^2 - \sqrt{2}$ for some positive a. If $f(f(\sqrt{2})) = -\sqrt{2}$, then $a =$

(A) $\frac{2-\sqrt{2}}{2}$ **(B)** $\frac{1}{2}$ **(C)** $2 - \sqrt{2}$ **(D)** $\frac{\sqrt{2}}{2}$ **(E)** $\frac{2+\sqrt{2}}{2}$

Solution:

Using the definition of $f(x)$, we find

$$f(\sqrt{2}) = 2a - \sqrt{2},$$

and

$$f(f(\sqrt{2})) = a(2a - \sqrt{2})^2 - \sqrt{2}.$$

Hence, $a(2a - \sqrt{2})^2 - \sqrt{2} = -\sqrt{2}$. Then

$$a(2a - \sqrt{2})^2 = 0,$$

so $a = 0$ or $2a - \sqrt{2} = 0$. Since a is positive, we must have $2a - \sqrt{2} = 0$, so $a = \boxed{\sqrt{2}/2}$. The answer is (D).

Your Response(s):

D

Problem 4 – Correct! – Score: 6 / 6 (2868)

Problem:

Report Error

Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. The number of possible values of k is

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** more than four

Solution:

Let the roots of the quadratic equation $x^2 - 63x + k = 0$ be the primes p and q. Then by Vieta's formulas, $p + q = 63$ and $pq = k$.

The only even prime is 2. Since $p + q = 63$, both p and q cannot be odd, and both p and q cannot be even, so exactly one of p and q is even. This means p and q must be 2 and 61 in some order, so the only possible value of k is $2 \cdot 61 = 122$. The answer is (B).

Your Response(s):

B

Problem 5 – Correct! – Score: 6 / 6 (2869)

Problem:

Report Error

Let @ denote the "averaged with" operation: $a@b = \dfrac{a+b}{2}$. Which of the following distributive laws holds for all numbers x, y, and z?

I. $x@(y+z) = (x@y) + (x@z)$

II. $x + (y@z) = (x+y)@(x+z)$

III. $x@(y@z) = (x@y)@(x@z)$

(A) I only **(B)** II only **(C)** III only **(D)** I and III only **(E)** II and III only

Solution:

In I, the left-hand side is

$$x@(y+z) = \frac{x+y+z}{2},$$

and the right-hand side is

$$(x@y) + (x@z) = \frac{x+y}{2} + \frac{x+z}{2} = \frac{2x+y+z}{2},$$

so I does not hold.

In II, the left-hand side is

$$x + (y@z) = x + \frac{y+z}{2} = \frac{2x+y+z}{2},$$

and the right-hand side is

$$(x+y)@(x+z)=\frac{(x+y)+(x+z)}{2}=\frac{2x+y+z}{2},$$

so II does hold.

In III, the left-hand side is

$$x@(y@z)=x@\frac{y+z}{2}=\frac{x+(y+z)/2}{2}=\frac{2x+y+z}{4},$$

and the right-hand side is

$$(x@y)@(x@z)=\frac{x+y}{2}@\frac{x+z}{2}=\frac{(x+y)/2+(x+z)/2}{2}=\frac{2x+y+z}{4},$$

so III does hold.

The answer is (E).

Your Response(s):
E

Problem 6 – Correct! – Score: 6 / 6 (2870)

Problem:
Report Error
If $f(x)=ax^4-bx^2+x+5$ and $f(-3)=2$, then $f(3)=$

(A) -5 **(B)** -2 **(C)** 1 **(D)** 3 **(E)** 8

Solution:
Substituting $x=-3$, we get

$$f(-3)=81a-9b-3+5=81a-9b+2.$$

But $f(-3)=2$, so $81a-9b+2=2$, which means $81a-9b=0$. Then

$$f(3)=81a-9b+3+5=0+3+5=\boxed{8}.$$

The answer is (E).

