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Problem 1

Question

The "Manhattan distance" between two cells is the shortest distance between those cells when traveling up, down, left, or right, as if one were traveling along city blocks rather than as the crow flies.
Place numbers from 1-6 in some cells so the following criteria are satisfied:

1. A cell contains at most one number. Cells can be left empty.
2. For each cell containing a number N in the grid, exactly two other cells containing N are at a Manhattan distance of N.
3. For each cell containing a number N in the grid, no other cells containing N are at a Manhattan distance less than N.

		4					
	5	2				3	
3			1				
				1			4
	2				6	5	
					4		

There is a unique solution, but you do not need to prove that your answer is the only one possible.

Solution:

	3	4	6	3	5	4	
	5	2		2		3	
3	2		1	1	2		
6		3	1	1	3	2	4
	2	4	2		6	5	
		5		2	4	2	

Problem 1: Discussion

The solution can be found by trial and error. Writing a good algorithm for solving the problem is challenging. Ideas that I found interesting and motivated me to write code to investigate the problem:

- the effect of shuffling the numbers in the starting grid.
- the conditions under which the solution is unique.
- find situations where the completed grid has no holes.
- change the condition "for each cell containing a number N in the grid, exactly two other cells containing N are at a Manhattan distance of N" to "at most two other cells" or to "at least two other cells" or to some other quantity.
- measure the level of difficulty of a starting grid.

I coded a basic algorithm and developed a few tools to assist in solving. A naive algorithm is:

1. For each number N in the grid, identify all the cells that are at a distance N. These cells are potential "connections".
2. Among these cells, if there are fewer than 2 with the number N, then create a list of the empty cells: the "candidates".

3. Iterate through the lits of candidates: insert the value if the insertion is consistent with the rules. Keep a record of the sequence of insertions.

4. At the end of the iteration, check if the condition "for each cell containing a number N in the grid, exactly two other cells containing N are at a Manhattan distance of N" is satisfied. If the condition is not satisfied, go back to the start and attempt another sequence of insertions.

This algorithm is very expensive. A single iteration yields the outcome shown in Figure 1. To get the solution requires multiple iterations.

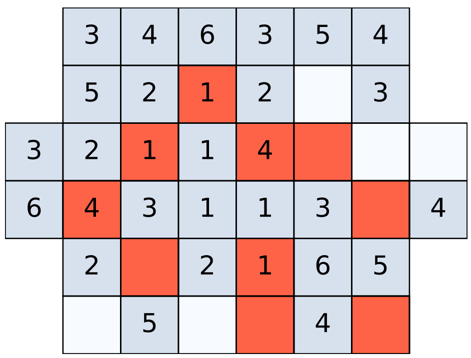


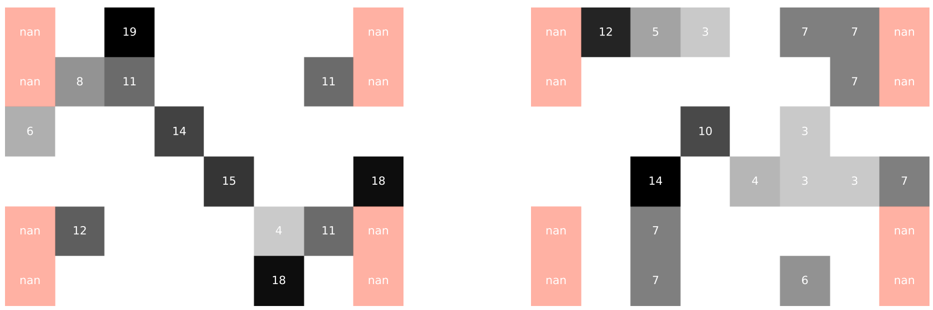
Figure 1: A single iteration of the naive algorithm yields this outcome. Subsequent iterations have to backtrack to the initial state. Incorrect insertions and omissions are highlighted in color.

I coded the following improvements:

- Before inserting a value in one of the candidate cells, order the candidates by the number of distinct values it could be filled with ("potential insertions"). To reduce the risk of a future clash, start inserting into candidate cells that have the smallest number of potential insertions (least attractive "targets").

- For each cell in the grid, calculate the number of potential connections to it. Update the "heatmap" before every step in the iteration. Start inserting values into cells that have the smallest "potential degree", as they are least likely to result in an incorrect insertion.

