2021 Fall AMC 12B Problems/Problem 20

The following problem is from both the 2021 Fall AMC 12B #20 and 2021 Fall AMC 10B #24, so both problems redirect to this page.

Contents

Problem

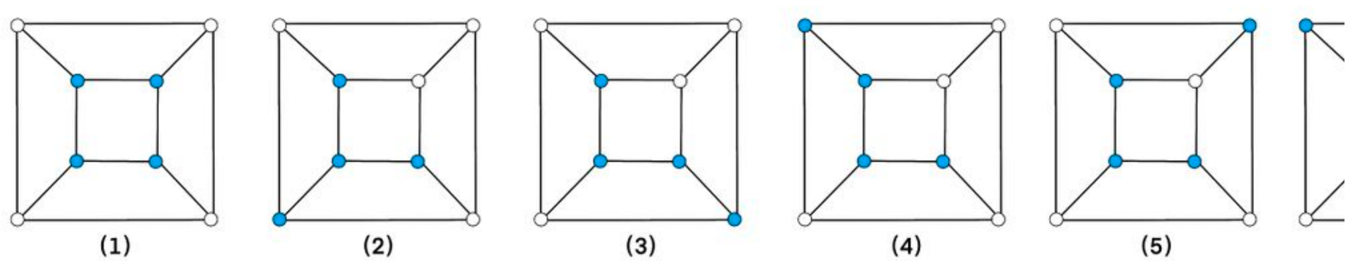
A cube is constructed from 4 white unit cubes and 4 black unit cubes. How many different ways are there to construct the $2 \times 2 \times 2$ cube using these smaller cubes? (Two constructions are considered the same if one can be rotated to match the other.)

$\textbf{(A)}\ 7 \qquad\textbf{(B)}\ 8 \qquad\textbf{(C)}\ 9 \qquad\textbf{(D)}\ 10 \qquad\textbf{(E)}\ 11$

Solution 1

To make drawing diagrams easier, we can represent each smaller cube as a dot, colored black or white - the 8 dots form the vertices of a large cube, and each edge of the large cube represents the shared face between 2 neighboring unit cubes.

We can now draw out all the possibilities we can, making sure that no two possibilities are rotations of one another.



Here is how the 4 blue unit cubes are arranged:

In Figure (1): 4 blue unit cubes are on the same layer (horizontal or vertical).

In Figure (2): 4 blue unit cubes are in T shape.

In Figure (3) and (4): 4 blue unit cubes are in S shape.

In Figure (5): 3 blue unit cubes are in L shape, and the other is isolated without a shared face.

In Figure (6): 2 pairs of neighboring blue unit cubes are isolated from each other without a shared face.

In Figure (7): 4 blue unit cubes are isolated from each other without a shared face.

So the answer is $\boxed{\textbf{(A) } 7}$

~isabelchen (https://artofproblemsolving.com/wiki/index.php/User:Isabelchen)

Solution 2 (Simple Casework)

Let's split the cube into two layers; a bottom and top. Note that there must be four of each color, so however many number of one color are in the bottom, there will be four minus that number of the color on the top. We do casework on the color distribution of the bottom layer.

$\textbf{Case 1:}$ 4, 0

In this case, there is only one possibility for the top layer - all of the other color - $\binom{4}{4}$. Therefore there is 1 construction from this case.

$\textbf{Case 2:}$ 3, 1

In this case, the top layer has four possibilities, because there are four different ways to arrange it so that it also has a 3, 1 color distribution - $\binom{4}{3}$. Therefore there are 4 constructions from this case.

$\textbf{Case 3:}$ 2, 2

In this case, the top layer has six possibilities of arrangement - $\binom{4}{2}$. However, having adjacent colors one way can be rotated to having adjacent colors any other way, so there is only one construction for the adjacent colors subcase and similarly, only one for the diagonal color subcase. Therefore the total number of constructions for this case is 2.

The total number of constructions for the cube is thus $1+4+2=7=\boxed{A}$

~KingRavi

Solution 3 (Direct Counting)

Divide the $2\times2\times2$ cube into two layers, say, front and back. Any possible construction can be rotated such that the front layer has the same or greater number of white cubes than black cubes, so we only need to count the number of cases given that is true.

1. **Case 1: Each layer contains 2 cubes of each color.** Note that we only need to consider the configuration of the white cubes because all the other cubes will be black cubes. There are 2 ways that the 2 white cubes in each layer can be arranged: adjacent or diagonal to each other.
    1. **Case 1.1: Both layers have 2 white cubes adjacent to each other.** Rotate the cube such that there are white cubes along the top edge of the front layer. Now, the white cubes in the back layer can be along the top, bottom, right, or left edges.[See note 1.] So, case 1.1 results in 4 constructions.
    2. **Case 1.2: One layer has 2 white cubes adjacent to each other, and the other has 2 white cubes diagonal from each other.** Rotate the cube such that there are white cubes along the top of the front layer. The white cubes in the back layer can be at the top-left and bottom-right or at the top-right and bottom-left. If we rotate the latter case by 90 degrees clockwise, it becomes the same as the former case. So, case 1.2 results in 1 additional construction.
    3. **Case 1.3: Both layers have white cubes diagonal from each other.** Rotate the cube such that there is a white cube at the top-left and bottom-right of the front layer. The back layer could also have white cubes at the top-left and bottom-right, but this is the same as case 1.1 with the white cubes in the back layer along the bottom edge. Alternatively, the back layer could have white cubes at the top-right and the bottom-left. This is a distinct case. So, case 1.3 results in 1 additional construction.
    4. So, case 1 results in $4+1+1=6$ distinct constructions.
2. **Case 2: The front layer contains 3 white cubes.** In this case, unless the sole black and white cubes in the front and back

