2021 AMC 10A Problems/Problem 24

Contents

Problem

The interior of a quadrilateral is bounded by the graphs of $(x+ay)^2=4a^2$ and $(ax-y)^2=a^2$, where a is a positive real number. What is the area of this region in terms of a, valid for all $a>0$?

$$\textbf{(A)} ~\frac{8a^2}{(a+1)^2}\qquad\textbf{(B)} ~\frac{4a}{a+1}\qquad\textbf{(C)} ~\frac{8a}{a+1}\qquad\textbf{(D)} ~\frac{8a^2}{a^2+1}\qquad\textbf{(E)} ~\frac{8a}{a^2+1}$$

Diagram

Graph in Desmos: https://www.desmos.com/calculator/satawguqsc

~MRENTHUSIASM

Solution 1 (Generalized Value of a)

The cases for $(x+ay)^2=4a^2$ are $x+ay=\pm2a,$ or two parallel lines. We rearrange each case and construct the table below:

Case	Line's Equation	x-intercept	y-intercept	Slope
1	$x+ay-2a=0$	$2a$	2	$-\frac{1}{a}$
2	$x+ay+2a=0$	$-2a$	-2	$-\frac{1}{a}$

The cases for $(ax-y)^2=a^2$ are $ax-y=\pm a,$ or two parallel lines. We rearrange each case and construct the table below:

Case	Line's Equation	x-intercept	y-intercept	Slope
$1*$	$ax-y-a=0$	1	$-a$	a

$2*$	$ax-y+a=0$	-1	a	a

Since the slopes of intersecting lines $(1)\cap(1*)$, $(1)\cap(2*)$, $(2)\cap(1*)$, and $(2)\cap(2*)$ are negative reciprocals, we get four right angles, from which the quadrilateral is a rectangle.

Two solutions follow from here:

Solution 1.1 (Distance Between Parallel Lines)

Recall that for constants A, B, C_1 and C_2, the distance d between parallel lines $\begin{cases} Ax+By+C_1=0 \\ Ax+By+C_2=0 \end{cases}$ is

$$d=\frac{|C_2-C_1|}{\sqrt{A^2+B^2}}.$$

From this formula:

- The distance between lines (1) and (2) is $\dfrac{4a}{\sqrt{1+a^2}}$, the length of this rectangle.
- The distance between lines $(1*)$ and $(2*)$ is $\dfrac{2a}{\sqrt{a^2+1}}$, the width of this rectangle.

The area we seek is

$$\frac{4a}{\sqrt{1+a^2}}\cdot\frac{2a}{\sqrt{a^2+1}}=\boxed{\textbf{(D)}\ \frac{8a^2}{a^2+1}}.$$

~MRENTHUSIASM

Solution 1.2 (Distance Between Points)

The solutions to systems of equations $(1)\cap(1*)$, $(1)\cap(2*)$, $(2)\cap(2*)$, $(2)\cap(1*)$ are

$$(x,y)=\left(\frac{a(a+2)}{a^2+1},\frac{a(2a-1)}{a^2+1}\right),\left(-\frac{a(a-2)}{a^2+1},\frac{a(2a+1)}{a^2+1}\right),\left(-\frac{a(a+2)}{a^2+1},-\frac{a(2a-1)}{a^2+1}\right),\left(\frac{a(a-2)}{a^2+1},-\frac{a(2a+1)}{a^2+1}\right),$$

respectively, which are the consecutive vertices of this rectangle.

By the Distance Formula, the length and width of this rectangle are $\dfrac{4a\sqrt{a^2+1}}{a^2+1}$ and $\dfrac{2a\sqrt{a^2+1}}{a^2+1}$, respectively.

The area we seek is

$$\frac{4a\sqrt{a^2+1}}{a^2+1}\cdot\frac{2a\sqrt{a^2+1}}{a^2+1}=\boxed{\textbf{(D)}\ \frac{8a^2}{a^2+1}}.$$

~MRENTHUSIASM

Solution 2 (Specified Value of a)

In this solution, we will refer to equations (1), (2), $(1*)$, and $(2*)$ from Solution 1.

Substituting $a=2$ into the answer choices gives

$$\textbf{(A)}\ \frac{32}{9}\qquad\textbf{(B)}\ \frac{8}{3}\qquad\textbf{(C)}\ \frac{16}{3}\qquad\textbf{(D)}\ \frac{32}{5}\qquad\textbf{(E)}\ \frac{16}{5}$$

At $a=2,$ the solutions to systems of equations $(1)\cap(1*), (1)\cap(2*), (2)\cap(2*), (2)\cap(1*)$ are

$$(x,y)=\left(\frac{8}{5},\frac{6}{5}\right), (0,2), \left(-\frac{8}{5},-\frac{6}{5}\right), (0,-2),$$

respectively, which are the consecutive vertices of the quadrilateral.

