

# UCLA Math Circle

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## Abstract

Notes on some problems from the UCLA Math Circle Intermediate-2 Assessment for Summer Session 2020. Comments and corrections welcome.

## 1. Random Integer

Sarah chose a random integer between 1000 and 9999, inclusive. What is the probability that the integer is odd and all of its digits are distinct?

A  $\frac{14}{75}$    B  $\frac{56}{225}$    C  $\frac{107}{400}$    D  $\frac{7}{25}$    E  $\frac{9}{25}$

We are looking for

$$P = \frac{\text{Count of the odd integer with distinct digits}}{\text{Count of all possible integers}}$$

The total number of possible 4-digit integers is readily calculated. The first digit may be any natural except 0 (because otherwise it would have fewer than 4 digits), so that leaves a choice of 9 integers. The second, third and fourth digits may be any of the 10 integers from 0 to 9.

$$\text{Count of all possible 4-digit integers} = 9 \times 10 \times 10 \times 10 = 9,000$$

The number of ways to pick 4 distinct integers to form a 4-digit number is  $9 \times 9 \times 8 \times 7 = 4,536$ . It is also clear that half of these are odd, so the probability we are looking for is about 25.2% ( $2,268/9,000$ ) give or take a couple of percents. We have:

A  $\frac{14}{75} \simeq 19\%$    B  $\frac{56}{225} \simeq 25\%$    C  $\frac{107}{400} \simeq 27\%$    D  $\frac{7}{25} \simeq 28\%$    E  $\frac{9}{25} \simeq 36\%$

Right away we can rule out answers A and E and we are particularly attracted to B. But let's do the math.

A 4-digit integer may be represented by the 4 boxes below, with the letters  $a, b, c, d$  placeholders for the first, second, third, and fourth digits.



with the following constraints:

$$\begin{aligned}
 a &\in \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \\
 b &\in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \\
 c &\in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \\
 d &\in \{1, 3, 5, 7, 9\}
 \end{aligned}$$

To randomly draw an odd number  $abcd$  from  $[1000, 9999]$  is like drawing  $a$  from  $\{1, 2, \dots, 9\}$ , drawing  $b, c$  from  $\{0, 1, \dots, 9\}$ , with replacement, and drawing  $d$  from  $\{1, 3, 5, 7, 9\}$ . Without the requirement of distinct digits, you could imagine drawing from the 4 sets simultaneously. But the requirement of distinct digits puts a constraint on the drawing. You can no longer think of it as a simultaneous draw. But with some care, you can think of it as a sequential draw of the 4 digits. Care is needed. If you first pick the last digit,  $d$ , then whichever digit you draw is no longer “available” when you go on to draw, say, the second digit  $b$ . On the other hand, if you draw  $b$  first, it may or may not impact the probability of drawing  $d$ . That is, if you draw a 0 for  $b$ , you still have 5 choices for  $d$ , but if you draw a 1 for  $b$ , you are left with only 4 choices for  $d$ . Similarly, if you draw a 0 for the third digit,  $c$ , say, it still leaves 9 choices for the first digit  $a$ , whereas if you draw a 1, it leaves only 8 choices for  $a$ . Thus, to get the calculation right, you have to imagine a predetermined order of digit selection that “works”. An order that works is  $d, a, b, c$ . Equivalently  $d, a, c, b$ . For the analogy with a sequential draw to work, you must draw  $d$  first and  $a$  second. So let’s suppose you are drawing  $d$  first (5 choices), that leaves one less digit for  $a$  (8 choices), and subsequently one less digit for  $b$  (8 choices), and lastly one less digit for  $c$  (7 choices).

And thus, the probability is:

$$\begin{aligned}
 P &= \frac{5 \times 8 \times 8 \times 7}{9 \times 10 \times 10 \times 10} \\
 &= \frac{8 \times 7}{9 \times 5 \times 5} \\
 &= \frac{56}{225} \\
 &\simeq 24.9\%
 \end{aligned}$$

The correct answer is B. Our rough estimate of 25.2%, it turns out, was pretty accurate.

## 2. No Sport Days!

John plays soccer once every three days, plays basketball once every four days and plays volleyball every five days. He played all three sports on December 31st, 2018. How many days in 2019 did John not play any of the sports?

