Art Of Problem Solving - AMC 10
Week 4

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Abstract

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1.

For how many integers n between 1 and 100 does $x^2 + x - n$ factor into the product of two linear factors with integer coefficients?

(A) 0	(B) 1	(C) 2	(D) 9	(E) 10

Does "between" mean $n \in [1, 100]$ or $n \in (1, 100)$?

The quadratic factors as:

$$\begin{aligned} x^2 + x - n &= \left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - n \\ &= \left(x + \frac{1}{2}\right)^2 - \frac{1 + 4n}{2^2} \\ &= \left(x + \frac{1}{2} - \frac{\sqrt{1+4n}}{2}\right)\left(x + \frac{1}{2} + \frac{\sqrt{1+4n}}{2}\right) \\ &= \left(x + \frac{1 - \sqrt{1+4n}}{2}\right)\left(x + \frac{1 + \sqrt{1+4n}}{2}\right) \end{aligned}$$

or skip these steps and write down the quadratic formula.

Both factors have integer coefficients if

$$\frac{1 - \sqrt{1+4n}}{2}, \qquad \frac{1 + \sqrt{1+4n}}{2}$$

are integers. Equivalently, if

$$1 - \sqrt{1+4n}, \qquad 1 + \sqrt{1+4n}$$

are even integers. Equivalently, if there is an integer k such that

$$\begin{aligned} \sqrt{1+4n} &= 2k + 1 \\ \implies 1 + 4n &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ \implies 4n &= 4k^2 + 4k \\ n &= k(k+1) \end{aligned}$$

The condition is satisfied for any $n \in [1, 100]$ that is the product of two successive integers.

$$n = 1 \times 2, \qquad 2 \times 3, \qquad 3 \times 4, \qquad \ldots \qquad 9 \times 10$$

since $10 \times 11 = 110 > 100$. Thus, there are $1 \to 9$ cases:

$$\boxed{9}$$

2.

Suppose that a and b are nonzero real numbers, and that the equation $x^2 + ax + b = 0$ has solutions a and b. Then the pair (a, b) is

(A) $(-2, 1)$	(B) $(-1, 2)$	(C) $(1, -2)$	(D) $(2, -1)$	(E) $(4, 4)$

The pair (a, b) can be used in the factorization $(x - a)(x - b)$, so that:

$$\begin{aligned} x^2 + ax + b &= (x - a)(x - b) \\ &= x^2 - (a + b)x + ab \end{aligned}$$

Equating the coefficients of the polynomial yields the linear system in (a, b):

$$\begin{aligned} a &= -(a + b) \\ b &= ab \end{aligned}$$

With solution $(1, -2)$ and $(0, 0)$, but only non-zero solutions are acceptable.

$$\boxed{(1, -2)}$$

3.

Let f be the function defined by $f(x) = ax^2 - \sqrt{2}$ for some positive a. If $f(f(\sqrt{2})) = -\sqrt{2}$, then $a =$

(A) $\frac{2-\sqrt{2}}{2}$ (B) $\frac{1}{2}$ (C) $2-\sqrt{2}$ (D) $\frac{\sqrt{2}}{2}$ (E) $\frac{2+\sqrt{2}}{2}$

Apply function f twice to the $\sqrt{2}$ argument:

$$\begin{aligned}
f(x) &= ax^2 - \sqrt{2} \\
\implies f(\sqrt{2}) &= a(\sqrt{2})^2 - \sqrt{2} = 2a - \sqrt{2} \\
f(f(\sqrt{2})) &= a\left(2a - \sqrt{2}\right)^2 - \sqrt{2} \\
&= a(4a^2 - 4\sqrt{2}a + 2) - \sqrt{2} \\
&= 4a^3 - 4\sqrt{2}a^2 + 2a - \sqrt{2}
\end{aligned}$$

We equate with the given value:

$$\begin{aligned}
4a^3 - 4\sqrt{2}a^2 + 2a - \sqrt{2} = -\sqrt{2}& \\
\implies 4a^3 - 4\sqrt{2}a^2 + 2a &= 0 \\
a\left(a^2 - \sqrt{2}a + \frac{1}{2}\right) &= 0 \\
\left(a - \frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2} &= 0 \qquad \text{because } a > 0 \\
\left(a - \frac{\sqrt{2}}{2}\right)^2 &= 0
\end{aligned}$$

$$\boxed{a = \frac{\sqrt{2}}{2}}$$

4.

Both roots of the quadratic equation $x^2 - 63x + k = 0$ are prime numbers. The number of possible values of k is

(A) 0	(B) 1	(C) 2	(D) 4	(E) more than four

Viète's formula for two roots r, s is

$$(x - r)(x - s) = x^2 - (r + s)x + rs$$

In the present case, the implication is that the sum of the roots is 63 and their product is k. For the sum of the roots to be odd, one of them must be odd and the other one must be even (since odd+odd=even, even+even=even). The only even prime is 2. Thus, the roots must be 61 and 2, which gives the only possible value of k, $k = 122$.

$$\boxed{\text{1 possible value of } k}$$

5.

Let @ denote the "averaged with" operation: $a@b = \frac{a+b}{2}$. Which of the following distributive laws holds for all numbers x, y, and z?

