Art Of Problem Solving - AMC 10
Week 6

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Abstract

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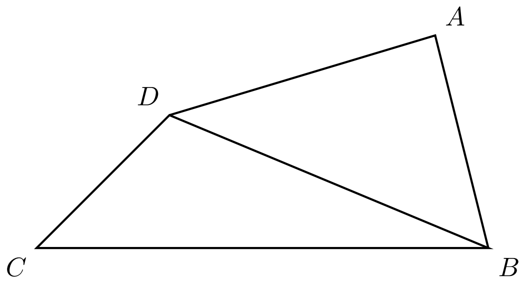
1.

In quadrilateral $ABCD$, $AB = 5$, $BC = 17$, $CD = 5$, $DA = 9$, and BD is an integer. What is BD?



(A) 11 (B) 12 (C) 13 (D) 14 (E) 15

By the triangle inequality,

$$\begin{aligned} BD &< DA + AB \\ BD + DC &> BC \end{aligned}$$

And therefore

$$\begin{aligned} BC - DC &< BD < DA + AB \\ 12 = 17 - 5 &< BD < 9 + 5 = 14 \end{aligned}$$

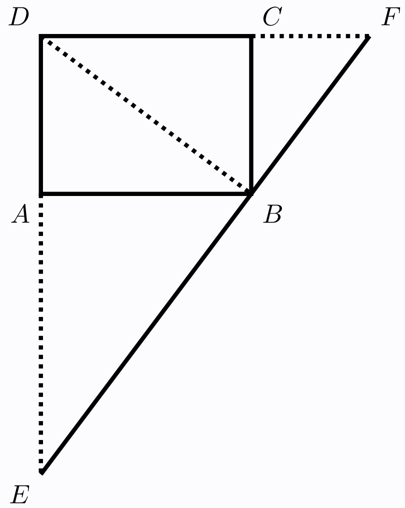
The answer is therefore

$$\boxed{13}$$

2.

Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment EF is constructed through B so that $EF \perp DB$, and A and C lie on DE and DF, respectively. What is EF?

(A) 9 (B) 10 (C) $\frac{125}{12}$ (D) $\frac{103}{9}$ (E) 12



Solution 1

Let D be the origin $(0, 0)$ of a Cartesian system. The other points have the following coordinates: $A : (0, -3)$, $B : (4, -3)$, $C : (3, 0)$. Line DB has a zero-intercept and equation

$$y = -\frac{3x}{4}$$

Since EF is orthogonal to DB, it has slope $4/3$ ("minus the inverse"). And, since it goes through point $B(4, -3)$, EF has intercept $-3 - 16/3$, and equation:

$$y = -\frac{25}{3} + \frac{4x}{3}$$

Point E is on the y axis, with coordinates $\left(0, -\frac{25}{3}\right)$. Point F is on the x axis, with coordinates $\left(\frac{25}{4}, 0\right)$. The length of segment EF is then

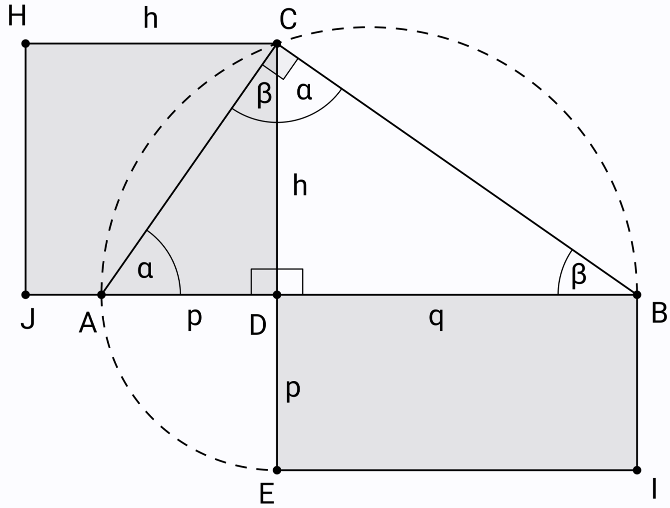
$$\left(\frac{25}{4}\right)^2 + \left(\frac{25}{3}\right)^2 = \frac{125}{12}$$

Solution 2

The *right-triangle altitude theorem* also known as the *geometric mean theorem* states that the altitude h is related to the p and q segments on the hypotenuse by

$$h = \sqrt{pq}$$

In the figure, the area of the square, h^2, is equal to the area of the rectangle, pq.