A 78   B 146   C 144   D 80   E 152

It helps to visualize the problem with a timeline. Figure 1 shows the first 60 days. A brute-force strategy would be to extend the timeline to 365 days and count the days where John played no sports. But it would be nicer to find another method. Note that on day 60, all three sports are played. In a sense, everything “resets” on day 60: day 61 is like 1st January. Since  $6 \times 60 = 360$ , days 361 and 362 are no-sport days to be added to six times the count from 1 to 60. Looking at the figure, we count 24 within the 60-day period and so the answer is:

$$6 \times 24 + 2 = 146$$

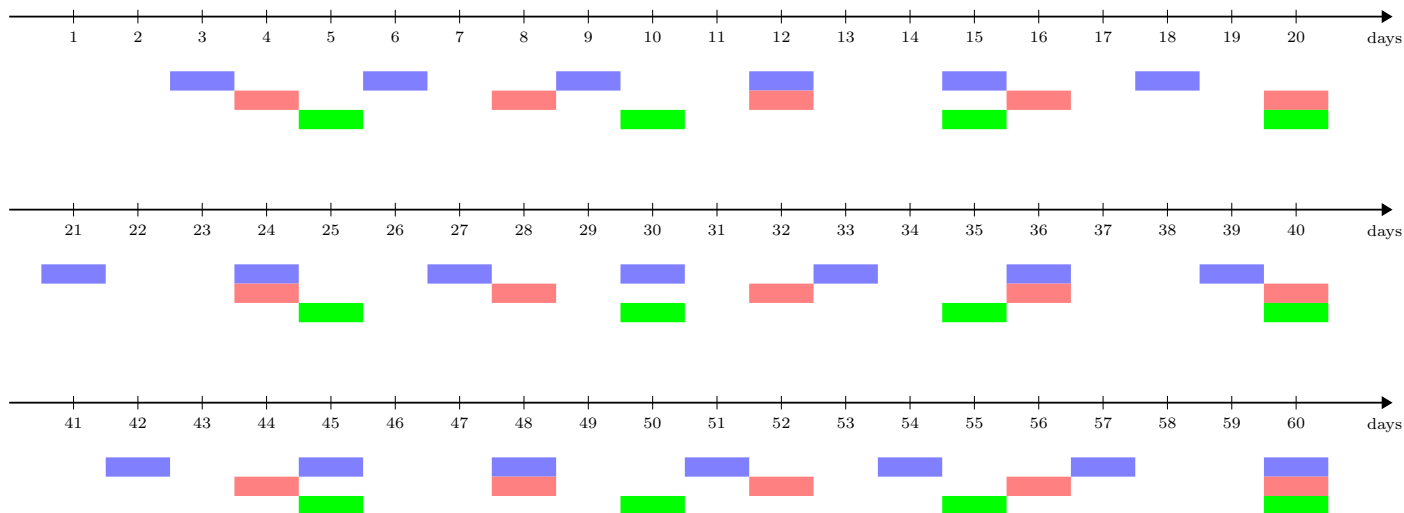


Figure 1: The first 60 days.

If you can sketch a timeline fast enough, this could be a reasonable approach. Is there a more subtle approach? It is obvious that there will be no sports on prime-number days greater than 5. It often helps to have memorized a bunch of prime numbers. Here you would need to quickly figure out this list:

7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359

But even that wouldn't be enough, because numbers of the form  $2p$ , for  $p$  prime, are also no-sports day. For instance, 14. So are numbers of the form  $p_a \times p_b$ , for  $p_a, p_b$  prime. For instance  $7 \times 11 = 77$ .

