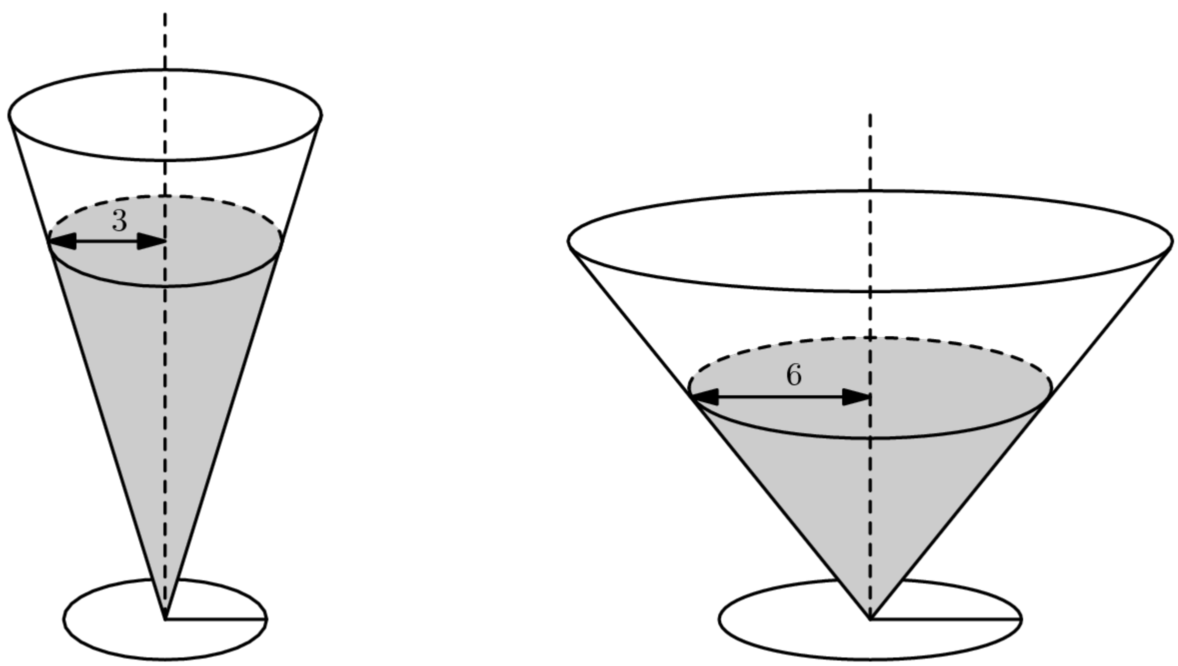
2021 AMC 12A Problems/Problem 10

The following problem is from both the 2021 AMC 10A #12 and 2021 AMC 12A #10, so both problems redirect to this page.

Contents

Problem

Two right circular cones with vertices facing down as shown in the figure below contains the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?



$\textbf{(A) }1:1 \qquad \textbf{(B) }47:43 \qquad \textbf{(C) }2:1 \qquad \textbf{(D) }40:13 \qquad \textbf{(E) }4:1$

Solution 1 (Algebra)

Initial Scenario

Let the heights of the narrow cone and the wide cone be h_1 and h_2, respectively. We have the following table:

	Base Radius	Height	Volume
Narrow Cone	3	h_1	$\frac{1}{3}\pi(3)^2h_1 = 3\pi h_1$
Wide Cone	6	h_2	$\frac{1}{3}\pi(6)^2h_2 = 12\pi h_2$

Equating the volumes gives $3\pi h_1 = 12\pi h_2,$ which simplifies to $\frac{h_1}{h_2} = 4.$

Furthermore, by similar triangles:

- For the narrow cone, the ratio of the base radius to the height is $\frac{3}{h_1},$ which always remains constant.
- For the wide cone, the ratio of the base radius to the height is $\frac{6}{h_2},$ which always remains constant.

Two solutions follow from here:

Solution 1.1 (Properties of Fractions)

Final Scenario

For the narrow cone and the wide cone, let their base radii be $3x$ and $6y$ (for some $x, y > 1$), respectively. By the similar triangles discussed above, their heights must be h_1x and $h_2y,$ respectively. We have the following table:

	Base Radius	Height	Volume
Narrow Cone	$3x$	h_1x	$\frac{1}{3}\pi(3x)^2(h_1x) = 3\pi h_1x^3$
Wide Cone	$6y$	h_2y	$\frac{1}{3}\pi(6y)^2(h_2y) = 12\pi h_2y^3$

Recall that $\frac{h_1}{h_2} = 4.$ Equating the volumes gives $3\pi h_1x^3 = 12\pi h_2y^3,$ which simplifies to $x^3 = y^3,$ or $x = y.$

Finally, the requested ratio is

$$\frac{h_1x - h_1}{h_2y - h_2} = \frac{h_1(x-1)}{h_2(y-1)} = \frac{h_1}{h_2} = \boxed{\textbf{(E) } 4:1}.$$

Remarks

1. This solution uses the following property of fractions:

   For unequal positive numbers a, b, c and $d,$ if $\frac{a}{b} = \frac{c}{d} = k,$ then $\frac{a \pm c}{b \pm d} = \frac{bk \pm dk}{b \pm d} = \frac{(b \pm d)k}{b \pm d} = k.$

2. This solution shows that, regardless of the shape or the volume of the solid dropped into each cone, the requested ratio stays the same as long as the solid sinks to the bottom and is completely submerged without spilling any liquid.

