MathCounts 2021
Art of Problem Solving
Mock National Competition

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Abstract

Answers collected in 2021 from https://artofproblemsolving.com/community/.

2021-Sprint-01

What is the value of $\frac{2021}{20}$, rounded to the nearest whole number?

2021-Sprint-02

Grizz can type 2500 characters per minute, and Kre can type 1000 characters per minute. After 1 hour of typing, how many more characters does Grizz type than Kre?

2021-Sprint-03

Define the operation $a\%b$ to be $ab + a + b$. What is the value of $(3\%3) + (5\%5)$.

2021-Sprint-04

All of the birds in a tree are sparrows or chickadees. If there were three more sparrows, the number of sparrows would be twice the number of chickadees. If there were six more chickadees, the number of chickadees would be twice the number of sparrows. How many birds are in the tree?

2021-Sprint-05

Pengu writes the number 8 on a chalkboard. Every minute, Pengu either subtracts 1 from his current number or divides his current number by 2, each with equal probability. What is the probability that the number on the chalkboard will be 1 after exactly four minutes? Express your answer as a common fraction.

2021-Sprint-06

A 2021-digit positive integer is chosen at random. What is the probability that it begins with the digits 2, 0, 2, and 1, not necessarily in that order? Express your answer as a common fraction.

2021-Sprint-07

In a set of five positive integers, the median is twice the mode and the mean is twice the median. Given that the mode is unique, what is the minimum possible range?

2021-Sprint-08

What is the sum of all positive integers x less than 100 such that the base 2 and base 3 representations of x end in 0 and 1, respectively?

2021-Sprint-09

For a positive integer x, let $f(x)$ denote the absolute difference between x and the closest perfect square to x. For how many positive integers n less than 1000 is $f(n) \leq 2$?

2021-Sprint-10

An increasing arithmetic sequence of 10 positive integers sums to 145 times the first term. What is the ratio of the ninth term to the second term? Express your answer as a common fraction.

2021-Sprint-11

In Mathland's Marching band, Jafko finds himself in the the middle of a rectangular array of students. He notices that in the array, 315 students are in a different row than him, and 308 are in a different column than him. Given this, how many students are in the formation?

2021-Sprint-12

What is the smallest positive integer x for which there exists an integer a such that

$$97 + \sqrt{98 + \sqrt{99 + \sqrt{a + x}}} = x?$$

2021-Sprint-13

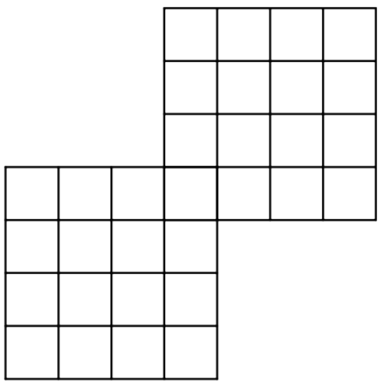
For how many positive integers x less than 30 is $x^2 + 5x + 6$ is divisible by 30?

2021-Sprint-14

John is trapped in a 2×2 grid of cells. Each cell in this grid has a 1 way portal which leads to a randomly chosen fixed different cell on the grid. If John begins in the lower left cell and enters the portal there, what is the probability that he can eventually get back to the lower left cell? Express you answer as a common fraction.

2021-Sprint-15

How many rectangles are formed by the line segments below?



2021-Sprint-16

A farmer tethers his horse to a fence of length 12 meters with a leash of the length of 18 meters, on flat ground. This leash is connected to the midpoint of one side of a fence. Given that the area of the land that the horse can walk on can be expressed as $a\pi + b\sqrt{c}$ where a, b, and c are positive integers such that c is not divisible by the square of any prime, what is the value of $a + b + c$? Assume that the horse cannot jump over the fence, and that the fence and the leash are the only objects that obstruct the horse's movement.

2021-Sprint-17

For how many positive integers n is the sum of the digits of n is equal to $\lfloor \frac{n}{10} \rfloor$? Note: $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

2021-Sprint-18

For positive integers n, let $\varphi(n)$ be the number of positive integers less than or equal to n which are relatively prime to n. For how many ordered pairs (a, b) of positive integers not exceeding 20 is $\varphi(a) + \varphi(b)$ is odd?

