AMC 10 Problem Series (2804)

Jon Joseph

Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Overview

Lesson 8 (Jul 23) Class Transcript - Polygons and Three-Dimensional Geometry

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jonjoseph 2021-07-23 19:30:49
There! See @ amips2000!! You won't fail.

jonjoseph 2021-07-23 19:31:51
Sometimes when the last post and the moderation happen at the same time it squeezes through.

jonjoseph 2021-07-23 19:32:00
Anyway, lots of claps.

jonjoseph 2021-07-23 19:32:15
Hmmmm.... Maybe Hooray?

jonjoseph 2021-07-23 19:32:32
Funny. Yes. That's it.

jonjoseph 2021-07-23 19:32:43
AMC 10 Problem Series
Week 8: Polygons and Three-Dimensional Geometry

jonjoseph 2021-07-23 19:32:53
In today's class, we continue our look at Euclidean geometry by dealing with problems involving polygons and three-dimensional geometry. In both cases, we draw on the concepts that we have seen in the previous classes.

jonjoseph 2021-07-23 19:33:02
POLYGONS

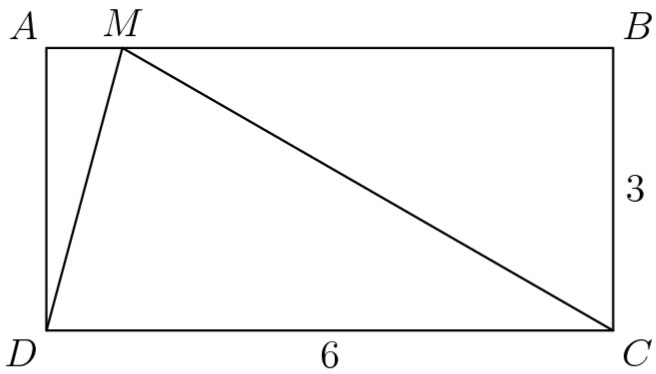
jonjoseph 2021-07-23 19:33:08
In most problems involving polygons, we depend on the same techniques that we have used before in other geometry problems, such as angle chasing and using right triangles and similar triangles.

jonjoseph 2021-07-23 19:33:19
Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

$(\text{A})\ 15 \quad (\text{B})\ 30 \quad (\text{C})\ 45 \quad (\text{D})\ 60 \quad (\text{E})\ 75$

jonjoseph 2021-07-23 19:33:26
First, let's draw a diagram.

jonjoseph 2021-07-23 19:33:29



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What can we say about the diagram?

jonjoseph 2021-07-23 19:34:46

Yep. Sort of an open ended question. Here is what I was after:

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We know that $\angle AMD = \angle CMD$. When we have equal angles, it is usually a good idea to mark them.

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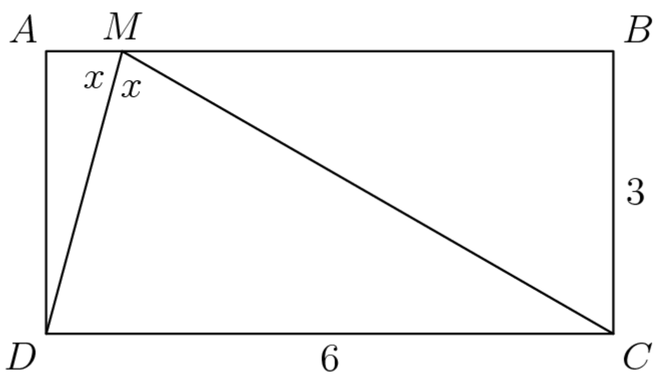

jonjoseph 2021-07-23 19:35:11

Marking the angles makes it easier to refer to them in your work.

jonjoseph 2021-07-23 19:35:17

Are there any other angles we can express in terms of x?

jonjoseph 2021-07-23 19:37:03

Hmmm... You missed the big one:

jonjoseph 2021-07-23 19:37:08

Since $\overline{AB}$ and $\overline{CD}$ are parallel, $\angle CDM = \angle AMD = x$.

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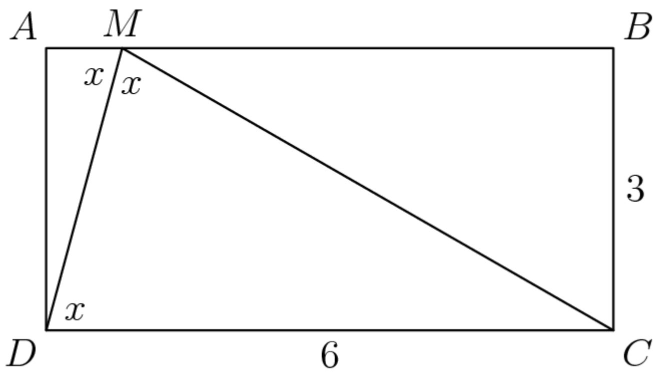

jonjoseph 2021-07-23 19:37:39
What does this say about $\triangle MCD$?

jonjoseph 2021-07-23 19:38:18
Since $\angle CDM = \angle CMD = x$, $\triangle CDM$ is isosceles with $CD = CM = 6$.

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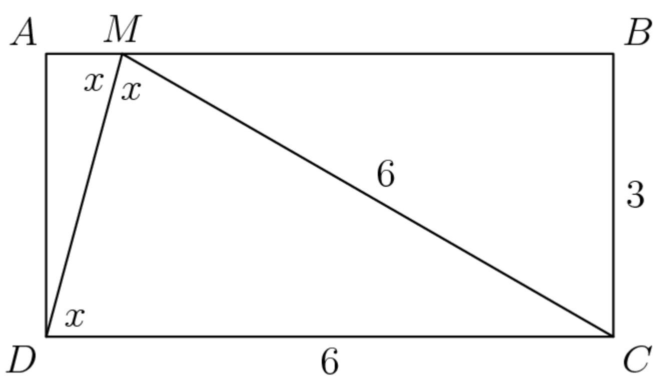

jonjoseph 2021-07-23 19:38:55
Something else going on. What can you say about $\triangle MCB$?

jonjoseph 2021-07-23 19:40:03
Since $\dfrac{CM}{BC} = \dfrac{6}{3} = 2$, $\triangle MCB$ is 30-60-90. How does this help us find $\angle AMD = x$?