Figure 2 shows the heatmap at the first and last iteration.



(a) Initial grid.　　**(b)** Final grid.

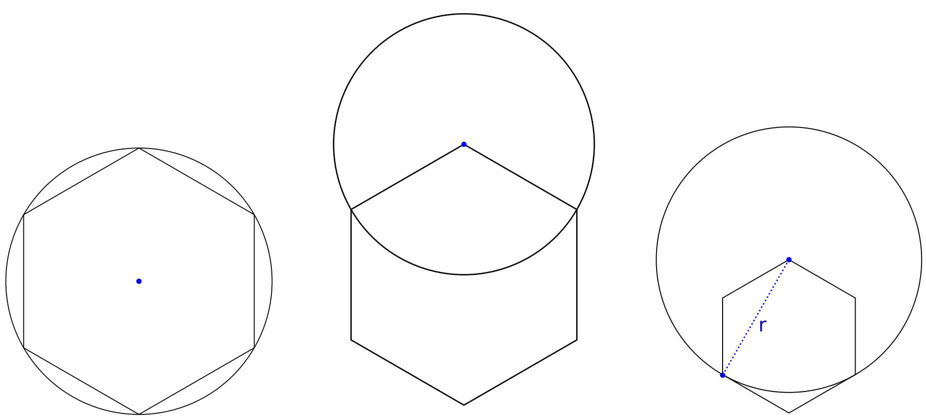
Figure 2: A heatmap of the potential number of connections to each cell in the grid. This information can be used to improve the speed of the algorithm.

Problem 2

Question

> A regular hexagon is placed on top of a unit circle such that one vertex coincides with the center of the circle, exactly two vertices lie on the circumference of the circle, and exactly one vertex lies outside of the circle. Determine the area of the hexagon.

A natural starting point for this problem is to draw a hexagon inscribed in a circle (Figure 3a) and to translate it so that one of its vertices coincides with the center of the circle (Figure 3b). It is then clear that the hexagon must be shrunk to the point where the short-diagonal of the hexagon coincides with the radius r of the circle (Figure 3c).



(a) **(b)** **(c)**

Figure 3: There exist two polygons with one vertex at the center of the circle and two vertices on the circumference of the circle: see panel (b) and (c). Only one of them is such that only one vertex is outside the circle: see panel (c).

In hexagon $ABCDEF$ (Figure 4), consider first the triangle ACE inscribed inside the hexagon. Claim: *Triangle ACE is equilateral.* Proof: Since A is the center of the circle and C and E are on the circumference, $AC = r$ and $AE = r$. By the half-angle formula, the chord length CE is equal to $2r\sin(\theta/2)$, where $\theta = \angle CAE = \pi/3$. Calculating $\sin(\theta/2) = \sin(\pi/6) = 1/2$ and substituting back gives a chord length $CE = 2r\sin(\theta/2) = r$. It follows that all three sides of the triangle have length r (it is actually obvious by considerations of symmetry).

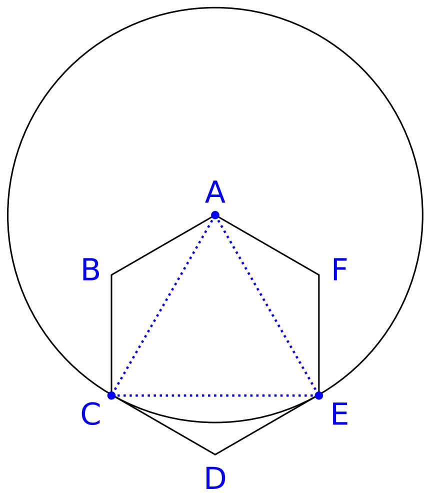


Figure 4: The area of the polygon may be calculated as the sum of areas of the equilateral triangle ACE and of the isosceles triangles ABC, CDE, and EFA.

Since triangle ACE is equilateral, its area is $\frac{\sqrt{3}}{4}r^2$ (the side lengths of the equilateral triangle coincide with the radius of the circle). This well-known formula can be proved by applying the Pythagorean theorem: See Figure 5.