Another strategy is, instead, to count all sports days and deduct from 365. That is more manageable. To calculate how many multiples of 3 there are in 365, divide the nearest, smaller multiple of 3:  $363/3 = 121$  are all soccer days. For soccer, basketball, volleyball, one gets:

$$\begin{aligned}\frac{363}{3} &= 121 \\ \frac{364}{4} &= 91 \\ \frac{365}{5} &= 73\end{aligned}$$

However, multiples of  $3 \times 4 = 12$  have been counted twice. And so have multiples of  $3 \times 5$  and  $4 \times 5$ .

$$\begin{aligned}\frac{360}{12} &= 30 \\ \frac{360}{15} &= 24 \\ \frac{360}{20} &= 18\end{aligned}$$

If we add up the first three numbers and subtract the last three, we are getting closer, but there is still one problem. Consider  $3 \times 4 \times 5 = 60$ . The number 60 is included among the 121 multiples of 3, among the 91 multiples of 4, and among the 73 multiples of 5. It is also among the 30 multiples of 12, among the 24 multiples of 15, and among the 18 multiples of 20. Thus by adding and subtracting we have in effect removed all the multiples of 60. Since  $360/60 = 6$ , we must add those 6 back in. In total,

$$\begin{aligned}121 + 91 + 73 - 30 - 24 - 18 + 6 &= 219 \\ 365 - 219 &= 146\end{aligned}$$

as we had found earlier by a direct counting method.

The correct answer is B.

### 3. Congruent Circles and Angle

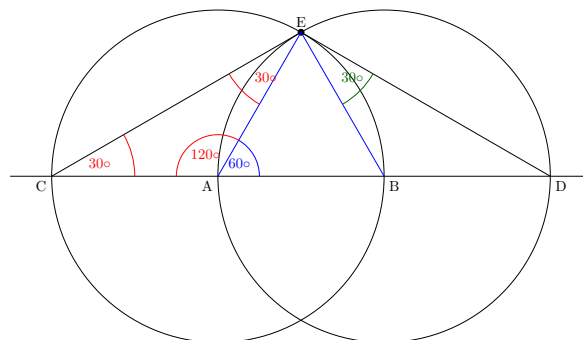
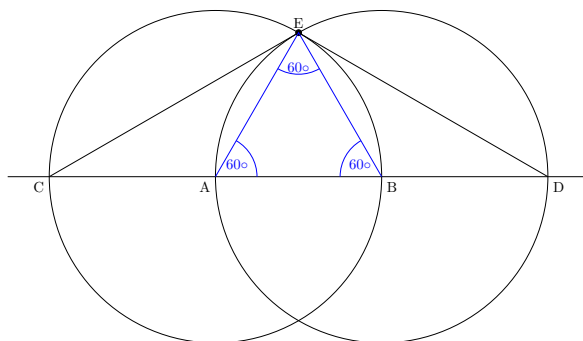
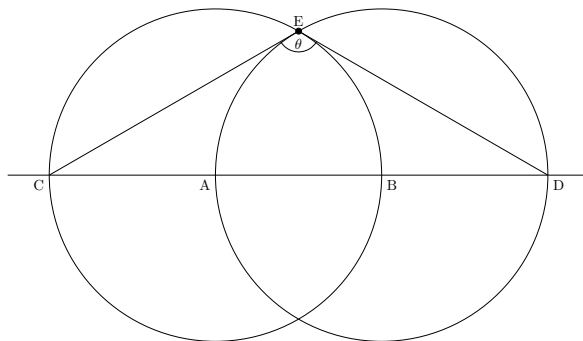
Two congruent circles centered at points  $A$  and  $B$  each pass through the other circle's center. The line containing both  $A$  and  $B$  is extended to intersect the circles at points  $C$  and  $D$ . The circles intersect at two points, one of which is  $E$ . What is the degree measure of  $\angle CED$ ?

A  $100^\circ$    B  $120^\circ$    C  $130^\circ$    D  $135^\circ$    E  $150^\circ$

Refer to the figures. Triangle  $AEB$  is equilateral, so  $\angle AEB = 60^\circ$ . Angle  $\angle CAE$  is the complement of  $\angle BAE$ , so  $\angle CAE = 180 - 60 = 120^\circ$ . Triangle  $CAE$  is isosceles, so  $\angle CEA = 30^\circ$ . Putting it together,

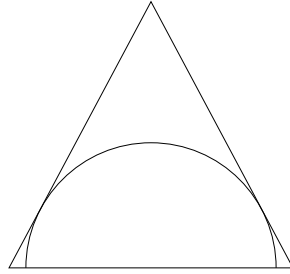
$$\begin{aligned}\angle CED &= \angle CEA + \angle AEB + \angle BED \\ &= 30^\circ + 60^\circ + 30^\circ \\ &= 120^\circ\end{aligned}$$

The correct answer is B.



