AMC 10 Problem Series (2804)

Jon Joseph

Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Overview

Week 7 (Jul 16) Class Transcript - Circles

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jonjoseph 2021-07-16 19:30:52
Nice @ EZ588!!! You get the extra points tonight.

jonjoseph 2021-07-16 19:31:35
Funny. Okay.

jonjoseph 2021-07-16 19:31:48
Uhm...

jonjoseph 2021-07-16 19:31:53
AMC 10 Problem Series
Week 7: Circles

jonjoseph 2021-07-16 19:32:08
Last week, we looked at one of the fundamental figures in geometry, namely triangles. This week, we look at the other fundamental figure in geometry: the circle.

jonjoseph 2021-07-16 19:32:16
AREAS

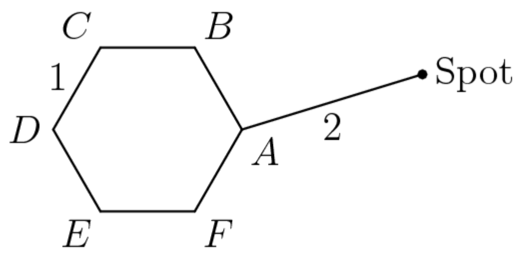
jonjoseph 2021-07-16 19:32:25
We start by looking at problems involving the areas of circles.

jonjoseph 2021-07-16 19:32:28
Spot's doghouse has a regular hexagonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the area, in square yards, of the region outside the doghouse that Spot can reach?

$(\text{A})\ \frac{2}{3}\pi \quad (\text{B})\ 2\pi \quad (\text{C})\ \frac{5}{2}\pi \quad (\text{D})\ \frac{8}{3}\pi \quad (\text{E})\ 3\pi$

jonjoseph 2021-07-16 19:32:44
How about a picture?

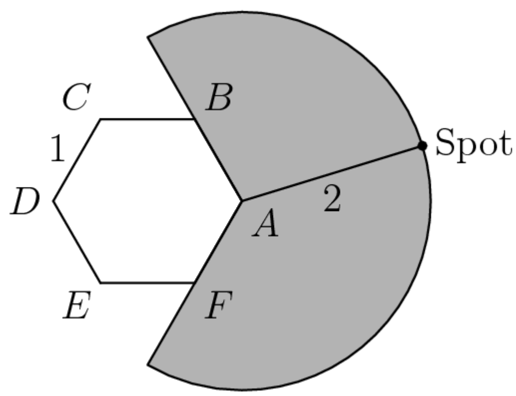
jonjoseph 2021-07-16 19:32:54



jonjoseph 2021-07-16 19:33:06
As a start, let's think about what area Spot can reach if his rope doesn't bend around corners. What does this area look like?

jonjoseph 2021-07-16 19:33:51
If Spot's rope doesn't bend, then the area that Spot can reach is a circular sector, bounded by radii along $\overline{AB}$ and $\overline{AF}$.

jonjoseph 2021-07-16 19:33:57



jonjoseph 2021-07-16 19:34:03
(It's not a whole circle because to get any further around, his rope would have to bend at B or F.)

jonjoseph 2021-07-16 19:34:08
What information do we need to compute the area of this circular sector?

jonjoseph 2021-07-16 19:35:07
We need the length of its radius and also the measure of the angle at its center.

jonjoseph 2021-07-16 19:35:22
What's the angle?

jonjoseph 2021-07-16 19:35:44
Careful.

jonjoseph 2021-07-16 19:36:14
This circular sector has radius 2 and contains an angle of

$$360^\circ - 120^\circ = 240^\circ,$$

so its area is $\dfrac{240}{360} = \dfrac{2}{3}$ the area of a circle with radius 2.

jonjoseph 2021-07-16 19:36:22
What is the area of the sector then?

jonjoseph 2021-07-16 19:37:00
Hence, the area of this circular sector is

$$\frac{2}{3} \cdot \pi \cdot 2^2 = \frac{8}{3}\pi.$$

jonjoseph 2021-07-16 19:37:05
Is this our answer?

jonjoseph 2021-07-16 19:37:28
Correct.

jonjoseph 2021-07-16 19:37:59
Will the answer be bigger or smaller than $8/3\pi$?

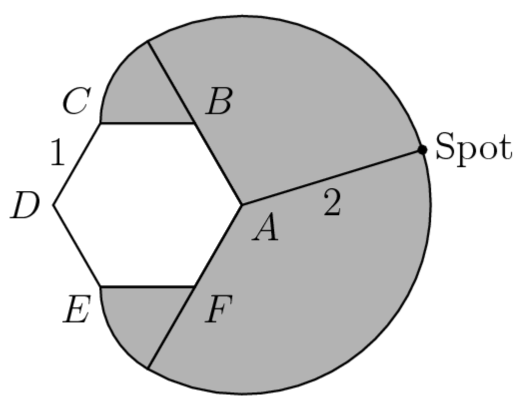
jonjoseph 2021-07-16 19:38:56
Yep. So, since this a contest I would suggest you don't even bother figuring it out.

jonjoseph 2021-07-16 19:39:08
Exactly.

jonjoseph 2021-07-16 19:39:20
But since this is also a math class let's figure it out.

jonjoseph 2021-07-16 19:39:38
We get two additional smaller circular sectors, one bounded by $\overline{AB}$ and $\overline{BC}$, and one bounded by $\overline{AF}$ and $\overline{EF}$. (Spot has just enough rope to reach C and E.)

jonjoseph 2021-07-16 19:39:41



jonjoseph 2021-07-16 19:39:45
What is the area of each of those smaller circular sectors?

jonjoseph 2021-07-16 19:40:38
Each of the smaller circular sectors has radius 1 and contains an angle of 60 degrees, so its area is $\dfrac{60}{360} = \dfrac{1}{6}$ the area of a circle with radius 1.

jonjoseph 2021-07-16 19:40:42
Hence, the area of each circular sector is

$$\frac{1}{6} \cdot \pi \cdot 1^2 = \frac{1}{6}\pi.$$

So what is the total area that Spot can reach?

jonjoseph 2021-07-16 19:41:22
The total area that Spot can reach is

$$\frac{8}{3}\pi + 2 \cdot \frac{1}{6}\pi = 3\pi.$$

The answer is (E).

