AMC 10 Problem Series (2804)

Jon Joseph

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Overview

Lesson 10 (Aug 6) Class Transcript - Probability

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jonjoseph 2021-08-06 19:30:35
Nice one @ hongs24514!! Your first win?

jonjoseph 2021-08-06 19:31:12
AMC 10 Problem Series
Week 10: Probability

jonjoseph 2021-08-06 19:31:18
In today's lesson, we look at various types of probability problems and the techniques for solving them. We start with the most common definition of probability.

jonjoseph 2021-08-06 19:31:38
You may find tonight's problems a little tougher than normal. Be sure to ask questions.

jonjoseph 2021-08-06 19:31:49
PROBABILITY BY COUNTING

jonjoseph 2021-08-06 19:31:53
In probability, we typically work with *events*. For example, flipping a coin and getting three heads in a row can be an event.

jonjoseph 2021-08-06 19:32:03
In situations where all outcomes are regarded as equally likely, the probability of an event is the number of ways that event can occur divided by the total number of possible outcomes. Thus, we use counting when applying this definition.

jonjoseph 2021-08-06 19:32:31
Do you see that definition: Part/Total?

jonjoseph 2021-08-06 19:32:57
That's almost always the starting point.

jonjoseph 2021-08-06 19:33:01
What is the probability that an integer in the set $\{1, 2, 3, \ldots, 100\}$ is divisible by 2 and not divisible by 3?

$$\text{(A) } \frac{1}{6} \quad \text{(B) } \frac{33}{100} \quad \text{(C) } \frac{17}{50} \quad \text{(D) } \frac{1}{2} \quad \text{(E) } \frac{18}{25}$$

jonjoseph 2021-08-06 19:33:13
We need to compute the number of integers between 1 and 100 that are divisible by 2 but not divisible by 3.

jonjoseph 2021-08-06 19:33:21
How many of these numbers are divisible by 2?

jonjoseph 2021-08-06 19:33:56

We can see that $\frac{100}{2} = 50$ of these numbers are divisible by 2. These numbers are $\{2, 4, 6, \ldots, 100\}$.

jonjoseph 2021-08-06 19:34:03
It is not so easy to determine how many of the numbers $\{2, 4, 6, \ldots, 100\}$ are not divisible by 3. Instead of trying to find how many are not divisible by 3, what else can we ask?

jonjoseph 2021-08-06 19:35:03
We can ask how many of the numbers $\{2, 4, 6, \ldots, 100\}$ *are* divisible by 3.

jonjoseph 2021-08-06 19:35:08
A number is divisible by both 2 and 3 if and only if it is divisible by 6. How many numbers between 1 and 100 are divisible by 6?

jonjoseph 2021-08-06 19:35:44
When we divide 100 by 6, we get 16 with a remainder of 4, so there are 16 multiples of 6 between 1 and 100.

jonjoseph 2021-08-06 19:35:54
So how many of the numbers $\{2, 4, 6, \ldots, 100\}$ are *not* divisible by 3?

jonjoseph 2021-08-06 19:36:30
There are 50 numbers in the set $\{2, 4, 6, \ldots, 100\}$. Since 16 of these numbers are divisible by 3, $50 - 16 = 34$ of these numbers are not divisible by 3. So what is the answer?

jonjoseph 2021-08-06 19:37:21
The probability that a number from 1 to 100 is divisible by 2 and not divisible by 3 is $\dfrac{34}{100} = \dfrac{17}{50}$. The answer is (C).

jonjoseph 2021-08-06 19:37:32
Slips of paper containing the numbers $1, 2, \ldots, 10$ are put in a hat. Two slips are drawn at random without replacement. What's the probability that their sum is 5?

$\text{(A) } \frac{1}{45} \quad \text{(B) } \frac{1}{25} \quad \text{(C) } \frac{2}{45} \quad \text{(D) } \frac{1}{30} \quad \text{(E) } \frac{2}{55}$

jonjoseph 2021-08-06 19:37:46
Each pair of slips is equally likely. That means we can use counting.

jonjoseph 2021-08-06 19:37:52
We're going to need to find the total number of outcomes (the denominator) and the number of successful outcomes (the numerator).

jonjoseph 2021-08-06 19:38:02
It turns out that in problems like this, we have a choice to make. We can either keep track of the pair of slips we picked AND the order which they were drawn, or we can just keep track of the pair we got at the end.

jonjoseph 2021-08-06 19:38:17
It's generally safer to keep track of order than not, so that's what we'll do today. That is, we'll denote picking a slip with a 1 and then a slip with a 2 by $(1, 2)$ and we'll denote a picking a slip with a 2 and then a slip with a 1 by a $(2, 1)$.

jonjoseph 2021-08-06 19:38:58
(I don't really agree with this last statement. You can choose when and if keeping track of order matters. Often it does not.)

jonjoseph 2021-08-06 19:39:10
Let's figure out the number of successful outcomes. Can you write them all down for me, using the convention above?

jonjoseph 2021-08-06 19:39:18
Don't forget that we are keeping track of order! So we're treating picking the same two slips in a different order as a different outcome.

jonjoseph 2021-08-06 19:40:12
There are 4 possible successful outcomes. Here they are:

$$(1,4),(4,1),(2,3),(3,2).$$

Remember, $(1,4)$ means that we picked a 1 and then a 4.

jonjoseph 2021-08-06 19:40:16
Now let's figure out how many total outcomes there are. Since we chose to keep track of order for the successful outcomes, we're going to have to keep track of order for total outcomes.

