Art Of Problem Solving - AMC 10
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Abstract

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1.

The number of real values of x that satisfy the equation

$$\left(2^{6x+3}\right)\left(4^{3x+6}\right) = 8^{4x+5}$$

is:

zero,	one,	two,	three,	greater than 3

We bring all exponentials to the same basis:

$$2^{6x+3} \cdot 2^{2(3x+6)} = 2^{3(4x+5)}$$

$$6x + 3 + 2(3x + 6) = 3(4x + 5)$$

$$12x + 15 = 12x + 15$$

which is always true. Thus, there are an infinity of solutions. And since $\infty > 3$

$$\boxed{\text{The number of real values of } x \text{ is greater than 3.}}$$

Note that we knew right away that the equation could be made linear by a change of the basis and that therefore the number of solutions had to be either 0, 1 or ∞, with 1 the "non-pathological" case. So if we had been in a hurry, a good bet would have been 1. Unfortunately, we would have lost the best in this case.

2.

For which of the following values of k does the equation

$$\frac{x-1}{x-2} = \frac{x-k}{x-6}$$

have no solution for x?

$\boxed{k = 1, \quad k = 2, \quad k = 3, \quad k = 4, \quad k = 5}$

We immediately note that the case $k = 1$ is special. Substituting $k = 1$ yields:

$$\frac{x-1}{x-2} = \frac{x-1}{x-6}$$

At this point it is tempting to cancel out the $x - 1$. But remember the question: when is there *no* solution? Thus if we find one solution for some k, we can move on to the next value of k. We do not need to find all the solutions! And since $x = 1$ is a solution of the above, we can rule out $k = 1$. There are still four more cases to consider.

We could go on substituting each possible value of k and looking for solutions. Let's do it for one more case, as an exercise. Let $k = 2$.

$$\begin{aligned}
\frac{x-1}{x-2} &= \frac{x-2}{x-6} \\
(x-1)(x-6) &= (x-2)^2 \\
x^2 - 7x + 6 &= x^2 - 4x + 2 \\
3x &= 4 \\
x &= 4/3
\end{aligned}$$

Thus, there is a valid solution for $k = 2$ (that solution is $x = 4/3$).

The other cases could be resolved with similar steps. But trying five cases will take a lot of work. So is there a quicker method? Let's be general:

$$\begin{aligned}
\frac{x-1}{x-2} &= \frac{x-k}{x-6} \\
(x-1)(x-6) &= (x-2)(x-k) \\
x^2 - 7x + 6 &= x^2 - (2+k)x + 2k \\
-7x + 6 &= -(2+k)x + 2k
\end{aligned}$$

Now note that if $7 = 2 + k$, the x term cancels out of the equation. And thus, there is no solution (the equation is impossible) if the following holds:

$$\begin{aligned}
&7 = 2 + k \\
&6 \neq 2k
\end{aligned}$$

which is true for $k = 5$.

$$\boxed{k = 5}$$

3.

How many ordered triples (a, b, c) of nonzero real numbers have the property that each number is the product of the other two?

1,	2,	3,	4,	5

We are looking for (a, b, c), where $a \neq 0$, $b \neq 0$, $c \neq 0$, such that:

$$a = bc$$
$$b = ac$$
$$c = ab$$

We notice that this implies:

$$abc = 1$$
$$\frac{abc}{bc} = \frac{1}{a}$$
$$a^2 = 1$$
$$a = \pm 1$$

By symmetry, we also have $b = \pm 1$, $c = \pm 1$. So the candidate unordered-triples are $(1, 1, 1)$, $(1, 1, -1)$, $(1, -1, -1)$, and $(-1, -1, -1)$. But clearly we need to be able to permute the symbols pairwise without violating the equality, which means only these are candidates: $(1, 1, 1)$ and $(1, -1, -1)$. That is triples. But the question asks for **ordered** triples. If the triple is ordered, $(+1, -1, -1)$, $(-1, +1, -1)$, and $(-1, -1, +1)$ are distinct. So altogether that gives 4 ordered triples

$$\boxed{\text{2 non-ordered triples, 4 ordered triples}}$$

4.

Two non-zero real numbers, a and b, satisfy $ab = a - b$. Which of the following is a possible value of $\dfrac{a}{b} + \dfrac{b}{a} - ab$?

