

Art Of Problem Solving - AMC 10

Week 1

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Abstract

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1.

The number of real values of x that satisfy the equation

$$(2^{6x+3}) (4^{3x+6}) = 8^{4x+5}$$

is:

zero,	one,	two,	three,	greater than 3
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We bring all exponentials to the same basis:

$$2^{6x+3} \cdot 2^{2(3x+6)} = 2^{3(4x+5)}$$

$$6x + 3 + 2(3x + 6) = 3(4x + 5)$$

$$12x + 15 = 12x + 15$$

which is always true. Thus, there are an infinity of solutions. And since $\infty > 3$

The number of real values of x is greater than 3.

Note that we knew right away that the equation could be made linear by a change of the basis and that therefore the number of solutions had to be either 0, 1 or ∞ , with 1 the “non-pathological” case. So if we had been in a hurry, a good bet would have been 1. Unfortunately, we would have lost the best in this case.

2.

For which of the following values of k does the equation

$$\frac{x-1}{x-2} = \frac{x-k}{x-6}$$

have no solution for x ?

$$k = 1, \quad k = 2, \quad k = 3, \quad k = 4, \quad k = 5$$

We immediately note that the case $k = 1$ is special. Substituting $k = 1$ yields:

$$\frac{x-1}{x-2} = \frac{x-1}{x-6}$$

At this point it is tempting to cancel out the $x - 1$. But remember the question: when is there *no* solution? Thus if we find one solution for some k , we can move on to the next value of k . We do not need to find all the solutions! And since $x = 1$ is a solution of the above, we can rule out $k = 1$. There are still four more cases to consider.

We could go on substituting each possible value of k and looking for solutions. Let's do it for one more case, as an exercise. Let $k = 2$.

$$\begin{aligned}\frac{x-1}{x-2} &= \frac{x-2}{x-6} \\ (x-1)(x-6) &= (x-2)^2 \\ x^2 - 7x + 6 &= x^2 - 4x + 2 \\ 3x &= 4 \\ x &= 4/3\end{aligned}$$

Thus, there is a valid solution for $k = 2$ (that solution is $x = 4/3$).

The other cases could be resolved with similar steps. But trying five cases will take a lot of work. So is there a quicker method? Let's be general:

$$\begin{aligned}\frac{x-1}{x-2} &= \frac{x-k}{x-6} \\ (x-1)(x-6) &= (x-2)(x-k) \\ x^2 - 7x + 6 &= x^2 - (2+k)x + 2k \\ -7x + 6 &= -(2+k)x + 2k\end{aligned}$$

Now note that if $7 = 2 + k$, the x term cancels out of the equation. And thus, there is no solution (the equation is impossible) if the following holds:

$$\begin{aligned}7 &= 2 + k \\ 6 &\neq 2k\end{aligned}$$

which is true for $k = 5$.

$$k = 5$$

3.

How many ordered triples (a, b, c) of nonzero real numbers have the property that each number is the product of the other two?

1,	2,	3,	4,	5
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We are looking for (a, b, c) , where $a \neq 0$, $b \neq 0$, $c \neq 0$, such that:

$$a = bc$$

$$b = ac$$

$$c = ab$$

We notice that this implies:

$$abc = 1$$

$$\frac{abc}{bc} = \frac{1}{a}$$

$$a^2 = 1$$

$$a = \pm 1$$

By symmetry, we also have $b = \pm 1$, $c = \pm 1$. So the candidate unordered-triples are $(1, 1, 1)$, $(1, 1, -1)$, $(1, -1, -1)$, and $(-1, -1, -1)$. But clearly we need to be able to permute the symbols pairwise without violating the equality, which means only these are candidates: $(1, 1, 1)$ and $(1, -1, -1)$. That is triples. But the question asks for **ordered** triples. If the triple is ordered, $(+1, -1, -1)$, $(-1, +1, -1)$, and $(-1, -1, +1)$ are distinct. So altogether that gives 4 ordered triples

2 non-ordered triples, 4 ordered triples
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4.

Two non-zero real numbers, a and b , satisfy $ab = a - b$. Which of the following is a possible value of $\frac{a}{b} + \frac{b}{a} - ab$?