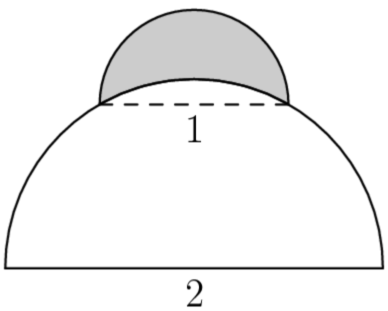
jonjoseph 2021-07-16 19:41:43
How many of you have already taken AMC 10?

jonjoseph 2021-07-16 19:42:15
Those that have know you are always pressed for time. So shortcuts like this can be really helpful.

jonjoseph 2021-07-16 19:42:45
A semicircle of diameter 1 sits at the top of a semicircle of diameter 2, as shown. The shaded area inside the smaller semicircle and outside the larger semicircle is called a *lune*. Determine the area of this lune.

$$\text{(A) } \frac{1}{6}\pi - \frac{\sqrt{3}}{4} \quad \text{(B) } \frac{\sqrt{3}}{4} - \frac{1}{12}\pi \quad \text{(C) } \frac{\sqrt{3}}{4} - \frac{1}{24}\pi \quad \text{(D) } \frac{\sqrt{3}}{4} + \frac{1}{24}\pi \quad \text{(E) } \frac{\sqrt{3}}{4} + \frac{1}{12}\pi$$

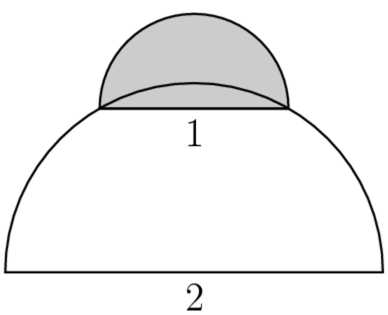
jonjoseph 2021-07-16 19:42:50



jonjoseph 2021-07-16 19:43:08
We want to find the area of the lune. Let's start simple. Is there an easy shape that we can relate the lune to?

jonjoseph 2021-07-16 19:43:55
We can relate the lune to the semicircle with diameter 1.

jonjoseph 2021-07-16 19:44:00



jonjoseph 2021-07-16 19:44:03
What is the area of this semicircle?

jonjoseph 2021-07-16 19:44:07
(Remember that 1 is the diameter, not the radius!)

jonjoseph 2021-07-16 19:44:50
Hint: it's a semicircle - not a whole circle.

jonjoseph 2021-07-16 19:45:31

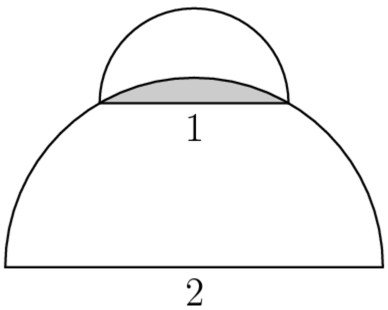
This semicircle is half of a circle with radius $\frac{1}{2}$, so the area of the semicircle is

$$\frac{1}{2} \cdot \pi \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{8}\pi.$$

jonjoseph 2021-07-16 19:45:35

Now, to find the area of the lune, we must subtract the area of the circular segment.

jonjoseph 2021-07-16 19:45:37



jonjoseph 2021-07-16 19:45:43

How can we go about finding the area of this circular segment?

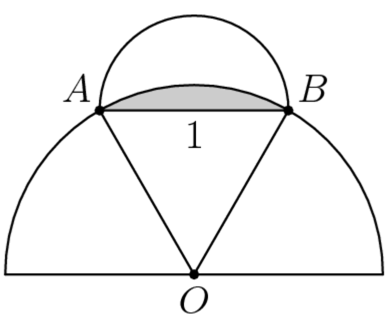
jonjoseph 2021-07-16 19:46:16

Okay. What should we do with it/

jonjoseph 2021-07-16 19:46:57

First, we can draw the lines joining the center of the large semicircle to the endpoints of the circular segment.

jonjoseph 2021-07-16 19:47:03



jonjoseph 2021-07-16 19:47:08

We can then find the area of the circular segment by finding the area of the circular sector OAB, and subtracting the area of triangle OAB.

jonjoseph 2021-07-16 19:47:32

What kind of triangle is this?

jonjoseph 2021-07-16 19:48:37

More than isosceles. It must be equilateral.

jonjoseph 2021-07-16 19:49:07

Who can explain why it is equilateral?

jonjoseph 2021-07-16 19:51:00

Good. The radius of the large circle is 1. The diameter of the small circle is 1. As @ Sipdip says 1=1=1

