AMC 10 Problem Series (2804)

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Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Homework

Lesson: 1 2 3 4 5 6 7 8 9 10 11 **12**

Homework: Lesson 12

You have completed **10** of **10** challenge problems.
Due **Aug 28**.

Challenge Problems

Total Score: 60 / 60

Problem 1 – Correct! – Score: 6 / 6 (3482)

Problem: Report Error

A parabola with equation $y = x^2 + bx + c$ passes through the points $(2, 3)$ and $(4, 3)$. What is c?

(A) 2 **(B)** 5 **(C)** 7 **(D)** 10 **(E)** 11

Solution:

Substituting the points $(2, 3)$ and $(4, 3)$ into $y = x^2 + bx + c$, we obtain the system of equations

$$\begin{aligned} 4 + 2b + c &= 3, \\ 16 + 4b + c &= 3. \end{aligned}$$

These equations simplify to

$$\begin{aligned} 2b + c &= -1, \\ 4b + c &= -13. \end{aligned}$$

Multiplying the first equation by 2, we get $4b + 2c = -2$. Subtracting the equation $4b + c = -13$, we find $c = \boxed{11}$. The answer is (E).

Your Response(s):

E

Problem 2 – Correct! – Score: 6 / 6 (3483)

Problem: Report Error

If $a, b > 0$ and the triangle in the first quadrant bounded by the coordinate axes and the graph of $ax + by = 6$ has area 6, then $ab =$

(A) 3 **(B)** 6 **(C)** 12 **(D)** 108 **(E)** 432

Solution:

Setting $y = 0$ in the equation $ax + by = 6$, we get $ax = 6$, so the x-intercept of the line is $(6/a, 0)$. Setting

$x = 0$, we get $by = 6$, so the y-intercept is $(0, 6/b)$. Therefore, the area of the triangle is

$$\frac{1}{2} \cdot \frac{6}{a} \cdot \frac{6}{b} = \frac{18}{ab}.$$

This is equal to 6, so $18/(ab) = 6$, which means $ab = \boxed{3}$. The answer is (A).

Your Response(s):

A

Problem 3 – Correct! – Score: 6 / 6 (3484)

Problem: Report Error

The lines $x = \frac{1}{4}y + a$ and $y = \frac{1}{4}x + b$ intersect at the point $(1, 2)$. What is $a + b$?

(A) 0 **(B)** $\frac{3}{4}$ **(C)** 1 **(D)** 2 **(E)** $\frac{9}{4}$

Solution:

Substituting the point $(1, 2)$ into the equation $x = \frac{1}{4}y + a$ and $y = \frac{1}{4}x + b$, we obtain the equations $1 = 1/2 + a$ and $2 = 1/4 + b$, so $a = 1/2$ and $b = 7/4$. Then $a + b = \boxed{9/4}$. The answer is (E).

Your Response(s):

E

Problem 4 – Correct! – Score: 6 / 6 (3485)

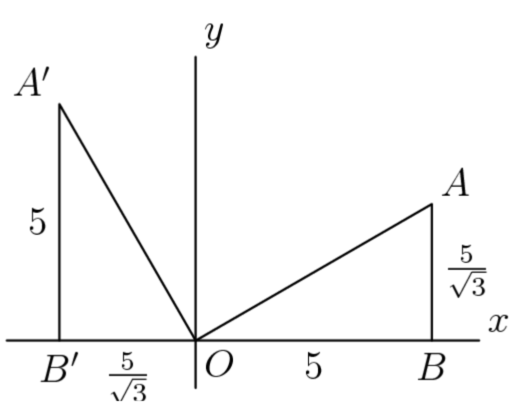
Problem: Report Error

Triangle OAB has $O = (0, 0)$, $B = (5, 0)$, and A in the first quadrant. In addition, $\angle ABO = 90^\circ$ and $\angle AOB = 30^\circ$. Suppose that $\overline{OA}$ is rotated 90° counterclockwise about O. What are the coordinates of the image of A?

