AMC 10 Problem Series (2804)

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Overview

Week 2 (Jun 11) Class Transcript - Word Problems

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jonjoseph 2021-06-11 19:30:41
And that's our winner: @ Sbpejp!!! Nicely done.

jonjoseph 2021-06-11 19:31:06
Yes. Many claps.

jonjoseph 2021-06-11 19:31:20
AMC 10 Problem Series
Week 2: Word Problems

jonjoseph 2021-06-11 19:31:25
Last week, we saw how to solve different kinds of equations. Today, we're going to apply our equation-solving techniques to word problems. In word problems, we must take information given in the problem and convert it into a mathematical form that we can work with.

jonjoseph 2021-06-11 19:31:40
When I give you a word problem (or any problem), I want you to read it and think about what it says. So I will wait a few moments before I ask you a question about the problem.

jonjoseph 2021-06-11 19:31:51
Here's our first problem:

jonjoseph 2021-06-11 19:31:52
The sum of three numbers is 20. The first is four times the sum of the other two. The second is seven times the third. What is the product of all three?

$$\text{(A) } 28 \quad \text{(B) } 40 \quad \text{(C) } 100 \quad \text{(D) } 400 \quad \text{(E) } 800$$

jonjoseph 2021-06-11 19:32:15
What can we do first?

jonjoseph 2021-06-11 19:33:13
I see several ideas. Let me remind people we can't really set up our equations until we have defined our variables.

jonjoseph 2021-06-11 19:33:29
It's always a good idea to explicitly define your variables at the beginning of a problem. That can eliminate a lot of confusion.

jonjoseph 2021-06-11 19:33:40
In this case, the variables we should define are easy to identify. Let the numbers be x, y, and z.

jonjoseph 2021-06-11 19:33:51
We have three separate conditions on x, y and z. What does the first condition tell us?

jonjoseph 2021-06-11 19:34:35
We are told that the sum of the three numbers is 20, so $x+y+z=20$.

jonjoseph 2021-06-11 19:34:43
And what do the other two conditions say?

jonjoseph 2021-06-11 19:35:50
They say that $x=4(y+z)$ and $y=7z$.

jonjoseph 2021-06-11 19:35:51
All right, we have the three equations:

$$\begin{aligned} x+y+z &= 20, \\ 4(y+z) &= x, \\ y &= 7z. \end{aligned}$$

jonjoseph 2021-06-11 19:35:59
There are several approaches we can take to solve the system. Let's start by substituting the last equation into the second equation. What do we get?

jonjoseph 2021-06-11 19:37:19
That's right, we get $x=32z$. We can now substitute all the expressions we've found into the first equation, and solve for the three numbers. What do you get for x, y and z?

jonjoseph 2021-06-11 19:38:52
Substituting, we get

$$32z+7z+z=20,$$

which gives $40z=20$ and $z=\frac{1}{2}$.

jonjoseph 2021-06-11 19:38:58
Plugging that in, we get

$$x=16, y=\frac{7}{2}, z=\frac{1}{2}.$$

So what's the answer to the problem?

jonjoseph 2021-06-11 19:39:56
We see that $xyz=16\cdot\frac{7}{2}\cdot\frac{1}{2}=28$. The answer is (A).

jonjoseph 2021-06-11 19:40:05
Patty has 20 coins consisting of nickels and dimes. If her nickels were dimes and her dimes were nickels, she would have 70 cents more. How much are her coins worth?

$\text{(A) } \$1.15 \quad \text{(B) } \$1.20 \quad \text{(C) } \$1.25 \quad \text{(D) } \$1.30 \quad \text{(E) } \$1.35$

jonjoseph 2021-06-11 19:40:12
Think for a moment.

jonjoseph 2021-06-11 19:40:37
What are our variables?

jonjoseph 2021-06-11 19:41:03
If you just say d and n does that mean daylight and nighttime?

jonjoseph 2021-06-11 19:42:13
For instance, we can't just say "let n equal nickels" because then it's not clear if you mean "number of nickels Patty has" or "the total number of cents Patty has in nickels" or maybe even something silly like "the total weight of the nickels Patty has."

