AMC 10 Problem Series (2804)

Jon Joseph

Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Homework

Homework: Lesson 6

You have completed **10** of **10** challenge problems.
Past Due **Jul 17**.

Readings

Lesson 6 Transcript: Fri, Jul 9

Challenge Problems

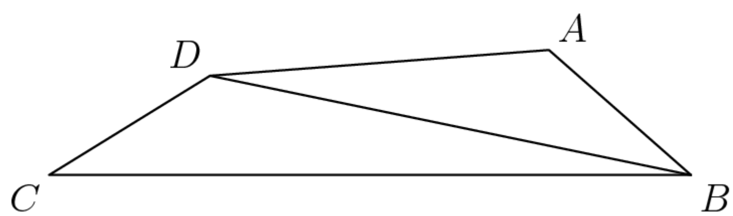
Total Score: 60 / 60

Problem 1 – Correct! – Score: 6 / 6 (2805)

Problem: Report Error

In quadrilateral $ABCD$, $AB = 5$, $BC = 17$, $CD = 5$, $DA = 9$, and BD is an integer. What is BD?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15



Solution:

By the triangle inequality on triangle ABD, $BD < AB + AD = 5 + 9 = 14$. By the triangle inequality on triangle BCD, $BD + CD > BC$, so $BD > BC - CD = 17 - 5 = 12$. The only integer that is less than 14 and greater than 12 is $\boxed{13}$. The answer is (C).

Your Response(s):

C

Problem 2 – Correct! – Score: 6 / 6 (2806)

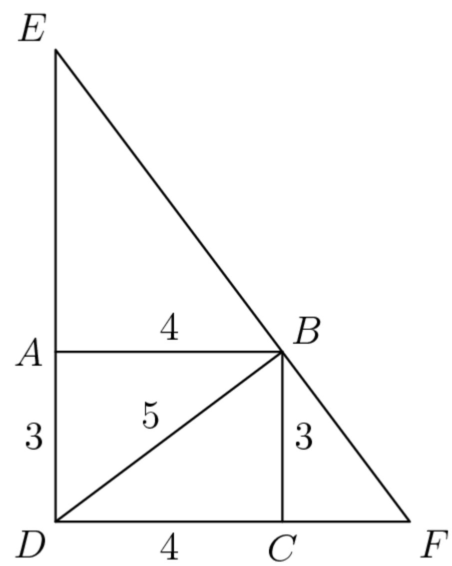
Problem: Report Error

Rectangle $ABCD$ has $AB = 4$ and $BC = 3$. Segment $\overline{EF}$ is constructed through B so that $\overline{EF} \perp \overline{DB}$, and A and C lie on $\overline{DE}$ and $\overline{DF}$, respectively. What is EF?

(A) 9 **(B)** 10 **(C)** $\dfrac{125}{12}$ **(D)** $\dfrac{103}{9}$ **(E)** 12

Solution:

Since $\angle BEA = 90^\circ - \angle ABE = \angle DBA$, right triangles BEA and DBA are similar. Hence, $BE/5 = 4/3$, so $BE = 20/3$.



Likewise, triangles BFC and DBC are similar. Hence, $BF/5 = 3/4$, so $BF = 15/4$. Then $EF = BE + BF = 20/3 + 15/4 = \boxed{125/12}$. The answer is (C).

Your Response(s):

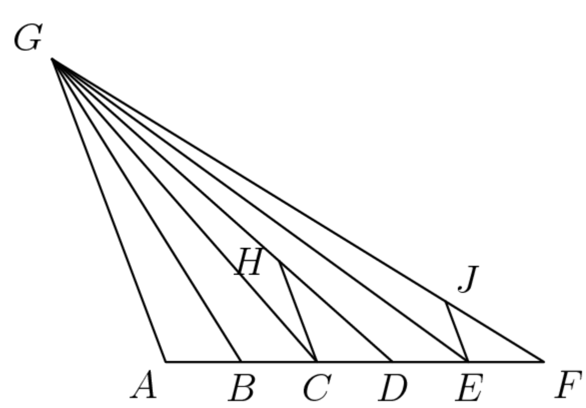
C

Problem 3 – Correct! – Score: 6 / 6 (2807)

Problem: Report Error

Points A, B, C, D, E, and F lie, in that order, on $\overline{AF}$, dividing it into five segments, each of length 1. Point G is not on line AF. Point H lies on $\overline{GD}$, and point J lies on $\overline{GF}$. The line segments $\overline{HC}$, $\overline{JE}$, and $\overline{AG}$ are parallel. Find HC/JE.