jonjoseph 2021-08-06 19:40:28
How many choices are there for the first slip?

jonjoseph 2021-08-06 19:40:47
There are 10 choices for the first slip, since we can pick any of the 10 slips.

jonjoseph 2021-08-06 19:40:52
Don't forget, we're picking without replacement. So how many choices do we have for the second slip?

jonjoseph 2021-08-06 19:41:17
We have 9 choices for the second slip, since it must be different from the first slip. So how many total outcomes are there?

jonjoseph 2021-08-06 19:42:02
Using the product rule from last class, there are $10 \cdot 9 = 90$. Here's why that works: How many outcomes are there starting with a 1?

jonjoseph 2021-08-06 19:42:27
That's right, there are 9: $(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(1,9),(1,10)$. There are similarly 9 outcomes starting with a 2, with a 3, and so on. That means that there's a total of $10 \cdot 9 = 90$ outcomes.

jonjoseph 2021-08-06 19:42:31
All right, so what's our answer?

jonjoseph 2021-08-06 19:43:17
That's right, it's $\dfrac{4}{90} = \dfrac{2}{45}$, which is (C).

jonjoseph 2021-08-06 19:43:41
Slips of paper containing the numbers $1, 2, \ldots, 10$ are put in a hat. Two slips are drawn at random with replacement. What's the probability that their sum is 5?

$\text{(A)}\ \frac{1}{45} \quad \text{(B)}\ \frac{1}{25} \quad \text{(C)}\ \frac{2}{45} \quad \text{(D)}\ \frac{1}{30} \quad \text{(E)}\ \frac{2}{55}$

jonjoseph 2021-08-06 19:43:53
Hey, this looks like almost the same question! Except we are drawing with replacement: after we choose a slip, we put it back in the hat.

jonjoseph 2021-08-06 19:44:01
Let's again figure this out first by keeping track of order. Again, we treat picking 1 then picking 2 as different from picking 2 then picking 1. The former we write as $(1,2)$ and the latter we write as $(2,1)$.

jonjoseph 2021-08-06 19:44:07
We have the same 4 successful outcomes as last time. They are

$$(1,4),(4,1),(2,3),(3,2).$$

jonjoseph 2021-08-06 19:44:18
How many total outcomes?

jonjoseph 2021-08-06 19:45:24
We put the first slip back in the hat. Using the product rule again, the number of total outcomes is $10 \cdot 10 = 100$.

jonjoseph 2021-08-06 19:45:29
What's the final answer?

jonjoseph 2021-08-06 19:46:15

With 100 total outcomes and 4 successful outcomes, the answer is $\frac{4}{100}=\frac{1}{25}$, which is (B).

jonjoseph 2021-08-06 19:46:51

An envelope contains eight bills: Two $\$1$ bills, two $\$5$ bills, two $\$10$ bills, and two $\$20$ bills. Two bills are drawn at random without replacement. What is the probability that their sum is $\$20$ or more?

$(\mathrm{A})\ \frac{1}{4} \quad (\mathrm{B})\ \frac{2}{5} \quad (\mathrm{C})\ \frac{3}{7} \quad (\mathrm{D})\ \frac{1}{2} \quad (\mathrm{E})\ \frac{2}{3}$

jonjoseph 2021-08-06 19:47:20

Like I mentioned above, we're going to keep track of order. We need to calculate the number of successful outcomes and the number of total outcomes.

jonjoseph 2021-08-06 19:47:28

Let's start out by calculating the number of successful outcomes. Here, success means getting a sum of 20 dollars or more. What are some ways that can happen?

jonjoseph 2021-08-06 19:48:42

Yes, any pair containing a 20 dollar bill works, and so does a pair consisting of the two 10 dollar bills.

jonjoseph 2021-08-06 19:48:47

Before we get started on counting, though, let's make sure that all our outcomes are equally likely. Each bill is equally likely, so let's pretend to write down letters on each of the bills so we can tell them apart.

jonjoseph 2021-08-06 19:48:56

We're going to pretend that the envelope contains a $1_A, 1_B, 5_A, 5_B, 10_A, 10_B, 20_A$ and 20_B. In other words, imagine that we wrote down a little "A" or "B" on the corner of each bill.

jonjoseph 2021-08-06 19:49:10

Let's start by counting the pairs containing a 20 dollar bill. Since we're keeping track of order, that means any pair of the form $(20_A, -), (20_B, -), (-, 20_A), (-, 20_B)$.

jonjoseph 2021-08-06 19:49:17

Since we're keeping track of order, the pair $(20_A, -)$ means that we first picked the 20 dollar bill marked with an A, and then any other bill.

jonjoseph 2021-08-06 19:49:20

How many pairs of the form $(20_A, -)$ are there?

jonjoseph 2021-08-06 19:50:12

There are 7. Same goes for each of the other possibilities. That gives $4 \cdot 7 = 28$ pairs containing a 20 dollar bill... actually, wait, we overcounted! We counted some pairs twice. Who can see which ones?

jonjoseph 2021-08-06 19:50:29

Do you see why there are 7 pairs?

jonjoseph 2021-08-06 19:51:13

We counted each of $(20_A, 20_B)$ and $(20_B, 20_A)$ twice. That means we have just 26 different pairs containing a 20 dollar bill.

jonjoseph 2021-08-06 19:51:30

The only other possibility is a pair of 10 dollar bills. How many ways are there to choose them?

