Ask Math Anything

Study at Home with Po-Shen Loh

4 July 2020

Abstract

Professor Po-Shen Loh solves problems on his YouTube channel. A selection for practice.
Reference: Ask Math Anything - Daily Challenge with Po-Shen Loh

A Triangle Puzzle

A triangle has side lengths 7, 7, and x. Its area is the same as a different triangle with sides 7, 7, and 10.
Find x.

Hint 1: Area of a triangle:

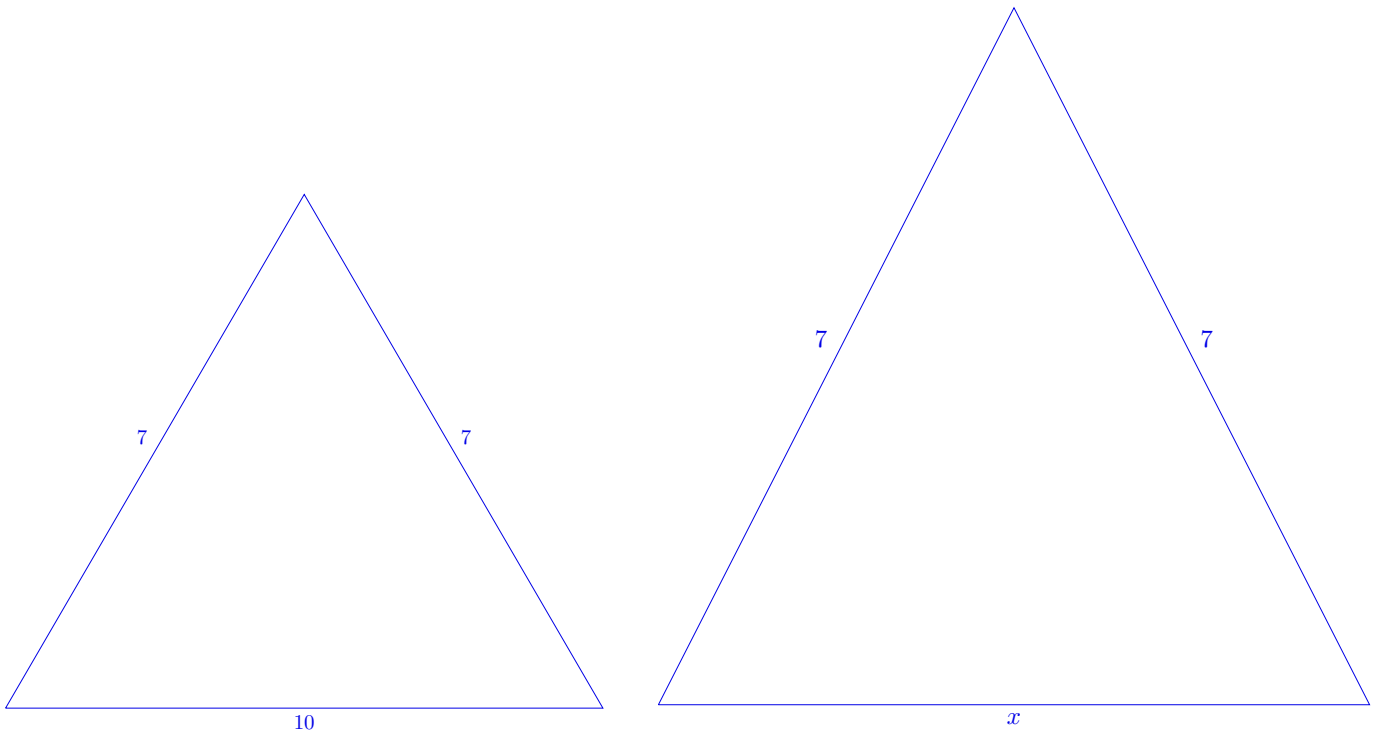$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

Hint 2: Heron's Formula:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, c are the lengths of the three sides of the triangle, and s is the semi-perimeter,

$$s = \frac{a+b+c}{2}$$

Consider the first triangle. The semi-perimeter is:

$$s = \frac{a+b+c}{2} = \frac{7+7+10}{2} = 12$$

Applying Heron's formula yields:

$$\begin{aligned}
\text{Area } == \sqrt{s(s-a)(s-b)(s-c)} \\
= \sqrt{12(12-7)(12-7)(12-10)} \\
= \sqrt{12 \times 5 \times 5 \times 2}
\end{aligned}$$

We may not need to calculate this square-root, so let's now consider the second triangle:

$$\begin{aligned}
\text{Area } &= \sqrt{s(s-a)(s-b)(s-c)} \\
&= \sqrt{\left(7 + \frac{x}{2}\right)\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)\left(7 - \frac{x}{2}\right)} \\
&= \sqrt{\left(\frac{x}{2}\right)^2 \left(49 - \left(\frac{x}{2}\right)^2\right)}
\end{aligned}$$

In the above, let $u = \left(\frac{x}{2}\right)^2$, we can rewrite the area as:

$$\text{Area } = \sqrt{u(49-u)}$$

Now set the two areas equal to each other:

$$\text{Area } = \sqrt{12 \times 5 \times 5 \times 2} = \sqrt{u(49-u)}$$

Squaring both sides of the equality:

$$\begin{aligned}
12 \times 5 \times 5 \times 2 &= u(49-u) \\
u^2 - 49u + 12 \times 5 \times 5 \times 2 &= 0 \\
(u-25)(u-24) &= 0
\end{aligned}$$

And thus we get $u = 24$, or

$$\begin{aligned}
u = \left(\frac{x}{2}\right)^2 &= 24 \\
x &= 2\sqrt{24} \\
&= 4\sqrt{6}
\end{aligned}$$