Your Response(s):
E

Problem 7 – Correct! – Score: 6 / 6 (2871)

Problem:
Report Error
What is the sum of the reciprocals of the roots of the equation

$$\frac{2003}{2004}x+1+\frac{1}{x}=0?$$

(A) $-\dfrac{2004}{2003}$ **(B)** -1 **(C)** $\dfrac{2003}{2004}$ **(D)** 1 **(E)** $\dfrac{2004}{2003}$

Solution:
Multiplying both sides of the given equation by $2004x$, we get

$$2003x^2 + 2004x + 2004 = 0.$$

Let the roots of this quadratic equation be a and b. Then by Vieta's formulas, $a + b = -2004/2003$ and $ab = 2004/2003$. Then the sum of the reciprocals of a and b is

$$\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{-2004/2003}{2004/2003} = \boxed{-1}.$$

The answer is (B).

Your Response(s):

B

Problem 8 – Correct! – Score: 6 / 6 (2872)

Problem: Report Error

Let f be a polynomial function such that, for all real x,

$$f(x^2+1) = x^4 + 5x^2 + 3.$$

For all real x, $f(x^2 - 1)$ is

(A) $x^4 + 5x^2 + 1$ **(B)** $x^4 + x^2 - 3$ **(C)** $x^4 - 5x^2 + 1$ **(D)** $x^4 + x^2 + 3$ **(E)** none of these

Solution:
Let $y = x^2 + 1$. Then $x^2 = y - 1$, so we can write the given equation as

$$\begin{aligned} f(y) &= x^4 + 5x^2 + 3 \\ &= (x^2)^2 + 5x^2 + 3 \\ &= (y-1)^2 + 5(y-1) + 3 \\ &= y^2 - 2y + 1 + 5y - 5 + 3 \\ &= y^2 + 3y - 1. \end{aligned}$$

Then substituting $x^2 - 1$, we get

$$\begin{aligned} f(x^2-1) &= (x^2-1)^2 + 3(x^2-1) - 1 \\ &= x^4 - 2x^2 + 1 + 3x^2 - 3 - 1 \\ &= \boxed{x^4 + x^2 - 3}. \end{aligned}$$

The answer is (B).

Your Response(s):

B

Problem 9 – Correct! – Score: 6 / 6 (2873)

Problem: Report Error

The polynomial $x^3 - ax^2 + bx - 2010$ has three positive integer roots. What is the smallest possible value of a?

(A) 78 **(B)** 88 **(C)** 98 **(D)** 108 **(E)** 118

Solution:

Let the positive integer zeros be r, s, and t. Then by Vieta's formulas,

$$\begin{aligned} r + s + t &= a, \\ rs + rt + st &= b, \\ rst &= 2010. \end{aligned}$$

The prime factorization of 2010 is $2 \cdot 3 \cdot 5 \cdot 67$, so one of the zeros must be a multiple of 67. If one of the zeros is 67, then the product of the other two zeros is $2010/67 = 30$. Then the sum of these two zeros is minimized when they are 5 and 6, and the sum of all three zeros is $5 + 6 + 67 = 78$.

Otherwise, if one of the zeros is not 67, then the zero that is a multiple of 67 must be at least $2 \cdot 67 = 134$, which is greater than 78. Therefore, the smallest possible value of a is $\boxed{78}$. The answer is (A).

Your Response(s):

A

Problem 10 – Correct! – Score: 6 / 6 (2874)

Problem: Report Error

Let f be a function for which $f(x/3) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

(A) -1/3 **(B)** -1/9 **(C)** 0 **(D)** 5/9 **(E)** 5/3

Solution:

Setting $x = 9z$, we get $f(3z) = (9z)^2 + 9z + 1 = 81z^2 + 9z + 1$. Hence, the equation $f(3z) = 7$ becomes $81z^2 + 9z + 1 = 7$, or

$$81z^2 + 9z - 6 = 0.$$

By Vieta's formulas, the sum of the roots of this quadratic equation is $-9/81 = \boxed{-1/9}$. The answer is (B).

Your Response(s):

B