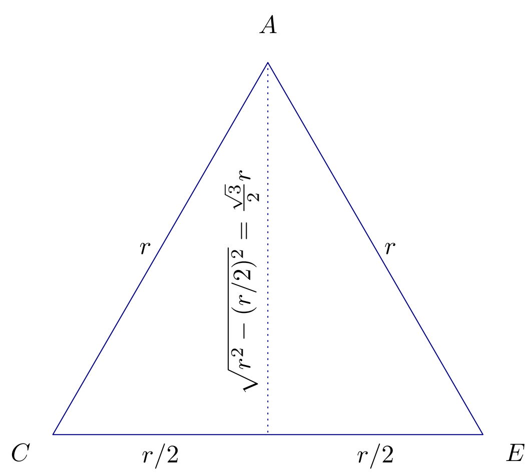


Figure 5: The area of equilateral triangle ACE is $\frac{\sqrt{3}}{4}r^2$.

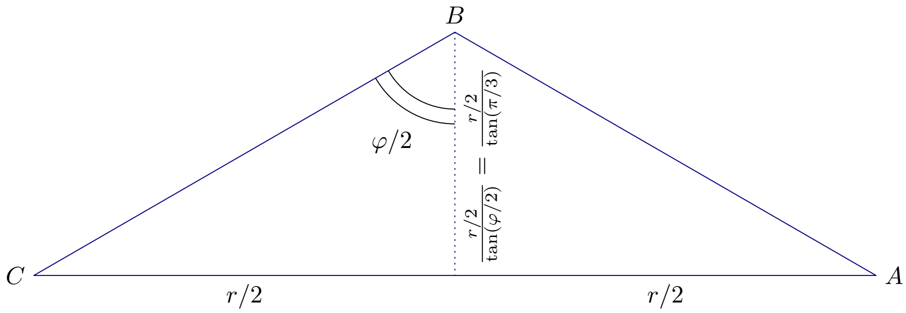


Figure 6: The area of isosceles triangle ABC is $\frac{\sqrt{3}}{12}r^2$.

Consider now the isosceles triangle ABC in hexagon $ABCDEF$ (Figure 4). The base of triangle ABC coincides with the chord $AC = r$. The angle at the vertex is $\varphi = \angle ABC = \frac{2\pi}{3}$. The angle φ follows from the sum-of-angles formula. The sum of the angles of a regular polygon with n sides is $(n-2)\pi$. For a hexagon, $n = 6$, the angle is therefore $\varphi = \frac{(6-2)\pi}{6} = \frac{2\pi}{3}$. The isosceles triangle ABC fits exactly three times into the equilateral triangle ACE. And therefore its area is $\frac{\sqrt{3}}{12}r^2$. As a sanity check, this area may also be calculated from trigonometry:

$$\frac{r^2}{4\tan(\varphi/2)} = \frac{r^2}{4\tan(\pi/3)} = \frac{r^2}{4\sqrt{3}} = \frac{\sqrt{3}}{12}r^2.$$

Adding together the area of triangle ACE and triangles ABC, AEF, and CDE gives the area of hexagon $ABCDEF$:

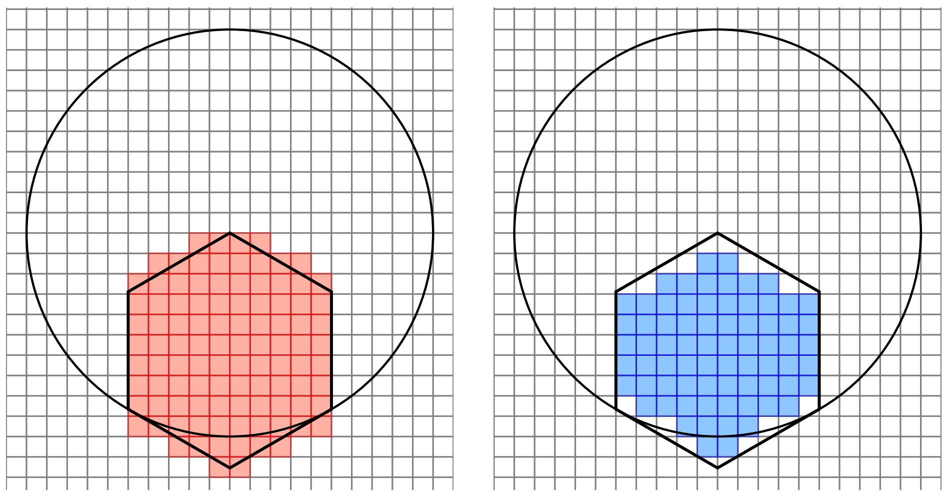
$$\frac{\sqrt{3}}{4}r^2 + 3\ \frac{\sqrt{3}}{12}r^2 = \frac{\sqrt{3}}{2}$$

where we have substituted $r = 1$, the radius of the unit circle.