Here $h = DB$, $p = BF$, $q = BE$ and thus

$$DB^2 = BF \times BE$$

Triangles EBD and DCB are similar, implying

$$\frac{BA}{BC} = \frac{BE}{BD} \quad \Rightarrow \quad BE = \frac{BD \times BA}{BC} = \frac{5 \times 4}{3} = \frac{20}{3}$$

where $BD = \sqrt{3^2 + 4^2} = 5$ follows from the well-known Pythagorean triple. Now BF follows from the geometric mean theorem,

$$BF = \frac{DB^2}{BE} = 5^2 \times \frac{3}{20} = \frac{15}{4}$$

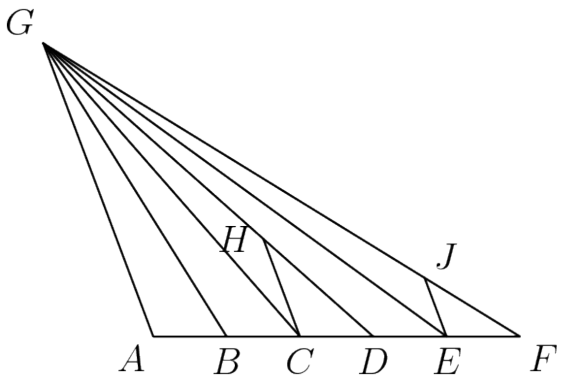
And lastly,

$$EF = BE + BF = \frac{20}{3} + \frac{15}{4} = \frac{80 + 45}{12} = \frac{125}{12}$$

$$\boxed{\frac{125}{12}}$$

3.

Points A, B, C, D, E, and F lie, in that order, on AF, dividing it into five segments, each of length 1. Point G is not on line AF. Point H lies on GD, and point J lies on GF. The line segments HC, JE, and AG are parallel. Find HC/JE.



(A) 5/4	(B) 4/3	(C) 3/2	(D) 5/3	(E) 2

Since AG and CH are parallel, triangles $\triangle GAD$ and $\triangle HCD$ are similar, implying

$$\frac{CH}{AG} = \frac{CD}{AD} = \frac{1}{3}$$

Since AG and JE are parallel, triangles $\triangle GAF$ and $\triangle JEF$ are similar, implying

$$\frac{EJ}{AG} = \frac{EF}{AF} = \frac{1}{5}$$

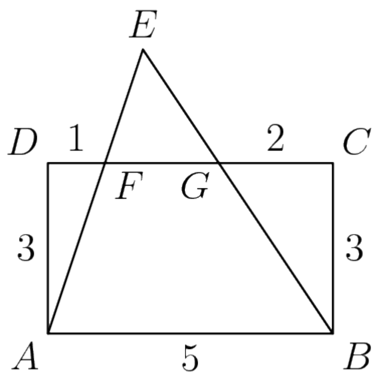
Putting it together,

$$\frac{CH}{EJ} = \frac{CH}{AG} \times \frac{AG}{EJ} = \frac{5}{3}$$

$$\boxed{\frac{5}{3}}$$

4.

In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. Find the area of triangle AEB.



(A) 10	(B) $\frac{21}{2}$	(C) 12	(D) $\frac{25}{2}$	(E) 15

Solution 1

Since FG and AB are parallel, triangles $\triangle EFG$ and $\triangle EAB$ are similar, implying

$$\frac{\triangle EFG}{\triangle EAB} = \frac{FG}{AB} = \frac{2}{5}$$

Let h denote the height of $\triangle AEB$. Since h is perpendicular to FG and AB, we have

$$\frac{h-3}{h} = \frac{2}{5} \quad \Rightarrow \quad 2h = 5h - 15 \quad \Rightarrow \quad h = 5$$

The height is 5 so the area of $\triangle EAB$ is

$$\frac{1}{2} \times 5 \times 5 = \frac{25}{2}$$

Solution 2

Let A be the origin $(0,0)$ of a Cartesian system. Segments EA and EB have equation

$$y = 3x$$
$$y = \frac{15}{2} - \frac{3}{2}x$$

The x coordinate of point E solves the system:

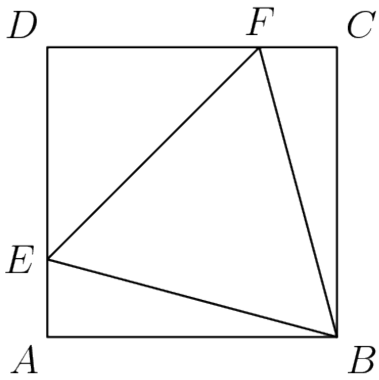
$$y = 3x = \frac{15}{2} - \frac{3}{2}x \quad \Rightarrow \quad x = \frac{5}{3}, \quad y = 5$$

Thus, the area of $\triangle EAB$ is

$$\boxed{\frac{25}{2}}$$

5.

Points E and F are located on square $ABCD$ so that triangle BEF is equilateral. What is the ratio of the area of triangle DEF to that of triangle ABE?



(A) $\frac{4}{3}$	(B) $\frac{3}{2}$	(C) $\sqrt{3}$	(D) 2	(E) $1+\sqrt{3}$

Without loss of generality, suppose the side length of square $ABCD$ is 1. Triangles $\triangle ABE$ and $\triangle CBF$ are congruent and since $BE = BF$, it follows that $CF = AE$ and $DE = DF$. Let $DE = x$. EF is the diagonal of a square of side length x, so that

$$EF = x\sqrt{2}$$

Consider now $\triangle ABE$. Its side lengths are $AB = 1$, $BE = EF = x\sqrt{2}$, $AE = 1 - x$. By the Pythagorean Theorem,

$$AE^2 + AB^2 = BE^2$$
$$(1-x)^2 + 1 = 2x^2$$

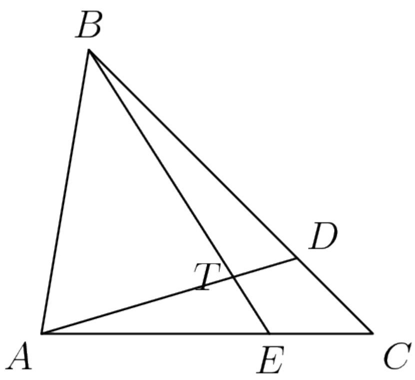
This gives $x^2 = 2(1-x)$ and the implied ratio

$$\frac{[DEF]}{[ABE]} = \frac{\frac{x^2}{2}}{\frac{1-x}{2}} = \frac{x^2}{1-x} = 2$$

$$\boxed{\frac{[DEF]}{[ABE]} = 2}$$

6.

In triangle ABC points D and E lie on BC and AC, respectively. If AD and BE intersect at T so that $AT/DT = 3$ and $BT/ET = 4$, what is CD/BD?



(A) $\frac{1}{8}$	(B) $\frac{2}{9}$	(C) $\frac{3}{10}$	(D) $\frac{4}{11}$	(E) $\frac{5}{12}$

Since triangles $\triangle ADC$ and $\triangle ADB$ have segment AD in common, the ratio of segments CD and BD is equal to the ratio of the areas:

$$\frac{CD}{BD} = \frac{[\triangle ADC]}{[\triangle ADB]}$$

These triangles are made up of several pieces. First, express $\triangle ADB$ in terms of $[\triangle BTD]$.

$$[\triangle ATB] = 3\,[\triangle BTD]$$

Since we are interested in ratios, we let $[\triangle BTD] = 1$ without loss of generality.

$$[\triangle ADB] = [\triangle ATB] + [\triangle BTD] = 4\,[\triangle BTD] = 4$$

Secondly, express $[\triangle ADC]$ in terms of $[\triangle BTD]$.

$$\begin{aligned}[\triangle ADC] &= [\triangle ATE] + [\triangle TDCE] \\ &= [\triangle ATE] + [\triangle TCD] + [\triangle TCE] \\ &= [\triangle ATE] + [\triangle TCD] + [\triangle TCE]\end{aligned}$$

$[\triangle ATE]$ is readily calculated, with more work needed for $[\triangle TCD]$ and $[\triangle TCE]$.

