# AMC 10 Problem Series (2804)

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**Friday**
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

## Homework

**Lesson:** **1** 2 3 4 5 6 7 8 9 10 11 12

### Homework: Lesson 1

You have completed **10** of **10** challenge problems.
Past Due **Jun 12**.

### Readings

**Lesson 1 Transcript:** Fri, Jun 4

### Challenge Problems

**Total Score: 60 / 60**

Announcement (28052)

Don't forget about Office Hours: an AoPS staff member will be on the message board to answer your questions in real time every day from 4:00 - 5:30 PM ET (1:00 - 2:30 PM PT) and 7:30-8:30 PM ET (4:30-5:30 PM PT).

Problem 1 – Correct! – Score: 6 / 6 (2835)

**Problem:** Report Error

The number of real values of $x$ that satisfy the equation

$$(2^{6x+3})(4^{3x+6}) = 8^{4x+5}$$

is:

**(A)** zero **(B)** one **(C)** two **(D)** three **(E)** greater than 3

**Solution:**

Replacing 4 with $2^2$ and 8 with $2^3$, we get

$$2^{6x+3} \cdot (2^2)^{3x+6} = (2^3)^{4x+5},$$

which simplifies to

$$2^{12x+15} = 2^{12x+15}.$$

This equation holds for $\boxed{\text{all real values of } x}$. The answer is (E).

**Your Response(s):**

- E

Problem 2 – Correct! – Score: 6 / 6 (2836)

**Problem:** Report Error

For which of the following values of $k$ does the equation

$$\frac{x-1}{x-2}=\frac{x-k}{x-6}$$

have no solution for $x$?

**(A)** 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

**Solution:**
Assuming $x \neq 2$ and $x \neq 6$ (neither 2 nor 6 can be solutions for $x$, since that would give division by 0), we multiply both sides by $(x-2)(x-6)$ to get

$$(x-1)(x-6)=(x-2)(x-k).$$

Expanding both sides and simplifying, we get

$$(k-5)x-(2k-6)=0,$$

so

$$x=\frac{2k-6}{k-5}.$$

This expression is not defined when $\boxed{k=5}$. The answer is (E).

**Your Response(s):**

- E

Problem 3 – Correct! – Score: 6 / 6 (2837)

**Problem:** Report Error

How many ordered triples $(a,b,c)$ of nonzero real numbers have the property that each number is the product of the other two?

**(A)** 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

**Solution:**
We want to solve the system of equations

$$\begin{aligned} ab &= c, \\ ac &= b, \\ bc &= a. \end{aligned}$$

Multiplying all these equations, we get $a^2b^2c^2=abc$. Since $a$, $b$, and $c$ are nonzero, we can divide both sides by $abc$ to get $abc=1$. Substituting $bc=a$, we get $a^2=1$, so $a=1$ or $a=-1$.

If $a=1$, then the given equations become $b=c$ and $bc=1$. Substituting $b=c$, we get $b^2=1$, so $b=1$ or $b=-1$. If $b=1$, then $c=1$, and if $b=-1$, then $c=-1$.

If $a=-1$, then the given equations become $b=-c$ and $bc=-1$. Substituting $b=-c$, we get $b^2=1$, so

$b = 1$ or $b = -1$. If $b = 1$, then $c = -1$, and if $b = -1$, then $c = 1$.

Hence, we have the four solutions $\boxed{(a, b, c) = (1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)}$. The answer is (D).

**Your Response(s):**

D

Problem 4 – Correct! – Score: 6 / 6 (2838)

**Problem:**

Report Error

Two non-zero real numbers, $a$ and $b$, satisfy $ab = a - b$. Which of the following is a possible value of $\frac{a}{b} + \frac{b}{a} - ab$?

**(A)** $-2$ **(B)** $-\frac{1}{2}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** 2

**Solution:**
The given expression simplifies to

$$\begin{aligned} \frac{a}{b} + \frac{b}{a} - ab &= \frac{a^2 + b^2 - a^2b^2}{ab} \\ &= \frac{a^2 + b^2 - (ab)^2}{ab}. \end{aligned}$$

Since $ab = a - b$, it then follows that

$$\begin{aligned} \frac{a^2 + b^2 - (ab)^2}{ab} &= \frac{a^2 + b^2 - (a - b)^2}{ab} \\ &= \frac{a^2 + b^2 - (a^2 - 2ab + b^2)}{ab} \\ &= \frac{2ab}{ab} \\ &= \boxed{2}. \end{aligned}$$

The answer is (E).

**Your Response(s):**

E

Problem 5 – Correct! – Score: 6 / 6 (2839)

**Problem:**

Report Error

If $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$, then $a + b + c + d$ is

**(A)** $-5$ **(B)** $-10/3$ **(C)** $-7/3$ **(D)** $5/3$ **(E)** $5$

**Solution:**
Let $s = a + b + c + d$. Then from the given equation, $a = s + 4$, $b = s + 3$, $c = s + 2$, and $d = s + 1$. Adding these equations, we get $a + b + c + d = 4s + 10$, or $s = 4s + 10$. Then $3s = -10$, so

$s = \boxed{-10/3}$. The answer is (B).