Conclusion: The area of the hexagon is $\boxed{\frac{\sqrt{3}}{2}}$.

Problem 2: Sanity Check

A quick check is shown in Figure 7.



(a) Outer grid.

(b) Inner grid.

Figure 7: Place a regular grid on top of the figure such that the radius of the circle is divided into 10 squares. The outer grid in Figure 7a covers 100 squares. The inner grid in Figure 7b covers 72 squares. The polygon's area is strictly inside $(0.72, 1)$ and approximately 0.86. This is consistent with $\frac{\sqrt{3}}{2} \approx 0.866$.

Problem 3

Question

> A sequence of integers $x_1, x_2, ..., x_k$ is called *fibtastic* if the difference between any two consecutive elements in the sequence is a Fibonacci number.
> The integers from 1 to 2024 are split into two groups, each written in increasing order. Group A is $a_1, a_2, ..., a_m$ and Group B is $b_1, b_2, ..., b_n$.
> Find the largest integer M such that we can guarantee that we can pick M consecutive elements from either Group A or Group B which form a fibtastic sequence.

The question does not state who splits the integers into two groups: If the split is engineered to maximize M, the answer is $M = 1012$. If the split is engineered to minimize M, the answer is $M = 2$. By another interpretation, the answer is $M = 14$ or $M = 15$, as shown below.

General Considerations

The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively, for $n \geq 1$, by

$$F_0 = 0,$$
$$F_1 = 1,$$
$$F_{n+1} = F_n + F_{n-1}.$$

The first few terms of the Fibonacci sequence are:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597.$$

where $F_{17} = 1597$.

Clearly, the largest possible difference $|a_{k+1} - a_k|$, calculated from consecutive terms in any subsequence $a_1, a_2, \ldots, a_m$, cannot exceed 1597, since $m < 2024$, and likewise for $b_1, b_2, \ldots, b_n$. But $F_{17} = 1597$ is unreachable if the subsequences are split into 1012 elements each.

The question does not explicitly state a constraint on the Fibonacci numbers, but there may be an implicit understanding that they must form an increasing sequence. We consider several interpretations.

Amicable Split

Consider the arithmetic sequence with first term 1 and common difference F_1. Split the integers as

$$\begin{aligned} &\text{Group A}: \quad a_1, a_2, a_3 \dots, a_{1012} &&= 1, 2, 3 \dots, 1012 \\ &\text{Group B}: \quad b_1, b_2, b_3 \dots, b_{1012} &&= 1013, 1014, 1015 \dots, 2024 \end{aligned}$$

where $m = n = 1012$. The difference between any two consecutive elements in the sequence $a_1, a_2, a_3 \dots, a_{1012}$ is the Fibonacci number F_1. Likewise for $b_1, b_2, b_3 \dots, b_{1012}$. Every consecutive element from either Group A or Group B is a fibtastic sequence. The stated conditions are therefore satisfied and the maximum (max-max) is $M = 1012$.

Adversarial Split

If m and n are chosen to minimize M, we must have $m \geq 2$ and $n \geq 2$ to ensure that the difference between any two consecutive numbers is defined. An adversarial split that minimizes M is:

$$\begin{aligned} &\text{Group A}: \quad a_1, a_2 &&= 1, 2 \\ &\text{Group B}: \quad b_1, b_2, b_3 \dots, b_{2022} &&= 3, 4, 5 \dots, 2024 \end{aligned}$$

These sequences are fibtastic, as shown above. The stated conditions are therefore satisfied and the maximum (max-min) is $M = 2$.

Increasing Fibonacci Numbers

Consider the revised definition: A sequence of integers $x_1, x_2, ..., x_k$ is called *increasing fibtastic* if the difference between any two consecutive elements in the sequence is a Fibonacci number ***and if these Fibonacci numbers form an increasing sequence***. Consider both "strictly increasing" and "monotonically increasing" fibtastic sequences.

