2021 AMC 10A Problems/Problem 20

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Problem

In how many ways can the sequence $1, 2, 3, 4, 5$ be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

$\textbf{(A)}\ 10 \qquad \textbf{(B)}\ 18 \qquad \textbf{(C)}\ 24 \qquad \textbf{(D)}\ 32 \qquad \textbf{(E)}\ 44$

Solution 1 (Enumeration)

We write out the $5! = 120$ cases, then filter out the valid ones:

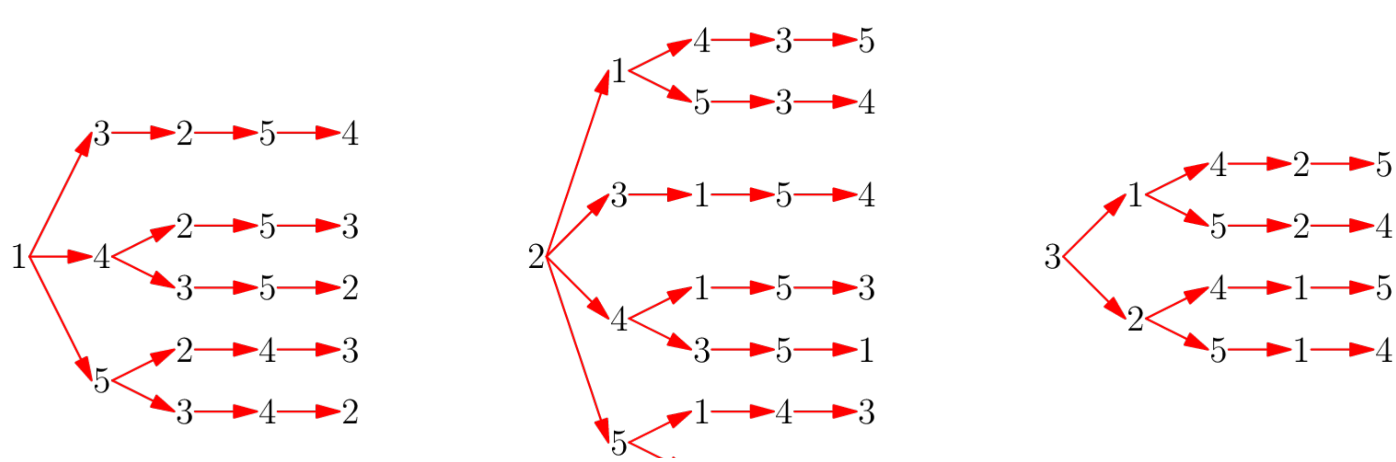
$$13254, 14253, 14352, 15243, 15342, 21435, 21534, 23154, 24153, 24351, 25143, 25341,$$
$$31425, 31524, 32415, 32514, 34152, 34251, 35142, 35241, 41325, 41523, 42315, 42513,$$
$$43512, 45132, 45231, 51324, 51423, 52314, 52413, 53412.$$

We count these out and get $\boxed{\textbf{(D)}\ 32}$ permutations that work.

~contactbibliophile

Solution 2 (Enumeration by Symmetry)

By symmetry with respect to $3,$ note that $(x_1, x_2, x_3, x_4, x_5)$ is a valid sequence if and only if $(6-x_1, 6-x_2, 6-x_3, 6-x_4, 6-x_5)$ is a valid sequence. We enumerate the valid sequences that start with $1, 2, 31,$ or $32,$ as shown below:



$$\to 3 \to 4 \to 1$$

There are 16 valid sequences that start with $1, 2, 31,$ or $32.$ By symmetry, there are 16 valid sequences that start with $5, 4, 35,$ or $34.$ So, the answer is $16+16=\boxed{\textbf{(D)}\ 32}.$

~MRENTHUSIASM (inspired by Snowfan)

Solution 3 (Casework on the Consecutive Digits)

Reading the terms from left to right, we have two cases for the consecutive digits, where $+$ means increase and $-$ means decrease:

$$\textbf{Case \#1:}\ +,-,+,-$$

$$\textbf{Case \#2:}\ -,+,-,+$$

For $\text{Case \#1},$ note that for the second and fourth terms, one term must be $5,$ and the other term must be either 3 or $4.$ We have four subcases:

$$(1)\ \_3\_5\_$$

$$(2)\ \_5\_3\_$$

$$(3)\ \_4\_5\_$$

$$(4)\ \_5\_4\_$$

For $(1),$ the first two blanks must be 1 and 2 in some order, and the last blank must be $4.$ So, we get 2 possibilities. Similarly, (2) also has 2 possibilities.