(A) 5/4 **(B)** 4/3 **(C)** 3/2 **(D)** 5/3 **(E)** 2



Solution:

Since AG and CH are parallel, triangles GAD and HCD are similar. Hence, $CH/AG = CD/AD = 1/3$.

Since AG and JE are parallel, triangles GAF and JEF are similar. Hence, $EJ/AG = EF/AF = 1/5$. Therefore, $CH/EJ = (CH/AG)/(EJ/AG) = (1/3)/(1/5) = \boxed{5/3}$. The answer is (D).

Your Response(s):

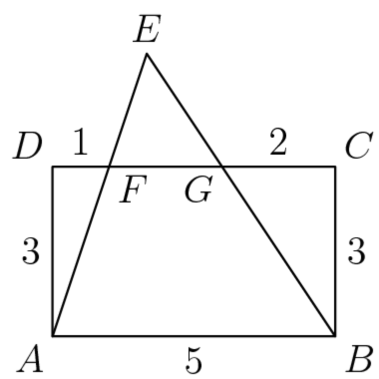
- D

Problem 4 – Correct! – Score: 6 / 6 (2808)

Problem: Report Error

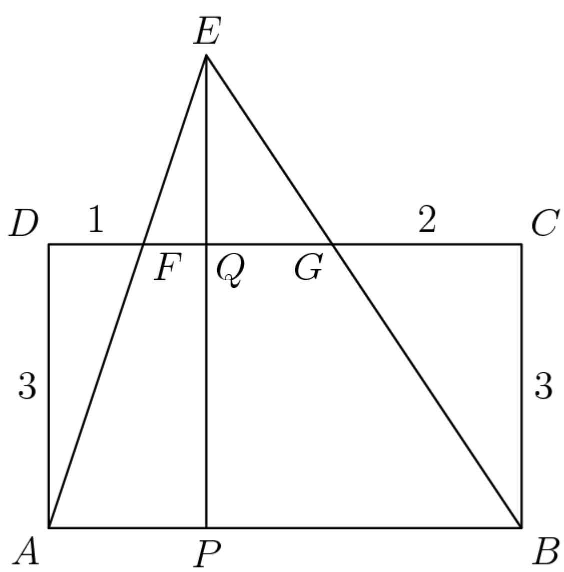
In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on $\overline{CD}$ so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. Find the area of triangle AEB.

(A) 10 **(B)** $\frac{21}{2}$ **(C)** 12 **(D)** $\frac{25}{2}$ **(E)** 15



Solution:

Let P and Q be the projections of E onto AB and FG, respectively.



We see that triangles EFG and EAB are similar, so $EQ/EP = FG/AB = 2/5$. But $EP = EQ + QP = EQ + 3$, so $EQ/(EQ + 3) = 2/5$. Solving for EQ, we find $EQ = 2$. Then

$EP = EQ + 3 = 2 + 3 = 5$, so the area of triangle AEB is $1/2 \cdot AB \cdot EP = 1/2 \cdot 5 \cdot 5 = \boxed{25/2}$. The answer is (D).

Your Response(s):

D

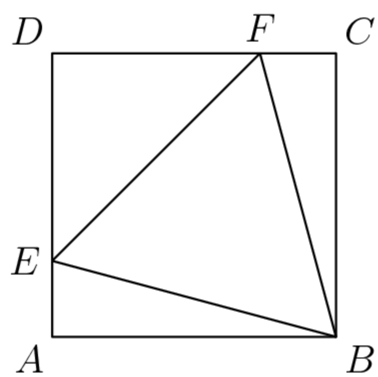
Problem 5 – Correct! – Score: 6 / 6 (2809)

Problem:

Report Error

Points E and F are located on square $ABCD$ so that triangle BEF is equilateral. What is the ratio of the area of triangle DEF to that of triangle ABE?

(A) $\frac{4}{3}$ **(B)** $\frac{3}{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** $1 + \sqrt{3}$



Solution:

First, we claim that $DE = DF$. Since AB and BC are sides of the same square, $AB = BC$. Since BE and BF are sides of the same equilateral triangle, $BE = BF$. Also, $\angle BAE = \angle BCF = 90^\circ$. Hence, triangles BAE and BCF are congruent, which means that $AE = CF$. Therefore, $DE = DF$.

Let $x = DE = DF$, so $EF = DE\sqrt{2} = x\sqrt{2}$. Since triangle BEF is equilateral, $BE = EF = x\sqrt{2}$.

