

Art Of Problem Solving - AMC 10

Week 3

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Abstract

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1.

Consider the set of numbers $\{1, 10, 10^2, 10^3, \dots, 10^{10}\}$. The ratio of the largest element of the set to the sum of the other ten elements of the set is closest to which integer?

- (A) 1 (B) 9 (C) 10 (D) 11 (E) 101

The ratio is:

$$\frac{10^{10}}{10^0 + 10^1 + \dots + 10^8 + 10^9}$$

The denominator is a geometric sum with ten terms, which may be written in closed-form as

$$10^0 + 10^1 + \dots + 10^8 + 10^9 = \frac{1 - 10^{10}}{1 - 10} = \frac{10^{10}(10^{-10} - 1)}{-9} = \frac{10^{10}(1 - 10^{-10})}{9}$$

Plugging this back into the original fraction and simplifying the 10^{10} term:

$$\frac{10^{10}}{10^0 + 10^1 + \dots + 10^8 + 10^9} = \frac{9}{1 - 10^{-10}} \approx 9$$

where the 10^{-10} term is so small relative to 1 that it may be neglected

9

2.

For each positive integer n , the mean of the first n terms of a sequence is n . What is the 2008th term of the sequence?

- (A) 2008 (B) 4015 (C) 4016 (D) 4,030,056 (E) 4,032,064

Let a_n denote the general term of the sequence and S_n its sum:

$$S_n = a_1 + a_2 + a_3 + \dots a_n$$

The mean of the first n terms is equal to n , so:

$$\frac{S_n}{n} = n \implies S_n = n^2$$

From the sums sequence we can recover the a_n sequence as follows:

$$S_{n+1} - S_n = a_{n+1} = (n+1)^2 - n^2 = 2n + 1$$

The general term is $a_{n+1} = 2n + 1$, so we get

$$a_{2008} = 2 \times 2007 + 1 = 4015$$

4015

3.

On Monday, Millie puts a quart of seeds, 25% of which are millet, into a bird feeder. On each successive day she adds another quart of the same mix of seeds without removing any seeds that are left. Each day the birds eat only 25% of the millet in the feeder, but they eat all of the other seeds. On which day, just after Millie has placed the seeds, will the birds find that more than half the seeds in the feeder are millet?

(A) Tuesday (B) Wednesday (C) Thursday (D) Friday (E) Saturday

Suppose the millet and the rest of the seeds weigh the same. Each day the amount added to the feeder weighs one unit (one quart). Let m_t denote the amount of millet seeds in the bird feeder on day t , immediately after refill. Let a_t denote the total amount of seeds in the bird feeder on day t , immediately after refill.

The amount of millet seeds evolves according to:

$$\begin{aligned} m_1 &= 1/4 \\ m_2 &= 1/4 + (3/4) \cdot (1/4) \\ m_3 &= 1/4 + (3/4)^2 \cdot (1/4) \\ m_t &= 1/4 + (3/4)^t \cdot (1/4) = 0.25 + 0.25(0.75)^t \end{aligned}$$

The total amount of seeds (including millet seeds) evolves according to:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 1 + (3/4) \cdot (1/4) \\ a_3 &= 1 + (3/4)^2 \cdot (1/4) \\ a_t &= 1 + (3/4)^t \cdot (1/4) = 1 + 0.25(0.75)^t \end{aligned}$$

The percentage of millet in the bird feeder immediately after refill must exceed 0.5:

$$\begin{aligned} \frac{0.25 + 0.25(0.75)^t}{1 + 0.25(0.75)^t} &\geq 0.5 \\ (0.75)^t &\geq 0.25 \\ (3/4)^t &\geq 1/4 \\ (4/3)^t &\leq 4 \end{aligned}$$

Now we check some values:

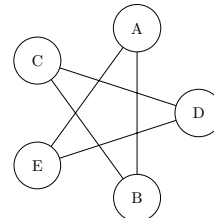
$$\begin{aligned} (4/3)^1 &\approx 1.333 < 4 \\ (4/3)^2 &\approx 1.778 < 4 \\ (4/3)^3 &\approx 2.370 < 4 \\ (4/3)^4 &\approx 3.160 < 4 \\ (4/3)^5 &\approx 4.214 > 4 \quad \checkmark \end{aligned}$$

Thus, $t = 5$, which corresponds to Friday.

Friday

4.

In the five-sided star shown, the letters A , B , C , D , and E are replaced by the numbers 3, 5, 6, 7, and 9, although not necessarily in this order. The sums of the numbers at the ends of the line segments AB , BC , CD , DE , and EA form an arithmetic sequence, although not necessarily in this order. What is the middle term of the arithmetic sequence?



