AMC 10 Problem Series (2804)

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Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Homework

Lesson: 1 2 3 4 5 6 7 8 9 **10** 11 12

Homework: Lesson 10

You have completed **10** of **10** challenge problems.
Past Due **Aug 14**.

Readings

Lesson 10 Transcript: Fri, Aug 6

Challenge Problems

Total Score: 60 / 60

Problem 1 – Correct! – Score: 6 / 6 (2993)

Problem: Report Error
Two fair coins are to be tossed once. For each head that results, one fair die is to be rolled. What is the probability that the sum of the die rolls is odd? (Note that if no die is rolled, the sum is 0.)

(A) $\frac{3}{8}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{43}{72}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{2}{3}$

Solution:
Among two coin tosses, the probability of getting 0, 1, and 2 heads are 1/4, 1/2, and 1/4, respectively.

For 0 dice rolls, the probability of getting an odd sum is 0. For 1 and 2 dice rolls, the probability of getting an odd sum is 1/2. Therefore, the probability that the sum is odd is

$$\frac{1}{4} \cdot 0 + \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \boxed{\frac{3}{8}}.$$

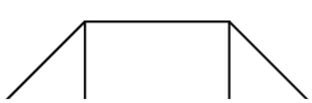
The answer is (A).

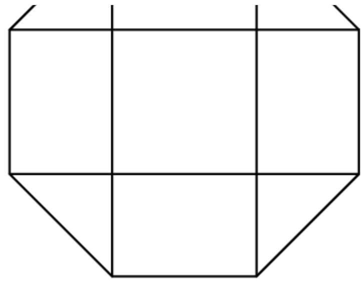
Your Response(s):
A

Problem 2 – Correct! – Score: 6 / 6 (2995)

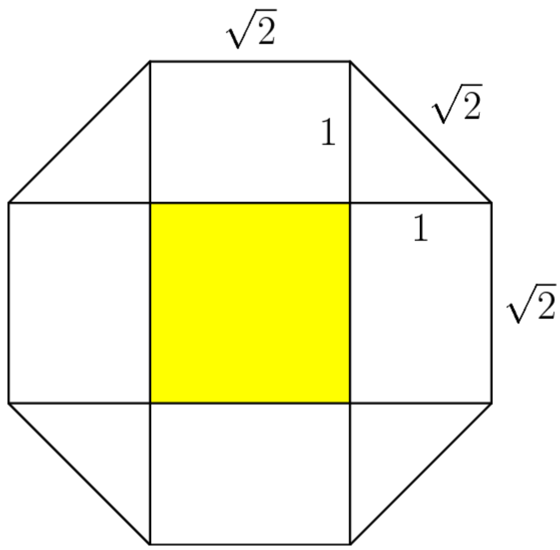
Problem: Report Error
A dart board is a regular octagon divided into regions as shown. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. What is the probability that the dart lands within the center square?

(A) $\frac{\sqrt{2}-1}{2}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{2-\sqrt{2}}{2}$ **(D)** $\frac{\sqrt{2}}{4}$ **(E)** $2-\sqrt{2}$





Solution:
The regular octagon consists of four right isosceles triangles. Let the legs of these triangles be 1. Then the hypotenuse of each triangle is $\sqrt{2}$, so the side length of the regular octagon is $\sqrt{2}$.



Then the area of the center square is $(\sqrt{2})^2 = 2$. The regular octagon consists of the four triangles (each has area $1/2 \cdot 1 \cdot 1 = 1/2$), four $1 \times \sqrt{2}$ rectangles, and the center square, so the area of the regular octagon is

$$4 \cdot \frac{1}{2} + 4 \cdot \sqrt{2} + 2 = 4 + 4\sqrt{2}.$$

Therefore, the probability that the dart lands within the center square is

$$\begin{aligned} \frac{2}{4+4\sqrt{2}} &= \frac{2}{4(\sqrt{2}+1)} \\ &= \frac{1}{2(\sqrt{2}+1)} \\ &= \frac{\sqrt{2}-1}{2(\sqrt{2}+1)(\sqrt{2}-1)} \\ &= \frac{\sqrt{2}-1}{2(2-1)} \\ &= \boxed{\frac{\sqrt{2}-1}{2}}. \end{aligned}$$

$\boxed{2}$

The answer is (A).

Your Response(s):

A

Problem 3 – Correct! – Score: 6 / 6 (2996)

Problem: Report Error

Bernardo randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and arranges them in descending order to form a 3-digit number. Silvia randomly picks 3 distinct numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8\}$ and also arranges them in descending order to form a 3-digit number. What is the probability that Bernardo's number is larger than Silvia's number?

