Russian School of Math Test

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Abstract

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

Pell's Equation

All the integer solutions (x, y) of the Pell's equation $x^2 - 2y^2 = 1$
are given by $(x_0, y_0) = (\pm 1, 0)$, $(x_1, y_1) = (\pm 3, \pm 2)$, and

$$x_n + \sqrt{2} y_n = \pm(3 + 2\sqrt{2})^n, \quad n \in \mathbb{Z}^+$$

or

$$x_n = \pm \frac{(3 + 2\sqrt{2})^n + (3 - 2\sqrt{2})^n}{2}$$
$$y_n = \pm \frac{(3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n}{2\sqrt{2}}$$

Example: $(x_2, y_2) = (\pm 17, \pm 12)$, etc.
$n \in \mathbb{Z}^+$, $(x_0, y_0) = (\pm 1, 0)$.
In general, for some $D \in \mathbb{R}$, $x^2 - Dy^2 = 1$,

$$x_n = \pm \frac{(x_1 + y_1\sqrt{D})^n + (x_1 - y_1\sqrt{D})^n}{2}$$
$$y_n = \pm \frac{(x_1 + y_1\sqrt{D})^n - (x_1 - y_1\sqrt{D})^n}{2\sqrt{D}}$$

These solutions hold for $x^2 - Dy^2 = -1$, except that n can take on only odd values, i.e.

$$x_n = \pm \frac{(x_1 + y_1\sqrt{D})^{2n-1} + (x_1 - y_1\sqrt{D})^{2n-1}}{2}$$
$$y_n = \pm \frac{(x_1 + y_1\sqrt{D})^{2n-1} - (x_1 - y_1\sqrt{D})^{2n-1}}{2\sqrt{D}}$$

Using the relevant recurrence relations, we have:
The solutions of $a_n = Aa_{n-1} + Ba_{n-2}$ are given by $a_n = C\lambda_1^n + D\lambda_2^n$ if $\lambda_1 \neq \lambda_2$, where C, D are constants created by a_0, a_1, and λ_1, λ_2 are the solutions of $\lambda^2 - A\lambda - B = 0$ (the characteristic polynomial), and $a_n = C\lambda^n + Dn\lambda^n$ if $\lambda_1 = \lambda_2 = \lambda$.
In this case, we want $\lambda_1 = 3 + 2\sqrt{2}$, $\lambda_2 = 3 - 2\sqrt{2}$, C_1, D_1 created by $x_0 = 1$, $x_1 = 3$, C_2, D_2 created by $y_0 = 0$, $y_1 = 2$.
Apply Vieta's formulas:
$\lambda_1 + \lambda_2 = 6 = A$, $\lambda_1\lambda_2 = 1 = -B$.
The characteristic polynomial is $\lambda^2 - 6\lambda + 1 = 0$.
The recurrence relations are $x_n = 6x_{n-1} - x_{n-2}$, $y_n = 6y_{n-1} - y_{n-2}$ with $x_0 = 1$, $x_1 = 3$, $y_0 = 0$, $y_1 = 2$.

Euler's Totient Function

Euler's totient function $\varphi(n)$ counts the positive integers up to a given integer n that are relatively prime to n. In other words, it is the number of integers k in the range $1 \leq k \leq n$ for which the greatest common divisor $\gcd(n, k)$ is equal to 1. If p is prime, $\varphi(p) = p - 1$, because $1, 2, \ldots, p - 1$ are all relatively prime to p. Euler's totient function is a multiplicative function, meaning that if two numbers m and n are relatively prime, then $\varphi(mn) = \varphi(m)\varphi(n)$. Another useful result is: If p is prime, then $\varphi(p^a) = p^a - p^{a-1}$ for any $a > 0$.
Examples:

$$\varphi(200) = \varphi(25)\varphi(8) = \varphi(5^2)\varphi(2^3) = (5^2 - 5^1) \times (2^3 - 2^2) = 20 \times 4 = 80$$
$$\varphi(2^3 3^4 7^2) = \varphi(2^3)\varphi(3^4)\varphi(7^2) = (2^3 - 2^2) \times (3^4 - 3^3) \times (7^2 - 7^1) = 4 \times 54 \times 42 = 9072$$

The first values of Euler's totient function are:
1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, 6, 8, 8, 16, 6, 18, 8, 12, 10, 22, 8, 20, 12, 18, 12, 28, 8, 30, 16, 20, 16, 24, 12, 36, 18, 24, 16, 40, 12, 42, 20, 24, 22, 46, 16, 42, 20, 32, 24, 52, 18, 40, 24, 36, 28, 58, 16, 60, 30, 36, 32, 48, 20, 66, 32, 44
$\varphi(n)$ does not attain all (even) positive integer values. The first values of even numbers which are not attained are:
14, 26, 34, 38, 50, 62, 68, 74, 76, 86, 90, 94, 98, 114, 118, 122, 124, 134, 142,
Odd values > 1 are never attained, because $\phi(n)$ has a factor 2 if $n \geq 3$, because ϕ is multiplicative and $\phi(p^n) = p^n - p^{n-1}$ is even for an odd prime p, as well as $\phi(2^n) = 2^n - 2^{n-1}$ for $n \geq 2$.