Art Of Problem Solving - AMC 10
Week 12

Patrick & James Toche

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Abstract

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1.

A parabola with equation $y = x^2 + bx + c$ passes through the points $(2, 3)$ and $(4, 3)$. What is c?

(A) 2	(B) 5	(C) 7	(D) 10	(E) 11

Substituting the points $(2, 3)$ and $(4, 3)$ into $y = x^2 + bx + c$, we obtain the system of equations

$$\begin{aligned} 4 + 2b + c &= 3 \\ 16 + 4b + c &= 3 \end{aligned}$$

These equations simplify to

$$\begin{aligned} 2b + c &= -1 \\ 4b + c &= -13 \end{aligned}$$

Multiplying the first equation by 2, we get $4b+2c = -2$. Subtracting the equation $4b+c = -13$, we get $c = 11$.

$$\boxed{c = 11}$$

2.

If a, $b > 0$ and the triangle in the first quadrant bounded by the coordinate axes and the graph of $ax + by = 6$ has area 6, then $ab =$

(A) 3	(B) 6	(C) 12	(D) 108	(E) 432

Setting $y = 0$ we have that the $x-$intercept of the line is $x = 6/a$. Similarly setting $x = 0$ we find the $y-$intercept to be $y = 6/b$. Then

$$\frac{1}{2} \cdot \frac{6}{a} \cdot \frac{6}{b} = \frac{18}{ab} = 6 \implies ab = 3$$

$$\boxed{ab = 3}$$

3.

The lines $x = \frac{1}{4}y + a$ and $y = \frac{1}{4}x + b$ intersect at the point $(1, 2)$. What is $a + b$?

(A) 0 (B) $\frac{3}{4}$ (C) 1 (D) 2 (E) $\frac{9}{4}$

$$\begin{cases} x = \frac{1}{4}y + a \\ y = \frac{1}{4}x + b \end{cases}$$

Add both equations and rearrange:

$$x + y = \frac{1}{4}(x + y) + a + b \implies \frac{3}{4}(x + y) = a + b$$

Substitute the intersection point $(1, 2)$:

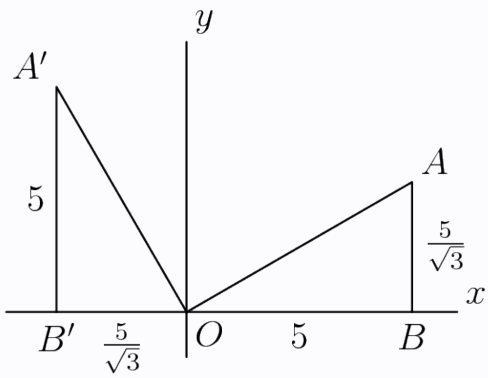
$$a + b = \frac{3}{4} \cdot (1 + 2) = \frac{9}{4}$$

$$\boxed{a + b = \frac{9}{4}}$$

4.

Triangle OAB has $O = (0,0)$, $B = (5,0)$, and A in the first quadrant. In addition, $\angle ABO = 90°$ and $\angle AOB = 30°$. Suppose that $\overline{OA}$ is rotated $90°$ counterclockwise about O. What are the coordinates of the image of A?

(A) $\left(-\frac{10}{3}\sqrt{3}, 5\right)$ (B) $\left(-\frac{5}{3}\sqrt{3}, 5\right)$ (C) $\left(\sqrt{3}, 5\right)$ (D) $\left(\frac{5}{3}\sqrt{3}, 5\right)$ (E) $\left(\frac{10}{3}\sqrt{3}, 5\right)$



Triangle $\triangle ABO$ is a special 30-60-90 triangle, with $\angle ABO = 90°$, and $\angle AOB = 30°$. Since B has coordinates $(5,0)$, we have $OB = 5$. The triangle's proportions imply

$$\frac{5}{\sqrt{3}} = \frac{AB}{1} \implies AB = \frac{5\sqrt{3}}{3}$$

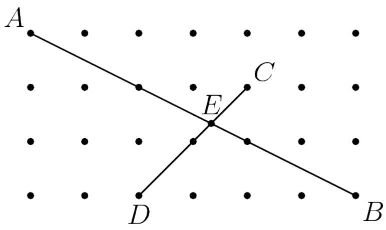
A has coordinates $\left(5, \frac{5\sqrt{3}}{3}\right)$.

Rotating triangle $\triangle ABO$ by $90°$ counterclockwise around O takes A to:

$$\boxed{\left(-\frac{5\sqrt{3}}{3}, 5\right)}$$

5.