(A) $\frac{47}{72}$ **(B)** $\frac{37}{56}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{49}{72}$ **(E)** $\frac{39}{56}$

Solution:

The only difference between Bernardo's set and Silvia's set is that Bernardo's set contains the digit 9. If Bernardo picks the number 9, then his number is automatically greater than Silvia's number.

Since Bernardo picks three numbers out of nine, and every number has an equal chance of appearing, the probability that Bernardo picks the number 9 is 3/9 = 1/3. Hence, the probability that Bernardo does not pick the number 9 is 1 - 1/3 = 2/3.

If Bernardo does not pick the number 9, then Bernardo and Silvia are essentially picking from the same set of numbers. The probability that Bernardo and Silvia pick the same three numbers is

$$\frac{1}{\binom{8}{3}} = \frac{1}{56}.$$

Then the probability that Bernardo and Silvia do not pick the same three numbers is $1 - 1/56 = 55/56$. Therefore, if Bernardo does not pick the number 9, then the probability that Bernardo's number is greater than Silvia's number is $1/2 \cdot 55/56 = 55/112$.

Hence, the overall probability that Bernardo's number is greater than Silvia's number is

$$\frac{1}{3} \cdot 1 + \frac{2}{3} \cdot \frac{55}{112} = \boxed{\frac{37}{56}}.$$

The answer is (B).

Your Response(s):

B

Problem 4 – Correct! – Score: 6 / 6 (2997)

Problem: Report Error

Positive integers a, b, and c are randomly and independently selected with replacement from the set $\{1, 2, 3, \dots, 2010\}$. What is the probability that $abc + ab + a$ is divisible by 3?

(A) $\frac{1}{3}$ **(B)** $\frac{29}{81}$ **(C)** $\frac{31}{81}$ **(D)** $\frac{11}{27}$ **(E)** $\frac{13}{27}$

Solution:

To determine when $abc + ab + a$ is divisible by 3, it suffices to consider their residues modulo 3. Since 2010 is divisible by 3, each of the numbers a, b, and c has an equal chance of having any of the residues modulo 3. We compute $abc + ab + a$ for each of the $3^3 = 27$ possible combinations of residues.

a	b	c	$abc + ab + a \pmod 3$
0	0	0	0
0	0	1	0
0	0	2	0
0	1	0	0
0	1	1	0
0	1	2	0
0	2	0	0
0	2	1	0
0	2	2	0
1	0	0	1
1	0	1	1
1	0	2	1
1	1	0	2
1	1	1	0
1	1	2	1
1	2	0	0
1	2	1	2
1	2	2	1
2	0	0	2
2	0	1	2
2	0	2	2
2	1	0	1
2	1	1	0
2	1	2	2
2	2	0	0
2	2	1	1
2	2	2	2

From the table, we see that there are 13 combinations of residues in which $abc + ab + a$ is divisible by 3. Therefore, the probability that $abc + ab + a$ is divisible by 3 is $\boxed{13/27}$. The answer is (E).

Your Response(s):

E

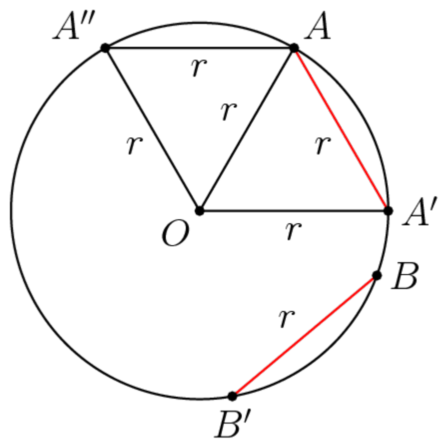
Problem 5 – Correct! – Score: 6 / 6 (2998)

Problem: Report Error

Two points on the circumference of a circle of radius r are selected independently and at random. From each point a chord of length r is drawn in a clockwise direction. What is the probability that the two chords intersect?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{5}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{1}{2}$

Solution:
Let O be the center of the circle. Let A and B be the points chosen, and let $\overline{AA'}$ and $\overline{BB'}$ be the chords drawn. Let A'' be the point on the circle such that $\overline{AA''}$ is a chord of length r, drawn in a counterclockwise direction from A.



First, we choose point A on the circle. Then chord $\overline{BB'}$ intersects chord $\overline{AA'}$ if and only if B lies on arc $A''A'$.

