Math Bits

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First Few Squares

$$\begin{aligned}
1^2 &= 1 \\
2^2 &= 4 \\
3^2 &= 9 \\
4^2 &= 16 \\
5^2 &= 25 \\
6^2 &= 36 \\
7^2 &= 49 \\
8^2 &= 64 \\
9^2 &= 81 \\
10^2 &= 100 \\
11^2 &= 121 \\
12^2 &= 144 \\
13^2 &= 169 \\
14^2 &= 196 \\
15^2 &= 225 \\
16^2 &= 256 \\
17^2 &= 289 \\
18^2 &= 324 \\
19^2 &= 361 \\
20^2 &= 400 \\
21^2 &= 441 \\
22^2 &= 484 \\
23^2 &= 529 \\
24^2 &= 576 \\
25^2 &= 625 \\
30^2 &= 900 \\
35^2 &= 1225 \\
40^2 &= 1600 \\
45^2 &= 2025 \\
50^2 &= 2500 \\
55^2 &= 3025 \\
60^2 &= 3600 \\
65^2 &= 4225 \\
70^2 &= 4900 \\
75^2 &= 5625 \\
80^2 &= 6400 \\
85^2 &= 7225 \\
90^2 &= 8100 \\
95^2 &= 9025 \\
100^2 &= 10000
\end{aligned}$$

Useful Cubes

$$
\begin{aligned}
2^3 &= 8 \\
3^3 &= 27 \\
4^3 &= 64 \\
5^3 &= 125 \\
6^3 &= 216 \\
7^3 &= 343 \\
8^3 &= 512 \\
9^3 &= 729 \\
10^3 &= 1000 \\
11^3 &= 1331 \\
12^3 &= 1728
\end{aligned}
$$

Useful Fourth Powers

$$
\begin{aligned}
2^4 &= 16 \\
3^4 &= 81 \\
4^4 &= 256 \\
5^4 &= 625 \\
6^4 &= 1296 \\
7^4 &= 2401 \\
8^4 &= 4096 \\
9^4 &= 6561 \\
10^4 &= 10000
\end{aligned}
$$

More Useful Powers

$$
\begin{aligned}
2^2 &= 4 \\
2^3 &= 8 \\
2^4 &= 16 \\
2^5 &= 32 \\
2^6 &= 64 \\
2^7 &= 128 \\
2^8 &= 256 \\
2^9 &= 512 \\
2^{10} &= 1024 \\
2^{11} &= 2048 \\
2^{12} &= 4096
\end{aligned}
$$

Highly Composite Numbers

A positive integer with more divisors than any smaller positive integer has. Plato set $5,040$ as the ideal number of citizens in a city. In bold, superior highly composite numbers.

2	2
4	2^2
6	$2 \cdot 3$
12	$2^2 \cdot 3$
24	$2^3 \cdot 3$
36	$2^2 \cdot 3^2$
48	$2^4 \cdot 3$
60	$2^2 \cdot 3 \cdot 5$
120	$2^3 \cdot 3 \cdot 5$
180	$2^2 \cdot 3^2 \cdot 5$
240	$2^4 \cdot 3 \cdot 5$
360	$2^3 \cdot 3^2 \cdot 5$
720	$2^4 \cdot 3^2 \cdot 5$
840	$2^3 \cdot 3 \cdot 5 \cdot 7$
$1,260$	$2^2 \cdot 3^2 \cdot 5 \cdot 7$
$1,680$	$2^4 \cdot 3^1 \cdot 5 \cdot 7$
$\mathbf{2,520}$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$
$\mathbf{5,040}$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$
$7,560$	$2^3 \cdot 3^3 \cdot 5 \cdot 7$
$10,080$	$2^5 \cdot 3^2 \cdot 5 \cdot 7$
$15,120$	$2^4 \cdot 3^3 \cdot 5 \cdot 7$
$20,160$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
$25,200$	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7$
$27,720$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
$45,360$	$2^4 \cdot 3^4 \cdot 5 \cdot 7$
$50,400$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7$
$\mathbf{55,440}$	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
$83,160$	$2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
$110,880$	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
$166,320$	$2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
$221,760$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
$277,200$	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$
$332,640$	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
$498,960$	$2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$
$554,400$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$
$665,280$	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$
$\mathbf{720,720}$	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$

Superior Highly Composite Numbers

2	2
6	$2 \cdot 3$
12	$2^2 \cdot 3$
60	$2^2 \cdot 3 \cdot 5$
120	$2^3 \cdot 3 \cdot 5$
360	$2^3 \cdot 3^2 \cdot 5$
$2,520$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$
$5,040$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$
$55,440$	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
$720,720$	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$

Perfect numbers

A number equal to the sum of its proper divisors.

6	$2^1(2^2-1)$
28	$2^2(2^3-1)$
496	$2^4(2^5-1)$
8128	$2^6(2^7-1)$
33550336	$2^{12}(2^{13}-1)$
8589869056	$2^{16}(2^{17}-1)$
137438691328	$2^{18}(2^{19}-1)$

Euclid's Perfect Number Theorem: If 2^p-1 is prime, then $2^{p-1}(2^p-1)$ is perfect. Euler's Perfect Number Theorem: If n is an even perfect number, it is of the form $n=2^{p-1}(2^p-1)$ for some prime p and Mersenne prime 2^p-1.