jonjoseph 2021-08-06 19:52:27

We can either get $(10_A, 10_B)$ or $(10_B, 10_A)$. So there are 2 possibilities here. So what's our total number of successful outcomes?

jonjoseph 2021-08-06 19:53:12

Adding together our two cases we have $26 + 2 = 28$ successful outcomes.

jonjoseph 2021-08-06 19:53:17
Now let's count the total number of outcomes. That's pretty easy. How many choices do we have for the first bill?

jonjoseph 2021-08-06 19:54:02
We have 8 choices for the first bill. How about for the second bill?

jonjoseph 2021-08-06 19:54:25
We have 7 choices for the second bill. Now we can calculate the number of total outcomes and the probability. What do you get?

jonjoseph 2021-08-06 19:55:29
Using the product principle, there are $8 \cdot 7 = 56$ total pairs. Therefore, the probability of success is $\frac{28}{56} = \frac{1}{2}$, which is (D).

jonjoseph 2021-08-06 19:55:42
Forty slips are placed into a hat, each bearing a number $1, 2, 3, 4, 5, 6, 7, 8, 9,$ or 10, with each number entered on four slips. Four slips are drawn from the hat at random and without replacement. Let p be the probability that all four slips bear the same number. Let q be the probability that two of the slips bear a number a and the other two bear a number $b \neq a$. What is the value of $\frac{q}{p}$?

$(\text{A})\ 162 \quad (\text{B})\ 180 \quad (\text{C})\ 324 \quad (\text{D})\ 360 \quad (\text{E})\ 720$

jonjoseph 2021-08-06 19:56:15
Since we have 4 slips with each numerical value, let's do the same thing as last time and label our slips as $1_A, 1_B, 1_C, 1_D$, then $2_A, 2_B, 2_C, 2_D$, and so on until $10_A, 10_B, 10_C, 10_D$.

jonjoseph 2021-08-06 19:56:27
Again, imagine you took the 4 slips with the number 1, and wrote an A in the corner for one of them, a B in the corner for the the second one, a C in the corner for the third one, and a D in the corner for the fourth one. And same for all the other numbers!

jonjoseph 2021-08-06 19:56:48
Since we've written down letters on our slips to distinguish them, let's think about how many ways there are to get the same number.

jonjoseph 2021-08-06 19:56:58
Let's start by figuring out the number of ways to get four 1s. Here's one successful quadruple: $(1_A, 1_B, 1_C, 1_D)$. That means we first pick a 1 with an A in the corner, then a 1 with a B in the corner, then a 1 with a C in the corner, and finally a 1 with a D in the corner.

jonjoseph 2021-08-06 19:57:25
What is the total number of successful quadruples consisting of only picking the number 1?

jonjoseph 2021-08-06 19:58:18
There are $4! = 24$ total quadruples which correspond to only picking slips with the number 1 on them, one for each ordering of the four slips with the number 1.

jonjoseph 2021-08-06 19:58:27
Of course, there are also 24 total quadruples corresponding to only picking slips with the number 2 on them, and same goes for any other number. That means that there are $10 \cdot 24 = 240$ total successful quadruples for the probability p.

jonjoseph 2021-08-06 19:58:40
Is that clear? We good?

jonjoseph 2021-08-06 19:58:52
Now for the total number of possible quadruples. We're picking one of forty slips, then another one without replacement, and so on until we get 4. How many total possibilities are there?

jonjoseph 2021-08-06 19:59:12

Hint: Do NOT multiply your result out.

jonjoseph 2021-08-06 20:00:15
There are $40 \cdot 39 \cdot 38 \cdot 37$ possible quadruples. Therefore,

$$p = \frac{240}{40 \cdot 39 \cdot 38 \cdot 37}.$$

We won't simplify this yet, since we need to divide q by p, and things might cancel.

jonjoseph 2021-08-06 20:01:06
This is a good rule of thumb. When doing a counting or probability problem usually you do not want to multiply things out.

jonjoseph 2021-08-06 20:01:12
Now for q. Let's count the number of successful outcomes. By definition, q is the probability that we draw 2 slips with one number, and 2 slips with another.

jonjoseph 2021-08-06 20:01:22
We've made our slips distinguishable, and we are also keeping track of order. There are going to be lots of successful outcomes! Let's think about how to count them.

jonjoseph 2021-08-06 20:01:31
Before we do so, though, let's figure out what they look like. Here's a successful outcome in which we see two slips with a 1 and two slips with a 2: $(1_A, 1_B, 2_A, 2_B)$. What's another successful outcome where we see two slips with a 1 and two slips with a 2?

jonjoseph 2021-08-06 20:02:21
Those work! Here's another one I like: $(1_A, 1_C, 2_D, 2_C)$.

jonjoseph 2021-08-06 20:02:28
Both of the last two I mentioned consisted of first picking two 1s, then two 2s. Let's think about how many ways there are to do that.

jonjoseph 2021-08-06 20:02:32
How many choices are there for the first 1?

jonjoseph 2021-08-06 20:02:55
There are 4: $1_A, 1_B, 1_C$ and 1_D. And for the second 1?

jonjoseph 2021-08-06 20:03:17
There are 3 possibilities remaining. That means there are 12 choices for the 1s. And how about for the pair of 2s?

jonjoseph 2021-08-06 20:03:51
There are similarly 12 choices. That means there are $12 \cdot 12 = 144$ successful pairs that consist of picking a 1 twice, then a 2 twice.