Two solutions follow from here:

Solution 2.1 (Area of a Rectangle)

From the tables in Solution 1, we conclude that the quadrilateral is a rectangle.

By the Distance Formula, the length and width of this rectangle are $\frac{8\sqrt5}{5}$ and $\frac{4\sqrt5}{5},$ respectively.

The area we seek is

$$\frac{8\sqrt5}{5}\cdot\frac{4\sqrt5}{5}=\frac{32}{5},$$

from which the answer is $\boxed{\textbf{(D)}\ \frac{8a^2}{a^2+1}}.$

~MRENTHUSIASM

Solution 2.2 (Area of a General Quadrilateral)

Even if we do not recognize that the quadrilateral is a rectangle, we can apply the Shoelace Theorem to its ***consecutive*** vertices

$$\begin{aligned}
(x_1,y_1)&=\left(\frac{8}{5},\frac{6}{5}\right), \\
(x_2,y_2)&=(0,2), \\
(x_3,y_3)&=\left(-\frac{8}{5},-\frac{6}{5}\right), \\
(x_4,y_4)&=(0,-2).
\end{aligned}$$

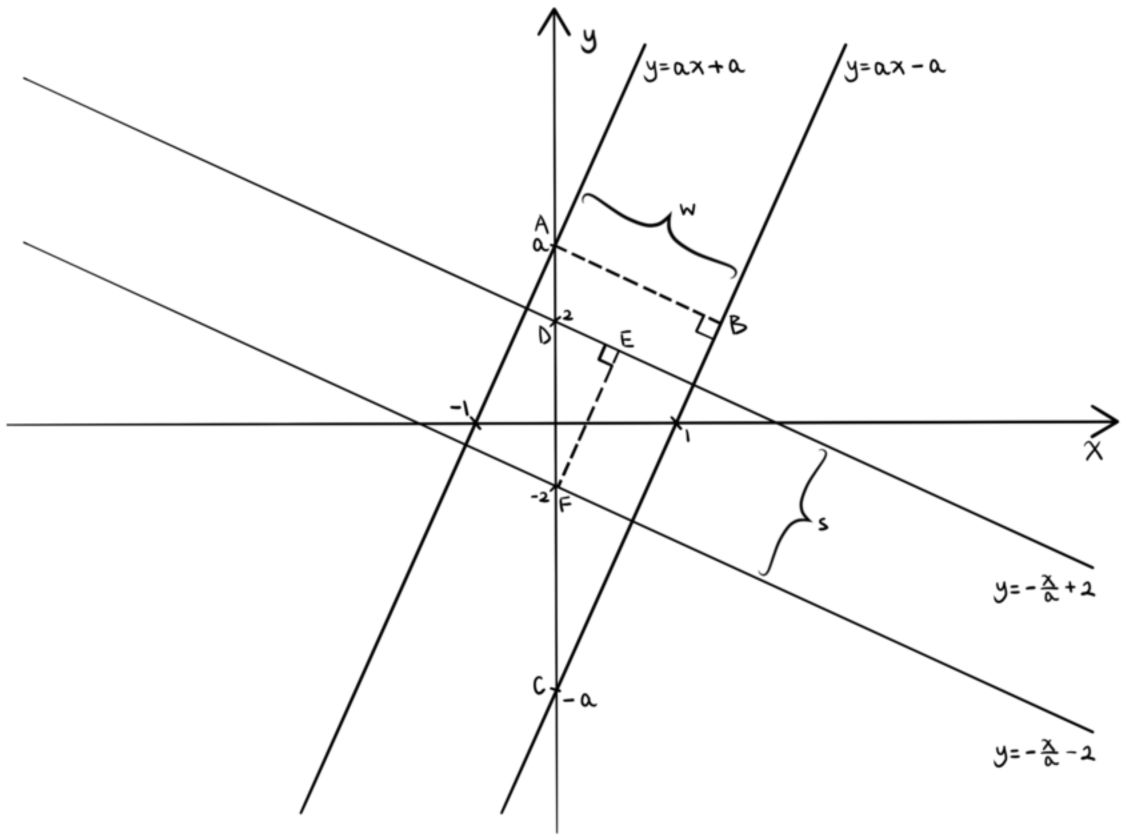
The area we seek is

$$\frac{1}{2}\left|(x_1y_2+x_2y_3+x_3y_4+x_4y_1)-(y_1x_2+y_2x_3+y_3x_4+y_4x_1)\right|=\frac{32}{5}.$$

from which the answer is $\boxed{\textbf{(D)}\ \frac{8a^2}{a^2+1}}.$

~MRENTHUSIASM

Solution 3 (Slopes and Intercepts)



The quadrilateral is enclosed by four lines. Similar to Solution 1, we will use the equations from the four cases:

1. $x + ay = 2a.$ This is a line with x-intercept $2a$, y-intercept 2, and slope $-\frac{1}{a}$.
2. $x + ay = -2a.$ This is a line with x-intercept $-2a$, y-intercept -2, and slope $-\frac{1}{a}$.
3. $ax - y = a.$ This is a line with x-intercept 1, y-intercept $-a$, and slope a.
4. $ax - y = -a.$ This is a line with x-intercept -1, y-intercept a, and slope a.

