AMC 10 Problem Series (2804)

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Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Homework

Lesson: 1 2 3 4 **5** 6 7 8 9 10 11 12

Homework: Lesson 5

You have completed **10** of **10** challenge problems.
Past Due **Jul 10**.

Readings

Lesson 5 Transcript: Fri, Jul 2

Challenge Problems

Total Score: 60 / 60

Problem 1 – Correct! – Score: 6 / 6 (2972)

Problem: Report Error

In a collection of red, blue, and green marbles, there are 25% more red marbles than blue marbles, and there are 60% more green marbles than red marbles. Suppose that there are r red marbles. What is the total number of marbles in the collection?

(A) $2.85r$ **(B)** $3r$ **(C)** $3.4r$ **(D)** $3.85r$ **(E)** $4.25r$

Solution:

Let b and g and be the number of blue and green marbles, respectively. There are 25% more red marbles than blue marbles, so

$$r = \frac{125}{100}b = \frac{5}{4}b,$$

which means $b = 4r/5$.

There are 60% more green marbles than red marbles, so

$$g = \frac{160}{100}r = \frac{8}{5}r.$$

Therefore, the total number of marbles is

$$r + b + g = r + \frac{4}{5}r + \frac{8}{5}r = \frac{17}{5}r = \boxed{3.4r}.$$

The answer is (C).

Your Response(s):

- C

Problem 2 – Correct! – Score: 6 / 6 (2973)

Problem:

Report Error

At the beginning of the school year, 50% of all students in Mr. Wells' math class answered "Yes" to the question "Do you love math", and 50% answered "No." At the end of the school year, 70% answered "Yes" and 30% answered "No." Altogether, $x\%$ of the students gave a different answer at the beginning and end of the school year. What is the difference between the maximum and the minimum possible values of x?

(A) 0 **(B)** 20 **(C)** 40 **(D)** 60 **(E)** 80

Solution:

First, we compute the maximum value of x. At the start of the year, 50% of the students said "yes," and at the end of the year, 30% of the students said "no," so only 30% of the students can change from "yes" to "no" - the remaining 20% must stay "yes."

At the start of the year, 50% of the students said "no," and all of them can change to "yes" at the end of the year. Thus, the maximum value of x is $30 + 50 = 80$.

Now we compute the minimum value of x. At the start of the year, 50% of the students said "no," and at the end of the year, 30% of the students said "no," so only 30% of the students can stay "no" - the remaining 20% must become "yes."

At the start of the year, 50% of the students said "yes," and all of them can stay "yes" at the end of the year. Thus, the minimum value of x is 20.

Hence, the difference between the maximum and minimum possible values of x is $80 - 20 = \boxed{60}$. The answer is (D).

Your Response(s):

D

Problem 3 – Correct! – Score: 6 / 6 (2974)

Problem:

Report Error

The average of the numbers 1, 2, 3, . . ., 98, 99, and x is $100x$. What is x?

(A) $\frac{49}{101}$ **(B)** $\frac{50}{101}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{51}{101}$ **(E)** $\frac{50}{99}$

Solution:

The average of the 100 numbers 1, 2, 3, . . ., 98, 99, and x is

$$\frac{1+2+3+\dots+98+99+x}{100} = \frac{\frac{1+99}{2}\cdot 99 + x}{100} = \frac{4950+x}{100}.$$

Hence,

$$\frac{4950+x}{100} = 100x.$$

Multiplying both sides by 100, we get $4950 + x = 10000x$. Then $9999x = 4950$, so

$$x = \frac{4950}{9999} = \boxed{\frac{50}{101}}.$$

The answer is (B).

Your Response(s):

B

Problem 4 – Correct! – Score: 6 / 6 (2975)

Problem: Report Error

A teacher gave a test to a class in which 10% of the students are juniors and 90% are seniors. The average score on the test was 84. The juniors all received the same score, and the average score of the seniors was 83. What score did each of the juniors receive on the test?

(A) 85 **(B)** 88 **(C)** 93 **(D)** 94 **(E)** 98

Solution:

Assume that there are 100 students in the class, so there are 10 juniors and 90 seniors. The average score of the seniors is 83, so the sum of their scores is $90 \cdot 83 = 7470$.