jonjoseph 2021-08-06 20:04:00
Instead of picking the 1s first and the 2s at the end, we could pick them in a different order. For instance, we could pick a 2 twice then a 1 twice. How many ways are there to do that?

jonjoseph 2021-08-06 20:04:41
There are still 144. We choose them in the same way as above.

jonjoseph 2021-08-06 20:04:46
But we could also pick a 1, then a 2, then a 1, then a 2!

jonjoseph 2021-08-06 20:04:50
Hmmm. I'm getting tired of listing the possible orders. What we're really doing here is choosing at which points exactly we're picking the 1s. How many ways are there to specify that?

jonjoseph 2021-08-06 20:05:00
Hint: we're writing down lists of two 1s and two 2s in some order.

jonjoseph 2021-08-06 20:05:54

Hint: We have 4 things and we want to choose 2.

jonjoseph 2021-08-06 20:06:16

There are $\binom{4}{2} = 6$ ways to do this, since we can pick two of the four positions to be the 1s and then have only one way to put the 2s in the remaining spots. We can actually write them all down:

$$1122, 2211, 1212, 1221, 2112, 2121.$$

jonjoseph 2021-08-06 20:06:31

Each of these possible orders has 144 successful outcomes associated to it: there 12 ways to pick the 1s, and 12 ways to pick the 2s.

jonjoseph 2021-08-06 20:06:41

That means there are $6 \cdot 144$ ways to pick a quadruple in which we see two 1s and two 2s.

jonjoseph 2021-08-06 20:06:48

How many ways are there to pick a quadruple in which we see two 1s and two 3s?

jonjoseph 2021-08-06 20:07:24

It's still $6 \cdot 144$, of course. The logic is identical. In fact, the same goes for seeing two 1s and two 4s... or two 7s and two 9s.

jonjoseph 2021-08-06 20:07:35

OK. But how many possible pairs of numbers are there that we might see?

jonjoseph 2021-08-06 20:08:16

Hint: We have 10 numbers and we are choosing two of them.

jonjoseph 2021-08-06 20:08:55

There are $\binom{10}{2} = 45$. Here, we are just talking about the pair of numbers we see, not their order!

jonjoseph 2021-08-06 20:09:02

That means the total number of successful quadruples must be $45 \cdot 6 \cdot 144$, since for each pair of numbers, there are $6 \cdot 144$ successful quadruples corresponding to seeing two of one number, and two of the other.

jonjoseph 2021-08-06 20:09:19

That was a lot of work. Luckily, the total number of possible outcomes is easy. We're picking 4 slips without replacement and keeping track of order. How many ways are there to do that?

jonjoseph 2021-08-06 20:10:45

Yes, it's the same as last time, which is $40 \cdot 39 \cdot 38 \cdot 37$. That means that

$$q = \frac{45 \cdot 6 \cdot 144}{40 \cdot 39 \cdot 38 \cdot 37}.$$

jonjoseph 2021-08-06 20:10:49

All right, now to calculate q/p! We see that

$$\frac{q}{p} = \frac{\frac{45 \cdot 6 \cdot 144}{40 \cdot 39 \cdot 38 \cdot 37}}{\frac{240}{40 \cdot 39 \cdot 38 \cdot 37}} = \frac{45 \cdot 6 \cdot 144}{240}.$$

jonjoseph 2021-08-06 20:10:51

What does that simplify to?

jonjoseph 2021-08-06 20:12:13

That simplifies as $\dfrac{45 \cdot 6 \cdot 12^2}{12 \cdot 20} = \dfrac{45 \cdot 6 \cdot 12}{20} = 9 \cdot 6 \cdot 3 = 162$**. The answer is** (A).

jonjoseph 2021-08-06 20:12:23

ALGEBRAIC PROBABILITY

jonjoseph 2021-08-06 20:12:31

In some probability problems, we must use the algebraic properties of probability. For example, if we toss a coin and roll a die, then what is the probability that we get heads and we roll a number that is less than or equal to 2**?**

jonjoseph 2021-08-06 20:13:21

The probability that we get heads is $\frac{1}{2}$**, and the probability that we roll a number that is less than or equal to** 2 **is** $\frac{2}{6} = \frac{1}{3}$**, so the probability that both occur is** $\dfrac{1}{2} \cdot \dfrac{1}{3} = \dfrac{1}{6}$.

jonjoseph 2021-08-06 20:13:28

In this example, we are using the *independence* property of probability: if two events are independent, then the probability that both occur is the product of their probabilities.

jonjoseph 2021-08-06 20:13:49

A positive integer n **not exceeding** 100 **is chosen in such a way that if** $n \le 50$**, then the probability of choosing** n **is** p**, and if** $n > 50$**, then the probability of choosing** n **is** $3p$**. The probability that a perfect square is chosen is**

$\text{(A) } 0.05 \quad \text{(B) } 0.065 \quad \text{(C) } 0.08 \quad \text{(D) } 0.09 \quad \text{(E) } 0.1$

jonjoseph 2021-08-06 20:14:21

First of all, let's figure out what the problem is asking. We're picking a number from 1 **to** 100**, but the numbers are *not* equally likely. The probability of each number between** 1 **and** 50 **is** p**, and the probability of each number between** 51 **and** 100 **is** $3p$**.**

jonjoseph 2021-08-06 20:14:50

One way to model this process would be to put a bunch of slips of paper in a bag -- 1 **slip with each of the numbers** 1 **to** 50**, and** 3 **slips with each of the numbers between** 51 **and** 100**. That's a lot easier to visualize!**

jonjoseph 2021-08-06 20:15:16

Great! We want to find the probability that a perfect square is chosen. What are the positive perfect squares not exceeding 100**?**

jonjoseph 2021-08-06 20:16:36

They are $1, 4, 9, 16, 25, 36, 49, 64, 81$ **and** 100.