layers are on opposite corners of the $2 \times 2 \times 2$ cube, then the $2 \times 2 \times 2$ cube can be split into left and right layers with 2 cubes of each color in each (these constructions were counted in case 1). So, case 2 results in 1 additional construction.

3. **Case 3: The front layer contains 4 white cubes.** Only 1 construction can result from this case. However, if we split this contsruction into its left and right layers, then each layer will have 2 cubes of each color. So, this construction is covered in case 1, and case 3 results in 0 additional constructions.

Therefore, our answer is $6+1+0=\boxed{\textbf{(A) } 7}$.

Notes

1: To prove the 3rd and 4th cases distinct, we can model them with our hands. Extend our thumbs and pointer fingers into an L. These fingers represent the three white cubes on the top layer. Our left and right hands represent the 3rd and 4th cases respectively. The 4th white cube in each case extends down from the tip of each pointer finger towards the rest of each hand. If we overlap our thumbs and pointer fingers, then the 4th cube in each situation will extend outwards in opposite directions, so these cases are distinct.

Solution 4 (Burnside Lemma)

Burnside lemma is used to counting number of orbit where the element on the same orbit can be achieved by the defined operator, naming rotation, reflection and etc.

The fact for Burnside lemma are

1. the sum of stablizer on the same orbit equals to the # of operators;

2. the sum of stablizer can be counted as $fix(g)$

3. the sum of the $fix(g)/|G|$ equals the # of orbit.

Let's start with defining the operator for a cube,

1. $\mathbf{e}$ $\textbf{(identity)}$

For identity, there are $\dfrac{8!}{4!4!} = 70$

2. $\mathbf{r^1}, \mathbf{r^2}, \mathbf{r^3}$ to be the rotation axis along three pair of opposite face,

each contains $r^i_{90}, r^i_{180}, r^i_{270}$ where $i = 1, 2, 3$

$$fix(r^i_{90}) = fix(r^i_{270}) = 2 \cdot 1 = 2$$

$$fix(r^i_{180}) = \frac{4!}{2! \cdot 2!} = 6$$

therefore $fix(\mathbf{r^i}) = \mathbf{2+2+6=10}$, and $fix(\mathbf{r^1}) + \mathbf{fix(r^2)} + \mathbf{fix(r^3)} = \mathbf{30}$

3. $\mathbf{r^4}, \mathbf{r^5}, \mathbf{r^6}, \mathbf{r^7}$ to the rotation axis along four cube diagnals.

each contains r^i_{120}, r^i_{240} where $i = 4, 5, 6, 7$

$$fix(r^i_{120}) = fix(r^i_{240}) = 2 \cdot 1 \cdot 2 \cdot 1 = 4$$

therefore $fix(\mathbf{r^i}) = \mathbf{4+4=8}$, and $fix(\mathbf{r^4}) + \mathbf{fix(r^5)} + \mathbf{fix(r^6)} + \mathbf{fix(r^7)} = \mathbf{32}$

4. $\mathbf{r^8}, \mathbf{r^9}, \mathbf{r^{10}}, \mathbf{r^{11}}, \mathbf{r^{12}}, \mathbf{r^{13}}$ to be the rotation axis along 6 pairs of diagnally opposite sides

each contains r^i_{180} where $i = 8, 9, 10, 11, 12, 13$

$$fix(r^i_{180}) = \frac{4!}{2! \cdot 2!} = 6$$

therefore $fix(\mathbf{r^8}) + \mathbf{fix}(\mathbf{r^9}) + \mathbf{fix}(\mathbf{r^{10}}) + \mathbf{fix}(\mathbf{r^{11}}) + \mathbf{fix}(\mathbf{r^{12}}) + \mathbf{fix}(\mathbf{r^{13}}) = \mathbf{36}$

5. The total number of operators are

$$|G| = 1 + 3 \cdot 3 + 4 \cdot 2 + 6 \cdot 1 = 24$$

Based on 1, 2, 3, 4 the total number of stablizer is $70 + 30 + 32 + 36 = 168$

therefore the number of orbit $= \frac{168}{G = 24} = \boxed{7}$

~wwei.yu

Video Solution

https://youtu.be/Khnq5ZMTwXQ

See Also

2021 Fall AMC 12B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
Preceded by **Problem 19**	Followed by **Problem 21**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 12 Problems and Solutions	

2021 Fall AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
Preceded by **Problem 23**	Followed by **Problem 25**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	