2021 Fall AMC 10B Problems/Problem 18

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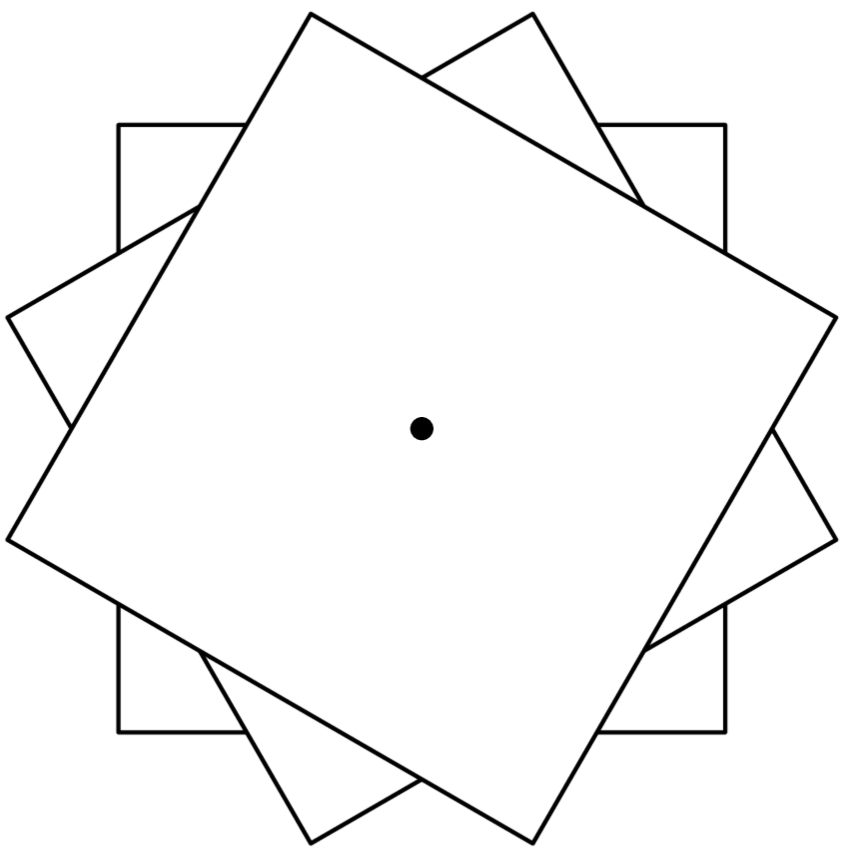
Problem

Three identical square sheets of paper each with side length 6 are stacked on top of each other. The middle sheet is rotated clockwise 30° about its center and the top sheet is rotated clockwise 60° about its center, resulting in the 24-sided polygon shown in the figure below. The area of this polygon can be expressed in the form $a-b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. What is $a+b+c?$

$\textbf{(A)}\ 75 \qquad\textbf{(B)}\ 93 \qquad\textbf{(C)}\ 96 \qquad\textbf{(D)}\ 129 \qquad\textbf{(E)}\ 147$



