AMC 10 Problem Series (2804)

Jon Joseph

Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Overview

Week 1 (Jun 4) Class Transcript - Solving Equations and Systems

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jonjoseph 2021-06-04 19:30:58
Nicely done @ chrischor!! Last on our first night.

jonjoseph 2021-06-04 19:31:11
Claps!!

jonjoseph 2021-06-04 19:31:19
Hello and welcome to the AMC 10 Problem Series class!

jonjoseph 2021-06-04 19:31:25
My name is Jon Joseph and I am the instructor for this course.

jonjoseph 2021-06-04 19:31:31
I earned my PhD in solid state physics from the University of Wisconsin, Madison in 1987 and was on the faculty at Madison until 1990. I joined Nicolet Biomedical where, as Vice President of Engineering and Chief Technology Officer, I oversaw the development of many medical devices. I joined a start-up technology company, Cyberkinetics, in 2003 to design and manufacture brain implants. Yes, electrodes that stick in your brain (not your brain) and listen to the neurons talk to each other. I then spent 5 years teaching AP courses in Calculus, Physics and Computer Science. While teaching I also worked to prepare many students for the AMC 8/10/12 and Mathcounts competitions.

jonjoseph 2021-06-04 19:32:02
Before we get started, I'd like to discuss a few details about our clean and shiny classroom!

jonjoseph 2021-06-04 19:32:22
The classroom is now moderated. This means that the messages you type will come to the instructors rather than going directly into the room. We'll choose some of the messages to share with all the students.

jonjoseph 2021-06-04 19:32:41
Types something interesting.

jonjoseph 2021-06-04 19:32:44
*** Type**

jonjoseph 2021-06-04 19:33:33
Hah. All very funny.

jonjoseph 2021-06-04 19:33:50
Once class begins, please feel free to ask questions at any time. Rather than going into the room, your questions will go directly to me or one of the assistants, and we'll help you out!

jonjoseph 2021-06-04 19:34:12
As you might have just experienced if you asked a question, sometimes we'll whisper quick comments to you. If you have a bigger question that requires a back-and-forth conversation, we'll open a private message window with you and chat one-on-one.

jonjoseph 2021-06-04 19:34:25
Most of the time those whispers will come from your class assistants. Let's say hi to them now!

jonjoseph 2021-06-04 19:34:35
Your assistants for this course will be Simon Bohun (simbo) and Kristiyan Vladimirov Vasilev (krismath).

jonjoseph 2021-06-04 19:34:40
simbo: Simon joined Art of Problem Solving in 2018 as an assistant / grader. He is currently an undergraduate at the University of New South Wales in Sydney, where he studies mathematics and aerospace engineering. He found a love for math at an early age, and is particularly interested in abstract algebra. He is currently the secretary for the UNSW rocketry team, where he helps build and launch model rockets around Australia. In his spare time he likes to run, swim and go rock climbing.

simbo 2021-06-04 19:35:10
Hello!

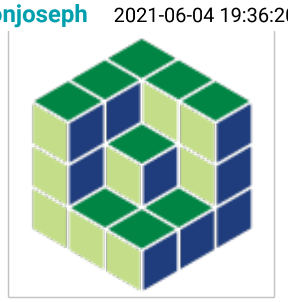
jonjoseph 2021-06-04 19:35:13
krismath: Kristiyan Vasilev is currently a first year Maths student at University of Oxford. He is also a former math competitor, who has participated twice in the IMO and has won a bronze medal and an honourable mention. His favourite field of mathematics is number theory, but he used to like olympiad geometry during his high school years.

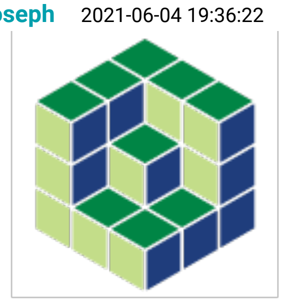
krismath 2021-06-04 19:35:21
Hello!

jonjoseph 2021-06-04 19:35:31
Guess what?

jonjoseph 2021-06-04 19:36:05
AoPS gave us two of their best assistants. We are very lucky. Ask them lots of questions.

jonjoseph 2021-06-04 19:36:18
Sometimes we like to show images in the classroom! Let's test out the software real quick to make sure everyone can see them.

jonjoseph 2021-06-04 19:36:20

cdn.artofproblemsolving.com

jonjoseph 2021-06-04 19:36:22

classroom.artofproblemsolving.com

jonjoseph 2021-06-04 19:36:26
What's the difference between the two images?

jonjoseph 2021-06-04 19:37:02
You got it! They're the same except for the labels. Good eye!

jonjoseph 2021-06-04 19:37:09
Actually, we have one more image for you. If you can see it, then the software is working.

jonjoseph 2021-06-04 19:37:10

jonjoseph 2021-06-04 19:37:12
If you don't see a smiley-face image, just say something and one of us can help you fix the issue.

