Rationalization & Irrationality

Typeset by Patrick Toche

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Reductio ad Absurdum

If $\sqrt{2}$ were rational, there would exist a pair $(a, b) \in \mathbb{N} \times \mathbb{N}^*$ such that:

$$\sqrt{2} = \frac{a}{b}$$

A proof by contradiction — *reductio ad absurdum* — establishes deductions that yield to a contradiction with the assumptions. Examples: Deduce that the fraction is even or greater than 2 or smaller than 1 or that a and b have opposite signs, etc..

Contradicting Parity

Let $\nu_2(n)$ denote the 2-order valuation operator on integer n, the largest natural number ν such that 2ν divides n. Square the equality, rearrange the terms, and apply the 2-order operator:

$$\begin{aligned} 2b^2 &= a^2 \\ \nu_2(2b^2) &= \nu_2(a^2) \\ \nu_2(b^2) + 1 &= \nu_2(a^2) \\ 2\nu_2(b) + 1 &= 2\nu_2(a) \\ \nu_2(a) - \nu_2(b) &= \frac{1}{2} \end{aligned}$$

where we have used $\nu_2(2n) = \nu_2(n) + 1$ and $\nu_2(n^2) = 2\nu_2(n)$.

Since the 2-order operator maps integers to integers, the difference cannot be equal to the rational $\frac{1}{2}$. By contradiction, $\sqrt{2}$ is not rational.

Contradicting Reducibility

A variant of the proof is to restrict attention to cases where a and b have no common factors and a/b is therefore an irreducible fraction. This is without loss of generality since any fraction can be simplified to an irreducible form by division of common factors. From $2b^2 = a^2$, it follows that a^2 is even. Since the square of odd integers is always odd, it follows that a cannot be odd and must therefore be even. Equivalently, a^2 must be a multiple of 4 ("doubly even"). From $a^2 = 4a' = 2b^2$, it follows that b^2 must be even and, by the same earlier reasoning, b must be even. But a and b cannot both be even, otherwise the fraction would be reducible further. By contradiction, $\sqrt{2}$ is not rational.

This proof relies on the intermediary result that the square of any odd integer is odd,

$$(2k+1)^2 = 4k^2 + 4k + 1 = 4(k^2 + k) + 1$$

for any odd integer $2k + 1 \in \mathbb{N}$.

Contradicting Finite Denominator

$$\sqrt{2} - \frac{a}{b} = \frac{\left(\sqrt{2} - \frac{a}{b}\right)\left(\sqrt{2} + \frac{a}{b}\right)}{\sqrt{2} + \frac{a}{b}} = \frac{2 - \left(\frac{a}{b}\right)^2}{2\sqrt{2}} = \frac{2b^2 - a^2}{2\sqrt{2}b^2} \geq \frac{1}{2\sqrt{2}b^2}$$

where we have used $\dfrac{a}{b} = \sqrt{2}$ and $2b^2 - a^2 \geq 1$. Thus, for finite b, the gap between $\sqrt{2}$ and $\dfrac{a}{b}$ remains bounded above zero.

Contradicting the Fundamental Theorem of Arithmetic

By the *fundamental theorem of arithmetic*, any natural number admits a unique decomposition as a product of primes, and its square admits a similar decomposition with each prime represented twice. Thus any square of a natural number can be decomposed uniquely as a product of an even number of primes. Since 2 is prime, $2b^2$ admits a decomposition into an odd number of primes, while a^2 admits an even number, contradicting the assumed equality:

$$2b^2 = a^2$$