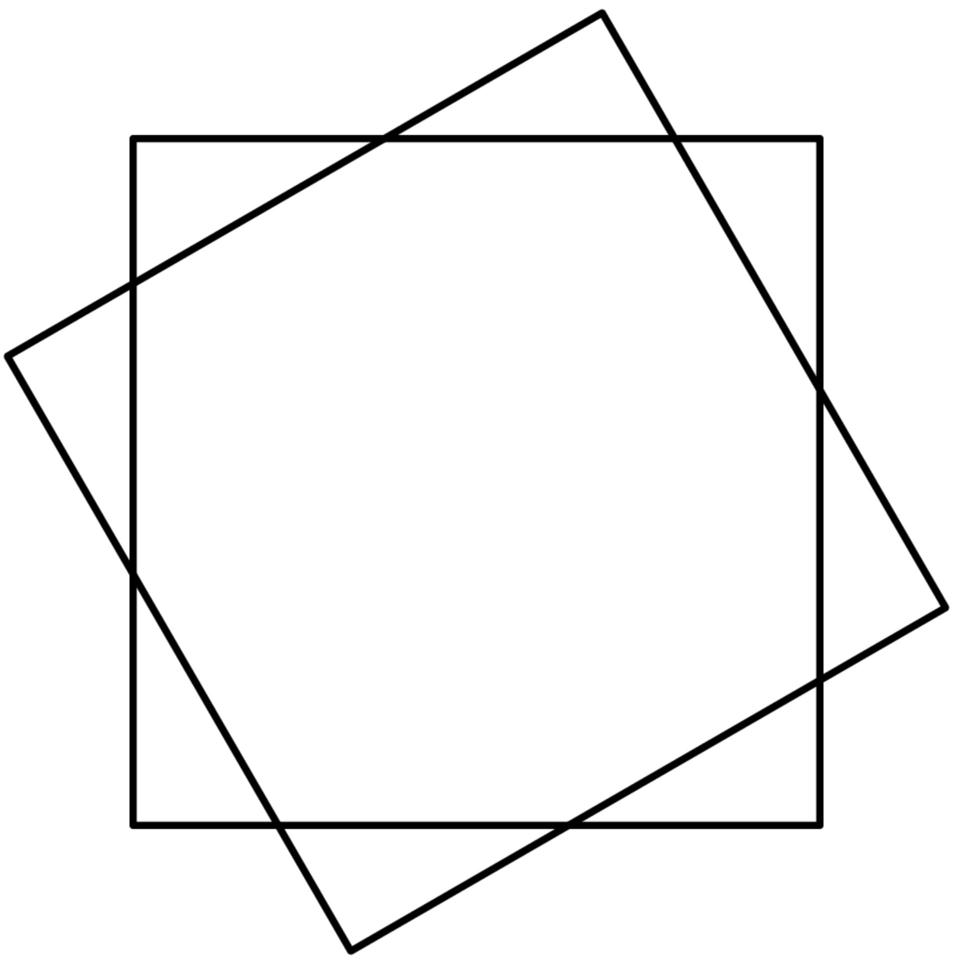
Solution 1

First note the useful fact that if R is the circumradius of a dodecagon (12-gon) the area of the figure is $3R^2.$ If we connect the vertices of the 3 squares we get a dodecagon. The radius of circumcircle of the dodecagon is simply half the diagonal of the square, which is $3\sqrt{2}.$ Thus the area of the dodecagon is $3 \cdot (3\sqrt{2})^2 = 3 \cdot 18 = 54.$ But, the problem asks for the area of figure of rotated squares. This area is the area of the dodecagon, which was found, subtracting the 12 isosceles triangles, which are formed when connecting the vertices of the squares to created the dodecagon. To find this area, we need to know the base of the isosceles triangle, call this $x.$ Then, we can use Law of Cosines, on the triangle that is formed from the two vertices of the square and the center of the square. After computing, we get that $x = 3\sqrt{3} - 3.$ Realize that the 12 isosceles are congruent with an angle measure of $120^\circ,$ this means that we can create 4 congruent equilateral triangles with side length of $3\sqrt{3} - 3.$ The area of the equilateral triangle is $\frac{\sqrt{3}}{4} \cdot (3\sqrt{3} - 3)^2 = \frac{\sqrt{3}}{4} \cdot (36 - 18\sqrt{3}) = \frac{36\sqrt{3} - 54}{4}.$ Thus, the area of all the twelve small equilateral traingles are $4 \cdot \frac{36\sqrt{3} - 54}{4} = 36\sqrt{3} - 54$. Thus, the requested area is $54 - (36\sqrt{3} - 54) = 108 - 36\sqrt{3}.$ Thus, $a + b + c = 108 + 36 + 3 = 147.$ Thus, the answer is $\boxed{\textbf{(E.)}}$.