jonjoseph 2021-06-04 19:37:21
What happens when you double-click on the smiley-face?

jonjoseph 2021-06-04 19:38:14
Yep. This helps especially if you don't want a picture to scroll off the screen.

jonjoseph 2021-06-04 19:38:26
Also, students can use LaTeX in the classroom, just like on the message board. Specifically, place your math LaTeX code inside dollar signs. For example, type:

jonjoseph 2021-06-04 19:38:28
We know that `$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$`.

jonjoseph 2021-06-04 19:38:29
This should give you:

jonjoseph 2021-06-04 19:38:30
We know that $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

jonjoseph 2021-06-04 19:38:39
If you are unfamiliar with LaTeX and would like to learn more about how to use it, we have a LaTeX Guide in the Resources area of our site. The Message Board is also a good place to practice using LaTeX.

jonjoseph 2021-06-04 19:39:00
Yep. It really works.

jonjoseph 2021-06-04 19:39:39
Sure. But if you plan to go on in math a little latex is a good idea.

jonjoseph 2021-06-04 19:39:49
Your AMC 10 Problem Series course Message Board can be found here:

http://www.artofproblemsolving.com/class/2804/forum

There's also a link to the message board on the course homepage.

jonjoseph 2021-06-04 19:39:59
Additionally, we have Office Hours: an AoPS staff member will be on the message board to answer questions in real time every day from 4:00 - 5:30 PM ET (1:00 - 2:30 PM PT) and 7:30-8:30 PM ET (4:30-5:30 PM PT).

jonjoseph 2021-06-04 19:40:07
You can also change the classroom size, font size, colors, and other settings in the Settings Menu at the top-right of your screen. "Flashing notifications" determines if your private message window will blink/flash with a number of unread messages when you receive one, or if it should only display the static number.

jonjoseph 2021-06-04 19:40:22
We will also post "sticky" problems and important comments to the top of the room.

jonjoseph 2021-06-04 19:40:30

You can slide the bar between the "sticky" portion of the room on the top of the classroom and the rest of the classroom to make it larger or smaller. Doing this will help you read the sticky area and the classroom.

jonjoseph 2021-06-04 19:40:40

At the end of class, we'll talk about homework and other class procedures. But for now, let's get going on the math.

jonjoseph 2021-06-04 19:40:56

AMC 10 Problem Series Week 1: Solving Equations and Systems

jonjoseph 2021-06-04 19:40:58

Today, we will be looking at different techniques for solving equations and systems.

jonjoseph 2021-06-04 19:41:01

SOLVING EQUATIONS

jonjoseph 2021-06-04 19:41:06

Many problems in mathematics reduce to solving an equation, so it is important to have a good understanding of how to handle different kinds of equations. We start with a warm-up problem.

jonjoseph 2021-06-04 19:41:12

Some formatting reminders for your responses: If you're using LaTeX, and you have an exponent with multiple characters, put curly brackets around it. So \$5^{-4},\$ not \$5^-4\$: the latter only puts the minus in the exponent.

jonjoseph 2021-06-04 19:41:24

To express it in plain text, use normal parentheses and write 5^(-4). This won't work in LaTeX, though. You'll get $5^(-4)$ if you use parentheses instead of curly brackets.

jonjoseph 2021-06-04 19:41:34

Find the value of x that satisfies the equation

$$25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}}.$$

$\text{(A) } 2 \quad \text{(B) } 3 \quad \text{(C) } 5 \quad \text{(D) } 6 \quad \text{(E) } 9$

jonjoseph 2021-06-04 19:41:43

How can we start?

jonjoseph 2021-06-04 19:42:53

Yes, this equation would be simpler if we wrote everything as a power of 5.

jonjoseph 2021-06-04 19:42:58

Replacing every appearance of 25 with 5^2, we get

$$(5^2)^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot (5^2)^{17/x}}.$$

jonjoseph 2021-06-04 19:43:12

This equation becomes

$$5^{-4} = \frac{5^{48/x}}{5^{26/x} \cdot 5^{34/x}}.$$

What does the right-hand side simplify to?

jonjoseph 2021-06-04 19:43:43

Careful with adding and subtracting exponents.

jonjoseph 2021-06-04 19:45:15

Good. Lots of you. I passed some answer that did not use parentheses - they are needed for the exponent to make sense so be careful.

jonjoseph 2021-06-04 19:45:24

The right-hand side simplifies to

$$\frac{5^{48/x}}{5^{26/x} \cdot 5^{34/x}} = 5^{48/x - 26/x - 34/x} = 5^{-12/x}.$$

Hence, $5^{-4} = 5^{-12/x}$.

jonjoseph 2021-06-04 19:45:29

So what is the value of x?

jonjoseph 2021-06-04 19:46:29

If $5^{-4} = 5^{-12/x}$**, then the exponents on both sides must be equal, so**

$$-4 = -\frac{12}{x}.$$

Then $-4x = -12$**, so** $x = \dfrac{-12}{-4} = 3$.

jonjoseph 2021-06-04 19:46:45

The answer is (B).

jonjoseph 2021-06-04 19:46:47

One of the first steps in solving an equation is often to simplify the equation, if possible. After simplifying the equation, then you can worry about isolating the variable.

jonjoseph 2021-06-04 19:47:02

What is the sum of all the solutions of $x = |2x - |60 - 2x||$?

