2021 Fall AMC 12B Problems/Problem 7

The following problem is from both the 2021 Fall AMC 10B #12 and 2021 Fall AMC 12B #7, so both problems redirect to this page.

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Problem

Which of the following conditions is sufficient to guarantee that integers x, y, and z satisfy the equation

$$x(x-y)+y(y-z)+z(z-x) = 1?$$

$\textbf{(A)}\ x>y$ and $y=z$

$\textbf{(B)}\ x=y-1$ and $y=z-1$

$\textbf{(C)}\ x=z+1$ and $y=x+1$

$\textbf{(D)}\ x=z$ and $y-1=x$

$\textbf{(E)}\ x+y+z=1$

Solution 1

Plugging in every choice, we see that choice $\textbf{(D)}$ works.

We have $y=x+1, z=x$, so

$x(x-y)+y(y-z)+z(z-x) = x(x-(x+1))+(x+1)((x+1)-x)+x(x-x) = x(-1)+(x+1)(1) = 1.$

Our answer is $\textbf{(D)}$.

~kingofpineapplz

Solution 2 (Bash)

Just plug in all these options one by one, and one sees that all but D fails to satisfy the equation.

For D, substitute $z=x$ and $y=x+1$:

$$LHS = x(x-(x+1))+(x+1)(x+1-x)+x(x-x) = (-x)+(x+1) = 1 = RHS$$

Hence the answer is $\boxed{\textbf{(D)}}$.

~Wilhelm Z

Solution 3 (Strategy)

Looking at the answer choices and the question, the simplest ones to plug in would be equalities because it would make one term of the equation become zero. We see that answer choices A and D have the simplest equalities in them. However, A has an inequality too, so it would be simpler to plug in D which has another equality. We see that $x = z$ and $y - 1 = x$ means the equation becomes $x(x - (x + 1)) + (x + 1)(x + 1 - x) = 1 \implies -x + x + 1 = 1 \implies 1 = 1,$ which is always true, so the answer is $\boxed{D}$

~KingRavi

Solution 4 (Completing the Square)

It is obvious x, y, and z are symmetrical. We are going to solve the problem by Completing the Square.

$$x^2 + y^2 + z^2 - xy - yz - zx = 1$$

$$2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx = 2$$

$$(x - y)^2 + (y - z)^2 + (z - x)^2 = 2$$

Because x, y, z are integers, $(x - y)^2$, $(y - z)^2$, and $(z - x)^2$ can only equal $0, 1, 1$. So one variable must equal another, and 2 variables are 1 larger than the other.

~isabelchen (https://artofproblemsolving.com/wiki/index.php/User:Isabelchen)

Video Solution by Interstigation

https://www.youtube.com/watch?v=lJ-RHZXPV_E

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