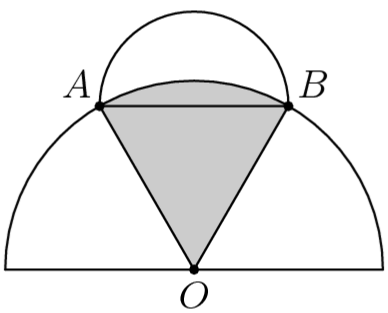
jonjoseph 2021-07-16 19:51:21

The radius of the large semicircle is 1, so $OA = OB = 1$. But $AB = 1$, so triangle OAB is equilateral. In particular, $\angle AOB$ is 60°.4

jonjoseph 2021-07-16 19:51:32

So what is the area of circular sector OAB?

jonjoseph 2021-07-16 19:51:35



jonjoseph 2021-07-16 19:52:27

Since $\angle AOB = 60^\circ$, the area of circular sector OAB is $\frac{60}{360} = \frac{1}{6}$ of the area of a circle with radius 1.

jonjoseph 2021-07-16 19:52:31

Hence, the area of circular sector OAB is $\frac{1}{6}\pi$.

jonjoseph 2021-07-16 19:52:34

What is the area of triangle OAB?

jonjoseph 2021-07-16 19:53:40

Nice

jonjoseph 2021-07-16 19:53:42

Triangle OAB is equilateral with side length 1, so its area is $\frac{\sqrt{3}}{4}$.

jonjoseph 2021-07-16 19:53:45

Then the area of the circular segment is

$$\frac{1}{6}\pi - \frac{\sqrt{3}}{4}.$$

jonjoseph 2021-07-16 19:53:49

We subtract this from the area of the semicircle to find that the area of the lune is

$$\frac{1}{8}\pi - \left(\frac{1}{6}\pi - \frac{\sqrt{3}}{4}\right).$$

What does this simplify to?

jonjoseph 2021-07-16 19:55:27

This simplifies to

$$\frac{\sqrt{3}}{4} - \frac{1}{24}\pi.$$

The answer is (C).

jonjoseph 2021-07-16 19:55:36

This solution strategy is typical for strangely shaped areas. If we have an odd shape defined in terms of arcs and/or lines, then we think to look for a way to relate the shape to areas we already know how to compute.

jonjoseph 2021-07-16 19:55:58

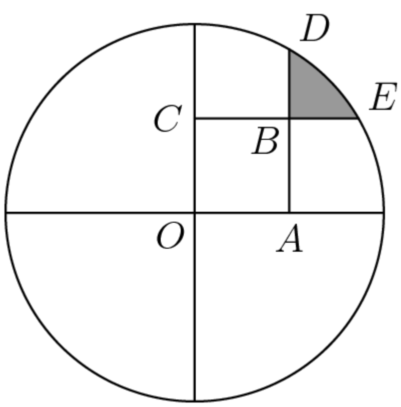
Here's another problem where you can use that strategy.

jonjoseph 2021-07-16 19:56:00

A circle of radius 2 is centered at O. Square $OABC$ has side length 1. Sides $\overline{AB}$ and $\overline{CB}$ are extended past B to meet the circle at D and E, respectively. What is the area of the shaded region in the figure, which is bounded by $\overline{BD}$, $\overline{BE}$, and the minor arc connecting D and E?

$$\text{(A) } \frac{\pi}{3} + 1 - \sqrt{3} \quad \text{(B) } \frac{\pi}{2}(2 - \sqrt{3}) \quad \text{(C) } \pi(2 - \sqrt{3}) \text{(D) } \frac{\pi}{6} + \frac{\sqrt{3}-1}{2} \quad \text{(E) } \frac{\pi}{3} - 1 + \sqrt{3}$$

jonjoseph 2021-07-16 19:56:10



jonjoseph 2021-07-16 19:56:37

I'll give you a moment to read it.

jonjoseph 2021-07-16 19:56:56

Before we get started, let's see if we can make any interesting observations about the diagram. We want to add some shapes whose areas we know. Can we draw any lines to help us with that?

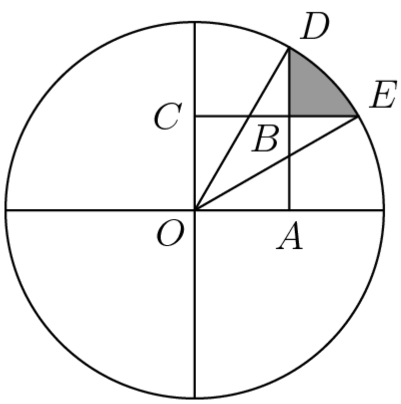
jonjoseph 2021-07-16 19:58:17

Remember, the only shapes whose areas we can handle are circular sectors and figures with straight edges!

jonjoseph 2021-07-16 19:58:22

Let's add OD and OE and see what happens. In general, when you have points on a circle, drawing radii to them is usually a good idea. Here's our diagram:

jonjoseph 2021-07-16 19:58:29



jonjoseph 2021-07-16 19:58:33
Do any of the triangles in this diagram look "nice"?

jonjoseph 2021-07-16 19:58:52
Hint: look for right angles.

jonjoseph 2021-07-16 19:59:26
Hint: What are OD and OA equal to?