I. $x@(y+z) = (x@y) + (x@z)$

III. $x + (y@z) = (x+y)@(x+z)$

III. $x@(y@z) = (x@y)@(x@z)$

(A) I only	(B) II only	(C) III only	(D) I and III only	(E) II and III only

Apply the rule to the left-hand side of I:

$$\begin{aligned} x@(y+z) &= \frac{x+(y+z)}{2} \\ &= \frac{x+y}{2} + \frac{z}{2} \\ &= x@y + 0@z \\ &\neq x@y + x@z \end{aligned}$$

so I is false.

Apply the rule to the left-hand side of II:

$$\begin{aligned} x + (y@z) &= \frac{2x}{2} + \frac{y+z}{2} \\ &= \frac{x+y}{2} + \frac{x+z}{2} \\ &= (x+y)@(x+z) \end{aligned}$$

so II is true.

Apply the rule to the left-hand side of III:

$$\begin{aligned} x@(y@z) &= \frac{x + \frac{y+z}{2}}{2} \\ &= \frac{2x+y+z}{4} \\ &= \frac{\frac{x+y}{2} + \frac{x+z}{2}}{2} \\ &= (x@y)@(x@z) \end{aligned}$$

so III is true.

$$\boxed{\text{II and III only}}$$

6.

If $f(x) = ax^4 - bx^2 + x + 5$ and $f(-3) = 2$, then $f(3) =$

(A) -5 (B) -2 (C) 1 (D) 3 (E) 8

Calculating $f(3)$ and $f(-3)$ yields:

$$f(-3) = a(-3)^4 - b(-3)^2 - 3 + 5 = 81a - 9b + 2 = 2$$
$$f(+3) = a(+3)^4 - b(+3)^2 + 3 + 5 = 81a - 9b + 8 = 81a - 9b + 2 + 6 = 2 + 6 = 8$$

$$\boxed{f(3) = 8}$$

7.

What is the sum of the reciprocals of the roots of the equation

$$\frac{2003}{2004}x + 1 + \frac{1}{x} = 0?$$

(A) $-\frac{2004}{2003}$	(B) -1	(C) $\frac{2003}{2004}$	(D) 1	(E) $\frac{2004}{2003}$

Let r, s denote the roots. The sum of the reciprocals is:

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs}$$

We can find the sum and product of the roots by rearranging the equation and applying Viète's formula. Rearranging:

$$x^2 - \frac{-2004}{2003}\,x + \frac{2004}{2003} = 0$$

The roots must satisfy

$$\begin{aligned} r + s &= -\frac{2004}{2003} \\ rs &= +\frac{2004}{2003} \end{aligned}$$

$$\implies \frac{r+s}{rs} = -1$$

And thus,

$$\boxed{\frac{1}{r} + \frac{1}{s} = -1}$$

8.

Let f be a polynomial function such that, for all real x,

$$f(x^2+1) = x^4 + 5x^2 + 3$$

For all real x, $f(x^2-1)$ is

(A) x^4+5x^2+1	(B) x^4+x^2-3	(C) x^4-5x^2+1	(D) x^4+x^2+3	(E) none of these

Let $X = x^2+1$ for any real x. This implies:

$$x^2 = X-1 \implies x^4 = (X-1)^2$$

Substituting these back into the polynomial function gives:

$$\begin{aligned} f(x^2+1) &= x^4+5x^2+3 \\ f(X) &= (X-1)^2 + 5(X-1) + 3 \\ &= X^2 + 3X - 1 \end{aligned}$$

This is true for any $X \geq 1$. Now substitute x^2-1 for X and simplify:

$$\begin{aligned} f(x^2-1) &= (x^2-1)^2 + 3(x^2-1) - 1 \\ &= x^4 + x^2 - 3 \end{aligned}$$

$$\boxed{f(x^2-1) = x^4 + x^2 - 3}$$

9.

The polynomial $x^3 - ax^2 + bx - 2010$ has three positive integer roots. What is the smallest possible value of a?

(A) 78	(B) 88	(C) 98	(D) 108	(E) 118

The smallest root must be at least 0. Viète's formula for a cubic with roots r, s, t may be written as:

$$x^3 - (r+s+t)x^2 + (rs+rt+st)x - rst = 0$$

As with the quadratic equation, the formula features the sum and the product of the roots — but also the sum of the products of each pair. Note also the alternating signs. The product of the roots is

$$rst = 2010$$

Since r, s, t are integers, we consider all the possible combination of factors of 2010:

$$2010 = 2 \times 3 \times 5 \times 67$$

We seek the smallest possible sum formed by any three combinations of these factors. To this effect, we multiply the smallest numbers together, 2 and 3 and keep the other two:

$$6, \quad 5, \quad 67 \quad \rightarrow \quad 6 + 5 + 67 = 78$$

$$\boxed{a \rightarrow 78}$$

10.

Let f be a function for which $f(x/3) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

(A) $-1/3$	(B) $-1/9$	(C) 0	(D) $5/9$	(E) $5/3$

Let $x = 9z$ and substitute:

$$\begin{aligned} f(x/3) &= x^2 + x + 1 \\ \implies f(9z/3) &= (9z)^2 + 9z + 1 \\ f(3z) &= 81z^2 + 9z + 1 \end{aligned}$$

Solve the following quadratic equation for z:

$$\begin{aligned} f(3z) &= 7 \\ 81z^2 + 9z + 1 &= 7 \\ 81z^2 + 9z - 6 &= 0 \\ z^2 + \frac{1}{9}z - \frac{2}{27} &= 0 \end{aligned}$$

By Viète's formula, the sum of the roots is $-1/9$.

$$\boxed{-\frac{1}{9}}$$