$(\mathrm{A})\ 32 \quad (\mathrm{B})\ 60 \quad (\mathrm{C})\ 92 \quad (\mathrm{D})\ 120 \quad (\mathrm{E})\ 124$

jonjoseph 2021-06-04 19:47:22

Yech. Multiple absolute values. Ideas?

jonjoseph 2021-06-04 19:48:11

Our right hand side is the expression $|2x - |60 - 2x||$.

jonjoseph 2021-06-04 19:48:16

Since we're seeing absolute values, let's review them for a moment. For a number y**, what are the possible values of** $|y|$?

jonjoseph 2021-06-04 19:49:00

Yes, $|y|$ **is either** y **or** $-y$**, with** $|y| = y$ **whenever** $y \geq 0$**, and** $|y| = -y$ **whenever** $y \leq 0$.

jonjoseph 2021-06-04 19:49:06

Let's think about $|2x - |60 - 2x||$**. Let's first deal with the outer absolute value -- that is, we'll just leave the inner** $|60 - 2x|$ **be for now.**

jonjoseph 2021-06-04 19:49:13

What are the two possible values of $|2x - |60 - 2x||$**?**

jonjoseph 2021-06-04 19:49:29

(Leave the inner absolute value on)

jonjoseph 2021-06-04 19:50:52

This expression is either:

Case 1: $2x - |60 - 2x|$

or

Case 2: $|60 - 2x| - 2x$.

jonjoseph 2021-06-04 19:51:16
Notice in case 2 the minus must distribute to both terms.

jonjoseph 2021-06-04 19:51:25
We haven't yet gotten rid of all the absolute values. Let's start with Case 1**. What are the two possibilities for** $2x - |60 - 2x|$**? Please simplify your answer as much as you can.**

jonjoseph 2021-06-04 19:52:26
Hint: Careful with that minus sign again!

jonjoseph 2021-06-04 19:53:20
The two possibilities for Case 1 are:

Case 1.1: $2x - (60 - 2x) = 4x - 60$

or

Case 1.2: $2x - (2x - 60) = 60$.

jonjoseph 2021-06-04 19:53:26
And what are the two possibilities for Case 2, $|60 - 2x| - 2x$**?**

jonjoseph 2021-06-04 19:54:46
The two possibilities for Case 2 are:

Case 2.1: $(60 - 2x) - 2x = 60 - 4x$

or

Case 2.2: $(2x - 60) - 2x = -60$.

jonjoseph 2021-06-04 19:54:57
Now we set them equal to x **and solve the resulting equations. What value of** x **do we get from Case** 1.1**?**

jonjoseph 2021-06-04 19:55:47
We get $x = 4x - 60$**, and solving, we get** $x = 20$**. Let's check -- does this work?**

jonjoseph 2021-06-04 19:55:52
Here, the right-hand side works out to $|40 - |60 - 40|| = 20$**, so it works! By the way, why are we checking our answers?**

jonjoseph 2021-06-04 19:56:31
Because the right-hand side isn't *always* equal to $4x - 60$**. There are certain assumptions (which we could unpack, but won't at the moment) we have to make for that to be true. It's possible that the** x **we solve for won't actually satisfy those assumptions, and so it may not be a solution to the original equation.**

jonjoseph 2021-06-04 19:57:01
And yes, that would be called an extraneous solution.

jonjoseph 2021-06-04 19:57:22
Also, some of you said to check our work. This is also always a good idea.

jonjoseph 2021-06-04 19:57:25
Continuing, what do we get from Case 1.2**?**

jonjoseph 2021-06-04 19:58:31
We get $x = 60$. Does it work?

jonjoseph 2021-06-04 19:59:26
Here, the right-hand side works out to $|120 - |60 - 120|| = 60$, so this one works, too!

jonjoseph 2021-06-04 19:59:30
What solutions do we get from Cases 2.1 and 2.2?

jonjoseph 2021-06-04 19:59:40
Start with 2.1

jonjoseph 2021-06-04 20:00:17
From Case 2.1, we get $x = 60 - 4x$, which gives

$$5x = 60,$$

or

$$x = 12.$$

From Case 2.2, we get $x = -60$. Do these work?

jonjoseph 2021-06-04 20:01:02
$x = 12$ works, but $x = -60$ doesn't! So we do in fact get an extraneous solution.