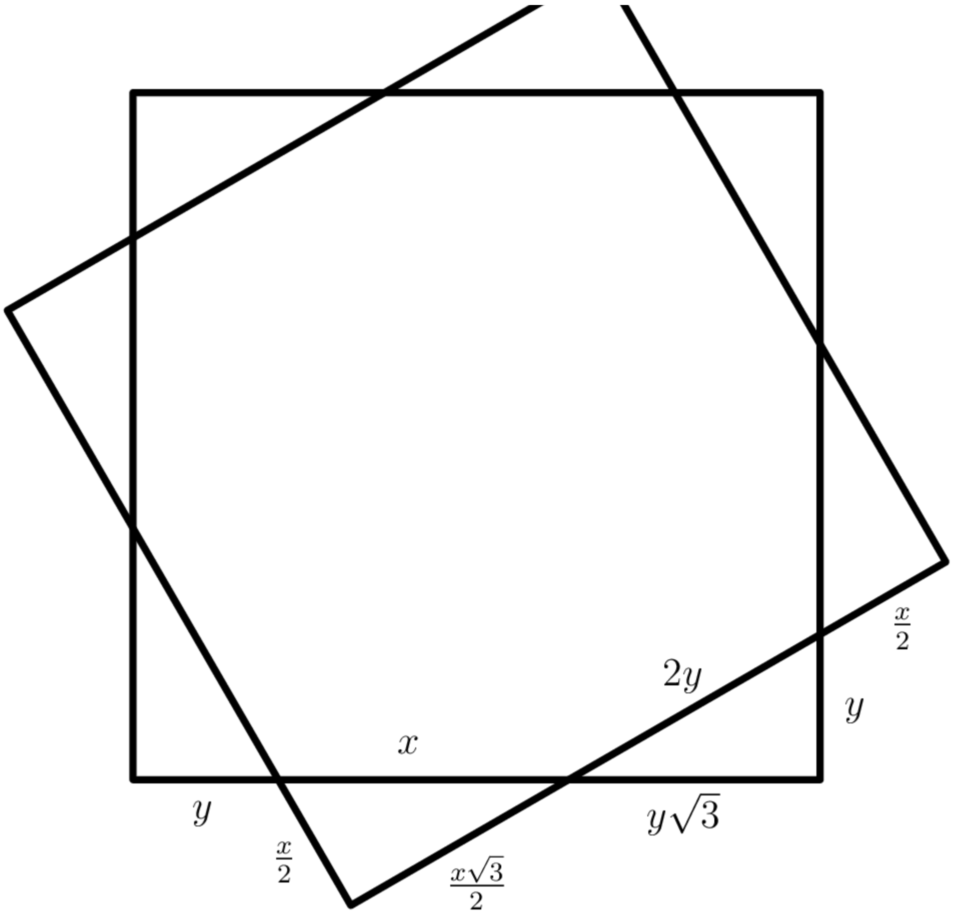
~NH14

Solution 2 (30-60-90 Triangles)

To make things simpler, let's take only the original sheet and the 30 degree rotated sheet. Then the diagram is this;



The area of this diagram is the original square plus the area of the four triangles that 'jut' out of the square. Because the square is rotated 30°, each triangle is a 30-60-90 triangle. Similarly, the triangles that are bounded on the inside of the original square outside of the rotated square are also congruent 30-60-90 triangles. Noting this, we can do some labelling:



Since the side lengths of the squares must be the same, and they are both 6, we have a system of equations;

$$y + x + y\sqrt{3} = 6$$

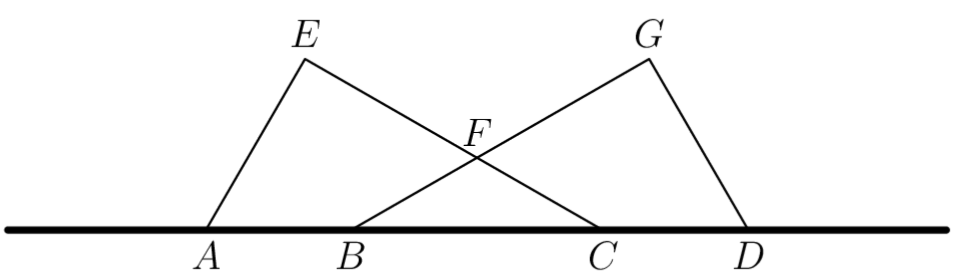
$$\frac{x\sqrt{3}}{2} + 2y + \frac{x}{2} = 6$$

We solve this to get $x = 6 - 2\sqrt{3}$ and $y = 3 - \sqrt{3}$.

