AMC 10 Problem Series (2804)

Jon Joseph

Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Overview

Lesson 12 (Aug 20) Class Transcript - Coordinates and Graphs

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jonjoseph 2021-08-20 19:30:58
And who is the winner on our last night??? It goes to @ jamesl123456!! Nicely done.

jonjoseph 2021-08-20 19:31:27
AMC 10 Problem Series
Week 12: Coordinates and Graphs

jonjoseph 2021-08-20 19:31:38
Say hello to Julie Zhu (lizhufan) who will be helping us today!

jonjoseph 2021-08-20 19:31:41
Julie (Li) Zhu has a PhD in biostatistics from University of Minnesota and has taught undergraduate and graduate level statistics courses. She is also a proud math mom, having trained her four children in math since they started talking. Julie enjoys playing mathematical games with her family and learning together. She volunteers for a variety of local and national math competitions. She first heard of AoPS when her daughter took a class years ago. She joined the AoPS staff in 2019. She enjoys gardening and hiking while not doing math.

lizhufan 2021-08-20 19:31:45
Hi everybody!

jonjoseph 2021-08-20 19:32:28
Really? Lucky you. She's one of the best.

jonjoseph 2021-08-20 19:32:43
As this is the last class, please be sure to fill out your feedback form on the class homepage! Your ideas and suggestions help us make AoPS even more amazing than it already is.

lizhufan 2021-08-20 19:32:43
I'll do my best. Thanks Jon!

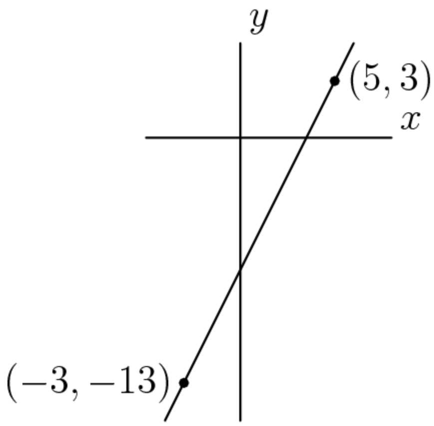
jonjoseph 2021-08-20 19:32:58
Today, we will look at problems involving the coordinate plane. We have saved this topic for last because a proper study of coordinate geometry requires a good handle on both algebra and geometry.

jonjoseph 2021-08-20 19:33:06
LINES

jonjoseph 2021-08-20 19:33:14
We start by looking at problems involving lines, so let's make sure we understand how to find the equations of lines.

jonjoseph 2021-08-20 19:33:20
Suppose we want to find the equation of the line passing through $(-3,-13)$ and $(5,3)$.

jonjoseph 2021-08-20 19:33:29



jonjoseph 2021-08-20 19:33:41
(Always draw a picture if you can)

jonjoseph 2021-08-20 19:33:46
How could we start working towards the equation of the line?

jonjoseph 2021-08-20 19:34:14
We can find the slope of this line. What is the slope of this line?

jonjoseph 2021-08-20 19:34:44
The slope of this line is $\frac{\text{rise}}{\text{run}} = \frac{3-(-13)}{5-(-3)} = \frac{16}{8} = 2.$

jonjoseph 2021-08-20 19:34:50
So what is an equation of the line that we can write down?

jonjoseph 2021-08-20 19:35:58
Since the line has slope 2 and passes through $(5,3)$, an equation of the line can be written as

$$y - 3 = 2(x-5).$$

jonjoseph 2021-08-20 19:36:08
Indeed, as we can immediately read off from how the equation has just been written, this is the equation of a line with slope 2, and when x is equal to 5, y is equal to 3, so the line passes through $(5,3)$.

jonjoseph 2021-08-20 19:36:26
Since the line also passes through $(-3,-13)$, we can also write $y-(-13) = 2(x-(-3))$, which simplifies to: $y+13 = 2(x+3)$.

jonjoseph 2021-08-20 19:36:38
(In case you're puzzled by this assertion, take a moment to solve for y in both equations. You will find that, after simplifying, the two equations reduce to saying the same thing.)

jonjoseph 2021-08-20 19:36:55
In general, the line with slope m passing through (x_0, y_0) has the equation

$$y - y_0 = m(x - x_0).$$

After we have $y-3 = 2(x-5)$, we can simplify this to $y = 2x-7$.

jonjoseph 2021-08-20 19:37:16
The form $y = mx+b$ is called the *slope-intercept* form of a line, because from it you can immediately read off the slope m of the line and the y-coordinate of the y-intercept of the line, which is the point $(0,b)$.

jonjoseph 2021-08-20 19:37:35

We can see that our line intersects the y-axis at $(0, -7)$.

jonjoseph 2021-08-20 19:37:43

There are several forms for writing down the equation of a line, and the slope-intercept form is *often* the most useful form, but not always! The form you want to use depends on the needs of the problem.

jonjoseph 2021-08-20 19:38:02

A line with slope 3 intersects a line with slope 5 at the point $(10, 15)$. What is the distance between the x-intercepts of these two lines?

$\text{(A) } 2 \quad \text{(B) } 5 \quad \text{(C) } 7 \quad \text{(D) } 12 \quad \text{(E) } 20$

jonjoseph 2021-08-20 19:38:13

How could we start?

jonjoseph 2021-08-20 19:39:23

I agree with the drawing a picture comments. Although that may difficult at this stage.

jonjoseph 2021-08-20 19:39:27

We could start by figuring out the equations of the lines.

jonjoseph 2021-08-20 19:39:32

We have a line with slope 3 that passes through the point $(10, 15)$. What is an equation for this line?

jonjoseph 2021-08-20 19:39:53

See if you can write your answer in slope-intercept form.

jonjoseph 2021-08-20 19:40:50

An equation for this line is $y - 15 = 3(x - 10)$, which we could simplify to $y = 3x - 15$ if we want. How do we find the x-intercept of this line?

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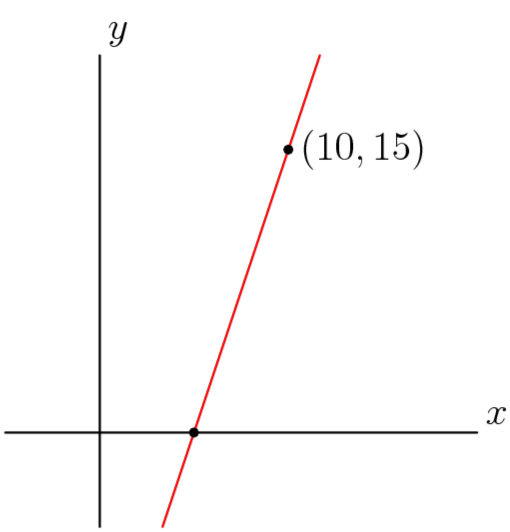


jonjoseph 2021-08-20 19:41:20

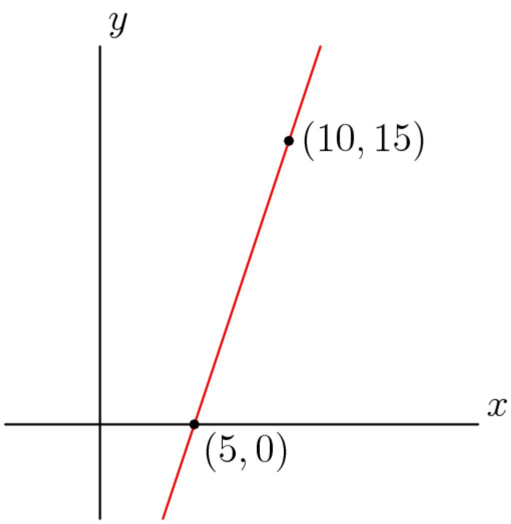
Good. Go. What do you find?