$-2, \quad -1/2, \quad 1/3, \quad 1/2, \quad 2$

Addition/subtraction is typically simpler than multiplication, so we are tempted to replace every instance of ab with $a - b$ and hope that it will yield easy simplifications. We also see that if we bring the fractions to the same denominator, the product ab will appear:

$$\frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab}$$

The appearance of $a^2 + b^2$ is not very encouraging. However since we know something about $a - b$, we are reminded that $(a - b)^2$ generates $a^2 + b^2$. So we take it from here:

$$\begin{aligned}(a - b)^2 &= a^2 + b^2 - 2ab \\ a^2 + b^2 &= (a - b)^2 + 2ab \\ a^2 + b^2 &= (ab)^2 + 2ab\end{aligned}$$

Substituting back in:

$$\begin{aligned}\frac{a}{b} + \frac{b}{a} &= \frac{a^2 + b^2}{ab} \\ &= \frac{(ab)^2 + 2ab}{ab} \\ &= ab + 2\end{aligned}$$

And thus

$$\frac{a}{b} + \frac{b}{a} - ab = 2$$

5.

If $a + 1 = b + 2 = c + 3 = d + 4 = a + b + c + d + 5$, then $a + b + c + d$ is

$-5, \quad -10/3, \quad -7/3, \quad 5/3, \quad 5$

It pays to rewrite these for clarity:

$$a + 1 = b + 2$$

$$b + 2 = c + 3$$

$$c + 3 = d + 4$$

$$d + 4 = a + b + c + d + 5$$

This shows that we have 4 linear equations in 4 unknowns, which we expect to be able to solve. But can we get the sum faster than that? If we add up the equations or multiply them, we get a lot of cancellations, but the answer does not come out right away. Let's introduce new variables to see the pattern:

$$s = a + b + c + d$$

$$\alpha = a + 1$$

$$\beta = b + 2$$

$$\gamma = c + 3$$

$$\delta = d + 4$$

We want to find $s = \delta - 5$, where

$$\alpha = \beta$$

$$\beta = \gamma$$

$$\gamma = \delta$$

$$\delta = \alpha + \beta + \gamma + \delta - 5$$

where the last equation comes from:

$$\begin{aligned} \delta &= a + b + c + d + 5 \\ &= (a + 1) + (b + 2) + (c + 3) + (d + 4) + 5 - 1 - 2 - 3 - 4 \\ &= \alpha + \beta + \gamma + \delta - 5 \end{aligned}$$

Now we have a reasonably simple way to get the answer. Since $\alpha = \beta = \gamma = \delta$,

$$\delta = \alpha + \beta + \gamma + \delta - 5$$

$$\delta = \delta + \delta + \delta + \delta - 5$$

$$\delta = 5/3$$

$$\Rightarrow s = 5/3 - 5 = -10/3$$

$a + b + c + d = \frac{-10}{3}$

6.

Let a , b , c , and d be real numbers with $|a - b| = 2$, $|b - c| = 3$, and $|c - d| = 4$. What is the sum of all possible values of $|a - d|$?

9, 12, 15, 18, 24

The problem is to solve for $x = |a - d|$, where:

$$|a - b| = 2$$

$$|b - c| = 3$$

$$|c - d| = 4$$

We have several cases to consider:

$$\begin{cases} a - b = 2 \\ a - b = -2 \end{cases}$$

$$\begin{cases} b - c = 3 \\ b - c = -3 \end{cases}$$

$$\begin{cases} c - d = 4 \\ c - d = -4 \end{cases}$$

We can get $a - d$ by adding the three cases. For instance:

$$a - b = 2$$

$$b - c = 3$$

$$c - d = 4$$

$$a - d = 2 + 3 + 4 = 9$$

So now we repeat the pattern for all possible values of $a - d$:

$$\begin{aligned} a - d &= +2 + 3 + 4 = +9 \\ &= -2 + 3 + 4 = +5 \\ &= +2 - 3 + 4 = +3 \\ &= +2 + 3 - 4 = +1 \\ &= -2 - 3 + 4 = -1 \\ &= -2 + 3 - 4 = -3 \\ &= +2 - 3 - 4 = -5 \\ &= -2 - 3 - 4 = -9 \end{aligned}$$

But we are looking for $|a - d|$, so adding up the positive values gives:

$$|a - d| = 1 + 3 + 5 + 9 = 18$$

$$|a - d| = 18$$

7.