The area of each triangle is $\frac{x}{2} \cdot \frac{x\sqrt{3}}{2} \cdot \frac{1}{2} = 6\sqrt{3} - 9$ by plugging in x.

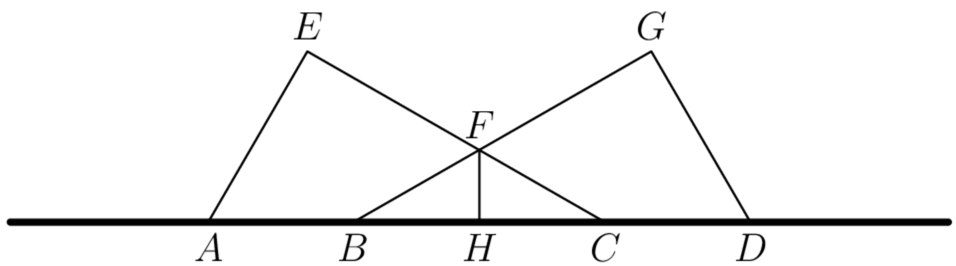
The rotated 60 degree square is the same thing as rotating it 30 degrees counterclockwise, so it's triangles that jut out of the square will be congruent to the triangles we have found, and therefore they will have the same area.

Unfortunately, when drawing all three squares, we see the two triangles overlap; take the very top for example.



The area of this shape is twice the area of each of the triangles that we have already found minus the area of the small triangle that is overlapped by the two by PIE. Now we only need to find the area of $\triangle BCF$.

$\angle GBD \cong \angle ECA \cong 30^\circ$ and by symmetry $\triangle BCF$ is isosceles, so it is a 30-30-120 triangle. If we draw a perpendicular, we split it into two 30-60-90 triangles;



By symmetry, the distance from A to the edge of the square is equal to the distance from D to the edge of the square is equal to y. AC = BD = x, and the side length of the square is 6, so we use PIE to obtain

$$x + x - BC = 6 - y - y \implies BC = 12 - 6\sqrt{3}$$

To find the height of $\triangle BFC$, we see that $HC = \frac{BC}{2} = 6 - 3\sqrt{3}$. Then by 30-60-90 triangles, $HF = \frac{HC}{\sqrt{3}} = 2\sqrt{3} - 3$. Finally, the area of $\triangle BFC = \frac{BC \cdot HF}{2} = 21\sqrt{3} - 36$.

Putting it all together, the area of the entire diagram is the area of the square plus four of these triangle-triangle intersections. The area of these intersections by PIE is $2 \cdot [ACE] - [BFC] = 12\sqrt{3} - 18 - (21\sqrt{3} - 36) = 18 - 9\sqrt{3}$. Therefore the total area is $36 + 4 \cdot (18 - 9\sqrt{3}) = 36 + 72 - 36\sqrt{3} = 108 - 36\sqrt{3}$.

Thus $a + b + c = 108 + 36 + 3 = 147 = \boxed{E}$

~KingRavi

Solution 3

As shown in Image:2021_AMC_12B_(Nov)_Problem_15,_sol.png, all 12 vertices of three squares form a regular dodecagon (12-gon). Denote by O the center of this dodecagon.

Hence, $\angle AOB = \frac{360^\circ}{12} = 30^\circ$.

Because the length of a side of a square is 6, $AO = 3\sqrt{2}$.

Hence, $AB = 2AO \sin \frac{\angle AOB}{2} = 3\left(\sqrt{3} - 1\right)$.

We notice that $\angle MAB = \angle MBA = 30^\circ$. Hence, $AM = \frac{AB}{2\cos \angle MAB} = 3 - \sqrt{3}$.