Triangles OAA' and OAA'' are both equilateral, so $\angle A'OA'' = 120^\circ$. Therefore, the probability that chords $\overline{AA'}$ and $\overline{BB'}$ intersect is $120/360 = \boxed{1/3}$. The answer is (D).

Your Response(s):
D

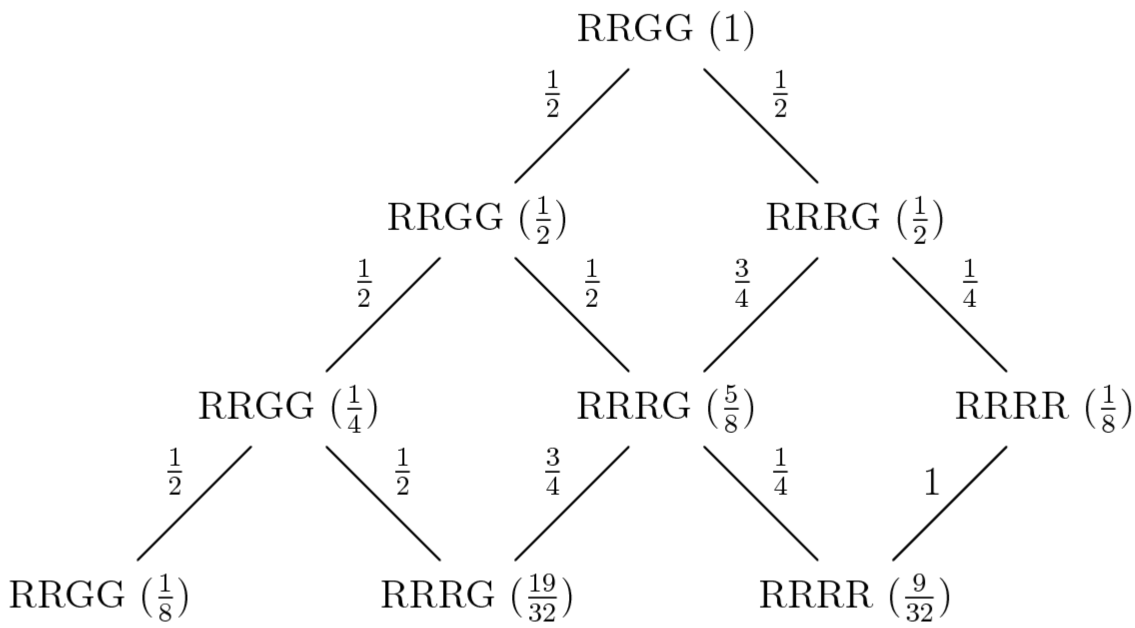
Problem 6 – Correct! – Score: 6 / 6 (2999)

Problem: Report Error
A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a new red bead regardless of the color you pulled out. What is the probability that all beads in the bag are red after three such replacements?

(A) $\frac{1}{8}$ **(B)** $\frac{5}{32}$ **(C)** $\frac{9}{32}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{7}{16}$

Solution:
We can compute the possible outcomes and their probabilities with a tree.

At the start, we have two red beads and two green beads, which we denote by the string RRGG. The probability of drawing a red bead is $2/4 = 1/2$, and we end up with RRGG. The probability of drawing a green bead is $2/4 = 1/2$, and we end up with RRRG. We then proceed through the tree. (We go left if we draw a red bead, and right if we draw a green bead. We mark each line segment with the probability that we move along that line segment. Note that the probability of drawing a red bead depends on the number of red beads.)



For each possible outcome, we label the outcome with the probability. For example, the probability of being at RRRG after two draws is 5/8. We can compute this as follows: The only way to arrive at RRRG after two draws is if we are at RRGG after one draw (which occurs with probability 1/2) and we draw a green bead (which occurs with probability 1/2), or we are at RRRG after one draw (which occurs with probability 1/2) and we draw a red bead (which occurs with probability 3/4). Therefore, the probability of arriving at RRRG after two draws is

$$\frac{1}{2}\cdot\frac{1}{2}+\frac{3}{4}\cdot\frac{1}{2}=\frac{5}{8}.$$

Reading off the tree, we see that the probability of arriving at RRRR after three draws is $\boxed{9/32}$. The answer is (C).

Your Response(s):

C

Problem 7 – Correct! – Score: 6 / 6 (3000)

Problem: Report Error

For a particular peculiar pair of dice, the probabilities of rolling 1, 2, 3, 4, 5, and 6 on each die are in the ratio $1:2:3:4:5:6$. What is the probability of rolling a total of 7 on the two dice?