jonjoseph 2021-06-11 19:42:26
We're going to let n and d be the number of nickels and dimes that Patty has, respectively. What can we say about n and d?

jonjoseph 2021-06-11 19:43:41
Since Patty has 20 coins, $n + d = 20$. How much money does Patty have, in cents?

jonjoseph 2021-06-11 19:45:12
Patty has $5n + 10d$ in cents.

jonjoseph 2021-06-11 19:45:14
We are also told that if Patty had n dimes and d nickels instead, then she would have 70 cents more.

jonjoseph 2021-06-11 19:45:28
And how many cents are there in n dimes and d nickels?

jonjoseph 2021-06-11 19:46:02
There are $10n + 5d$ cents. We know that this is 70 cents more than what she actually has. What equation does this give us?

jonjoseph 2021-06-11 19:47:17
It gives us the equation

$$10n + 5d = 5n + 10d + 70.$$

jonjoseph 2021-06-11 19:47:21
Bringing all the variables to the left side of the equation, we get

$$5n - 5d = 70,$$

which simplifies to $n - d = 14$.

jonjoseph 2021-06-11 19:47:25
We also have $n + d = 20$. So what are n and d?

jonjoseph 2021-06-11 19:48:32
Adding the two equations and solving, we get $n = 17$, and substituting back in, we get $d = 3$. So what's our answer?

jonjoseph 2021-06-11 19:49:34
Patty has $5n + 10d = 5 \cdot 17 + 10 \cdot 3 = 115$ in cents, or $\$1.15$. The answer is (A).

jonjoseph 2021-06-11 19:49:45
By the way, as is often the case in AMC questions, there's a clever trick you can use to figure out the answer faster!

jonjoseph 2021-06-11 19:49:51
We know that if we replaced Patty's nickels with dimes, and Patty's dimes with nickels, she's have 70 cents more. Do you think she has more nickels or more dimes?

jonjoseph 2021-06-11 19:50:21
She clearly has more nickels. Otherwise, swapping would result in less money, not more.

jonjoseph 2021-06-11 19:50:28
We can imagine swapping the nickels for dimes one by one. Imagine that she puts down one nickel and picks up one dime, until the numbers are flipped. By how much does her total change if she does that once?

jonjoseph 2021-06-11 19:50:55
Yes, every time she swaps a coin, she gets 5 cents more. We know that after she's done swapping, she has 70 cents more. So how many nickels did she exchange for dimes?

jonjoseph 2021-06-11 19:51:47
She exchanged 14 nickels! That shows that she must have had 14 more nickels than she had dimes.

jonjoseph 2021-06-11 19:51:53
Since we know the total number of nickels and dimes is 20, it's easy to spot that she must have had 17 nickels and 3 dimes.

jonjoseph 2021-06-11 19:52:10
Clear?

jonjoseph 2021-06-11 19:53:00
Good. Takes a little practice to "see" these shortcuts. But once you know about them you'll always be on the lookout.

jonjoseph 2021-06-11 19:53:14
Some freshmen and sophomores are having a car wash to raise money for a class trip to China. Initially 40% of the group are sophomores. Shortly thereafter two sophomores leave and two freshmen arrive, and then 30% of the group are sophomores. How many sophomores were initially in the group?

$(\mathrm{A})\ 4 \quad (\mathrm{B})\ 6 \quad (\mathrm{C})\ 8 \quad (\mathrm{D})\ 10 \quad (\mathrm{E})\ 12$

jonjoseph 2021-06-11 19:53:45
As usual, we start by defining variables. Tell me what they should be, and please be specific!

jonjoseph 2021-06-11 19:55:10
Let f and s be the original numbers of freshmen and sophomores, respectively.

jonjoseph 2021-06-11 19:55:20
We are told that initially, 40% of the group are sophomores. What equation does this give us?

jonjoseph 2021-06-11 19:55:57
Hint: Careful here. the whole group is $s+f$.

