AMC 10 Problem Series (2804)

Jon Joseph

Friday
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7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Overview

Week 4 (Jun 25) Class Transcript - Functions and Polynomials

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jonjoseph 2021-06-25 19:30:38
And there it is!! Nice one @ BearandBear!!!

jonjoseph 2021-06-25 19:31:06
You sure did. Perfect timing. Claps all around

jonjoseph 2021-06-25 19:31:27
Should we do a little problem solving?

jonjoseph 2021-06-25 19:31:38
AMC 10 Problem Series
Week 4: Functions and Polynomials

jonjoseph 2021-06-25 19:31:43
Say hello to Pavle Martinovic (pavleto00) who will be helping us today!

jonjoseph 2021-06-25 19:31:46
Pavle joined AoPS in 2020, while searching for an adventure. He represented Serbia at the 2017, 2018 and 2019 International Mathematical Olympiads, winning a silver and 2 gold medals, respectively. He has also participated at Romanian Masters of Mathematics 2017, 2018 and 2019, winning one gold and 2 bronze medals. A native of Belgrade, Serbia, Pavle is earning a Bachelor's degree in mathematics and informatics from the University of Belgrade. His interests include playing football, volleyball, and all kinds of sports with a ball. He is currently participating in bridge tournaments and enjoys the game a lot.

pavleto00 2021-06-25 19:32:09
Hello! 🙂

jonjoseph 2021-06-25 19:32:35
Hah! Lots of quadratics tonight.

jonjoseph 2021-06-25 19:32:48
In today's class, we will look at problems involving functions, and in particular the class of functions known as polynomials. We commonly work with functions and function notation in mathematics, so it is important to have a good understanding of them.

jonjoseph 2021-06-25 19:33:00
FUNCTIONS

jonjoseph 2021-06-25 19:33:02
We start with a warm-up problem.

jonjoseph 2021-06-25 19:33:05
A function f has the property that $f(3x-1)=x^2+x+1$ for all real numbers x. What is $f(5)$?

$(\mathrm{A})\ 7 \quad (\mathrm{B})\ 13 \quad (\mathrm{C})\ 31 \quad (\mathrm{D})\ 111 \quad (\mathrm{E})\ 211$

jonjoseph 2021-06-25 19:33:22
Go ahead and try to solve it. Then we will go over together.

jonjoseph 2021-06-25 19:34:45
Nice. Lots of you. Let's go over it.

jonjoseph 2021-06-25 19:34:52
We want $f(5)$. So do we substitute $x = 5$?

jonjoseph 2021-06-25 19:35:23
We don't substitute $x = 5$, because the function is defined in terms of $f(3x-1)$, not $f(x)$.

jonjoseph 2021-06-25 19:35:27
What do we need to substitute for x?

jonjoseph 2021-06-25 19:36:22
We want to substitute the value of x that makes $3x - 1 = 5$, which is $x = 2$. So what is $f(5)$?

jonjoseph 2021-06-25 19:36:51
Setting $x = 2$, we get $f(5) = 2^2 + 2 + 1 = 7$. The answer is (A).

jonjoseph 2021-06-25 19:36:55
When you are working with a function, make sure you pay attention to the definition of the function. Not all functions are defined in terms of $f(x)$, as in the problem above. Thus, if you want to find the function at a certain value, this is not necessarily the value that you want to plug as the given variable.

jonjoseph 2021-06-25 19:37:08
Let

$$f(t) = \frac{t}{1-t},$$

where $t \neq 1$. If $y = f(x)$, then x can be expressed as:

$(\mathrm{A})\ f\left(\frac{1}{y}\right) \quad (\mathrm{B})\ -f(y) \quad (\mathrm{C})\ -f(-y) \quad (\mathrm{D})\ f(-y) \quad (\mathrm{E})\ f(y)$

jonjoseph 2021-06-25 19:37:28
Since $y = f(x)$, what equation can we write down?

jonjoseph 2021-06-25 19:38:20
We can rewrite $y = f(x)$ as

$$y = \frac{x}{1-x}.$$

jonjoseph 2021-06-25 19:38:26
What do we want to do with this equation?

jonjoseph 2021-06-25 19:39:33
The problem asks how x can be expressed as a function of y. So, let's solve for x. How can we do that?

jonjoseph 2021-06-25 19:40:04
Go ahead and solve for x. What do you find?

jonjoseph 2021-06-25 19:41:42
First, we multiply both sides by $1 - x$, to get $y - xy = x$.

jonjoseph 2021-06-25 19:42:07
Then we move all the terms involving x to one side of the equation, and move all other terms to the other side. This gives us

