Russian School of Math: Lesson 12

James & Patrick

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Abstract

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

1

Hexagon $ABCDEF$ is given. It is known that $AB = AC$, $CD = DE$, $EF = FA$ and angles $\angle A$ and $\angle C$ are right angles. Find the angle between the lines FD and BE.

Solution

To find the angle between lines FD and BE in hexagon ABCDEF with the given properties, let's analyze the configuration step by step.
Understanding the Properties: - The hexagon has pairs of equal sides: AB=BC
, CD=DE, and EF=FA. - Angles A and C are right angles (i.e., A=90 and C=90). Positioning the Hexagon: - Let's place point A at the origin (0, 0) in a Cartesian coordinate system. - Since A is a right angle, we can place B at (a,0) and C at (a,b) where AB=BC=a
. - Thus, we have:
A(0,0)
B(a,0) C(a,b)
Finding Points D, E, and F: - Since C
is also a right angle, we can place point D directly below point C at (ac,b) where CD=DE=c. - Point E would then be at (ac,bc). - Finally, point F would be directly left of point E at (0,bc) since EF=FA=d . Calculating Slopes: - The slope of line FD from F(0,bc) to D(ac,b) is: slope of FD=b(bc)(ac)0=cac
The slope of line BE from B(a,0)
to E(ac,bc) is: slope of BE=(bc)0(ac)a=bcc=bcc
Finding the Angle Between the Lines: - The angle
between two lines with slopes m1 and m2 can be found using the formula: tan=m1m21+m1m2 - Here, m1=cac and m2=bcc . Calculating the Tangent: - Substitute the slopes into the formula: tan=cac+bcc1+cacbcc Final Angle Calculation: - After simplifying the expression for tan , we can find that the angle is 45
.
Thus, the angle between lines FD and BE is 45 degrees.

2

The perimeter of triangle $\triangle ABC$ is equal to 4. The points X and Y are marked on rays AB and AC respectively, so that $AX = AY = 1$. The segments BC and XY intersect at point M. Find the minimum of two perimeters: The triangle $\triangle ABM$ or the triangle $\triangle ACM$.

Solution

Let us reflect the point A over both X and Y to two points U and V so that $AU = AV = 2$. This seems slightly better, because $AU = AV = 2$ now, and the "two" in the perimeter is now present. But what do we do? Recalling that $s = 2$ in the triangle, we find that U and V is the tangency points of the excircle, call it a. Set IA the excenter, tangent to BC at TA. See Figure.
We have now encoded the $AX = AY = 1$ condition as follows: X and Y are the midpoints of the tangents to the A-excircle. We need to show that one of ABM or ACM has a perimeter equal to the length of the tangent.
What would have to be true in order to obtain the relation $AB + BM + MA = AU$? Write $AU = AB + BU = AB + BT$. We need $BM + MA = BT$, or $MA = MT$. Points X, M, Y have the property that their distance to A equals the length of their tangents to the A-excircle. This

motivates the last addition to our diagram: construct a circle of radius zero at A, say 0. Then X and Y lie on the radical axis of 0 and Ta; hence so does M. Now we have $MA = MT$, as required. It reflects whether T lies on BM or CM. (It must lie in at least one, because we are told that M lies inside the segment BC, and the tangency points of the A-excircle to BC always lie in this segment as well.) This completes the solution, which we present concisely below.

Let IA be the center of the A-excircle, tangent to BC at T and to the extensions of AB and AC at U and V. We see that $AU = AV = s = 2$. Then XY is the radical axis of the A-excircle and the circle of radius zero at A. Therefore $AM = MT$.

Assume without loss of generality that T lies on MC, as opposed to MB. Then $AB + BM + MA = AB + BM + MT = AB + BT = AB + BU = AU = 2$ as desired.

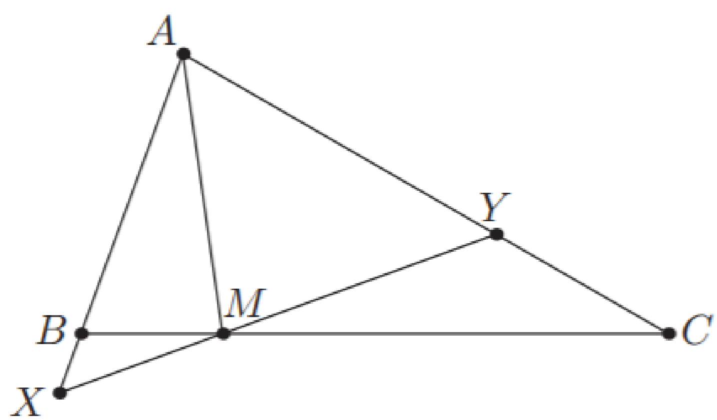

Figure 2.7D. A problem from the All-Russian MO 2010.

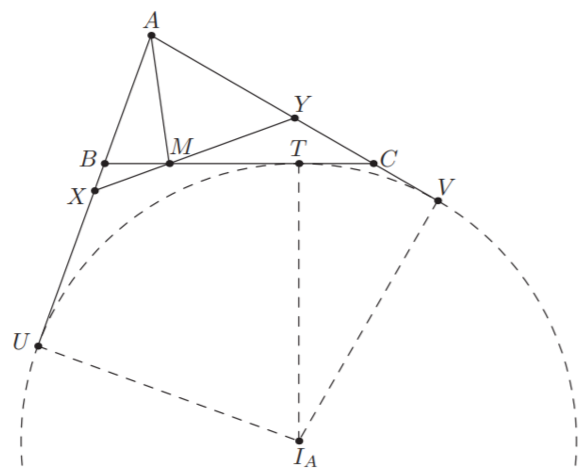

Figure 2.7E. Adding an excircle to handle the conditions.

3

Triangle $\triangle ABC$ is inscribed in circle σ with $AB = 5$, $BC = 7$, and $AC = 3$. The bisector of angle $\angle A$ meets side BC at D and circle σ at a second point E. Let γ be the circle with diameter DE. Circles σ and γ meet at E and second point F. Then $F^2 = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

Solution

4

Circles σ_1 and σ_2 intersect at point X and Y. Line l is tangent to σ_1 and σ_2 at A and B respectively, with line AB closer to point X than to Y. Circle σ passes through A and B intersecting σ_1 again at $D \neq A$ and intersecting σ_2 again at $C \neq B$. The three points C, Y, D are collinear, and $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

Solution