$$[\triangle ATE] = \frac{3}{4}\,[\triangle BTD] = \frac{3}{4}$$

Let $x = [\triangle TCD]/[\triangle BTD]$ and $y = [\triangle TCE]/[\triangle BTD]$.

$$\frac{[\triangle BTC]}{[\triangle TCE]} = \frac{1+x}{y} = 4 \quad \Rightarrow \quad x - 4y = -1$$

$$\frac{[\triangle ATC]}{[\triangle TCD]} = \frac{\frac{3}{4}+y}{x} = 3 \quad \Rightarrow \quad 12x - 4y = 3$$

Solving the system for x and y yields

$$[\triangle TCD]/[\triangle BTD] = \frac{4}{11}$$

$$[\triangle TCE]/[\triangle BTD] = \frac{15}{44}$$

Putting it together:

$$[\triangle ADC]/[\triangle BTD] = \frac{3}{4} + \frac{4}{11} + \frac{15}{44} = \frac{33+16+15}{44} = \frac{64}{44}$$

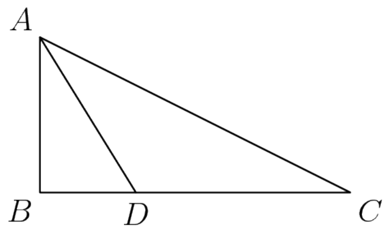
and thus,

$$\frac{CD}{BD} = \frac{64}{44} \div 4 = \frac{4}{11}$$

$$\boxed{\frac{4}{11}}$$

7.

Triangle ABC has a right angle at B, $AB = 1$, and $BC = 2$. The bisector of $\angle BAC$ meets BC at D. What is BD?



(A) $\frac{\sqrt{3}-1}{2}$	(B) $\frac{\sqrt{5}-1}{2}$	(C) $\frac{\sqrt{5}+1}{2}$	(D) $\frac{\sqrt{6}+\sqrt{2}}{2}$	(E) $2\sqrt{3}-1$

By the Pythagorean Theorem,

$$AC = \sqrt{5}$$

By the Angle Bisector Theorem,

$$\frac{BD}{AB} = \frac{DC}{AC}$$

Substituting the known lengths,

$$\frac{BD}{1} = \frac{2 - BD}{\sqrt{5}}$$

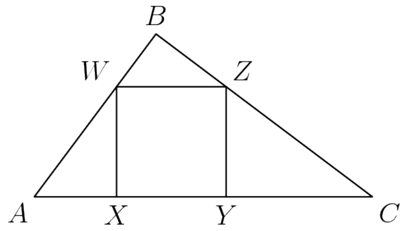
$$\left(1 + \frac{1}{\sqrt{5}}\right) BD = \frac{2}{\sqrt{5}}$$

$$BD = \frac{2}{\sqrt{5}+1} = \frac{2(\sqrt{5}-1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{\sqrt{5}-1}{2}$$

$$\boxed{\frac{\sqrt{5}-1}{2}}$$

8.

Right triangle ABC has $AB = 3$, $BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in triangle ABC with X and Y on AC, W on AB, and Z on BC. What is the side length of the square?



(A) $\frac{3}{2}$	(B) $\frac{60}{37}$	(C) $\frac{12}{7}$	(D) $\frac{23}{13}$	(E) 2

Let a be the side length of the inscribed square.

Triangles $\triangle ACB$ and $\triangle WZB$ are congruent, implying

$$\frac{ZB}{ZW} = \frac{CB}{CA} = \frac{4}{5} \quad \Rightarrow \quad ZB = \frac{4a}{5}$$

Triangles $\triangle CBA$ and $\triangle ZYC$ are congruent, implying

$$\frac{ZC}{ZY} = \frac{AC}{AB} = \frac{5}{3} \quad \Rightarrow \quad ZC = \frac{5a}{3}$$