$y = xy + x$. Now a tricky step. What's next?

jonjoseph 2021-06-25 19:43:19
We factor $xy + x$ as $x(1 + y)$ and then divide both sides by $1 + y$ to see that

$$x = \frac{y}{1+y}.$$

jonjoseph 2021-06-25 19:43:34
It's not obvious which answer choice this is equivalent to! Let's expand each one to see what we have.

jonjoseph 2021-06-25 19:43:41
Let's start with (A). What is $f(1/y)$?

jonjoseph 2021-06-25 19:44:06
Hint: Simplify as much as you can.

jonjoseph 2021-06-25 19:45:23
We see that

$$f\left(\frac{1}{y}\right) = \frac{1/y}{1-1/y} = \frac{1}{y-1}.$$

This is not equal to $\dfrac{y}{1+y}$, so this answer choice is incorrect.

jonjoseph 2021-06-25 19:45:25
Next up is (B). What is $-f(y)$?

jonjoseph 2021-06-25 19:46:26
Careful. Many of you have an extra minus sign.

jonjoseph 2021-06-25 19:47:44
We see that

$$-f(y) = -\frac{y}{1-y} = \frac{y}{y-1}.$$

This is not equal to $\dfrac{y}{1+y}$, so this answer choice is incorrect.

jonjoseph 2021-06-25 19:48:18
The algebra was a little tricky. First just find $f(y)$ then apply the minus sign.

jonjoseph 2021-06-25 19:48:21
How about (C)? What is $-f(-y)$?

jonjoseph 2021-06-25 19:50:00
We see that

$$-f(-y) = -\frac{-y}{1-(-y)} = \frac{y}{1+y}.$$

This is equal to our expression for x!

jonjoseph 2021-06-25 19:50:08
Therefore, the correct answer is (C).

jonjoseph 2021-06-25 19:50:52
Again, the algebra (all the signs really), through many of your answers off. You may want to come back later and see if you can find the wayward minus.

jonjoseph 2021-06-25 19:51:04

***threw**

jonjoseph 2021-06-25 19:51:13
You may be wondering if there's a faster way to do this problem. There isn't really any way to avoid evaluating the different options and comparing them to $\dfrac{y}{1+y}$, but what might suggest to us that we try options (C) or (D) first?

jonjoseph 2021-06-25 19:52:17
Yes. Nice.

jonjoseph 2021-06-25 19:52:20
Notice that the denominator of $f(t)$ is $1-t$. We want an expression with $1+y$ in the denominator, so of the given answer choices, it's likely that the correct answer involves $f(-y)$.

jonjoseph 2021-06-25 19:52:30
The function f from the integers to the integers is defined as follows:

$$f(n) = \begin{cases} n+3 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

Suppose k is odd and $f(f(f(k))) = 27$. What is the sum of the digits of k?

$\text{(A) } 3 \quad \text{(B) } 6 \quad \text{(C) } 9 \quad \text{(D) } 12 \quad \text{(E) } 15$

jonjoseph 2021-06-25 19:52:50
Here, we have a function that is called a *piecewise defined* function, because the formula for $f(n)$ is different for different values of n. To see which formula applies, we must check the conditions.

jonjoseph 2021-06-25 19:53:05
To get to $f(f(f(k)))$, we apply f one step at a time.

jonjoseph 2021-06-25 19:53:09
What is $f(k)$?

jonjoseph 2021-06-25 19:54:24
Since k is odd, $f(k) = k+3$.

jonjoseph 2021-06-25 19:54:49
Don't miss the given that k starts life as odd. Or you'll really be stuck.

jonjoseph 2021-06-25 19:55:03
Then what is $f(f(k))$?

jonjoseph 2021-06-25 19:55:47
Hint: k is odd. So what must $k+3$ be?

jonjoseph 2021-06-25 19:56:24
To evaluate $f(f(k))$, we must check if $f(k) = k+3$ is even or odd. Which is it?

jonjoseph 2021-06-25 19:56:52
Since k is odd, $f(k) = k+3$ is even. So $f(f(k)) = f(k+3) = \dfrac{k+3}{2}$.

jonjoseph 2021-06-25 19:56:56
We're almost there. We just need to apply f one more time.

jonjoseph 2021-06-25 19:56:58
Since $f(f(k)) = \dfrac{k+3}{2}$, we can write

$$f(f(f(k))) = f\left(\frac{k+3}{2}\right).$$

jonjoseph 2021-06-25 19:57:02
What is this equal to, in terms of k?