jonjoseph 2021-06-11 19:57:01
Since $\dfrac{40}{100}=\dfrac{2}{5}$, this gives us $s=\dfrac{2}{5}(f+s)$. How does this simplify?

jonjoseph 2021-06-11 19:58:40
We can multiply both sides by 5 to get rid of the fraction. Multiplying both sides by 5, we get $5s=2f+2s$, so $3s=2f$.

jonjoseph 2021-06-11 19:58:47
We are also told that after two sophomores leave and two freshmen arrive, 30% of the group are sophomores. What equation does this give us?

jonjoseph 2021-06-11 19:58:59
Don't simplify yet.

jonjoseph 2021-06-11 20:00:44
Since $\dfrac{30}{100}=\dfrac{3}{10}$, this gives us $s-2=\dfrac{3}{10}(f+s)$.

jonjoseph 2021-06-11 20:00:54
Note that the total number of students is still $f+s$, since the two added freshmen cancel out the two subtracted sophomores. How does this equation simplify?

jonjoseph 2021-06-11 20:02:27
We can multiply both sides by 10 to get rid of the fraction. Multiplying both sides by 10, we get

$$10s-20=3f+3s,$$

so $7s=3f+20$.

jonjoseph 2021-06-11 20:02:37

Hence, we have the system of equations

$$\begin{aligned} 3s &= 2f, \\ 7s &= 3f + 20. \end{aligned}$$

jonjoseph 2021-06-11 20:02:56
You finish.

jonjoseph 2021-06-11 20:04:43
From the first equation, $f = \dfrac{3s}{2}$. Substituting into the second equation, we get

$$7s = 3\left(\frac{3s}{2}\right) + 20.$$

Multiplying both sides by 2, we see that $14s = 9s + 40$, which gives $s = 8$. The answer is (C).

jonjoseph 2021-06-11 20:05:00
In retrospect, this is another one with a cleverer way! Note that two sophomores left, and two freshmen replaced them. So what didn't change?

jonjoseph 2021-06-11 20:05:31
Yes, the total number of people didn't change.

jonjoseph 2021-06-11 20:05:33
But we also know that the percentage of sophomores decreased by 10%! Hmmmm. Can anyone spot what the total number of people must be?

jonjoseph 2021-06-11 20:06:11
We see that 10% of the total number must be 2, which means there must have been 20 people total.

jonjoseph 2021-06-11 20:06:24
So how many sophomores were there?

jonjoseph 2021-06-11 20:06:54
Since 40% of them were sophomores, there were 8 sophomores to start with. That's quite a bit faster.

jonjoseph 2021-06-11 20:07:08
Looking for what stays the same in a problem (like the total number of coins in the last problem and the total number of people in this one) can be a very powerful strategy.

jonjoseph 2021-06-11 20:07:31
Speed counts on both AMC's.

jonjoseph 2021-06-11 20:07:50
The area of a rectangle remains unchanged when it is made $2\frac{1}{2}$ inches longer and $\frac{2}{3}$ inches narrower, or when it is made $2\frac{1}{2}$ inches shorter and $\frac{4}{3}$ inches wider. Its area, in square inches, is:

$\text{(A) } 30 \quad \text{(B) } \frac{80}{3} \quad \text{(C) } 24 \quad \text{(D) } \frac{45}{2} \quad \text{(E) } 20$

jonjoseph 2021-06-11 20:08:15
Unsurprisingly, we start out by defining variables. What should they be here?

jonjoseph 2021-06-11 20:09:04
Let L and W be the original length and width of the rectangle in inches, respectively.

jonjoseph 2021-06-11 20:09:42
(And it really doesn't matter if you use L and W or x and y as long as you back them up with definition in English.

jonjoseph 2021-06-11 20:09:55
We are told that the area of a rectangle remains unchanged when it is made $2\frac{1}{2}$ inches longer and $\frac{2}{3}$ inches narrower.

jonjoseph 2021-06-11 20:09:59
What equation does this give us?