$\boxed{-2, \quad -1/2, \quad 1/3, \quad 1/2, \quad 2}$

Addition/subtraction is typically simpler than multiplication, so we are tempted to replace every instance of ab with $a - b$ and hope that it will yield easy simplifications. We also see that if we bring the fractions to the same denominator, the product ab will appear:

$$\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab}$$

The appearance of $a^2 + b^2$ is not very encouraging. However since we know something about $a - b$, we are reminded that $(a - b)^2$ generates $a^2 + b^2$. So we take it from here:

$$\begin{aligned}(a-b)^2 &= a^2 + b^2 - 2ab \\ a^2 + b^2 &= (a-b)^2 + 2ab \\ a^2 + b^2 &= (ab)^2 + 2ab\end{aligned}$$

Substituting back in:

$$\begin{aligned}\frac{a}{b} + \frac{b}{a} &= \frac{a^2 + b^2}{ab} \\ &= \frac{(ab)^2 + 2ab}{ab} \\ &= ab + 2\end{aligned}$$

And thus

$$\boxed{\frac{a}{b} + \frac{b}{a} - ab = 2}$$

5.

If $a+1=b+2=c+3=d+4=a+b+c+d+5$, then $a+b+c+d$ is

$\boxed{-5, \quad -10/3, \quad -7/3, \quad 5/3, \quad 5}$

It pays to rewrite these for clarity:

$$\begin{aligned}
a+1 &= b+2 \\
b+2 &= c+3 \\
c+3 &= d+4 \\
d+4 &= a+b+c+d+5
\end{aligned}$$

This shows that we have 4 linear equations in 4 unknowns, which we expect to be able to solve. But can we get the sum faster than that? If we add up the equations or multiply them, we get a lot of cancellations, but the answer does not come out right away. Let's introduce new variables to see the pattern:

$$\begin{aligned}
s &= a+b+c+d \\
\alpha &= a+1 \\
\beta &= b+2 \\
\gamma &= c+3 \\
\delta &= d+4
\end{aligned}$$

We want to find $s=\delta-5$, where

$$\begin{aligned}
\alpha &= \beta \\
\beta &= \gamma \\
\gamma &= \delta \\
\delta &= \alpha+\beta+\gamma+\delta-5
\end{aligned}$$

where the last equation comes from:

$$\begin{array}{ccccccccccc}
\delta & = & a & + & b & + & c & + & d & + & 5 \\
 & = & (a+1) & + & (b+2) & + & (c+3) & + & (d+4) & + & 5-1-2-3-4 \\
 & = & \alpha & + & \beta & + & \gamma & + & \delta & - & 5
\end{array}$$

Now we have a reasonably simple way to get the answer. Since $\alpha=\beta=\gamma=\delta$,

$$\begin{aligned}
\delta &= \alpha+\beta+\gamma+\delta-5 \\
\delta &= \delta+\delta+\delta+\delta-5 \\
\delta &= 5/3
\end{aligned}$$

$$\Rightarrow \quad s = 5/3-5 = -10/3$$

$$\boxed{a+b+c+d = \frac{-10}{3}}$$

6.

Let a, b, c, and d be real numbers with $|a-b|=2$, $|b-c|=3$, and $|c-d|=4$. What is the sum of all possible values of $|a-d|$?

9,	12,	15,	18,	24

The problem is to solve for $x = |a-d|$, where:

$$\begin{aligned} |a-b| &= 2 \\ |b-c| &= 3 \\ |c-d| &= 4 \end{aligned}$$

We have several cases to consider:

$$\begin{cases} a-b=2 \\ a-b=-2 \end{cases}$$
$$\begin{cases} b-c=3 \\ b-c=-3 \end{cases}$$
$$\begin{cases} c-d=4 \\ c-d=-4 \end{cases}$$

We can get $a-d$ by adding the three cases. For instance:

$$\begin{aligned} a-b &= 2 \\ b-c &= 3 \\ c-d &= 4 \end{aligned}$$

$$a-d = 2+3+4 = 9$$

So now we repeat the pattern for all possible values of $a-d$:

$$\begin{aligned} a-d &= +2+3+4 = +9 \\ &= -2+3+4 = +5 \\ &= +2-3+4 = +3 \\ &= +2+3-4 = +1 \\ &= -2-3+4 = -1 \\ &= -2+3-4 = -3 \\ &= +2-3-4 = -5 \\ &= -2-3-4 = -9 \end{aligned}$$

But we are looking for $|a-d|$, so adding up the positive values gives:

$$|a-d| = 1+3+5+9 = 18$$

$$\boxed{|a-d| = 18}$$

7.