jonjoseph 2021-07-23 19:41:00
Adding up $\angle AMD$, $\angle CMD$, and $\angle BMC$, we get a straight angle. Hence, $2x + 30 = 180$.

jonjoseph 2021-07-23 19:41:04
Solving for x, we find $x = 75$. The answer is (E).

jonjoseph 2021-07-23 19:41:50
Let me show you a different way to solve this problem It's a bit of an advanced technique which you'd see often in Olympiad Geometry.

jonjoseph 2021-07-23 19:42:04
Let $\overline{MF}$ be the reflection of $\overline{MC}$ over the line MD. Then $\angle FMD = \angle CMD = \angle AMD$, and so $\overline{MF}$ must lie on the line AB:

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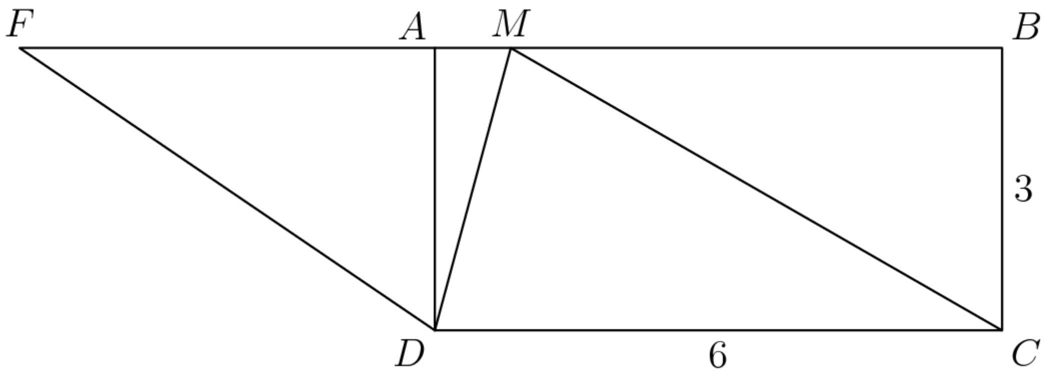

jonjoseph 2021-07-23 19:42:42

Do you see why MF is the reflection of MC?

jonjoseph 2021-07-23 19:43:12

Right.

jonjoseph 2021-07-23 19:43:14

Note that $FD = DC = 6$ and $AD = BC = 3$.

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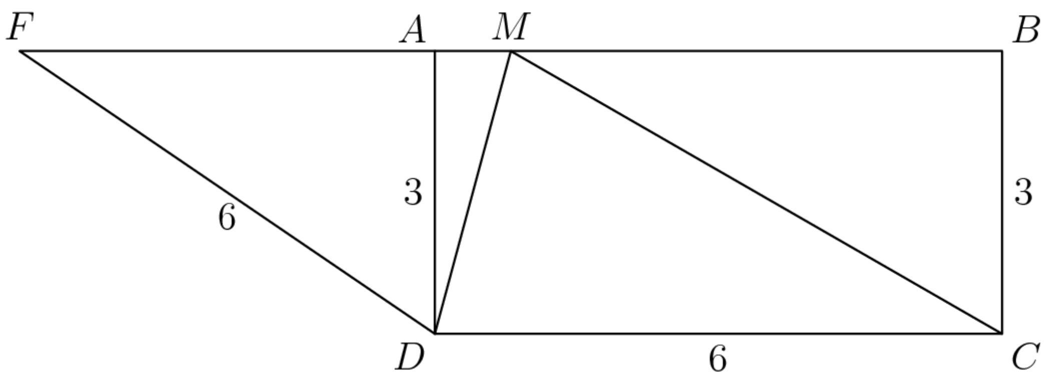

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Since $FD/AD = 2$ and $\angle FAD = 90^\circ$, it follows that $\triangle FDA$ is a $30-60-90$ triangle. From the congruent triangles $\triangle FMD$ and $\triangle CMD$, it follows that $\angle MCD = \angle AFD = 30^\circ$.

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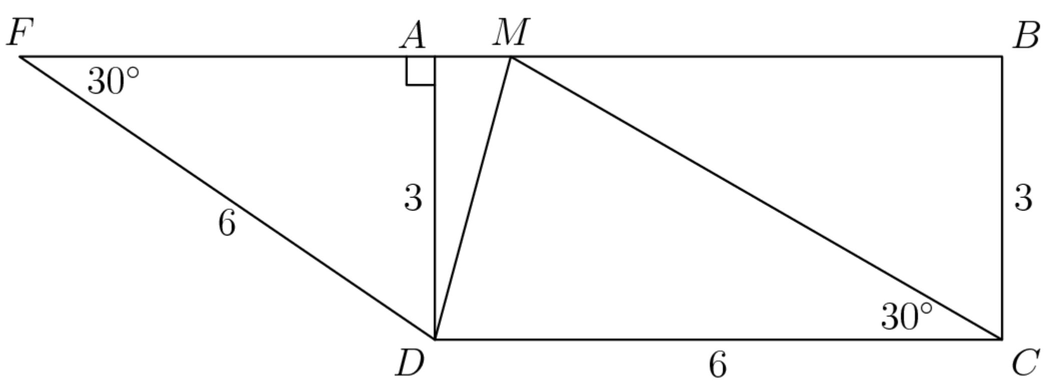

jonjoseph 2021-07-23 19:44:06

From the angle sum of the quadrilateral $AMCD$ (or the straight angle $\angle AMB$, or the parallelogram $FMCD$), it follows that $\angle AMC = 150^\circ$, and so $\angle AMD = 75^\circ$.

jonjoseph 2021-07-23 19:45:21

Reflections, rotations, translations and dilatations can make short work of many geometry problems.

jonjoseph 2021-07-23 19:45:43

In trapezoid $ABCD$, $\overline{AB}$ and $\overline{CD}$ are perpendicular to $\overline{AD}$, with $AB + CD = BC$, $AB < CD$, and $AD = 7$. What is $AB \cdot CD$?