jonjoseph 2021-06-25 19:57:51
We can't say, because we don't know whether $\frac{k+3}{2}$ is even or odd. So what do we do?

jonjoseph 2021-06-25 19:57:56
We could do casework depending on whether $\frac{k+3}{2}$ is odd or even; however this could double the amount of work we need to do.

jonjoseph 2021-06-25 19:57:59
Problem solving tip: is there any information we haven't used yet?

jonjoseph 2021-06-25 19:58:53
We're told that $f(f(f(k))) = 27$, which tells us that

$$f\left(\frac{k+3}{2}\right) = 27.$$

How does this help us?

jonjoseph 2021-06-25 20:00:44
Look at the definition of our function. For odd n, is $f(n)$ even or odd?

jonjoseph 2021-06-25 20:01:16
It's even. Since 27 is odd, it follows that the above equation is impossible if $\frac{k+3}{2}$ is odd. Therefore, $\frac{k+3}{2}$ must be even. This means that

$$f\left(\frac{k+3}{2}\right) = \frac{k+3}{4}.$$

jonjoseph 2021-06-25 20:01:31
What's our answer?

jonjoseph 2021-06-25 20:02:00
Hint: Be sure to answer the question.

jonjoseph 2021-06-25 20:03:09
We can now straightforwardly solve our equation. We have that $\frac{k+3}{4} = 27$. Solving, we get that $k = 105$.

jonjoseph 2021-06-25 20:03:15
We want the sum of the digits of $k = 105$, which is $1 + 0 + 5 = 6$. The answer is (B).

jonjoseph 2021-06-25 20:03:59
Yep. AMC always does this. They want just a little more. That's why it's important to really read the problem.

jonjoseph 2021-06-25 20:04:11
Annoying but that's the game.

jonjoseph 2021-06-25 20:04:55
There is a very special class of functions known as polynomials, which we will now look at.

jonjoseph 2021-06-25 20:05:04
POLYNOMIALS

jonjoseph 2021-06-25 20:05:10
To discuss polynomials rigorously, we must first introduce some notation and terminology.

jonjoseph 2021-06-25 20:05:41
A polynomial in a single variable is any expression of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_1$, and a_0 are constants and x is a variable.

jonjoseph 2021-06-25 20:06:11
For example, $7x^4 - x + \sqrt{3}$ is a polynomial.

jonjoseph 2021-06-25 20:06:17
What about $x^2 - \frac{2}{\pi}x$?

jonjoseph 2021-06-25 20:07:17
Yes. This is still a polynomial because a polynomial's coefficients are allowed to be irrational.

jonjoseph 2021-06-25 20:07:30
Is $3x^2 + \frac{1}{x}$ a polynomial?

jonjoseph 2021-06-25 20:08:41
No, it's not. This is because only non-negative integer exponents are allowed in polynomials.

jonjoseph 2021-06-25 20:09:31
Is this clear? All exponents on the variable must be integers and ≥ 0.

jonjoseph 2021-06-25 20:09:53
Or it is NOT a polynomial.

jonjoseph 2021-06-25 20:10:11
For each term

$$a_kx^k,$$

we say that the constant a_k is the coefficient for that term, and we say that k is the degree (of that term).

jonjoseph 2021-06-25 20:10:34
A term a_kx^k is said to be "nonzero" when its coefficient a_k is nonzero.

jonjoseph 2021-06-25 20:10:40
The degree of a polynomial overall is, by definition, the largest degree that appears among all the nonzero terms that make up the polynomial.

jonjoseph 2021-06-25 20:10:58
For example, the polynomial $7x^4 - x + \sqrt{3}$ has degree 4 (because the nonzero term with largest degree is the term $7x^4$, which has degree equal to 4).

jonjoseph 2021-06-25 20:11:15
What is the degree of the polynomial $x^2 - \frac{2}{\pi}x$?

jonjoseph 2021-06-25 20:12:03
This polynomial has degree 2 because the nonzero term with largest degree is the term x^2, which has degree equal to 2.

jonjoseph 2021-06-25 20:12:06
If a polynomial has degree n, then we call the term a_nx^n the leading term and we call its coefficient a_n the leading coefficient.

jonjoseph 2021-06-25 20:12:19
We call a_0 either the constant term or the constant coefficient, depending on which aspect of it we want to emphasize.

jonjoseph 2021-06-25 20:12:51
(That is the variable is $x^0 = 1$)

jonjoseph 2021-06-25 20:13:00
For example, the leading coefficient of $7x^4 - x + \sqrt{3}$ is 7, and the constant coefficient is $\sqrt{3}$.