jonjoseph 2021-06-11 20:11:48
This gives us the equation

$$LW = \left(L + \frac{5}{2}\right)\left(W - \frac{2}{3}\right).$$

jonjoseph 2021-06-11 20:11:52
We can expand the right side, to get

$$LW = LW - \frac{2}{3}L + \frac{5}{2}W - \frac{5}{3}.$$

Subtracting LW from both sides and moving the terms around, we get

$$\frac{5}{2}W - \frac{2}{3}L = \frac{5}{3}.$$

How can we make this equation even nicer?

jonjoseph 2021-06-11 20:12:48
Multiplying both sides by 6 to get rid of the fractions, we get $15W - 4L = 10$. Things cleaned up nicely!

jonjoseph 2021-06-11 20:12:55
We are also told that the area of a rectangle remains unchanged when it is made $2\frac{1}{2}$ inches shorter and $\frac{4}{3}$ inches wider.

jonjoseph 2021-06-11 20:12:59
Write down an equation, then expand and simplify it yourself! What do you get?

jonjoseph 2021-06-11 20:15:24
This gives us the equation

$$LW = \left(L - \frac{5}{2}\right)\left(W + \frac{4}{3}\right).$$

Doing the exact same manipulations as before, we get the equation $8L - 15W = 20$.

jonjoseph 2021-06-11 20:15:27
Hence, we have the system of equations

$$\begin{aligned} 15W - 4L &= 10, \\ 8L - 15W &= 20. \end{aligned}$$

Now what?

jonjoseph 2021-06-11 20:16:27
We can add them to cancel the W terms. What do we get for L when we add the equations and solve?

jonjoseph 2021-06-11 20:17:25
Adding the equations gives $4L = 30$, so $L = \dfrac{30}{4} = \dfrac{15}{2}$. And what do we get for W?

jonjoseph 2021-06-11 20:18:30

Substituting into the first equation, we get $15W - 30 = 10$. That means that $W = \frac{8}{3}$. What's the final answer?

jonjoseph 2021-06-11 20:19:15

The area of the rectangle is $LW = \frac{15}{2} \cdot \frac{8}{3} = 20$. The answer is (E).

jonjoseph 2021-06-11 20:19:20

The number of geese in a flock increases so that the difference between the populations in year $n+2$ and year n is directly proportional to the population in year $n+1$. If the populations in the years $1994, 1995$, and 1997 were $39, 60$, and 123, respectively, then the population in 1996 was

$\text{(A) } 81 \quad \text{(B) } 84 \quad \text{(C) } 87 \quad \text{(D) } 90 \quad \text{(E) } 102$

jonjoseph 2021-06-11 20:19:29

Take a second to read this one. It's a bit long!

jonjoseph 2021-06-11 20:20:04

In case you have not heard the term before, two quantities are directly proportional if one is a constant multiple of the other.

jonjoseph 2021-06-11 20:20:36

You might see notation like this: $x \propto y$ where $\propto$ means proportional to.

jonjoseph 2021-06-11 20:20:52

It means this: $x = ky$ where k is the constant of proportionality.

jonjoseph 2021-06-11 20:21:01

Is that clear?

jonjoseph 2021-06-11 20:21:34

Yes. It can be any CONSTANT number.

jonjoseph 2021-06-11 20:22:10

In other words, the ratio $\frac{x}{y}$ is always constant.

jonjoseph 2021-06-11 20:22:30

I knew you'd spot it.

jonjoseph 2021-06-11 20:22:38

Here, the difference between the populations in year $n+2$ and year n is directly proportional to the population in year $n+1$. In other words, if we take the difference between two terms that are separated by a single term, that's proportional to that single (middle) term.

jonjoseph 2021-06-11 20:22:53

For example, here's a sequence that satisfies this:

$$1, 2, 5, 12, 29.$$

What's the constant ratio of the difference between the populations in year $n+2$ and year n and the population in year $n+1$ here?

jonjoseph 2021-06-11 20:24:01

Yes, it's 2, because we have

$$2 = \frac{5-1}{2} = \frac{12-2}{5} = \frac{29-5}{12}.$$

jonjoseph 2021-06-11 20:24:14

Here's another sequence:

$$1, 2, 3, 6, 8.$$

Does it also satisfy this property?