jonjoseph 2021-08-20 19:42:11

The x-intercept of this line is the point where the line crosses the x-axis, so we can find the x-intercept of this line by setting $y = 0$. This gives us $-15 = 3(x - 10)$.

jonjoseph 2021-08-20 19:42:20
Solving for x, we find $x = 5$.

jonjoseph 2021-08-20 19:42:22



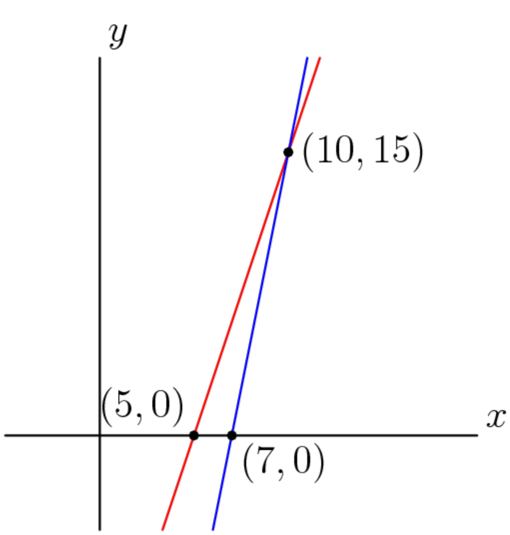
jonjoseph 2021-08-20 19:42:28
We also have a line with slope 5 that passes through $(10, 15)$. What is an equation for this line?

jonjoseph 2021-08-20 19:44:01
An equation for this line is $y - 15 = 5(x - 10)$, which we could simplify to $y = 5x - 35$.

jonjoseph 2021-08-20 19:44:08
Again, we can find the x-intercept of this line by setting $y = 0$. This gives us $-15 = 5(x - 10)$. So what is the corresponding x-value?

jonjoseph 2021-08-20 19:45:08
Solving for x, we find $x = 7$. So what is the distance between the x-intercepts?

jonjoseph 2021-08-20 19:45:18



jonjoseph 2021-08-20 19:45:48
The distance between the x-intercepts is $7 - 5 = 2$. The answer is (A).

jonjoseph 2021-08-20 19:45:57

Not too hard. Any questions?

jonjoseph 2021-08-20 19:47:13

I almost always use slope-intercept because the important information is right there. However, when you take calculus and especially if you take AP Calculus then point slope is better because it is usually faster.

jonjoseph 2021-08-20 19:48:10

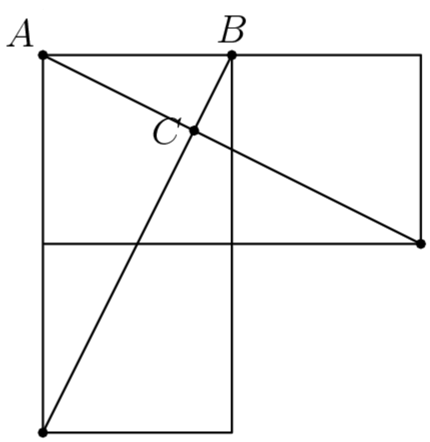
Also, when working with lines the biggest mistake I see is getting your signs messed up when calculating the slope. So watch for that.

jonjoseph 2021-08-20 19:48:34

Three unit squares and two line segments connecting two pairs of vertices are shown. What is the area of triangle ABC?

$(\text{A})\ \frac{1}{6} \quad (\text{B})\ \frac{1}{5} \quad (\text{C})\ \frac{2}{9} \quad (\text{D})\ \frac{1}{3} \quad (\text{E})\ \frac{\sqrt{2}}{4}$

jonjoseph 2021-08-20 19:48:40



jonjoseph 2021-08-20 19:49:00

"Unit" squares suggest what approach?

jonjoseph 2021-08-20 19:49:51

The three unit squares create a grid which relates nicely to coordinates.

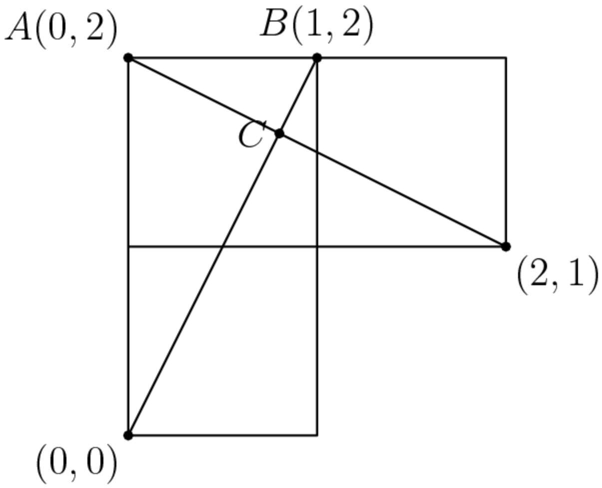
jonjoseph 2021-08-20 19:50:01

Also, we are interested in an intersection of lines, and lines are easy to describe using coordinates. So coordinates are probably a good choice for this problem.

jonjoseph 2021-08-20 19:50:07

Let's place the diagram in the coordinate plane as follows:

jonjoseph 2021-08-20 19:50:09



jonjoseph 2021-08-20 19:50:14

What is the equation of the line passing through $(0,0)$ and $B(1,2)$?

jonjoseph 2021-08-20 19:51:06

The equation of the line passing through $(0,0)$ and $B(1,2)$ is $y = 2x$.

jonjoseph 2021-08-20 19:51:11

Next, we find the equation of the line passing through $A(0,2)$ and $(2,1)$. First, what is the slope of this line?

jonjoseph 2021-08-20 19:52:15

The slope of the line passing through $A(0,2)$ and $(2,1)$ is $\dfrac{1-2}{2-0} = -\dfrac{1}{2}$. So what is an equation of this line?

jonjoseph 2021-08-20 19:53:17

An equation of this line is $y - 2 = -\dfrac{x}{2}$.

jonjoseph 2021-08-20 19:53:26

So we have the system of equations $y = 2x$ and $y - 2 = -\frac{x}{2}$. (As the intersection of the lines, the point C corresponds to the solution of this system.) What can we do with these equations?

jonjoseph 2021-08-20 19:54:23

We can substitute $y = 2x$ into the equation $y - 2 = -\frac{x}{2}$, to get $2x - 2 = -\frac{x}{2}$. So what is the value of x?

jonjoseph 2021-08-20 19:55:25

Solving for x, we find $x = \frac{4}{5}$. Then what is the value of y?

jonjoseph 2021-08-20 19:56:12

We see that $y = 2x = \frac{8}{5}$.

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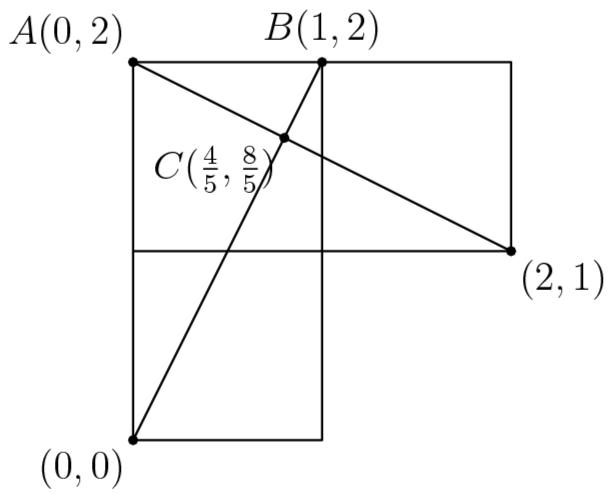

jonjoseph 2021-08-20 19:56:25
So what is the area of $\triangle ABC$?

jonjoseph 2021-08-20 19:56:45
Hint: There are several hard ways and one really easy way to find the area.