The best we found is:

- strictly increasing fibtastic sequence: $M = 14$.

- monotonically increasing fibtastic sequence: $M = 15$.

Method of construction:

- Construct a fibtastic sequence of length n starting from 3 (strict) or 2 (not strict), that will be used for $(a)_k$.
- Construct another fibtastic sequence of length n starting after the nth term of the fibtastic sequence created for $(a)_k$.
- For each fibtastic sequence, place all the integers that fall in the "gaps" into the other subsequence.
- Any integer greater or lower than the fibtastic sequences can be placed into any one of the subsequences.

Solution for "strictly increasing" fibtastic sequence:

$M = 14$: The terms in red form a strictly increasing fibtastic sequence. The terms in blue could be moved from subsequence $(b)_k$ to subsequence $(a)_k$ without altering the solution.

$$(a)_k = 3, 4, 6, 9, 14, 22, 35, 56, 90, 145, 234, 378, 611, 988,$$
$$993, 995, 996, \ldots, 1973, 1974, 1975.$$

$$(b)_k = 1, 2, 5, 7, 8, 10, 11, \ldots, 985, 986, 987, 989, 990,$$
$$991, 992, 994, 997, 1002, 1010, 1023, 1044, 1078, 1133, 1222, 1366, 1599, 1976,$$
$$1977, 1978, 1979, 1980, 1981, \ldots, 2022, 2023, 2024.$$

The difference between consecutive terms in the fibtastic sequence is:

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377$$

The difference sequence is the same for $(a)_k$ and $(b)_k$. It is strictly increasing.

Solution for "monotonically increasing" fibtastic sequence:

$M = 15$: The terms in red form a monotonically increasing fibtastic sequence. The terms in blue could be moved from subsequence $(b)_k$ to subsequence $(a)_k$ without altering the solution.

$$(a)_k = 2, 3, 4, 6, 9, 14, 22, 35, 56, 90, 145, 234, 378, 611, 988,$$
$$993, 995, 996, \ldots, 1973, 1974, 1975.$$

$$(b)_k = 1, 5, 7, 8, 10, 11, \ldots, 985, 986, 987, 989, 990,$$
$$990, 991, 992, 994, 997, 1002, 1010, 1023, 1044, 1078, 1133, 1222, 1366, 1599, 1976,$$
$$1977, 1978, 1979, 1980, 1981, \ldots, 2022, 2023, 2024.$$

The difference between consecutive terms in the fibtastic sequence is:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377$$

The difference sequence is the same for $(a)_k$ and $(b)_k$. It is monotonically increasing.

The next Fibonacci number after $F_{14} = 377$ is $F_{15} = 610$. Since $1976+610 > 2024$, it is not possible to insert F_{15}.

Problem 4

Question

> During a lecture, each of 26 mathematicians falls asleep exactly once, and stays asleep for a nonzero amount of time. Each mathematician is awake at the moment the lecture starts, and the moment the lecture finishes. Prove that there are either 6 mathematicians such that no two are asleep at the same time, or 6 mathematicians such that there is some point in time during which all 6 are asleep.

Let A and B denote the following statements:

A There exist a subset of 6 mathematicians such that all 6 were asleep simultaneously at some time.

B There exist a subset of 6 mathematicians such that no two were asleep at the same time.

We prove the statement "A IS TRUE OR B IS TRUE" by contradiction. We suppose "A IS FALSE AND B IS FALSE" and derive a contradiction from the premises.

Proof:

We suppose "A IS FALSE AND B IS FALSE". If "A IS FALSE", then there does not exist a subset of 6 mathematicians such that all 6 were asleep at the same time. At most, there could be 5 mathematicians asleep at the same time. To ensure that requires at least 6 non-overlapping intervals (5 intervals would guarantee 6 clashes, but we can have at most 5). During each one of these 6 non-overlapping intervals, a different mathematician is asleep: in other words, there exist a subset of 6 mathematicians such that no two are asleep at the same time: "B IS TRUE".

"A IS FALSE" implies "B IS TRUE" and therefore contradicts the premise "A IS FALSE AND B IS FALSE". Since statements A and B cannot be both false, one of them must be true. Conclusion: "A OR B IS TRUE", as claimed.

