AMC 10 Problem Series (2804)

Jon Joseph

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Overview

Week 3 (Jun 18) Class Transcript - Sequences and Series

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jonjoseph 2021-06-18 19:30:31
And @ m25jlm takes it!!!

jonjoseph 2021-06-18 19:31:15
I edit everything but the winner out.

jonjoseph 2021-06-18 19:31:35
Cool dude (dudess).

jonjoseph 2021-06-18 19:31:54
Or that.

jonjoseph 2021-06-18 19:31:56
AMC 10 Problem Series
Week 3: Sequences and Series

jonjoseph 2021-06-18 19:32:03
Today we will look at problems involving sequences and series. We start by looking at two common kinds of sequences, namely arithmetic and geometric sequences, and move on to more general sequences.

jonjoseph 2021-06-18 19:32:11
ARITHMETIC SEQUENCES AND SERIES

jonjoseph 2021-06-18 19:32:20
An arithmetic sequence is a sequence where the difference between every pair of consecutive terms is equal.

jonjoseph 2021-06-18 19:32:29
Let's do a quick example to remind ourselves how this works. Here's an arithmetic sequence:

$$3, 5, 7, 9, \ldots.$$

jonjoseph 2021-06-18 19:32:51
Who can tell me a formula for nth term of the sequence above? Test your formula using $n = 1$ to be sure it works!

jonjoseph 2021-06-18 19:33:48
Yes, the nth term is $3 + 2(n-1)$, or if you prefer, $2n + 1$ (although this form makes it less clear where the numbers come from).

jonjoseph 2021-06-18 19:33:54
Now, let me remind you of a nifty trick for finding the sum of a arithmetic sequence. Let's do an example with the sequence above to see how it works.

jonjoseph 2021-06-18 19:34:02
Say we want to figure out the sum of the first 6 terms. That is, we want to calculate

$$3 + 5 + 7 + 9 + 11 + 13.$$

jonjoseph 2021-06-18 19:34:07
Then we can write this sum above itself, except backwards:

$$\begin{array}{ccccccccccc} 3 & + & 5 & + & 7 & + & 9 & + & 11 & + & 13 \\ 13 & + & 11 & + & 9 & + & 7 & + & 5 & + & 3 \end{array}$$

jonjoseph 2021-06-18 19:34:09
What's the sum of every column above?

jonjoseph 2021-06-18 19:34:40
Yes, it's 16! That means that if we sum up all the numbers above, we get $16 \cdot 6$. But of course, if we sum up all the numbers above, we get twice the sum of numbers in our arithmetic series.

jonjoseph 2021-06-18 19:34:50
In other words, $2S = 6 \cdot 16 = 96$, which means that $S = 48$.

jonjoseph 2021-06-18 19:35:03
Here's a good way to remember this intuitively: "The sum of an arithmetic series is the average of the first and last terms, multiplied by the number of terms."

jonjoseph 2021-06-18 19:35:17
All right, enough examples! Let's do a general arithmetic sequence. This is a sequence of the form $a, a+d, a+2d$, and so on. In other words, we have a first term a, and add a common difference d to each term.

jonjoseph 2021-06-18 19:35:40
What is the formula for the n^{th} term? Again, be careful that your formula works for $n = 1$!

jonjoseph 2021-06-18 19:36:51
The n^{th} term is $a + (n-1)d$.

jonjoseph 2021-06-18 19:37:01
We start with a and we take $n-1$ steps of size d to get from term 1 to term n.

jonjoseph 2021-06-18 19:37:08
Now let's generalize our work for the sum of the first n terms. Here's the sum written down twice again:

jonjoseph 2021-06-18 19:37:10

$$\begin{array}{ccccccc} a & + & a+d & + & \dots & + & a+(n-1)d \\ a+(n-1)d & + & a+(n-2)d & + & \dots & + & a \end{array}$$

jonjoseph 2021-06-18 19:37:13
What's the sum of each column?

jonjoseph 2021-06-18 19:38:18
It's $2a + (n-1)d$. Again, that means that twice the sum is $n(2a + (n-1)d)$, which means that

$$a + (a+d) + \cdots + a + (n-1)d = n \cdot \frac{2a + (n-1)d}{2}.$$

jonjoseph 2021-06-18 19:38:31
I don't recommend memorizing this formula. Instead, just remember that the sum is the average of the first and last terms, multiplied by the number of terms!

jonjoseph 2021-06-18 19:38:45
In a given arithmetic sequence, the first term is 2, the last term is 29, and the sum of all the terms is 155. The common difference is:

(A) 3 (B) 2 (C) $\frac{27}{19}$ (D) $\frac{13}{9}$ (E) $\frac{23}{38}$

jonjoseph 2021-06-18 19:38:59
Hmmmm. We know the first and the last term, and the sum of all the terms. We want to know the common difference.