jonjoseph 2021-06-04 20:01:20
Hence, we have the solutions $x = 12, 20,$ and 60. But, what is our answer?

jonjoseph 2021-06-04 20:02:18
Their sum is $12 + 20 + 60 = 92$. The answer is (C).

jonjoseph 2021-06-04 20:02:22
If you have an equation that contains absolute value signs, then a good way to deal with them is to consider cases.

jonjoseph 2021-06-04 20:02:27
What is the product of all the roots of the equation

$$\sqrt{5|x| + 8} = \sqrt{x^2 - 16}?$$

$\text{(A)} -64 \quad \text{(B)} -24 \quad \text{(C)} -9 \quad \text{(D)}\ 24 \quad \text{(E)}\ 576$

jonjoseph 2021-06-04 20:02:36
This equation contains both absolute value signs and square roots. Let's see if we can deal with both.

jonjoseph 2021-06-04 20:02:41
How can we deal with the square roots?

jonjoseph 2021-06-04 20:03:14
Conveniently, both sides are the square root of something, so we can square both sides. We get $5|x| + 8 = x^2 - 16$. This rearranges to

$$x^2 - 5|x| - 24 = 0.$$

jonjoseph 2021-06-04 20:03:23
Now how can we deal with the absolute value sign?

jonjoseph 2021-06-04 20:04:08
Yes. Casework will work. Let's do that first but then I'll show a second way.

jonjoseph 2021-06-04 20:04:17
Yes, we know that $|x|$ is either x or $-x$.

jonjoseph 2021-06-04 20:04:23
If $x \geq 0$, then $|x| = x$, and we obtain the equation $x^2 - 5x - 24 = 0$. What are the solutions to this equation?

jonjoseph 2021-06-04 20:05:26
This equation factors as $(x-8)(x+3) = 0$, so $x = 8$ or $x = -3$. Are both of these valid solutions?

jonjoseph 2021-06-04 20:06:28
We are considering the case $x \geq 0$, so only $x = 8$ is a solution. Alternatively, plugging in $x = -3$, we see that the right-hand side $\sqrt{(-3)^2 - 16}$ is not real, while the left hand side, $\sqrt{5 \cdot 3 + 8}$, is real!

jonjoseph 2021-06-04 20:06:44
Now we look at the case $x < 0$, which means that $|x| = -x$.

jonjoseph 2021-06-04 20:06:55
We obtain the equation $x^2 + 5x - 24 = 0$. What are the solutions to this equation?

jonjoseph 2021-06-04 20:07:55
This equation factors as $(x+8)(x-3) = 0$, so $x = -8$ or $x = 3$. Are both of these valid solutions?

jonjoseph 2021-06-04 20:08:52
We are considering the case $x < 0$, so only $x = -8$ is a solution.

jonjoseph 2021-06-04 20:08:55
We can also again spot that the $x = 3$ solution doesn't work by plugging it into the original equation: $\sqrt{3^2 - 16}$ is once more not defined.

jonjoseph 2021-06-04 20:09:01
Hence, the solutions are $x = 8$ and -8. What is the product of the solutions?

jonjoseph 2021-06-04 20:09:57
The product of the solutions is $8 \cdot (-8) = -64$. The answer is (A).

jonjoseph 2021-06-04 20:10:23
Now let's look at interesting way to handle problems like this.

jonjoseph 2021-06-04 20:10:38
Let's come back to the equation $x^2 - 5|x| - 24 = 0$.

jonjoseph 2021-06-04 20:10:49
Yep. Smart. Let's see how this goes.

jonjoseph 2021-06-04 20:10:54
This expression has some plain xs and some $|x|$s.

jonjoseph 2021-06-04 20:11:00
We can't rewrite it all in terms of x alone without cases. Can we rewrite this as a quadratic in $|x|$, though? If yes, tell me how!

jonjoseph 2021-06-04 20:12:13
Note that $x^2 = |x|^2$, since $(-x)^2 = x^2$. Therefore, we can rewrite the equation as $|x|^2 - 5|x| - 24 = 0$.

jonjoseph 2021-06-04 20:12:18
This is a quadratic in $|x|$. How does it factor?

jonjoseph 2021-06-04 20:12:56
Hint: Treat $|x|$ like any other variable at this point.

jonjoseph 2021-06-04 20:13:30
It factors as $(|x| - 8)(|x| + 3) = 0$. What possibilities for $|x|$ does this give?

jonjoseph 2021-06-04 20:14:10

Second hint: Don't drop the absolute value just yet.

jonjoseph 2021-06-04 20:15:13
This gives us $|x| = 8$ or $|x| = -3$.

jonjoseph 2021-06-04 20:15:17
$|x| = -3$ is impossible, because absolute values are never negative. So $|x| = 8$, which gives the two possibilities $x = 8$ and $x = -8$.