$\text{(A) } 12 \quad \text{(B) } 12.25 \quad \text{(C) } 12.5 \quad \text{(D) } 12.75 \quad \text{(E) } 13$

jonjoseph 2021-07-23 19:45:52

Nope.

jonjoseph 2021-07-23 19:46:07

First, let's draw a diagram. What are the parallel sides in trapezoid $ABCD$?

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Since both $\overline{AB}$ and $\overline{CD}$ are perpendicular to $\overline{AD}$, $\overline{AB}$ and $\overline{CD}$ are parallel.

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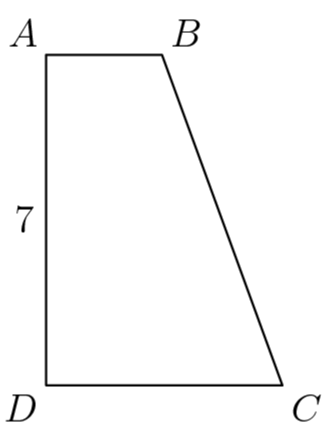

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We want to find the product $AB \cdot CD$. We don't see any way of calculating AB or CD directly, so what can we do?

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Hint: What's a good, simplifying step (like our last problem).

jonjoseph 2021-07-23 19:47:59

Nice.

jonjoseph 2021-07-23 19:48:01

We can use variables. Let $x = AB$ and $y = CD$.

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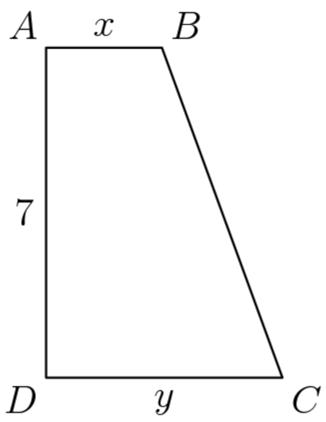

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What else can we express in terms of x and y?

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Use capital letters for points.

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We are given that $BC = AB + CD$, so $BC = x + y$.

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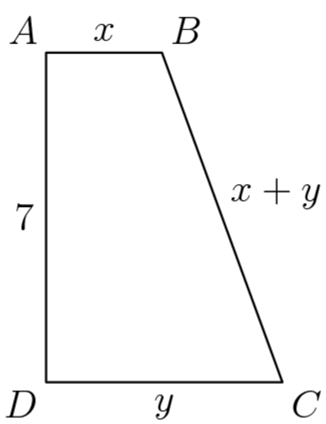

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We haven't used the fact that $AD = 7$. How can we use this?

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We can drop a perpendicular from B to $\overline{CD}$.

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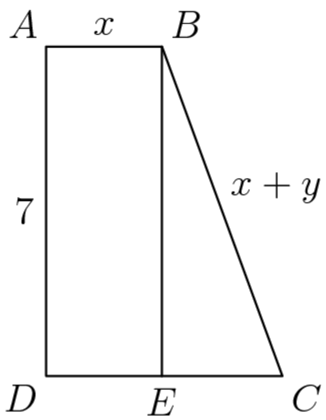

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What can we say about CE?

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Good. And the problem is really starting to crack.

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Since $ABED$ is a rectangle, $BE = AD = 7$ and $DE = AB = x$. Then $CE = CD - DE = y - x$.

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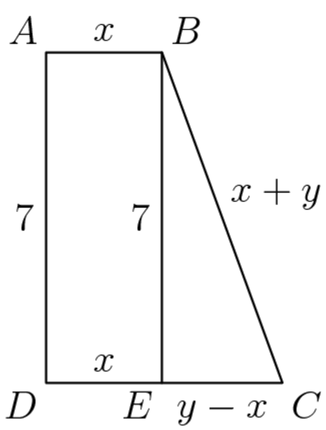

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Can you finish?

jonjoseph 2021-07-23 19:51:54

Good. Go.

jonjoseph 2021-07-23 19:53:31

By Pythagoras on right triangle BEC, $(y-x)^2 + 49 = (x+y)^2$.

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A whole bunch of terms can cancel out, and it simplifies to just $4xy = 49$. So what is $AB \cdot CD$?

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We have that $AB \cdot CD = xy = \dfrac{49}{4} = 12.25$. The answer is (B).

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Sometimes the right choice of variables can help a lot. In this case, since we wanted $AB \cdot CD$, it made sense to use x for AB and y for CD, because then the thing we wanted was something we could easily spot (xy). If we'd used a different setup,

say letting y be EC, then it would have been harder to see the expression we want.

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In trapezoid $ABCD$ we have $\overline{AB}$ parallel to $\overline{DC}$, E as the midpoint of $\overline{BC}$, and F as the midpoint of $\overline{DA}$. The area of $ABEF$ is twice the area of $FECD$. What is $\dfrac{AB}{DC}$?

$\text{(A) } 2 \quad \text{(B) } 3 \quad \text{(C) } 5 \quad \text{(D) } 6 \quad \text{(E) } 8$

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Let's start by drawing a diagram:

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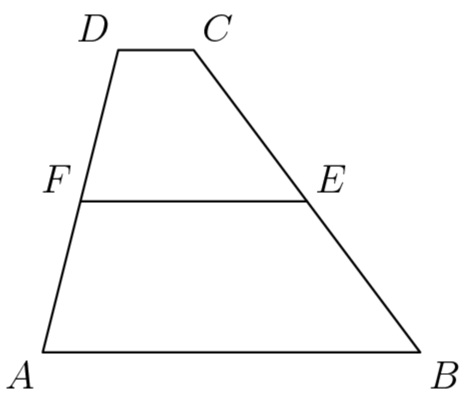

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How can we find the areas of trapezoids $ABEF$ and $FECD$?

jonjoseph 2021-07-23 19:55:33

Since we're interested in $\dfrac{AB}{DC}$, we want to write the areas in terms of the bases AB and CD. Let's see if we can do that using the trapezoid area formula. First, let's label the base lengths with variables to make them easier to talk about.