We are computing the ratio of two areas, so we may assume that the side length of the square is 1. Then $AE = 1 - x$. By Pythagoras on right triangle ABE, $(1 - x)^2 + 1^2 = (x\sqrt{2})^2$, which simplifies to $x^2 + 2x - 2 = 0$.

The ratio of the area of triangle DEF to the area of triangle ABE is

$$\frac{[DEF]}{[ABE]} = \frac{1/2 \cdot DE \cdot DF}{1/2 \cdot AE \cdot AB} = \frac{x^2}{1 - x}.$$

From the equation $x^2 + 2x - 2 = 0$, $x^2 = 2 - 2x$. Therefore,

$$\frac{[DEF]}{[ABE]} = \frac{x^2}{1 - x} = \frac{2 - 2x}{1 - x} = \frac{2(1 - x)}{1 - x} = \boxed{2}.$$

The answer is (D).

Your Response(s):

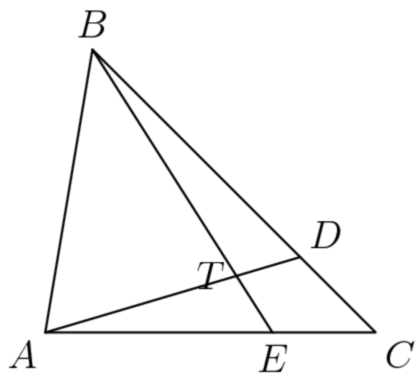
D

Problem 6 – Correct! – Score: 6 / 6 (2810)

Problem: Report Error

In triangle ABC points D and E lie on $\overline{BC}$ and $\overline{AC}$, respectively. If $\overline{AD}$ and $\overline{BE}$ intersect at T so that $AT/DT = 3$ and $BT/ET = 4$, what is CD/BD?

(A) $\frac{1}{8}$ **(B)** $\frac{2}{9}$ **(C)** $\frac{3}{10}$ **(D)** $\frac{4}{11}$ **(E)** $\frac{5}{12}$



Solution:

Let $[ABT] = 12k$. Then

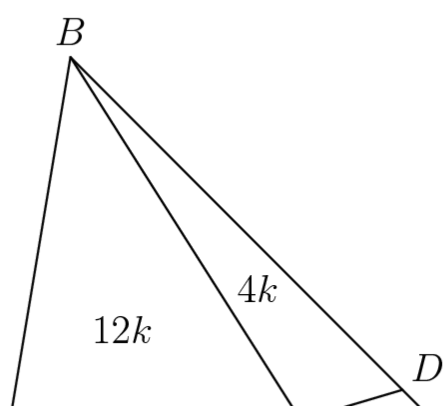
$$\frac{[ABT]}{[BDT]} = \frac{AT}{DT} = 3,$$

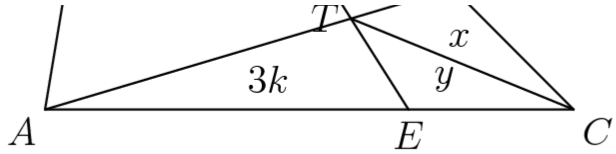
so $[BDT] = [ABT]/3 = 12k/3 = 4k$. Also,

$$\frac{[ABT]}{[AET]} = \frac{BT}{ET} = 4,$$

so $[AET] = [ABT]/4 = 12k/4 = 3k$.

Let $x = [CDT]$ and $y = [CET]$.





Then

$$\frac{[ACT]}{[CDT]} = \frac{AT}{DT} = 3,$$

which gives us the equation $(y + 3k)/x = 3$. Also,

$$\frac{[BCT]}{[CET]} = \frac{BT}{ET} = 4,$$

which gives us the equation $(x + 4k)/y = 4$. Hence, we have the system of equations

$$\begin{aligned} y + 3k &= 3x, \\ x + 4k &= 4y. \end{aligned}$$

Multiplying the first equation by 4, we get $4y + 12k = 12x$, so $4y = 12x - 12k$. Then by the second equation, $x + 4k = 12x - 12k$, so $11x = 16k$, or $x = 16k/11$. Therefore,

$$\frac{CD}{BD} = \frac{[CDT]}{[BDT]} = \frac{x}{4k} = \frac{16k/11}{4k} = \boxed{\frac{4}{11}}.$$

The answer is (D).