#### 4. Semi-Circle Inscribed in Triangle

A semicircle is inscribed in an isosceles triangle with base 16 and height 15 so that the diameter of the semicircle is contained in the base of the triangle as shown. What is the radius of the semicircle?



- A  $4\sqrt{3}$    B  $17\sqrt{2}$    C  $17\sqrt{3}$    D 10   E  $\frac{120}{17}$

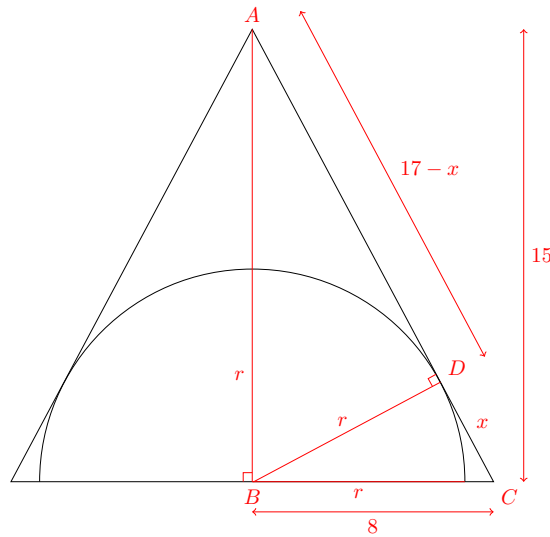


Figure 2: **Semi-Circle Inscribed in a Triangle.**

Refer to Figure 2. Triangles  $ADB$  and  $BDC$  are right-angled, i.e.  $\angle ADB = 90^\circ$  and  $\angle BDC = 90^\circ$ . The known lengths are  $AB = 15$  and  $BC = 8$ . Applying Pythagoras to  $ABC$  yields:

$$AC = \sqrt{15^2 + 8^2} = \sqrt{289} = 17$$

Applying Pythagoras to  $ADB$  yields:

$$(17 - x)^2 + r^2 = 15^2$$

Applying Pythagoras to  $BDC$  yields:

$$r^2 + x^2 = 8^2$$

Subtracting the last two equations yields:

$$\begin{aligned}(17 - x)^2 - x^2 &= 15^2 - 8^2 \\ 17^2 - 2 \cdot 17 \cdot x + x^2 - x^2 &= 15^2 - 8^2 \\ x &= \frac{17^2 - 15^2 + 8^2}{2 \cdot 17}\end{aligned}$$

which we can use in  $r^2 = 8^2 - x^2$ :

$$\begin{aligned}r^2 &= 8^2 - \left( \frac{17^2 - 15^2 + 8^2}{2 \cdot 17} \right)^2 \\ &= 64 - \left( \frac{289 - 225 + 64}{2 \cdot 17} \right)^2 \\ &= 64 - \left( \frac{64}{17} \right)^2 \\ &= \frac{64 \cdot 17^2 - 64^2}{17^2} \\ &= \frac{64(289 - 64)}{17^2} \\ &= \frac{64 \cdot 225}{17^2} \\ &= \frac{8^2 \cdot 15^2}{17^2} \\ \Rightarrow \quad r &= \sqrt{r^2} = \frac{8 \cdot 15}{17} \\ &= \frac{120}{17}\end{aligned}$$

The correct answer is E.

This was a long and tedious calculation. There may be a shorter way ...

## 5. Ordering Big Numbers

What is the correct ordering of the three numbers  $10^8$ ,  $5^{12}$ , and  $2^{24}$ .

A  $2^{24} < 10^8 < 5^{12}$    B  $2^{24} < 5^{12} < 10^8$    C  $5^{12} < 2^{24} < 10^8$    D  $10^8 < 5^{12} < 2^{24}$    E  $10^8 < 2^{24} < 5^{12}$

We have  $10 = 2 \times 5$ , so we can rewrite  $10^8$  in terms of  $2^{24}$  and  $5^{12}$  and check whether the multiplier is greater or smaller than 1.

$$\begin{aligned} 10^8 &= 5^8 2^{-16} \times 2^{24} \\ &= \left(\frac{5}{4}\right)^8 \times 2^{24} \\ &> 2^{24} \end{aligned}$$

since  $\frac{5}{4} > 1$  (any number greater than 1 raised to some positive power is greater than 1).

