MathCounts Competition Practice IV, January 2021 Target Round

Patrick & James Toche

Revised: March 12, 2021

Abstract

Notes on Target Round of MathCounts Competition Practice IV, January 2021. Questions are from MathCounts Foundation (https://www.mathcounts.org/). Copyright restrictions may apply. Written for personal use. Please report typos and errors over at https://github.com/ptoche/Math/tree/master/mathcounts.

Target Round

1.

Washington, D.C. has a land area of 68 square miles and a population of $720,000$ people. What is the number of people per square mile in Washington, D.C.? Express your answer to the nearest hundred.

[ ] people per square mile

$$\frac{720000\,\text{people}}{68\,\text{mi}^2} \approx 10588.24\,\text{people/mi}^2$$

10600 people per square mile

2.

What is the maximum number of consecutive positive integers that can be added together before the sum exceeds 400?

[ ]

Minimize the sum of the first n integers minus 400:

$$\min_n \sum_{k=1}^{k=n} k - 400 = \min_n \frac{n(n+1)}{2}$$

To get a quick order of magnitude, get an approximate solution to the associated quadratic equation:

$$\begin{aligned} n^2 + n - 800 &= \left(n + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 800 \\ &= \left(n + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 - 784 - 6 \\ &= \left(n + \frac{1}{2} - 28\right)\left(n + \frac{1}{2} + 28\right) - 6 \end{aligned}$$

So the solution is less than 28. We can check that 27 gives:

$$\frac{27(27+1)}{2} = 27 \times 14 = 378$$

The maximum number of consecutive positive integers is:

27

3.

Clifton left 280 acres of land to be divided among his sons Al and Bob in the ratio 4:3 respectively. How many acres should Al receive?

$\square$ acres

Al would receive

$$\frac{4}{4+3} \times 280 = 160$$

160 acres

4.

George and Lea each toss a tetrahedral die with faces numbered 1 through 4, and then multiply the two resulting numbers. If the product is less than 9, George wins; otherwise, Lea wins. What is the probability that Lea will win? Express your answer as a common fraction.

$\frac{\square}{\square}$

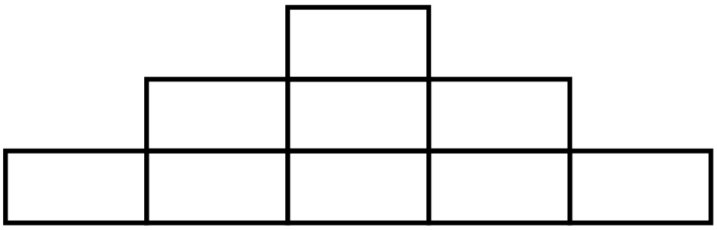
The product is less than 9 if the faces show $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,2)$, $(2,3)$, $(2,4)$. The number of ordered pairs is twice as many, or 14, out of a total of $4^2 = 16$. The probability is therefore

$$\frac{14}{16} = \frac{7}{8}$$

$\frac{7}{8}$

5.

How many rectangles of any size are in the diagram?



☐ rectangles

There are 9 rectangles of length 1. There are 6 rectangles of length 2. There are 4 rectangles of length 3. There are 2 rectangles of length 4. There is 1 rectangle of length 5. There are 4 rectangles of height 2. There is 1 rectangles of height 3. There are 2 rectangles of length 2 and height 2. There is 1 rectangle of length 3 and height 2.

$$9 + 6 + 2 + 1 + 4 + 1 + 2 + 1 = 26$$

26 rectangles

6.

At the Word Store, each vowel sells for a different price, but all consonants are free. The word "triangle" sells for \$6, "square" sells for \$9, "pentagon" sells for \$7, "cube" sells for \$7 and "tetrahedron" sells for \$8. What is the cost of the word "octahedron"?

\$ ☐

Counting the vowels in each word and assigning the word's price yields the system:

$$\begin{aligned}
1\,i + 1\,a + 1\,e &= 6 \quad \text{(triangle)} && (1)\\
1\,u + 1\,a + 1\,e &= 9 \quad \text{(square)} && (2)\\
1\,o + 1\,a + 1\,e &= 7 \quad \text{(pentagon)} && (3)\\
1\,u + 1\,e &= 7 \quad \text{(cube)} && (4)\\
1\,o + 1\,a + 2\,e &= 8 \quad \text{(tetrahedron)} && (5)\\
2\,o + 1\,a + 1\,e &= C \quad \text{(octahedron)} && (6)
\end{aligned}$$

where C, the unknown, denotes the cost of "octahedron". Combining equation (3) with (6) yields $C = 1\,o + 7$. To find C, we now solve the system for o. Subtracting (3) from (5) gives $e = 1$. Substituting into (4) gives $u = 6$. Substituting into (2) gives $a = 7$. Finally substituting into (5) gives $o = -1$ (a negative price is a subsidy, a little unexpected). And thus $C = 6$

\$ 6

7.

All six faces of a large wooden cube are painted blue, and then the cube is divided into smaller unit cubes. If 486 of the unit cubes have exactly one blue face, how many unit cubes make up the original large cube?

[ ] unit cubes

Take a cube made up of n^3 cubelets. The cubelets with exactly one painted side are found on the surface, excluding edges and corners. Each of the six faces has $(n-2)^2$ "inner" cubelets. Now solve for n:

$$\begin{aligned} 6(n-2)^2 &= 486 \\ (n-2)^2 &= 81 \\ (n-2) &= \pm 9 \end{aligned}$$

The solution is the positive root, so $n = 9+2 = 11$ and the total number of cubelets $11^3 = 1331$

1331 unit cubes

8.

Darla can complete a job in 8 hours, while Lonnie can complete the same job in 6 hours. Both Darla and Lonnie begin working on the job together. After working together for 3 hours, how much of the job is left to be completed? Express your answer as a common fraction.

[ ]

[ ]

First note that if two Lonnies worked for 3 hours, that would be equivalent to Lonnie working 6 hours and completing exactly one job. Since Darla is less productive than Lonnie, it follows that together they will complete (a little) less than one job. Let h denote hours and p productivity so that ph denotes the output. Darla produces one unit of output (a "job") in 8 hours, so her productivity is 1/8th of a job per hour. Likewise Lonnie has productivity 1/6 (Lonnie's productivity is greater than Darla's). Together, Darla and Lonnie work for $h = 3$ hours and produce:

$$3 \times \frac{1}{8} + 3 \times \frac{1}{6} = 3 \times \frac{3+4}{3 \times 8} = \frac{7}{8}$$

which is indeed a little less than 1 job.

$\frac{7}{8}$