jonjoseph 2021-06-18 19:39:07
So what should we try to find? Remember to be specific and describe your answer in words: don't just give me a letter!

jonjoseph 2021-06-18 19:40:00
Yes, we should find the number of elements in this sequence. Let's call it n.

jonjoseph 2021-06-18 19:40:08
Wait a minute. We know the sum of all the terms! What did we just learn about the sum of an arithmetic sequence?

jonjoseph 2021-06-18 19:41:10
Right, the sum is the number of terms, multiplied by the average of the first and last term. So what equation do we get?

jonjoseph 2021-06-18 19:41:40
Remember, an equation has an equal sign in it.

jonjoseph 2021-06-18 19:42:20
The average of the first and last term is $\dfrac{2+29}{2}$. That means that we have

$$n \cdot \frac{2+29}{2} = 155.$$

So what's n equal to?

jonjoseph 2021-06-18 19:42:58
We see that $n = \dfrac{2 \cdot 155}{31} = 10.$

jonjoseph 2021-06-18 19:43:01
All right, we have an arithmetic sequence with 10 terms, whose first element is 2 and last element is 29.

jonjoseph 2021-06-18 19:43:07
Is the common difference equal to $\dfrac{29-2}{10}$?

jonjoseph 2021-06-18 19:43:56
It isn't! To get from the first to the tenth term, we only add the common difference 9 times. So what's the common difference actually equal to?

jonjoseph 2021-06-18 19:45:01
It's equal to $\dfrac{29-2}{9} = 3$. The answer is (A).

jonjoseph 2021-06-18 19:45:04
If you aren't sure, here's our sequence:

$$2, 5, 8, 11, 14, 17, 20, 23, 26, 29.$$

jonjoseph 2021-06-18 19:45:10
And indeed, if we add it up in our usual way, we get

$$\begin{array}{ccccccccccccccccccc} 2 & + & 5 & + & 8 & + & 11 & + & 14 & + & 17 & + & 20 & + & 23 & + & 26 & + & 29 \\ 29 & + & 26 & + & 23 & + & 20 & + & 17 & + & 14 & + & 11 & + & 8 & + & 5 & + & 2 \end{array}$$

and we can see that twice the sum is $31 \cdot 10 = 310$, which means the sum is 155, like we wanted.

jonjoseph 2021-06-18 19:45:21
If the sum of the first 10 terms and the sum of the first 100 terms of a given arithmetic progression are 100 and 10, respectively, then the sum of the first 110 terms is

$$\text{(A) } 90 \quad \text{(B) } -90 \quad \text{(C) } 110 \quad \text{(D) } -110 \quad \text{(E) } -100$$

jonjoseph 2021-06-18 19:45:36
As usual, we're going to start by defining variables.

jonjoseph 2021-06-18 19:45:42
Think about what variables we're going to need. Make sure to describe them in words: don't just give me letters. For all I know, a stands for the number of hairs on your head...

jonjoseph 2021-06-18 19:46:19
Wait. What?

jonjoseph 2021-06-18 19:47:00
Let's use the first term and common difference of the arithmetic sequence. Let those be a and d, respectively.

jonjoseph 2021-06-18 19:47:07
In this problem, we are told the sum for certain values of n (namely $n = 10$ and $n = 100$), and want to find the sum when $n = 110$. We can start by using the information we're given to write some equations.

jonjoseph 2021-06-18 19:47:25
What is the 10^{th} term, in terms of a and d?

jonjoseph 2021-06-18 19:48:23
The 10^{th} term is $a + 9d$. So what is an expression for the sum of the first 10 terms?

jonjoseph 2021-06-18 19:49:39
The sum of the first 10 terms is the average of the first and 10^{th} term, multiplied by 10, which is

$$\frac{a + (a + 9d)}{2} \cdot 10 = 5(2a + 9d) = 10a + 45d.$$

jonjoseph 2021-06-18 19:49:43
Hence, $10a + 45d = 100.$

jonjoseph 2021-06-18 19:49:44
What is the 100^{th} term, in terms of a and d?

jonjoseph 2021-06-18 19:50:53
The 100^{th} term is $a + 99d$. So what is an expression for the sum of the first 100 terms?

jonjoseph 2021-06-18 19:52:16
The sum of the first 100 terms is the average of the first and 100^{th} term, multiplied by 100, which is

$$\frac{a + (a + 99d)}{2} \cdot 100 = 50(2a + 99d) = 100a + 4950d.$$

jonjoseph 2021-06-18 19:52:22
Hence, $100a + 4950d = 10.$

jonjoseph 2021-06-18 19:52:24
At this point, we could just solve this system of two equations for a and d, but before jumping into that, let's remember what we're looking for.