jonjoseph 2021-08-20 19:57:48
$\triangle ABC$ has base $AB = 1$ and height $2 - \frac{8}{5} = \frac{2}{5}$, so its area is $\frac{1}{2} \cdot b \cdot h = \frac{1}{2} \cdot 1 \cdot \frac{2}{5} = \frac{1}{5}$. The answer is (B).

jonjoseph 2021-08-20 19:58:54
Yep. Agreed. Your choice.

jonjoseph 2021-08-20 19:59:03
A triangle has vertices $(0,0)$, $(1,1)$, and $(6m,0)$, and the line $y = mx$ divides the triangle into two triangles of equal area. What is the sum of all possible values of m?

$$\text{(A)}\ -\frac{1}{3} \quad \text{(B)}\ -\frac{1}{6} \quad \text{(C)}\ \frac{1}{6} \quad \text{(D)}\ \frac{1}{3} \quad \text{(E)}\ \frac{1}{2}$$

jonjoseph 2021-08-20 19:59:26
First, let's draw a diagram. To make things easier to talk about, let's label the points. Let $A = (0,0)$, $B = (1,1)$, and $C = (6m,0)$.

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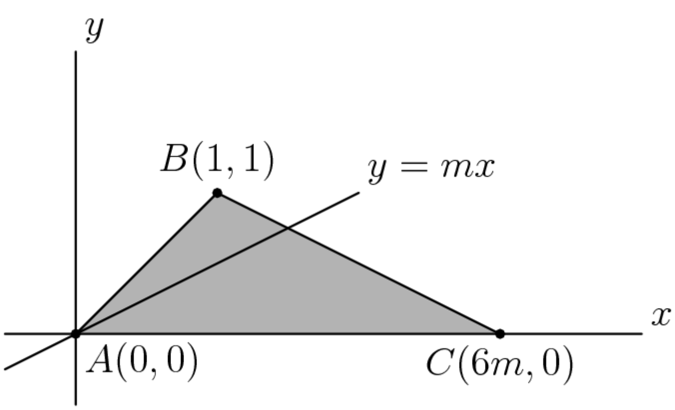


jonjoseph 2021-08-20 19:59:48

When does the line $y = mx$ bisect the area of triangle ABC?

jonjoseph 2021-08-20 20:01:11

The line $y = mx$ bisects the area of $\triangle ABC$ if and only if it passes through the midpoint of $\overline{BC}$.

jonjoseph 2021-08-20 20:01:20

That's because this line will divide $\triangle ABC$ into two triangles. Those two triangles have a common height from point A to $\overleftrightarrow{BC}$, so their bases on $\overleftrightarrow{BC}$ must be equal.

jonjoseph 2021-08-20 20:01:51

Thus the intersection point must cut segment $\overline{BC}$ in half.

jonjoseph 2021-08-20 20:01:55

What is the midpoint of $\overline{BC}$?

jonjoseph 2021-08-20 20:03:27

The midpoint of $\overline{BC}$ is the average of the endpoints:

$$\left(\frac{6m+1}{2}, \frac{1}{2}\right).$$

jonjoseph 2021-08-20 20:04:21

Just to get our definitions right is the segment BC called the median or is the line segment from the origin to the midpoint of BC called the median?

jonjoseph 2021-08-20 20:05:25

Perfect.

jonjoseph 2021-08-20 20:05:32

We want this point to lie on the line $y = mx$. What equation does this give us?

jonjoseph 2021-08-20 20:07:01

We substitute the point that is on the line into the equation of the desired line, $y = mx$. This gives us the equation

$$\frac{1}{2} = m \cdot \frac{6m+1}{2}.$$

jonjoseph 2021-08-20 20:07:13

This equation simplifies to $6m^2 + m - 1 = 0$. What are the solutions?

jonjoseph 2021-08-20 20:09:25

This equation factors as $(3m-1)(2m+1) = 0$, so the solutions are $m = \frac{1}{3}$ and $m = -\frac{1}{2}$.

jonjoseph 2021-08-20 20:09:38

That took quite some time. Do you have questions?

jonjoseph 2021-08-20 20:09:57

Or just one of those things?

jonjoseph 2021-08-20 20:10:22

Okay. I get it. Are both solutions valid?

jonjoseph 2021-08-20 20:11:11

Hmmmm??? Let's dig deeper.

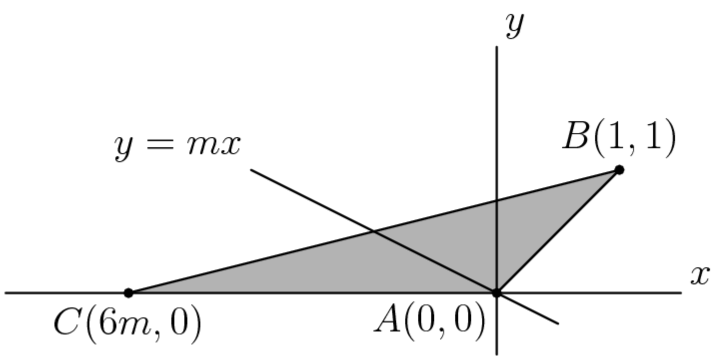
jonjoseph 2021-08-20 20:11:23

Almost everyone was saying no. In the diagram above, it sure looks like a negative slope will only intersect the triangle at point A and it can't possibly be split in two. What is wrong with that logic? Why does the negative value of m actually work?

jonjoseph 2021-08-20 20:12:45
Right. Don't be confused by our first drawing.

jonjoseph 2021-08-20 20:12:53
The negative solution works here. If $m = -\frac{1}{2}$, then the point $C(6m, 0)$ is to the left of the y-axis, and $y = mx$ will still bisect the area of $\triangle ABC$. The diagram will look something like this:

jonjoseph 2021-08-20 20:12:58



jonjoseph 2021-08-20 20:13:56
Yes. Exactly my point. And I'm a terrible picture maker. Never take geometry from me (kidding!!)

jonjoseph 2021-08-20 20:13:59
So, what is the sum of the solutions?

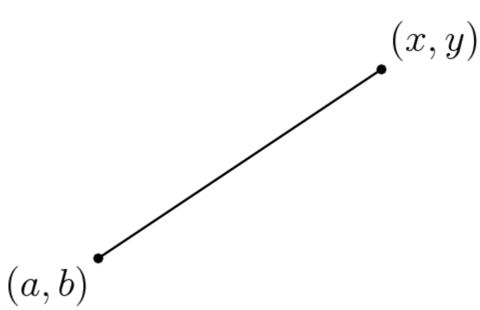
jonjoseph 2021-08-20 20:14:35
Hint: Careful when you add!!

jonjoseph 2021-08-20 20:15:14
The sum of the solutions is $-\frac{1}{6}$. The answer is (B).

jonjoseph 2021-08-20 20:16:00
Before we move on to the next problem, let's review how to compute distances in coordinates.

jonjoseph 2021-08-20 20:16:02
What is the distance between points (a, b) and (x, y)?

jonjoseph 2021-08-20 20:16:05



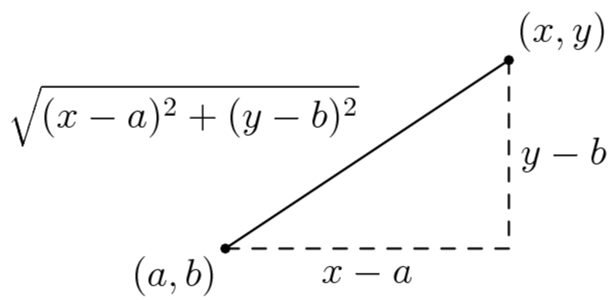
jonjoseph 2021-08-20 20:17:50
The distance between (a, b) and (x, y) is

$$\sqrt{(x-a)^2 + (y-b)^2}.$$

jonjoseph 2021-08-20 20:17:53

This follows from the Pythagorean Theorem. We can build a right triangle with legs of length $x-a$ and $y-b$, and the length of the hypotenuse is the distance between the two points.