jonjoseph 2021-06-25 20:13:11
One reason why polynomials are a useful class of functions is that it's relatively easy to check when two polynomials are equal.

jonjoseph 2021-06-25 20:13:24
One of the most common methods, which we'll use several times today, is that two polynomials are equal at every point if and only if they have the same degree and the same coefficients.

jonjoseph 2021-06-25 20:13:42
If you want to learn more, AoPS proves this and other polynomial theorems in the Intermediate Algebra book and course.

jonjoseph 2021-06-25 20:13:51
For us, the practical use of this theorem is as follows. Given two polynomials

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = b_nx^n + b_{n-1}x^{n-1} + \cdots + b_1x + b_0$$

which we know to be equal for all x, but whose coefficients we don't know yet, we can solve for the coefficients by setting up a system of equations

$$\begin{aligned} a_n &= b_n \\ a_{n-1} &= b_{n-1} \\ &\vdots \\ a_0 &= b_0. \end{aligned}$$

jonjoseph 2021-06-25 20:14:22
Let's see how this works with a warm-up problem.

jonjoseph 2021-06-25 20:14:33
Suppose that $f(x+3) = 3x^2 + 7x + 4$ and $f(x) = ax^2 + bx + c$. What is $a + b + c$?

$\text{(A)} \ -1 \quad \text{(B)} \ 0 \quad \text{(C)} \ 1 \quad \text{(D)} \ 2 \quad \text{(E)} \ 3$

jonjoseph 2021-06-25 20:14:44
We are given a polynomial in two different forms, namely $f(x)$ and $f(x+3)$. How can we use these forms to find a, b, and c?

jonjoseph 2021-06-25 20:17:20
Okay. You went in a little different direction than I did but your ideas will work.

jonjoseph 2021-06-25 20:17:30
Here is what I did:

jonjoseph 2021-06-25 20:17:33
We can substitute $x + 3$ into $f(x) = ax^2 + bx + c$.

jonjoseph 2021-06-25 20:17:51
(I used the other equation in the same way you proposed.)

jonjoseph 2021-06-25 20:17:56

This gives us

$$f(x+3) = a(x+3)^2 + b(x+3) + c.$$

jonjoseph 2021-06-25 20:18:29

Expanding, we get

$$\begin{aligned} f(x+3) &= a(x^2+6x+9) + b(x+3) + c \\ &= ax^2 + 6ax + 9a + bx + 3b + c. \end{aligned}$$

jonjoseph 2021-06-25 20:18:40

Collecting like terms, we get

$$f(x+3) = ax^2 + (6a+b)x + 9a + 3b + c.$$

jonjoseph 2021-06-25 20:18:51

What can we do from here?

jonjoseph 2021-06-25 20:19:20

Hint: When are two polynomials equal?

jonjoseph 2021-06-25 20:19:56

We also know that $f(x+3) = 3x^2 + 7x + 4$. So, we can write

$$ax^2 + (6a+b)x + 9a + 3b + c = 3x^2 + 7x + 4.$$

What equations can we write?

jonjoseph 2021-06-25 20:21:10

Equating the coefficients of the powers of x, we get

$$\begin{aligned} a &= 3, \\ 6a + b &= 7, \\ 9a + 3b + c &= 4. \end{aligned}$$

jonjoseph 2021-06-25 20:21:14

This tells us a right off the bat. What is b?

jonjoseph 2021-06-25 20:21:58

Substituting $a = 3$ into the second equation, we have $18 + b = 7$. So, $b = 7 - 18 = -11$. What is c?

jonjoseph 2021-06-25 20:23:03

Substituting $a = 3$ and $b = -11$ into the third equation, we have $27 - 33 + c = 4$. So, $c = 4 + 6 = 10$. So what is $a + b + c$?

jonjoseph 2021-06-25 20:23:51

We see that $a + b + c = 3 + (-11) + 10 = 2$. The answer is (D).

jonjoseph 2021-06-25 20:24:12

Here is something to remember on your way to being future IMO winners.

jonjoseph 2021-06-25 20:24:58

When you have a polynomial $f(x) = ax^2 + bx + c$ what is $f(1)$?

jonjoseph 2021-06-25 20:25:45
Right. So there is a shortcut lurking here.

jonjoseph 2021-06-25 20:25:52
We are given that $f(x+3) = 3x^2 + 7x + 4$, so we can find $f(1)$ by setting $x = -2$.

jonjoseph 2021-06-25 20:26:01
What do you find?