Problem 5

Question

> Let $f(x) = x^2 + bx + 1$ for some real number b. Across all possible values of b, find all possible values for the number of integers x that satisfy $f(f(x) + x) < 0$.
> That is, if there are some values of b that give us 180 integer solutions for x and there are other values of b that give us 314 integer solutions for x (and these are the only possibilities), the answer would be $\boxed{180,\ 314}$.

Let $g(x) = f(f(x) + x)$. Substituting $f(x) = x^2 + bx + 1$ into $f(f(x) + x)$ yields a polynomial of degree 4. Except for special values of b, this polynomial has two turning points and could satisfy $g(x) < 0$ on two disconnected intervals. Figure 8 shows the graph of $g(x)$ for $b = 5$. We analyze $f(x)$ and used its special properties to analyze the nested expression $g(x)$.

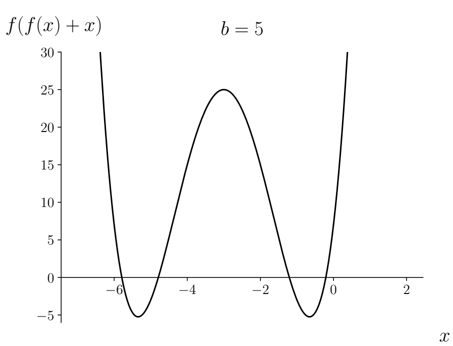


Figure 8: $g(x) = f(f(x) + x)$.

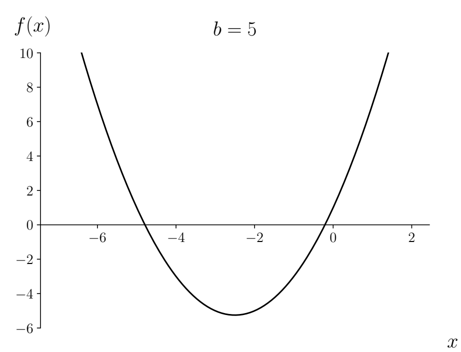


Figure 9: $f(x) = x^2 + bx + 1$.

The graph of $f(x)$ is a U-shaped parabola whose vertex is located at point $(0, -b)$. Figure 9 shows the graph of $f(x)$ for $b = 5$. The discriminant associated with the equation $f(x) = 0$ is:

$$\delta = (b^2 - 4) = (b - 2)(b + 2).$$

It follows that:

$$
\begin{aligned}
b < 2 &\implies \delta < 0 \implies f(x) > 0 \quad \forall x \in \mathbb{R} \\
b = 2 &\implies \delta = 0 \implies f(x) = x^2 + 2x + 1 = (x+1)^2 \geq 0 \quad \forall x \in \mathbb{R} \\
b > 2 &\implies \delta > 0 \implies f(x) > 0 \quad \forall x \in (-\infty, \tfrac{-b-\sqrt{\delta}}{2}) \cup (\tfrac{-b+\sqrt{\delta}}{2}, +\infty) \\
& \implies f(x) = 0 \quad \text{for } x = -1 \\
& \implies f(x) < 0 \quad \forall x \in (\tfrac{-b-\sqrt{\delta}}{2}, \tfrac{-b+\sqrt{\delta}}{2})
\end{aligned}
$$

If $\delta > 0$, f and g can be factorized as:

$$f(x) = \left(x + \tfrac{b+\sqrt{\delta}}{2}\right)\left(x + \tfrac{b-\sqrt{\delta}}{2}\right)$$

$$
\begin{aligned}
g(x) &= f(f(x) + x)) \\
&= \left(f(x) + x + \tfrac{b+\sqrt{\delta}}{2}\right)\left(f(x) + x + \tfrac{b-\sqrt{\delta}}{2}\right) \\
&= \left(x^2 + (b+1)x + \tfrac{(b+2)+\sqrt{\delta}}{2}\right)\left(x^2 + (b+1)x + \tfrac{(b+2)-\sqrt{\delta}}{2}\right) \\
&= \ell(x)\ h(x), \quad \text{where}
\end{aligned}
$$

$$
\begin{aligned}
\ell(x) &= x^2 + (b+1)x + \tfrac{(b+2)+\sqrt{\delta}}{2} \\
h(x) &= x^2 + (b+1)x + \tfrac{(b+2)-\sqrt{\delta}}{2}
\end{aligned}
$$

The sign of $g(x)$ depends on the sign of the product of the two quadratic factors: $g(x) < 0$ if and only if $\ell(x)h(x) < 0$.