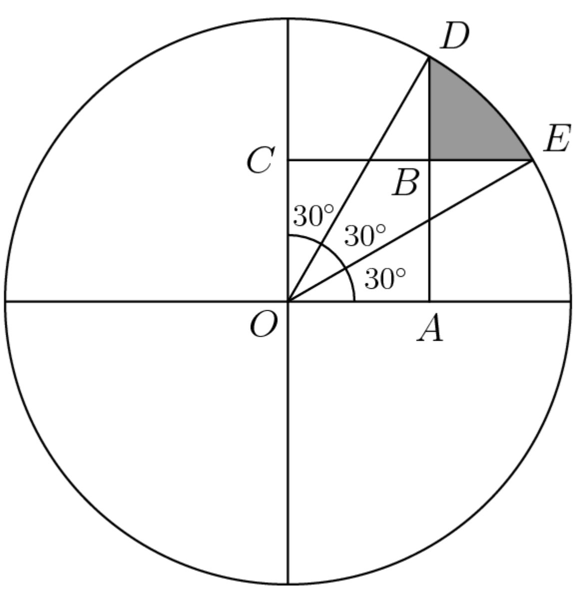
jonjoseph 2021-07-16 20:00:23
We have $OD = 2$ and $OA = 1$. What does that tell us?

jonjoseph 2021-07-16 20:01:07
Since $OABC$ is a square, $\angle OAD = 90^\circ$, so $\triangle DOA$ is a 30-60-90 triangle. That means that $\angle DOA = 60^\circ$, and therefore, $\angle COD = 30^\circ$. What's $\angle DOE$?

jonjoseph 2021-07-16 20:02:17
Using the same reasoning with $\triangle COE$, $\angle COE = 60^\circ$, so $\angle EOA = 30^\circ$. Therefore,

$$\angle DOE = 90^\circ - \angle EOA - \angle COD = 30^\circ.$$

jonjoseph 2021-07-16 20:02:26



jonjoseph 2021-07-16 20:02:39
We good? Questions?

jonjoseph 2021-07-16 20:03:20
Funny. A certain economy to that answer.

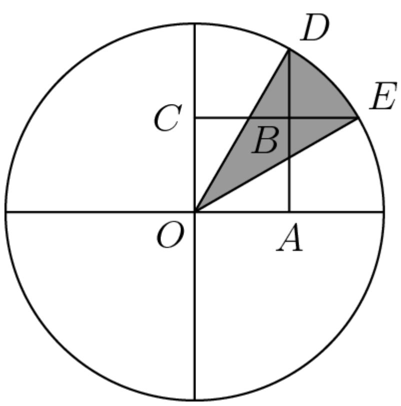
jonjoseph 2021-07-16 20:03:28
All right, now let's see if we can figure out the area of our shaded figure. We want to relate it to areas that we know.

jonjoseph 2021-07-16 20:03:42
We're going to need to subtract. One thing we could do is draw line OB, and subtract $\triangle OBE$ and $\triangle OBD$ from the circular sector ODE. However, there's another way that does not involve drawing new lines!

jonjoseph 2021-07-16 20:03:53
This is a handy trick -- we're going to subtract away more area than we need to, and that add the extra area we subtracted away back on. What's an easy area to start with that contains our shaded figure?

jonjoseph 2021-07-16 20:04:40
We can start with circular sector DOE. Let's keep track -- what's the area of the current shaded figure?

jonjoseph 2021-07-16 20:04:42



jonjoseph 2021-07-16 20:05:40
Circular sector DOE has radius 2 and contains 30 degrees, so its area is $\frac{30}{360} = \frac{1}{12}$ the area of a circle of radius 2.

jonjoseph 2021-07-16 20:05:46
Hence, the area of circular sector DOE is

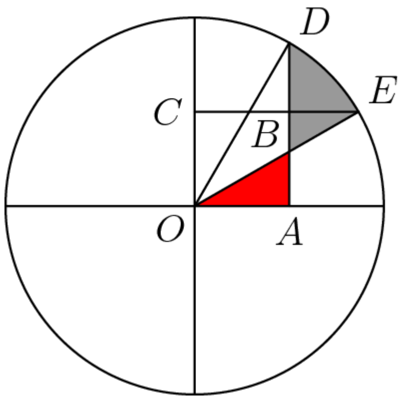
$$\frac{1}{12} \cdot \pi \cdot 2^2 = \frac{\pi}{3}.$$

jonjoseph 2021-07-16 20:05:57
Now we should take away some shapes that border our region. Any ideas?

jonjoseph 2021-07-16 20:07:02
Remember, it's OK to take away too much area, as long as it can be added back on later!

jonjoseph 2021-07-16 20:07:30
We can take away $\triangle OAD$. We'll put the extra area we're taking away in red:

jonjoseph 2021-07-16 20:07:34



jonjoseph 2021-07-16 20:07:53
What's our current area? (We're now keeping track of the area of the gray region minus the red region.)

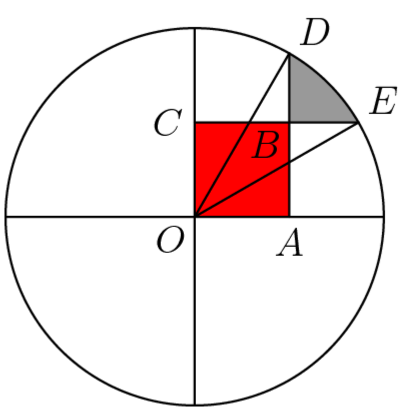
jonjoseph 2021-07-16 20:08:49

We took away a triangle of height $\sqrt{2^2-1^2}=\sqrt{3}$ and base 1, so we subtracted $\frac{\sqrt{3}}{2}$ from our previous area. Therefore, we're currently at

$$\frac{\pi}{3}-\frac{\sqrt{3}}{2}.$$

jonjoseph 2021-07-16 20:09:05
Let's take away $\triangle OCE$. It's looking very close!

