2021 AMC 12A Problems/Problem 17

The following problem is from both the 2021 AMC 10A #17 and 2021 AMC 12A #17, so both problems redirect to this page.

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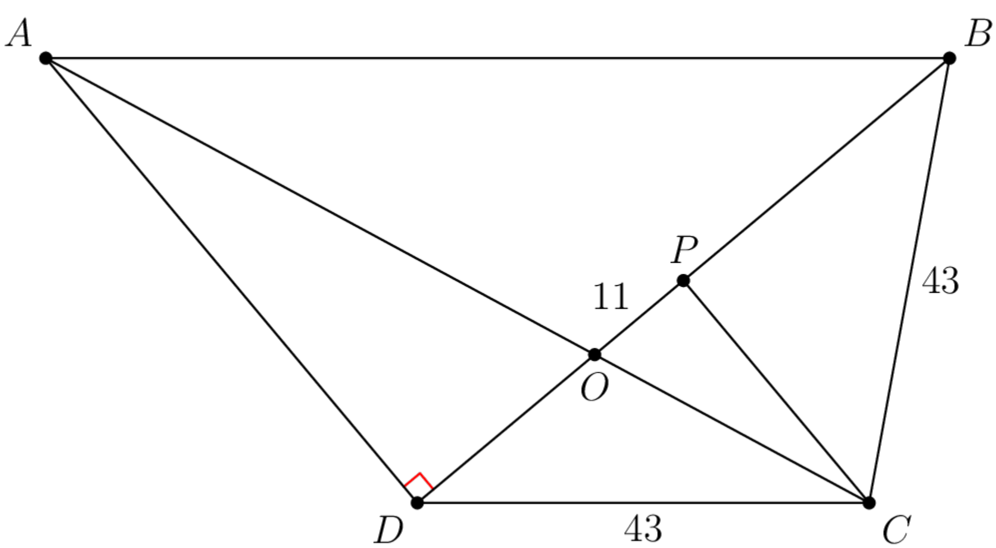
Problem

Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}, BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length of AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m + n$?

$\textbf{(A) }65 \qquad \textbf{(B) }132 \qquad \textbf{(C) }157 \qquad \textbf{(D) }194 \qquad \textbf{(E) }215$

Diagram



~MRENTHUSIASM

Solution 1 (Similar Triangles and Pythagorean Theorem)

Angle chasing* reveals that $\triangle BPC \sim \triangle BDA$, therefore

$$2 = \frac{\overline{BD}}{} = \frac{\overline{AB}}{} = \frac{AB}{}.$$

$$BP \qquad BC \qquad 43,$$

or $AB = 86$.

Additional angle chasing shows that $\triangle ABO \sim \triangle CDO$, therefore

$$2 = \frac{AB}{CD} = \frac{BO}{OD} = \frac{BP + 11}{BP - 11},$$

or $BP = 33$ and $BD = 66$.

Since $\triangle ADB$ is right, the Pythagorean theorem implies that

$$AD = \sqrt{86^2 - 66^2} = 4\sqrt{190}.$$

The answer is $4 + 190 = \boxed{\textbf{(D) } 194}$.

- Angle Chasing: If we set $\angle DBC = \alpha$, then we know that $\angle DCB = 180^\circ - 2\alpha$ because $\triangle DBC$ is isosceles. And because $\overline{AB} \parallel \overline{DC}$, we conclude that $\angle ABD = \alpha$ too. Lastly, because $\triangle BPC$ and $\triangle BDA$ are both right triangles, they are similar by AA.

~mn28407 (Solution)

~mm (Angle Chasing Remark)

~eagleye ~MRENTHUSIASM (Minor Edits)

Solution 2 (Similar Triangles, Areas, Pythagorean Theorem)

Since $\triangle BCD$ is isosceles with base $\overline{BD}$, it follows that median $\overline{CP}$ is also an altitude. Let $OD = x$ and $CP = h$, so $PB = x + 11$.

Since $\angle AOD = \angle COP$ by vertical angles, we conclude that $\triangle AOD \sim \triangle COP$ by AA, from which $\frac{AD}{CP} = \frac{OD}{OP}$, or

$$AD = CP \cdot \frac{OD}{OP} = h \cdot \frac{x}{11}.$$

Let the brackets denote areas. Notice that $[AOD] = [BOC]$ (By the same base and height, we deduce that $[ACD] = [BDC]$. Subtracting $[OCD]$ from both sides gives $[AOD] = [BOC]$.). Doubling both sides produces

$$\begin{aligned} 2[AOD] &= 2[BOC] \\ OD \cdot AD &= OB \cdot CP \\ x\left(\frac{hx}{11}\right) &= (x+22)h \\ x^2 &= 11(x+22). \end{aligned}$$

Rearranging and factoring result in $(x-22)(x+11) = 0$, from which $x = 22$.

Applying the Pythagorean Theorem to right $\triangle CPB$, we have

$$h = \sqrt{43^2 - 33^2} = \sqrt{(43+33)(43-33)} = \sqrt{760} = 2\sqrt{190}.$$

Finally, we get

$$AD = h \cdot \frac{x}{11} = 4\sqrt{190},$$

so the answer is $4+190=\boxed{\textbf{(D) }194}$.

~MRENTHUSIASM

Solution 3 (Short)

Let $CP=y$. CP a is perpendicular bisector of DB. Then, let $DO=x$, thus $DP=PB=11+x$.