jonjoseph 2021-06-11 20:24:51

No, it doesn't. For instance, $\frac{3-1}{2} \neq \frac{6-2}{3}$.

jonjoseph 2021-06-11 20:24:55

All right, back to our problem! It's a word problem. We're definitely going to start by defining variables.

jonjoseph 2021-06-11 20:24:57

What's one variable we're certainly going to need? As usual, be specific!

jonjoseph 2021-06-11 20:26:12

We are definitely going to need a variable for the population of geese in 1996, since that's what we're trying to find. Let's call it x.

jonjoseph 2021-06-11 20:26:18

That means the populations of geese in $1994, 1995, 1996$ and 1997 were $39, 60, x$ and 123, respectively.

jonjoseph 2021-06-11 20:26:31

We could also define a variable for the constant of proportionality, but it turns out we don't need to! Instead, we're going to use our ratio formulation above. Often you can write an equation for the ratios being equal and never need to know what the ratio actually is.

jonjoseph 2021-06-11 20:26:49

We know that if we take the difference between two terms that are separated by a single term, that's proportional to that single (middle) term.

jonjoseph 2021-06-11 20:26:56

We only have two ratios we can make, because we need a middle term. The middle term can be 60 or x and we get the same ratio either way. What equation does that give us?

jonjoseph 2021-06-11 20:28:28

The difference between the 1996 and 1994 populations is $x-39$, and the population in year 1995 is 60. The difference between the 1997 and 1995 populations is 63 and the population in year 1996 is x.

jonjoseph 2021-06-11 20:28:33

Therefore, we get the equation

$$\frac{x-39}{60} = \frac{63}{x}.$$

jonjoseph 2021-06-11 20:28:49

We could cross-multiply, getting the quadratic $x^2 - 39x = 3780$, and move everything to one side, getting

$$x^2 - 39x - 3780 = 0.$$

jonjoseph 2021-06-11 20:28:55

That looks a little painful... we could factor it, but it'd take time.

jonjoseph 2021-06-11 20:29:04

We could also use the quadratic formula, but that involves the expression $\sqrt{39^2 + 4 \cdot 3780}$, which I think we can all agree we'd prefer to avoid, even if it does turn out to be an integer.

jonjoseph 2021-06-11 20:29:26

Smart.

jonjoseph 2021-06-11 20:29:30

Let's just try plugging in the answer choices we were given and see which one works! Should we plug into the quadratic, or our

stickied equation (the one with the fractions)?

jonjoseph 2021-06-11 20:30:31
Let's plug into our stickied equation, since the fractions are easier to work with than the big numbers in the quadratic.

jonjoseph 2021-06-11 20:30:35
Let's start with (A), which is $x = 81$.

jonjoseph 2021-06-11 20:30:37
We want to know whether $\frac{81-39}{60}$ is equal to $\frac{63}{81}$. How do those fractions simplify?

jonjoseph 2021-06-11 20:30:52
Does it work?

jonjoseph 2021-06-11 20:31:34
The right hand side simplifies to $\frac{42}{60} = \frac{7}{10}$ and the left hand side simplifies to $\frac{7}{9}$. So this doesn't work.

jonjoseph 2021-06-11 20:31:36
Now let's try (B), which is $x = 84$. This time, we want to know whether $\frac{84-39}{60}$ is equal to $\frac{63}{84}$. What do you get?

jonjoseph 2021-06-11 20:32:55
The right hand side simplifies to $\frac{45}{60} = \frac{3}{4}$ and the left hand side becomes $\frac{9 \cdot 7}{12 \cdot 7} = \frac{9}{12} = \frac{3}{4}$. This one works! So the answer is (B).

jonjoseph 2021-06-11 20:33:17
It can be a pain to plug in answer but in this case not so bad.

jonjoseph 2021-06-11 20:33:35
Elmo makes N sandwiches for a fundraiser. For each sandwich he uses B globs of peanut butter at 4 cents per glob and J globs of jam at 5 cents per glob. The cost of the peanut butter and jam to make all the sandwiches is $\$2.53$. Assume that B, J, and N are all positive integers with $N > 1$. What is the cost of the jam Elmo uses to make the sandwiches?