Prime Quadruplets

The first eight prime quadruplets are:

$$
\begin{aligned}
5, 7, 11, 13\\
11, 13, 17, 19\\
101, 103, 107, 109\\
191, 193, 197, 199\\
821, 823, 827, 829\\
1481, 1483, 1487, 1489\\
1871, 1873, 1877, 1879\\
2081, 2083, 2087, 2089
\end{aligned}
$$

Prime Quintuplets

The first few quintuplets that are not also sextuplets are:

$$
\begin{aligned}
5, 7, 11, 13, 17\\
1481, 1483, 1487, 1489, 1493\\
3457, 3461, 3463, 3467, 3469\\
5647, 5651, 5653, 5657, 5659\\
15727, 15731, 15733, 15737, 15739\\
21011, 21013, 21017, 21019, 21023\\
22271, 22273, 22277, 22279, 22283\\
55331, 55333, 55337, 55339, 55343\\
79687, 79691, 79693, 79697, 79699\\
88807, 88811, 88813, 88817, 88819
\end{aligned}
$$

Prime Sextuplets

The first five sextuplets are:

$$
\begin{aligned}
7, 11, 13, 17, 19, 23\\
97, 101, 103, 107, 109, 113\\
16057, 16061, 16063, 16067, 16069, 16073\\
19417, 19421, 19423, 19427, 19429, 19433\\
43777, 43781, 43783, 43787, 43789, 43793
\end{aligned}
$$

Mersenne Primes

Prime numbers of the form $2^p - 1$, where p is some prime number. Examples:

p	$2^p - 1$
2	3
3	7
5	31
7	127
11	23×89
13	$8,191$
17	$131,071$
19	$524,287$
23	47×178481

51 Mersenne primes are known. The largest known prime number is a Mersenne prime with $p = 82,589,933$.

Rectangle

Any rectangle of sides a and b:

$$P = 2 \times (a + b)$$
$$A = a \times b$$

Triangle

Any triangle of height h and base b:

$$A = \frac{1}{2}(h \times b)$$

Pythagorean Theorem

For any right-triangle with hypotenuse length c and legs a, b:

$$c^2 = a^2 + b^2$$

Example:

$$5^2 = 3^2 + 4^2$$

Pythagorean Triples

Famous Pythagorean triple:

$$(3, 4, 5)$$

More triples:

$$\begin{array}{rrr} (5, 12, 13) & (8, 15, 17) & (7, 24, 25) \\ (20, 21, 29) & (12, 35, 37) & (9, 40, 41) \\ (28, 45, 53) & (11, 60, 61) & (16, 63, 65) \\ (33, 56, 65) & (48, 55, 73) & (13, 84, 85) \\ (36, 77, 85) & (39, 80, 89) & (65, 72, 97) \end{array}$$

Non-Pythagorean triple: $(1, 1, \sqrt{2})$ ($\sqrt{2}$ is irrational!)

Circle

Any circle of radius r (or diameter $d = 2r$):

$$P = \tau r = 2\pi r = \pi 2r = \pi d \quad \approx 6.28r$$
$$A = \pi r^2 = \pi \left(\frac{d}{2}\right)^2 \quad \approx 3.14r^2$$

Interesting Natural Numbers

$$3435 = 3^3 + 4^4 + 3^3 + 5^5$$

Interesting Irrational Numbers

$$\begin{aligned}
\pi &\approx 3.14159265358979311600\ldots \\
\varphi &\approx 1.61803398874989490253\ldots \\
\psi &\approx 3.35988566624317755317\ldots \\
\sqrt{2} &\approx 1.41421356237309514548\ldots \\
\sqrt{3} &\approx 1.73205080756887719318\ldots \\
\sqrt{5} &\approx 2.23606797749978980505\ldots \\
\sqrt{6} &\approx 2.44948974278317788134\ldots \\
\sqrt{7} &\approx 2.64575131106459071617\ldots \\
\sqrt{8} &\approx 2.82842712474619029095\ldots \\
e &\approx 2.71828182845904509080\ldots \\
\gamma &\approx 0.57721566490153286061\ldots
\end{aligned}$$

More Interesting Numbers

$$\begin{aligned}
\tau &= 2\pi \\
\varphi &= \frac{\sqrt{5}}{2} \\
\psi &= \frac{1}{\varphi} \\
F_n &= \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} \\
e &= \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \times 2} + \frac{1}{1 \times 2 \times 3} + \frac{1}{1 \times 2 \times 3 \times 4} + \ldots = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \\
i &= \sqrt{-1} \\
\text{googol} &= 10^{100} \\
\text{googolplex} &= 10^{\text{googol}} = 10^{10^{100}} \\
C_{10} &= 0.12345678910111213141516171819202122232425\ldots
\end{aligned}$$

Riemann zeta Numbers

$$
\begin{aligned}
\zeta(s) &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \dots \\
\zeta(1) &= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \to \infty \\
\zeta(2) &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots \approx 1.64493406684822643647 \\
\zeta(3) &= \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \dots \approx 1.20205690315959428540 \\
\zeta(4) &= \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots \approx 1.08232323371113819152 \\
\zeta(1/2) &= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \dots - 1.46035450880958681289 \\
\zeta(0) &= 1 + 1 + 1 + 1 + 1 + \dots = -\frac{1}{2} \\
\zeta(-1) &= 1 + 2 + 3 + 4 + 5 + \dots = -\frac{1}{12}
\end{aligned}
$$