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Problem 1

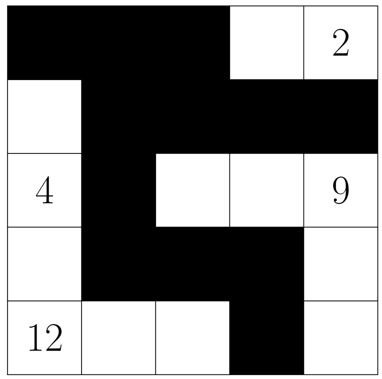
Question

Shade some squares in the grid so that:

1. Squares with numbers are unshaded.
2. Each number is equal to the product of the number of unshaded squares it can "see" in its row and column. (A square can see another square if they're in the same row or column and the sight line between them doesn't have any shaded squares. Each square can see itself.)
3. The shaded squares must make one connected group. Two squares are considered to be connected if they share an edge.

									4
			6			6			
		36	24						
				16					
					24				
						36	18		
			6			12			
36									

The following is an example of a completed puzzle to clarify the rules.

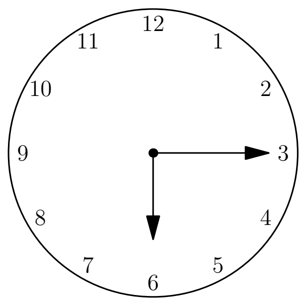


There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Problem 2

Question

Calamitous Clod deceives the math beasts by changing a clock at Beast Academy. First, he removes both the minute and hour hands, then places each of them back in a random position, chosen uniformly along the circle. Professor Grok notices that the clock is not displaying a valid time. That is, the hour and minute hands are pointing in an orientation that a real clock would never display. One such example is the hour hand pointed at 6 and the minute hand pointed at 3.



The math beasts can fix this, though. They can turn both hands by the same number of degrees clockwise. On average, what is the minimal number of degrees they must turn the hands so that they display a valid time?[b].

[b]Assume that after Calamitous Clod replaces the hands, they don't move again until the math beasts adjust their position.

Problem 3

Question

> Let a, b be positive integers such that $a^2 \geq b$. Let
>
> $$x = \sqrt{a + \sqrt{b}} - \sqrt{a - \sqrt{b}}$$
>
> 1. Prove that for all integers $a \geq 2$, there exists a positive integer b such that x is also a positive integer.
>
> 2. Prove that for all sufficiently large a, there are at least two b such that x is a positive integer.

Problem 4

Question

> $ABCD$ is a convex quadrilateral where $\angle A = 45°$ and $\angle C = 135°$. P is a point inside $\triangle ABC$ such that $\angle BAP = \angle CAD$ and $\angle BCP = \angle ACD$. Prove that $\overline{PB} \perp \overline{PD}$ if and only if $\overline{AC} \perp \overline{BD}$.

Problem 5

Question

> Find all ordered triples of non-negative integers (a, b, c) satisfying
>
> $$2^a \cdot 5^b - 3^c = 1.$$

Acknowledgments

Problem 1

I used trial and error to find the solution.

Problem 2

Problem 3

Problem 4

Problem 5

References