$(\text{A})\ \$1.05 \quad (\text{B})\ \$1.25 \quad (\text{C})\ \$1.45 \quad (\text{D})\ \$1.65 \quad (\text{E})\ \$1.85$

jonjoseph 2021-06-11 20:33:57
Super cool dude.

jonjoseph 2021-06-11 20:34:31
This time we don't have to define variables, because the problem already defines all the variables for us!

jonjoseph 2021-06-11 20:34:39
We're going to need to use the fact that all the sandwiches together cost $\$2.53$, or in other words, 253 cents. Let's start by figuring out how much a single sandwich costs in terms of our variables.

jonjoseph 2021-06-11 20:34:45
A single sandwich uses B globs of peanut butter at 4 cents a glob, and J globs of jam at 5 cents a glob. How much does a single sandwich cost, then?

jonjoseph 2021-06-11 20:35:55
A single sandwich costs $4B + 5J$ cents. We know that Elmo makes N sandwiches.

jonjoseph 2021-06-11 20:36:05
So what's an expression for how much all the N sandwiches cost together, in terms of B, J and N?

jonjoseph 2021-06-11 20:36:26
Hint: You'll need an equal sign in this answer.

jonjoseph 2021-06-11 20:36:50
Careful. Some of you are mixing dollars and cents.

jonjoseph 2021-06-11 20:37:04
Hint: Make it all cents (sense?)

jonjoseph 2021-06-11 20:38:00
They cost $N(4B+5J)$ cents. Therefore, we know that

$$N(4B+5J)=253.$$

jonjoseph 2021-06-11 20:38:05
OK... we have a single equation, and 3 variables! That does not seem like enough information.

jonjoseph 2021-06-11 20:38:09
Wait, there's some extra information we have besides this equation. What else do we know about B, J and N?

jonjoseph 2021-06-11 20:39:05
We know that they are positive integers, with $N>1$. That seems like it might be useful! What can you tell me about N from the above equation?

jonjoseph 2021-06-11 20:40:11
It must be an integer divisor of 253. What's the prime factorization of 253?

jonjoseph 2021-06-11 20:40:56
We have that $253=11\cdot 23$. That means that N can either be $1, 11, 23,$ or 253. We can rule two of those out immediately. Which ones?

jonjoseph 2021-06-11 20:42:17
We can rule out $N=1$ and $N=253$. We know that $N=1$ is not allowed by the conditions of the problem, and if $N=253$, we'd get $4B+5J=1$, which is impossible for positive integers B and J.

jonjoseph 2021-06-11 20:42:24
Therefore, the two possibilities are $N=11$ and $N=23$. Let's start with $N=11$. Then what must $4B+5J$ be?

jonjoseph 2021-06-11 20:42:47
It must be 23. Is there a solution to $4B+5J=23$?

jonjoseph 2021-06-11 20:43:58
Yes, there's the unique solution $B=2$ and $J=3$. That means that we've found the solution $N=11, B=2$ and $J=3$.

jonjoseph 2021-06-11 20:44:05
If this was a short answer problem, we'd have to keep going and see if $N=23$ yields a solution. (Spoiler: it doesn't.)

jonjoseph 2021-06-11 20:44:13
But this is the AMC, so this must be our solution! We need the total cost of the jam that Elmo used. What do we get?

jonjoseph 2021-06-11 20:45:23
We have N sandwiches, each using J globs of jam, and each glob costs 5 cents. That means the total cost is $5NJ$. This is equal to $5\cdot 11\cdot 3=165$ cents, otherwise known as $\$1.65$. The answer is (D).

jonjoseph 2021-06-11 20:45:39
We good? Questions?

jonjoseph 2021-06-11 20:46:21
It will never be horrible. Remember your tricks for divisibility.

jonjoseph 2021-06-11 20:46:42
RATE PROBLEMS

jonjoseph 2021-06-11 20:46:52
In certain word problems, we must work with objects and processes that are moving at a constant rate, such as a car driving

down a road or a tap filling a bathtub. In these problems, the fundamental equation that links the relevant variables is given by DISTANCE = RATE x TIME.

jonjoseph 2021-06-11 20:47:20
Mr. Earl E. Bird leaves his house for work at exactly 8:00 A.M. every morning. When he averages 40 miles per hour, he arrives at his workplace three minutes late. When he averages 60 miles per hour, he arrives three minutes early. At what average speed, in miles per hour, should Mr. Bird drive to arrive at his workplace precisely on time?

