Math Competition Tricks

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Abstract

This note reviews a selection of tricks that may be useful in math competitions.

Mental Arithmetic

It is useful to add/multiply/divide fast. There are too many tricks to review, but here are a few basic ones. With practice you will be able to use these tricks while calculating in your head.

Add numbers by grouping them:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 4 \times 10 + 5 = 45$$

Add numbers by rounding them:

$$978 + 237 = (980 - 2) + (220 + 17) = 1200 + 15 = 1215$$

Multiply by 5:

$$978 \times 5 = (1000 - 20 - 2) \times \frac{10}{2} = \frac{500 - 10 - 1}{2} \times 10 = 4890$$

where we have also decomposed 978 to make the division by 2 even easier (skip this step if you can quickly halve 978).

Multiply numbers by decomposing them:

$$\begin{aligned} 14 \times 16 &= (15 - 1) \times (15 + 1) \\ &= 15^2 - 1 = 224 \end{aligned}$$

where we have used $15^2 = 225$ and the difference-of-squares formula:

$$(a + b)(a - b) = a^2 - b^2$$

Similarly, $13 \times 17 = 15^2 - 4 = 221$ (if hesitant, check that the last digit matches: $3 \times 7 = 21$, so the last digit 1 is indeed correct). The difference-of-squares formula can always be applied when multiplying numbers that differ by a multiple of 2 (multiplying two even numbers or multiplying two odd numbers).

Multiply numbers by rounding up:

$$\begin{aligned} 19 \times 18 &= 20 \times 18 - 18 \\ &= 360 - 18 = 342 \end{aligned}$$

Multiply numbers by rounding up and down:

$$\begin{aligned} 19 \times 23 &= (20 - 1) \times (20 + 3) \\ &= 20^2 + (3 - 1) \times 20 - 3 = 400 + 40 - 3 = 437 \end{aligned}$$

Square numbers by rounding up:

$$\begin{aligned} 99^2 &= (100 - 1)^2 \\ &= 10000 - 200 + 1 = 9801 \end{aligned}$$

where we have used:

$$(a+b)^2 = a^2 + 2ab + b^2$$

Square numbers by rounding up or down:

$$\begin{aligned} 13^2 &= (15-2)^2 \\ &= 200 + 25 - 60 + 4 = 140 + 29 = 169 \end{aligned}$$

where we suppose you have memorized $15^2 = 225 = 200 + 25$ (but forgotten 13^2). Because it is easier to subtract 60 from 200 than from 225, we also split 225 as $200 + 25$. These manipulations are to be done in your head or very quickly on a scrap of paper.

Useful Sums

The sum of the first n natural numbers:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
$$1 + 2 + 3 + \ldots + 10 = 55$$
$$1 + 2 + 3 + \ldots + 100 = 505$$

The sum of the first $(2n-1)$ odd numbers:

$$1 + 3 + 5 + \ldots + (2n-1) = n^2$$
$$1 + 3 + 5 + \ldots + 9 = 1 + 3 + 5 + \ldots + (2 \times 5 - 1) = 5^2 = 25$$
$$1 + 3 + 5 + \ldots + 99 = 1 + 3 + 5 + \ldots + (2 \times 50 - 1) = 50^2 = 2500$$

The sum of the first $2n$ even numbers:

$$2 + 4 + 6 + \ldots + (2n) = n(n+1)$$
$$2 + 4 + 6 + \ldots + 10 = 2 + 4 + 6 + \ldots + (2 \times 5) = 5 \times 6 = 30$$
$$2 + 4 + 6 + \ldots + 100 = 2 + 4 + 6 + \ldots + (2 \times 50) = 50 \times 51 = 2550$$

The sum of the first n squares formula and first ten sums:

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1, 5, 14, 30, 55, 91, 140, 204, 285, 385.$$

The sum of the first n cubes formula and first ten sums:

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^3$$
$$1, 9, 36, 100, 225, 441, 784, 1296, 2025, 3025.$$

The sum of n terms of a geometric series:

$$1 + a + a^2 + \ldots + a^n = \frac{1-a^n}{1-a}$$

The Fibonacci numbers are the sum of the two preceding numbers in the Fibonacci sequence:

$$F_n = F_{n-1} + F_{n-2}$$

where the first two numbers in the sequence are typically 0 and 1:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

The Lucas numbers are Fibonacci numbers with starting values 2 and 0:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \ldots$$

Prime Numbers

The first few prime numbers with their index:

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
p	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71

i	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
p	73	79	83	89	97	101	103	107	109	113	127	131	137	139	149	151	157	163	167	173

i	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
p	179	181	191	193	197	199	211	223	227	229	233	239	241	251	257	263	269	271	277	281

Prime numbers of the twentieth and twenty-first centuries (problems involving numbers close to the current year are popular: the closest prime numbers to 2020 are 2017 and 2027):

$$1901, 1907, 1913, 1931, 1933, 1949, 1951, 1973, 1979, 1987, 1993, 1997, 1999,$$
$$2003, 2011, 2017, 2027, 2029, 2039, 2053, 2063, 2069, 2081, 2083, 2087, 2089, 2099$$