jonjoseph 2021-06-18 19:52:32

We're looking for the sum of the first 110 terms, so let's write an expression for that in terms of a and d.

jonjoseph 2021-06-18 19:52:44

The 110^{th} term is $a + 109d$, so the sum of the first 110 terms is

$$\frac{a + (a + 109d)}{2} \cdot 110 = 55(2a + 109d) = 110a + 5995d.$$

jonjoseph 2021-06-18 19:52:54

Let's recap. We know that $10a + 45d = 100$ and $100a + 4950d = 10$. We want $110a + 5995d$.

jonjoseph 2021-06-18 19:53:02

What can we do to help us find $110a + 5995d$?

jonjoseph 2021-06-18 19:53:52

There are a number of ways we can handle this. One way is to add the first two equations, because this will give us an expression containing $10a + 100a = 110a$.

jonjoseph 2021-06-18 19:54:21

(Yes. We could just solve the system. And on the AMC you might do that but in this case it is a little messy.)

jonjoseph 2021-06-18 19:54:33

We want $110a + 5995d$, so we need to find d (more specifically, $1000d$). How can we use the equations $10a + 45d = 100$ and $100a + 4950d = 10$ to find d?

jonjoseph 2021-06-18 19:55:23

Hint: NOW it does help to eliminate a variable. Which one?

jonjoseph 2021-06-18 19:56:12

We can eliminate a.

jonjoseph 2021-06-18 19:56:17

We can multiply the first equation by 10, to get $100a + 450d = 1000$.

jonjoseph 2021-06-18 19:56:21

Then we subtract this equation from the equation $100a + 4950d = 10$ to get $4500d = -990$. So what is the value of $1000d$?

jonjoseph 2021-06-18 19:57:23

Remember, we want $1000d$.

jonjoseph 2021-06-18 19:58:21

We see that

$$1000d = -\frac{990}{4500} \cdot 1000 = -220.$$

So what is the value of $110a + 5995d$?

jonjoseph 2021-06-18 19:59:34

We see that

$$\begin{aligned} 110a + 5995d &= (110a + 4995d) + 1000d \\ &= 110 - 220 \\ &= -110. \end{aligned}$$

The answer is (D).

jonjoseph 2021-06-18 19:59:49

Notice how we used the $1000d$.

jonjoseph 2021-06-18 20:00:03

This problem shows the importance of remembering what you're trying to solve for. Instead of doing the trick with adding the two equations together, we could have just solved for a and d immediately.

jonjoseph 2021-06-18 20:00:18
It is entirely possible to solve the problem that way, but it's a lot messier. We would have gotten quite ugly values for a and d (neither is an integer), and there's a serious risk we might have made an arithmetic mistake trying to plug them in and calculate $110a + 5995d$.

jonjoseph 2021-06-18 20:00:33
The measures of the interior angles of a convex polygon of n sides are in arithmetic progression. If the common difference is 5° and the largest angle is 160°, then n equals:

$\text{(A) } 9 \quad \text{(B) } 10 \quad \text{(C) } 12 \quad \text{(D) } 16 \quad \text{(E) } 32$

jonjoseph 2021-06-18 20:00:50
We could, as usual, define variables and solve. Here, we could let a be the first term of the sequence, and we could let d be the common difference. Then we could plug and chug, as usual.

jonjoseph 2021-06-18 20:01:01
But as we saw last class, sometimes thinking a bit leads to a cleverer, quicker solution! And this problem is amenable to that.

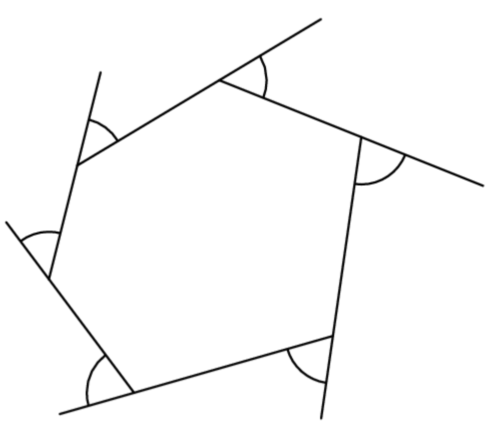
jonjoseph 2021-06-18 20:01:11
One thing that makes this problem tricky is that we don't know the sum of the interior angles, since we aren't told how many sides there are. So what might be easier than working with interior angles?

jonjoseph 2021-06-18 20:01:54
Smart.

jonjoseph 2021-06-18 20:01:57
We could try working with exterior angles instead!