(A) $\left(-\frac{10}{3}\sqrt{3}, 5\right)$ **(B)** $\left(-\frac{5}{3}\sqrt{3}, 5\right)$ **(C)** $(\sqrt{3}, 5)$ **(D)** $\left(\frac{5}{3}\sqrt{3}, 5\right)$ **(E)** $\left(\frac{10}{3}\sqrt{3}, 5\right)$

Solution:

We see that triangle OAB is 30°-60°-90° triangle, so $AB = OB/\sqrt{3} = 5/\sqrt{3}$.



Let A' be the image of A under the rotation, and let B' be the projection of A' onto the x-axis. Then triangles $A'B'O$ and OBA are congruent, so $OB' = AB = 5/\sqrt{3} = 5\sqrt{3}/3$, and $A'B' = OB = 5$. Therefore, the coordinates of A' are

$$\boxed{\left(-\frac{5}{3}\sqrt{3}, 5\right)}.$$

The answer is (B).

Your Response(s):

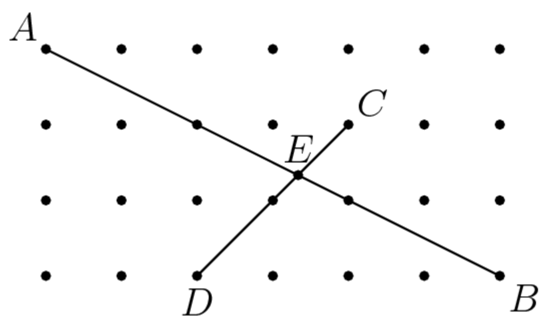
B

Problem 5 – Correct! – Score: 6 / 6 (3486)

Problem: Report Error

The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE.

(A) $4\sqrt{5}/3$ **(B)** $5\sqrt{5}/3$ **(C)** $12\sqrt{5}/7$ **(D)** $2\sqrt{5}$ **(E)** $5\sqrt{65}/9$



Solution:

Solution 1: We place the diagram in the coordinate plane so that $A = (0, 3)$, $B = (6, 0)$, $C = (4, 2)$, and $D = (2, 0)$.

The slope of AB is $3/(-6) = -1/2$, so an equation for line AB is

$$y = -\frac{1}{2}x + 3.$$

The slope of CD is $2/2 = 1$, so an equation for line CD is

$$y = x - 2.$$

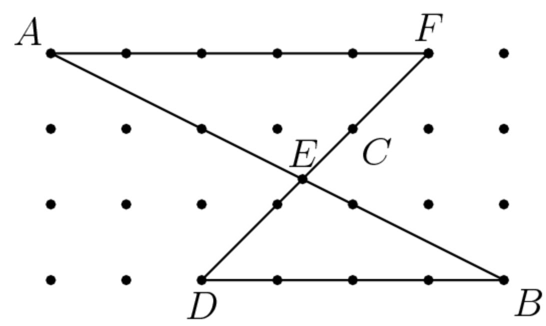
Setting these equations equal and solving for x, we find $x = 10/3$. Then $y = 10/3 - 2 = 4/3$.

Hence,

$$AE = \sqrt{\left(0 - \frac{10}{3}\right)^2 + \left(3 - \frac{4}{3}\right)^2} = \sqrt{\frac{125}{9}} = \boxed{\frac{5\sqrt{5}}{3}}.$$

The answer is (B).

Solution 2: Let CD and the line through A parallel to BD intersect at F.



Then triangles AEF and BED are similar, so

$$\frac{AE}{BE} = \frac{AF}{BD},$$

so

$$\frac{AE}{AE + BE} = \frac{AF}{AF + BD}.$$

But $AE + BE = AB = \sqrt{6^2 + 3^2} = \sqrt{45} = 3\sqrt{5}$, so

$$AE = AB \cdot \frac{AF}{AF + BD} = 3\sqrt{5} \cdot \frac{5}{5 + 4} = \boxed{\frac{5\sqrt{5}}{3}}.$$

Your Response(s):