Your Response(s):

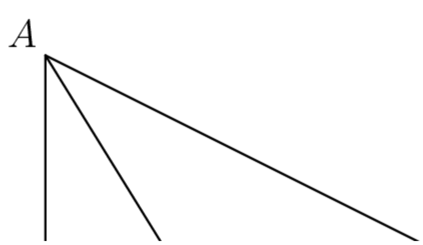
D

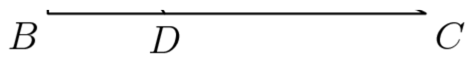
Problem 7 – Correct! – Score: 6 / 6 (2811)

Problem: Report Error

Triangle ABC has a right angle at B, $AB = 1$, and $BC = 2$. The bisector of $\angle BAC$ meets $\overline{BC}$ at D. What is BD?

(A) $\frac{\sqrt{3}-1}{2}$ **(B)** $\frac{\sqrt{5}-1}{2}$ **(C)** $\frac{\sqrt{5}+1}{2}$ **(D)** $\frac{\sqrt{6}+\sqrt{2}}{2}$ **(E)** $2\sqrt{3}-1$





Solution:
By Pythagoras, $AC = \sqrt{AB^2 + BC^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$. Let $x = BD$ and $y = CD$, so $x + y = BC = 2$.

By the angle bisector theorem,

$$\frac{x}{y} = \frac{AB}{AC} = \frac{1}{\sqrt{5}},$$

so $y = x\sqrt{5}$. Substituting into the equation $x + y = 2$, we get $x + x\sqrt{5} = 2$, so

$$x = \frac{2}{1+\sqrt{5}} = \frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} = \frac{2(1-\sqrt{5})}{-4} = \boxed{\frac{\sqrt{5}-1}{2}}.$$

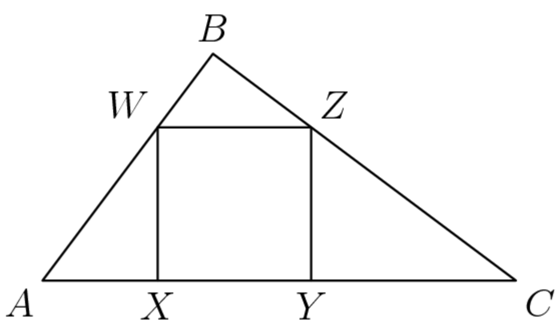
The answer is (B).

Your Response(s):
B

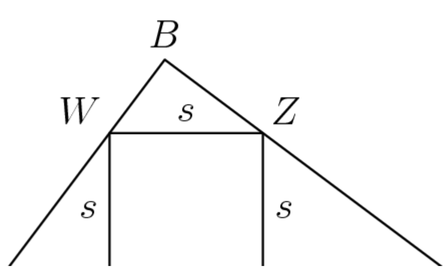
Problem 8 – Correct! – Score: 6 / 6 (2812)

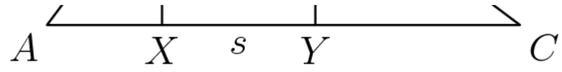
Problem: Report Error
Right triangle ABC has $AB = 3$, $BC = 4$, and $AC = 5$. Square $XYZW$ is inscribed in triangle ABC with X and Y on AC, W on AB, and Z on BC. What is the side length of the square?

(A) $\frac{3}{2}$ **(B)** $\frac{60}{37}$ **(C)** $\frac{12}{7}$ **(D)** $\frac{23}{13}$ **(E)** 2



Solution:
Let s be the side length of square $WXYZ$.





Triangles BWZ and BAC are similar, so $BW/3 = s/5$, which means $BW = 3s/5$. Triangles WAX and CAB are similar, so $AW/5 = s/4$, which means $AW = 5s/4$.

We see that $AW + BW = AB$, so $3s/5 + 5s/4 = 3$. Solving for s, we find $s = \boxed{60/37}$. The answer is (B).

Your Response(s):

B

Problem 9 – Correct! – Score: 6 / 6 (2813)

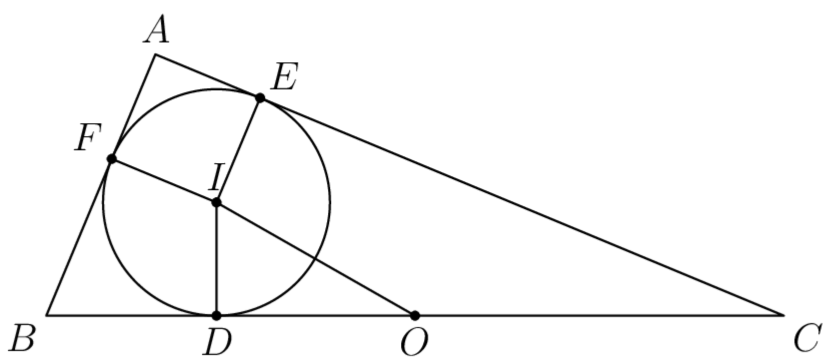
Problem: Report Error

A triangle with sides of 5, 12, and 13 has both an inscribed and a circumscribed circle. What is the distance between the centers of those circles?