$\text{(A)}\ 45 \quad \text{(B)}\ 48 \quad \text{(C)}\ 50 \quad \text{(D)}\ 55 \quad \text{(E)}\ 58$

jonjoseph 2021-06-11 20:47:39
It's probably tempting to guess 50; he needs to go at the average of the speeds to get there on time! Why doesn't that work?

jonjoseph 2021-06-11 20:48:28
Nice. We spend more time traveling slowly.

jonjoseph 2021-06-11 20:48:40
There are a bunch of variables we might make. Let's just define D to be the distance in miles between his house and his workplace. I like that variable, since the distance is always the same for all the different speeds Mr. Bird might go.

jonjoseph 2021-06-11 20:48:51
Let's now express the given information in terms of D. We know that if he averages 60 miles per hour, he arrives three minutes early. How long does it take him to get from his house to work at this speed?

jonjoseph 2021-06-11 20:49:39
Yes, it takes him $\frac{D}{60}$ hours. We also know that if he averages 40 miles per hour, he arrives three minutes late. How long does the trip take him at this speed?

jonjoseph 2021-06-11 20:50:03
That's right, it takes $\frac{D}{40}$ hours.

jonjoseph 2021-06-11 20:50:04
Wait a minute. We know that one of those times is 6 minutes longer than the other! That means that we can write an equation.

jonjoseph 2021-06-11 20:50:09
However, we have to be careful. What's wrong with the equation below?

$$\frac{D}{60} + 6 = \frac{D}{40}.$$

jonjoseph 2021-06-11 20:50:18
Hint: this is subtle.

jonjoseph 2021-06-11 20:50:45
Hint: What are the units of $D/60$?

jonjoseph 2021-06-11 20:51:16
HInt: What are the units of the 6?

jonjoseph 2021-06-11 20:51:44
Good we need to fix that.

jonjoseph 2021-06-11 20:51:54
What is 6 minutes in hours/

jonjoseph 2021-06-11 20:52:35
Yes, it's $\frac{1}{10}$th of an hour. That means our equation becomes

$$\frac{D}{60}+\frac{1}{10}=\frac{D}{40}.$$

So what's D?

jonjoseph 2021-06-11 20:53:09
Multiplying both sides by 120 gives $2D+12=3D$, and therefore $D=12$ miles.

jonjoseph 2021-06-11 20:53:13
We're almost done! All we need to know now is how long his trip ought to be for him to be precisely on time.

jonjoseph 2021-06-11 20:53:16
We know that he's 3 minutes late going at 40 miles an hour. That means the trip at this speed is 3 minutes too long.

jonjoseph 2021-06-11 20:53:21
How long does Mr. Earl E. Bird take at this speed? Tell me whether your answer is in minutes or in hours!

jonjoseph 2021-06-11 20:54:25
He takes $\frac{12\text{ miles}}{40\text{ mph}}=\frac{3}{10}$ hours, which is 18 minutes. So how long would the optimal trip take?

jonjoseph 2021-06-11 20:54:44
It would take 15 minutes, or $\frac{1}{4}$ of an hour. We know that he has to go 12 miles, so what is the optimal speed?

jonjoseph 2021-06-11 20:55:19
It's equal to $\frac{12}{1/4}=48$ miles per hour. The answer is (B).

jonjoseph 2021-06-11 20:55:35
Why is the answer a little bit closer to the original 40 mph?

jonjoseph 2021-06-11 20:56:22
Exactly correct. Last problem:

jonjoseph 2021-06-11 20:56:34
Doug can paint a room in 5 hours. Dave can paint the same room in 7 hours. Doug and Dave paint the room together and take a one-hour break for lunch. Let t be the total time, in hours, required for them to complete the job working together, including lunch. Which of the following equations is satisfied by t?