jonjoseph 2021-06-18 20:02:03
OK! What do we know about the exterior angles of a polygon? They are these angles, if you don't remember:

jonjoseph 2021-06-18 20:02:04



jonjoseph 2021-06-18 20:03:39
They always add to 360°.

jonjoseph 2021-06-18 20:03:42
And if the interior angles of a polygon are in an arithmetic sequence with a common difference of 5°, what can you tell me about the exterior angles?

jonjoseph 2021-06-18 20:04:54

They are also in an arithmetic progression with a common difference of 5°!

jonjoseph 2021-06-18 20:05:00
Now, we know that the largest interior angle is 160°. What can you tell me about the exterior angles?

jonjoseph 2021-06-18 20:05:51
That's right, the smallest exterior angle is $180^\circ - 160^\circ = 20^\circ$.

jonjoseph 2021-06-18 20:05:53
All right! So the exterior angles are

$$20^\circ, 25^\circ, 30^\circ, \ldots$$

How do we know when our sequence ends?

jonjoseph 2021-06-18 20:06:42
It ends when we get to a sum of 360°! Very cool.

jonjoseph 2021-06-18 20:06:50
All right, now let's be clever and not forget that this is an AMC question! What's the smallest possible value of n we might get?

jonjoseph 2021-06-18 20:07:33
Looking at the answer choices, the smallest n might be is 9. So we should start calculating our sums from $n = 9$ and keep going until we hit 360°.

jonjoseph 2021-06-18 20:07:39
That means we start out by calculating

$$20^\circ + 25^\circ + 30^\circ + 35^\circ + 40^\circ + 45^\circ + 50^\circ + 55^\circ + 60^\circ.$$

What do we get?

jonjoseph 2021-06-18 20:08:29
We know that the sum of the sequence is the number of terms multiplied by the average of the first and last term. Therefore, this sum is equal to

$$9 \cdot \frac{20^\circ + 60^\circ}{2} = 9 \cdot 40^\circ = 360^\circ.$$

jonjoseph 2021-06-18 20:08:35
Wait a minute... that's very nice! We can immediately spot our final answer. What is it?

jonjoseph 2021-06-18 20:09:12
Yes, our final answer is $n = 9$, or (A). That was lucky!

jonjoseph 2021-06-18 20:09:17
The sum of the first m positive odd integers is 212 more than the sum of the first n positive even integers. What is the sum of all possible values of n?

$(\text{A})\ 255 \quad (\text{B})\ 256 \quad (\text{C})\ 257 \quad (\text{D})\ 258 \quad (\text{E})\ 259$

jonjoseph 2021-06-18 20:09:38
The first m positive odd integers are $1, 3, 5, \ldots, (2m-1)$. Let's figure out their sum.

jonjoseph 2021-06-18 20:09:53
Anyone know what the sum is?

jonjoseph 2021-06-18 20:10:15
Nice. Let's go through it.

jonjoseph 2021-06-18 20:10:27

Let's do our usual trick:

$$\begin{array}{ccccccc} 1 & + & 3 & + & \dots & + & 2m-1 \\ 2m-1 & + & 2m-3 & + & \dots & + & 1 \end{array}$$

So what's the sum of $1, 3, 5, \dots, 2m-1$?

jonjoseph 2021-06-18 20:11:12
That's right, the sum of the first m positive odd integers is

$$\frac{1+(2m-1)}{2} \cdot m = \frac{2m}{2} \cdot m = m^2.$$

jonjoseph 2021-06-18 20:11:22
The first n positive even integers are $2, 4, 6, \dots, 2n$. What is their sum?

jonjoseph 2021-06-18 20:12:36
The sum of the first n positive even integers is

$$\frac{2+2n}{2} \cdot n = n^2 + n.$$

jonjoseph 2021-06-18 20:12:42
From what we are told in the problem, we can write

$$m^2 = n^2 + n + 212.$$

Somehow we must find all solutions to this equation in positive integers.

jonjoseph 2021-06-18 20:13:01
Here's a tricky idea!

jonjoseph 2021-06-18 20:13:05
We see that the equation $m^2 = n^2 + n + 212$ is quadratic in n, so let's write it in the form

$$n^2 + n + (212 - m^2) = 0.$$

jonjoseph 2021-06-18 20:13:14
Now, how can we solve for n in terms of m? (That is, we treat m as a constant, and solve for n.)

jonjoseph 2021-06-18 20:14:03
Nice

jonjoseph 2021-06-18 20:14:05
Yes, we can use the quadratic formula! That means we get

$$n = \frac{-1 \pm \sqrt{1^2 - 4(212 - m^2)}}{2} = \frac{-1 \pm \sqrt{4m^2 - 847}}{2}.$$

jonjoseph 2021-06-18 20:14:14
Hmmm. We're interested in solutions where both m and n are positive integers. We've solved for n in terms of m.

