Art Of Problem Solving - AMC 10
Week 11

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Abstract

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1.

Mrs. Walter gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Walter noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Walter entered?

(A) 71	(B) 76	(C) 80	(D) 82	(E) 91

The first number is divisible by 1.

The sum of the first two numbers is even.

The sum of the first three numbers is divisible by 3.

The sum of the first four numbers is divisible by 4.

The sum of the first five numbers is 400.

Since 400 is divisible by 4. the last score must also be divisible by 4. Therefore, the last score is either 76 or 80.

Case 1: 76 is the last number entered.

Since $400 \equiv 76 \equiv 1 \pmod 3$, the fourth number must be divisible by 3, but none of the scores are divisible by 3.

Case 2: 80 is the last number entered.

Since $80 \equiv 2 \pmod 3$, the fourth number must be $2 \pmod 3$. That number is 71 and 71 only. The next number must be 91 since the sum of the first two numbers is even. So the only arrangement of the scores $76, 82, 91, 71, 80$

$\boxed{80}$

2.

The digits 1, 2, 3, 4, 5, 6, 7, and 9 are used to form four two-digit prime numbers, with each digit used exactly once. What is the sum of these four primes?

(A) 150	(B) 160	(C) 170	(D) 180	(E) 190

Since a multiple-digit prime number is not divisible by either 2 or 5, it must end with 1, 3, 7, or 9 in the units place. The remaining digits given must therefore appear in the tens place.

$$20 + 40 + 50 + 60 + 1 + 3 + 7 + 9 = 190$$

$$\boxed{190}$$

3.

Suppose that m and n are positive integers such that $75m = n^3$. What is the minimum possible value of $m + n$?

(A) 15	(B) 30	(C) 50	(D) 60	(E) 5700

$3 \cdot 5^2 m$ must be a perfect cube, so each power of a prime in the factorization for $3 \cdot 5^2 m$ must be divisible by 3. Thus the minimum value of m is $3^2 \cdot 5 = 45$, which makes

$$n = \sqrt[3]{3^3 \cdot 5^3} = 15$$

The minimum possible value for the sum of m and n is 60.

$$\boxed{60}$$

4.

What is the largest integer that is a divisor of $(n+1)(n+3)(n+5)(n+7)(n+9)$ for all positive even integers n?

(A) 3	(B) 5	(C) 11	(D) 15	(E) 165

For all consecutive odd integers, one of every five is a multiple of 5 and one of every three is a multiple of 3. The answer is $3 \cdot 5 = 15$.

$\boxed{15}$

5.

Let S be the set of the 2005 smallest positive multiples of 4, and let T be the set of the 2005 smallest positive multiples of 6. How many elements are common to S and T?

(A) 166	(B) 333	(C) 500	(D) 668	(E) 1001

Since the least common multiple $\text{lcm}(4, 6) = 12$, the elements that are common to S and T must be multiples of 12.

Since $4 \cdot 2005 = 8020$ and $6 \cdot 2005 = 12030$, several multiples of 12 that are in T won't be in S, but all multiples of 12 that are in S will be in T. So we just need to find the number of multiples of 12 that are in S.

Since $4 \cdot 3 = 12$, every 3rd element of S will be a multiple of 12.

Therefore the answer is

$$\left\lfloor \frac{2005}{3} \right\rfloor = 668$$

$$\boxed{668}$$

6.

For how many positive integers n less than or equal to 24 is $n!$ evenly divisible by $1+2+\cdots+n$?

(A) 8	(B) 12	(C) 16	(D) 17	(E) 21

Since $1+2+\cdots+n = \frac{n(n+1)}{2}$, the condition is equivalent to having an integer value for

$$\frac{n!}{\dfrac{n(n+1)}{2}}$$

This reduces, when $n \geq 1$, to having an integer value for

$$\frac{2(n-1)!}{n+1}$$

This fraction is an integer unless $n+1$ is an odd prime. There are 8 odd primes less than or equal to 24, so there are $24-8=16$ numbers less than or equal to 24 that satisfy the condition.

$$\boxed{16}$$

7.

A finite sequence of three-digit integers has the property that the tens and units digits of each term are, respectively, the hundreds and tens digits of the next term, and the tens and units digits of the last term are, respectively, the hundreds and tens digits of the first term. For example, such a sequence might begin with terms 247, 475, and 756 and end with the term 824. Let S be the sum of all the terms in the sequence. What is the largest prime number that always divides S?