Similarly,

$$\begin{aligned} 10^8 &= 2^8 5^{-4} \times 5^{12} \\ &= \left(\frac{4}{5}\right)^4 \times 5^{12} \\ &< 5^{12} \end{aligned}$$

since  $\frac{4}{5} < 1$  (the inverse of the fraction above).

Putting it together,

$$2^{24} < 10^8 < 5^{12}$$

The correct answer is A.

A trick we learned at the *UCLA Math Circle* that simplifies the problem is to note that the exponents are all multiples of 4:

$$\begin{aligned} 8 &= 2 \times 4 \\ 12 &= 3 \times 4 \\ 24 &= 6 \times 4 \end{aligned}$$

For any  $0 < a < b$ , we have

$$0 < a < b \iff 0 < a^4 < b^4$$

Thus the question reduces to comparing  $10^2 = 100$ ,  $5^3 = 125$ , and  $2^6 = 64$ , which is straightforward!



## 6. Moduli

What is the tens digit of  $7^{2011}$

A 0   B 1   C 3   D 4   E 7

If you have access to a computer/calculator, you can make the following table. However, try  $7^{2011}$  and it most likely will give you either an error or something like **Inf**. That is because the number  $7^{2011}$  is large beyond imagination. The largest power of 7 we were able to compute/estimate was  $7^{364} \simeq 4.1 \times 10^{307}$ , which doesn't help with the tens digit!

Exponent	Number	10-Digit
0	1	0
1	7	0
2	49	4
3	343	4
4	2,401	0
5	16,807	0
6	117,649	4
7	823,543	4
8	5,764,801	0
9	40,353,607	0

without a calculator and no special tricks, the largest powers we can compute are the first 5 or 6. It's a bit of a stretch from that to guess that the tens digits come in sequences of 0, 0, 4, 4, 0, 0, 4, 4, ... At this point, you're faced with a toss-up between 0 and 4 or you can work out whether 2011 maps to exponent 0, 1, 2, or 3.

**Definition** (Modular Arithmetic).

$$a \pmod{b} = \text{remainder of } a/b$$

In general, we have  $a \pmod{b} \in [0, b-1]$ . For instance,

$$11 \pmod{3} \equiv 2$$

which may be read “eleven mod three is congruent to two.” The word comes from the latin “modulus”, whose plural form is “moduli.”

An analogy to give intuition about modular arithmetic is the hour system. Some countries in Europe use a 24 hour clock, while other countries like the United States more commonly use a 12 hour clock (except the military, who also use the 24 hour clock!). Thus 8:00pm would be equivalent to 20:00 (In France, the popular evening news slot is called “the twenty-hour” — “le vingt-heure”). In modular arithmetic, that is

$$20 \pmod{12} \equiv 8$$

We all do modular arithmetic without thinking (too hard). For instance, if it takes about 50 hours to watch all James Bond movies, what time will it be when you finish, if you start at 10pm? The answer is midnight because

$$50 \pmod{24} \equiv 2$$

In Python and the C languages, the `mod` command is the percentage sign, that is the first line returns the second:

$$\begin{aligned} 50 \% 24 \\ 2 \end{aligned}$$

There are online modulo calculators that you can play with too.

From the pattern detected above, it is clear that  $7^n$  has a tens digit of 0 whenever  $n$  is a multiple of 4. A candidate for a multiple of 4 near 2011 is 2010, but it is easily verified that it's not. It follows immediately that 2008 is a multiple of 4, which can also be verified easily. To sum up,

ten digit =	0	0	4	④
modulo value =	0	1	2	③
	...	...	...	...
other values of $n$	2004	2005	2006	2007
	2008	2009	2010	②011
	2012	2013	2014	2015

In `mod` notation,

$$\begin{aligned} 7^{2008} \pmod{4} &\equiv 0 \\ 7^{2011} \pmod{4} &\equiv 3 \end{aligned}$$

which implies that the digit is 4.