~MRENTHUSIASM

Solution 1.2 (Bash)

Final Scenario

For the narrow cone and the wide cone, let their base radii be r_1 and $r_2,$ respectively; let their rises of the liquid levels be Δh_1 and $\Delta h_2,$ respectively. We have the following table:

Base Radius Height Volume

	Base Radius	Height	Volume
Narrow Cone	r_1	$h_1+\Delta h_1$	$\frac{1}{3}\pi r_1^2(h_1+\Delta h_1)$
Wide Cone	r_2	$h_2+\Delta h_2$	$\frac{1}{3}\pi r_2^2(h_2+\Delta h_2)$

By the similar triangles discussed above, we get

$$\begin{aligned}\frac{3}{h_1} &= \frac{r_1}{h_1+\Delta h_1} &\implies&& r_1 &= \frac{3}{h_1}(h_1+\Delta h_1), &&(1)\\ \frac{6}{h_2} &= \frac{r_2}{h_2+\Delta h_2} &\implies&& r_2 &= \frac{6}{h_2}(h_2+\Delta h_2). &&(2)\end{aligned}$$

The volume of the marble dropped into each cone is $\frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi.$

Now, we set up an equation for the volume of the narrow cone, then express Δh_1 in terms of h_1 :

$$\begin{aligned}\frac{1}{3}\pi r_1^2(h_1+\Delta h_1) &= 3\pi h_1 + \frac{4}{3}\pi\\ \frac{1}{3}r_1^2(h_1+\Delta h_1) &= 3h_1 + \frac{4}{3}\\ \frac{1}{3}\left(\frac{3}{h_1}(h_1+\Delta h_1)\right)^2(h_1+\Delta h_1) &= 3h_1 + \frac{4}{3} && \text{by } (1)\\ \frac{3}{h_1^2}(h_1+\Delta h_1)^3 &= 3h_1 + \frac{4}{3}\\ (h_1+\Delta h_1)^3 &= h_1^3 + \frac{4h_1^2}{9}\\ \Delta h_1 &= \sqrt[3]{h_1^3 + \frac{4h_1^2}{9}} - h_1.\end{aligned}$$

Next, we set up an equation for the volume of the wide cone, then express Δh_2 in terms of h_2 :

$$\frac{1}{3}\pi r_2^2(h_2+\Delta h_2) = 12\pi h_2 + \frac{4}{3}\pi.$$

Using a similar process from above, we get

$$\Delta h_2 = \sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - h_2.$$

Recall that $\frac{h_1}{h_2} = 4.$ Therefore, the requested ratio is

$$\begin{aligned}\frac{\Delta h_1}{\Delta h_2} &= \frac{\sqrt[3]{h_1^3 + \frac{4h_1^2}{9}} - h_1}{\sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - h_2}\\ &= \frac{\sqrt[3]{(4h_2)^3 + \frac{4(4h_2)^2}{9}} - 4h_2}{\sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - h_2}\end{aligned}$$

$$= \frac{\sqrt[3]{4^3\left(h_2^3 + \frac{h_2^2}{9}\right)} - 4h_2}{\sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - h_2}$$

$$= \frac{4\sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - 4h_2}{\sqrt[3]{h_2^3 + \frac{h_2^2}{9}} - h_2}$$

$$= \boxed{\textbf{(E) } 4:1}.$$

~MRENTHUSIASM

Solution 2 (Quick and Dirty)

The heights of the cones are not given, so suppose the heights are very large (i.e. tending towards infinity) in order to approximate the cones as cylinders with base radii 3 and 6 and infinitely large height. Then the base area of the wide cylinder is 4 times that of the narrow cylinder. Since we are dropping a ball of the same volume into each cylinder, the water level in the narrow cone/cylinder should rise $\boxed{\textbf{(E) } 4}$ times as much.

-scrabbler94

Solution 3

Since the radius of the narrow cone is 1/2 the radius of the wider cone, the ratio of their areas is $\frac{1}{4}$. Therefore, the ratio of the height of the narrow cone to the height of the wide cone must be $\frac{4}{1}$. Note that this ratio is constant, regardless of how much water is dropped as long as it is an equal amount for both cones. See Solution 2 for another explanation.

Video Solution (Simple and Quick)

https://youtu.be/TgjvviBALac

~ Education, the Study of Everything

Video Solution by Aaron He (Algebra)

https://www.youtube.com/watch?v=xTGDKBthWsw&t=10m20s

Video Solution by OmegaLearn (Similar Triangles, 3D Geometry - Cones)

https://youtu.be/4Iuo7cvGJr8

~ pi_is_3.14

Video Solution by TheBeautyofMath

First-this is not the most efficient solution. I did not perceive the shortcut before filming though I suspected it.

https://youtu.be/t-EEP2V4nAE?t=231 (for AMC 10A)

https://youtu.be/cckGBU2x1zg?t=814 (for AMC 12A)

~IceMatrix

Video Solution by WhyMath

https://youtu.be/c-5-8PnCvCk

~savannahsolver

See also

2021 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
Preceded by **Problem 11**	Followed by **Problem 13**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	

2021 AMC 12A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
Preceded by **Problem 9**	Followed by **Problem 11**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

The problems on this page are copyrighted by the Mathematical Association of America (http://www.maa.org)'s American Mathematics Competitions (http://amc.maa.org).

Retrieved from "https://artofproblemsolving.com/wiki/index.php?title=2021_AMC_12A_Problems/Problem_10&oldid=169359"

Copyright © 2022 Art of Problem Solving