2021-Sprint-19

Compute $ab + bc + ca$, given that a, b, c are positive real numbers satisfying

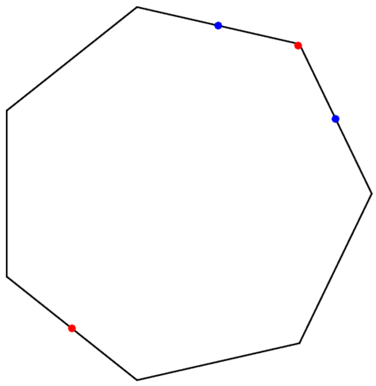
$$a + \frac{14}{b} = b + \frac{36}{c} = c + \frac{153}{a} = \frac{720}{a + b + c}$$

2021-Sprint-20

In an infinite sequence of positive integers, let $a_1 = 1$, and let a_{n+1} be the sum of a_n and the largest odd factor of a_n for all $n \geq 2$. What is the sum of the reciprocals of the terms in this sequence? Express your answer as a common fraction.

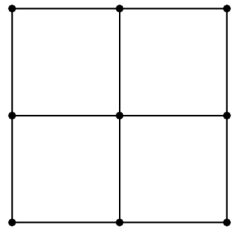
2021-Sprint-21

Jeremy places two red flags, one on the midpoint of a side and one on a vertex, and two blue flags, on the midpoints of two sides, of a regular heptagon, as shown. Then, he selects a point at random in the interior of the heptagon. What is probability that that point is closer to one of the red flags than either of the blue flags? Express your answer as a common fraction.



2021-Sprint-22

How many distinct trapezoids have all four vertices in the 3 by 3 grid of points shown below? (Here we define a trapezoid to be a convex quadrilateral with at least one pair of parallel sides).



2021-Sprint-23

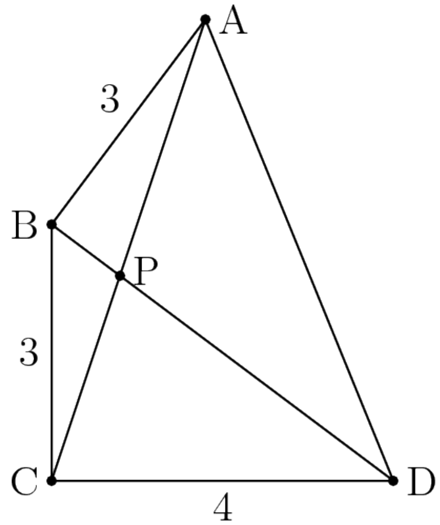
Let $ABCD$ be a rectangle such that $AB = 75$ and $BC = 100$. Let E be a point such that $AEDC$ is a convex isosceles trapezoid. What is the area of pentagon $ABCDE$?

2021-Sprint-24

Niugnep flips a fair coin 5 times. Given that he flipped at least one pair of consecutive heads, what is the probability that he flipped at least one pair of consecutive tails? Express your answer as a common fraction.

2021-Sprint-25

Quadrilateral $ABCD$ has side lengths $AB = 3$, $BC = 3$, and $CD = 4$, as shown. Furthermore, $\angle ABD = \angle BCD = 90°$. If P is the intersection of diagonals AC and BD, what is the length of AP? Express your answer in simplest radical form.

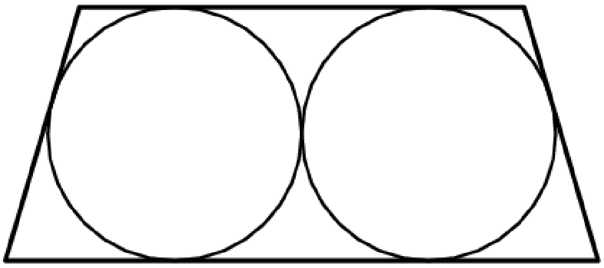


2021-Sprint-26

A positive integer n is selected at random from the first 17000 positive integers. What is the probability that $\frac{n+n^2+...+n^{15}+n^{16}}{1+2+3+\cdots+16}$ is an integer? Express your answer as a common fraction.

