UCLA Math Circle

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Abstract

Notes on homework problems from the UCLA Math Circle Intermediate-2 for Summer Session 2020, July 26th.

1. Show by Induction that

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n \qquad \forall n \geq 1$$

Recall the following definition of the Fibonacci numbers:

$$\begin{aligned} &F(0) = 0 \\ &F(1) = 1 \\ &F(n) = F(n-1) + F(n-2) \qquad \forall n \geq 2 \end{aligned}$$

Let $P(n)$ denote the equality for *some* fixed value $n \in \mathbb{N}$. We have:

$$P(n+1): \quad F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}$$

<u>Base Cases:</u>

$$\begin{aligned} P(1): &\quad F_0F_2 - F_1^2 = 0 \cdot 1 - 1^2 = (-1)^1 = -1 \qquad \checkmark \\ P(2): &\quad F_1F_3 - F_2^2 = 1 \cdot 2 - 1^2 = (-1)^2 = +1 \qquad \checkmark \end{aligned}$$

<u>Induction Step:</u> As usual, we start with the left-hand side of $P(n+1)$ and aim to equate it to the right-hand side after substituting $P(n)$, perhaps also $P(n-1)$, the definition of Fibonacci numbers, and inspired manipulations. With the right inspiration, you can get from the left-hand side to the right-hand side in about three steps, but off the direct path it can take longer. Below is the derivation followed by details and comments.

$$\begin{aligned} \texttt{lhs} &= F_nF_{n+2} - F_{n+1}^2 \\ &= F_n(F_{n+1} + F_n) - F_{n+1}^2 \\ &= F_nF_{n+1} + F_n^2 - F_{n+1}^2 \\ &= F_{n+1}(F_n - F_{n+1}) + F_n^2 \\ &= -F_{n-1}F_{n+1} + F_n^2 \\ &= -(-1)^n \\ &= (-1)^{n+1} \\ &= \texttt{rhs} \qquad \square \end{aligned}$$

Comments

The same derivation with comments:

$$\begin{aligned} \texttt{lhs} &= F_n \times F_{n+2} - F_{n+1}^2 \\ &= F_n \times \overbrace{(F_{n+1} + F_n)} - F_{n+1}^2 && \text{by definition of } F_{n+2} \\ &= F_n \times F_{n+1} + F_n^2 - F_{n+1}^2 \\ &= \underbrace{(F_n - F_{n+1})} \times F_{n+1} + F_n^2 \\ &= -F_{n-1} \times F_{n+1} + F_n^2 && \text{by definition of } F_{n+1} \\ &= -\underbrace{(F_{n-1}F_{n+1} - F_n^2)} \\ &= -(-1)^n && \text{by proposition } P(n) \\ &= (-1)^1 \times (-1)^n \\ &= (-1)^{n+1} \\ &= \texttt{rhs} \quad \square \end{aligned}$$

The derivation above is easy enough to follow, much less to discover. The first insight in the induction step is to note that the `lhs` of $P(n+1)$ contains the "future" term F_{n+2}, which appears neither on the `rhs` of $P(n+1)$ nor in $P(n)$, and is therefore a strong candidate for a substitution. The second insight is to note that, after the substitution, the term F_n^2, present in $P(n)$, has appeared, while the term F_{n+1} may be factorized. The third insight is to note that the left-hand side of $P(n)$ has almost appeared, except for the term multiplying F_{n+1}, which it turns out can be simplified by using the definition of the Fibonacci number F_{n+1}:

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ \implies -F_{n-1} &= F_n - F_{n+1} \end{aligned}$$

If you hadn't noticed that F_{n+1} could be factorized, you would have been stuck here:

$$F_n F_{n+1} + F_n^2 - F_{n+1}^2$$

How to get unstuck? In order to use $P(n)$ in our proof, we seek to replace the product $F_n F_{n+1}$ by $F_{n-1}F_{n+1}$. This suggests using a definition of the Fibonacci numbers to write F_n in terms of F_{n-1}. A natural temptation here is to use:

$$F_n = F_{n-1} + F_{n-2}$$

which gives $F_{n-1}F_{n+1} + F_n^2 +$ etc.. The problem with this is that the sign is wrong. What we would have needed instead is $-F_{n-1}F_{n+1} + F_n^2 +$ etc.. This definition of the Fibonacci numbers delivers:

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ \implies F_n &= F_{n+1} - F_{n-1} \end{aligned}$$

so that:

$$\begin{aligned} & F_n \quad F_{n+1} + F_n^2 - F_{n+1}^2 \\ = \; & \overbrace{(F_{n+1} - F_{n-1})}^{F_n} \quad F_{n+1} + F_n^2 - F_{n+1}^2 \\ = \; & - F_{n-1}F_{n+1} + F_n^2 \end{aligned}$$

as before.

The key takeaway of this exercise is that we typically need to make $P(n)$ appear out of the left-hand side of $P(n+1)$. In simpler problems involving sums, we usually simply truncate the sum at the nth term and substitute it with the right-hand side of $P(n)$. In this problem, we need to use definitions of the Fibonacci numbers and some factorization to make the left-hand side of $P(n)$ appear.

2. Show by Induction that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\dots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n} \qquad \forall n \geq 2$$

Let $P(n)$ denote the equality for *some* fixed value $n \in \mathbb{N}$. We have:

$$P(n+1): \quad \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\dots\left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{(n+1)^2}\right)=\frac{n+2}{2(n+1)}$$

Base Case:

$$P(2): \quad \left(1-\frac{1}{2^2}\right)=\frac{2+1}{2\cdot 2}=\frac{3}{4} \qquad \checkmark$$

Induction Step: As usual, we start with the left-hand side of $P(n+1)$ and aim to equate it to the right-hand side after substituting $P(n)$ and some manipulations.

$$\begin{aligned}
\texttt{lhs} &= \underbrace{\left(1-\frac{1}{2^2}\right)\dots\left(1-\frac{1}{n^2}\right)}\left(1-\frac{1}{(n+1)^2}\right)\\
&= \left(\frac{n+1}{2n}\right)\left(1-\frac{1}{(n+1)^2}\right)\\
&= \frac{n+1}{2n}\cdot\frac{(n+1)^2-1}{(n+1)^2}\\
&= \frac{n+1}{2n}\cdot\frac{(n+1+1)(n+1-1)}{(n+1)^2}\\
&= \frac{n+1}{2n}\cdot\frac{(n+2)n}{(n+1)^2}\\
&= \frac{n+2}{2(n+1)}\\
&= \texttt{rhs} \qquad \square
\end{aligned}$$

Conclusion: $P(n)$ implies $P(n+1)$ and $P(1)$ is true, so $P(n)$ true for all $n \geq 2$.