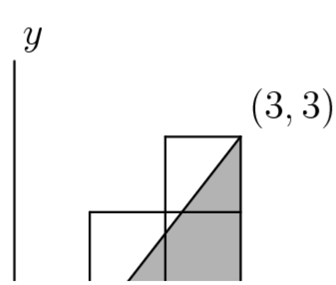
B

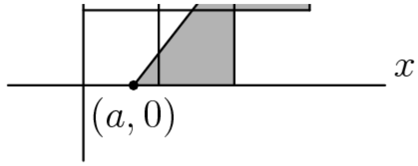
Problem 6 – Correct! – Score: 6 / 6 (3487)

Problem: Report Error

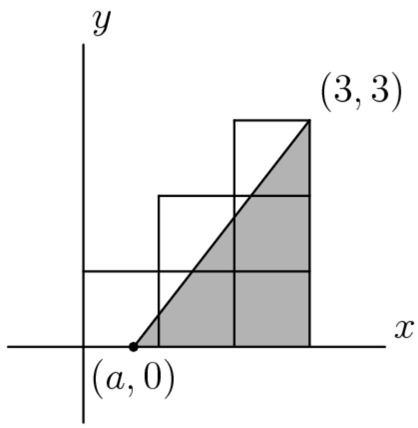
Five unit squares are arranged in the coordinate plane as shown, with the lower left corner at the origin. The slanted line, extending from $(a, 0)$ to $(3, 3)$, divides the entire region into two regions of equal area. What is a?

(A) $\frac{1}{2}$ **(B)** $\frac{3}{5}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{3}{4}$ **(E)** $\frac{4}{5}$





Solution:
There are five squares, so the area of the shaded region is 5/2. If we add another square, as shown, then the area of the shaded region becomes $5/2 + 1 = 7/2$.



The height of this triangle is 3, so the base has length

$$\frac{2 \cdot 7/2}{3} = \frac{7}{3}.$$

The base is also $3 - a$, so $3 - a = 7/3$, which means $a = \boxed{2/3}$. The answer is (C).

Your Response(s):
C

Problem 7 – Correct! – Score: 6 / 6 (3488)

Problem: Report Error
In rectangle $ABCD$, we have $A = (6, -22)$, $B = (2006, 178)$, and $D = (8, y)$, for some integer y. What is the area of rectangle $ABCD$?

(A) 4000 **(B)** 4040 **(C)** 4400 **(D)** 40,000 **(E)** 40,400

Solution:
The slope of AB is $[178 - (-22)]/(2006 - 6) = 200/2000 = 1/10$. Then the slope of AD must be -10. But the slope of AD is

$$\frac{y + 22}{8 - 6} = \frac{y + 22}{2},$$

so $(y + 22)/2 = -10$, which means $y = -42$.

Then $AB = \sqrt{(2006-6)^2 + (178+22)^2} = \sqrt{4040000} = 100\sqrt{404}$, and $AD = \sqrt{(6-8)^2 + (-22+42)^2} = \sqrt{404}$, so the area of rectangle $ABCD$ is $AB \cdot AD = 100\sqrt{404} \cdot \sqrt{404} = \boxed{40400}$. The answer is (E).

Your Response(s):

E

Problem 8 – Correct! – Score: 6 / 6 (3489)

Problem:

Report Error

If (a, b) and (c, d) are two points on the line whose equation is $y = mx + k$, then the distance between (a, b) and (c, d), in terms of a, c, and m, is

(A) $|a-c|\sqrt{1+m^2}$ **(B)** $|a+c|\sqrt{1+m^2}$ **(C)** $\dfrac{|a-c|}{\sqrt{1+m^2}}$ **(D)** $|a-c|(1+m^2)$ **(E)** $|a-c||m|$

Solution:

Since (a, b) and (c, d) lie on the line $y = mx + k$, $b = am + k$ and $d = cm + k$. Then the distance between (a, b) and (c, d) is given by

$$\begin{aligned}\sqrt{(a-c)^2 + (b-d)^2} &= \sqrt{(a-c)^2 + [(am+k) - (cm+k)]^2} \\ &= \sqrt{(a-c)^2 + (am-cm)^2} \\ &= \sqrt{(a-c)^2 + m^2(a-c)^2} \\ &= \sqrt{(a-c)^2}\sqrt{1+m^2} \\ &= \boxed{|a-c|\sqrt{1+m^2}}.\end{aligned}$$

The answer is (A).