jonjoseph 2021-06-18 20:14:21
We want n to be a positive integer. Look at our formula for n above. Which part looks problematic for getting a positive integer result?

jonjoseph 2021-06-18 20:15:16
The square root $\sqrt{4m^2 - 847}$ does, of course! So what has to be true for n to be a positive integer?

jonjoseph 2021-06-18 20:16:33

If the square root comes out an integer what does that mean?

jonjoseph 2021-06-18 20:17:05
Yes, $4m^2 - 847$ has to be a perfect square.

jonjoseph 2021-06-18 20:17:07
In other words, there has to exist a non-negative integer k such that

$$4m^2 - 847 = k^2.$$

jonjoseph 2021-06-18 20:17:21
OK, but it's still not obvious at all how to get the answer. We can't just plug in every single possible value of m and see when we get squares!

jonjoseph 2021-06-18 20:17:34
Let's see if the above equation can give us some information about m, though. Let's rewrite the above as

$$4m^2 - k^2 = 847.$$

Wait a minute! I see something nice on the left-hand side. What is it?

jonjoseph 2021-06-18 20:18:29
This equation factors as a difference of squares! We now have

$$(2m - k)(2m + k) = 847.$$

jonjoseph 2021-06-18 20:18:34
This is now approachable! There are only so many ways to factor 847. What's the prime factorization of 847?

jonjoseph 2021-06-18 20:19:23
The prime factorization of 847 is $7 \cdot 11^2$. That means that the only factorizations of 847 into two positive integers are the following:

$$847 = 1 \cdot 847 = 7 \cdot 121 = 11 \cdot 77.$$

jonjoseph 2021-06-18 20:19:30
For each factorization, we can let $2m - k$ be one factor, and we can let $2m + k$ be the other factor. Can we figure out which one is which? Or do we have to try both ways?

jonjoseph 2021-06-18 20:20:19
Ah, we chose k to be positive, so $2m - k$ always has to be the smaller factor.

jonjoseph 2021-06-18 20:20:24
All right, now let's go through the possibilities. If we have $2m - k = 1$ and $2m + k = 847$, then what are m and k?

jonjoseph 2021-06-18 20:20:46
(This is a bit of a bash from here on out)

jonjoseph 2021-06-18 20:21:54
We get $m = \dfrac{847 + 1}{4} = 212$ and $k = \dfrac{847 - 1}{2} = 423$.

jonjoseph 2021-06-18 20:22:01
Substituting into $4m^2 - 847 = k^2$, we have

$$4 \cdot 212^2 - 847 = 423^2.$$

jonjoseph 2021-06-18 20:22:11
Of course, we don't actually want k! We want to figure out n. We know that

$$n = \frac{-1 \pm \sqrt{4m^2 - 847}}{2} = \frac{-1 \pm \sqrt{4 \cdot 212^2 - 847}}{2}.$$

Do we use the plus or the minus from the $\pm$?

jonjoseph 2021-06-18 20:22:47
Since we want a positive integer solution, we see that

$$n = \frac{-1 + \sqrt{4 \cdot 212^2 - 847}}{2}.$$

jonjoseph 2021-06-18 20:22:50
Of course, that square root is rather ugly! Luckily, along the way we figured out not only the fact that $4 \cdot 212^2 - 847$ is a perfect square, but we also figured out what perfect square it is.

jonjoseph 2021-06-18 20:22:59
So what is $\sqrt{4 \cdot 212^2 - 847}$? You might want to check our calculations above.

jonjoseph 2021-06-18 20:23:49
That's right, it's equal to 423. So what's n in this case?

jonjoseph 2021-06-18 20:24:33
Correct, $n = \dfrac{-1 + 423}{2} = 211$. We will need to remember that.

jonjoseph 2021-06-18 20:24:39
Let's do the next possibility. If we have $2m - k = 7$ and $2m + k = 121$, then what are m and k?

jonjoseph 2021-06-18 20:25:46
We get $m = \dfrac{121 + 7}{4} = 32$ and $k = \dfrac{121 - 7}{2} = 57$. Therefore,

$$4 \cdot 32^2 - 847 = 57^2.$$

jonjoseph 2021-06-18 20:25:50
Plugging in again, and using the fact that we want to get a positive integer, we see that

$$n = \frac{-1 + \sqrt{4 \cdot 32^2 - 847}}{2}.$$

What do we get?

jonjoseph 2021-06-18 20:26:40
We see that

$$n = \frac{-1 + 57}{2} = 28.$$

jonjoseph 2021-06-18 20:26:44
Finally, if we have $2m - k = 11$ and $2m + k = 77$, then we have that $m = 22$ and $k = 33$. That means that

$$4 \cdot 22^2 - 847 = 33^2.$$

jonjoseph 2021-06-18 20:26:48
What do we get for n, then?