(1) $\triangle CPO\sim\triangle ADO$, so we get $\dfrac{AD}{x}=\dfrac{y}{11}$, or $AD=\dfrac{xy}{11}$.

(2) Applying Pythagorean Theorem on $\triangle CDP$ gives $(11+x)^2+y^2=43^2$.

(3) $\triangle BPC\sim\triangle BDA$ with ratio $1:2$, so $AD=2y$ using the fact that P is the midpoint of BD.

Thus, $\dfrac{xy}{11}=2y$, or $x=22$. And $y=\sqrt{43^2-33^2}=2\sqrt{190}$, so $AD=4\sqrt{190}$ and the answer is $4+190=\boxed{\textbf{(D) }194}$.

~ ccx09

Solution 4 (Extending the Line)

Observe that $\triangle BPC$ is congruent to $\triangle DPC$; both are similar to $\triangle BDA$. Let's extend $\overline{AD}$ and $\overline{BC}$ past points D and C respectively, such that they intersect at a point E. Observe that $\angle BDE$ is 90 degrees, and that $\angle DBE\cong\angle PBC\cong\angle DBA\implies\angle DBE\cong\angle DBA$. Thus, by ASA, we know that $\triangle ABD\cong\triangle EBD$, thus, $AD=ED$, meaning D is the midpoint of AE. Let M be the midpoint of $\overline{DE}$. Note that $\triangle CME$ is congruent to $\triangle BPC$, thus $BC=CE$, meaning C is the midpoint of $\overline{BE}$.

Therefore, $\overline{AC}$ and $\overline{BD}$ are both medians of $\triangle ABE$. This means that O is the centroid of $\triangle ABE$; therefore, because the centroid divides the median in a 2:1 ratio, $\dfrac{BO}{2}=DO=\dfrac{BD}{3}$. Recall that P is the midpoint of BD; $DP=\dfrac{BD}{2}$. The question tells us that $OP=11$; $DP-DO=11$; we can write this in terms of DB; $\dfrac{DB}{2}-\dfrac{DB}{3}=\dfrac{DB}{6}=11\implies DB=66$.

We are almost finished. Each side length of $\triangle ABD$ is twice as long as the corresponding side length $\triangle CBP$ or $\triangle CPD$, since those triangles are similar; this means that $AB=2\cdot 43=86$. Now, by Pythagorean theorem on $\triangle ABD$, $AB^2-BD^2=AD^2\implies 86^2-66^2=AD^2\implies AD=\sqrt{3040}\implies AD=4\sqrt{190}$.

The answer is $4+190=\boxed{\textbf{(D) }194}$.

~ ihatemath123

Solution 5

Since P is the midpoint of isosceles triangle BCD, it would be pretty easy to see that $CP\perp BD$. Since $AD\perp BD$ as well, $AD\parallel CP$. Connecting AP, it's obvious that $[ADC]=[ADP]$. Since $DP=BP$, $[APB]=[ADC]$.

Since P is the midpoint of BD, the height of $\triangle APB$ on side AB is half that of $\triangle ADC$ on CD. Since $[APB]=[ADC]$, $AB=2CD$.

As a basic property of a trapezoid, $\triangle AOB\sim\triangle COD$, so $\dfrac{OB}{OD}=\dfrac{AB}{CD}=2$, or $OB=2OD$. Letting $OD=x$, then $PB=DP=11+x$, and $OB=22+x$. Hence $22+x=2x$ and $x=22$.

Since $\triangle AOD \sim \triangle COP$, $\dfrac{AD}{PC} = \dfrac{OD}{OP} = 2$. Since $PD = 11 + 22 = 33$, $PC = \sqrt{43^2 - 33^2} = \sqrt{760}$.

So, $AD = 2\sqrt{760} = 4\sqrt{190}$. The correct answer is $\boxed{\textbf{(D)}\ 194}$

Solution 6 (Coordinate Bash)

Let D be the origin of the cartesian coordinate plane, B lie on the positive x axis, and A lie on the negative y axis. Then let the coordinates of $B = (2a, 0), A = (0, -2b)$. Then the slope of AB is $\dfrac{b}{a}$. Since $AB \parallel CD$ the slope of CD is the same. Note that as $\triangle DCB$ is isosceles C lies on $x = a$. Thus since CD has equation $y = \dfrac{b}{a}x$ (D is the origin), $C = (a, b)$. Therefore AC has equation $y = \dfrac{3b}{a}x - 2b$ and intersects BD (x axis) at $O = (\dfrac{2}{3}a, 0)$. The midpoint of BD is $P = (a, 0)$ thus $OP = \dfrac{a}{3} = 11 \implies a = 33$. Then by Pythagorean theorem on $\triangle DPC$ ($\triangle DBC$ is isosceles) we have $b = \sqrt{44^2 - 33^2} = 4\sqrt{190} \implies \boxed{\textbf{(D)}\ 194}$.

~Aaryabhatta1

Video Solution (Using Similar Triangles, Pythagorean Theorem)

https://youtu.be/gjeSGJy_ld4

~ pi_is_3.14

Video Solution by Punxsutawney Phil

https://youtube.com/watch?v=rtdovluzgQs

Video Solution by Mathematical Dexterity

https://www.youtube.com/watch?v=QzAVdsgBBqg

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