(A) 3	(B) 7	(C) 13	(D) 37	(E) 43

A given digit appears as the hundreds digit, the tens digit, and the units digit of a term the same number of times. Let k be the sum of the units digits in all the terms. Then

$$S = 111k = 3 \cdot 37k$$

so S must be divisible by 37. To see that it need not be divisible by any larger prime, the sequence 123, 231, 312 gives.

$$S = 666 = 2 \cdot 3^2 \cdot 37$$

$$\boxed{37}$$

8.

Sally has five red cards numbered 1 through 5 and four blue cards numbered 3 through 6. She stacks the cards so that the colors alternate and so that the number on each red card divides evenly into the number on each neighboring blue card. What is the sum of the numbers on the middle three cards?

(A) 8	(B) 9	(C) 10	(D) 11	(E) 12

Let R_i and B_j designate the red card numbered i and the blue card numbered j, respectively.

B_5 is the only blue card that R_5 evenly divides, so R_5 must be at one end of the stack and B_5 must be the card next to it.

R_1 is the only other red card that evenly divides B_5, so R_1 must be the other card next to B_5.

B_4 is the only blue card that R_4 evenly divides, so R_4 must be at one end of the stack and B_4 must be the card next to it.

R_2 is the only other red card that evenly divides B_4, so R_2 must be the other card next to B_4.

R_2 doesn't evenly divide B_3, so B_3 must be next to R_1, B_6 must be next to R_2, and R_3 must be in the middle.

This yields the following arrangement from top to bottom: $\{R_5, B_5, R_1, B_3, R_3, B_6, R_2, B_4, R_4\}$

Therefore, the sum of the numbers on the middle three cards is $3 + 3 + 6 = 12$.

$\boxed{12}$

9.

Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers x and y satisfy $x^2 - y^2 = m^2$ for some positive integer m. What is $x + y + m$?

(A) 88	(B) 112	(C) 116	(D) 144	(E) 154

Let $x = 10a + b, y = 10b + a$. The given conditions imply $x > y$, which implies $a > b$, and they also imply that both a and b are nonzero. Then

$$\begin{aligned} x^2 - y^2 &= (x - y)(x + y) \\ &= (9a - 9b)(11a + 11b) \\ &= 99(a - b)(a + b) \\ &= m^2 \end{aligned}$$

Since this must be a perfect square, all the exponents in its prime factorization must be even. 99 factorizes into $3^2 \cdot 11$, so $11|(a-b)(a+b)$. However, the maximum value of $a-b$ is $9-1 = 8$, so $11|a+b$. The maximum value of $a+b$ is $9+8 = 17$, so $a+b = 11$. Then we have $33^2(a-b) = m^2$, so $a - b$ is a perfect square, but the only perfect squares that are within our bound on $a - b$ are 1 and 4. We know $a + b = 11$, and, for $a - b = 1$, adding equations to eliminate b gives us $2a = 12 \implies a = 6, b = 5$. Testing $a - b = 4$ gives us $2a = 15 \implies a = \frac{15}{2}, b = \frac{7}{2}$, which is impossible, as a and b must be digits. Therefore, $(a, b) = (6, 5)$, and

$$x + y + m = 65 + 56 + 33 = 154$$

$$\boxed{154}$$

10.

A high school basketball game between the Raiders and the Wildcats was tied at the end of the first quarter. The number of points scored by the Raiders in each of the four quarters formed an increasing geometric sequence, and the number of points scored by the Wildcats in each of the four quarters formed an increasing arithmetic sequence. At the end of the fourth quarter, the Raiders had won by one point. Neither team scored more than 100 points. What was the total number of points scored by the two teams in the first half?

(A) 30	(B) 31	(C) 32	(D) 33	(E) 34

Let a, ar, ar^2, ar^3 be the quarterly scores for the Raiders. We know that the Raiders and Wildcats both scored the same number of points in the first quarter so let $a, a+d, a+2d, a+3d$ be the quarterly scores for the Wildcats. The sum of the Raiders scores is $a(1+r+r^2+r^3)$ and the sum of the Wildcats scores is $4a+6d$. Now we can narrow our search for the values of a, d, and r. Because points are always measured in positive integers, we can conclude that a and d are positive integers. We can also conclude that r is a positive integer by writing down the equation:

$$a(1+r+r^2+r^3) = 4a+6d+1$$

Now we can start trying out some values of r. We try $r=2$, which gives

$$15a = 4a+6d+1$$
$$11a = 6d+1$$

We need the smallest multiple of 11 (to satisfy the ¡100 condition) that is $\equiv 1 \pmod 6$. We see that this is 55, and therefore $a=5$ and $d=9$. So the Raiders' first two scores were 5 and 10 and the Wildcats' first two scores were 5 and 14.

$$5+10+5+14 = 34$$

$$\boxed{34}$$