For $(3),$ there are no restrictions for the numbers $1, 2,$ and $3.$ So, we get $3!=6$ possibilities. Similarly, (4) also has 6 possibilities.

Together, $\text{Case \#1}$ has $2+2+6+6=16$ possibilities. By symmetry, $\text{Case \#2}$ also has 16 possibilities.

Finally, the answer is $16+16=\boxed{\textbf{(D)}\ 32}.$

Remark

This problem is somewhat similar to 2004 AIME I Problem 6.

~MRENTHUSIASM

Solution 4 (Casework Similar to Solution 3)

Like Solution 3, we have two cases. Due to symmetry, we just need to count one of the cases. For the purpose of this solution, we will be doing $-, +, -, +$. Instead of starting with 5, we start with 1.

There are two ways to place it:

_1_ _ _

_ _ _1_

Now we place 2, it can either be next to 1 and on the outside, or is place in where 1 would go in the other case. So now we have another two "sub case":

_1_2_(case 1)

21_ _ _(case 2)

There are 3! ways to arrange the rest for case 1, since there is no restriction.

For case 2, we need to consider how many ways to arrange 3,4,5 in a a>b<c fashion. It should seem pretty obvious that b has to be

3, so there will be 2! way to put 4 and 5.

Now we find our result, times 2 for symmetry, times 2 for placement of 1 and times (3!+2!) for the two different cases for placement of 2. This give us $2*2*(3!+2!)=4*(6+2)=\boxed{\textbf{(D) } 32}$.

~~Xhte

Solution 5 (Casework on the Position of 5)

We only need to find the # of rearrangements when 5 is the 4th digit and 5th digit. Find the total, and multiply by 2. Then we can get the answer by adding the case when 5 is the third digit.

Case 1**:** 5 is the 5th digit. __ __ __ __ 5

Then 4 can only be either 1st digit or the 3rd digit.

4 __ __ __ 5, then the only way is that 3 is the 3rd digit, so it can be either 231 or 132, give us 2 results.

__ __ 4 __ 5, then the 1st digit must be 2 or 3, 2 gives us 1 way, and 3 gives us 2 ways. (Can't be 1 because the first digit would increasing). Therefore, 4 in the middle and 5 in the last would result in 3 ways.

Case 2**:** 5 is the fourth digit. __ __ __ 5 __

Then the last digit can be all of the 4 numbers 1, 2, 3, and 4. Let's say if the last digit is 4, then the 2nd digit would be the largest for the remaining digits to prevent increasing order or decreasing order. Then the remaining two are interchangeable, give us $2!$ ways. All of the 4 can work, so case 2 would result in $2!+2!+2!+2!=8$ ways.

Case 3**:** 5 is in the middle. __ __ 5 __ __

Then there are only two cases: 1. 42513, then 4 and 3 are interchangeable, which results in $2!*2!$. Or it can be 43512, then 4 and 2 are interchangeable, but it can not be 23514, so there can only be 2 possible ways: 43512, 21534.

Therefore, case 3 would result in $4+2=6$ ways.

$8+3+2=13$, so the total ways for case 1 and case 2 with both increasing and decreasing would be $13*2=26.$

Finally, we have $26+6=\boxed{\textbf{(D) } 32}$.

~Michael595

Video Solution by OmegaLearn (Using PIE - Principle of Inclusion Exclusion)

https://youtu.be/Fqak5BArpdc

~ pi_is_3.14

Video Solution by Power of Logic (Using Idea of Symmetrically Counting)

https://youtu.be/ZLQ8KYtai_M

Video Solution by TheBeautyofMath

https://youtu.be/UZZoSYHBJlI

~IceMatrix

See Also

2021 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
Preceded by **Problem 19**	Followed by **Problem 21**
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