jonjoseph 2021-06-18 20:26:52
(last one)

jonjoseph 2021-06-18 20:27:29

We get that

$$n=\frac{-1+\sqrt{4\cdot 22^2-847}}{2}=\frac{-1+33}{2}=16.$$

So what's the final answer?

jonjoseph 2021-06-18 20:28:48
The answer is $211+28+16=255$, which is (A).

jonjoseph 2021-06-18 20:29:40
A busy work problem with important trick - using the quadratic formula even though it looks as if we have two variables. Questions?

jonjoseph 2021-06-18 20:30:07
GEOMETRIC SEQUENCES AND SERIES

jonjoseph 2021-06-18 20:30:12
A geometric sequence is a sequence of the form $a, ar, ar^2, \dots$. In other words, we have a first term a, and multiply each term by a common ratio r.

jonjoseph 2021-06-18 20:30:18
What is the formula for the n^{th} term?

jonjoseph 2021-06-18 20:31:40
The n^{th} term is ar^{n-1}. We start with a and take $n-1$ steps, multiplying by r at each step.

jonjoseph 2021-06-18 20:32:18
Remember, when you latex a multi-term exponent you need braces: ar^{n-1}

jonjoseph 2021-06-18 20:32:30
The sum of the geometric series with n terms is

$$a+ar+\dots+ar^{n-1}=\frac{a(r^n-1)}{r-1}=\frac{a(1-r^n)}{1-r}.$$

jonjoseph 2021-06-18 20:32:52
The sum of an infinite geometric series converges (that is to say, it has a numerical value) only for r between -1 and 1, and it is given by the formula:

$$a+ar+ar^2+ar^3+\dots=\frac{a}{1-r}.$$

jonjoseph 2021-06-18 20:33:09
****** These are two very important results ******

jonjoseph 2021-06-18 20:33:20
Define a sequence of real numbers a_1, a_2, a_3, $\dots$ by $a_1=1$ and $a_{n+1}^3=99a_n^3$ for all $n\geq 1$. Then a_{100} equals

$\text{(A)}\ 33^{33}\qquad\text{(B)}\ 33^{99}\qquad\text{(C)}\ 99^{33}\qquad\text{(D)}\ 99^{99}\qquad\text{(E)}\ \text{None of these}$

jonjoseph 2021-06-18 20:33:37
We are given that

$$a_{n+1}^3=99a_n^3$$

for all $n\geq 1$. What can we do with this formula?

jonjoseph 2021-06-18 20:34:03

Hint: The cube is annoying.

jonjoseph 2021-06-18 20:34:31
We can take the cube root of both sides, to get

$$a_{n+1} = a_n \sqrt[3]{99}.$$

jonjoseph 2021-06-18 20:34:35
Each term is multiplied by a constant (namely the cube root of 99) to get the next term, so the sequence is a geometric sequence with common ratio $r = \sqrt[3]{99}$.

jonjoseph 2021-06-18 20:34:48
We're looking for a_{100}. What is the 100^{th} term in this sequence in terms of r?

jonjoseph 2021-06-18 20:35:16
Hint: We know a_1.

jonjoseph 2021-06-18 20:35:56
The 100^{th} term is

$$a_{100} = a_1 \cdot r^{99} = r^{99}.$$

jonjoseph 2021-06-18 20:36:02
What is this equal to?

jonjoseph 2021-06-18 20:37:08
This is equal to

$$r^{99} = (\sqrt[3]{99})^{99} = 99^{33}.$$

jonjoseph 2021-06-18 20:37:10
The answer is (C).

jonjoseph 2021-06-18 20:37:18
This problem is a lesson in the fact that when there is complicated notation on the AMC, it is often hiding a relatively easy problem. So be brave and read and understand it!

jonjoseph 2021-06-18 20:37:38
A sequence of three real numbers forms an arithmetic progression with a first term of 9. If 2 is added to the second term and 20 is added to the third term, the three resulting numbers form a geometric progression. What is the smallest possible value for the third term of the geometric progression?