jonjoseph 2021-07-16 20:09:07



jonjoseph 2021-07-16 20:09:14
Our current area is $\frac{\pi}{3}-2\frac{\sqrt{3}}{2}=\frac{\pi}{3}-\sqrt{3}.$

jonjoseph 2021-07-16 20:09:16
What do we need to add back on?

jonjoseph 2021-07-16 20:10:13
We need to add the square $OABC$ back on. What's our final answer?

jonjoseph 2021-07-16 20:11:12

Our final answer is

$$\frac{\pi}{3}-\sqrt{3}+1,$$

or (A).

jonjoseph 2021-07-16 20:11:24

Note that it might have been very tempting to calculate BD and then treat the area in question as a quarter circle with radius $\overline{BD}$. But that wouldn't have worked since the region is not a quarter circle! You have be careful with making sure you don't assume your shapes are nicer than they are.

jonjoseph 2021-07-16 20:11:44

If you had made this mistake you would have ended up with answer B. The clever folks at AMC know what traps to set! Any questions before we move on to the next type of circle problem?

jonjoseph 2021-07-16 20:13:12

We could have found the area directly as a second solution. It's more complicated than the method we used.

jonjoseph 2021-07-16 20:13:54

In fact I can see three ways to do this problem (probably more!). You might want to come back and see if you can rework it a different way.

jonjoseph 2021-07-16 20:14:54

I didn't answer your question I know. However, I wanted you to see this method of "over" subtracting. Have you seen PIE in counting problems?

jonjoseph 2021-07-16 20:15:44

Similar to PIE. You take away more than you need and then add back the "overcount".

jonjoseph 2021-07-16 20:16:09

Yes.

jonjoseph 2021-07-16 20:16:22

TANGENT PROBLEMS

jonjoseph 2021-07-16 20:16:31

We now look at problems involving circles tangent to lines or other circles. Whenever you have circles that are tangent to other geometrical objects, it is almost always useful to join the centers of the circles to the points of tangency.

jonjoseph 2021-07-16 20:16:55

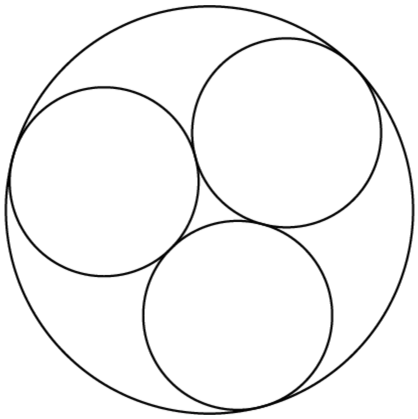
Do you see that? * Join centers!!!*****

jonjoseph 2021-07-16 20:17:06

Three circles of radius 1 are externally tangent to each other and internally tangent to a larger circle. What is the radius of the large circle?

$$\text{(A) } \frac{2+\sqrt{6}}{3} \quad \text{(B) } 2 \quad \text{(C) } \frac{2+3\sqrt{2}}{3} \quad \text{(D) } \frac{3+2\sqrt{3}}{3} \quad \text{(E) } \frac{3+\sqrt{3}}{2}$$

jonjoseph 2021-07-16 20:17:10

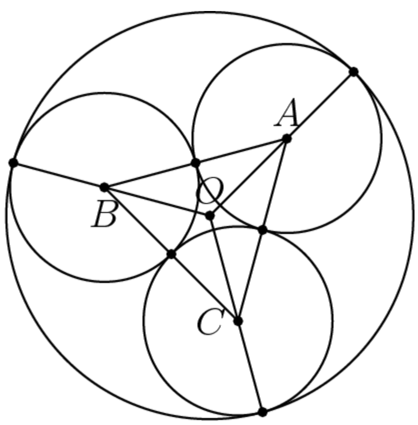


jonjoseph 2021-07-16 20:17:26
What should we do with the diagram first?

jonjoseph 2021-07-16 20:18:09
We'll let O be the center of the big circle and A, B, C the centers of the small circles.

jonjoseph 2021-07-16 20:18:11
We can join the centers of the circles to the points of tangency.

jonjoseph 2021-07-16 20:18:13



jonjoseph 2021-07-16 20:18:28
What kind of triangle is $\triangle ABC$?

jonjoseph 2021-07-16 20:19:18
Each of the small circles has radius 1, so each side of $\triangle ABC$ has length 2. Therefore, $\triangle ABC$ is equilateral.

jonjoseph 2021-07-16 20:19:27
What is the point O with respect to $\triangle ABC$?

jonjoseph 2021-07-16 20:20:07
Hint: What is the proper name for O? The geometric name.