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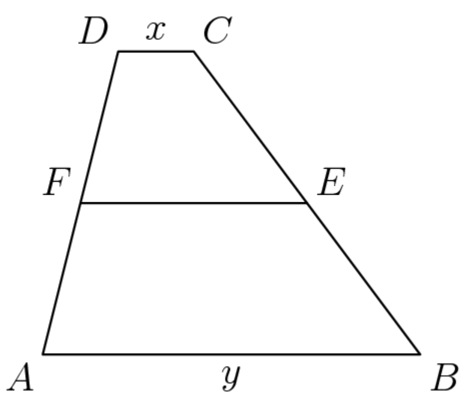

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What is EF in terms of x and y?

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Since E and F are the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively, it turns out that $EF = \frac{x+y}{2}$.

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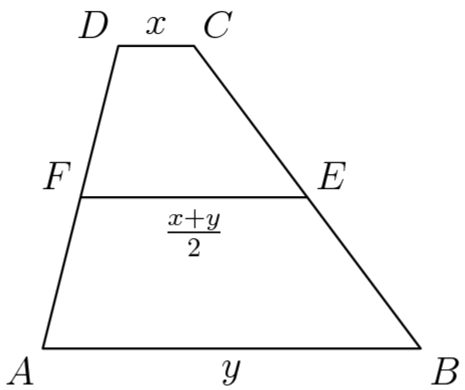

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What can we say about the heights of trapezoids $ABEF$ and $FECD$?

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They are equal, since E and F are each halfway from $\overline{AB}$ to $\overline{CD}$. Let's call this common height h.

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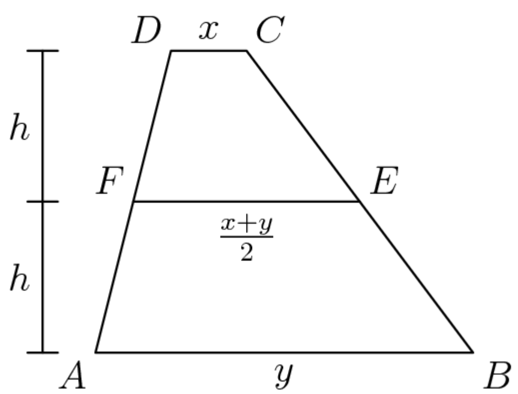

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What is the area of trapezoid $ABEF$?

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Hint: Try to simplify.

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The area of trapezoid $ABEF$ is

$$\frac{EF + AB}{2} \cdot h = \frac{\frac{x+y}{2} + y}{2} \cdot h = \frac{(x+3y)h}{4}.$$

jonjoseph 2021-07-23 20:00:31

Notice the denominator is a 4.

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What is the area of trapezoid $FECD$?

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The area of trapezoid $FECD$ is

$$\frac{EF + CD}{2} \cdot h = \frac{\frac{x+y}{2} + x}{2} \cdot h = \frac{(3x + y)h}{4}.$$

jonjoseph 2021-07-23 20:02:18

Since we know that the area of $ABEF$ is twice that of $FECD$, this gives us the equation

$$\frac{(x + 3y)h}{4} = 2 \cdot \frac{(3x + y)h}{4}.$$

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We can cancel the factors of $\frac{h}{4}$. This gives us

$$\frac{x + 3y}{3x + y} = 2.$$

Multiplying both sides by $3x + y$, we get $x + 3y = 6x + 2y$. So what is $\frac{AB}{DC}$?

jonjoseph 2021-07-23 20:04:00

$x + 3y = 6x + 2y$ simplifies to $y = 5x$. Therefore $\frac{AB}{DC} = \frac{y}{x} = 5$. The answer is (C).

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The formula for the area of a trapezoid is worth committing to memory.

jonjoseph 2021-07-23 20:04:46

A regular octagon $ABCDEFGH$ has an area of one square unit. What is the area of the rectangle $ABEF$?

$(\text{A})\ 1 - \frac{\sqrt{2}}{2} \quad (\text{B})\ \frac{\sqrt{2}}{4} \quad (\text{C})\ \sqrt{2} - 1 \quad (\text{D})\ \frac{1}{2} \quad (\text{E})\ \frac{1+\sqrt{2}}{4}$

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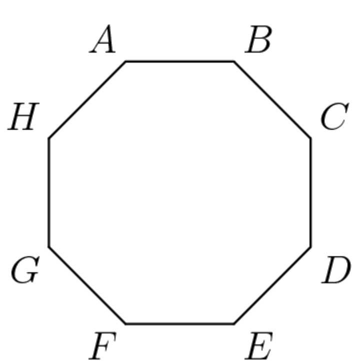

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Let's add the rectangle.

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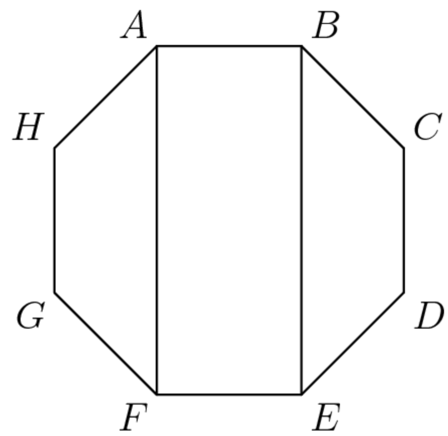

jonjoseph 2021-07-23 20:05:43
Should we grind or solve the cool way?

jonjoseph 2021-07-23 20:06:10
Great. I was hoping for that.

jonjoseph 2021-07-23 20:06:25
And you'd be rightr.

jonjoseph 2021-07-23 20:06:28
*** right**

jonjoseph 2021-07-23 20:06:53
Therectangle doesn't relate very naturally to the octagon.

jonjoseph 2021-07-23 20:07:04
Let's forget about it for a little while and look at some other shapes that do relate!

jonjoseph 2021-07-23 20:07:10
What would be the area of the following triangle?

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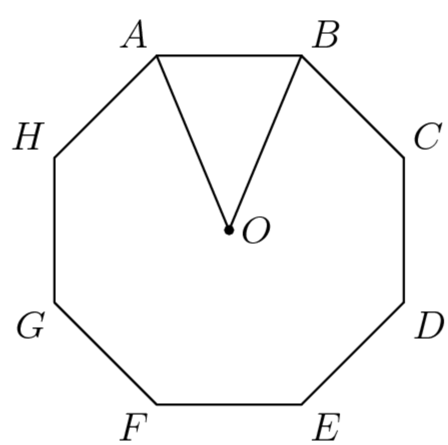

jonjoseph 2021-07-23 20:07:22
Hint: Symmetry!!!