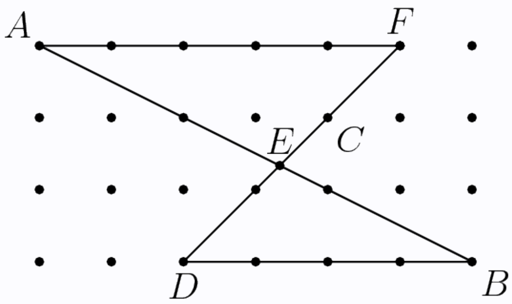
The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE.



(A) $4\sqrt{5}/3$ (B) $5\sqrt{5}/3$ (C) $12\sqrt{5}/7$ (D) $2\sqrt{5}$ (E) $5\sqrt{65}/9$

Solution 1

Let CD and the line through A parallel to BD intersect at F.

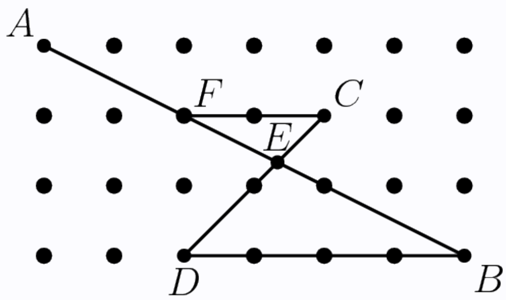


Then triangles AEF and BED are similar, so

$$\frac{AE}{BE} = \frac{AF}{BD} \implies \frac{AE}{AE+BE} = \frac{AF}{AF+BD}$$
$$\implies AE + BE = AB = \sqrt{6^2+3^2} = \sqrt{45} = 3\sqrt{5}$$
$$\implies AE = AB \cdot \frac{AF}{AF+BD} = 3\sqrt{5} \cdot \frac{5}{5+4} = \frac{5\sqrt{5}}{3}$$

$$\boxed{AE = \frac{5\sqrt{5}}{3}}$$

Solution 2



Draw segment BD and parallel line CF. Since triangles $\triangle FCE$ and $\triangle BDE$ are similar,

$$\frac{FE}{EB} = \frac{FC}{DB} = \frac{2}{4} = \frac{1}{2} \implies \frac{EB + FE}{FE} = 2 + 1 \implies FE = \frac{1}{3}FB$$

By construction, $FD = 2$. Applying the Pythagorean Theorem to $\triangle BDF$,

$$FB = \sqrt{2^2 + 4^2} = 2\sqrt{5} \implies FE = \frac{1}{3}FB = \frac{2\sqrt{5}}{3}$$

Applying the Pythagorean Theorem,

$$AF = \sqrt{1^2 + 2^2} = \sqrt{5}$$

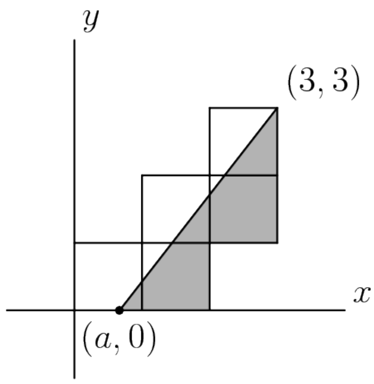
Adding it up,

$$AE = AF + FE = \sqrt{5} + \frac{2\sqrt{5}}{3} = \frac{5\sqrt{5}}{3}$$

$$\boxed{AE = \frac{5\sqrt{5}}{3}}$$

6.

Five unit squares are arranged in the coordinate plane as shown, with the lower left corner at the origin. The slanted line, extending from $(a,0)$ to $(3,3)$, divides the entire region into two regions of equal area. What is a?



(A) $\frac{1}{2}$	(B) $\frac{3}{5}$	(C) $\frac{2}{3}$	(D) $\frac{3}{4}$	(E) $\frac{4}{5}$

If one unit square is added to the bottom-right corner, the shaded area has the shape of a triangle with base length $3-a$, height 3, and therefore area $(9-3a)/2$.

If three unit squares are added to the top-left corner, the unshaded area has the shape of a trapezoid with area $(9+3a)/2$.

The shaded and unshaded areas are equal:

$$\frac{9-3a}{2} - 1 = \frac{9+3a}{2} - 3 \implies a = \frac{2}{3}$$

$$\boxed{a = \frac{2}{3}}$$

7.

In rectangle $ABCD$, we have $A = (6, -22)$, $B = (2006, 178)$, and $D = (8, y)$, for some integer y. What is the area of rectangle $ABCD$?