jonjoseph 2021-06-04 20:15:26
Plugging those back into the original equation, we can see that both work. The product of the solutions is therefore again $8 \cdot (-8) = -64$, giving the answer of (A).

jonjoseph 2021-06-04 20:15:46
How many of you have seen complex numbers before?

jonjoseph 2021-06-04 20:16:37
So, this same kind of trick works with a complex number z and its conjugate $\bar{z}$.

jonjoseph 2021-06-04 20:16:50
Something to keep in mind for AMC 12!

jonjoseph 2021-06-04 20:17:56
Quickly if we have complex number $z = x + iy$ then $\bar{z} = x - iy$.

jonjoseph 2021-06-04 20:18:20
Don't worry about this for AMC 10. A look a head.

jonjoseph 2021-06-04 20:18:38
Only flip the sign of the imaginary part.

jonjoseph 2021-06-04 20:18:49
By the way, the moral of the last two questions is to be very careful about extraneous solutions!

jonjoseph 2021-06-04 20:19:03
We would have gotten wrong answers for both of them if we didn't check that our answers actually worked.

jonjoseph 2021-06-04 20:19:12
So whenever you perform an operation that can't be reversed, like squaring both sides of an equation, make sure that you check your solutions at the end.

jonjoseph 2021-06-04 20:19:58
There is another point that I want to mention. The previous two equations had absolute value signs and square roots, which complicate the equation.

jonjoseph 2021-06-04 20:20:09
Whenever you see something complicated that makes the equation more difficult to work with, try to deal with the complicated part head-on by getting rid of it. Just as casework is good for dealing with absolute value signs, squaring can be a good way of dealing with square roots.

jonjoseph 2021-06-04 20:20:19
How many positive integers n satisfy the following condition:

$$(130n)^{50} > n^{100} > 2^{200}?$$

$(\mathrm{A})\ 0 \quad (\mathrm{B})\ 7 \quad (\mathrm{C})\ 12 \quad (\mathrm{D})\ 65 \quad (\mathrm{E})\ 125$

jonjoseph 2021-06-04 20:20:25
What makes this inequality difficult to deal with?

jonjoseph 2021-06-04 20:21:15
The large exponents make this inequality difficult to deal with. What could we do with the large exponents?

jonjoseph 2021-06-04 20:21:56

The exponents are 50, 100, and 200. They are all multiples of 50. Thus, we can raise every expression to the power of $\frac{1}{50}$.

jonjoseph 2021-06-04 20:22:01

To make this more precise, we can write the given inequality as

$$(130n)^{50} > (n^2)^{50} > (2^4)^{50}.$$

We can then take every expression to the power of $\frac{1}{50}$ (or equivalently, take the 50^{th} root of every expression).

jonjoseph 2021-06-04 20:22:12

Does raising all three expressions to the $1/50$th power mess up the inequality at all?

jonjoseph 2021-06-04 20:22:58

No. The inequalities $x^{50} > y^{50}$ and $x > y$ are not necessarily equivalent -- for instance, think about $x = -1$ and $y = -2$. However, when x and y are positive, as all three expressions in our problem statement are, the two inequalities are equivalent.

jonjoseph 2021-06-04 20:23:07

So we get

$$130n > n^2 > 16.$$

What does the left inequality $130n > n^2$ say about n?

jonjoseph 2021-06-04 20:24:13

Dividing both sides by n, we get $130 > n$, or $n < 130$.

jonjoseph 2021-06-04 20:24:35

By the way, be careful with this step: we're only allowed to do this because we're given that n is positive! Dividing both sides by a negative number actually flips the inequality sign.

jonjoseph 2021-06-04 20:24:46

What does the right side inequality $n^2 > 16$ say about n?

jonjoseph 2021-06-04 20:25:45

Taking the square root, we get $n > 4$.

jonjoseph 2021-06-04 20:25:47

Wait a minute, don't we also need to take $n < -4$?

jonjoseph 2021-06-04 20:26:22

Ah, right, n is positive. So we indeed only get $n > 4$.

jonjoseph 2021-06-04 20:26:27

Let's count the number of positive integers n which satisfy $4 < n < 130$.

jonjoseph 2021-06-04 20:26:35

So how many integers n are there that satisfy $4 < n < 130$?

jonjoseph 2021-06-04 20:28:07

Yes, there are 125! You can see that because if we subtract 4 from each entry, we get the list $1, 2, \ldots, 125$, which has 125 numbers in it.

jonjoseph 2021-06-04 20:28:30

Of course, in this question, the only answer anywhere near the right one is 125, so the multiple choice helps you out, too! Either way, the answer is (E).

jonjoseph 2021-06-04 20:29:11

There is a little formula for figuring how many consecutive numbers. Its $\text{high} - \text{low} + 1$.

jonjoseph 2021-06-04 20:29:31

So, we have 129 -5 + 1 = 125.

