

# Math Competition Tricks

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## **Abstract**

This note reviews a selection of tricks that may be useful in math competitions.

# Mental Arithmetic

It is useful to add/multiply/divide fast. There are too many tricks to review, but here are a few basic ones. With practice you will be able to use these tricks while calculating in your head.

Add numbers by grouping them:

$$\textcircled{1} + \textcircled{2} + \textcircled{3} + \textcircled{4} + 5 + \textcircled{6} + \textcircled{7} + \textcircled{8} + \textcircled{9} = 4 \times 10 + 5 = 45$$

Add numbers by rounding them:

$$978 + 237 = (980 - 2) + (220 + 17) = 1200 + 15 = 1215$$

Multiply by 5:

$$978 \times 5 = (1000 - 20 - 2) \times \frac{10}{2} = \frac{500 - 10 - 1}{2} \times 10 = 4890$$

where we have also decomposed 978 to make the division by 2 even easier (skip this step if you can quickly halve 978).

Multiply numbers by decomposing them:

$$\begin{aligned} 14 \times 16 &= (15 - 1) \times (15 + 1) \\ &= 15^2 - 1 = 224 \end{aligned}$$

where we have used  $15^2 = 225$  and the difference-of-squares formula:

$$(a + b)(a - b) = a^2 - b^2$$

Similarly,  $13 \times 17 = 15^2 - 4 = 221$  (if hesitant, check that the last digit matches:  $3 \times 7 = 21$ , so the last digit 1 is indeed correct). The difference-of-squares formula can always be applied when multiplying numbers that differ by a multiple of 2 (multiplying two even numbers or multiplying two odd numbers).

Multiply numbers by rounding up:

$$\begin{aligned} 19 \times 18 &= 20 \times 18 - 18 \\ &= 360 - 18 = 342 \end{aligned}$$

Multiply numbers by rounding up and down:

$$\begin{aligned} 19 \times 23 &= (20 - 1) \times (20 + 3) \\ &= 20^2 + (3 - 1) \times 20 - 3 = 400 + 40 - 3 = 437 \end{aligned}$$

Square numbers by rounding up:

$$\begin{aligned} 99^2 &= (100 - 1)^2 \\ &= 10000 - 200 + 1 = 9801 \end{aligned}$$

where we have used:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Square numbers by rounding up or down:

$$\begin{aligned} 13^2 &= (15 - 2)^2 \\ &= 200 + 25 - 60 + 4 = 140 + 29 = 169 \end{aligned}$$

where we suppose you have memorized  $15^2 = 225 = 200 + 25$  (but forgotten  $13^2$ ). Because it is easier to subtract 60 from 200 than from 225, we also split 225 as  $200 + 25$ . These manipulations are to be done in your head or very quickly on a scrap of paper.

## Useful Sums

The sum of the first  $n$  natural numbers:

$$\begin{aligned} 1 + 2 + 3 + \dots + n &= \frac{n(n+1)}{2} \\ 1 + 2 + 3 + \dots + 10 &= 55 \\ 1 + 2 + 3 + \dots + 100 &= 505 \end{aligned}$$

The sum of the first  $(2n - 1)$  odd numbers:

$$\begin{aligned} 1 + 3 + 5 + \dots + (2n - 1) &= n^2 \\ 1 + 3 + 5 + \dots + 9 &= 1 + 3 + 5 + \dots + (2 \times 5 - 1) = 5^2 = 25 \\ 1 + 3 + 5 + \dots + 99 &= 1 + 3 + 5 + \dots + (2 \times 50 - 1) = 50^2 = 2500 \end{aligned}$$

The sum of the first  $2n$  even numbers:

$$\begin{aligned} 2 + 4 + 6 + \dots + (2n) &= n(n+1) \\ 2 + 4 + 6 + \dots + 10 &= 2 + 4 + 6 + \dots + (2 \times 5) = 5 \times 6 = 30 \\ 2 + 4 + 6 + \dots + 100 &= 2 + 4 + 6 + \dots + (2 \times 50) = 50 \times 51 = 2550 \end{aligned}$$

The sum of the first  $n$  squares formula and first ten sums:

$$\begin{aligned} 1^2 + 2^2 + \dots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\ 1, 5, 14, 30, 55, 91, 140, 204, 285, 385. \end{aligned}$$

The sum of the first  $n$  cubes formula and first ten sums:

$$\begin{aligned} 1^3 + 2^3 + \dots + n^3 &= \left( \frac{n(n+1)}{2} \right)^3 \\ 1, 9, 36, 100, 225, 441, 784, 1296, 2025, 3025. \end{aligned}$$

The sum of  $n$  terms of a geometric series:

$$1 + a + a^2 + \dots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

The Fibonacci numbers are the sum of the two preceding numbers in the Fibonacci sequence:

$$F_n = F_{n-1} + F_{n-2}$$

where the first two numbers in the sequence are typically 0 and 1:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

The Lucas numbers are Fibonacci numbers with starting values 2 and 0:

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \dots$$

# Prime Numbers

The first few prime numbers with their index:

$i$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$p$	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71

  