If x and y are nonzero numbers such that $x = 1 + \frac{1}{y}$ and $y = 1 + \frac{1}{x}$, then y equals

$\boxed{x - 1, \quad 1 - x, \quad 1 + x, \quad -x, \quad x}$

$$\begin{aligned} x &= 1 + \frac{1}{y} \\ y &= 1 + \frac{1}{x} \end{aligned}$$

The symmetry of the problem is obvious and indeed:

$$\begin{aligned} xy &= y + 1 \\ yx &= x + 1 \\ \Rightarrow \quad x &= y \qquad \square \end{aligned}$$

$$\boxed{y = x}$$

Out of curiosity, let's solve the quadratic equation:

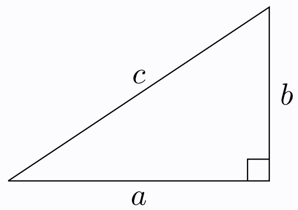
$$\begin{aligned} x &= 1 + \frac{1}{x} \\ x^2 - x - 1 &= 0 \\ \left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 1 &= 0 \\ \left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 &= 0 \\ (x - \frac{1+\sqrt{5}}{2})(x - \frac{1-\sqrt{5}}{2}) &= 0 \\ y &= \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2} \end{aligned}$$

8.

A right triangle has perimeter 32 and area 20. What is the length of its hypotenuse?

$$\boxed{\frac{57}{4},\quad \frac{59}{4},\quad \frac{61}{4},\quad \frac{63}{4},\quad \frac{65}{4}}$$

Let a, b denote the lengths of the legs and c the length of the hypotenuse.



We know these facts:

$$\begin{aligned} \text{area:} \quad \frac{ab}{2} &= 20 \\ \text{perimeter:} \quad a+b+c &= 32 \end{aligned}$$

We want to solve for c, which is related to a and b by the Pythagorean triangle identity:

$$c = \sqrt{a^2 + b^2}$$

To get an expression involving a^2 and b^2 , we square the perimeter formula:

$$\begin{aligned} a + b &= 32 - c \\ (a+b)^2 &= (32-c)^2 \\ a^2 + b^2 + 2ab &= 32^2 - 64c + c^2 \\ c^2 + 80 &= 1024 - 64c + c^2 \\ 64c &= 944 \\ c &= \frac{944}{64} = \frac{59}{4} \end{aligned}$$

$$\boxed{\text{length of hypotenuse: } \frac{59}{4}}$$

9.

Let a, b, c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

0,	1,	4,	7,	8

We have a linear system of two equations in three unknowns a, b, c:

$$\begin{aligned} a - 7b + 8c &= 4 \\ 8a + 4b - c &= 7 \end{aligned}$$

We notice a $1, 4, 7, 8$ pattern. The key insight is that the first equation has $(1a, 8c)$ and the second $(8a, -1c)$, while b appears with different coefficients. This suggests squaring both equations after a clever rearrangement:

$$\begin{aligned} (a + 8c)^2 &= (4 + 7b)^2 \\ (8a - c)^2 &= (7 - 4b)^2 \end{aligned}$$

Distributing:

$$\begin{aligned} a^2 + 16ac + 64c^2 &= 16 + 56b + 49b^2 \\ 64a^2 - 16ac + c^2 &= 49 - 56b + 16b^2 \end{aligned}$$

Adding up:

$$\begin{aligned} & 65a^2 + 65c^2 = 65 + 65b^2 \\ \Rightarrow \quad & a^2 - b^2 + c^2 = 1 \end{aligned}$$

$$\boxed{a^2 - b^2 + c^2 = 1}$$

10.

Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

164,	172,	192,	194,	212

Let's expand the binomial. The coefficients of Pascal's triangle are 1, 4, 6, 4, 1.

$$\begin{aligned}\left(a + a^{-1}\right)^4 &= a^4 \cdot \left(a^{-1}\right)^0 + 4a^3 \cdot \left(a^{-1}\right)^1 + 6a^2 \cdot \left(a^{-1}\right)^2 + 4a^1 \cdot \left(a^{-1}\right)^3 + a^0 \cdot \left(a^{-1}\right)^4 \\ &= a^4 + 4a^2 + 6 + 4a^{-2} + a^{-4} \\ &= a^4 + a^{-4} + 6 + 4(a^2 + a^{-2})\end{aligned}$$

Rearranging and setting $(a + a^{-1})^4 = 4^4$,

$$a^4 + a^{-4} = 4^4 - 6 - 4(a^2 + a^{-2})$$

Expanding the binomial again:

$$\begin{aligned}\left(a + a^{-1}\right)^2 &= a^2 \cdot \left(a^{-1}\right)^0 + 2a^1 \cdot \left(a^{-1}\right)^1 + a^0 \cdot \left(a^{-1}\right)^2 \\ &= a^2 + 2 + a^{-2}\end{aligned}$$

Rearranging and setting $(a + a^{-1})^2 = 4^2$,

$$a^2 + a^{-2} = 4^2 - 2 = 14$$

Plugging this back into the previous expression:

$$\begin{aligned}a^4 + a^{-4} &= 4^4 - 6 - 4 \cdot 14 \\ &= 194\end{aligned}$$

$$\boxed{a^4 + a^{-4} = 194}$$