Math Competition Tricks

Clairbourn School Grade 7/8

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Abstract

This note reviews a selection of tricks that may be useful in math competitions.

Mental Arithmetic

It is useful to add/multiply/divide fast. There are too many tricks to review, but here are a few basic ones. With practice you will be able to use these tricks while calculating in your head.

Add numbers by grouping them:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 4 \times 10 + 5 = 45$$

Add numbers by rounding them:

$$978 + 237 = (980 - 2) + (220 + 17) = 1200 + 15 = 1215$$

Multiply by 5:

$$978 \times 5 = (1000 - 20 - 2) \times \frac{10}{2} = \frac{500 - 10 - 1}{2} \times 10 = 4890$$

where we have also decomposed 978 to make the division by 2 even easier (skip this step if you can quickly halve 978).

Multiply numbers by decomposing them:

$$\begin{aligned} 14 \times 16 &= (15 - 1) \times (15 + 1) \\ &= 15^2 - 1 = 224 \end{aligned}$$

where we have used $15^2 = 225$ and the difference-of-squares formula:

$$(a + b)(a - b) = a^2 - b^2$$

Similarly, $13 \times 17 = 15^2 - 4 = 221$ (if hesitant, check that the last digit matches: $3 \times 7 = 21$, so the last digit 1 is indeed correct). The difference-of-squares formula can always be applied when multiplying numbers that differ by a multiple of 2 (multiplying two even numbers or multiplying two odd numbers).

Multiply numbers by rounding up:

$$\begin{aligned} 19 \times 18 &= 20 \times 18 - 18 \\ &= 360 - 18 = 342 \end{aligned}$$

Multiply numbers by rounding up and down:

$$\begin{aligned} 19 \times 23 &= (20 - 1) \times (20 + 3) \\ &= 20^2 + (3 - 1) \times 20 - 3 = 400 + 40 - 3 = 437 \end{aligned}$$

Square numbers by rounding up:

$$\begin{aligned} 99^2 &= (100-1)^2 \\ &= 10000 - 200 + 1 = 9801 \end{aligned}$$

where we have used:

$$(a+b)^2 = a^2 + 2ab + b^2$$

Square numbers by rounding up or down:

$$\begin{aligned} 13^2 &= (15-2)^2 \\ &= 200 + 25 - 60 + 4 = 140 + 29 = 169 \end{aligned}$$

where we suppose you have memorized $15^2 = 225 = 200 + 25$ (but forgotten 13^2). Because it is easier to subtract 60 from 200 than from 225, we also split 225 as $200 + 25$. These manipulations are to be done in your head or very quickly on a scrap of paper.

Useful Sums

The sum of the first n natural numbers:

$$\begin{aligned} &1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \\ &1 + 2 + 3 + \ldots + 10 = 55 \\ &1 + 2 + 3 + \ldots + 100 = 505 \end{aligned}$$

The sum of the first $(2n-1)$ odd numbers:

$$\begin{aligned} &1 + 3 + 5 + \ldots + (2n-1) = n^2 \\ &1 + 3 + 5 + \ldots + 9 = 1 + 3 + 5 + \ldots + (2 \times 5 - 1) = 5^2 = 25 \\ &1 + 3 + 5 + \ldots + 99 = 1 + 3 + 5 + \ldots + (2 \times 50 - 1) = 50^2 = 2500 \end{aligned}$$

The sum of the first $2n$ even numbers:

$$\begin{aligned} &2 + 4 + 6 + \ldots + (2n) = n(n+1) \\ &2 + 4 + 6 + \ldots + 10 = 2 + 4 + 6 + \ldots + (2 \times 5) = 5 \times 6 = 30 \\ &2 + 4 + 6 + \ldots + 100 = 2 + 4 + 6 + \ldots + (2 \times 50) = 50 \times 51 = 2550 \end{aligned}$$

The sum of the first n squares formula and first ten sums:

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$1, 5, 14, 30, 55, 91, 140, 204, 285, 385.$

The sum of the first n cubes formula and first ten sums:

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^3$$

$1, 9, 36, 100, 225, 441, 784, 1296, 2025, 3025.$

The sum of n terms of a geometric series:

$$1 + a + a^2 + \ldots + a^n = \frac{1 - a^n}{1 - a}$$

The Fibonacci numbers are the sum of the two preceding numbers in the Fibonacci sequence:

$$F_n = F_{n-1} + F_{n-2}$$

where the first two numbers in the sequence are typically 0 and 1:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

The Lucas numbers are Fibonacci numbers with starting values 2 and 0:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \ldots$$

Useful Products

These factorial products are worth remembering:

$$\begin{aligned} 3! &= 6 \\ 4! &= 24 \\ 5! &= 120 \\ 6! &= 720 \\ 7! &= 5040 \\ 8! &= 40320 \\ 9! &= 362880 \\ 10! &= 3628800 \end{aligned}$$

Prime Numbers

The first 25 prime numbers are:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$$

Prime numbers of the twentieth and twenty-first centuries (problems involving numbers close to the current year are popular: the closest prime numbers to 2020 are 2017 and 2027):

$$1901, 1907, 1913, 1931, 1933, 1949, 1951, 1973, 1979, 1987, 1993, 1997, 1999,$$
$$2003, 2011, 2017, 2027, 2029, 2039, 2053, 2063, 2069, 2081, 2083, 2087, 2089, 2099$$

Mersenne primes are prime numbers of the form $2^p - 1$, for some prime number p. The first few Mersenne primes are:

p	2	3	5	7	11	13	17	19	31
$2^p - 1$	3	7	31	127	8191	131071	524287	2147483647	

Some Mersenne numbers that are not prime include:

p	11	23	29
$2^p - 1$	$2047 = 23 \times 89$	$8388607 = 47 \times 178481$	$536870911 = 233 \times 1103 \times 2089$

Fermat primes are prime numbers of the form $2^{2^n} + 1$. There are only five known Fermat primes:

n	0	1	2	3	4
$2^{2^n} + 1$	3	5	17	257	65537

The famous mathematician Euler showed that

$$2^{2^5} + 1 = 2^{32} + 1 = 4294967297 = 641 \times 6700417$$

Some Fibonacci numbers are prime. Here are the first few:

$$2, 3, 5, 13, 89, 233, 1597, 28657$$

Useful Squares

These are the first ten squares: $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$. Here are a few more:

$$
\begin{aligned}
11^2 &= 121\\
12^2 &= 144\\
13^2 &= 169\\
14^2 &= 196\\
15^2 &= 225\\
16^2 &= 256\\
17^2 &= 289\\
18^2 &= 324\\
19^2 &= 361\\
20^2 &= 400\\
21^2 &= 441\\
22^2 &= 484\\
23^2 &= 529\\
24^2 &= 576\\
25^2 &= 625\\
30^2 &= 900\\
35^2 &= 1225\\
40^2 &= 1600\\
45^2 &= 2025\\
50^2 &= 2500\\
55^2 &= 3025\\
60^2 &= 3600\\
65^2 &= 4225\\
70^2 &= 4900\\
75^2 &= 5625\\
80^2 &= 6400\\
85^2 &= 7225\\
90^2 &= 8100\\
95^2 &= 9025\\
100^2 &= 10000
\end{aligned}
$$

Useful Cubes

$$
\begin{aligned}
2^3 &= 8 \\
3^3 &= 27 \\
4^3 &= 64 \\
5^3 &= 125 \\
6^3 &= 216 \\
7^3 &= 343 \\
8^3 &= 512 \\
9^3 &= 729 \\
10^3 &= 1000 \\
11^3 &= 1331 \\
12^3 &= 1728 \\
13^3 &= 2197 \\
14^3 &= 2744 \\
15^3 &= 3375 \\
16^3 &= 4096 \\
17^3 &= 4913 \\
18^3 &= 5832 \\
19^3 &= 6859 \\
20^3 &= 8000
\end{aligned}
$$