We first analyze $h(x)$. The discriminant associated with $h(x)$ is:

$$\alpha = (b+1)^2 - 4\ \tfrac{(b+2)-\sqrt{\delta}}{2} = b^2 - 3 + 2\sqrt{\delta}$$

where $b > 2$ implies $\alpha > 0$. The quadratic $h(x)$ may be factorized as follows:

$$h(x) = \left(x - \tfrac{-(b+1)-\sqrt{b^2-3+2\sqrt{\delta}}}{2}\right)\left(x - \tfrac{-(b+1)+\sqrt{b^2-3+2\sqrt{\delta}}}{2}\right)$$

The nested square-roots can be simplified (proof at the end of this section):

$$\sqrt{b^2 - 3 + 2\sqrt{(b-2)(b+2)}} = 1 + \sqrt{(b-2)(b+2)}$$

Going back to the factorization of $h(x)$, we have

$$
\begin{aligned}
h(x) &= \left(x - \tfrac{-(b+1)-(1+\sqrt{(b-2)(b+2)}\,)}{2}\right)\left(x - \tfrac{-(b+1)+(1+\sqrt{(b-2)(b+2)}\,)}{2}\right) \\
&= \left(x - \tfrac{-(b+2)-\sqrt{(b-2)(b+2)}\,)}{2}\right)\left(x - \tfrac{-b+\sqrt{(b-2)(b+2)}\,)}{2}\right)
\end{aligned}
$$

Both roots of $h(x) = 0$ are negative with $-(b+2) - \sqrt{(b-2)(b+2)}\,) < -b + \sqrt{(b-2)(b+2)}\,)$ for all $b > 2$. The problem reduces to counting the number of integer values $n \in \mathbb{N}$ such that:

$$n \in \left(\frac{-(b+2)-\sqrt{(b-2)(b+2)}\,)}{2}, \ \frac{-b+\sqrt{(b-2)(b+2)}\,)}{2} \right)$$

for $b > 2$. This is an interval of width

$$\frac{-b+\sqrt{(b-2)(b+2)}\,)}{2} - \frac{-(b+2)-\sqrt{(b-2)(b+2)}\,)}{2} = 1.$$

This interval contains typically exactly 1 integer value such that $g(x) < 0$. However, since the bounds of the interval are excluded, whenever $f(f(x) + x) = 0$ for some integer value of x, the interval contains exactly 0 integers.

The analysis of $\ell(x)$ is similar. The problem reduces to counting the number of integer values $n \in \mathbb{N}$ such that:

$$n \in \left(\frac{-b-\sqrt{(b-2)(b+2)}\,)}{2}, \ \frac{-(b+2)+\sqrt{(b-2)(b+2)}\,)}{2} \right)$$

for $b > 2$. This interval also has width of exactly 1.

Putting it together, the generic situation is 0 solutions or 2 solutions, with 1 solution occurring in cases where exactly one of $\ell(x)$ or $h(x)$ is zero at an integer value of x.

Conclusion: The solutions are $\boxed{0,1,2}$.

Proof: Simplifying the Nested Square-Roots

Suppose that there exist $x > 0$ and $y > 0$ such that:

$$\sqrt{b^2 - 3 + 2\sqrt{(b-2)(b+2)}} = \sqrt{x} + \sqrt{y}$$

Square both sides of the equality:

$$b^2 - 3 + 2\sqrt{(b-2)(b+2)} = x + y + 2\sqrt{xy}$$

Split the equality and equate each part:

$$\begin{aligned} x + y &= b^2 - 3 \\ xy &= (b-2)(b+2) \end{aligned}$$

Reduce the system to a single quadratic in x:

$$x^2 - (b^2 - 3)x + (b-2)(b+2) = 0$$

The discriminant of this quadratic is $(b^2-3)^2 - 4(b-2)(b+2) = (b^2-5)^2$.
The roots are 1 and $(b-2)(b+2)$. Substituting back for y gives

$$\sqrt{b^2 - 3 + 2\sqrt{(b-2)(b+2)}} = 1 + \sqrt{(b-2)(b+2)}$$