jonjoseph 2021-06-04 20:30:02
SOLVING SYSTEMS

jonjoseph 2021-06-04 20:30:16
In a system of equations, we must simultaneously deal with several variables that are linked by several equations. In these cases, we can typically use the familiar techniques of substitution and elimination.

jonjoseph 2021-06-04 20:30:22
Real numbers a and b satisfy the equations $3^a = 81^{b+2}$ and $125^b = 5^{a-3}$. What is ab?

$$\text{(A)}\ -60 \quad \text{(B)}\ -17 \quad \text{(C)}\ 9 \quad \text{(D)}\ 12 \quad \text{(E)}\ 60$$

jonjoseph 2021-06-04 20:30:31
How can we approach this question?

jonjoseph 2021-06-04 20:31:26
Yeah, let's rewrite both equations so that the bases on both sides match.

jonjoseph 2021-06-04 20:31:32
What equation do we get from $3^a = 81^{b+2}$ after doing so?

jonjoseph 2021-06-04 20:32:52
Rewriting this as $3^a = (3^4)^{b+2}$, we get the equation

$$a = 4(b+2) = 4b + 8.$$

jonjoseph 2021-06-04 20:32:55
And what do we get from $125^b = 5^{a-3}$?

jonjoseph 2021-06-04 20:34:00
Rewriting this as $(5^3)^b = 5^{a-3}$, we get

$$3b = a - 3.$$

jonjoseph 2021-06-04 20:34:02
We now have the following system of equations

$$\begin{cases} a = 4b + 8 \\ 3b = a - 3. \end{cases}$$

How can we solve for a and b?

jonjoseph 2021-06-04 20:34:12
You finish. Then we'll go over it.

jonjoseph 2021-06-04 20:34:35
Hint: Be sure you answer the question.

jonjoseph 2021-06-04 20:35:28
Nice work.

jonjoseph 2021-06-04 20:35:33
There are several approaches. Since the first equation gives us an explicit expression for a, we can easily substitute into the second equation.

jonjoseph 2021-06-04 20:35:42

This gives us $3b = 4b + 8 - 3$.

jonjoseph 2021-06-04 20:35:45
We find $b = -5$. So what is the value of a?

jonjoseph 2021-06-04 20:36:09
We find $a = 4b + 8 = 4 \cdot (-5) + 8 = -12$.

jonjoseph 2021-06-04 20:36:12
We see that $ab = (-5) \cdot (-12) = 60$. The answer is (E).

jonjoseph 2021-06-04 20:36:21
If x, y, and z are positive with $xy = 24$, $xz = 48$, and $yz = 72$, then $x + y + z$ is

$(\text{A})\ 18 \quad (\text{B})\ 19 \quad (\text{C})\ 20 \quad (\text{D})\ 22 \quad (\text{E})\ 24$

jonjoseph 2021-06-04 20:37:16
Hint: The variables are symmetric in the three equations. How can we make use of this fact?

jonjoseph 2021-06-04 20:38:10
Yep. When you have this kind of symmetry there is often this type of shortcut. Let's see what happens.

jonjoseph 2021-06-04 20:38:16
We can multiply all three equations together. The left-hand side becomes $(xy) \cdot (xz) \cdot (yz) = x^2y^2z^2$, so we have

$$x^2y^2z^2 = 24 \cdot 48 \cdot 72.$$

Then what?

jonjoseph 2021-06-04 20:39:43
We can take the square root of both sides of the equation. And be very smart about how you extract perfect squares from the RHS. It will save you unnecessary multiplying

jonjoseph 2021-06-04 20:39:49
On the left, we have xyz, and on the right, we have

$$\sqrt{24 \cdot 48 \cdot 72} = \sqrt{24 \cdot (24 \cdot 2) \cdot 72} = \sqrt{24^2 \cdot 144} = 24 \cdot 12 = 288.$$

So $xyz = 288$.

jonjoseph 2021-06-04 20:40:13
Do you see how we extracted the squares? Kept our numbers small?

jonjoseph 2021-06-04 20:40:33
So, we have the equation $xyz = 288$. How can we solve for each of the variables?

jonjoseph 2021-06-04 20:41:21
We divide this equation by each of our three original equations. For example, dividing $xyz = 288$ by $xy = 24$ gives us

$$\frac{xyz}{xy} = \frac{288}{24},$$

which simplifies to $z = 12$. What do we get for x and y?

jonjoseph 2021-06-04 20:42:09
In a similar manner, we have $x = \dfrac{xyz}{yz} = \dfrac{288}{72} = 4$ and $y = \dfrac{xyz}{xz} = \dfrac{288}{48} = 6$. So what's the answer?

jonjoseph 2021-06-04 20:43:13
Adding them up, the sum is $4 + 6 + 12 = 22$. The answer is (D).

jonjoseph 2021-06-04 20:43:54
You could have ground through each variable separately and it would have worked but watch for problems with a high degree of symmetry.

jonjoseph 2021-06-04 20:44:04
If (x, y) is a solution to the system $xy = 6$ and

$$x^2y + xy^2 + x + y = 63,$$

find $x^2 + y^2$.

