Slopes of Perpendicular Lines

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Abstract

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[In mathematics, perpendicular lines are more commonly called orthogonal lines. On a plane, the two concepts are equivalent: orthogonality is an extension of perpendicularity to spaces of higher dimension than the plane] The situation is depicted in Figure 1. Note that $a < 0$ and $b > 0$ in this example. Are the slopes always of opposite signs?

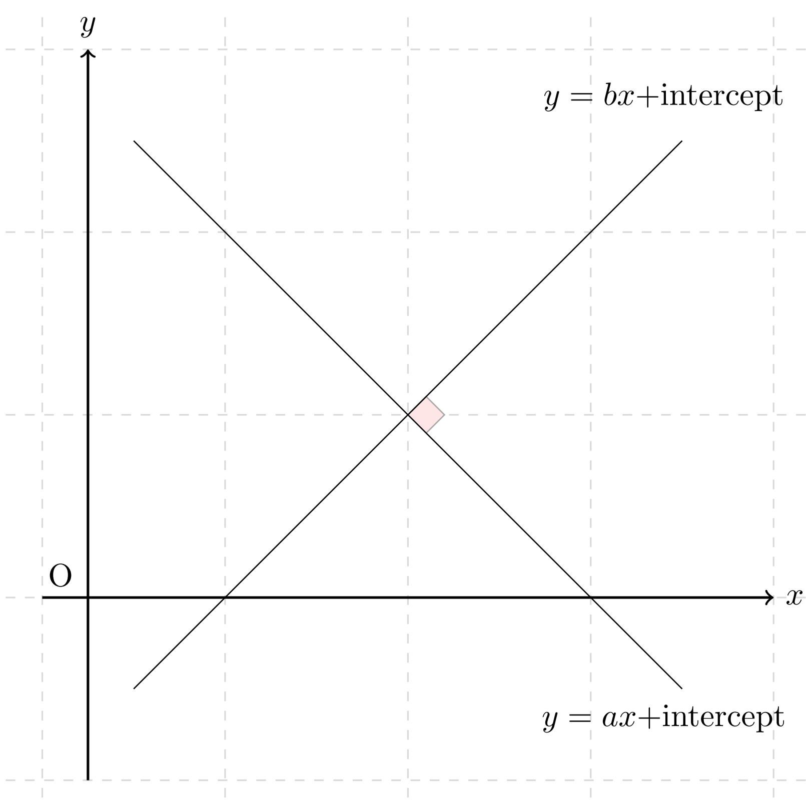

Figure 1: **Two Orthogonal Lines in a Cartesian Coordinate System**.

We have very little information to go by. Can we infer the slope b from a? Note that since only the slope matters in this problem, we can consider two lines that intersect at the origin. Figure 2 shows that we can also represent the slopes graphically. As we are now considering two lines that intersect at the origin, their equations are simply $y = ax$ and $y = bx$. And so for $x = 1$, say, we have $y = a$ on line OA and $y = b$ on line OB. Can we find an expression for b in terms of a? Amazingly we can! Thanks to several right triangles and the Pythagoras theorem.

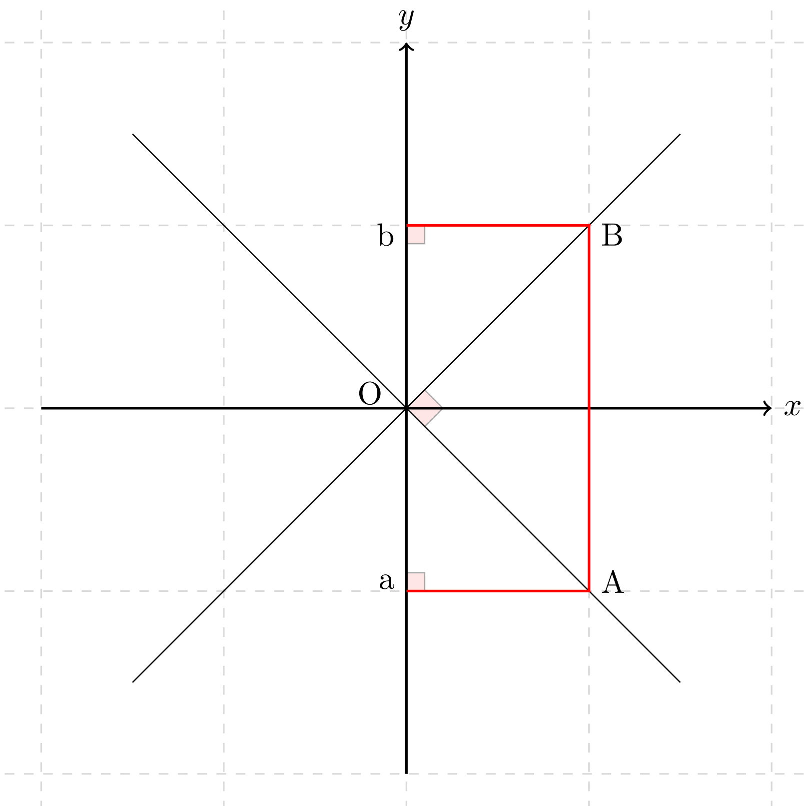

Figure 2: **Two Orthogonal Lines form Three Pythagorean Triangles**.

Triangle AOB yields:

$$(b-a)^2 = OA^2 + OB^2$$

Triangle OaA yields:

$$OA^2 = 1^2 + a^2$$

Triangle ObB yields:

$$OB^2 = 1^2 + b^2$$

Putting it all together gives:

$$\begin{aligned}
(b-a)^2 &= \quad OA^2 \quad + \quad OB^2 \\
&= 1^2 + a^2 \;\; + \;\; 1^2 + b^2 \\
b^2 - 2ab + a^2 &= a^2 + b^2 + 2 \\
-2ab &= 2 \\
b &= -\frac{1}{a}
\end{aligned}$$

The slope of any perpendicular line is therefore equal to the opposite of the inverse slope — or minus the inverse: $\boxed{b = -1/a}$. The special case $a = 1$ (the 45° degree line) is well known.