jonjoseph 2021-08-06 20:16:38

That means we have 7 **perfect squares between** 1 **and** 50**, and** 3 **perfect squares between** 51 **and** 100.

jonjoseph 2021-08-06 20:16:44

What's the probability that the number we pick is one of $1, 4, 9, 16, 25, 36$ **or** 49**?**

jonjoseph 2021-08-06 20:17:59

Each of these numbers has probability p**. The probabilities add up, so the probability of picking one of them is** $7p$.

jonjoseph 2021-08-06 20:18:21

Some of you have figured out p**. Excellent. I'll come back to that.**

jonjoseph 2021-08-06 20:18:26

What's the probability the number we pick is one of $64, 81$ **or** 100**?**

jonjoseph 2021-08-06 20:19:14

Each of these has probability $3p$**, so the probability of picking one of them is** $3 \cdot 3p = 9p$**. So what's the probability of choosing a square?**

jonjoseph 2021-08-06 20:20:17

Hint: The two outcomes are mutually exclusive. Either ≤ 50 OR > 50.

jonjoseph 2021-08-06 20:20:48

The probability of choosing a square is $7p + 9p = 16p$.

jonjoseph 2021-08-06 20:20:53

We need to find p.

jonjoseph 2021-08-06 20:20:57

Let's see how we could figure it out. What is the total probability of picking a 1 or a 2, or a 3, and so on until 100?

jonjoseph 2021-08-06 20:21:31

Both correct. What's p?

jonjoseph 2021-08-06 20:22:09

The probabilities for each of the numbers from 1 to 50 is p, and the probabilities for each of the numbers from 51 to 100 is $3p$. So the sum of all these probabilities is

$$50 \cdot p + 50 \cdot 3p = 50p + 150p = 200p.$$

jonjoseph 2021-08-06 20:22:27

We're always picking some number between 1 and 100! The above includes the whole universe of possibilities.

jonjoseph 2021-08-06 20:22:45

So this probability is also 1.

jonjoseph 2021-08-06 20:22:50

This means that we can write the equation $200p = 1$. So $p = \frac{1}{200}$.

jonjoseph 2021-08-06 20:22:52

So what is the probability of getting a perfect square?

jonjoseph 2021-08-06 20:23:53

The probability of choosing a perfect square is

$$16p = \frac{16}{200} = 0.08.$$

The answer is (C).

jonjoseph 2021-08-06 20:24:11

A poll shows that 70% of all voters approve of the mayor's work. On three separate occasions a pollster selects a voter at random. What is the probability that on exactly one of these three occasions the voter approves of the mayor's work?

$$\text{(A) } 0.063 \quad \text{(B) } 0.189 \quad \text{(C) } 0.233 \quad \text{(D) } 0.333 \quad \text{(E) } 0.441$$

jonjoseph 2021-08-06 20:24:28

In how many different ways can exactly one voter approve of the mayor's work?

jonjoseph 2021-08-06 20:25:32

It could be that the first voter approves and the other two disapprove, or the second voter approves and the other two disapprove, or the third voter approves and the other two disapprove.

jonjoseph 2021-08-06 20:25:54

Consider the case where the first voter approves the mayor's work, and the other two do not. What is the probability of this occurring?

jonjoseph 2021-08-06 20:27:18

If there is a 70% chance that a voter approves the mayor's work, then there is a 30% chance that a voter does not approve the

mayor's work.

jonjoseph 2021-08-06 20:27:23
Hence, the probability that the first voter approves the mayor's work and the other two do not is $0.7 \cdot 0.3 \cdot 0.3 = 0.063.$

jonjoseph 2021-08-06 20:27:26
The other two scenarios have the same probability of occurring. So what is the answer?

jonjoseph 2021-08-06 20:28:31
The probability that exactly one voter approves the mayor's work is $3 \cdot 0.063 = 0.189.$ The answer is (B).

jonjoseph 2021-08-06 20:28:38
TREE ANALYSIS

jonjoseph 2021-08-06 20:28:42
Many probability problems involve a process, such as a coin being flipped over and over again, or balls being drawn from an urn. One systematic way of dealing with such processes is constructing a tree, where the branches represent different possible outcomes.

jonjoseph 2021-08-06 20:28:57
Jacob uses the following procedure to write down a sequence of numbers. First he chooses the first term to be 6. To generate each succeeding term, he flips a fair coin. If it comes up heads, he doubles the previous term and subtracts 1. If it comes up tails, he takes half of the previous term and subtracts 1. What is the probability that the fourth term in Jacob's sequence is an integer?

$\text{(A) } \frac{1}{6} \quad \text{(B) } \frac{1}{3} \quad \text{(C) } \frac{1}{2} \quad \text{(D) } \frac{5}{8} \quad \text{(E) } \frac{3}{4}$

jonjoseph 2021-08-06 20:29:20
How can we use a tree to help solve this? How should we draw our tree? What should we put in it?

jonjoseph 2021-08-06 20:29:30
We'll work through Jacob's procedure step by step and keep track of all the possible outcomes in a tree. Each possibility in each step will be a new branch in the tree.