It follows that

$$CB = ZB + ZC$$

$$4 = \frac{4a}{5} + \frac{5a}{3} = \frac{37a}{15}$$

$$a = 4 \times \frac{15}{37} = \frac{60}{37}$$

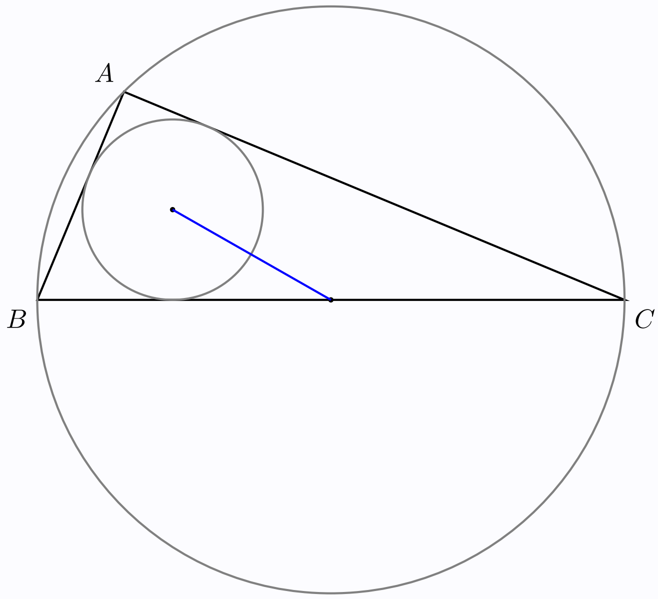
$$\boxed{\frac{60}{37}}$$

9.

A triangle with sides of 5, 12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

(A) $\frac{3\sqrt{5}}{2}$	(B) $\frac{7}{2}$	(C) $\sqrt{15}$	(D) $\frac{\sqrt{65}}{2}$	(E) $\frac{9}{2}$

Pick a coordinate system so that the right angle is at Let A be the center $(0, 0)$ of a coordinate system such that the vertices B and C have coordinates $(12, 0)$ and $(0, 5)$. This is a right triangle — it is a well-known Pythagorean triple. This means that the center of the circumscribed circle is on the hypotenuse at the middle point, at coordinates $(6, 2.5)$.



Let A be the area of the triangle, let P be its perimeter, and let r be the radius of the inscribed circle. These quantities are related by the identity

$$r = \frac{A}{{}^1\!/_2\, P}$$

For this particular triangle, we have

$$P = 5 + 12 + 13 = 30$$

$$A = \frac{5 \times 12}{2} = 30$$

implying $r = 2$. The coordinates of the center are therefore $(r, r) = (2, 2)$.

The distance between the center of the circumscribed circle $(6, 2.5)$ and the center of the inscribed circle $(2, 2)$ is

$$\sqrt{(6-2)^2 + (2.5-2)^2} = \sqrt{16.25} = \frac{\sqrt{65}}{2}$$

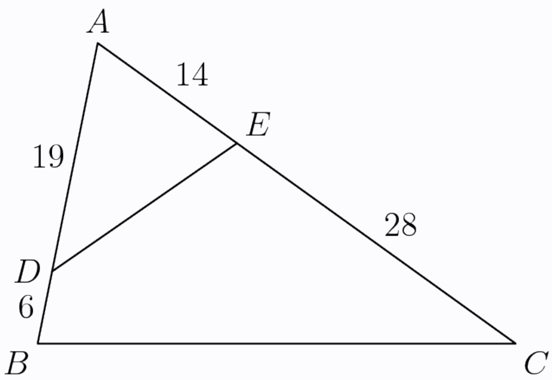
$$\boxed{\frac{\sqrt{65}}{2}}$$

10.

In triangle ABC we have $AB = 25$, $BC = 39$, and $AC = 42$. Points D and E are on AB and AC respectively, with $AD = 19$ and $AE = 14$. What is the ratio of the area of triangle ADE to the area of the quadrilateral $BCED$?

(A) $\frac{266}{1521}$	(B) $\frac{19}{75}$	(C) $\frac{1}{3}$	(D) $\frac{19}{56}$	(E) 1

Consider



We have that

$$\frac{[ADE]}{[ABC]} = \frac{AD}{AB} \cdot \frac{AE}{AC} = \frac{19}{25} \cdot \frac{14}{42} = \frac{19}{75}.$$

But $[BCED] = [ABC] - [ADE]$, so

$$\begin{aligned}\frac{[ADE]}{[BCED]} &= \frac{[ADE]}{[ABC] - [ADE]} \\ &= \frac{1}{[ABC]/[ADE] - 1} \\ &= \frac{1}{75/19 - 1} \\ &= \frac{19}{56}\end{aligned}$$

$$\boxed{\frac{19}{56}}$$