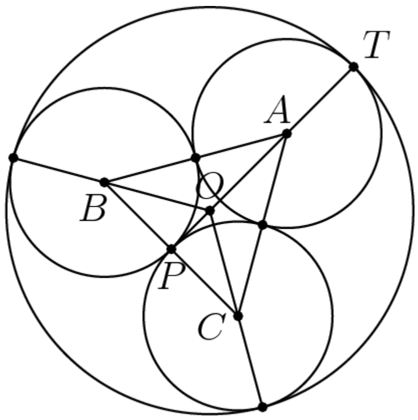
jonjoseph 2021-07-16 20:20:56
By symmetry, we intuitively suspect that the point O is the centroid of $\triangle ABC$. On a contest like the AMC, that's usually enough. We can return to the problem with more rigor if we have time later, but let's proceed for now.

jonjoseph 2021-07-16 20:21:25
I like that one!! Unfortunately that's where an earthquake starts.

jonjoseph 2021-07-16 20:21:42

It might help us to draw in the full height of $\triangle ABC$. Let's update our diagram.

jonjoseph 2021-07-16 20:21:47



jonjoseph 2021-07-16 20:22:23

(And in an equilateral triangle all of our special points come together as a single point.)

jonjoseph 2021-07-16 20:22:36

The radius of the large circle is $OT = OA + AT$. And we already know $AT = 1$. How can we find OA?

jonjoseph 2021-07-16 20:23:47

Hint: The centroid has some special properties. (the way it divides a height.)

jonjoseph 2021-07-16 20:24:30

The centroid O lies $\frac{1}{3}$ of the way between P and A. So $OA = \frac{2}{3}PA$.

jonjoseph 2021-07-16 20:24:33

How can we find PA?

jonjoseph 2021-07-16 20:25:14

Can you find $PA?$

jonjoseph 2021-07-16 20:26:12

Since $\triangle ABC$ is equilateral with side length 2, we have $PA = \sqrt{3}$.

jonjoseph 2021-07-16 20:26:16

Then what is OA?

jonjoseph 2021-07-16 20:27:31

We see that $OA = \frac{2}{3}PA = \frac{2}{3}\sqrt{3} = \frac{2\sqrt{3}}{3}$.

jonjoseph 2021-07-16 20:27:34

Finally, what is the answer?

jonjoseph 2021-07-16 20:28:43

The radius of the large circle is $OT = OA + AT = \frac{2\sqrt{3}}{3} + 1 = \frac{2\sqrt{3}+3}{3}$. The answer is (D).

jonjoseph 2021-07-16 20:29:35

There are other ways to solve this problem. However, I would recommend becoming comfortable with the special points of a triangle. The centroid, the circumcenter, the incenter.

jonjoseph 2021-07-16 20:29:59

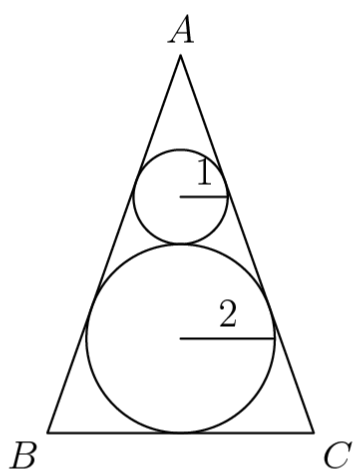
If you want a good reference to these points ask on the message board. I know a few.

jonjoseph 2021-07-16 20:30:24

A circle of radius 1 is tangent to a circle of radius 2. The sides of triangle ABC are tangent to the circles as shown, and the sides AB and AC are congruent. What is the area of triangle ABC?

$$\text{(A) } \frac{35}{2} \quad \text{(B) } 15\sqrt{2} \quad \text{(C) } \frac{64}{3} \quad \text{(D) } 16\sqrt{2} \quad \text{(E) } 24$$

jonjoseph 2021-07-16 20:30:46



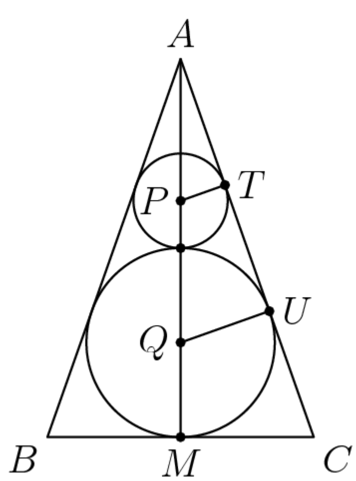
jonjoseph 2021-07-16 20:31:14

What do we always draw when we have circles and tangent points?

jonjoseph 2021-07-16 20:32:06

As in the previous problem, we can join the centers of the circles to the points of tangency. We can also draw an altitude from A to $\overline{BC}$.

jonjoseph 2021-07-16 20:32:08



jonjoseph 2021-07-16 20:32:15

Always!!!!

jonjoseph 2021-07-16 20:32:20

Do we see anything in the diagram that is notable or interesting?

jonjoseph 2021-07-16 20:33:59

We see that both $\overline{PT}$ and $\overline{QU}$ are perpendicular to $\overline{AC}$, so they are parallel.

jonjoseph 2021-07-16 20:34:01

This tells us that triangles APT and AQU are similar. Do we know any side lengths of triangles APT or AQU?

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Hint: We do.

jonjoseph 2021-07-16 20:35:08

We know that $PT = 1$ and $QU = 2$. This gives us the similarity ratio of $2 : 1$ right away, which is nice!

jonjoseph 2021-07-16 20:35:15

We also know that $PQ = 1 + 2 = 3$. Does this give us any other lengths in the diagram?

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We know by our similarity ratio that the sides of $\triangle AQU$ are double the corresponding sides of $\triangle APT$. Hence, P is the midpoint of $\overline{AQ}$.

jonjoseph 2021-07-16 20:37:00

As $PQ = 3$, we have $AP = PQ = 3$. How does this help in finding the area of $\triangle ABC$?

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We can say that the height of $\triangle ABC$ is $AM = AP + PQ + QM = 3 + 3 + 2 = 8$.

jonjoseph 2021-07-16 20:39:14

Some of you said $AM = 9$. I'm guessing you calculated each length as 3. Be careful.

jonjoseph 2021-07-16 20:39:27

To find the area of $\triangle ABC$, we still need the base BC. How can we find BC?