Your Response(s):

A

Problem 9 – Correct! – Score: 6 / 6 (3490)

Problem:

Report Error

A *lattice point* is a point in the plane with integer coordinates. How many lattice points are on the line segment whose endpoints are $(3, 17)$ and $(48, 281)$? (Include both endpoints of the segment in your count.)

(A) 2 **(B)** 4 **(C)** 6 **(D)** 16 **(E)** 46

Solution:

The slope of the line joining $(3, 17)$ and $(48, 281)$ is

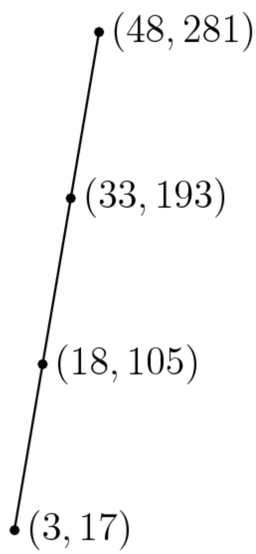
$$\frac{281-17}{48-3} = \frac{264}{45} = \frac{88}{15}.$$

Then for any point (x, y) on this line,

$$\frac{y-17}{x-3} = \frac{88}{15}.$$

If (x, y) is a lattice point, then x and y are integers, so $y - 17$ and $x - 3$ are integers. Since 88 and 15 are relatively prime, $y - 17$ must be a multiple of 88, and $x - 3$ must be a multiple of 15. Hence, $x - 3 = 15k$ for some integer k.

Furthermore, we want $3 \le x \le 48$. Therefore, the only possible values of k are 0, 1, 2, and 3, for a total of $\boxed{4}$ lattice points. The answer is (B).



Your Response(s):

B

Problem 10 – Correct! – Score: 6 / 6 (4514)

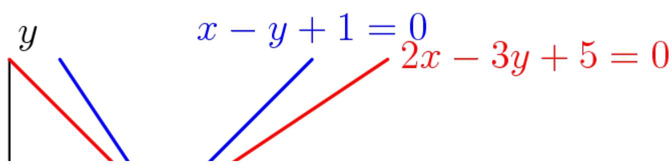
Problem: Report Error

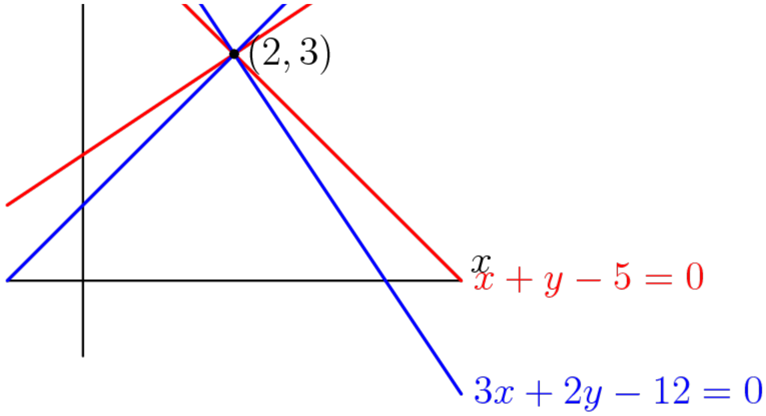
The number of distinct points in the xy-plane common to the graphs of $(x+y-5)(2x-3y+5)=0$ and $(x-y+1)(3x+2y-12)=0$ is

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution:

The graph of $(x+y-5)(2x-3y+5)=0$ is the union of the lines $x+y-5=0$ and $2x-3y+5=0$ (shown in red below). The graph of $(x-y+1)(3x+2y-12)=0$ is the union of the lines $x-y+1=0$ and $3x-2y-12=0$ (shown in blue below).





Every line passes through the point $(2, 3)$, so the intersection of the two graphs consists of exactly $\boxed{1}$ point. The answer is (B).

Your Response(s):

B