Let x be the score of every junior. Then the sum of the scores of all 100 students is $10x + 7470$. The average score is 84, so

$$\frac{10x + 7470}{100} = 84.$$

Solving for x, we find $x = \boxed{93}$. The answer is (C).

Your Response(s):

C

Problem 5 – Correct! – Score: 6 / 6 (2976)

Problem: Report Error

Let a and b be relatively prime integers with $a > b > 0$ and

$$\frac{a^3 - b^3}{(a-b)^3} = \frac{73}{3}.$$

What is $a - b$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Note: Two positive integers are *relatively prime* if their greatest common divisor is 1.

Solution:

Cross-multiplying, we get

$$3(a^3 - b^3) = 73(a - b)^3.$$

The expression $a^3 - b^3$ factors as $(a-b)(a^2 + ab + b^2)$. Since $a > b$, the quantity $a - b$ is non-zero, so we can divide both sides by $a - b$ to get

$$3(a^2+ab+b^2)=73(a-b)^2.$$

Expanding and putting all the terms on one side, we get

$$70a^2-149ab+70b^2=0,$$

which factors as

$$(7a-10b)(10a-7b)=0.$$

Therefore, $7a=10b$ or $10a=7b$. Since $a>b, 7a=10b$. Also, a and b are relatively prime, so $a=10$ and $b=7$. Then $a-b=\boxed{3}$. The answer is (C).

Your Response(s):

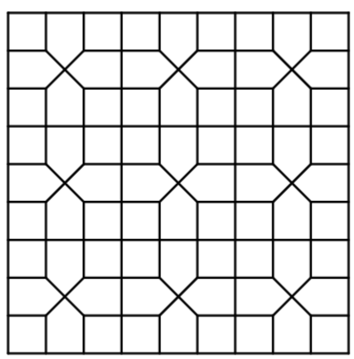
C

Problem 6 – Correct! – Score: 6 / 6 (2977)

Problem: Report Error

The plane is tiled by congruent squares of side length a and congruent pentagons of side lengths a and $\frac{a\sqrt{2}}{2}$, as arranged in the diagram below. The percent of the plane that is enclosed by the pentagons is closest to

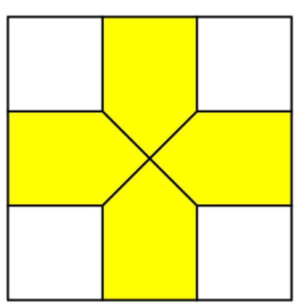
(A) 50 **(B)** 52 **(C)** 54 **(D)** 56 **(E)** 58



(You can assume that this diagram is drawn to scale.)

Solution:

We can think of the plane as being tiled by the following square tile.



The pentagons take up $5/9$ of the square tile, which is closest to $\boxed{56}$ percent. The answer is (D).

Your Response(s):

D

Problem 7 – Correct! – Score: 6 / 6 (2978)

Problem:
Report Error

Let A, M, and C be nonnegative integers such that $A+M+C=10$. What is the maximum value of $A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A$?

(A) 49 **(B)** 59 **(C)** 69 **(D)** 79 **(E)** 89

Solution:
In the spirit of Simon's Favorite Factoring Trick, we look for an expression that resembles $AMC+AM+MC+CA$ which factors. Note that

$$(A+1)(M+1)(C+1) = AMC+AM+AC+MC+A+M+C+1.$$

Since $A+M+C=10$ is a constant, maximizing this product is equivalent to maximizing the expression given in the problem.

Subject to the condition $A+M+C=10$, the product $(A+1)(M+1)(C+1)$ is maximized when the factors $A+1$, $M+1$, and $C+1$ are as close as possible. Thus, we can take $A=4$, $M=3$, and $C=3$. This gives us

$$(A+1)(M+1)(C+1) = 5 \cdot 4 \cdot 4 = 80,$$

so $AMC+AM+AC+MC = 80-(A+M+C)-1 = \boxed{69}$. The answer is (C).