jonjoseph 2021-08-20 20:17:59



jonjoseph 2021-08-20 20:18:08

This fact is known as the distance formula.

jonjoseph 2021-08-20 20:18:21

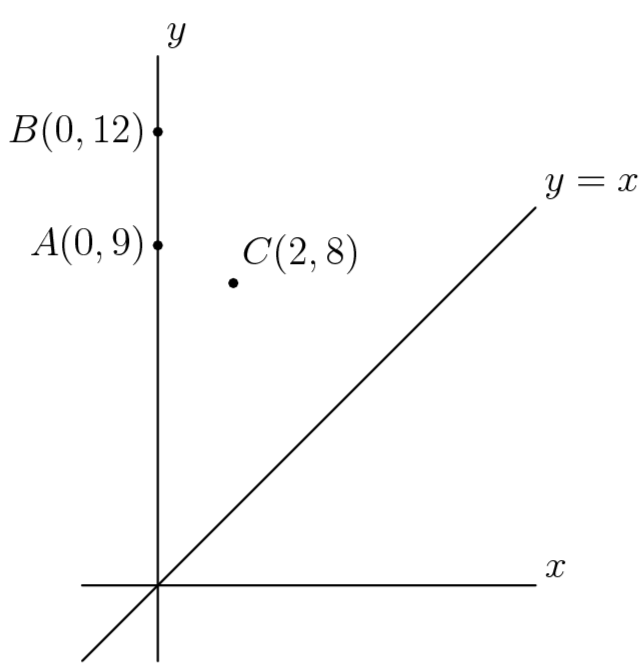
Let $A=(0,9)$ and $B=(0,12)$. Points A' and B' are on the line $y=x$, and $\overline{AA'}$ and $\overline{BB'}$ intersect at $C=(2,8)$. What is the length of $\overline{A'B'}$?

$(\mathrm{A})\ 2 \quad (\mathrm{B})\ 2\sqrt{2} \quad (\mathrm{C})\ 3 \quad (\mathrm{D})\ 2+\sqrt{2} \quad (\mathrm{E})\ 3\sqrt{2}$

jonjoseph 2021-08-20 20:18:44

We can begin by drawing a diagram.

jonjoseph 2021-08-20 20:18:46



jonjoseph 2021-08-20 20:18:56

We need to find points A' and B'. How can we find A'?

jonjoseph 2021-08-20 20:19:41

We are told that A' lies on the line $y = x$.

jonjoseph 2021-08-20 20:19:44

We are also told that $\overline{AA'}$ goes through $C(2, 8)$ (where it intersects $\overline{BB'}$).

jonjoseph 2021-08-20 20:19:51

Thus, A' is the intersection of the line $y = x$ and the line passing through $A(0, 9)$ and $C(2, 8)$.

jonjoseph 2021-08-20 20:19:55

What is the slope of the line passing through $A(0, 9)$ and $C(2, 8)$?

jonjoseph 2021-08-20 20:20:45

The slope of the line passing through $A(0, 9)$ and $C(2, 8)$ is $\dfrac{8-9}{2-0} = -\dfrac{1}{2}$. So what is an equation of this line?

jonjoseph 2021-08-20 20:21:35

The line has a slope of $-\frac{1}{2}$ and a y-intercept of 9, so an equation for the line is $y = -\frac{1}{2}x + 9$.

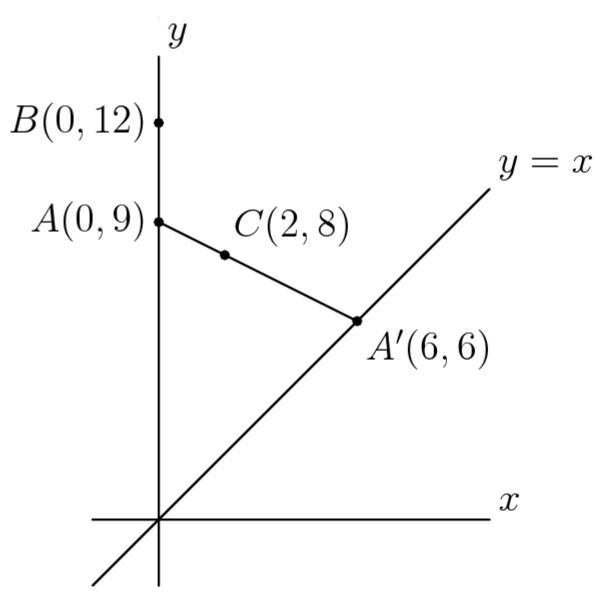
jonjoseph 2021-08-20 20:21:43

We want the intersection of this line with the line $y = x$, so we can substitute $y = x$ to get $x = -\frac{1}{2}x + 9$. So what is x?

jonjoseph 2021-08-20 20:22:28

Solving for x, we find $x = 6$. Hence, $A' = (6, 6)$.

jonjoseph 2021-08-20 20:22:31



jonjoseph 2021-08-20 20:22:44

(Always update your picture if you can)

jonjoseph 2021-08-20 20:22:47

Next, B' is the intersection of the line $y = x$ and the line passing through $B(0, 12)$ and $C(2, 8)$.

jonjoseph 2021-08-20 20:22:56

What is the slope of the line passing through $B(0, 12)$ and $C(2, 8)$?

jonjoseph 2021-08-20 20:23:57

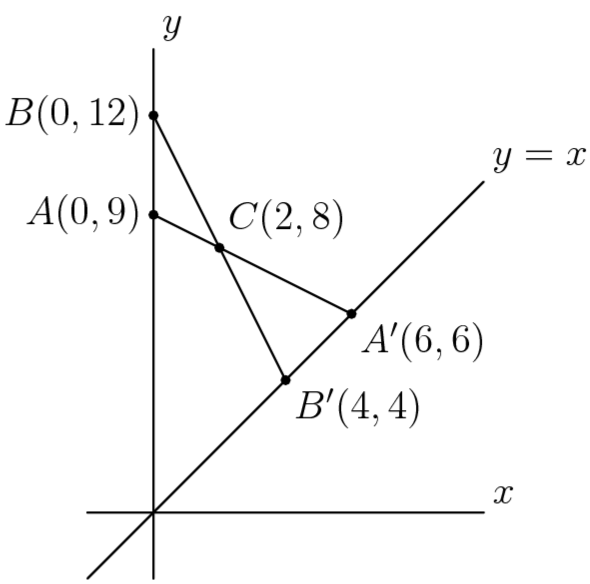
The slope of the line passing through $B(0, 12)$ and $(2, 8)$ is $\dfrac{8-12}{2-0} = -2$. So what is an equation of this line?

jonjoseph 2021-08-20 20:24:58
The line has a slope of -2 and a y-intercept of 12, so an equation of this line is $y = -2x + 12$.

jonjoseph 2021-08-20 20:25:05
We want the intersection of this line with the line $y = x$, so we can substitute $y = x$ to get $x = -2x + 12$. So what is x?

jonjoseph 2021-08-20 20:25:52
Solving for x, we find $x = 4$. Hence, $B' = (4, 4)$.

jonjoseph 2021-08-20 20:25:56



jonjoseph 2021-08-20 20:26:02
So what is the distance $A'B'$?

jonjoseph 2021-08-20 20:27:10
The distance $A'B'$ is

$$\sqrt{(6-4)^2 + (6-4)^2} = \sqrt{8} = 2\sqrt{2}.$$

The answer is (B).