Therefore, the area of the region that three squares cover is

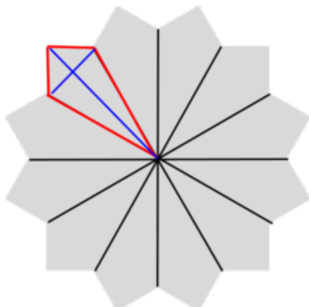
$$\begin{aligned}
&\text{Area } ABCDEFGHIJKL - 12\text{Area } \triangle MAB \\
&= 12\text{Area } \triangle OAB - 12\text{Area } \triangle MAB \\
&= 12 \cdot \frac{1}{2} OA \cdot OB \sin \angle AOB - 12 \cdot \frac{1}{2} MA \cdot MB \sin \angle AMB \\
&= 6OA^2 \sin \angle AOB - 6MA^2 \sin \angle AMB
\end{aligned}$$

$$= 108 - 36\sqrt{3}$$

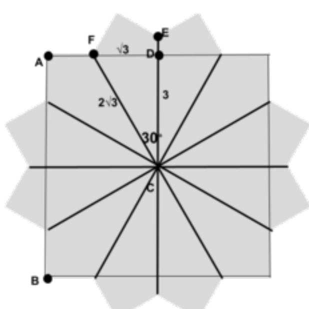
Therefore, the answer is $\boxed{\textbf{(E) }147}$.

~Steven Chen (www.professorchenedu.com)

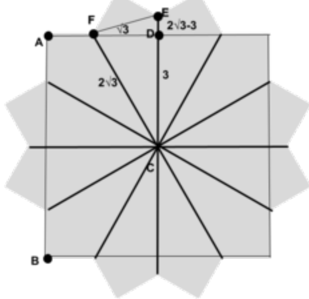
Solution 4



First, we can separate the shape into 12 congruent kites. The area of the figure can be determined by finding the area of one kite and multiplying it by 12. In order to to get the area of one kite, we need to find its diagonals, shown in blue.

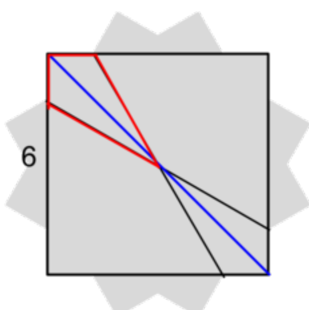


We notice that angle FCE is $\frac{360^\circ}{12} = 30^\circ$. Also, we know that CD is half of AB, so it has a length of 3. Now, we can find the lengths of FC and FD using the 30-60-90 triangle. We find that FC is $2\sqrt{3}$ and FD is $\sqrt{3}$. Since FC is congruent to CE, CE is also $2\sqrt{3}$. Using this information, we can conclude that ED is $2\sqrt{3} - 3$.



Now, we can find the shorter diagonal by using the Pythagorean theorem:

$$\mathrm{FC}^2 = \sqrt{3}^2 + (2\sqrt{3} - 3)^2$$
$$\mathrm{FC} = \sqrt{24 - 12\sqrt{3}}$$



We can find the longer diagonal of the kite by looking at one of the square sheets of paper. We know that the side of the square has a length of 6, so the diagonal of the square must be $6\sqrt{2}$. The longer diagonal of the kite is half of this length, so it has a length of $3\sqrt{2}$.

The area of the entire figure is

$$= 12 \cdot \frac{\mathrm{d}_1 * \mathrm{d}_2}{2}$$
$$= 12 \cdot \frac{3\sqrt{2} \cdot \sqrt{24 - 12\sqrt{3}}}{2}$$
$$= 12 \cdot \frac{6\sqrt{12 - 6\sqrt{3}}}{2}$$
$$= 36\sqrt{12 - 6\sqrt{3}}$$

Now we can use algebra to make our answer look a little nicer.

$$a-\sqrt{b}=\sqrt{12-6\sqrt{3}}$$

$$(a-\sqrt{b})^2=(\sqrt{12-6\sqrt{3}})^2$$

$$a^2-2a\sqrt{b}+b=12-6\sqrt{3}$$

$$a^2+b=12$$

$$2a\sqrt{b}=6\sqrt{3}$$

$$a=3, b=3$$

$$a-\sqrt{b}=\sqrt{12-6\sqrt{3}}=3-\sqrt{3}$$

The area of the entire region is $36(3-\sqrt{3})$, or $108-36\sqrt{3}$.

Therefore, $a+b+c=108+36+3=147=\boxed{e}$.

~JavaWhiz12

Video Solution by Power of Logic(math2718281828459)

https://youtu.be/i8h8cq_WBjA

~math2718281828459

See Also

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