Your Response(s):

C

Problem 8 – Correct! – Score: 6 / 6 (2979)

Problem:
Report Error

The mean, median, unique mode, and range of a collection of eight integers are all equal to 8. The largest integer that can be an element of this collection is

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Note: The *mode* is the most frequent number, and the *range* is the difference between the lowest number and the highest number. For example, the mode of the numbers 2, 2, 7, 7, 7, 7, 13, 13, 13 is 7, and the range is $13-2=11$.

Solution:
We claim that the largest possible integer is 14.

Suppose the largest number is at least 16. The range is 8, so the smallest number must be at least $16-8=8$. But if the smallest number is at least 8 and the largest number is at least 16, then the average of the numbers must be greater than 8, contradiction, so the largest number is at most 15.

Suppose the largest number is 15. The range is 8, so the smallest number must be $15 - 8 = 7$. Also, the median is 8, so the average of the two middle numbers must be 8. The lowest number is 7, so the two middle numbers are either 7 and 9, or 8 and 8. If the two middle numbers are 7 and 9, then there are no 8s in the collection. But the mode is 8, so the two middle numbers are 8 and 8. Then the numbers, in increasing order, must be of the form

$$7, a, b, 8, 8, c, d, 15.$$

The average is 8, so the sum of all eight numbers is $8 \cdot 8 = 64$. Therefore, the sum of a, b, c, and d is $64 - 7 - 8 - 8 - 15 = 26$. But a, b, c, and d are all at least 7, so their sum must be at least $4 \cdot 7 = 28$, contradiction.

Hence, the largest number is at most 14. The numbers 6, 6, 6, 8, 8, 8, 8, 14 show that the value $\boxed{14}$ is achievable. The answer is (D).

Your Response(s):

D

Problem 9 – Correct! – Score: 6 / 6 (2980)

Problem: Report Error

A rectangular floor measures a feet by b feet, where a and b are positive integers with $b > a$. An artist paints a rectangle on the floor with the sides of the rectangle parallel to the sides of the floor. The unpainted part of the floor forms a border of width 1 foot around the painted rectangle and occupies half the area of the entire floor. How many possibilities are there for the ordered pair (a, b)?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Solution:

The border has width 1, so the dimensions of the painted rectangle are $(a - 2) \times (b - 2)$. Hence,

$$(a - 2)(b - 2) = \frac{ab}{2}.$$

Multiplying both sides by 2, expanding, and putting all the terms on one side, we get

$$ab - 4a - 4b + 8 = 0.$$

Applying Simon's Favorite Factoring Trick, we add 8 to both sides of the equation, to get

$$ab - 4a - 4b + 16 = 8.$$

The left-hand side factors as

$$(a - 4)(b - 4) = 8.$$

Since $a < b$, the possible pairs $(a - 4, b - 4)$ are $(1, 8)$ and $(2, 4)$, which leads to the $\boxed{2}$ solutions (a, b) of $(5, 12)$ and $(6, 8)$. The answer is (B).

Your Response(s):

B

Problem 10 – Correct! – Score: 6 / 6 (2981)

Problem: Report Error

When 15 is appended to a list of integers, the mean is increased by 2. When 1 is appended to the enlarged list, the mean of the enlarged list is decreased by 1. How many integers were in the original list?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Solution:

Let n be the number of integers in the original list, let m be the original mean, so the sum of the original n numbers is mn.

When we add the number 15, the sum of the $n+1$ numbers is now $mn+15$, and the average is increased by 2, so

$$\frac{mn+15}{n+1}=m+2.$$

Multiplying both sides by $n+1$ and simplifying, we get $m+2n=13$.

When we further add the number 1, the sum of the $n+2$ numbers is now $mn+16$, and the average decreases by 1, so

$$\frac{mn+16}{n+2}=m+1.$$

Multiplying both sides by $n+2$ and simplifying, we get $2m+n=14$.

Multiplying the equation $m+2n=13$ by 2, we get $2m+4n=26$. Subtracting the equation $2m+n=14$, we get $3n=12$, so $n=\boxed{4}$. The answer is (A).

Your Response(s):

A

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