jonjoseph 2021-06-25 20:27:13
Setting $x = -2$, we get $f(1) = 3(-2)^2 + 7(-2) + 4 = 2$, and, BAM, we're done!

jonjoseph 2021-06-25 20:27:40
In general, if $p(x)$ is a polynomial, then the sum of the coefficients is equal to $p(1)$.

jonjoseph 2021-06-25 20:28:16
Funny. Nope. Not this time!

jonjoseph 2021-06-25 20:28:20
If the function f defined by

$$f(x) = \frac{cx}{2x+3},$$

for all $x \neq -\frac{3}{2}$, where c is a constant, satisfies $f(f(x)) = x$ for all real numbers where $f(f(x))$ is defined, then c is

$\text{(A)} \ -3 \quad \text{(B)} \ -\frac{3}{2} \quad \text{(C)} \ \frac{3}{2} \quad \text{(D)} \ 3 \quad \text{(E)}$ Not uniquely determined by the given information

jonjoseph 2021-06-25 20:28:42
How could we start this problem?

jonjoseph 2021-06-25 20:29:22
Hint: What equation must we build?

jonjoseph 2021-06-25 20:29:46
We can use the definition of $f(x)$ to find the formula for $f(f(x))$.

jonjoseph 2021-06-25 20:30:21
With careful algebra find $f(f(x))$. I said careful algebra.

jonjoseph 2021-06-25 20:30:53
It does simplify nicely. I'll give you a moment.

jonjoseph 2021-06-25 20:32:52
It is a bit of mess to start. Let's go over it.

jonjoseph 2021-06-25 20:32:56
Using the definition of $f(x)$, we find

$$f(f(x)) = f\left(\frac{cx}{2x+3}\right) = \frac{c \cdot \dfrac{cx}{2x+3}}{2 \cdot \dfrac{cx}{2x+3} + 3}.$$

jonjoseph 2021-06-25 20:33:07
Let's clean this up a bit. Multiplying the numerator and denominator by $2x+3$, we get

$$f(f(x)) = \frac{c^2 x}{2cx + 3(2x+3)} = \frac{c^2 x}{(2c+6)x + 9}.$$

jonjoseph 2021-06-25 20:33:29

What is this equal to?

jonjoseph 2021-06-25 20:33:53
We're told that $f(f(x)) = x$, so we can write

$$\frac{c^2x}{(2c+6)x+9} = x.$$

jonjoseph 2021-06-25 20:33:56
Multiplying both sides by $(2c+6)x+9$, we get

$$c^2x = (2c+6)x^2 + 9x.$$

jonjoseph 2021-06-25 20:34:01
Just as a reminder, we want this to be true as an equality of functions, so it needs to hold for all x (except maybe $x = -3/2$).

jonjoseph 2021-06-25 20:34:05
How can we solve for c?

jonjoseph 2021-06-25 20:34:33
Hint: True for *all* x!!!!

jonjoseph 2021-06-25 20:35:53
In order for these polynomials to be equal for all x, we need the corresponding coefficients to be equal.

jonjoseph 2021-06-25 20:36:44
Yes. That will work. One warning however, it will generate an extraneous solution so I'm going to go that way.

jonjoseph 2021-06-25 20:37:08
***I'm NOT going that way**

jonjoseph 2021-06-25 20:37:12
The coefficient of x^2 on the right is $2c+6$. What is the coefficient of x^2 on the left?

jonjoseph 2021-06-25 20:38:48
Be careful not to confuse c^2 with x^2!!!

jonjoseph 2021-06-25 20:38:58
There is a hidden $0x^2$ term on the left hand side, so the coefficient is 0.

jonjoseph 2021-06-25 20:39:05
We can write the following system of equations:

$$\begin{aligned} 0 &= 2c+6 \\ c^2 &= 9. \end{aligned}$$

jonjoseph 2021-06-25 20:39:16
So what is the constant c?

jonjoseph 2021-06-25 20:39:54
Hint: If you're worried about ± 3 be sure to check both!!!

jonjoseph 2021-06-25 20:40:54
We see that the constant c must be -3. The answer is (A).

jonjoseph 2021-06-25 20:41:20
And our favorite:

jonjoseph 2021-06-25 20:41:23

Quadratic Polynomials

jonjoseph 2021-06-25 20:41:32

We'll now take a close look at quadratic polynomials.

jonjoseph 2021-06-25 20:41:38

When we deal with quadratic polynomials, it is often because we want to find the solutions. When possible, we factor. What if we can't factor?

jonjoseph 2021-06-25 20:42:19

Then we can use the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This is one of the few formulas in math you should really memorize. It's not too hard to derive via completing the square, but you don't want to do that over and over again!

