Stanford University Mathematics Camp (SUMaC) 2025

Admission Exam

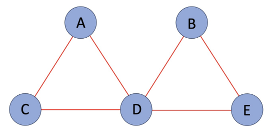
For use by SUMaC 2025 applicants only. Not for distribution.

- Solve as many of the following problems as you can. Your work on these problems together with your grades in school, teacher recommendations, and answers to the questions on the application form are all used to evaluate your SUMaC application. Although SUMaC is very selective with a competitive applicant pool, correct answers on every problem are not required for admission.
- There is no time limit for this exam other than the application deadline.
- Please include clear, detailed explanations for all of your solutions; numerical answers or formulas with no explanation are not useful for evaluating your application.
- In the event you are unable to solve a problem completely, you are encouraged to write up any partial progress that you feel captures your ideas leading toward a solution.
- You will need to create a separate document with your solutions and explanations. This document may be typed or handwritten, as long as the final document you upload is legible for our review.
- None of these problems require a calculator or computer, and they are all designed so that they can be done without computational tools.
- You are expected to do your own work without the use of any outside source (books, teacher or parent help, internet search, etc). If you recognize one of the problems from another source, or if you receive any assistance, please indicate this in your write up.
- **Please do not share these problems or your solutions with anyone.**

1. Find all positive integers x and y such that $x^2 - y^2 = 2025$.

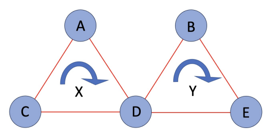
2. Form a set of positive integers S by seeding S with two positive integers a and b such that $a \neq b$. Then if x and y are in S where $x \neq y$, and g is the greatest common divisor of x and y, include z in S where
$$z = \frac{x}{g} + \frac{y}{g}.$$
For example if we start with $a = 6$ and $b = 10$, then 8 is also in S since $\gcd(6, 10) = 2$, where $\gcd(6, 10)$ is the greatest common divisor of 6 and 10, and
$$8 = 3 + 5 = \frac{6}{2} + \frac{10}{2}.$$
   (a) Show that with seeds 6 and 10, the set S is infinite.
   (b) Is it possible to seed S with distinct positive integers a and b such that S is a finite set?

3. Consider the following puzzle. Five coins labeled A, B, C, D, and E are arranged around two triangles as follows:

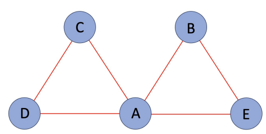


There are two allowable moves that can be applied repeatedly in any order:

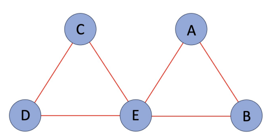
- Move X: you can rotate the three coins on the triangle on the left 120° clockwise around the center of the triangle on the left.
- Move Y: you can rotate the three coins on the triangle on the right 120° clockwise around the center of the triangle on the right.



For example, one application of move X from the starting position results in

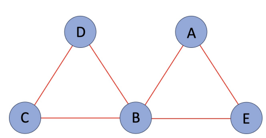


Then one application of move Y to the above results in

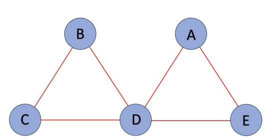


For each of the following, show that the given configuration can be obtained with a combination of moves X and Y, or prove no combination of moves X and Y lead to the given configuration:

(a)



(b)



4. Every positive integer N can be represented in base 2 by $(N)_2 = b_n b_{n-1} \dots b_1 b_0$, where

$$N = b_0 + b_1 \cdot 2 + \cdots + b_{n-1} 2^{n-1} + b_n 2^n$$

such that for each $i = 0, \dots, n$, $b_i = 0$ or $b_i = 1$, and $b_n \neq 0$. In this problem, we consider writing numbers in an alternative form of base 2 where $(N)_b^{\text{alt}} = b_n b_{n-1} \dots b_1 b_0$ if

$$N = b_0 + b_1 \cdot (2^0 + 1) + \cdots + b_{n-1}(2^{n-2} + 1) + b_n \cdot (2^{n-1} + 1)$$

such that for each $i = 0, \dots, n$, $b_i = 0$ or $b_i = 1$ and $b_n \neq 0$. Noting