jonjoseph 2021-08-20 20:27:16
CIRCLES

jonjoseph 2021-08-20 20:27:20
The distance formula also gives us the general equation of a circle.

jonjoseph 2021-08-20 20:27:27
What is an equation for a circle with center (a, b) and radius r?

jonjoseph 2021-08-20 20:28:25
If (x, y) lies on the circle with center (a, b) and radius r, then by the distance formula,

$$\sqrt{(x-a)^2 + (y-b)^2} = r.$$

jonjoseph 2021-08-20 20:28:31

Squaring both sides, we find that an equation for a circle with center (a, b) and radius r is

$$(x-a)^2 + (y-b)^2 = r^2.$$

jonjoseph 2021-08-20 20:28:43

In the xy-plane, the segment with endpoints $(-5, 0)$ and $(25, 0)$ is the diameter of a circle. If the point $(x, 15)$ is on the circle, then x equals

(A) 10 (B) 12.5 (C) 15 (D) 17.5 (E) 20

jonjoseph 2021-08-20 20:28:54

To find the equation of the circle, what do we need?

jonjoseph 2021-08-20 20:29:20

Good. What else?

jonjoseph 2021-08-20 20:29:56

We need the center and radius of the circle. What is the center of the circle?

jonjoseph 2021-08-20 20:31:05

Since the segment from $(-5, 0)$ to $(25, 0)$ is a diameter of the circle, the center is their midpoint, which is $\left(\frac{-5+25}{2}, \frac{0}{2}\right) = (10, 0).$

jonjoseph 2021-08-20 20:31:13

What is the length of the circle's radius?

jonjoseph 2021-08-20 20:31:54

The points $(-5, 0)$ and $(25, 0)$ both lie on the x-axis and the distance between them is 30, so the radius has length $\frac{30}{2} = 15.$

jonjoseph 2021-08-20 20:32:06

So what is the equation of the circle?

jonjoseph 2021-08-20 20:33:34

The equation of the circle is $(x-10)^2 + y^2 = 15^2.$

jonjoseph 2021-08-20 20:33:44

Is that clear?

jonjoseph 2021-08-20 20:33:54

We want to find x when $y = 15$. What is x when $y = 15$?

jonjoseph 2021-08-20 20:34:58

Well done.

jonjoseph 2021-08-20 20:35:01

Substituting $y = 15$, we get $(x-10)^2 + 15^2 = 15^2$, so $(x-10)^2 = 0.$

jonjoseph 2021-08-20 20:35:09

The only possible value of x is 10. The answer is (A).

jonjoseph 2021-08-20 20:35:17

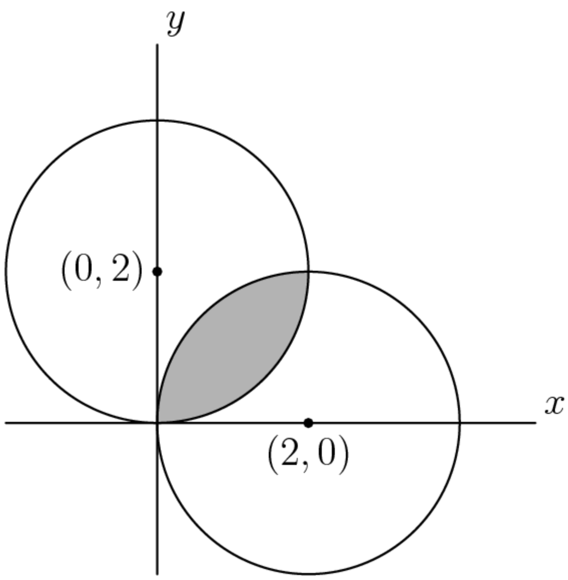
Two circles of radius 2 are centered at $(2, 0)$ and at $(0, 2)$. What is the area of the intersection of the interiors of the two circles?

(A) $\pi - 2$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi\sqrt{3}}{3}$ (D) $2(\pi - 2)$ (E) π

jonjoseph 2021-08-20 20:35:24

We can start by graphing the two circles.

jonjoseph 2021-08-20 20:35:25



jonjoseph 2021-08-20 20:35:32

What are the two points where the circles intersect?

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Hint: Guess!

jonjoseph 2021-08-20 20:36:27

Wow. Nice guesses.

jonjoseph 2021-08-20 20:36:31

It looks like the two circles intersect at $(0,0)$ and $(2,2)$. We can verify that this guess is correct by observing that each of those points is a distance of 2 from both $(2,0)$ and $(0,2)$, so that they lie on both circles.

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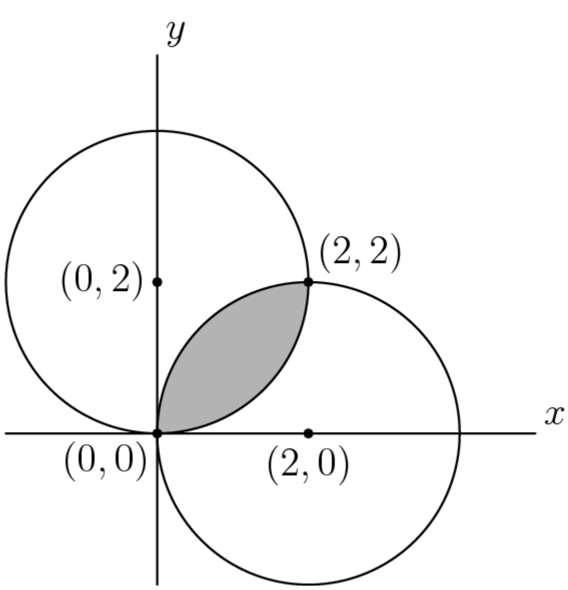


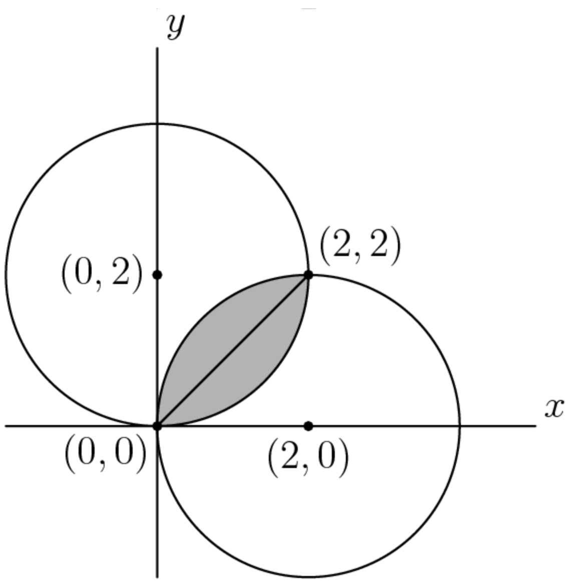
jonjoseph 2021-08-20 20:36:57

How can we find the area of the intersection?

jonjoseph 2021-08-20 20:37:49

We can split the intersection into two circular segments.

jonjoseph 2021-08-20 20:37:51



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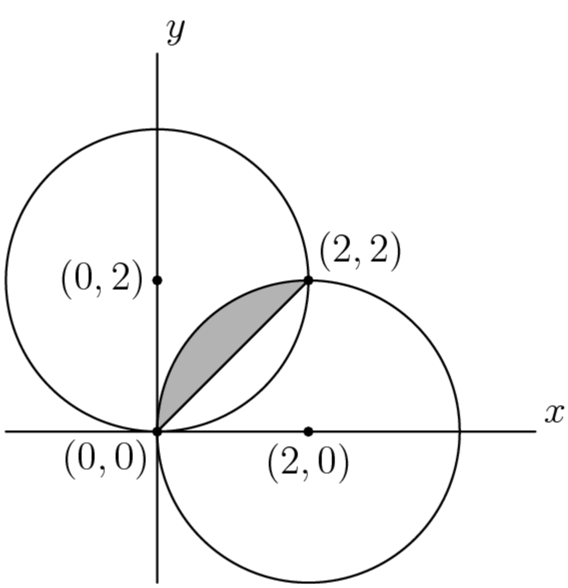
How can we find the area of one of these circular segments?

jonjoseph 2021-08-20 20:38:04



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We can think of one circular segment as the difference between a circular sector and a triangle.