Mersenne primes are prime numbers of the form $2^p - 1$, for some prime number p. The first few Mersenne primes are:

p	2	3	5	7	11	13	17	19	31
$2^p - 1$	3	7	31	127	8191	131071	524287	2147483647	

Some Mersenne numbers that are not prime include:

p	11	23	29
$2^p - 1$	$2047 = 23 \times 89$	$8388607 = 47 \times 178481$	$536870911 = 233 \times 1103 \times 2089$

Fermat primes are prime numbers of the form $2^{2^n} + 1$. There are only five known Fermat primes:

n	0	1	2	3	4
$2^{2^n} + 1$	3	5	17	257	65537

The famous mathematician Euler showed that

$$2^{2^5} + 1 = 2^{32} + 1 = 4294967297 = 641 \times 6700417$$

Some Fibonacci numbers are prime. Here are the first few:

$$2, 3, 5, 13, 89, 233, 1597, 28657$$

Pythagorean Triples

A selection of Pythagorean triples:

$(3, 4, 5)$	$(5, 12, 13)$	$(8, 15, 17)$	$(7, 24, 25)$	$(20, 21, 29)$	$(12, 35, 37)$
$(9, 40, 41)$	$(28, 45, 53)$	$(11, 60, 61)$	$(16, 63, 65)$	$(33, 56, 65)$	$(48, 55, 73)$
$(13, 84, 85)$	$(36, 77, 85)$	$(39, 80, 89)$	$(65, 72, 97)$	$(20, 99, 101)$	$(60, 91, 109)$

Useful Factorials

These factorial products are worth remembering:

$$
\begin{aligned}
3! &= 6 \\
4! &= 24 \\
5! &= 120 \\
6! &= 720 \\
7! &= 5,040 \\
8! &= 40,320 \\
9! &= 362,880 \\
10! &= 3,628,800
\end{aligned}
$$

Useful Squares

These are the first ten squares: $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$. Here are a few more:

$11^2 = 121$	$12^2 = 144$	$13^2 = 169$	$14^2 = 196$	$15^2 = 225$
$16^2 = 256$	$17^2 = 289$	$18^2 = 324$	$19^2 = 361$	$20^2 = 400$
$21^2 = 441$	$22^2 = 484$	$23^2 = 529$	$24^2 = 576$	$25^2 = 625$
$26^2 = 676$	$27^2 = 729$	$28^2 = 784$	$29^2 = 841$	$30^2 = 900$
$31^2 = 961$	$32^2 = 1,024$	$33^2 = 1,089$	$34^2 = 1,156$	$35^2 = 1,225$
$36^2 = 1,296$	$37^2 = 1,369$	$38^2 = 1,444$	$39^2 = 1.521$	$40^2 = 1,600$
$41^2 = 1,681$	$42^2 = 1,764$	$43^2 = 1,849$	$44^2 = 1,936$	$45^2 = 2,025$
$50^2 = 2,500$	$55^2 = 3,025$	$60^2 = 3,600$	$65^2 = 4,225$	$70^2 = 4,900$
$75^2 = 5,625$	$80^2 = 6,400$	$85^2 = 7,225$	$90^2 = 8,100$	$95^2 = 9,025$
$96^2 = 9,216$	$97^2 = 9,409$	$98^2 = 9,604$	$99^2 = 9,801$	$100^2 = 10,000$

Useful Cubes

$$
\begin{aligned}
2^3 &= 8\\
3^3 &= 27\\
4^3 &= 64\\
5^3 &= 125\\
6^3 &= 216\\
7^3 &= 343\\
8^3 &= 512\\
9^3 &= 729\\
10^3 &= 1000\\
11^3 &= 1331\\
12^3 &= 1728\\
13^3 &= 2197\\
14^3 &= 2744\\
15^3 &= 3375\\
16^3 &= 4096\\
17^3 &= 4913\\
18^3 &= 5832\\
19^3 &= 6859\\
20^3 &= 8000
\end{aligned}
$$

Useful Fourth Powers

$$
\begin{aligned}
2^4 &= 16\\
3^4 &= 81\\
4^4 &= 256\\
5^4 &= 625\\
6^4 &= 1296\\
7^4 &= 2401\\
8^4 &= 4096\\
9^4 &= 6561\\
10^4 &= 10000
\end{aligned}
$$

Some Useful Powers

Memorizing powers can come in handy:

$$
\begin{aligned}
2^2 &= 4\\
2^3 &= 8\\
2^4 &= 16\\
2^5 &= 32\\
2^6 &= 64\\
2^7 &= 128\\
2^8 &= 256\\
2^9 &= 512\\
2^{10} &= 1024\\
2^{11} &= 2048\\
2^{12} &= 4096\\
2^{13} &= 8192\\
2^{14} &= 16384\\
2^{15} &= 32768
\end{aligned}
$$