(A) 4000	(B) 4040	(C) 4400	(D) 40,000	(E) 40,400

Let the slope of AB be m_1 and the slope of AD be m_2.

$$m_1 = \frac{178 - (-22)}{2006 - 6} = \frac{1}{10}$$

$$m_2 = \frac{y - (-22)}{8 - 6} = \frac{y + 22}{2}$$

Since AB and AD form a right angle:

$$\begin{aligned} m_2 &= -\frac{1}{m_1} \\ m_2 &= -10 \\ \frac{y+22}{2} &= -10 \\ y &= -42 \end{aligned}$$

Using the distance formula:

$$\begin{aligned} AB &= \sqrt{(2006-6)^2 + (178 - (-22))^2} \\ &= \sqrt{(2000)^2 + (200)^2} \\ &= 200\sqrt{101} \\ AD &= \sqrt{(8-6)^2 + (-42 - (-22))^2} \\ &= \sqrt{(2)^2 + (-20)^2} \\ &= 2\sqrt{101} \end{aligned}$$

Therefore the area of rectangle $ABCD$ is

$$200\sqrt{101} \cdot 2\sqrt{101} = 40,400$$

$$\boxed{\text{area } = 40,400}$$

8.

If (a, b) and (c, d) are two points on the line whose equation is $y = mx + k$, then the distance between (a, b) and (c, d), in terms of a, c, and m, is

(A) $\lvert a-c\rvert\sqrt{1+m^2}$	(B) $\lvert a+c\rvert\sqrt{1+m^2}$	(C) $\dfrac{\lvert a-c\rvert}{\sqrt{1+m^2}}$	(D) $\lvert a-c\rvert(1+m^2)$	(E) $\lvert a-c\rvert\lvert m\rvert$

Notice that, since (a, b) is on $y = mx + k$, we have $b = am + k$. Similarly, $d = cm + k$. Using the distance formula, the distance between the points (a, b) and (c, d) is

$$\begin{aligned}\sqrt{(a-c)^2 + (b-d)^2} &= \sqrt{(a-c)^2 + [(am+k) - (cm+k)]^2} \\ &= \sqrt{(a-c)^2 + m^2(a-c)^2} \\ &= |a-c|\sqrt{1+m^2}\end{aligned}$$

$$\boxed{|a-c|\sqrt{1+m^2}}$$

9.

A lattice point is a point in the plane with integer coordinates. How many lattice points are on the line segment whose endpoints are $(3, 17)$ and $(48, 281)$? (Include both endpoints of the segment in your count.)

(A) 2	(B) 4	(C) 6	(D) 16	(E) 46

The difference in the y-coordinates is $281 - 17 = 264$, and the difference in the x-coordinates is $48 - 3 = 45$. The gcd of 264 and 45 is 3, so the line segment joining $(3, 17)$ and $(48, 281)$ has slope $\frac{88}{15}$. The points on the line have coordinates

$$\left(3 + t,\ 17 + \frac{88}{15}t\right)$$

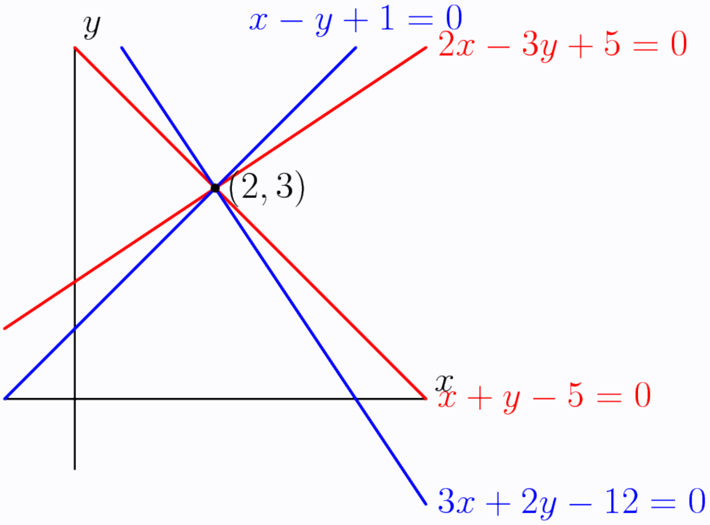
If t is an integer, the y-coordinate of this point is an integer if and only if t is a multiple of 15. The points where t is a multiple of 15 on the segment $3 \leq x \leq 48$ are 3, $3 + 15$, $3 + 30$, and $3 + 45$. There are 4 lattice points on this line.

$$\boxed{\text{4 lattice points}}$$

10.

The number of distinct points in the xy-plane common to the graphs of $(x+y-5)(2x-3y+5)=0$ and $(x-y+1)(3x+2y-12)=0$ is

(A) 0	(B) 1	(C) 2	(D) 3	(E) 4



The graph $(x+y-5)(2x-3y+5)=0$ is the combined graphs of $x+y-5=0$ and $2x-3y+5=0$. Likewise, the graph $(x-y+1)(3x+2y-12)=0$ is the combined graphs of $x-y+1=0$ and $3x+2y-12=0$. All these lines intersect at one point, $(2,3)$. Therefore, the answer is 1.

$\boxed{\text{1 distinct point}}$