$(\mathrm{A})\ \left(\frac{1}{5}+\frac{1}{7}\right)(t+1)=1 \quad (\mathrm{B})\ \left(\frac{1}{5}+\frac{1}{7}\right)t+1=1$

$(\mathrm{C})\ \left(\frac{1}{5}+\frac{1}{7}\right)t=1 \quad (\mathrm{D})\ \left(\frac{1}{5}+\frac{1}{7}\right)(t-1)=1$

$(\mathrm{E})\ (5+7)t=1$

jonjoseph 2021-06-11 20:57:01
Rather than solving for t, as you might expect, we're asked to find an equation that t satisfies.

jonjoseph 2021-06-11 20:57:12
So to find the answer, let's try writing down a work-rate equation. Since we see both a 5 and a 7 in the answers, we could guess that we want to write down the work rate equation for both Doug and Dave working together.

jonjoseph 2021-06-11 20:57:18
To calculate the combined work rate of Doug and Dave together, let's first ask what is Doug's work rate. How much of a room does Doug paint in an hour?

jonjoseph 2021-06-11 20:58:04

Doug can paint a room at the rate of $\dfrac{1}{5}$ rooms per hour.

jonjoseph 2021-06-11 20:58:15
(Notice the units - it's a rate!)

jonjoseph 2021-06-11 20:58:21
Dave can paint a room at the rate of $\dfrac{1}{7}$ rooms per hour.

jonjoseph 2021-06-11 20:58:27
Now let's figure out how quickly can Doug and Dave paint a room when working together. Note that given the answer choices, it's probably best not to combine the fractions! What do you get?

jonjoseph 2021-06-11 20:59:19
When working together, Doug and Dave can paint a room at the rate of $\left(\dfrac{1}{5}+\dfrac{1}{7}\right)$ rooms per hour.

jonjoseph 2021-06-11 20:59:26
(Again, a rate)

jonjoseph 2021-06-11 20:59:29
That number appears in four of the answer choices, so we're probably on the right track! (And E is probably not the correct answer.)

jonjoseph 2021-06-11 20:59:35
Remembering that $\text{WORK RATE} \times \text{TIME} = \text{OUTPUT}$, what is the output in all of the answer choices?

jonjoseph 2021-06-11 20:59:52
The OUTPUT is 1, or 1 room. So the TIME we're looking for is the time it takes to paint 1 room.

jonjoseph 2021-06-11 20:59:56
How long do Doug and Dave spend (*actually*) painting? (Don't forget that they take a lunch break!)

jonjoseph 2021-06-11 21:00:49
They spent one hour for lunch and t hours on the job in total, so they spent $t-1$ hours actually painting.

jonjoseph 2021-06-11 21:00:50
So what is the correct answer?

jonjoseph 2021-06-11 21:01:22
Our equation becomes

$$\text{WORK RATE} \times \text{TIME} = \left(\frac{1}{5}+\frac{1}{7}\right)(t-1) = 1.$$

The answer is (D).

jonjoseph 2021-06-11 21:01:28
Nicely done.

jonjoseph 2021-06-11 21:01:42
That's a wrap for today. Great work.

jonjoseph 2021-06-11 21:01:55
SUMMARY

jonjoseph 2021-06-11 21:01:56
When solving a word problem, the first step is to convert the information in the problem to mathematical form, such as a system of equations. Then you can apply the techniques you have learned to solve the equations.

jonjoseph 2021-06-11 21:02:00
Usually, which variables you want to define will be clear, but when it is not, it may help to think about what the problem is

asking for, and what will keep things simple. And when solving a rate problem, remember the fundamental equation $\text{DISTANCE} = \text{RATE} \times \text{TIME}.$

jonjoseph 2021-06-11 21:02:26

See you next week. Tear it up this week! Stay safe.

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