Powers of 3:

$$
\begin{aligned}
3^2 &= 9\\
3^3 &= 27\\
3^4 &= 81\\
3^5 &= 243\\
3^6 &= 729\\
3^7 &= 2187\\
3^8 &= 6561\\
3^9 &= 19683\\
3^{10} &= 59049
\end{aligned}
$$

Powers of 5:

$$
\begin{aligned}
5^2 &= 25\\
5^3 &= 125\\
5^4 &= 625\\
5^5 &= 3125\\
5^6 &= 15625\\
5^7 &= 78125\\
5^8 &= 390625
\end{aligned}
$$

Powers of 6:

$$6^2 = 36$$
$$6^3 = 216$$
$$6^4 = 1296$$
$$6^5 = 7776$$

Powers of 7:

$$7^2 = 49$$
$$7^3 = 343$$
$$7^4 = 2401$$
$$7^5 = 16807$$

Powers of 11:

$$11^2 = 121$$
$$11^3 = 1331$$
$$11^4 = 14641$$
$$11^5 = 161051$$

Useful Conversion Rates

1 ft	$=$	12 in
1 yd	$=$	3 ft
1 in	$=$	2.54 cm
1 m	$=$	3.28 ft
1 mi	$=$	1760 yd
1 mi	$=$	1609 m
1 ft^2	$=$	144 in^2
1 yd^2	$=$	9 ft^2
1 mi^2	$=$	2.59 km^2
1 mi^2	$=$	640 ac^2
1 ha	$=$	2.47 ac
1 ft^3	$=$	1728 in^3
1 yd^3	$=$	27 ft^3
1 in^3	$=$	16.39 cm^3
1 m^3	$=$	35.31 ft^3
1 qt	$=$	2 pt
1 pt	$=$	16 fl oz
1 gal	$=$	128 fl oz
1 gal	$=$	4 qt
1 pt	$=$	473.18 ml
1 fl oz	$=$	29.57 ml

Highly Composite Numbers

A positive integer that has more divisors than any smaller positive integer. A selection:

2	2
4	2^2
6	$2 \cdot 3$
12	$2^2 \cdot 3$
24	$2^3 \cdot 3$
36	$2^2 \cdot 3^2$
48	$2^4 \cdot 3$
60	$2^2 \cdot 3 \cdot 5$
120	$2^3 \cdot 3 \cdot 5$
180	$2^2 \cdot 3^2 \cdot 5$
240	$2^4 \cdot 3 \cdot 5$
360	$2^3 \cdot 3^2 \cdot 5$
720	$2^4 \cdot 3^2 \cdot 5$
840	$2^3 \cdot 3 \cdot 5 \cdot 7$
$1,260$	$2^2 \cdot 3^2 \cdot 5 \cdot 7$
$1,680$	$2^4 \cdot 3^1 \cdot 5 \cdot 7$
$2,520$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$
$5,040$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$

Useful Irrational Numbers

$$\begin{aligned}
\pi &\approx 3.14159\ldots \\
e &\approx 2.71828\ldots \\
\varphi &\approx 1.61803\ldots \\
\gamma &\approx 0.57722\ldots
\end{aligned}$$

Useful Square Roots

$$\begin{aligned}
\sqrt{2} &\approx 1.414214 \\
\sqrt{3} &\approx 1.732051 \\
\sqrt{4} &= 2 \\
\sqrt{5} &\approx 2.236068 \\
\sqrt{6} &\approx 2.449490 \\
\sqrt{7} &\approx 2.645751 \\
\sqrt{8} &\approx 2.828427 \\
\sqrt{9} &= 3 \\
\sqrt{10} &\approx 3.162278 \\
\sqrt{11} &\approx 3.316625 \\
\sqrt{12} &\approx 3.464102 \\
\sqrt{13} &\approx 3.605551 \\
\sqrt{14} &\approx 3.741657 \\
\sqrt{15} &\approx 3.872983 \\
\sqrt{16} &= 4 \\
\sqrt{17} &\approx 4.123106 \\
\sqrt{18} &\approx 4.242641 \\
\sqrt{19} &\approx 4.358899 \\
\sqrt{20} &\approx 4.472136
\end{aligned}$$

Simplifying Numbers

Bombelli, 1572 (« L'algebra »):

$$\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}=4$$

Leibniz, 1675:

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}$$

A simple complex example:

$$(1+i)(1-i)=2$$

where i is the imaginary unit and satisfies $i^2=-1$.

Trigonometric angles:

$$\begin{array}{rrr}
\cos(\pi)=-1 & \cos(\pi/2)=0 & \cos(\pi/3)=\frac{1}{2} \\
\cos(\pi/4)=\frac{\sqrt{2}}{2} & \cos(\pi/6)=\frac{\sqrt{3}}{2} & \cos(0)=1
\end{array}$$