Math Miscellani

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Revised: October 23, 2024

Abstract

This note reviews tips and tricks and selected problems to prepare for middle school math competitions. Written for personal use. Please report typos and errors over at `https://github.com/ptoche/Math/tree/master/mathcounts`.

Problem 1

Simplify $\sqrt{\sqrt{9}-\sqrt{8}}$.

Solution

$$\begin{aligned} x &= \sqrt{\sqrt{9}-\sqrt{8}} \\ \implies x^2 &= 3-2\sqrt{2} = (1)^2 - 2\sqrt{2} + (\sqrt{2})^2 = (1-\sqrt{2})^2 \\ \implies x &= |1-\sqrt{2}| = \sqrt{2}-1 \end{aligned}$$

Problem 2

If x satisfies $x^2 + \dfrac{1}{x^2} = \sqrt{2}$, evaluate $x^{2024} - \dfrac{1}{x^{2024}}$.

Solution

$$x^2 + \frac{1}{x^2} = \sqrt{2} \implies \left(x^2 + \frac{1}{x^2}\right)^2 = (\sqrt{2})^2 \implies x^4 + 2x^2\frac{1}{x^2} + \frac{1}{x^4} = 2 \implies x^4 + \frac{1}{x^4} = 0$$
$$\implies x^8 = -1 \implies (x^8)^{253} = (-1)^{253} \implies x^{2024} = -1 \implies x^{2024} - \frac{1}{x^{2024}} = -1 - \frac{1}{-1} = 0$$

Problem 3

Solve for $x \in \mathbb{R}$, where $x^2 - x^3 = 12$.

Solution

$$\begin{aligned} x^2 - x^3 &= 2^3 + 2^2 && \text{from } 12 = 2^3 + 2^2 \\ x^2 - 2^2 &= x^3 + 2^3 && \text{grouping} \\ (x-2)(x+2) &= (x+2)(x^2 - 2x + 2^2) && \text{factoring} \\ (x+2)(x^2 - 3x + 6) &= 0 && \text{grouping} \\ \implies x = \frac{3 \pm i\sqrt{15}}{2} &\text{ or } x = -2 \end{aligned}$$

Problem 4

Solve for $x \in \mathbb{R}$, where $3^x - 2^x = 65$.

Solution

$$\begin{aligned} 3^x - 2^x &= 65 \\ (3^{x/2})^2 - (2^{x/2})^2 &= 5 \times 13 \\ (3^{x/2} - 2^{x/2}) \times (3^{x/2} + 2^{x/2}) &= 5 \times 13 \end{aligned}$$

Equating the factors on each side of the equality

$$\begin{cases} 3^{x/2} - 2^{x/2} & = 5 \\ 3^{x/2} + 2^{x/2} & = 13 \end{cases} \implies 2 \times 2^{x/2} = 13 - 5 = 8 \implies 2^{x/2} = 2^2 \implies x/2 = 2 \implies x = 4$$

Problem 5

Solve for $x \in \mathbb{R}$, where $16^x + 20^x = 25^x$.

Solution

We notice that 4 and 5 can be factored

$$\begin{aligned} 16^x + 20^x &= 25^x \\ 4^{2x} + 5^x \cdot 4^x &= 5^{2x} \\ 1 + \frac{5^x}{4^x} \cdot \frac{4^x}{4^x} &= \frac{5^{2x}}{4^{2x}} \\ 1 + (5/4)^x &= (5/4)^{2x} \end{aligned}$$

Let $X = (5/4)^x$ and solve for $X > 0$:

$$X^2 - X - 1 = 0 \implies X = \frac{1 + \sqrt{5}}{2}$$

Now solve for x by taking the logarithm on both sides of the equality:

$$\left(\frac{5}{4}\right)^x = \frac{1+\sqrt{5}}{2} \implies x \ln\left(\frac{5}{4}\right) = \ln\left(1 + \sqrt{5}\right) - \ln(2) \implies x = \frac{\ln(1+\sqrt{5}) - \ln(2)}{\ln(5/4)}$$

Check the plausibility of the solution with coarse approximations:

$$\ln(2) \approx 0.69,\ \ln(3) \approx 1.10,\ \ln(5) \approx 1.61,\ \sqrt{5} \approx 2.24, \implies \ln(1+\sqrt{5}) \approx \ln(3.24) \approx 1.2$$

$$\frac{\ln(1+\sqrt{5}) - \ln(2)}{\ln(5) - 2\ln(2)} \approx \frac{1.2 - 0.69}{1.61 - 2 \times 0.69} \approx 2.2$$

$16^{2.2} + 20^{2.2} \approx 1174$

$25^{2.2} \approx 1190$