(A) $\frac{3\sqrt{5}}{2}$ **(B)** $\frac{7}{2}$ **(C)** $\sqrt{15}$ **(D)** $\frac{\sqrt{65}}{2}$ **(E)** $\frac{9}{2}$

Solution:

Let A, B, and C be the vertices of the triangle so that $AB = 5$, $AC = 12$, and $BC = 13$. Let I and O be the incenter and circumcenter of triangle ABC, respectively. Let the incircle of triangle ABC be tangent to sides BC, AC, and AB at D, E, and F, respectively.



Since $\angle BAC = 90^\circ$, the circumcenter O of triangle ABC is the midpoint of hypotenuse BC.

Since AE and AF are tangents from A to the same circle, $AE = AF$. Let $x = AE = AF$. Similarly, let $y = BD = BF$ and $z = CD = CE$. Then $x + y = AF + BF = AB = 5$, $x + z = AE + CE = AC = 12$, $y + z = BD + CD = BC = 13$. Solving this system of equations, we find $x = 2$, $y = 3$, and $z = 10$. Then $DO = BO - BD = BC/2 - y = 13/2 - 3 = 7/2$.

The inradius r of triangle ABC is given by $r = K/s$, where K is the area of triangle ABC, and s is the semi-perimeter. We see that $K = [ABC] = 1/2 \cdot AB \cdot AC = 1/2 \cdot 5 \cdot 12 = 30$, and $s = (AB + AC + BC)/2 = (5 + 12 + 13)/2 = 15$, so $r = 30/15 = 2$.

Hence, by Pythagoras on right triangle IDO,

$$IO = \sqrt{ID^2 + DO^2} = \sqrt{2^2 + \left(\frac{7}{2}\right)^2} = \sqrt{4 + \frac{49}{4}} = \sqrt{\frac{65}{4}} = \boxed{\frac{\sqrt{65}}{2}}.$$

The answer is (D).

Your Response(s):

- D

Problem 10 – Correct! – Score: 6 / 6 (2814)

Problem: Report Error

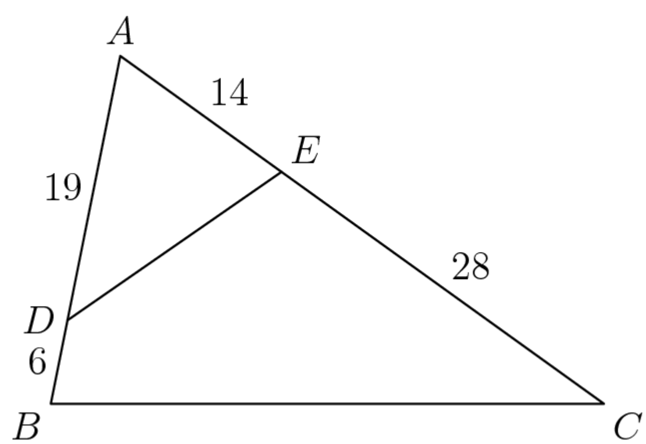
In triangle ABC we have $AB = 25$, $BC = 39$, and $AC = 42$. Points D and E are on $\overline{AB}$ and $\overline{AC}$ respectively, with $AD = 19$ and $AE = 14$. What is the ratio of the area of triangle ADE to the area of the quadrilateral $BCED$?

(A) $\frac{266}{1521}$ **(B)** $\frac{19}{75}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{19}{56}$ **(E)** 1

Solution:

We have that

$$\frac{[ADE]}{[ABC]} = \frac{AD}{AB} \cdot \frac{AE}{AC} = \frac{19}{25} \cdot \frac{14}{42} = \frac{19}{75}.$$



But $[BCED] = [ABC] - [ADE]$, so

$$\begin{aligned}\frac{[ADE]}{[BCED]} &= \frac{[ADE]}{[ABC] - [ADE]} \\ &= \frac{1}{[ABC]/[ADE] - 1} \\ &= \frac{1}{75/19 - 1}\end{aligned}$$

$$75/19 - 1$$

$$= \boxed{\frac{19}{56}}.$$

The answer is (D).

Your Response(s):

D

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