The correct answer is D.

## 7. Area of Triangle in Cartesian Coordinates

What is the area of the triangle with vertices  $A = (1, 3)$ ,  $B = (5, 1)$ ,  $C = (4, 4)$ .

A 5   B  $3\sqrt{2}$    C  $\frac{4}{3}\sqrt{2}$    D  $3\sqrt{3}$    E 6

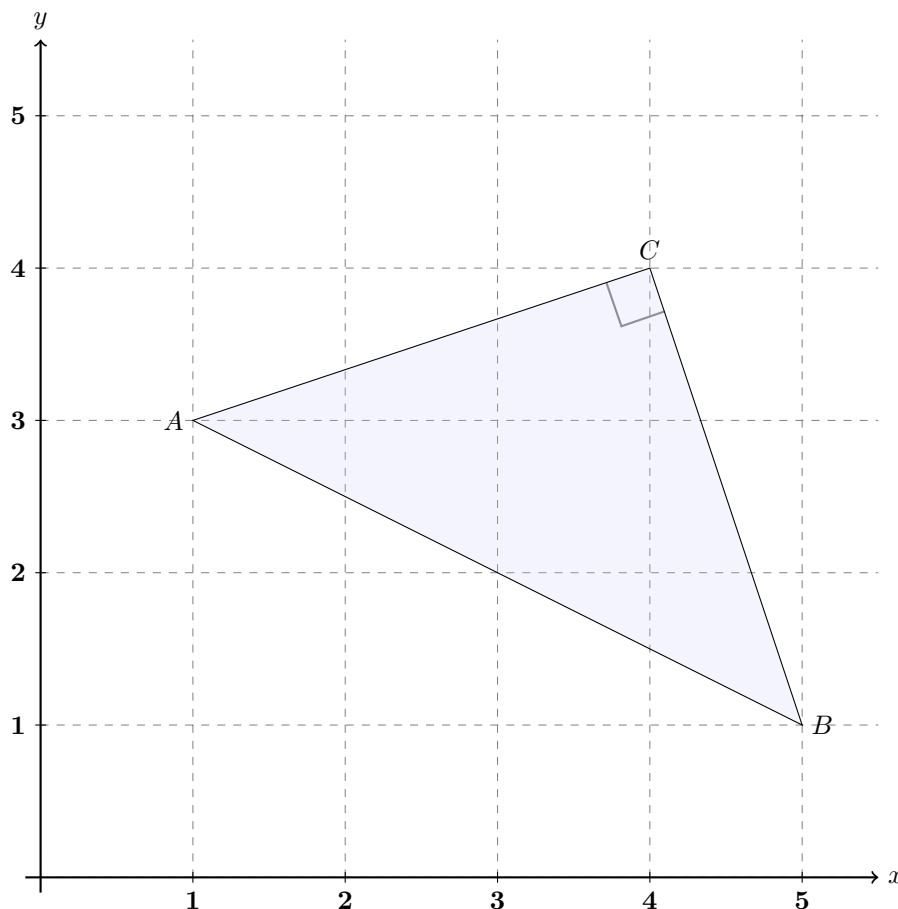


Figure 3: Area of Triangle in a Cartesian Coordinate System.

Refer to Figure 3. Before you draw a figure in a Cartesian coordinate system, check the largest and smallest values for each axis, to ensure that your figure will fit. Here all the coordinates are positive, ranging on  $[1, 5]$  on  $x$  and  $[1, 4]$  on  $y$ . Since 1 is close enough to 0, we will include the origin and, for symmetry, we'll set the limits to  $[0, 5]$  along both axes.

With an accurately drawn triangle, a few properties are immediately apparent. (1) The triangle is isosceles: length  $CA$  equals length  $CB$ . To see this, note that length  $CA$  spans three units horizontally for one unit vertically, while length  $CB$  spans three units vertically for one unit horizontally. (2) The angle  $\angle ACB$  is a right angle. This can be seen by rotating the triangle around point  $C$ , moving point  $A$  by one unit vertically and point  $B$  by one unit horizontally. Once you see this, you also see that the area of the triangle is a little more than half of  $3 \times 3$ , since the lengths  $CA$  and  $CB$  are a little longer than 3. So we are looking for an area approximately equal to 5.