jonjoseph 2021-06-25 20:42:22

One important part of the quadratic formula is the value

$$b^2 - 4ac,$$

which is called the discriminant. The discriminant is important because it tells us about the nature of the roots of a quadratic equation.

jonjoseph 2021-06-25 20:42:41

If the discriminant is positive, then the roots are real and distinct.

jonjoseph 2021-06-25 20:42:46

If the discriminant is zero, then the roots are real and equal. (We sometimes refer to this root as a double root.)

jonjoseph 2021-06-25 20:42:56

And if the discriminant is negative, then the roots are nonreal and distinct.

jonjoseph 2021-06-25 20:43:15

The thing to remember is that the discriminant contains important information.

jonjoseph 2021-06-25 20:43:27

There are two values of a for which the equation $4x^2 + ax + 8x + 9 = 0$ has only one solution for x. What is the sum of those values of a?

$(\text{A})\ -16 \quad (\text{B})\ -8 \quad (\text{C})\ 0 \quad (\text{D})\ 8 \quad (\text{E})\ 20$

jonjoseph 2021-06-25 20:43:38

What should we do first with the given equation?

jonjoseph 2021-06-25 20:44:14

First, let's group terms with the same powers of x together:

$$4x^2 + (a + 8)x + 9 = 0.$$

jonjoseph 2021-06-25 20:44:18

(We have written it this way so that the coefficient of x is clear. If you need to, you should always re-write your quadratic so that the coefficients are clear.)

jonjoseph 2021-06-25 20:44:27

When does a quadratic have only one solution?

jonjoseph 2021-06-25 20:45:12
A quadratic has only one solution if it its discriminant is 0. What is the discriminant of this quadratic equation?

jonjoseph 2021-06-25 20:45:35
Hint: Do not expand the $(a+8)^2$.

jonjoseph 2021-06-25 20:46:39
The discriminant of this quadratic equation is

$$(a+8)^2 - 4 \cdot 4 \cdot 9 = (a+8)^2 - 144,$$

so we want to solve $(a+8)^2 = 144$.

jonjoseph 2021-06-25 20:46:54
Solve for a. What do you find?

jonjoseph 2021-06-25 20:48:01
Taking the square root of both sides, we get

$$a + 8 = \pm 12.$$

(Don't forget about the negative square root!)

jonjoseph 2021-06-25 20:48:09
The possible values of a are $12 - 8 = 4$ and $-12 - 8 = -20$. Their sum is $4 - 20 = -16$. The answer is (A).

jonjoseph 2021-06-25 20:48:29
Vieta's Formulas

jonjoseph 2021-06-25 20:48:37
We know how to solve quadratic equations, but in some problems, we can still obtain relevant information about the roots without having to solve for them.

jonjoseph 2021-06-25 20:48:45
To see how, let r and s be the roots of the equation $ax^2 + bx + c = 0$.

jonjoseph 2021-06-25 20:48:54
The roots of the equation $(x-r)(x-s) = 0$ are also r and s. Does this mean that $ax^2 + bx + c$ and $(x-r)(x-s)$ are the same polynomial?

jonjoseph 2021-06-25 20:49:17
Remember, two polynomials are only equal if they have the exact same coefficients.

jonjoseph 2021-06-25 20:49:55
No, $ax^2 + bx + c$ and $(x-r)(x-s)$ may not be the same polynomial. The easiest way to see this is to compare the coefficients of x^2. The coefficient of x^2 in $ax^2 + bx + c$ is a, and the coefficient of x^2 in $(x-r)(x-s)$ is 1.

jonjoseph 2021-06-25 20:50:15
So how can we adjust the polynomials so that they have the same coefficient of x^2?

jonjoseph 2021-06-25 20:51:04
We multiply the polynomial $(x-r)(x-s)$ by a, to get

$$a(x-r)(x-s).$$

Then the coefficients of x^2 match and the roots are still r and s.

jonjoseph 2021-06-25 20:51:09
It turns out that a polynomial is uniquely determined by its roots (listed with multiplicity) once you correct the leading coefficient like this.

jonjoseph 2021-06-25 20:51:16
Expanding, we get

$$a(x-r)(x-s) = ax^2 - a(r+s)x + ars.$$

jonjoseph 2021-06-25 20:51:26
But this is the same polynomial as $ax^2 + bx + c$, so what equations can we write down?

jonjoseph 2021-06-25 20:52:45
Hint: Don't lose a minus sign!!