$\text{(A) } 1 \quad \text{(B) } 4 \quad \text{(C) } 36 \quad \text{(D) } 49 \quad \text{(E) } 81$

jonjoseph 2021-06-18 20:37:55
As usual, we want to start by converting the given information into algebra. We have three numbers that form an arithmetic progression, and the first term is 9. How can we express these numbers?

jonjoseph 2021-06-18 20:39:28
We can express these numbers as 9, $9 + d$, and $9 + 2d$, if we define d as our common difference.

jonjoseph 2021-06-18 20:39:35
If we add 2 to the second term and 20 to the third term, then we get 9, $11 + d$, and $29 + 2d$. These numbers form a geometric progression. How can we use this?

jonjoseph 2021-06-18 20:40:55
Yep. Some good ideas.

jonjoseph 2021-06-18 20:41:00

We can introduce a second variable, say r, and express the terms as 9, $9r$, and $9r^2$. However, there is another way that avoids introducing a new variable.

jonjoseph 2021-06-18 20:41:11

Since the sequence is geometric, the ratio between consecutive entries is always the same (and is r, in our usual notation). What equation does that get us?

jonjoseph 2021-06-18 20:41:18

We know the ratio of the first and second terms is equal to the ratio of the second and third terms. This gives us

$$\frac{11+d}{9} = \frac{29+2d}{11+d}.$$

jonjoseph 2021-06-18 20:41:33

What do we do now?

jonjoseph 2021-06-18 20:42:06

To solve the equation, let's cross multiply. This gets us $(11+d)^2 = 9(29+2d)$.

jonjoseph 2021-06-18 20:42:12

This equation simplifies to $d^2 + 4d - 140 = 0$.

jonjoseph 2021-06-18 20:42:15

What are the solutions to this quadratic equation?

jonjoseph 2021-06-18 20:43:22

The left hand side factors as $(d-10)(d+14) = 0$, so $d = 10$ or $d = -14$.

jonjoseph 2021-06-18 20:43:27

Huh, there are two different possibilities for d. Looking back at the problem statement, which one do we want?

jonjoseph 2021-06-18 20:44:18

We want the smallest possible value of the third term of the geometric progression, which is $29 + 2d$. That means we want d to be as small as possible. So what's the answer?

jonjoseph 2021-06-18 20:45:12

We take $d = -14$. This gives us $29 + 2d = 29 + 2(-14) = 1$. The answer is (A).

jonjoseph 2021-06-18 20:45:19

Both values of d gives us valid sequences. For $d = 10$ we get the arithmetic sequence $9, 19, 29$ and the geometric sequence $9, 21, 49$. For $d = -14$ we get the arithmetic sequence $9, -5, -19$ and the geometric sequence $9, -3, 1$.

jonjoseph 2021-06-18 20:45:26

Don't assume that a value doesn't work just because it's negative! Check and see. 🙂

jonjoseph 2021-06-18 20:45:49

GENERAL SEQUENCES AND SERIES

jonjoseph 2021-06-18 20:45:54

We now consider some problems about more general sequences (sequences that are neither arithmetic nor geometric). When dealing with a general sequence, one main technique is to compute the first few terms in the hopes of finding a pattern.

jonjoseph 2021-06-18 20:46:08

The first term of a sequence is 2005. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2005^{th} term of the sequence?

$(\text{A})\ 29 \quad (\text{B})\ 55 \quad (\text{C})\ 85 \quad (\text{D})\ 133 \quad (\text{E})\ 250$

jonjoseph 2021-06-18 20:46:17
What can we do here?

jonjoseph 2021-06-18 20:46:32
How? What should we do?

jonjoseph 2021-06-18 20:47:08
When you have a weird unfamiliar sequence like this, the best approach is usually to compute a few terms of it and hope to notice a pattern.

jonjoseph 2021-06-18 20:47:12
The first term is 2005. What is the second term?

jonjoseph 2021-06-18 20:48:05
The second term is $2^3 + 0^3 + 0^3 + 5^3 = 133$.

jonjoseph 2021-06-18 20:48:07
What is the third term?

jonjoseph 2021-06-18 20:48:36
The third term is $1^3 + 3^3 + 3^3 = 55$.

jonjoseph 2021-06-18 20:48:38
What is the fourth term?

jonjoseph 2021-06-18 20:49:02
The fourth term is $5^3 + 5^3 = 250$.

jonjoseph 2021-06-18 20:49:03
What is the fifth term?

jonjoseph 2021-06-18 20:49:42
Bingo!

jonjoseph 2021-06-18 20:49:52
The fifth term is $2^3 + 5^3 + 0^3 = 133$.

jonjoseph 2021-06-18 20:49:56
This is the same as the second term. What does this tell us about the sequence?

jonjoseph 2021-06-18 20:50:31
This tells us that starting with the second term, the sequence is periodic with period 3. In other words, the three numbers $133, 55, 250$ will repeat over and over starting from the second term.

jonjoseph 2021-06-18 20:51:10
Now, we need to be a little careful because the first term is not included in our repeats.

jonjoseph 2021-06-18 20:51:13
So what is the 2005^{th} term in the sequence? Explain how you found it!

jonjoseph 2021-06-18 20:52:41
Since 2005 and 4 differ by 2001, which is a multiple of 3, the 2005^{th} term is equal to the fourth term, which is 250. The answer is (E).