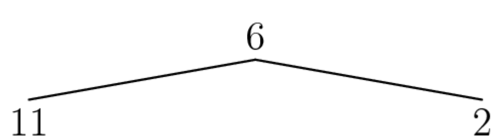
jonjoseph 2021-08-06 20:29:33
Jacob starts with a 6. What are the possible values of the second term?

jonjoseph 2021-08-06 20:30:15
Careful with your arithmetic.

jonjoseph 2021-08-06 20:30:52
The possible values of the second term are $2 \cdot 6 - 1 = 11$ and $\frac{6}{2} - 1 = 2.$

jonjoseph 2021-08-06 20:31:01
Accordingly, we draw a tree with a 6 at the top, branching to 11 and 2.

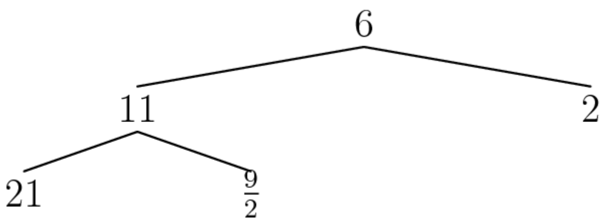
jonjoseph 2021-08-06 20:31:03



jonjoseph 2021-08-06 20:31:17
If 11 is the second term, then what are the possible values of the third term?

jonjoseph 2021-08-06 20:32:37
The possible values of the third term are $2 \cdot 11 - 1 = 21$ and $\frac{11}{2} - 1 = \frac{9}{2}$. We add these values to the tree.

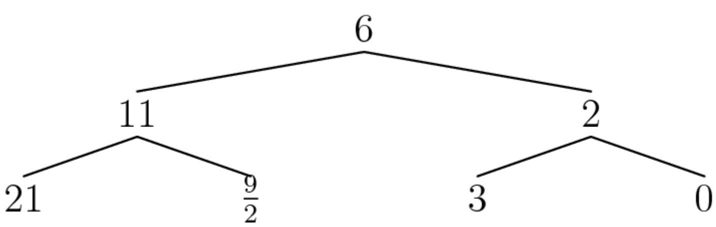
jonjoseph 2021-08-06 20:32:38



jonjoseph 2021-08-06 20:32:47

The possible values of the third term are $2 \cdot 2 - 1 = 3$ and $\frac{2}{2} - 1 = 0$.

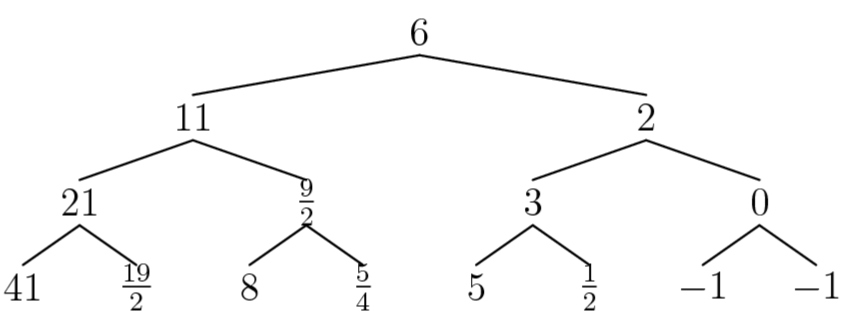
jonjoseph 2021-08-06 20:32:49



jonjoseph 2021-08-06 20:33:03

Then for each possible third term, we compute the possible fourth terms. When we fill them in, we get the following tree.

jonjoseph 2021-08-06 20:33:09



jonjoseph 2021-08-06 20:33:20

So what is the probability that Jacob's fourth term is an integer?

jonjoseph 2021-08-06 20:34:31

Of the eight possible ending points for the fourth term (all of which are equally likely), five are integers, so the probability is $\frac{5}{8}$. The answer is (D).

jonjoseph 2021-08-06 20:35:25

The one thing to notice is that the intermediate result of 9/2 became an integer again in the next row. So be sure to not let that type of result just drop.

jonjoseph 2021-08-06 20:35:46

CASEWORK

jonjoseph 2021-08-06 20:35:52

Since much of probability is based on counting (or at least the principles of counting), it should not be surprising that we must sometimes employ casework in probability.

jonjoseph 2021-08-06 20:36:09
A fair die is rolled six times. The probability of rolling at least a five at least five times is

$(\text{A})\ \frac{13}{729} \quad (\text{B})\ \frac{12}{729} \quad (\text{C})\ \frac{2}{729} \quad (\text{D})\ \frac{3}{729} \quad (\text{E})\ \text{none of these}$

jonjoseph 2021-08-06 20:36:20
A lot of this problem is in the wording! So let's make sure that we understand the problem.

jonjoseph 2021-08-06 20:36:30
We are interested in rolls that are at least a five, or in other words rolls that are fives or sixes, so let's call a five or a six a "high-roll," and everything else a "low-roll." What's the probability that we want to compute, in terms of high-rolls and low-rolls?

jonjoseph 2021-08-06 20:37:21
We want to compute the probability that we roll at least five high-rolls. In other words, we want to compute the probability that we roll five or six high-rolls. How can we compute this probability?

jonjoseph 2021-08-06 20:37:28
We can divide into the cases of rolling exactly five high-rolls, and exactly six high-rolls.

jonjoseph 2021-08-06 20:37:36
Let's start with exactly five high-rolls. First, in how many different ways can we roll exactly five high-rolls? (Here by "different ways", I mean "ways" in terms of high rolls and low rolls, not individual dice numbers.)