jonjoseph 2021-06-25 20:53:22
Equating the coefficients of the powers of x, we get

$$\begin{aligned} -a(r+s) &= b, \\ ars &= c. \end{aligned}$$

jonjoseph 2021-06-25 20:53:23
It follows that

$$\begin{aligned} r+s &= -\frac{b}{a}, \\ rs &= \frac{c}{a}. \end{aligned}$$

jonjoseph 2021-06-25 20:53:31
Hence, we can find the sum and product of the roots of a quadratic equation just by reading off the coefficients.

jonjoseph 2021-06-25 20:53:35
For example, what are the sum and product of the roots of $x^2 - 5x + 6 = 0$?

jonjoseph 2021-06-25 20:54:26
The sum of the roots is 5. (Always remember the minus sign when computing the sum of the roots.) The product of the roots is 6.

jonjoseph 2021-06-25 20:54:30
What are the sum and product of the roots of $2x^2 + 7x - 15 = 0$?

jonjoseph 2021-06-25 20:55:38
Good

jonjoseph 2021-06-25 20:55:39
The sum of the roots is $-7/2$, and the product of the roots is $-15/2$. (Always remember to divide by the coefficient of x^2.)

jonjoseph 2021-06-25 20:55:42
These formulas linking the roots and coefficients of a polynomial are known as Vieta's formulas.

jonjoseph 2021-06-25 20:55:49
One big benefit of Vieta's formulas is that they allow us to do things with roots without solving for them first. Let's see this in some problems.

jonjoseph 2021-06-25 20:55:54
Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + (1/b)$ and $b + (1/a)$ are the roots of the equation $x^2 - px + q = 0$. What is q?

$(\text{A})\ \frac{5}{2} \quad (\text{B})\ \frac{7}{2} \quad (\text{C})\ 4 \quad (\text{D})\ \frac{9}{2} \quad (\text{E})\ 8$

jonjoseph 2021-06-25 20:56:17
We are given that a and b are the roots of $x^2 - mx + 2 = 0$. So what equations can we write down?

jonjoseph 2021-06-25 20:57:04

By Vieta's formulas, $a+b=m$ and $ab=2$.

jonjoseph 2021-06-25 20:57:09
We are also given that $a+1/b$ and $b+1/a$ are the roots of x^2-px+q. So what is q equal to?

jonjoseph 2021-06-25 20:58:05
By Vieta's formulas,

$$q=\left(a+\frac{1}{b}\right)\left(b+\frac{1}{a}\right).$$

jonjoseph 2021-06-25 20:58:16
I distribute for you:

jonjoseph 2021-06-25 20:58:19
Distributing, we have

$$\begin{aligned} q &= (ab)+\left(a\cdot\frac{1}{a}\right)+\left(b\cdot\frac{1}{b}\right)+\left(\frac{1}{a}\cdot\frac{1}{b}\right) \\ &= ab+1+1+\frac{1}{ab} \\ &= 2+ab+\frac{1}{ab}. \end{aligned}$$

jonjoseph 2021-06-25 20:58:25
Can you finish?

jonjoseph 2021-06-25 20:59:14
Since $ab=2$, $q=2+2+\frac{1}{2}=\frac{9}{2}$. The answer is (D).

jonjoseph 2021-06-25 20:59:30
I want to mention one last thing about Vieta.

jonjoseph 2021-06-25 20:59:40
In the past couple problems, we've used Vieta's formulas for quadratic polynomials.

jonjoseph 2021-06-25 20:59:44
There are also versions of Vieta's formulas for higher degree polynomials, which can be derived the same way we derived Vieta's formulas for quadratic polynomials.

jonjoseph 2021-06-25 20:59:53
For example, if r, s, and t are the roots of $ax^3+bx^2+cx+d=0$, then $ax^3+bx^2+cx+d=a(x-r)(x-s)(x-t)$. Expanding and equating coefficients gives us

$$\begin{aligned} r+s+t &= -\frac{b}{a}, \\ rs+rt+st &= \frac{c}{a}, \\ rst &= -\frac{d}{a}. \end{aligned}$$

jonjoseph 2021-06-25 21:00:08
Be careful of the signs! Those are the most common mistakes when using Vieta's formulas.

jonjoseph 2021-06-25 21:00:19

One way you can keep track of signs is that the sum of the roots is always $-\frac{b}{a}$. From there the signs alternate, to the sum of the roots multiplied together two at a time to the sum multiplied together three at a time, etc. etc. until we get to the all of the roots multiplied together.

jonjoseph 2021-06-25 21:00:38

These formulas are particularly useful because we do not have nice formulas for roots of higher degree polynomials, unlike quadratics.