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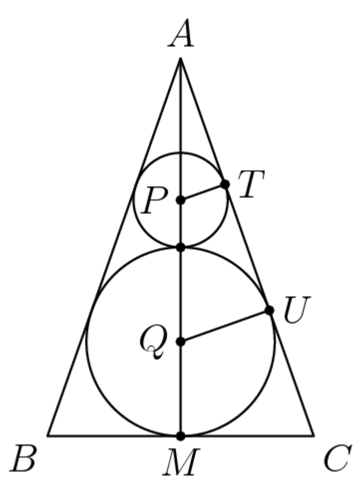

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Note that $\overline{MC}$ is a side of $\triangle AMC$. Is there anything we can compare $\triangle AMC$ to?

jonjoseph 2021-07-16 20:41:18

We see that triangles AMC and ATP are similar: they're right triangles that share $\angle A$. So what useful equation involving MC can we write down?

jonjoseph 2021-07-16 20:41:43

See if you can AT.

jonjoseph 2021-07-16 20:41:48
*** find**

jonjoseph 2021-07-16 20:42:54
We can write $\frac{AM}{MC} = \frac{AT}{TP}$.

jonjoseph 2021-07-16 20:42:57
Filling in what we know, we get $\frac{8}{MC} = AT$. **What is the length** AT?

jonjoseph 2021-07-16 20:44:00
By Pythagoras on right triangle ATP,

$$AT = \sqrt{AP^2 - PT^2} = \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}.$$

So what is the length MC?

jonjoseph 2021-07-16 20:44:15
Is that clear? Lots of wrong answers.

jonjoseph 2021-07-16 20:45:37
From $\frac{8}{MC} = AT$, **we get** $MC = \frac{8}{AT} = \frac{8}{2\sqrt{2}} = 2\sqrt{2}$.

jonjoseph 2021-07-16 20:45:42
So $BC = 2MC = 4\sqrt{2}$. **Then what is the area of** $\triangle ABC$?

jonjoseph 2021-07-16 20:46:51
The area of $\triangle ABC$ **is**

$$\frac{1}{2}AM \cdot BC = AM \cdot MC = 8 \cdot 2\sqrt{2} = 16\sqrt{2}.$$

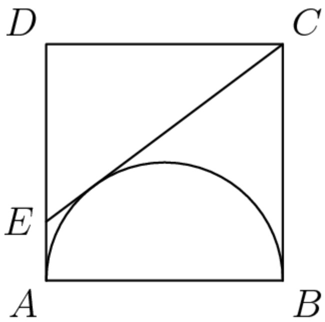
jonjoseph 2021-07-16 20:46:53
The answer is (D).

jonjoseph 2021-07-16 20:47:20
Pythagorus and repeated use of similar triangles.

jonjoseph 2021-07-16 20:47:38
Square $ABCD$ **has side length** 2. **A semicircle with diameter** $\overline{AB}$ **is constructed inside the square, and the tangent to the semicircle from** C **intersects side** $\overline{AD}$ **at** E. **What is the length of** $\overline{CE}$?

$$(\mathrm{A})\ \frac{2+\sqrt{5}}{2} \quad (\mathrm{B})\ \sqrt{5} \quad (\mathrm{C})\ \sqrt{6} \quad (\mathrm{D})\ \frac{5}{2} \quad (\mathrm{E})\ 5-\sqrt{5}$$

jonjoseph 2021-07-16 20:47:42



jonjoseph 2021-07-16 20:48:02
How can we start? How do we always start?

jonjoseph 2021-07-16 20:48:21
Which one?

jonjoseph 2021-07-16 20:49:13
We have a line tangent to a semicircle, so we draw the line joining the center to the point of tangency.

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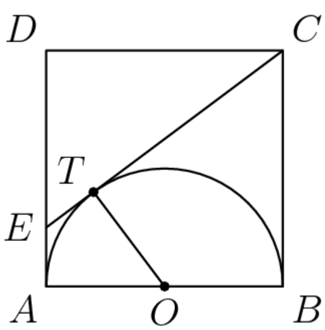

jonjoseph 2021-07-16 20:49:30
Since T lies on $\overline{CE}$, $CE = CT + ET$, so we can try to find CT and ET.

jonjoseph 2021-07-16 20:49:46
There's actually a really quick way to find CT. Anyone see it?

jonjoseph 2021-07-16 20:50:33
HInt: Point B is also a tangent.

jonjoseph 2021-07-16 20:51:22
Since $\overline{CT}$ and $\overline{CB}$ are tangents to the *same* (semi)circle from the *same* point, $CT = CB = 2$.

jonjoseph 2021-07-16 20:51:33
What about ET? Can we say something similar about it?

jonjoseph 2021-07-16 20:52:36
Since $\overline{ET}$ and $\overline{EA}$ are tangents to the same semicircle from the same point, $ET = EA$.

jonjoseph 2021-07-16 20:53:09
This "hidden" fact about tangents from the same point is very helpful. Easy to overlook.

jonjoseph 2021-07-16 20:53:24
We know that $AD = 2$, so if we knew DE, then we could find AE as $AD - DE = 2 - DE$. How might we find information about DE?

jonjoseph 2021-07-16 20:54:21
Good. Let's see how.

jonjoseph 2021-07-16 20:54:25
Let's see what we can learn from using the Pythagorean theorem on right triangle CDE.

jonjoseph 2021-07-16 20:54:28
Let's introduce a variable to make things easier to write down. Let $x = ET = EA$.