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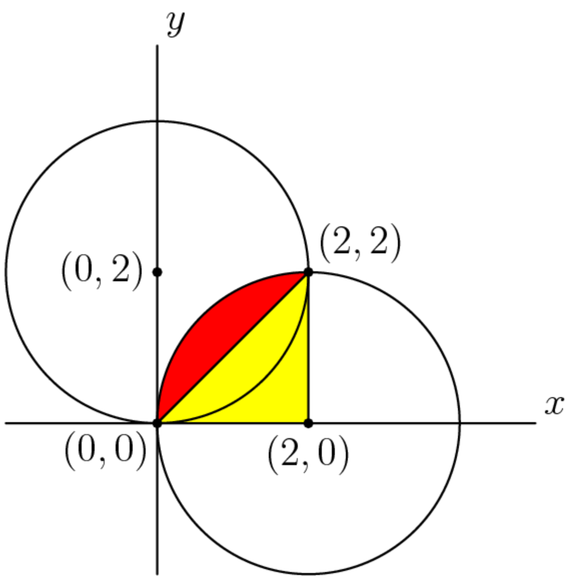


jonjoseph 2021-08-20 20:39:06
What is the area of the circular sector?

jonjoseph 2021-08-20 20:40:02
The area of the circle is $2\pi r = 2 \cdot 2\pi = 4\pi$, and the circular sector is $\frac{1}{4}$ of it, which is π.

jonjoseph 2021-08-20 20:40:08
What is the area of the triangle?

jonjoseph 2021-08-20 20:40:43
The area of the triangle is $\frac{1}{2} \cdot 2 \cdot 2 = 2$. So what is the area of the intersection of the two circles?

jonjoseph 2021-08-20 20:41:07
Hint: Careful. There are two of them.

jonjoseph 2021-08-20 20:41:51
The area of each circular segment is $\pi - 2$, so the area of the intersection is $2(\pi - 2)$. The answer is (D).

jonjoseph 2021-08-20 20:41:59
A square region $ABCD$ is externally tangent to the circle with equation $x^2 + y^2 = 1$ at the point $(0, 1)$ on the side CD. Vertices A and B are on the circle with equation $x^2 + y^2 = 4$. What is the side length of this square?

$\text{(A) } \frac{\sqrt{10}+5}{10} \quad \text{(B) } \frac{2\sqrt{5}}{5} \quad \text{(C) } \frac{2\sqrt{2}}{3} \quad \text{(D) } \frac{2\sqrt{19}-4}{5} \quad \text{(E) } \frac{9-\sqrt{17}}{5}$

jonjoseph 2021-08-20 20:42:26
Hmmm... Lots going on.

jonjoseph 2021-08-20 20:42:30
We can start by drawing a diagram.

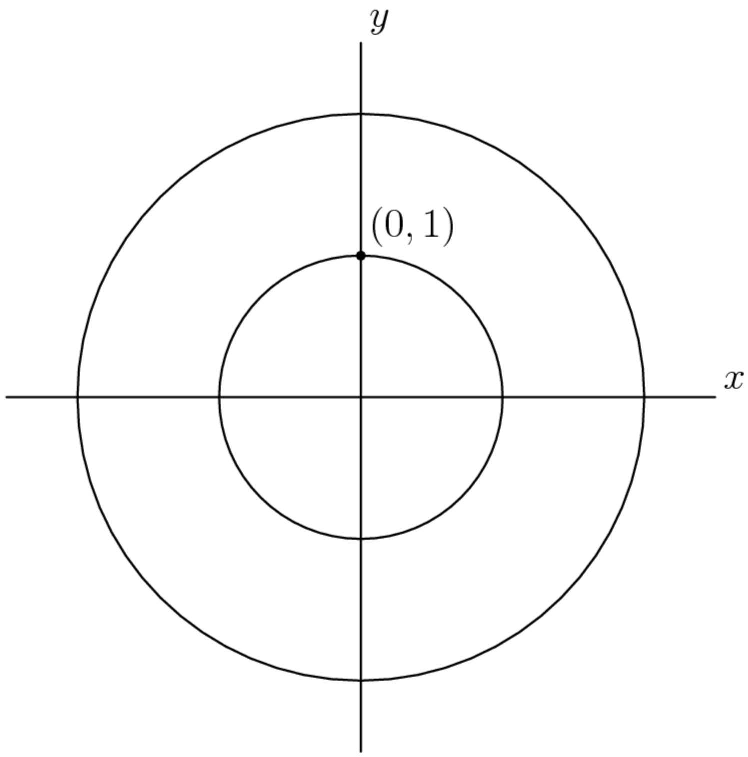
jonjoseph 2021-08-20 20:42:35
Let's draw the circles first. What are the center and radius of the circle $x^2 + y^2 = 1$?

jonjoseph 2021-08-20 20:43:07
This circle is centered at the origin, and its radius is 1. What about $x^2 + y^2 = 4$?

jonjoseph 2021-08-20 20:43:38

This circle is also centered at the origin, and its radius is 2. Here are the circles:

jonjoseph 2021-08-20 20:43:40



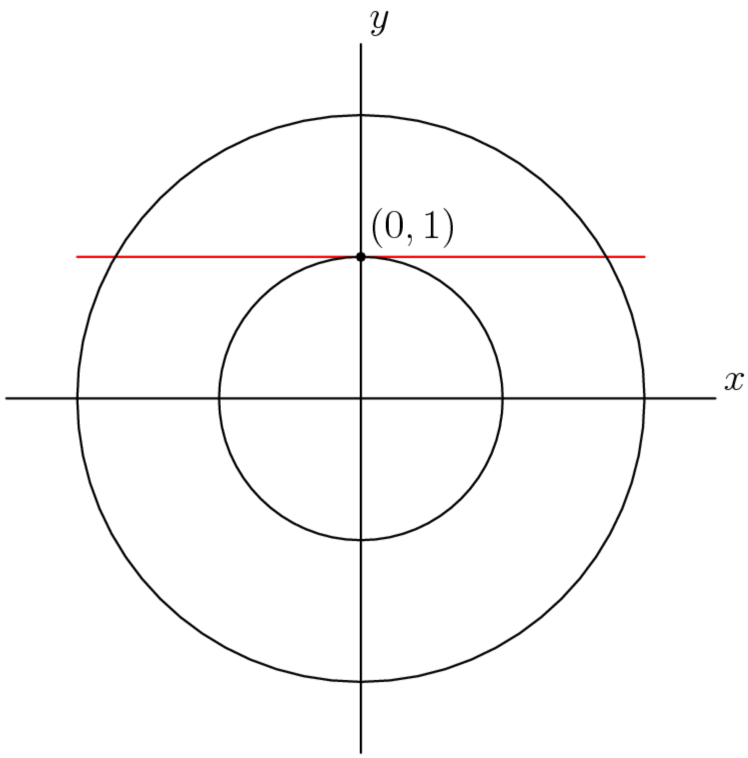
jonjoseph 2021-08-20 20:43:46

We now need to draw the square. We know that the square is externally tangent to the inner circle at $(0, 1)$ along side $\overline{CD}$, and that vertices A and B are on the outer circle.

jonjoseph 2021-08-20 20:44:01

We know C and D must be on the tangent line to the inner circle at $(0, 1)$. So C and D have to be somewhere on this red line:

jonjoseph 2021-08-20 20:44:03



jonjoseph 2021-08-20 20:44:13

Are A and B going to be above or below the red line?

