Russian School of Math Test 1

James & Patrick

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Abstract

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

1

The set S contains nine numbers. The mean of the [ILLEGIBLE] in S is 202. The mean of the five smallest of the numbers in S is 100. The mean of [ILLEGIBLE] largest numbers in S is 300. What is the median of the numbers in S? Solution: 182

2

The parabola $f(x) = 3x^2 + 2x - 6$ intersects the x-axis and y-axis at three different points. The area of the triangle formed by these points is equal to S. Find the least whole n such that $n \geq S$.

3

Find the sum of the digits in the decimal representation of the number $5^{2026} \cdot 16^{506}$.

$$5^{2026} \cdot 16^{506} = 5^{2026} \cdot 2^{2024} = 10^x \implies s = 1$$

4

Let a be the sum of the numbers:

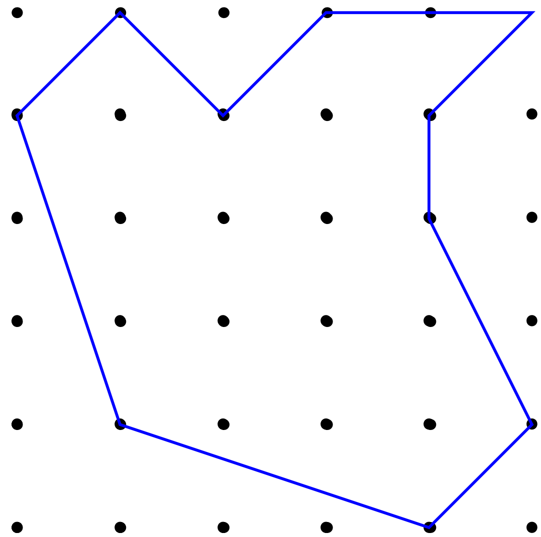
$$\begin{aligned} 99 \times 0.9 \\ 999 \times 0.9 \\ 9999 \times 0.9 \\ \ldots \times \ldots \\ 999\ldots9 \times 0.9 \end{aligned}$$

where the final number in the list is 0.9 times a number written as a string of 101 digits all equal to 9. Find the sum of the digits in the number a.

$$\begin{aligned} a &= 99 \cdot 0.9 + 999 \cdot 0.9 + \ldots + 99\ldots9 \cdot 0.9 \\ &= 0.9(99 + 999 + \ldots + 99\ldots9) \\ &= 0.9((10^2 - 1) + (10^3 - 1) + \ldots + (10^n - 1)) \\ &= (9/10) \cdot (10^2 + 10^3 + \ldots + 10^n - (n-1)) \\ &= 9 \cdot (10 + 10^2 + \ldots + 10^{n-1} - (n-1)/10) \\ n = 101 \to a &= 9 \cdot (10 + 10^2 + \ldots + 10^{100} - 100/10) \\ &= 9 \cdot (10^2 + \ldots + 10^{100}) \\ &= 900 \cdot (1 + 10 + \ldots + 10^{98}) \\ &= 900 \cdot \frac{1 - 10^{99}}{1 - 10} \\ &= 100 \cdot (10^{99} - 1) \\ &= 10^{101} - 100 \end{aligned}$$

5

The grid below contains six rows with six points in each row. Points that are adjacent either horizontally or vertically are a distance of two apart. Find the area of the irregularly shaped ten-sided figure shown.



6

Solve the equation

$$\operatorname{arccot} x = \operatorname{arccot}(-1) = \arctan 2 + \arctan 3 + \arctan 4$$

7

Find the number of pairs of interest (m, n) for which the equality $m^2 + 2^{2024} = n^2$ holds.

8

There are positive integers b and c such that the polynomial $2x^2 + bc = c$ has two real roots which differ by 30. find the least possible value of $b + c$.

9

Find the sum of all such values of a, for each of which equation

$$x^2 + x + a = 0$$

has two different real roots satisfying relation

$$x_1^4 + 2x_1x_2^2 - x_2 = 19.$$

Product: $x_1x_2 = a$. Sum: $x_1 + x_2 = -1$.

$$\begin{aligned} x^2 + x + a &= 0 \\ x_1^2x_1^2 + 2x_1x_2^2 - x_2 &= 19 \\ x_1^2 + 2ax_1 + a^2 - 2x_1(x_2 + a) - x_2 &= 19 \\ -(x_1 + a) + a^2 - 2x_1x_2 - x_2 &= 19 \\ -(x_1 + x_2) + a^2 - 3a &= 19 \\ -1 + a^2 - 3a &= 19 \\ a = \frac{3 \pm \sqrt{9 + 4 \cdot 18}}{2} = \frac{3 \pm \sqrt{81}}{2} &= \frac{3 \pm 9}{2} \\ a &\in (-3, 6) \end{aligned}$$

10

On the side AC of triangle ABC, points M and N are marked such that $\widehat{ABM} = 15°$, $\widehat{MBN} = 45°$, $\widehat{NBC} = 75°$, and the sum and product of the areas of triangle ABM and NBC are equal to 5 and 3 respectively. Find the area of triangle ABC.

11

Suppose that $2024\,x^2 + ax + b$ has 2 equal roots, where a and b are positive integers. Determine the smallest possible value of $a + b$.

$$2024x^2 + ax + b = 0$$

The roots are real if $\Delta = a^2 - 4 \cdot 2024b = 0$.

$$\begin{aligned}
\frac{a^2}{4 \cdot 2024} &= b \\
a + b = a &= \frac{a^2}{4 \cdot 2024} \\
\min(a, b) : 1 + \frac{2}{4 \cdot 2024} \cdot a &= 0 \\
a &= -1\frac{4 \cdot 2024}{2} = -4048 \\
\implies a + b &= -4048 + \frac{(-4048)^2}{4 \cdot 2024} \\
&= -4048 + \frac{4048}{2} \cdot \frac{4048}{4048} \\
&= -4048/2 \\
&= -2024
\end{aligned}$$

Solution: $\boxed{-2024.}$

12

In base b, we have $r = 0.\overline{57}_b$ and $3r = 1.\overline{06}_b$. What is the value of r in base 10? Express your answer as a common fraction.

13

Among the numbers greater than 2025, find the smallest integer N for which the fraction $\frac{15N-7}{22N-5}$ is reducible.

14

Find the largest natural number n for which the number $\dfrac{2024!}{2024^n}$ is whole. Here $2024! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2023 \cdot 2024$.

15

Triangle ABC has side lengths $AB = 71$, $BC = 75$, and $CA = 80$ as shown. Median AD is divided into three congruent segments by points E and F. Lines BE and BF intersect side AC at points G and H, respectively. Find the length of segment GH.