$\text{(A) } 13 \quad \text{(B) } \frac{1173}{32} \quad \text{(C) } 55 \quad \text{(D) } 69 \quad \text{(E) } 81$

jonjoseph 2021-06-04 20:44:13
Let's look at the second equation, $x^2y + xy^2 + x + y = 63$. What can we do with the expression on the left?

jonjoseph 2021-06-04 20:45:03
We can factor $x^2y + xy^2$ as $xy(x + y)$, so

$$xy(x + y) + (x + y) = 63.$$

What now?

jonjoseph 2021-06-04 20:46:22
We know that $xy = 6$, so we get $7(x + y) = 63$, which means $x + y = \dfrac{63}{7} = 9$.

jonjoseph 2021-06-04 20:46:35
If we wanted to, we could now use one equation to solve for y in terms of x and substitute into the other equation. We would end up with a quadratic equation which we could then try to solve for x.

jonjoseph 2021-06-04 20:46:44
However, if we try to do this, we end up getting a big mess, because we can't factor the quadratic and instead have to use the quadratic formula.

jonjoseph 2021-06-04 20:46:51
Specifically, we get $x^2 - 9x + 6 = 0$ and

$$x = \frac{9 \pm \sqrt{57}}{2}.$$

jonjoseph 2021-06-04 20:47:12
ick^2

jonjoseph 2021-06-04 20:47:39
Hmmmm... Let's explore.

jonjoseph 2021-06-04 20:47:51
We are looking for $x^2 + y^2$, so we don't necessarily have to find x and y, just the value of this expression.

jonjoseph 2021-06-04 20:48:02
We've already figured out that $x + y = 9$, and we're given that $xy = 6$. How can we get $x^2 + y^2$ from what we know?

jonjoseph 2021-06-04 20:48:57
We use $(x + y)^2$ and subtract away the extra xy's. So what's $x^2 + y^2$?

jonjoseph 2021-06-04 20:49:46
We have that

$$x^2+y^2=(x+y)^2-2xy=81-2xy=81-2\cdot 6=81-12=69.$$

The answer is (D).

jonjoseph 2021-06-04 20:50:16
Suppose that $4^a=5$, $5^b=6$, $6^c=7$, and $7^d=8$. What is $a\cdot b\cdot c\cdot d$?

$(\mathrm{A})\ 1 \quad (\mathrm{B})\ \frac{3}{2} \quad (\mathrm{C})\ 2 \quad (\mathrm{D})\ \frac{5}{2} \quad (\mathrm{E})\ 3$

jonjoseph 2021-06-04 20:50:36
We might think of solving for the variables. But what is the problem?

jonjoseph 2021-06-04 20:51:50
Ugh. You're right. Well, we won't score a perfect score on this test.

jonjoseph 2021-06-04 20:52:24
Maybe...

jonjoseph 2021-06-04 20:52:28
We can use logarithms (if you know them), but since logarithms are not covered on the AMC 10, there must be a solution that does not use logarithms. In any case, simplifying a logarithmic solution into one of the answer choices is a lot of work!

jonjoseph 2021-06-04 20:52:46
Let's take another look at the equations. We know that $4^a=5$ and $5^b=6$.

jonjoseph 2021-06-04 20:52:49
Both equations have the number 5 in them. How can we use that fact to our advantage?

jonjoseph 2021-06-04 20:53:58
Hah! The AMC problem designers are NOT smarter than us!

jonjoseph 2021-06-04 20:53:59
We can substitute 4^a for 5 in the second equation. This gives us

$$(4^a)^b=6.$$

So, $4^{ab}=6$. What can we do with this equation?

jonjoseph 2021-06-04 20:54:46
I sense some confidence now.

jonjoseph 2021-06-04 20:54:48
Once again, we can substitute. Replacing 6 with 4^{ab} in the equation $6^c=7$, we have

$$(4^{ab})^c=7.$$

Quickly simplifying, we have $4^{abc}=7$. What's our last step?

jonjoseph 2021-06-04 20:55:18
Rinse and repeat. Substituting 4^{abc} for 7 in the equation $7^d=8$, we have

$$4^{abcd}=8.$$

jonjoseph 2021-06-04 20:55:24
You finish now.

jonjoseph 2021-06-04 20:56:50
Both 4 and 8 are powers of 2. Replacing 4 with 2^2 and 8 with 2^3, we get

$$(2^2)^{abcd} = 2^3.$$

Then $2^{2abcd} = 2^3$.

jonjoseph 2021-06-04 20:56:58
From the equation $2^{2abcd} = 2^3$, $2abcd = 3$, so $abcd = \frac{3}{2}$.

