Math Miscellani

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Abstract

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1. Simplify $\sqrt{\sqrt{9}-\sqrt{8}}$.

$$\begin{aligned} & x=\sqrt{\sqrt{9}-\sqrt{8}} \\ \Longrightarrow\; & x^2 = 3-2\sqrt{2} = (1)^2 - 2\sqrt{2} + (\sqrt{2})^2 = (1-\sqrt{2})^2 \\ \Longrightarrow\; & x = |1-\sqrt{2}| = \sqrt{2}-1 \end{aligned}$$

2. If x satisfies $x^2+\dfrac{1}{x^2}=\sqrt{2}$, evaluate $x^{2024}-\dfrac{1}{x^{2024}}$.

$$x^2+\frac{1}{x^2}=\sqrt{2} \implies \left(x^2+\frac{1}{x^2}\right)^2 = (\sqrt{2})^2 \implies x^4+2x^2\frac{1}{x^2}+\frac{1}{x^4}=2 \implies x^4+\frac{1}{x^4}=0$$
$$\implies x^8=-1 \implies (x^8)^{253}=(-1)^{253} \implies x^{2024}=-1 \implies x^{2024}-\frac{1}{x^{2024}} = -1-\frac{1}{-1}=0$$

3. Solve for $x\in\mathbb{R}$, where $x^2-x^3=12$.

$$\begin{aligned} x^2-x^3 &= 2^3+2^2 && \text{from } 12 = 2^3+2^2 \\ x^2-2^2 &= x^3+2^3 && \text{grouping} \\ (x-2)(x+2) &= (x+2)(x^2-2x+2^2) && \text{factoring} \\ (x+2)(x^2-3x+6) &= 0 && \text{grouping} \end{aligned}$$
$$\implies x = \frac{3\pm i\sqrt{15}}{2} \text{ or } x=-2$$

4. Solve for $x\in\mathbb{R}$, where $3^x-2^x=65$.

$$\begin{aligned} 3^x-2^x &= 65 \\ (3^{x/2})^2-(2^{x/2})^2 &= 5\times 13 \\ (3^{x/2}-2^{x/2})\times(3^{x/2}+2^{x/2}) &= 5\times 13 \end{aligned}$$

Equating the factors on each side of the equality

$$\begin{cases} 3^{x/2}-2^{x/2} &= 5 \\ 3^{x/2}+2^{x/2} &= 13 \end{cases} \implies 2\times 2^{x/2} = 13-5=8 \implies 2^{x/2}=2^2 \implies x/2=2 \implies x=4$$

5. Solve for $x\in\mathbb{R}$, where $16^x+20^x=25^x$.

We notice that 4 and 5 can be factored

$$\begin{aligned} 16^x + 20^x &= 25^x \\ 4^{2x} + 5^x \cdot 4^x &= 5^{2x} \\ 1 + \frac{5^x}{4^x} \cdot \frac{4^x}{4^x} &= \frac{5^{2x}}{4^{2x}} \\ 1 + (5/4)^x &= (5/4)^{2x} \end{aligned}$$

Let $X = (5/4)^x$ and solve for $X > 0$:

$$X^2 - X - 1 = 0 \implies X = \frac{1+\sqrt{5}}{2}$$

Now solve for x by taking the logarithm on both sides of the equality:

$$\left(\frac{5}{4}\right)^x = \frac{1+\sqrt{5}}{2} \implies x \ln\left(\frac{5}{4}\right) = \ln\left(1+\sqrt{5}\right) - \ln(2) \implies x = \frac{\ln(1+\sqrt{5}) - \ln(2)}{\ln(5/4)}$$

Check the plausibility of the solution with coarse approximations:

$$\ln(2) \approx 0.69,\ \ln(3) \approx 1.1,\ \ln(5) \approx 1.61,\ \sqrt{5} \approx 2.24, \implies \ln(1+\sqrt{5}) \approx \ln(3.24) \approx 1.2$$

$$\frac{\ln(1+\sqrt{5}) - \ln(2)}{\ln(5) - 2\ln(2)} \approx \frac{1.2 - 0.69}{1.61 - 2 \times 0.69} \approx 2.2$$

$16^{2.2} + 20^{2.2} \approx 1174$

$25^{2.2} \approx 1190$