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By symmetry, $\triangle ABO$ must be exactly $1/8$ of the area of the entire octagon.

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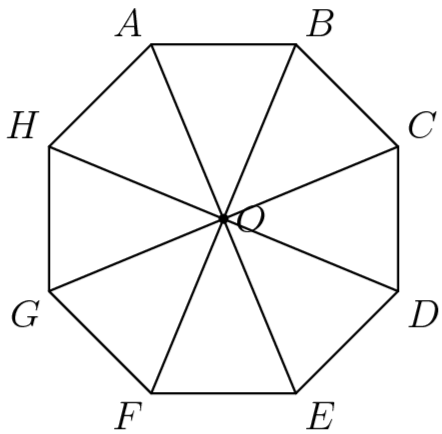

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Since the octagon has area 1, we see that $\triangle ABO$ has area $1/8$.

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Let's go back to our diagram with the rectangle.

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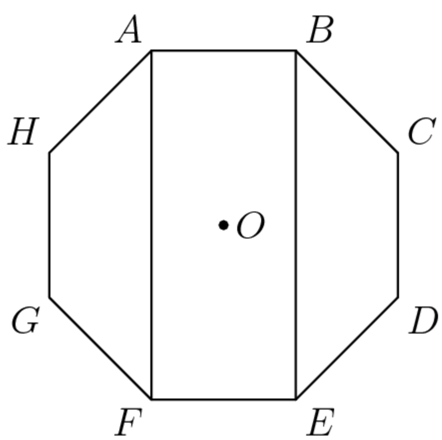

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Is there a way we can introduce $\triangle ABO$ into the mix here?

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We draw diagonals $\overline{AE}$ and $\overline{BF}$.

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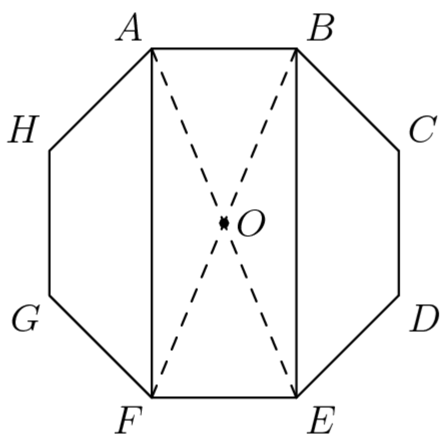

jonjoseph 2021-07-23 20:09:51
What can we say about the area of rectangle $ABEF$ now?

jonjoseph 2021-07-23 20:10:35
Clever. Let's break it down.

jonjoseph 2021-07-23 20:10:39
The area of rectangle $ABEF$ is four times the area of $\triangle ABO$.

jonjoseph 2021-07-23 20:10:43
We can make this extra clear by adding a few additional lines.

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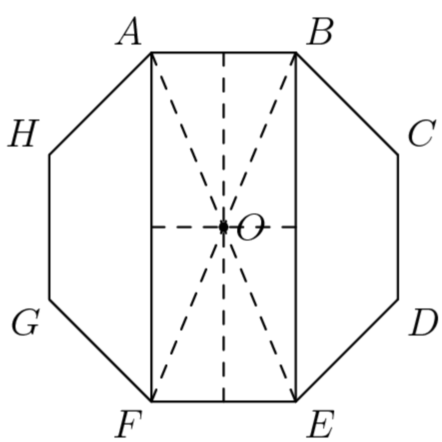

jonjoseph 2021-07-23 20:10:56
What's true about all of those triangles?

jonjoseph 2021-07-23 20:11:48
Then the area of rectangle $ABEF$ is divided into 8 congruent triangular pieces. Each piece has half the area of $\triangle ABO$.

jonjoseph 2021-07-23 20:11:52
So what is the area of rectangle $ABEF$?

jonjoseph 2021-07-23 20:12:59
Each of the 8 small triangles has area $\frac{1}{16}$, so the area of rectangle $ABEF$ is $8 \cdot \dfrac{1}{16} = \dfrac{1}{2}$. The answer is (D).

jonjoseph 2021-07-23 20:13:40
We found a way to exploit symmetry (= the cool way). Problem 3 or 4 bashy ways to solve this as well if you don't see this

method.

jonjoseph 2021-07-23 20:14:04

Equiangular hexagon $ABCDEF$ has side lengths $AB = CD = EF = 1$ and $BC = DE = FA = r$. The area of triangle ACE is 70% of the area of the hexagon. What is the sum of all possible values of r?

$(\mathrm{A})\ \frac{4\sqrt{3}}{3} \qquad (\mathrm{B})\ \frac{10}{3} \qquad (\mathrm{C})\ 4 \qquad (\mathrm{D})\ \frac{17}{4} \qquad (\mathrm{E})\ 6$

jonjoseph 2021-07-23 20:14:25

First, let's draw a diagram. What can we say about hexagon $ABCDEF$ that will help us draw it?

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Hint: it is NOT regular.

jonjoseph 2021-07-23 20:15:53

Yep. Equiangular. What is the angle measure?

jonjoseph 2021-07-23 20:16:43

We are given that hexagon $ABCDEF$ is equiangular, so every interior angle of the hexagon is 120 degrees.

jonjoseph 2021-07-23 20:16:48

We're also given that its side lengths alternate between 1 and r. With a bit of thought, you can come up with a diagram that looks something like this:

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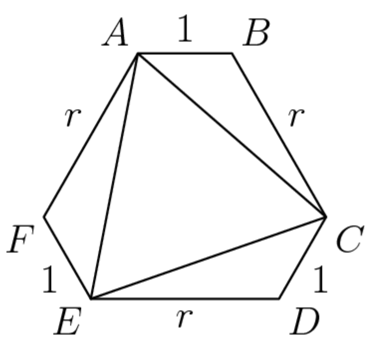

jonjoseph 2021-07-23 20:17:11

We are given that the area of triangle ACE is 70% of the area of hexagon $ABCDEF$. I don't really like working with (non-regular) hexagons, so let's see if we can transform this into a question about triangles.

jonjoseph 2021-07-23 20:17:40

Our hexagon is conveniently split into four triangles, so let's see how their areas compare.