- (A) 9 (B) 10 (C) 11 (D) 12 (E) 13

The sums of the numbers at the ends of the line segments are:

$$(A + B) + (B + C) + (C + D) + (D + E) + (E + A) = 2(A + B + C + D + E)$$

Letters A , B , C , D , E take values in $\{3, 5, 6, 7, 9\}$, so the sum is:

$$2(A + B + C + D + E) = 2(3 + 5 + 6 + 7 + 9) = 2 \cdot 30 = 60$$

In an arithmetic sequence, the middle term is equal to the average. Since there are five terms, the average is $60/5 = 12$.

$$\text{middle term} = 12$$

This reasoning does not give an explicit solution, so let's construct the actual solution. To find the solution, note that the common difference d can only be 1 or 2. If $d = 2$, then the sequence must be $[8, 10, 12, 14, 16]$, since 8 is the lowest possible sum ($3 + 5$) and 16 the largest possible sum ($7 + 9$). But this can be ruled out as follows: all the sums are even, but 6 is an isolated even integer and therefore cannot be paired with any of the other integers to yield an even sum. This leaves $d = 1$ as the only possibility and therefore the values must be one of $[8, 9, 10, 11, 12]$, $[9, 10, 11, 12, 13]$, $[10, 11, 12, 13, 14]$, $[11, 12, 13, 14, 15]$, $[12, 13, 14, 15, 16]$. Now 6 is not a problem since these sums contain odd values. Consider our integers and guess what pairing yields the admissible sequence. Here are the pairs and their sum:

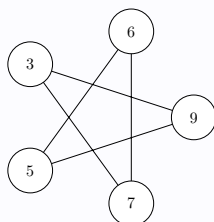
$$\begin{array}{llllll} (3,5) \rightarrow 8 & (3,6) \rightarrow 9 & (3,7) \rightarrow 10 & (3,9) \rightarrow 12 & (5,6) \rightarrow 11 & \\ (5,7) \rightarrow 12 & (5,9) \rightarrow 14 & (6,7) \rightarrow 13 & (6,9) \rightarrow 15 & (7,9) \rightarrow 16 & \end{array}$$

Since the sum 13 cannot be attained with any pair, that leaves $[10, 11, 12, 13, 14]$ as the only feasible sequence. And here are the relevant pairs:

$$10: (3,7) \quad 11: (5,6) \quad 12: (3,9) \text{ or } (5,7) \quad 13: (6,7) \quad 14: (5,9)$$

Further guesswork yields the star:

$$A = 6 \quad B = 7 \quad C = 3 \quad D = 9 \quad E = 5$$



5.

In the eight-term sequence A, B, C, D, E, F, G, H , the value of C is 5 and the sum of any three consecutive terms is 30. What is $A + H$?

- | | | | | |
|--------|--------|--------|--------|--------|
| (A) 17 | (B) 18 | (C) 25 | (D) 26 | (E) 43 |
|--------|--------|--------|--------|--------|

We have the following implications:

$A + B + C = 30,$	$C = 5$	$\implies B = 25 - A$
$B + C + D = 30,$	$B + C = 30 - A$	$\implies D = A$
$C + D + E = 30,$	$C + D = 5 + A$	$\implies E = 25 - A$
$D + E + F = 30,$	$D + E = 25$	$\implies F = 5$
$E + F + G = 30,$	$E + F = 30 - A$	$\implies G = A$
$F + G + H = 30,$	$F + G = 5 + A$	$\implies H = 25 - A$

The last line implies

$$A + H = 25$$

25

6.

Let a_1, a_2, \dots be a sequence for which $a_1 = 2$, $a_2 = 3$, and $a_n = a_{n-1}/a_{n-2}$ for each positive integer $n \geq 3$. What is a_{2006} ?

- | | | | | |
|-------------------|-------------------|-------------------|-------|-------|
| (A) $\frac{1}{2}$ | (B) $\frac{2}{3}$ | (C) $\frac{3}{2}$ | (D) 2 | (E) 3 |
|-------------------|-------------------|-------------------|-------|-------|

Calculating the first few terms of the sequence:

$$a_1 = 2$$

$$a_2 = 3$$

$$a_3 = 3/2$$

$$a_4 = 1/2$$

$$a_5 = 1/3$$

$$a_6 = 2/3$$

$$a_7 = 2$$

$$a_8 = 3$$

The sequence repeats every 6 terms: $a_7 = a_1$, $a_8 = a_2$, \dots . Since $2004 = 6 \times 334$, it follows that 2006 leaves a remainder of 2 when divided by 6. That is $2006 \equiv 2 \pmod{6}$. And thus,

$$a_{2006} = a_2$$

$a_{2006} = 3$

7.

Suppose that $\{a_n\}$ is an arithmetic sequence with $a_1 + a_2 + \cdots + a_{100} = 100$ and $a_{101} + a_{102} + \cdots + a_{200} = 200$. What is the value of $a_2 - a_1$?