- $2^0 + 1 = 2$
- $2^1 + 1 = 3$
- $2^2 + 1 = 5$
- $2^3 + 1 = 9$
- $2^4 + 1 = 17$

we get

$$(12)_2^{\text{alt}} = 10100 \text{ since } 12 = 9 + 3 = 1 \cdot 9 + 0 \cdot 5 + 1 \cdot 3 + 0 \cdot 2 + 0 \cdot 1,$$

and

$$(29)_2^{\text{alt}} = 110100 \text{ since } 29 = 17 + 9 + 3 = 1 \cdot 17 + 1 \cdot 9 + 0 \cdot 5 + 1 \cdot 3 + 0 \cdot 2 + 0 \cdot 1.$$

(a) Show that for any positive integer N there are $b_0, \dots, b_n$ such that $(N)_2^{\text{alt}} = b_n b_{n-1} \dots b_1 b_0$.

(b) Using this definition, 3 has two representations in alternative base 2:

$$(3)_2^{\text{alt}} = 11 \text{ and } (3)_2^{\text{alt}} = 100$$

However, some numbers, such as 4 and 7, have unique representations in alternative base 2. For example $(4)_2^{\text{alt}} = 101$ and $(7)_2^{\text{alt}} = 1010$, and neither of these numbers have a second representation in alternative base 2. Show that there are infinitely many positive integers that have a unique representation in alternative base 2, and there are infinitely many numbers that have at least two representations in alternative base 2.

5. A sequence of k distinct positive integers between 1 and n is a (k, n)-**tour** if the sequence begins with n ends with $n - 1$, and every number in the sequence is the difference of a pair of numbers that occur earlier in the sequence. More precisely, $(a_1, a_2, \dots, a_k)$ is a (k, n)-tour if

- $1 \leq a_i \leq n$ for $i = 1, \dots, k$.
- $a_1 = n$, and $a_k = n - 1$,
- $a_i \neq a_j$ whenever $i \neq j$,
- For $\ell \geq 3$, $a_\ell = a_i - a_j$ for some $i, j < \ell$ such that $i \neq j$.

For example, $(10, 1, 9)$ is a $(3, 10)$ tour, $(10, 3, 7, 4, 1, 9)$ is a $(6, 10)$-tour, and $(10, 7, 3, 4, 6, 2, 8, 1, 5, 9)$ is a $(10, 10)$-tour. Notice that the last example is an (k, n)-tour where $n = k$, and any (n, n)-tour must include all of the integers from 1 up to n.

(a) How many $(10, 10)$-tours can you find?

(b) How many $(16, 16)$-tours can you find?

(c) For what n does there exist an (n, n)-tour, and when one exists, what are the restrictions on a_2? (For example, a_2 cannot be even in any $(10, 10)$-tour of $\{1, 2, \dots, 10\}$.)

6. The pairs $(1, k_1), (2, k_2), \ldots (n, k_n)$ form a **one-lump snake** if they satisfy the following conditions.

   - $\{k_1, \ldots, k_n\}$ are exactly the integers from 1 through n.
   - **One lump condition:** The sequence $k_1, \ldots, k_n$ increases, and then decreases, without increasing again; i.e., for some i,

   $$k_1 < k_2 < \cdots < k_{i-1} < k_i \text{ and } k_i > k_{i+1} > \cdots > k_n.$$

   - **Snake condition:** The pairs $(1, k_1), (2, k_2), \ldots (n, k_n)$ can be rearranged in the following way:

   $$(a_1, a_2), (a_2, a_3), \ldots, (a_{n-1}, a_n), (a_n, a_1)$$

   For example, for $n = 3$, the sequence of pairs $(1, 2), (2, 3), (3, 1)$ satisfy the conditions of being a one-lump snake. And for $n = 4$ both $(1, 2), (2, 3), (3, 4), (4, 1)$ and $(1, 3), (2, 4), (3, 2), (4, 1)$ satisfy the conditions of being a one-lump snake (in the latter example, the pairs can be rearranged $(1, 3), (3, 2), (2, 4), (4, 1)$ to see they form a snake).

   (a) Find all one-lump snakes for $n = 4, 5, 6, 7$.

   (b) Find an upper bound to the number of one-lump snakes of length n. In particular, find a function f such that there are less than or equal to $f(n)$ one-lump snakes of length n. This problem is not asking for an exact answer; instead you are asked to find the best bound f that you can, and then justify your bound.