jonjoseph 2021-07-23 20:17:50

Let's find a relationship between the area of $\triangle ACE$ and the area of $\triangle CDE$. We see by SAS that $\triangle CDE$, $\triangle EFA$, and $\triangle ABC$ are all congruent triangles, and therefore their areas are the same. We also know that the area of $\triangle ACE$ is 70% of the total area of the hexagon.

jonjoseph 2021-07-23 20:18:14

Can we now figure out the ratio between $[ACE]$ and $[CDE]$?

jonjoseph 2021-07-23 20:19:24

Excellent. Here is the argument:

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In total, the three congruent triangles make up 30% of the area of the hexagon, so each of them must be 10% of the area of

the hexagon. Therefore, $[ACE] = 7[CDE]$.

jonjoseph 2021-07-23 20:19:46

Clear?

jonjoseph 2021-07-23 20:19:58

Let's see if we can express both of these areas in terms of r. We start with $\triangle CDE$. How might we calculate its area?

jonjoseph 2021-07-23 20:20:56

Hint: Several ways. If you know any trig there is a shortcut.

jonjoseph 2021-07-23 20:21:46

I see several ways. They will all work.

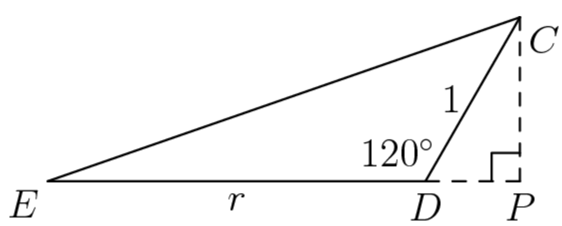
jonjoseph 2021-07-23 20:21:59

Since we know that $\angle CDE = 120^\circ$, either dropping the altitude from C or from E will give us a right triangle in which we know all the angles. Let's drop it from C, and let the foot of the altitude be P.

jonjoseph 2021-07-23 20:22:18

Dropping an altitude from D, as many of you want to do, destroys the one angle we know by cutting it into two unknown pieces! Sometimes it's better to use an altitude outside the triangle, and this illustrates one reason why.

jonjoseph 2021-07-23 20:22:33



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What is the height CP equal to?

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Hint: What kind of triangle is $\triangle CDP$?

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Since $\angle CDE = 120^\circ$, $\angle CDP = 60^\circ$. Therefore, $CP = \dfrac{\sqrt{3}}{2}$. So what's $[CDE]$ equal to?

jonjoseph 2021-07-23 20:25:16

$[CDE] = \dfrac{1}{2} CP \cdot DE = \dfrac{\sqrt{3}}{4} r.$

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For those of you with a little trig knowledge finding $[CDE]$ is even easier. We can use

$$[CDE] = \frac{1 \cdot r \sin(120^\circ)}{2}.$$

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The formula is $1/2a \cdot b \cdot \sin((\text{Included angle})$

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Great, we've gotten the area of $\triangle CDE$ in terms of r! Now we just need to find the area of $\triangle ACE$ in terms of r. Then we'll be able to use the fact that $[ACE] = 7[CDE]$ to solve for r.

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What useful fact do we observe about $\triangle ACE$ that will help us calculate its area?

jonjoseph 2021-07-23 20:27:57
Since $\triangle CDE$, $\triangle EFA$, and $\triangle ABC$ are congruent, $\triangle ACE$ has three equal sides. Therefore, it's equilateral, and to calculate its area, we just need the length of one of the sides.

jonjoseph 2021-07-23 20:28:06
How can we go about finding a side of $\triangle ACE$ in terms of r?

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HInt:

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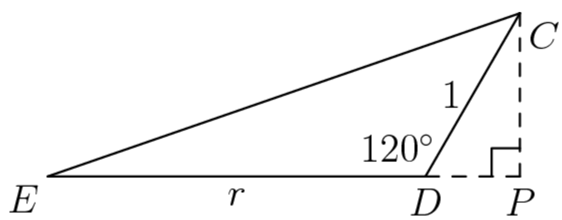

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The diagram we have above is helpful! Using Pythagoras, we see that

$$CE^2 = PE^2 + CP^2.$$

We already know CP, so we just need to find $PE = ED + PD$. What is PD?

jonjoseph 2021-07-23 20:31:16
Again using the 30-60-90 triangle, we see that $PD = \frac{1}{2}$. So what's CE^2 equal to? (Simplify the answer, please! It turns out very nicely.)

jonjoseph 2021-07-23 20:31:33
Take your time! No mistakes.

jonjoseph 2021-07-23 20:33:59
We get $CE^2 = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(r + \frac{1}{2}\right)^2 = r^2 + r + 1$. What does this give for the area of $\triangle ACE$?

jonjoseph 2021-07-23 20:35:20

$$[ACE] = \frac{\sqrt{3}}{4}CE^2 = \frac{\sqrt{3}}{4}(r^2 + r + 1).$$

And we know that $[ACE] = 7[CDE]$. Plugging in, we get

$$\frac{\sqrt{3}}{4}(r^2 + r + 1) = 7 \cdot \frac{\sqrt{3}}{4}r.$$

jonjoseph 2021-07-23 20:35:30
This simplifies to $r^2 - 6r + 1 = 0$.

jonjoseph 2021-07-23 20:35:44
What are the solutions to this equation?

jonjoseph 2021-07-23 20:36:16
Hint: Quadratic formula may be necessary.