It follows that $DF = 4$ and $DE = \sqrt{4^2 - s^2}$.

Because the slope of line $y = -\frac{x}{a} + 2$ is $-\frac{1}{a}$, $\frac{1}{a} = \frac{DE}{EF} = \frac{\sqrt{16 - s^2}}{s}$, $s^2(a^2 + 1) = 16a^2$, $s = \frac{4a}{\sqrt{a^2 + 1}}$.

It follows that $AC = 2a$ and $BC = \sqrt{(2a)^2 - w^2}$.

Because the slope of line $y = ax - a$ is a, $a = \frac{BC}{AB} = \frac{\sqrt{4a^2 - w^2}}{w}$, $w^2(a^2 + 1) = 4a^2$, $w = \frac{2a}{\sqrt{a^2 + 1}}$.

Therefore, the answer is

$$\text{Area} = s \cdot w = \frac{4a}{\sqrt{a^2 + 1}} \cdot \frac{2a}{\sqrt{a^2 + 1}} = \boxed{\textbf{(D)}\ \frac{8a^2}{a^2 + 1}}.$$

~isabelchen (https://artofproblemsolving.com/wiki/index.php/User:Isabelchen)

Solution 4 (Trigonometry)

Similar to Solution 1, we will use the equations from the four cases:

1. $x+ay=2a.$ This is a line with x-intercept $2a,$ y-intercept $2,$ and slope $-\frac{1}{a}.$
2. $x+ay=-2a.$ This is a line with x-intercept $-2a,$ y-intercept $-2,$ and slope $-\frac{1}{a}.$
3. $ax-y=a.$ This is a line with x-intercept $1,$ y-intercept $-a,$ and slope $a.$
4. $ax-y=-a.$ This is a line with x-intercept $-1,$ y-intercept $a,$ and slope $a.$

Let $\tan A=a.$ The area of the rectangle created by the four equations can be written as

$$\begin{aligned} 2a\cdot\cos A\cdot 4\sin A &= 8a\cos A\cdot\sin A \\ &= 8a\cdot\frac{1}{\sqrt{a^2+1}}\cdot\frac{a}{\sqrt{a^2+1}} \\ &= \boxed{\textbf{(D)}\ \frac{8a^2}{a^2+1}}. \end{aligned}$$

~fnothing4994 (Solution)

~MRENTHUSIASM (Code Adjustments)

Solution 5 (Observations)

The conditions $(x+ay)^2=4a^2$ and $(ax-y)^2=a^2$ give $|x+ay|=|2a|$ and $|ax-y|=|a|$ or $x+ay=\pm 2a$ and $ax-y=\pm a$. The slopes here are perpendicular, so the quadrilateral is a rectangle. Plug in $a=1$ and graph it. We quickly see that the area is $2\sqrt{2}\cdot\sqrt{2}=4$, so the answer can't be $\textbf{(A)}$ or $\textbf{(B)}$ by testing the values they give (test it!). Now plug in $a=2$. We see using a ruler that the sides of the rectangle are about $\frac{7}{4}$ and $\frac{7}{2}$. So the area is about $\frac{49}{8}=6.125$. Testing $\textbf{(C)}$, we get $\frac{16}{3}$ which is clearly less than 6, so it is out. Testing $\textbf{(D)}$, we get $\frac{32}{5}$ which is near our answer, so we leave it. Testing $\textbf{(E)}$, we get $\frac{16}{5}$, way less than 6, so it is out. So, the only plausible answer is $\boxed{\textbf{(D)}\ \frac{8a^2}{a^2+1}}$.

~firebolt360

Solution 6 (Observations)

Trying $a=1$ narrows down the choices to options $\textbf{(C)}$, $\textbf{(D)}$ and $\textbf{(E)}$. Trying $a=2$ and $a=3$ eliminates $\textbf{(C)}$ and $\textbf{(E)}$, to obtain $\boxed{\textbf{(D)}\ \frac{8a^2}{a^2+1}}$ as our answer. Refer to Solution 2 for a detailed explanation.

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Solution 7 (Observations: Cheap)

Note that $a=2$ yields different values for all answer choices. If we put in $a=2,$ we find that the area of the quadrilateral is

$\frac{32}{5}$. This means that the answer must be $\boxed{\textbf{(D)}\ \frac{8a^2}{a^2+1}}$. Refer to Solution 2 for a detailed explanation.

Video Solution by OmegaLearn (System of Equations and Shoelace Formula)

https://youtu.be/2iohPYkZpkQ

~ pi_is_3.14

Video Solution by MRENTHUSIASM (English & Chinese)

https://www.youtube.com/watch?v=oEY-kX4d87M

~MRENTHUSIASM

See also

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