Useful Fourth Powers

$$
\begin{aligned}
2^4 &= 16 \\
3^4 &= 81 \\
4^4 &= 256 \\
5^4 &= 625 \\
6^4 &= 1296 \\
7^4 &= 2401 \\
8^4 &= 4096 \\
9^4 &= 6561 \\
10^4 &= 10000
\end{aligned}
$$

More Useful Powers

Memorizing powers can come in handy:

$$
\begin{aligned}
2^2 &= 4\\
2^3 &= 8\\
2^4 &= 16\\
2^5 &= 32\\
2^6 &= 64\\
2^7 &= 128\\
2^8 &= 256\\
2^9 &= 512\\
2^{10} &= 1024\\
2^{11} &= 2048\\
2^{12} &= 4096\\
2^{13} &= 8192\\
2^{14} &= 16384\\
2^{15} &= 32768
\end{aligned}
$$

Powers of 3:

$$
\begin{aligned}
3^2 &= 9\\
3^3 &= 27\\
3^4 &= 81\\
3^5 &= 243\\
3^6 &= 729\\
3^7 &= 2187\\
3^8 &= 6561\\
3^9 &= 19683\\
3^{10} &= 59049
\end{aligned}
$$

Powers of 5:

$$
\begin{aligned}
5^2 &= 25\\
5^3 &= 125\\
5^4 &= 625\\
5^5 &= 3125\\
5^6 &= 15625\\
5^7 &= 78125\\
5^8 &= 390625
\end{aligned}
$$

Powers of 6:

$$6^2 = 36$$
$$6^3 = 216$$
$$6^4 = 1296$$
$$6^5 = 7776$$

Powers of 7:

$$7^2 = 49$$
$$7^3 = 343$$
$$7^4 = 2401$$
$$7^5 = 16807$$

Powers of 11:

$$11^2 = 121$$
$$11^3 = 1331$$
$$11^4 = 14641$$
$$11^5 = 161051$$

Highly Composite Numbers

A positive integer that has more divisors than any smaller positive integer. A selection:

2	2
4	2^2
6	$2 \cdot 3$
12	$2^2 \cdot 3$
24	$2^3 \cdot 3$
36	$2^2 \cdot 3^2$
48	$2^4 \cdot 3$
60	$2^2 \cdot 3 \cdot 5$
120	$2^3 \cdot 3 \cdot 5$
180	$2^2 \cdot 3^2 \cdot 5$
240	$2^4 \cdot 3 \cdot 5$
360	$2^3 \cdot 3^2 \cdot 5$
720	$2^4 \cdot 3^2 \cdot 5$
840	$2^3 \cdot 3 \cdot 5 \cdot 7$
$1,260$	$2^2 \cdot 3^2 \cdot 5 \cdot 7$
$1,680$	$2^4 \cdot 3^1 \cdot 5 \cdot 7$
$2,520$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$
$5,040$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$

Pythagorean Triples

Famous Pythagorean triples:

$$\begin{array}{r} (3,4,5) \quad (5,12,13) \quad (8,15,17) \quad (7,24,25) \\ (20,21,29) \quad (12,35,37) \quad (9,40,41) \\ (28,45,53) \quad (11,60,61) \quad (16,63,65) \\ (33,56,65) \quad (48,55,73) \quad (13,84,85) \\ (36,77,85) \quad (39,80,89) \quad (65,72,97) \end{array}$$

Useful Irrational Numbers

$$
\begin{aligned}
\pi &\approx 3.14159\ldots \\
\varphi &\approx 1.61803\ldots \\
\sqrt{2} &\approx 1.41421\ldots \\
\sqrt{3} &\approx 1.73205\ldots \\
\sqrt{5} &\approx 2.23607\ldots \\
\sqrt{6} &\approx 2.44949\ldots \\
\sqrt{7} &\approx 2.64575\ldots \\
\sqrt{8} &\approx 2.82843\ldots \\
e &\approx 2.71828\ldots \\
\gamma &\approx 0.57722\ldots
\end{aligned}
$$