jonjoseph 2021-06-04 20:57:02
The answer is (B).

jonjoseph 2021-06-04 20:57:18
We good? Did we get a perfect score and on to AIME?

jonjoseph 2021-06-04 20:57:57
$2^{2abcd} = 2^3$.

jonjoseph 2021-06-04 20:58:09
SUMMARY

jonjoseph 2021-06-04 20:58:12
In today's class, we saw how to solve equations and systems. When solving an equation that has something complicated or ugly, like absolute value signs or square roots, try getting rid of it. And whenever you perform an operation that can't be reversed, like squaring both sides of an equation, check your solutions.

jonjoseph 2021-06-04 20:58:26
When dealing with a system of equations, use substitution and elimination, and symmetry! Usually, the trick is to eliminate the variables one by one, in a systematic way, until you have an equation in one variable that you know how to solve.

jonjoseph 2021-06-04 20:58:32
Also, remember to simplify your equations. Since there are so many different kinds of equations, the right method for solving your equations may depend heavily on the nature of the equations you are dealing. And the best way to understand the nature of your equations is to just take a minute and look at them for anything interesting.

jonjoseph 2021-06-04 20:58:34
All right, now I want to take a few more minutes to tell you about the rest of the course procedures.

jonjoseph 2021-06-04 20:58:37
First, if you haven't been to the course homepage yet, check it out right after class. You can get to it by clicking "My Classes" on the top right of any page on the AoPS website when you're logged in.

jonjoseph 2021-06-04 20:58:46
The course homepage has 5 tabs: "Overview", "Homework", "Message Board", "Report," and "Practice AMC." Here's what each one is about:

jonjoseph 2021-06-04 20:58:50
Overview
This tab includes a Course Introduction document, which you should read once, and also the syllabus for the course. Also, within an hour or two after every class, we'll post a transcript of everything that happened here. You can use this to review anything from class that you might not have understood at the time.

jonjoseph 2021-06-04 20:58:58
Homework
This tab will include a few problems to solve between classes each week. The homework problems are multiple choice and you are given one chance to answer each problem (like the AMC). You will be given 6 points for a correct answer, 0 points for a wrong answer, and 1.5 points for giving up without answering.

jonjoseph 2021-06-04 20:59:06
Message Board

This tab connects you to the message board. The message board is where you can chat with the instructor and with other students during the week about anything class or homework related. Additionally, we have Office Hours on the message board: an AoPS staff member will be on the message board to answer questions in real time every day from 4:00 - 5:30 PM ET (1:00 - 2:30 PM PT) and 7:30-8:30 PM ET (4:30-5:30 PM PT).

jonjoseph 2021-06-04 20:59:24
***** USE THE MESSAGE BOARD *****

jonjoseph 2021-06-04 20:59:53
Hmmmm.... You won't and the even if you do the world will hold together.

jonjoseph 2021-06-04 20:59:59
We have integrated the message board into the transcripts and homework to make it easier to discuss the transcript and the homework -- even specific lines in the transcript or problems in the homework. If you see a speech balloon icon, clicking it will show you the threads already discussing that item.

jonjoseph 2021-06-04 21:00:31
Yes. Each class generates a complete transcript - including your answers.

jonjoseph 2021-06-04 21:00:35
Clicking a pencil and paper icon will create a new thread linked to that item. A linked thread will make it easier for everyone to see exactly what you want to discuss.

jonjoseph 2021-06-04 21:00:41
Whenever you create a new thread, it will also e-mail the instructors and assistants to let them know that a question has been asked. Also you have the option to ask questions that are anonymous to the other students (though you needn't ever be embarrassed to ask a question!).

jonjoseph 2021-06-04 21:00:49
You can also get to the course message board by clicking on the Message Board tab of the class homepage.

jonjoseph 2021-06-04 21:00:51
Report
This tab will tell you how you're doing in this course. It has magic bars that track your performance on the weekly Challenge Problems and in class. Green bars mean that you've passed the task and can move on. Blue bars mean you've mastered the task. Red and orange mean "Keep going!"

jonjoseph 2021-06-04 21:00:55
Video
This tab has links to videos to watch each week. The videos are generally pretty short, and go with the material in the class.

jonjoseph 2021-06-04 21:00:59
Practice AMC
Before the 11th class, we will post the practice exam. You can take the exam any time after it is posted, but you should not discuss the exam until the due date -- a couple days after the 12th class -- has passed. You will receive your score as soon as you submit your answers. After the due date, we will post solutions.

jonjoseph 2021-06-04 21:01:12
Throughout the course, you should visit the class homepage at least a couple of times a week, and use the Homework tab to make sure you're keeping up with your work.

jonjoseph 2021-06-04 21:01:35
Well done tonight. Great enthusiasm. Stay well. See you next week.