(A) $\frac{4}{63}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{8}{63}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{2}{7}$

Solution:

From the given ratio, the probabilities of rolling 1, 2, 3, 4, 5, and 6 are 1/21, 2/21, 3/21, 4/21, 5/21, and 6/21, respectively. The only ways of rolling a total of 7 are 1 and 6, 2 and 5, 3 and 4, 4 and 3, 5 and 2, and 6 and 1, so the probability of rolling a total of 7 is

$$\frac{1}{21}\cdot\frac{6}{21}+\frac{2}{21}\cdot\frac{5}{21}+\frac{3}{21}\cdot\frac{4}{21}+\frac{4}{21}\cdot\frac{3}{21}+\frac{5}{21}\cdot\frac{2}{21}+\frac{6}{21}\cdot\frac{1}{21}=\boxed{\frac{8}{63}}.$$

The answer is (C).

Your Response(s):

C

Problem 8 – Correct! – Score: 6 / 6 (3001)

Problem: Report Error

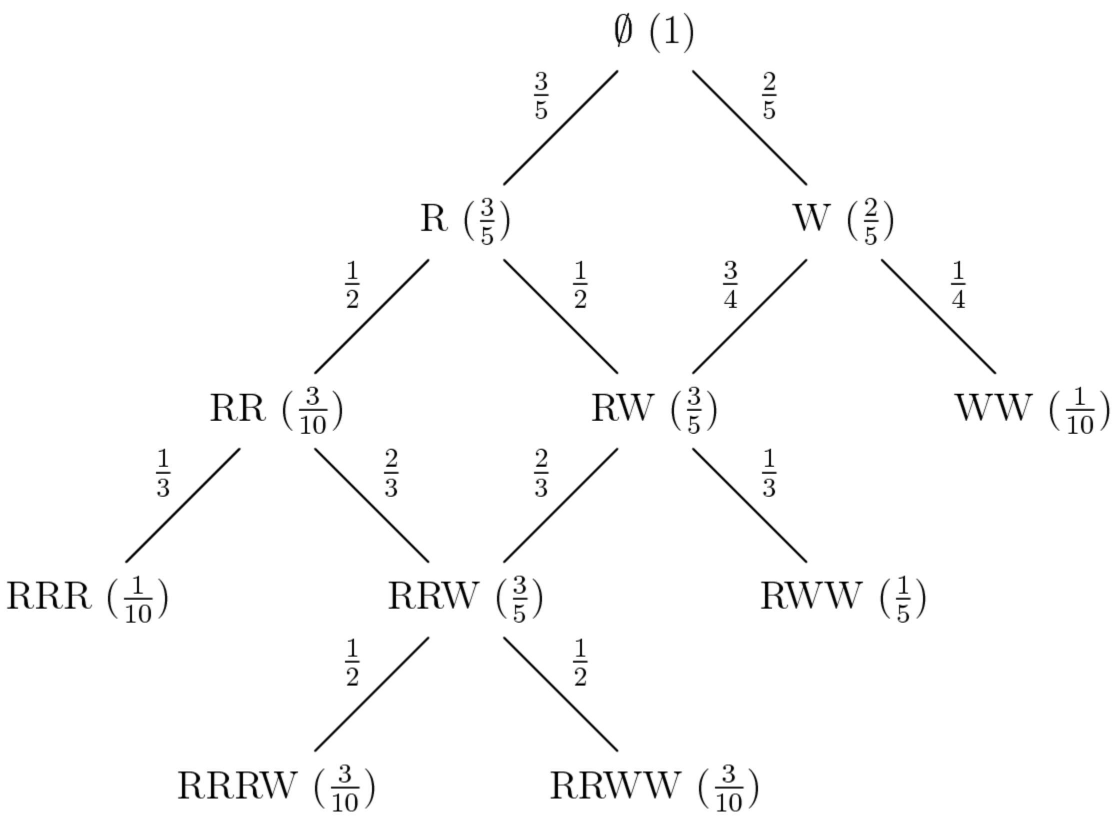
A box contains exactly five chips, three red and two white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

(A) $\frac{3}{10}$ **(B)** $\frac{2}{5}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{7}{10}$

Solution:

Solution 1: We can compute the possible outcomes and their probabilities with a tree.