jonjoseph 2021-08-06 20:38:32
We can roll exactly five high-rolls in 6 ways. If there are exactly five high-rolls, then there is exactly one low-roll. We choose one of the six rolls to be the low-roll, and then all the other rolls must be high-rolls.

jonjoseph 2021-08-06 20:38:39
What is the probability of rolling a high-roll?

jonjoseph 2021-08-06 20:39:10
The probability of rolling a high-roll is $\frac{2}{6} = \frac{1}{3}$, so the probability of rolling a low-roll is $1 - \frac{1}{3} = \frac{2}{3}$.

jonjoseph 2021-08-06 20:39:24
So our first case has 6 different subcases, depending on the order of the high and low rolls. Let's start by computing the probability of just one of the subcases.

jonjoseph 2021-08-06 20:39:29
For example, what's the probability we roll two highs, then a low, and then three highs?

jonjoseph 2021-08-06 20:40:41
The probability of a high roll is $\dfrac{1}{3}$, and the probability of a low roll is $\dfrac{2}{3}$, so our overall probability is

$$\left(\frac{1}{3}\right)^5 \cdot \frac{2}{3} = \frac{2}{3^6} = \frac{2}{9^3} = \frac{2}{81 \cdot 9} = \frac{2}{729}.$$

jonjoseph 2021-08-06 20:40:46
That was just one of our six subcases, though. What's the total probability of getting one of our six possibilities with five high rolls?

jonjoseph 2021-08-06 20:41:30
Good.

jonjoseph 2021-08-06 20:41:32
The probability for each is $\dfrac{2}{729}$, and there are six of these, so our overall probability is $\dfrac{2}{729} \cdot 6 = \dfrac{12}{729}$. (We could simplify it

to $\dfrac{4}{243}$, but seeing that the answer options all have 729 in the denominator, it's probably better to keep it as $\dfrac{12}{729}$.)

jonjoseph 2021-08-06 20:41:50
Now for our second case. What is the probability of rolling exactly six high-rolls?

jonjoseph 2021-08-06 20:42:44
There is only one way to roll exactly six high-rolls, and it occurs with probability $\left(\frac{1}{3}\right)^6 = \frac{1}{729}$.

jonjoseph 2021-08-06 20:42:50
So what is the probability of rolling at least five high-rolls?

jonjoseph 2021-08-06 20:43:36
The probability of rolling at least five high rolls is $\dfrac{12}{729} + \dfrac{1}{729} = \dfrac{13}{729}$. The answer is (A).

jonjoseph 2021-08-06 20:44:05
GEOMETRIC PROBABILITY

jonjoseph 2021-08-06 20:44:13
All the problems we have seen so far involve quantities that we can count, like the number of rolls of a die. But what if we have a probability problem involving a continuous quantity, like choosing a number from an interval? In these problems, it can be helpful to take a geometric point of view.

jonjoseph 2021-08-06 20:44:36
Two real numbers are selected independently at random from the interval $[-20, 10]$. What is the probability that the product of those numbers is greater than zero?

$(\text{A})\ \frac{1}{9} \quad (\text{B})\ \frac{1}{3} \quad (\text{C})\ \frac{4}{9} \quad (\text{D})\ \frac{5}{9} \quad (\text{E})\ \frac{2}{3}$

jonjoseph 2021-08-06 20:44:49
Let the two real numbers be x and y. When is their product xy greater than zero?

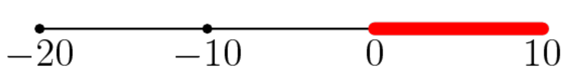
jonjoseph 2021-08-06 20:46:02
The product xy is greater than zero when x and y are both greater than zero, and when x and y are both less than zero.

jonjoseph 2021-08-06 20:46:29
(And neither is $= 0$.

jonjoseph 2021-08-06 20:46:34
What is the probability that a number chosen from the interval $[-20, 10]$ is greater than zero?

jonjoseph 2021-08-06 20:46:59
Good. Idea:

jonjoseph 2021-08-06 20:47:09



jonjoseph 2021-08-06 20:47:15
What is the probability that a number chosen from the interval $[-20, 10]$ is greater than zero?

jonjoseph 2021-08-06 20:48:09
The portion of the interval $[-20, 10]$ that is greater than 0 is $(0, 10]$. The length of the interval $(0, 10]$ is 10, and the length of the interval $[-20, 10]$ is 30, so the probability of choosing a number greater than zero is $\frac{10}{30} = \frac{1}{3}$.

jonjoseph 2021-08-06 20:48:35
Instead of counting successes and total, this time we measure the *lengths* of the successful interval and the total interval, and

divide those to get the probability.

jonjoseph 2021-08-06 20:48:46

It follows that the probability of choosing a number less than zero is $1-\frac{1}{3}=\frac{2}{3}$.

jonjoseph 2021-08-06 20:48:55

So what is the probability that both x and y are positive?

jonjoseph 2021-08-06 20:49:36

The probability that both x and y are positive is $\dfrac{1}{3}\cdot\dfrac{1}{3}=\dfrac{1}{9}$.

jonjoseph 2021-08-06 20:49:41

What is the probability that both x and y are negative?

jonjoseph 2021-08-06 20:50:29

The probability that both x and y are negative is $\dfrac{2}{3}\cdot\dfrac{2}{3}=\dfrac{4}{9}$.

jonjoseph 2021-08-06 20:50:59

Are these two possibilities Independent or Mutually Exclusive?