**Your Response(s):**

B

Problem 6 – Correct! – Score: 6 / 6 (2840)

**Problem:**

Report Error

Let $a$, $b$, $c$, and $d$ be real numbers with $|a-b|=2$, $|b-c|=3$, and $|c-d|=4$. What is the sum of all possible values of $|a-d|$?

**(A)** 9 **(B)** 12 **(C)** 15 **(D)** 18 **(E)** 24

**Solution:**

If we consider $a$, $b$, $c$, and $d$ as points on the real number line, then the given equations tell us that the distances between $a$ and $b$, $b$ and $c$, and $c$ and $d$ are 2, 3, and 4, respectively.

We have that $b = a - 2$ or $b = a + 2$. Then for each such value of $b$, we have that $c = b - 3$ or $c = b + 3$, and so on. We list the possibilities in a table.

| $b = a - 2$ | | | | $b = a + 2$ | | | |
|---|---|---|---|---|---|---|---|
| $c = a - 5$ | | $c = a + 1$ | | $c = a - 1$ | | $c = a + 5$ | |
| $d = a - 9$ | $d = a - 1$ | $d = a - 3$ | $d = a + 5$ | $d = a - 5$ | $d = a + 3$ | $d = a + 1$ | $d = a + 9$ |

We see that the possible values of $|a-d|$ are 1, 3, 5, and 9. Their sum is $1+3+5+9=\boxed{18}$. The answer is (D).

**Your Response(s):**

D

Problem 7 – Correct! – Score: 6 / 6 (2841)

**Problem:**

Report Error

If $x$ and $y$ are nonzero numbers such that $x = 1 + \frac{1}{y}$ and $y = 1 + \frac{1}{x}$, then $y$ equals

**(A)** $x - 1$ **(B)** $1 - x$ **(C)** $1 + x$ **(D)** $-x$ **(E)** $x$

**Solution:**

Multiplying the first equation by $y$, we get

$$xy = y + 1,$$

and multiplying the second equation by $x$, we get

$$xy = x + 1.$$

Hence, $y + 1 = x + 1$, or $y = \boxed{x}$. The answer is (E).

**Your Response(s):**

E

Problem 8 – Correct! – Score: 6 / 6 (2842)

**Problem:**

Report Error

A right triangle has perimeter 32 and area 20. What is the length of its hypotenuse?

**(A)** $\frac{57}{4}$ **(B)** $\frac{59}{4}$ **(C)** $\frac{61}{4}$ **(D)** $\frac{63}{4}$ **(E)** $\frac{65}{4}$

**Solution:**

Let the legs have lengths $a$ and $b$, and hypotenuse have length $c$, so that the Pythagorean Theorem gives

$$a^2 + b^2 = c^2.$$

From the information about the area, we have $ab = 40$. From the information about the perimeter, we have

$$a + b + c = 32.$$

Subtracting $c$ from both sides gives $a + b = 32 - c$, and squaring both sides gives

$$a^2 + 2ab + b^2 = c^2 - 64c + 1024.$$

Since $ab = 40$, we have $2ab = 80$. Moreover, $a^2 + b^2 = c^2$, so the equation above simplifies to

$$80 = -64c + 1024.$$

Solving for $c$ gives $c = \boxed{\frac{59}{4}}$.

*Solution by aninditg.*

**Your Response(s):**

B

Problem 9 – Correct! – Score: 6 / 6 (2843)

**Problem:**

Report Error

Let $a$, $b$, and $c$ be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

**(A)** 0 **(B)** 1 **(C)** 4 **(D)** 7 **(E)** 8

**Solution:**

Since we are given two equations, we can solve for two of the variables in terms of the third variable. For example, we can write

$$\begin{aligned} a + 8c &= 4 + 7b, \\ 8a - c &= 7 - 4b, \end{aligned}$$

and solve for $a$ and $c$ in terms of $b$.

Multiplying the second equation by 8, we get $64a - 8c = 56 - 32b$. Adding this to the first equation, we get $65a = 60 - 25b$. We can divide both sides by 5, to get $13a = 12 - 5b$.

Multiplying the first equation by 8, we get $8a + 64c = 32 + 56b$. Subtracting the second equation, we get $65c = 25 + 60b$. We can divide both sides by 5, to get $13c = 5 + 12b$.

Squaring the equations $13a = 12 - 5b$ and $13c = 5 + 12b$, we get $169a^2 = 144 - 120b + 25b^2$ and $169c^2 = 25 + 120b + 144b^2$. Adding these equations, we get $169a^2 + 169c^2 = 169 + 169b^2$. Dividing this equation by 169, we get $a^2 + c^2 = 1 + b^2$, so $a^2 - b^2 + c^2 = \boxed{1}$. The answer is (B).

**Your Response(s):**

B

Problem 10 – Correct! – Score: 6 / 6 (2844)

**Problem:**

Report Error

Suppose that the number $a$ satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

**(A)** 164 **(B)** 172 **(C)** 192 **(D)** 194 **(E)** 212

**Solution:**

Squaring the equation $a + \frac{1}{a} = 4$, we get

$$a^2 + 2 + \frac{1}{a^2} = 16,$$

or $a^2 + \frac{1}{a^2} = 14$.

Squaring this equation, we get

$$a^4 + 2 + \frac{1}{a^4} = 196,$$

so $a^4 + \frac{1}{a^4} = \boxed{194}$. The answer is (D).

**Your Response(s):**

D