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By the quadratic formula,

$$r = \frac{6 \pm \sqrt{6^2 - 4}}{2} = \frac{6 \pm \sqrt{32}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2}.$$

Are both roots actually possible values of r?

jonjoseph 2021-07-23 20:38:26
Yes! Both of these numbers are positive and are therefore acceptable. So what is the sum of the possible values of r?

jonjoseph 2021-07-23 20:38:52
(Important to know that $\sqrt{2} \approx 1.41$)

jonjoseph 2021-07-23 20:39:01
What's the answer?

jonjoseph 2021-07-23 20:39:50
The sum of the possible values of r is $(3 + 2\sqrt{2}) + (3 - 2\sqrt{2}) = 6.$ The answer is (E).

jonjoseph 2021-07-23 20:40:36
It is. But one possible, slightly risky shortcut, is Vieta's.

jonjoseph 2021-07-23 20:41:03
Notice our equation: $r^2 - 6r + 1 = 0.$

jonjoseph 2021-07-23 20:41:11
The 6 is sitting right there.

jonjoseph 2021-07-23 20:41:42
I know. What's the risk however?

jonjoseph 2021-07-23 20:42:39
That's right. Smart. Possible one of the solutions is negative. However, looking at our answer choices it is safe to exclude that possibility in this case.

jonjoseph 2021-07-23 20:43:29
THREE-DIMENSIONAL GEOMETRY

jonjoseph 2021-07-23 20:43:33
We start our exploration of three-dimensional geometry by working with common solids, such as pyramids, cones, and spheres.

jonjoseph 2021-07-23 20:43:37
Which of the cones below can be formed from a 252° sector of a circle of radius 10 by aligning the two straight sides?

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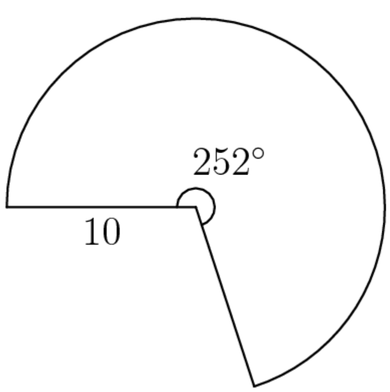

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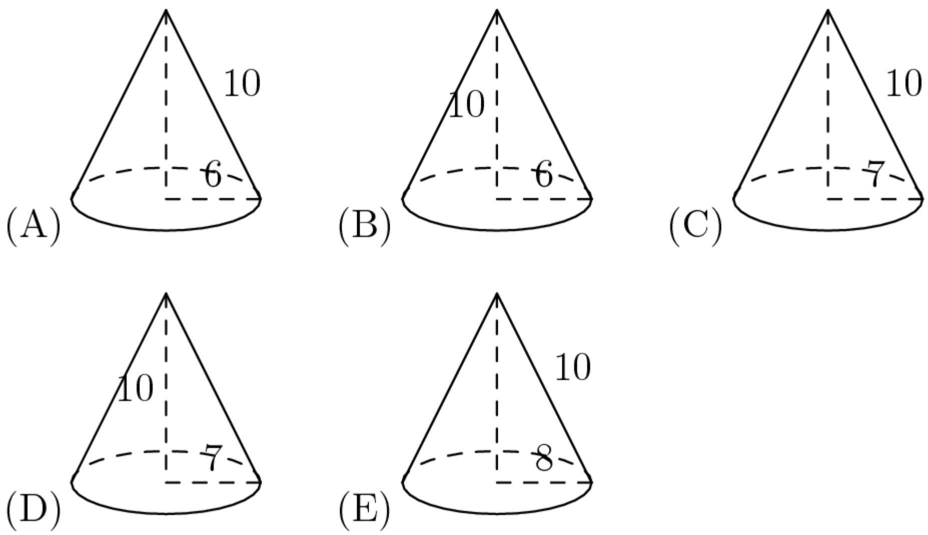

jonjoseph 2021-07-23 20:43:56
The circular sector has radius 10. What length does this correspond to in the cone?

jonjoseph 2021-07-23 20:45:03
This corresponds to the slant height of the cone.

jonjoseph 2021-07-23 20:45:07
So the answer is (A), (C), or (E). How can we figure out which one it is?

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We can compute the circumference of the circular sector, which then must be the same as the circumference of the base of the cone. What is the circumference of the circular sector?

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Hint: What's the circumference of the full circle?

jonjoseph 2021-07-23 20:47:04
The circumference of the full circle would be $2 \cdot 10 \cdot \pi = 20\pi$.

jonjoseph 2021-07-23 20:47:28
So what's the circumference of the circular sector?

jonjoseph 2021-07-23 20:48:23
The circular sector contains an angle of 252 degrees, which is $\dfrac{252}{360} = \dfrac{7}{10}$ of the full circle.

jonjoseph 2021-07-23 20:48:27
Therefore, the circumference of the circular sector is $\dfrac{7}{10} \cdot 20\pi = 14\pi$.

jonjoseph 2021-07-23 20:48:35
What does this correspond to in the cone?

jonjoseph 2021-07-23 20:49:38
This corresponds to the circumference of the base of the cone.

jonjoseph 2021-07-23 20:49:49
We know that $2\pi r = 14\pi$, so the radius of the base of the cone is $\frac{14\pi}{2\pi} = 7$.

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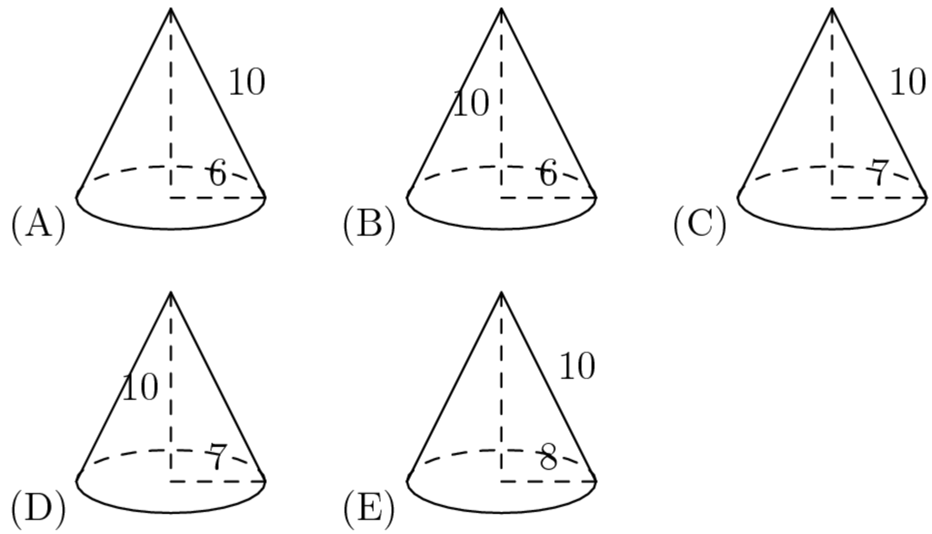

jonjoseph 2021-07-23 20:49:56
We are looking for a cone with slant height 10 and base radius 7, so the correct answer is (C).

jonjoseph 2021-07-23 20:50:12
A sphere is inscribed in a cube that has a surface area of 24 square meters. A second cube is then inscribed within the sphere. What is the surface area in square meters of the inner cube?