jonjoseph 2021-08-06 20:52:55

Good. It is either one or the other. They are about as dependent as two events can be. If x and y are >0 that means the possibility they are <0 is exactly 0! Meaning dependent.

jonjoseph 2021-08-06 20:53:12

Is that clear?

jonjoseph 2021-08-06 20:53:49

When two events are mutually exclusive what do we do with their respective probabilities?

jonjoseph 2021-08-06 20:54:36

Good. So what is the probability that the product xy is greater than zero?

jonjoseph 2021-08-06 20:55:29

The probability that the product xy is greater than zero is $\dfrac{1}{9}+\dfrac{4}{9}=\dfrac{5}{9}$. The answer is (D).

jonjoseph 2021-08-06 20:55:42

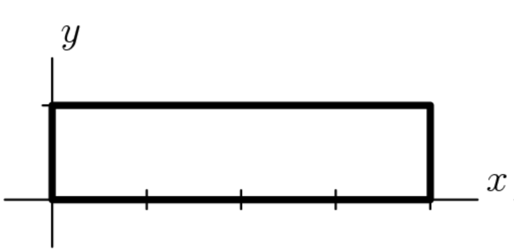
A point (x,y) is randomly picked from inside a rectangle with vertices $(0,0)$, $(4,0)$, $(4,1)$, and $(0,1)$. What is the probability that $x<y$?

$\mathrm{(A)}\ \frac{1}{8} \quad \mathrm{(B)}\ \frac{1}{4} \quad \mathrm{(C)}\ \frac{3}{8} \quad \mathrm{(D)}\ \frac{1}{2} \quad \mathrm{(E)}\ \frac{3}{4}$

jonjoseph 2021-08-06 20:55:54

Let's graph the rectangle with vertices $(0,0)$, $(4,0)$, $(4,1)$, and $(0,1)$.

jonjoseph 2021-08-06 20:55:56



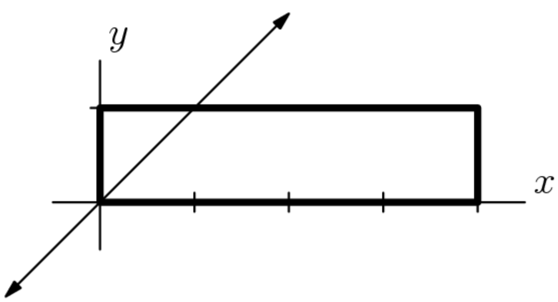
jonjoseph 2021-08-06 20:56:02
We choose a point (x, y) at random from this rectangle.

jonjoseph 2021-08-06 20:56:06
How do we compute the probability that $x < y$?

jonjoseph 2021-08-06 20:56:46
First, we determine the region of the rectangle where $x < y$. How can we find this region?

jonjoseph 2021-08-06 20:56:49
We can start by graphing the equation $x = y$.

jonjoseph 2021-08-06 20:56:51

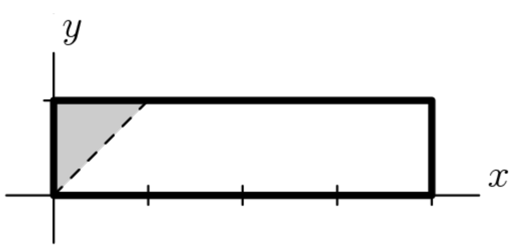


jonjoseph 2021-08-06 20:56:54
The region where $x < y$ lies on which side of this line?

jonjoseph 2021-08-06 20:58:07
If you're not sure try some experimental points. Probably one experiment will answer the question.

jonjoseph 2021-08-06 20:58:32
All points to the left of (or above) this line satisfy $x < y$.

jonjoseph 2021-08-06 20:58:36



jonjoseph 2021-08-06 20:58:42
So, the region of the rectangle where $x < y$ is the shaded triangle. What are the vertices of this triangle?

jonjoseph 2021-08-06 20:59:30
The vertices of the shaded triangle are $(0, 0)$, $(0, 1)$, and $(1, 1)$. So what is the area of the triangle?

jonjoseph 2021-08-06 20:59:48
The area of this triangle is $\frac{1}{2}$. Is this the probability we seek?

jonjoseph 2021-08-06 21:00:03
Good. You finish.

jonjoseph 2021-08-06 21:00:52
The denominator is the area of the entire rectangle, which is 4.

jonjoseph 2021-08-06 21:00:55

The probability that $x < y$ is $\frac{\frac{1}{2}}{4} = \frac{1}{8}$. The answer is (A).

jonjoseph 2021-08-06 21:01:01

Nicely done.

jonjoseph 2021-08-06 21:01:03

SUMMARY

jonjoseph 2021-08-06 21:01:07

When solving a probability problem, the first step should be to determine what kind of probability you are dealing with. In some cases, all you have to do is compute the number of "successful" outcomes, and divide by the total number of outcomes. However, in other cases, you may have to use other techniques, such as tree analysis or geometric probability.

jonjoseph 2021-08-06 21:01:16

Also, make sure you read the problem carefully. In probability, simply relying on intuition can easily lead to incorrect answers. Therefore, you should make sure you understand why your steps are correct. If you're counting something with multiple cases, make sure you aren't double-counting any overlap between the cases.

jonjoseph 2021-08-06 21:01:40

Well done. I thought these would be super hard but you smashed them. See you next week and stay safe!!

jonjoseph 2021-08-06 21:03:49

@ serenaliu: You add them. If Independent you multiply.

jonjoseph 2021-08-06 21:04:12

@ applepi3: Use student-services@aops.com