2021-Sprint-27

Two congruent, tangent circles are inscribed in an isosceles trapezoid with bases of length 5 and 8, as shown. What is the radius of these circles? Express your answer as a common fraction.



2021-Sprint-28

Every day, three penguins sit in a circle and are each either happy or sad for that day in a way such that the following are true:

1. If both of the neighbors of a penguin are happy, the penguin will be happy $\frac{2}{3}$ of the time.
2. If exactly 1 of the neighbors of a penguin are happy, the penguin will be happy $\frac{1}{2}$ of the time.
3. If none of the neighbors of a penguin are happy, the penguin will be happy $\frac{1}{3}$ of the time.

Given this, what is the probability that, on any given day, all three penguins are happy? Express your answer as a common fraction.

2021-Sprint-29

What is the smallest positive integer m so that $2^{2^{2^{2^2}}} - 2^{2^{2^2}}$ is NOT divisible by m? Note: The notation a^{b^c} means $a^{(b^c)}$.

2021-Sprint-30

Charlie chooses 10 different positive integers between 1 and 20, inclusive. Henry chooses 12 different positive integers, all different from Charlie's, between 1 an 24, inclusive. If no two of Charlie's numbers sum to 21 and no two of Henry's numbers sum to 25, in how many ways can Charlie and Henry choose their numbers? Note: The order in which they choose numbers does not matter, only the sets of numbers they each choose.

2021-Target-1

A special rock lies in the ocean with weight 20 pounds. Every minute, the rock loses 25 percent of its weight, and then gains 20 pounds. After 5 years, the weight of the rock is closest to what integer?

2021-Target-2

Jerry thinks of a finite arithmetic sequence of integers with first term 1 and last term 19. Then, Laura computes the sum of all of the numbers in Jerry's sequence. What is the sum of all possible numbers Laura could obtain?

2021-Target-3

How many ordered triples (a, b, c) of (not necessarily distinct) positive integers not exceeding 10 satisfy $\frac{a}{b} \times \frac{a}{c} = \frac{a}{b} - \frac{a}{c}$?

2021-Target-4

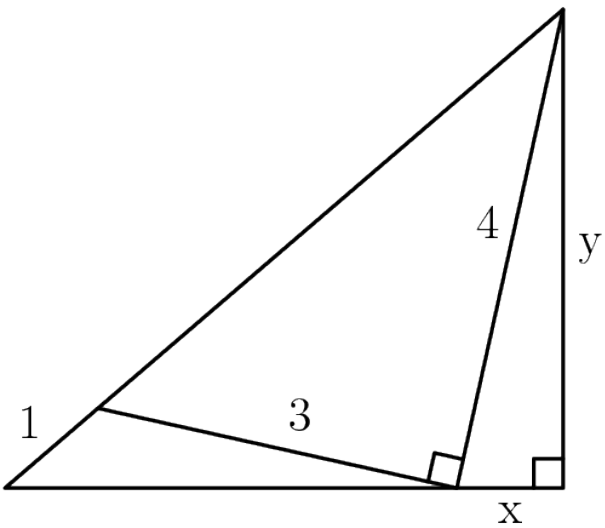
What is the sum of the three smallest positive, odd integers $n > 1$ for which there exists a non-negative integer k such that $2^k n$ has exactly n divisors?

2021-Target-5

An integer a is selected between 3 and 5, inclusive, and an integer b is selected between 30 and 50, inclusive. How many distinct possible values are there for the product ab?

2021-Target-6

n the figure shown, a right triangle with leg lengths 3 and 4 is inscribed in a larger right triangle. What is the value of $\frac{x}{y}$? Express your answer as a common fraction.



2021-Target-7

What is the sum of all positive integers n less than 50 such that the sum of the digits of the base 6 and base 9 representations of n are the same?

2021-Target-8

Four basketball teams with distinct fixed skill levels participate in a special tournament. The manager of the tournament knows that whenever two teams play, the team with the higher skill level always wins, but he has no prior knowledge of the skill levels of the four teams. Every day, the manager randomly chooses two teams to play each other and records the winner and loser. What is the expected number of days until the manager can determine with certainty the order of the four teams' skill levels? Note: Two teams might play each other more than once.