$(\text{A})\ 3 \quad (\text{B})\ 6 \quad (\text{C})\ 8 \quad (\text{D})\ 9 \quad (\text{E})\ 12$

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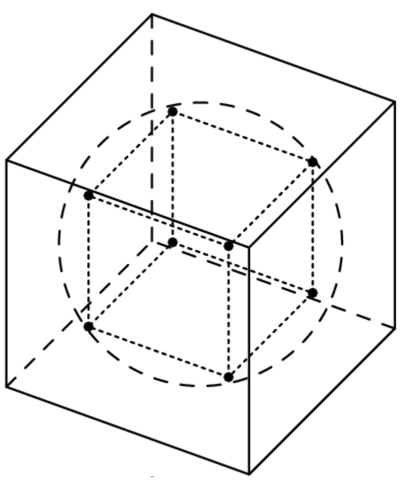

jonjoseph 2021-07-23 20:50:31
Let's start with the outer cube, since we're given its surface area. What is the side length of the outer cube?

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The outer cube has surface area 24 (in square meters), so each face has area $\frac{24}{6} = 4$. Hence, the side length of the outer cube is 2.

jonjoseph 2021-07-23 20:51:31
What does this tell us about the sphere?

jonjoseph 2021-07-23 20:51:59
Careful. Know if you're talking about the diameter or the radius.

jonjoseph 2021-07-23 20:52:39
The sphere is inscribed in a cube with side length 2. Then the diameter of the sphere is 2, so its radius is 1.

jonjoseph 2021-07-23 20:52:46
Now, we need the side length of the inner cube. Let x be the side length of the inner cube. How can we determine x?

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We know that every vertex of the inner cube lies on the sphere. How can we use this?

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Perhaps the easiest way is to consider a diagonal of the inner cube going from one vertex to an opposite vertex.

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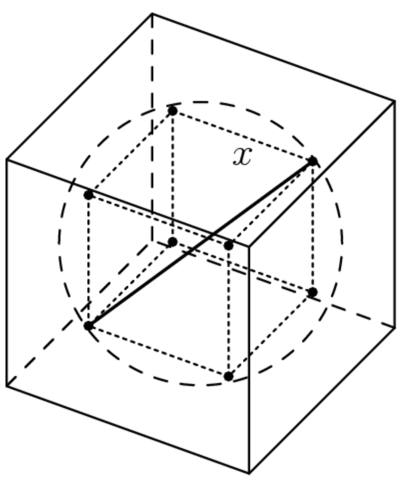

jonjoseph 2021-07-23 20:54:35
(This type of diagonal is called a "space diagonal".)

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What is the length of this diagonal in terms of x?

jonjoseph 2021-07-23 20:55:45
The length of this diagonal is $x\sqrt{3}$, from the distance formula in 3 dimensions. But what else is this diagonal equal to?

jonjoseph 2021-07-23 20:56:14
This diagonal is equal to a diameter of the sphere, which is 2.

jonjoseph 2021-07-23 20:56:18
Hence, $x\sqrt{3} = 2$.

jonjoseph 2021-07-23 20:56:19
Then $x = \dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$. So what is the surface area of the inner cube?

jonjoseph 2021-07-23 20:57:10
The surface area of the inner cube is $6x^2 = 6 \cdot (\frac{2}{\sqrt{3}})^2 = 6 \cdot \frac{4}{3} = 8$. The answer is (C).

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An ice cream cone consists of a sphere of vanilla ice cream and a right circular cone that has the same diameter as the sphere. If the ice cream melts, it will exactly fill the cone. Assume that the melted ice cream occupies 75% of the volume of the frozen ice cream. What is the ratio of the cone's height to its radius?

$\text{(A) } 2:1 \quad \text{(B) } 3:1 \quad \text{(C) } 4:1 \quad \text{(D) } 16:3 \quad \text{(E) } 6:1$

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jonjoseph 2021-07-23 20:57:51
Let the radius of the sphere of vanilla ice cream be r. What is its volume?

jonjoseph 2021-07-23 20:59:05
The volume of the sphere of vanilla ice cream is

$$\frac{4}{3}\pi r^3.$$

We know the radius of the cone is the same as the radius of the ice cream, namely r.

jonjoseph 2021-07-23 20:59:07
Let h be the height of the cone. Then what is the volume of the cone?

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The volume of the cone is

$$\frac{1}{3}\pi r^2 h.$$

We are told that when the ice cream melts, it fills the cone exactly. We are also told that the melted ice cream takes up 75% the volume of the frozen ice cream. So what equation can we write down?

jonjoseph 2021-07-23 21:01:18
Hint: The π's cancel too!

jonjoseph 2021-07-23 21:01:51
We can write

$$\frac{3}{4} \cdot \frac{4}{3}\pi r^3 = \frac{1}{3}\pi r^2 h.$$

jonjoseph 2021-07-23 21:02:09
Dividing both sides by πr^2, we get

$$3r = h.$$

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We see that $\frac{h}{r} = 3$. The answer is (B).

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SUMMARY

jonjoseph 2021-07-23 21:02:42
In today's class, we looked at techniques to solve problems involving polygons and three-dimensional geometry. However, none of these techniques were really new.

jonjoseph 2021-07-23 21:02:49
For example, when dealing with polygons, one useful approach is to break them up into triangles, and then apply what we know about triangle geometry. In three-dimensional geometry, we can take cross-sections, and use the usual two-dimensional geometry techniques, such as building right triangles and looking for similar triangles. We can also break complicated three-dimensional shapes into more familiar ones, just like we do in two dimensions.

jonjoseph 2021-07-23 21:03:18
We covered quite a bit tonight. Nice job. Stay safe and I'll see you next week!!

jonjoseph 2021-07-23 21:04:02
midnight your timezone.