$i$	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
$p$	73	79	83	89	97	101	103	107	109	113	127	131	137	139	149	151	157	163	167	173

  

$i$	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
$p$	179	181	191	193	197	199	211	223	227	229	233	239	241	251	257	263	269	271	277	281

Prime numbers of the twentieth and twenty-first centuries (problems involving numbers close to the current year are popular: the closest prime numbers to 2020 are 2017 and 2027):

$$1901, 1907, 1913, 1931, 1933, 1949, 1951, 1973, 1979, 1987, 1993, 1997, 1999, \\ 2003, 2011, 2017, 2027, 2029, 2039, 2053, 2063, 2069, 2081, 2083, 2087, 2089, 2099$$

Mersenne primes are prime numbers of the form  $2^p - 1$ , for some prime number  $p$ . The first few Mersenne primes are:

$$\begin{array}{c|cccccccc} p & 2 & 3 & 5 & 7 & 11 & 13 & 17 & 19 & 31 \\ 2^p - 1 & 3 & 7 & 31 & 127 & 8191 & 131071 & 524287 & 2147483647 \end{array}$$

Some Mersenne numbers that are not prime include:

$$\begin{array}{c|cccc} p & 11 & 23 & 29 \\ 2^p - 1 & 2047 = 23 \times 89 & 8388607 = 47 \times 178481 & 536870911 = 233 \times 1103 \times 2089 \end{array}$$

Fermat primes are prime numbers of the form  $2^{2^n} + 1$ . There are only five known Fermat primes:

$$\begin{array}{c|ccccc} n & 0 & 1 & 2 & 3 & 4 \\ 2^{2^n} + 1 & 3 & 5 & 17 & 257 & 65537 \end{array}$$

The famous mathematician Euler showed that

$$2^{2^5} + 1 = 2^{32} + 1 = 4294967297 = 641 \times 6700417$$

Some Fibonacci numbers are prime. Here are the first few:

$$2, 3, 5, 13, 89, 233, 1597, 28657$$

## Pythagorean Triples

Famous Pythagorean triples:

$$\begin{array}{cccc}(3, 4, 5) & (5, 12, 13) & (8, 15, 17) & (7, 24, 25) \\ & (20, 21, 29) & (12, 35, 37) & (9, 40, 41) \\ & (28, 45, 53) & (11, 60, 61) & (16, 63, 65) \\ & (33, 56, 65) & (48, 55, 73) & (13, 84, 85) \\ & (36, 77, 85) & (39, 80, 89) & (65, 72, 97)\end{array}$$

## Useful Products

These factorial products are worth remembering:

$$\begin{array}{l}3! = 6 \\4! = 24 \\5! = 120 \\6! = 720 \\7! = 5040 \\8! = 40320 \\9! = 362880 \\10! = 3628800\end{array}$$

## Useful Squares

These are the first ten squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100. Here are a few more:

$$\begin{array}{cccc}11^2 = 121 & 12^2 = 144 & 11^2 = 121 & 12^2 = 144 \\11^2 = 121 & 12^2 = 144 & 11^2 = 121 & 12^2 = 144\end{array}$$

$$\begin{aligned}11^2 &= 121 \\12^2 &= 144 \\13^2 &= 169 \\14^2 &= 196 \\15^2 &= 225 \\16^2 &= 256 \\17^2 &= 289 \\18^2 &= 324 \\19^2 &= 361 \\20^2 &= 400 \\21^2 &= 441 \\22^2 &= 484 \\23^2 &= 529 \\24^2 &= 576 \\25^2 &= 625 \\30^2 &= 900 \\35^2 &= 1225 \\40^2 &= 1600 \\45^2 &= 2025 \\50^2 &= 2500 \\55^2 &= 3025 \\60^2 &= 3600 \\65^2 &= 4225 \\70^2 &= 4900 \\75^2 &= 5625 \\80^2 &= 6400 \\85^2 &= 7225 \\90^2 &= 8100 \\95^2 &= 9025 \\100^2 &= 10000\end{aligned}$$

## Useful Cubes

$$2^3 = 8$$

$$3^3 = 27$$

$$4^3 = 64$$

$$5^3 = 125$$

$$6^3 = 216$$

$$7^3 = 343$$

$$8^3 = 512$$

$$9^3 = 729$$

$$10^3 = 1000$$

$$11^3 = 1331$$

$$12^3 = 1728$$

$$13^3 = 2197$$

$$14^3 = 2744$$

$$15^3 = 3375$$

$$16^3 = 4096$$

$$17^3 = 4913$$

$$18^3 = 5832$$

$$19^3 = 6859$$

$$20^3 = 8000$$

## Useful Fourth Powers

$$2^4 = 16$$

$$3^4 = 81$$

$$4^4 = 256$$

$$5^4 = 625$$

$$6^4 = 1296$$

$$7^4 = 2401$$

$$8^4 = 4096$$

$$9^4 = 6561$$

$$10^4 = 10000$$

# Some Useful Powers

Memorizing powers can come in handy:

$$2^2 = 4$$

$$2^3 = 8$$

$$2^4 = 16$$

$$2^5 = 32$$

$$2^6 = 64$$

$$2^7 = 128$$

$$2^8 = 256$$

$$2^9 = 512$$

$$2^{10} = 1024$$

$$2^{11} = 2048$$

$$2^{12} = 4096$$

$$2^{13} = 8192$$

$$2^{14} = 16384$$

$$2^{15} = 32768$$

Powers of 3:

$$3^2 = 9$$

$$3^3 = 27$$

$$3^4 = 81$$

$$3^5 = 243$$

$$3^6 = 729$$

$$3^7 = 2187$$

$$3^8 = 6561$$

$$3^9 = 19683$$

$$3^{10} = 59049$$

Powers of 5:

$$5^2 = 25$$

$$5^3 = 125$$

$$5^4 = 625$$

$$5^5 = 3125$$

$$5^6 = 15625$$

$$5^7 = 78125$$

$$5^8 = 390625$$



Powers of 6:

$$6^2 = 36$$

$$6^3 = 216$$

$$6^4 = 1296$$

$$6^5 = 7776$$

Powers of 7:

$$7^2 = 49$$

$$7^3 = 343$$

$$7^4 = 2401$$

$$7^5 = 16807$$

Powers of 11:

$$11^2 = 121$$

$$11^3 = 1331$$

$$11^4 = 14641$$

$$11^5 = 161051$$

## Useful Conversion Rates

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1 ft	=	12 in
1 yd	=	3 ft
1 in	=	2.54 cm
1 m	=	3.28 ft
1 mi	=	1760 yd
1 mi	=	1609 m

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1 ft <sup>2</sup>	=	144 in <sup>2</sup>
1 yd <sup>2</sup>	=	9 ft <sup>2</sup>
1 mi <sup>2</sup>	=	2.59 km <sup>2</sup>
1 mi <sup>2</sup>	=	640 ac <sup>2</sup>
1 ha	=	2.47 ac

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1 ft <sup>3</sup>	=	1728 in <sup>3</sup>
1 yd <sup>3</sup>	=	27 ft <sup>3</sup>
1 in <sup>3</sup>	=	16.39 cm <sup>3</sup>
1 m <sup>3</sup>	=	35.31 ft <sup>3</sup>

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1 qt	=	2 pt
1 pt	=	16 fl oz
1 gal	=	128 fl oz
1 gal	=	4 qt
1 pt	=	473.18 ml
1 fl oz	=	29.57 ml

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# Highly Composite Numbers

A positive integer that has more divisors than any smaller positive integer. A selection:

2

4

6

12

24

36

48

60

120

180

240

360

720

840

1,260

1,680

2,520

5,040

2

$2^2$

$2 \cdot 3$

$2^2 \cdot 3$

$2^3 \cdot 3$

$2^2 \cdot 3^2$

$2^4 \cdot 3$

$2^2 \cdot 3 \cdot 5$

$2^3 \cdot 3 \cdot 5$

$2^2 \cdot 3^2 \cdot 5$

$2^4 \cdot 3 \cdot 5$

$2^3 \cdot 3^2 \cdot 5$

$2^4 \cdot 3^2 \cdot 5$

$2^3 \cdot 3 \cdot 5 \cdot 7$

$2^2 \cdot 3^2 \cdot 5 \cdot 7$

$2^4 \cdot 3^1 \cdot 5 \cdot 7$

$2^3 \cdot 3^2 \cdot 5 \cdot 7$

$2^4 \cdot 3^2 \cdot 5 \cdot 7$

## Useful Irrational Numbers

$$\pi \approx 3.14159 \dots$$

$$e \approx 2.71828 \dots$$

$$\varphi \approx 1.61803 \dots$$

$$\gamma \approx 0.57722 \dots$$

## Useful Square Roots

$$\sqrt{2} \approx 1.414214$$

$$\sqrt{3} \approx 1.732051$$

$$\sqrt{4} = 2$$

$$\sqrt{5} \approx 2.236068$$

$$\sqrt{6} \approx 2.449490$$

$$\sqrt{7} \approx 2.645751$$

$$\sqrt{8} \approx 2.828427$$

$$\sqrt{9} = 3$$

$$\sqrt{10} \approx 3.162278$$

$$\sqrt{11} \approx 3.316625$$

$$\sqrt{12} \approx 3.464102$$

$$\sqrt{13} \approx 3.605551$$

$$\sqrt{14} \approx 3.741657$$

$$\sqrt{15} \approx 3.872983$$

$$\sqrt{16} = 4$$

$$\sqrt{17} \approx 4.123106$$

$$\sqrt{18} \approx 4.242641$$

$$\sqrt{19} \approx 4.358899$$

$$\sqrt{20} \approx 4.472136$$

## Simplifying Numbers

Bombelli, 1572 (« L'algebra »):

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} = 4$$

Leibniz, 1675:

$$\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}} = \sqrt{6}$$

$$\cos(\pi) = -1$$

$$\cos(\pi/2) = 0 \quad \cos(\pi/3)$$

$$\cos(\pi/6) = \frac{\sqrt{3}}{2}$$

$$\cos(0) = 1$$

$$= \frac{1}{2} \cos(\pi/4) = \frac{\sqrt{2}}{2}$$