jonjoseph 2021-06-18 20:52:56
Note that even though 2005 and 1 differ by 2004, which is also a multiple of 3, the 2005^{th} term is not equal to the first term because the period does not kick in until the second term of the sequence. This is why we compare the 2005^{th} term to the fourth term.

jonjoseph 2021-06-18 20:53:10
Periodic sequences appear frequently in mathematics, so always keep an eye out for them. You may have to compute many terms to find the period, but this kind of persistence usually pays off.

jonjoseph 2021-06-18 20:53:31
Suppose that (u_n) is a sequence of real numbers satisfying $u_{n+2} = 2u_{n+1} + u_n$, and that $u_3 = 9$ and $u_6 = 128$. What is u_5 ?

$\text{(A) } 40 \quad \text{(B) } 53 \quad \text{(C) } 68 \quad \text{(D) } 88 \quad \text{(E) } 104$

jonjoseph 2021-06-18 20:53:47
We say that the definition of this sequence is *recursive*. That means that we're given an expression for the next term using the previous terms.

jonjoseph 2021-06-18 20:53:54
We can use the recursion to get lots of equations with different sets of terms from the sequence by plugging in different values for n.

jonjoseph 2021-06-18 20:54:01
For instance, what equation do we get if we plug in $n = 1$?

jonjoseph 2021-06-18 20:54:48
For a subscript you can write u_3 or, in latex, u_3

jonjoseph 2021-06-18 20:55:21
We get the equation $u_3 = 2u_2 + u_1$. How about $n = 2$?

jonjoseph 2021-06-18 20:56:14
Right, we get $u_4 = 2u_3 + u_2$. In words, if you add a term in the sequence to twice the next term, you get the term after that.

jonjoseph 2021-06-18 20:56:21
We are given $u_3 = 9$ and $u_6 = 128$. We want to find u_5.

jonjoseph 2021-06-18 20:56:26
What values of n would it be helpful to consider?

jonjoseph 2021-06-18 20:57:50
We can take $n = 3$.

jonjoseph 2021-06-18 20:57:51
Taking $n = 3$, we get

$$u_5 = 2u_4 + u_3.$$

jonjoseph 2021-06-18 20:57:55
Since $u_3 = 9$, this equation becomes

$$u_5 = 2u_4 + 9.$$

jonjoseph 2021-06-18 20:58:08
We can take $n = 4$.

jonjoseph 2021-06-18 20:58:13
Taking $n = 4$, we get

$$u_6 = 2u_5 + u_4.$$

jonjoseph 2021-06-18 20:58:22
Since $u_6 = 128$, this equation becomes

$$128 = 2u_5 + u_4.$$

jonjoseph 2021-06-18 20:58:31
Thus, we have the system of equations

$$\begin{aligned} 2u_4 + 9 &= u_5, \\ u_4 + 2u_5 &= 128. \end{aligned}$$

jonjoseph 2021-06-18 20:59:08
You try to finish. Hint: Remember, we want u_5.

jonjoseph 2021-06-18 21:00:28
Well done.

jonjoseph 2021-06-18 21:00:31
We can multiply the second equation by 2, then subtract the resulting equation from the first equation.

jonjoseph 2021-06-18 21:00:37
Multiplying the second equation by 2, we get

$$2u_4 + 4u_5 = 256.$$

jonjoseph 2021-06-18 21:00:42
Subtracting this equation from the first equation, we get

$$9 - 4u_5 = u_5 - 256.$$

jonjoseph 2021-06-18 21:00:47
The equation above simplifies to $5u_5 = 265$, so $u_5 = 265/5 = 53$. The answer is (B).

jonjoseph 2021-06-18 21:00:52
BAM. Done.

jonjoseph 2021-06-18 21:01:23
How about if I put one more on the message board?

jonjoseph 2021-06-18 21:01:39
OKay. I've got a good one.

jonjoseph 2021-06-18 21:01:44
SUMMARY

jonjoseph 2021-06-18 21:01:47
In today's class, we saw how to solve problems involving sequences and series.

jonjoseph 2021-06-18 21:01:50
An arithmetic sequence is determined by its first term and common difference. Similarly, a geometric sequence is determined by its first term and common ratio. Hence, when dealing with these kinds of sequences, the approach is usually to define the appropriate variables and set up the equations.

jonjoseph 2021-06-18 21:01:59
If we are dealing with a more general sequence, then things get more tricky. Start by computing the first few terms, to see if you can find a pattern. Look for periodic behavior. Be persistent! Sometimes, it may take a few terms for the period or the pattern to kick in.

jonjoseph 2021-06-18 21:02:25
Nicely done tonight. Stay safe, stay cool, don't get wet (if you live FL). See you next week.