- (A) 0.0001 (B) 0.001 (C) 0.01 (D) 0.1 (E) 1

Subtract the two equations:

$$(a_{101} - a_1) + (a_{102} - a_2) + \cdots + (a_{200} - a_{100}) = 200 - 100 = 100$$

This gives the common difference of every hundred terms repeated one hundred times. Thus the common difference between two terms in the sequence separated by 100 positions is 1. The common difference of two sequential terms is one-hundredth of that:

$$d = 0.01$$

If you did not see that, perhaps you noticed that the sums satisfy

$$S_{100} = 100$$

$$S_{200} = 300$$

where S_{200} follows by adding the given sums. Any arithmetic sum satisfies

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ S_n &= a_1 + a_2 + \cdots + a_n \\ &= na_1 + (1 + 2 + \cdots + (n - 1))d \\ &= na_1 + \frac{(n - 1)n}{2}d \end{aligned}$$

We thus obtain a linear system of two equations in two unknowns, a_1 and d ,

$$\begin{aligned} S_{100} &= 100a_1 + \frac{99 \cdot 100}{2}d = 100 \\ S_{200} &= 200a_1 + \frac{199 \cdot 200}{2}d = 300 \end{aligned}$$

Eliminate a_1 and solve for the common difference d

$$\frac{(199 - 99) \cdot 200}{2}d = 300 - 200 \implies d = \frac{1}{100}$$

8.

Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$, for all positive integers m and n . Then a_{12} is

- | | | | | |
|--------|--------|--------|--------|--------|
| (A) 45 | (B) 56 | (C) 67 | (D) 78 | (E) 89 |
|--------|--------|--------|--------|--------|

Let $m = 1$. We have $a_{n+1} = 1 + a_n + n$. The first few terms in the sequence are:

$$a_1 = 1$$

$$a_2 = 1 + a_1 + 1$$

$$a_3 = 1 + a_2 + 2$$

$$\dots \quad \dots$$

$$a_{12} = 1 + a_{11} + 11$$

Adding up the left- and right-hand sides, all the a_n terms cancel out except a_{12} :

$$a_{12} = 12 + (1 + 2 + 3 + \dots + 11)$$

$$= 12 + \frac{11 \cdot 12}{2}$$

$$= 12 + 66$$

$$= 78$$

78

9.

The first four terms in an arithmetic sequence are $x + y$, $x - y$, xy , and x/y , in that order. What is the fifth term?

- (A) $-\frac{15}{8}$ (B) $-\frac{6}{5}$ (C) 0 (D) $\frac{27}{20}$ (E) $\frac{123}{40}$

The difference between consecutive terms, obtained from the first two terms, is $(x-y)-(x+y) = -2y$. We can also express the third and fourth terms by adding the common difference, which gives the equivalent sequence:

$$\begin{array}{cccc} x + y, & x - y, & xy, & x/y \\ x + y, & x - y, & x - 3y, & x - 5y \end{array}$$

Equating the third terms gives:

$$\begin{aligned} xy &= x - 3y \\ x(y - 1) &= -3y \\ x &= \frac{-3y}{y - 1} \end{aligned}$$

Equating the fourth terms and substituting the expression above to eliminate x :

$$\begin{aligned} \frac{x}{y} &= x - 5y \\ \frac{-3}{y - 1} &= \frac{-3y}{y - 1} - 5y \\ -3 &= -3y - 5y(y - 1) \\ 5y^2 - 2y - 3 &= 0 \end{aligned}$$

We have a quadratic in y , which may be factored as

$$(5y + 3)(y - 1) = 0$$

If $y = 1$, the sequence would be:

$$x + 1, \quad x - 1, \quad x, \quad x$$

which is not arithmetic. So we reject $y = 1$.

The correct solution is the pair:

$$x = -\frac{9}{8}, \quad y = -\frac{3}{5}$$

The fifth term in the sequence is therefore

$$\frac{x}{y} - 2y = \frac{-9/8}{-3/5} - 2\left(\frac{-3}{5}\right) = \frac{45}{24} + \frac{6}{5} = \frac{123}{40}$$

$$\frac{123}{40}$$

10.

Let a_1, a_2, \dots be a sequence with the following properties.

- (i) $a_1 = 1$, and
- (ii) $a_{2n} = n \cdot a_n$ for any positive integer n .

What is the value of $a_{2^{100}}$? (The subscript is 2^{100} .)

- | | | | | |
|-------|--------------|---------------|----------------|----------------|
| (A) 1 | (B) 2^{99} | (C) 2^{100} | (D) 2^{4950} | (E) 2^{9999} |
|-------|--------------|---------------|----------------|----------------|

Let $n = 2^{99}$ and use $a_{2n} = na_n$,

$$2^{100} = 2 \times 2^{99} \implies a_{2^{100}} = a_{2 \cdot 2^{99}} = 2^{99} a_{2^{99}}$$

Applying the same rule repeatedly gives:

$$a_{2^{100}} = 2^{99} 2^{98} a_{2^{98}} = 2^{99} 2^{98} 2^{97} a_{2^{97}} = 2^{99} 2^{98} 2^{97} \dots 2 a_2$$

where $a_2 = a_1 = 1$. So we have

$$\begin{aligned} a_{2^{100}} &= 2^{99+98+\dots+2+1} \\ &= 2^{\frac{99(100)}{2}} \\ &= 2^{4950} \end{aligned}$$

2^{4950}