If x and y are nonzero numbers such that $x = 1 + \frac{1}{y}$ and $y = 1 + \frac{1}{x}$, then y equals

$x - 1,$	$1 - x,$	$1 + x,$	$-x,$	x
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$$x = 1 + \frac{1}{y}$$

$$y = 1 + \frac{1}{x}$$

The symmetry of the problem is obvious and indeed:

$$xy = y + 1$$

$$yx = x + 1$$

$$\Rightarrow x = y \quad \square$$

$y = x$

Out of curiosity, let's solve the quadratic equation:

$$x = 1 + \frac{1}{x}$$

$$x^2 - x - 1 = 0$$

$$\left(x - \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 1 = 0$$

$$\left(x - \frac{1}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2 = 0$$

$$\left(x - \frac{1 + \sqrt{5}}{2}\right)\left(x - \frac{1 - \sqrt{5}}{2}\right) = 0$$

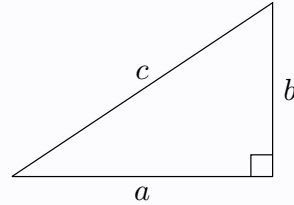
$$y = \frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}$$

8.

A right triangle has perimeter 32 and area 20. What is the length of its hypotenuse?

$\frac{57}{4},$	$\frac{59}{4},$	$\frac{61}{4},$	$\frac{63}{4},$	$\frac{65}{4}$
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Let a, b denote the lengths of the legs and c the length of the hypotenuse.



We know these facts:

$$\begin{aligned}\text{area:} \quad \frac{ab}{2} &= 20 \\ \text{perimeter:} \quad a + b + c &= 32\end{aligned}$$

We want to solve for c , which is related to a and b by the Pythagorean triangle identity:

$$c = \sqrt{a^2 + b^2}$$

To get an expression involving a^2 and b^2 , we square the perimeter formula:

$$\begin{aligned}a + b &= 32 - c \\ (a + b)^2 &= (32 - c)^2 \\ a^2 + b^2 + 2ab &= 32^2 - 64c + c^2 \\ c^2 + 80 &= 1024 - 64c + c^2 \\ 64c &= 944 \\ c &= \frac{944}{64} = \frac{59}{4}\end{aligned}$$

length of hypotenuse: $\frac{59}{4}$

9.

Let a, b, c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. Then $a^2 - b^2 + c^2$ is

0,	1,	4,	7,	8
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We have a linear system of two equations in three unknowns a, b, c :

$$a - 7b + 8c = 4$$

$$8a + 4b - c = 7$$

We notice a 1, 4, 7, 8 pattern. The key insight is that the first equation has $(1a, 8c)$ and the second $(8a, -1c)$, while b appears with different coefficients. This suggests squaring both equations after a clever rearrangement:

$$(a + 8c)^2 = (4 + 7b)^2$$

$$(8a - c)^2 = (7 - 4b)^2$$

Distributing:

$$a^2 + 16ac + 64c^2 = 16 + 56b + 49b^2$$

$$64a^2 - 16ac + c^2 = 49 - 56b + 16b^2$$

Adding up:

$$65a^2 + 65c^2 = 65 + 65b^2$$

$$\Rightarrow a^2 - b^2 + c^2 = 1$$

$a^2 - b^2 + c^2 = 1$

10.

Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$?

164, 172, 192, 194, 212

Let's expand the binomial. The coefficients of Pascal's triangle are 1, 4, 6, 4, 1.

$$\begin{aligned}(a + a^{-1})^4 &= a^4 \cdot (a^{-1})^0 + 4a^3 \cdot (a^{-1})^1 + 6a^2 \cdot (a^{-1})^2 + 4a^1 \cdot (a^{-1})^3 + a^0 \cdot (a^{-1})^4 \\ &= a^4 + 4a^2 + 6 + 4a^{-2} + a^{-4} \\ &= a^4 + a^{-4} + 6 + 4(a^2 + a^{-2})\end{aligned}$$

Rearranging and setting $(a + a^{-1})^4 = 4^4$,

$$a^4 + a^{-4} = 4^4 - 6 - 4(a^2 + a^{-2})$$

Expanding the binomial again:

$$\begin{aligned}(a + a^{-1})^2 &= a^2 \cdot (a^{-1})^0 + 2a^1 \cdot (a^{-1})^1 + a^0 \cdot (a^{-1})^2 \\ &= a^2 + 2 + a^{-2}\end{aligned}$$

Rearranging and setting $(a + a^{-1})^2 = 4^2$,

$$a^2 + a^{-2} = 4^2 - 2 = 14$$

Plugging this back into the previous expression:

$$\begin{aligned}a^4 + a^{-4} &= 4^4 - 6 - 4 \cdot 14 \\ &= 194\end{aligned}$$

$$a^4 + a^{-4} = 194$$