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What are the sides of triangle CDE, in terms of x?

jonjoseph 2021-07-16 20:55:47
We see that $DE = AD - AE = 2 - x$.

jonjoseph 2021-07-16 20:55:50
We also see that $CE = CT + ET = 2 + x$.

jonjoseph 2021-07-16 20:55:54
And we know that $CD = 2$. So what equation can we write down?

jonjoseph 2021-07-16 20:56:01
Can you finish?

jonjoseph 2021-07-16 20:56:54
By Pythagoras on right triangle CDE,

$$(2-x)^2 + 2^2 = (2+x)^2.$$

We see that $4 - 4x + x^2 + 4 = 4 + 4x + x^2$. So what is x?

jonjoseph 2021-07-16 20:57:59
This simplifies to $8x = 4$, so $x = \frac{1}{2}$.

jonjoseph 2021-07-16 20:58:01
So what is CE?

jonjoseph 2021-07-16 20:59:06
We get $CE = CT + TE = 2 + \frac{1}{2} = \frac{5}{2}$, so the answer is (D).

jonjoseph 2021-07-16 20:59:11
Note that $\triangle CDE$ has lengths $\frac{3}{2}, \frac{4}{2}, \frac{5}{2}$. It's a $3:4:5$ right triangle! Snazzy. This construction leads to beautiful integer ratios in a way we might not have expected.

jonjoseph 2021-07-16 20:59:41
I'll post one more problem. We won't have time to finish it but it's cool.

jonjoseph 2021-07-16 20:59:51
In the figure below, points B and C lie on line segment AD, and AB, BC, and CD are diameters of circles O, N, and P, respectively. Circles O, N, and P all have radius 15, and the line AG is tangent to circle P at G. If AG intersects circle N at points E and F, then chord EF has length

$\text{(A) } 20 \quad \text{(B) } 15\sqrt{2} \quad \text{(C) } 24 \quad \text{(D) } 25 \quad \text{(E) none of these}$

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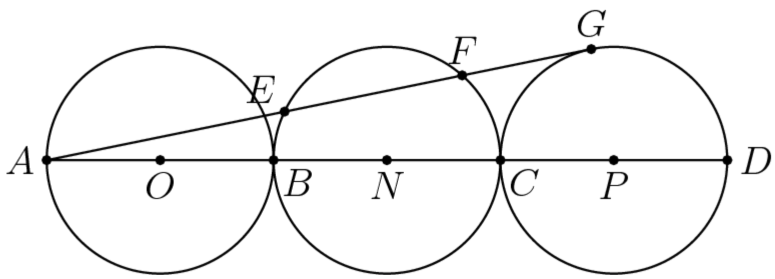

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Since $\overline{AG}$ is tangent to the circle centered at P, we can draw $\overline{PG}$.

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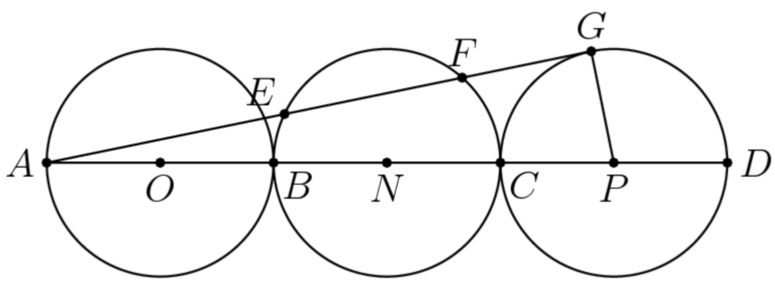

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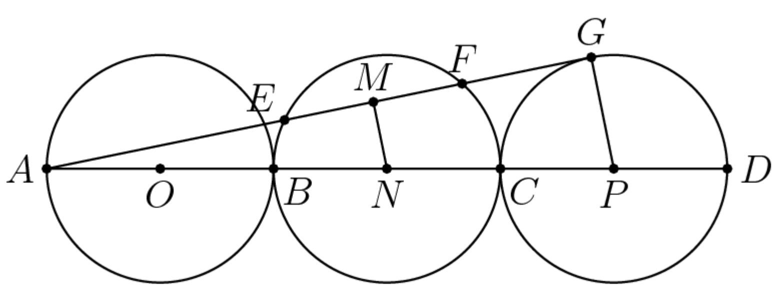
Let's also drop the perpendicular from center N to chord $\overline{EF}$.

jonjoseph 2021-07-16 21:00:49



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Okay. I leave it here. You are looking EF.

jonjoseph 2021-07-16 21:01:12

*** for**

jonjoseph 2021-07-16 21:01:39

I'll put this on the message board and hide the answer if you want to work on it.

jonjoseph 2021-07-16 21:02:03

SUMMARY

jonjoseph 2021-07-16 21:02:06

In today's class we looked at many problems involving the properties of circles, such as the areas of circles. If we have an area involving arcs of circles that we do not know how to compute directly, then the best strategy is often to compare the area to other figures where you know how to compute the area.

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We also saw that if we have circles that are tangents to lines or other circles, then it is often useful to join the centers to the points of tangency. Like joining the centers to midpoints of chords, constructing these additional lines can help you build right triangles, which will in turn help you find lengths.

jonjoseph 2021-07-16 21:02:40
See you next week!! And I forgot about the orthocenter!! Know that one too. Stay safe.