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Hint: the square is *externally* tangent to the circle.

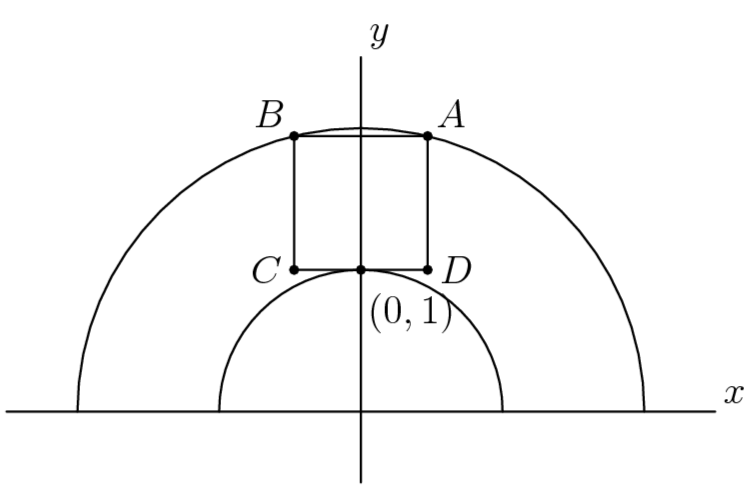
jonjoseph 2021-08-20 20:45:14

Yes, A and B must be above the red line. We're told the square is externally tangent to the inner circle, which means the square needs to be entirely outside the inner circle.

jonjoseph 2021-08-20 20:45:30

That enables us to finish the diagram. Let's just draw the top half of the circles, since that's all we need.

jonjoseph 2021-08-20 20:45:31



jonjoseph 2021-08-20 20:45:38

Clear?

jonjoseph 2021-08-20 20:46:04

We see that since $\overline{CD}$ is tangent to the inner circle at $(0, 1)$, it's perpendicular to the y-axis. That means it's parallel to the x-axis.

jonjoseph 2021-08-20 20:46:18

Can we figure out any of the coordinates of points A, B, C, D?

jonjoseph 2021-08-20 20:47:01

What must the y coordinates be for points C and D?

jonjoseph 2021-08-20 20:47:33

Yes, we know that the y-coordinates of C and D are 1, since $\overline{CD}$ passes through $(0, 1)$.

jonjoseph 2021-08-20 20:47:49

Do we immediately know the x-coordinates of C and D?

jonjoseph 2021-08-20 20:48:17

Good idea. Let's do that.

jonjoseph 2021-08-20 20:48:27

Let's say that $D = (a, 1)$. Do we now get an expression for C's coordinates?

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Hint: Think about the symmetry.

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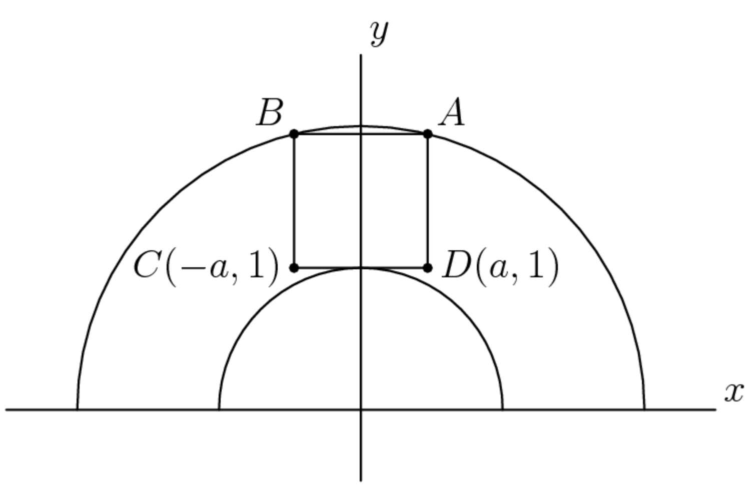
By symmetry, $C = (-a, 1)$. Let's label that in our diagram:

jonjoseph 2021-08-20 20:49:18



jonjoseph 2021-08-20 20:49:23

We can now get an expression for the side length of the square! What is it?

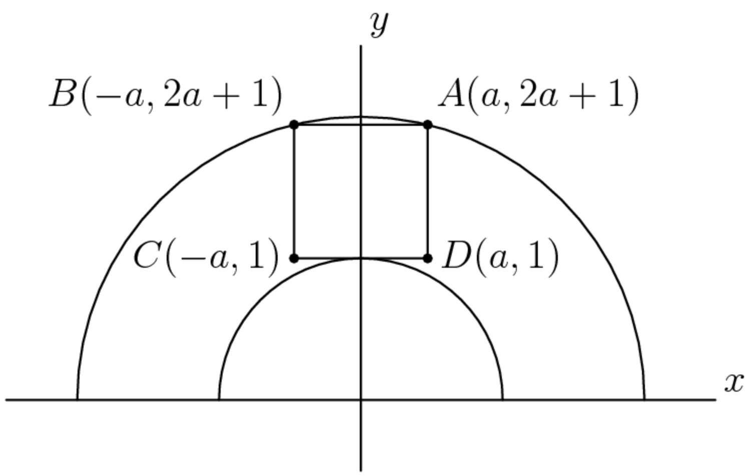
jonjoseph 2021-08-20 20:50:00

Since $CD = 2a$, the side length of the square is $2a$. Then what are the coordinates of A and B?

jonjoseph 2021-08-20 20:51:14

We have $A = (a, 2a + 1), B = (-a, 2a + 1)$:

jonjoseph 2021-08-20 20:51:18



jonjoseph 2021-08-20 20:51:25
Great, we've found coordinates for all our points. Now we just need to somehow solve for a, so we can find the side length of the square. How can we do that? What piece of information haven't we used yet?

jonjoseph 2021-08-20 20:52:07
We haven't used the fact that A and B are on the circle $x^2 + y^2 = 4$. So how do we figure out a?

jonjoseph 2021-08-20 20:53:12
We plug in the coordinates of A (or B, it doesn't matter!) into the equation. We get

$$a^2 + (2a+1)^2 = 4.$$

This simplifies to $5a^2 + 4a - 3 = 0$. How do we solve this?

jonjoseph 2021-08-20 20:54:08
We can use the quadratic formula. So what is a?

jonjoseph 2021-08-20 20:55:49
Yep. A bit nasty.

jonjoseph 2021-08-20 20:55:51
By the quadratic formula,

$$a = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5 \cdot (-3)}}{2 \cdot 5} = \frac{-4 \pm 2\sqrt{19}}{10} = \frac{-2 \pm \sqrt{19}}{5}.$$

Which of the two roots is the correct value of a?

jonjoseph 2021-08-20 20:56:06
What's our answer?

jonjoseph 2021-08-20 20:56:59
Since a is positive,

$$a = \frac{\sqrt{19} - 2}{5}.$$

jonjoseph 2021-08-20 20:57:03
The side length of the square is

$$2a = \frac{2\sqrt{19} - 4}{5}.$$

The answer is (D).

jonjoseph 2021-08-20 20:57:21
Okay. Last topic. We'll do one problem.

jonjoseph 2021-08-20 20:57:23
PARABOLAS

jonjoseph 2021-08-20 20:57:27
When we graph an equation of the form $y = ax^2 + bx + c$, where $a \neq 0$, b and c are constants, we obtain a parabola. For example, here is the graph of $y = x^2$.