jonjoseph 2021-06-25 21:00:55

I'll leave you with this problem to try on your own.

jonjoseph 2021-06-25 21:01:00

The roots of $64x^3 - 144x^2 + 92x - 15 = 0$ are in arithmetic progression. The difference between the largest and the smallest roots is:

$\text{(A) } 2 \quad \text{(B) } 1 \quad \text{(C) } 1/2 \quad \text{(D) } 3/8 \quad \text{(E) } 1/4$

jonjoseph 2021-06-25 21:01:14

(You can private message me if you want to go over it)

jonjoseph 2021-06-25 21:01:50

Hmmm... How many of you need to go?

jonjoseph 2021-06-25 21:02:23

Okay. Let's do it. If you need to leave please come back and review the transcript.

jonjoseph 2021-06-25 21:02:33

We can start by setting up the arithmetic sequence. How can we do that?

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We could write the terms of the arithmetic sequence as a, $a + d$, and $a + 2d$.

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However, we can exploit a little symmetry (since we have an odd number of terms) that might simplify things:

jonjoseph 2021-06-25 21:04:00

We can let the terms of the arithmetic sequence be $a - d$, a, and $a + d$. These terms are symmetric, so they usually give us equations that are easier to work with.

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Is this step clear?

jonjoseph 2021-06-25 21:04:30

Now we can apply Vieta's formulas. What equation do we get for the sum of the roots?

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The sum of the roots is

$$(a - d) + a + (a + d) = -\frac{(-144)}{64} = \frac{9}{4}.$$

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This gives us $3a = 9/4$, so $a = 3/4$.

jonjoseph 2021-06-25 21:06:20

(The d's canceled nicely which would not have happened with our initial definition)

jonjoseph 2021-06-25 21:06:29

What equation do we get for the product of the roots?

jonjoseph 2021-06-25 21:07:28
The product of the roots is

$$(a-d)a(a+d) = \frac{15}{64}.$$

jonjoseph 2021-06-25 21:07:33
Substituting $a = 3/4$, we get

$$\left(\frac{3}{4} - d\right) \cdot \frac{3}{4} \cdot \left(\frac{3}{4} + d\right) = \frac{15}{64}.$$

jonjoseph 2021-06-25 21:07:37
Then

$$\left(\frac{3}{4} - d\right)\left(\frac{3}{4} + d\right) = \frac{4}{3} \cdot \frac{15}{64} = \frac{5}{16}.$$

jonjoseph 2021-06-25 21:07:50
But the left hand side is nice. Why?

jonjoseph 2021-06-25 21:08:28
By difference of squares, the left-hand side is equal to

$$\left(\frac{3}{4} - d\right)\left(\frac{3}{4} + d\right) = \left(\frac{3}{4}\right)^2 - d^2 = \frac{9}{16} - d^2.$$

jonjoseph 2021-06-25 21:08:31
Hence,

$$\frac{9}{16} - d^2 = \frac{5}{16}.$$

So what is d?

jonjoseph 2021-06-25 21:09:03
Hint: Careful. Two solutions lurking.

jonjoseph 2021-06-25 21:09:54

$$d^2 = \frac{9}{16} - \frac{5}{16} = \frac{4}{16} = \frac{1}{4}.$$

jonjoseph 2021-06-25 21:09:58
Taking the square root of both sides, we get $d = \pm 1/2$.

jonjoseph 2021-06-25 21:10:01
So what is the difference between the smallest and largest root?

jonjoseph 2021-06-25 21:10:38
The difference between the smallest and largest root is $2 \cdot 1/2 = 1$. The answer is (B).

jonjoseph 2021-06-25 21:11:02
Good. Maybe your first cubic Vieta??

jonjoseph 2021-06-25 21:11:19
SUMMARY

jonjoseph 2021-06-25 21:11:23
In today's class, we saw how to work with functions and polynomials.

jonjoseph 2021-06-25 21:11:25
When working with functions, pay attention to the definition of the function, because the definition is not always in the standard form of "$f(x) = \dots.$"

jonjoseph 2021-06-25 21:11:31
When working with polynomials, it is often useful to keep in mind their basic anatomy, like terms, degrees, roots, and leading and constant coefficients. Also, remember that by using Vieta's formulas, you do not necessarily have to find the roots of a polynomial to work with it effectively.

jonjoseph 2021-06-25 21:11:50
Good class. Stay safe and see you next week!!

jonjoseph 2021-06-25 21:12:38
Thanks to our assistants tonight! Great job.