We start at $\emptyset$, where we have no chips. The probability of drawing a red chip is 3/5, and the probability of drawing a white chip is 2/5. We then proceed through the tree. (We go left if we draw a red chip, and right if we draw a white chip bead. We mark each line segment with the probability that we move along that line segment. Note that the probability of drawing a red chip depends on how many are left in the box.) We stop when we draw three red chips or two white chips.



The last chip drawn is white if and only if we end with WW, RWW, or RRWW. Reading off the table, we see that these probabilities are 1/10, 1/5, and 3/10, so the probability that the last chip drawn is white is $1/10+1/5+3/10=\boxed{3/5}$. The answer is (D).

Solution 2: Instead of stopping when we get three red chips or two white chips, let's consider what happens when we draw all five chips from the box. For example, we might draw them in the order W, R, R, W, R. If we were following the rules in the original problem, we would draw W, R, R, W, and then stop, because we draw both white chips. Note that we draw all white chips first before we draw all red chips if and only if the fifth chip is red.

There are three red chips and two white chips, so the probability that the fifth chip is red is $\frac{3}{5}$. The white chips are all drawn before the red chips if and only if the last chip is red, so the probability of drawing all of the white chips is likewise $\boxed{\frac{3}{5}}$.

Your Response(s):

D

Problem 9 – Correct! – Score: 6 / 6 (3002)

Problem:

Report Error

Tina randomly selects two distinct numbers from the set $\{1, 2, 3, 4, 5\}$, and Sergio randomly selects a number from the set $\{1, 2, \dots, 10\}$. The probability that Sergio's number is larger than the sum of the two numbers chosen by Tina is

(A) 2/5 **(B)** 9/20 **(C)** 1/2 **(D)** 11/20 **(E)** 24/25

Solution:

Let s be the sum of Tina's two numbers. Then s is always between 3 and 9, so the probability that Sergio's number is greater than s is

$$\frac{10 - s}{10}.$$

The probability that the sum of Tina's two numbers is s is as follows:

s	Probability that the sum is s
3	1/10
4	1/10
5	2/10
6	2/10
7	2/10
8	1/10
9	1/10

Therefore, the probability that Sergio's number is greater than s is

$$\frac{1}{10} \cdot \frac{7}{10} + \frac{1}{10} \cdot \frac{6}{10} + \frac{2}{10} \cdot \frac{5}{10} + \frac{2}{10} \cdot \frac{4}{10} + \frac{2}{10} \cdot \frac{3}{10} + \frac{1}{10} \cdot \frac{2}{10} + \frac{1}{10} \cdot \frac{1}{10} = \boxed{\frac{2}{5}}.$$

The answer is (A).

Your Response(s):

A

Problem 10 – Correct! – Score: 6 / 6 (3003)

Problem:

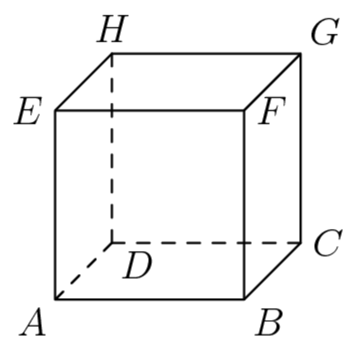
Report Error

A bug starts at one vertex of a cube and moves along the edges of the cube according to the following rule. At each vertex the bug will choose to travel along one of the three edges emanating from that vertex. Each edge has equal probability of being chosen, and all choices are independent. What is the probability that after seven moves the bug will have visited every vertex exactly once?

(A) $\frac{1}{2187}$ **(B)** $\frac{1}{729}$ **(C)** $\frac{2}{243}$ **(D)** $\frac{1}{81}$ **(E)** $\frac{5}{243}$

Solution:

Let the vertices of the cube be A, B, C, D, E, F, G, and H, as shown.



Without loss of generality, assume that the bug goes from A to B on the first move. The bug still has six moves left, and from each vertex, the bug has three possible vertices to move to, so there are a total of 3^6 total possible paths.

It is not hard to list all the possible paths that go through each vertex:

$$\begin{gathered}A \to B \to C \to D \to H \to G \to F \to E \\ A \to B \to C \to D \to H \to E \to F \to G \\ A \to B \to C \to G \to F \to E \to H \to D \\ A \to B \to F \to E \to H \to G \to C \to D \\ A \to B \to F \to E \to H \to D \to C \to G \\ A \to B \to F \to G \to C \to D \to H \to E\end{gathered}$$

Therefore, the probability that the bug goes visits every vertex is $6/3^6 = \boxed{2/243}$. The answer is (C).

Your Response(s):

C