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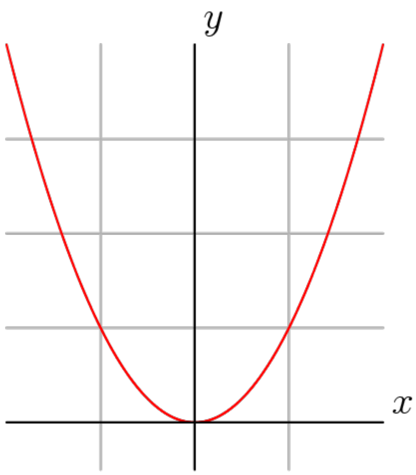
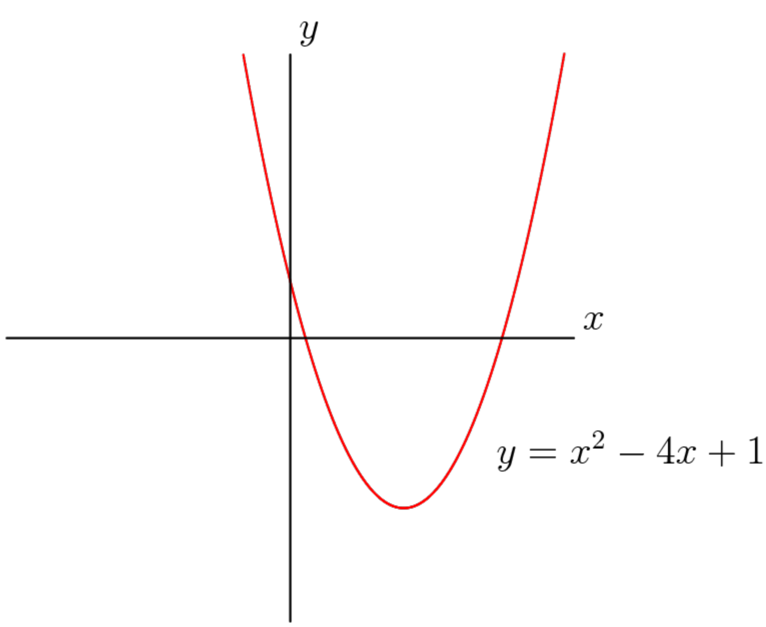


jonjoseph 2021-08-20 20:57:37
The lowest point or highest point of a parabola (depending on whether it faces upward or downward) is called the *vertex* of the parabola. We can use our knowledge of quadratic functions to find the vertex of a parabola.

jonjoseph 2021-08-20 20:57:45
For example, here is the graph of the parabola $y = x^2 - 4x + 1$.

jonjoseph 2021-08-20 20:57:49



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The lowest point on the parabola corresponds to the minimum value of $x^2 - 4x + 1$. How can we find the minimum value of $x^2 - 4x + 1$?

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Good. How do we do that?

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We can complete the square. What binomial can we square to produce the terms x^2 and $-4x$?

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If we square $x - 2$, then we get $(x-2)^2 = x^2 - 4x + 4$, which gives us the terms x^2 and $-4x$.

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Hence, we can write $x^2 - 4x + 1 = (x-2)^2 - 4 + 1 = (x-2)^2 - 3$. What is the minimum value of this quadratic?

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The $(x-2)^2$ term will always be positive or zero, so the minimum value occurs when it is zero. Hence, the minimum value of $(x-2)^2 - 3$ is -3, which occurs at $x = 2$.

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Note the minimum value is a value. In this case -3. And at what point does this minimum occur?

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Therefore, the vertex of the parabola is $(2, -3)$.

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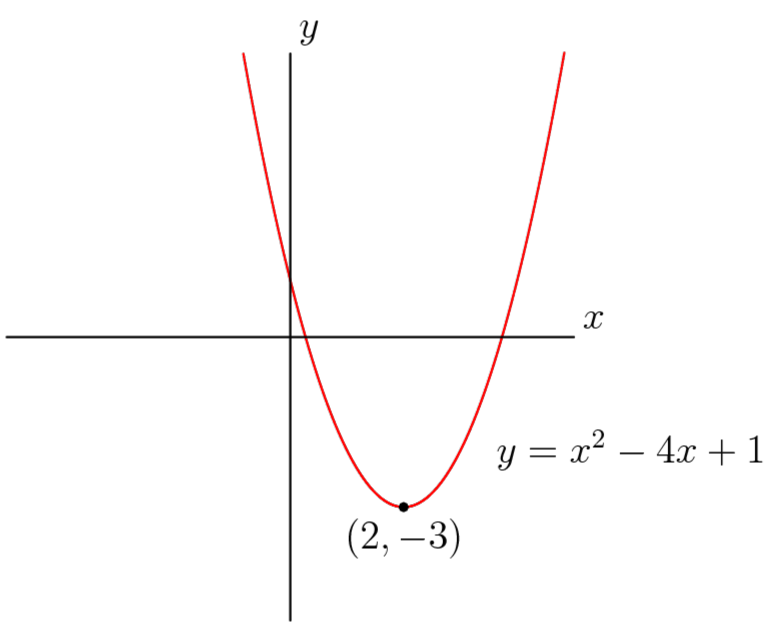

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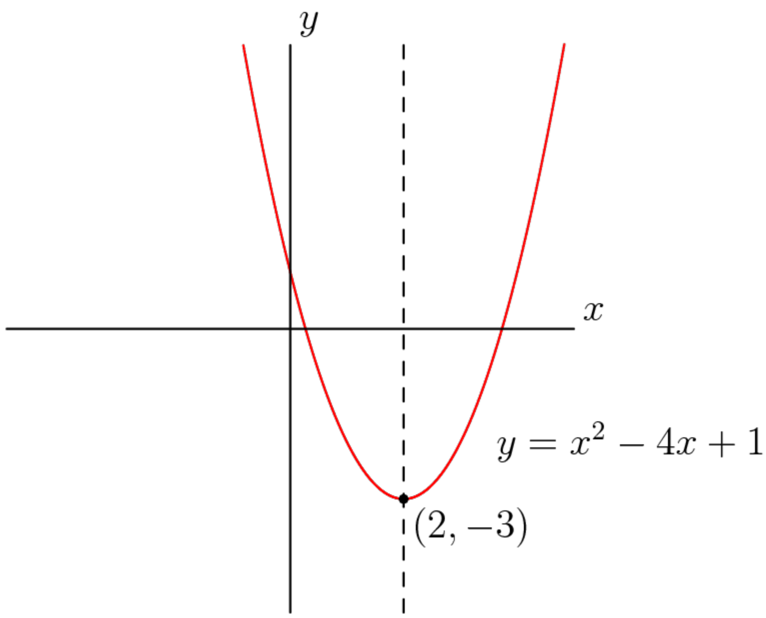
The parabola is symmetric around the vertical line passing through the vertex.

jonjoseph 2021-08-20 21:01:56



jonjoseph 2021-08-20 21:02:01
The vertical line is called the *axis of symmetry* of the parabola. Every parabola is symmetric around its axis of symmetry.

jonjoseph 2021-08-20 21:02:12
And that's a wrap!!

jonjoseph 2021-08-20 21:02:30
SUMMARY

jonjoseph 2021-08-20 21:02:32
In today's class, we looked at problems based on the coordinate plane. Many of these problems require a basic understanding of both algebra and geometry, so remember to use the principles that we have discussed when solving problems in these topics.

jonjoseph 2021-08-20 21:02:35
For example, on the geometry side, draw large diagrams and label what you know. On the algebra side, try to keep your equations as simple as possible.

jonjoseph 2021-08-20 21:02:51
****** Draw pictures!! ******

jonjoseph 2021-08-20 21:02:57
Also, please note that there is a Class Survey on the class home page. Please complete this survey; your responses help us to improve our classes.

jonjoseph 2021-08-20 21:03:34
Well done for the entire course and thanks to @ lizhufan for helping out tonight. Stay safe and I hope to see you in another AoPS course!!