It pays to know that  $\sqrt{2} \simeq 1.414$  and  $\sqrt{3} \simeq 1.732$ . From this we have the approximations:

**A** 5   **B**  $3\sqrt{2} \simeq 4.24$    **C**  $\frac{4}{3}\sqrt{2} \simeq 1.89$    **D**  $3\sqrt{3} \simeq 5.20$    **E** 6

From this, we can rule out  $B$ ,  $C$ ,  $E$ . We lean towards A, but let's do the math. We can apply Pythagoras.

Length  $AC$  may be computed from the triangle formed by points  $A$ ,  $(1, 4)$ , and  $C$ :

$$AC^2 = (4 - 1)^2 + (4 - 3)^2 = 10$$

Length  $CB$  may be computed from the triangle formed by points  $C$ ,  $(5, 4)$ , and  $B$ .

$$CB^2 = (5 - 4)^2 + (4 - 1)^2 = 10$$

Length  $AB$  may be computed from the triangle formed by points  $A$ ,  $(1, 1)$ , and  $B$ .

$$AB^2 = (3 - 1)^2 + (5 - 1)^2 = 20$$

We can verify that the angle is a right angle by checking the Pythagoras relation:

$$(\sqrt{10})^2 + (\sqrt{10})^2 = 20 = (\sqrt{20})^2$$

Finally, the area is given by:

$$\frac{1}{2}\sqrt{10} \times \sqrt{10} = 5$$

as we suspected.

The correct answer is A.
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## 8. A Quartic Equation!

How many different real numbers satisfy the equation  $(x^2 - 5)^2 = 16$  ?

A 0   B 1   C 2   D 4   E 8

This is a polynomial of degree 4 since  $(x^2 - 5)^2 = x^4 - 10x^2 + 25$ . This is also called a quartic equation.

**Theorem** (Fundamental Theorem of Algebra). *Every polynomial of degree  $n$  with complex coefficients has exactly  $n$  complex roots. That is, every equation of the form*

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x^1 + a_0 x^0 = 0,$$

*with  $a_n \neq 0$ , has exactly  $n$  roots. There is a catch: roots are counted with multiplicity. Example: The polynomial  $x^2 = 0$  has two roots, 0 and 0.*

The Fundamental Theorem of Algebra implies that we can rule out answer E, because there can be at most 4 distinct real roots.

**Theorem** (Complex Conjugate Root Theorem). *If a polynomial with real coefficients has a complex root  $a + bi$  ( $a$  and  $b$  are real numbers, with  $b \neq 0$ , and  $i$  is the imaginary unit), then the complex conjugate  $a - bi$  is also a root. In short, **complex roots come in pairs**.*

It follows that if the degree of a real polynomial is odd, it must have at least one real root. Since the polynomial has degree 4, it may have 0, 2 or 4 complex roots, implying that we can rule out answer B.

The above considerations become automatic once you are familiar with polynomials. Moreover inspection immediately suggests the existence of two real roots, 1 and  $-1$ . So now the remaining possibilities are C and D. In this example, we can actually solve the quartic equation by reducing it to a pair of quadratic equations:

$$(x^2 - 5)^2 = 16 \implies \begin{cases} (x^2 - 5) = 4 \implies x^2 = 9 \implies \begin{cases} x = 3 \\ x = -3 \end{cases} \\ -(x^2 - 5) = 4 \implies x^2 = 1 \implies \begin{cases} x = 1 \\ x = -1 \end{cases} \end{cases}$$

So there are 4 real roots.

The correct answer is D.

## 9. Sharing Markers

Ann has 24 markers. In how many ways can she share them with Bob and Chris so that each of the three has at least two markers?

A 105   B 114   C 190   D 210   E 380

The correct answer is C.
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## 10. Shopping for Socks

Ralph went to the store and bought 12 pairs of socks for a total of \$24. Some of the socks he bought cost \$1 a pair, some of the socks he bought cost \$3 a pair, and some of the socks he bought cost \$4 a pair. If he bought at least one pair of each type, how many pairs of \$1 socks did Ralph buy?

A 4   B 5   C 6   D 7   E 8

The correct answer is D.
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