Russian School of Math: Lesson 7

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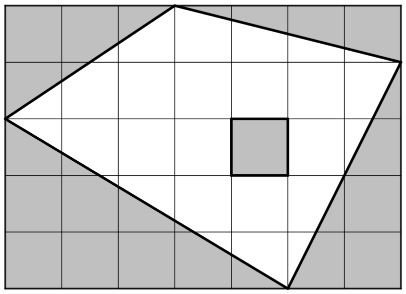
Abstract

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

1

Find the difference of the area of the shaded parts and the area of the white part, if the side length of each square is 1.



(a) -2
(b) -1
(c) 0
(d) 1
(e) 2

Solution

Area of the rectangle:

$$7 \times 5 = 35$$

Area of the 4 outer rectangles, starting from the bottom-left and going around clockwise

$$\frac{3 \times 5}{2} + \frac{2 \times 4}{2} + \frac{1 \times 4}{2} + \frac{2 \times 3}{2} = \frac{15 + 8 + 4 + 6}{2} = \frac{33}{2}$$

Area of the shaded part:

$$\frac{33}{2} + 1 = \frac{35}{2}$$

Area of the white part:

$$35 - \frac{35}{2} = \frac{35}{2}$$

Area of the difference:

$$\frac{35}{2} - \frac{35}{2} = 0$$

2

Triangle ABC has side lengths $AB = 5$, $BC = 6$, and $AC = 7$. Two bugs start simultaneously from A and crawl along the sides of the triangle in opposite directions at the same speed. They meet at point D. What is BD?

(a) 1

(b) 2

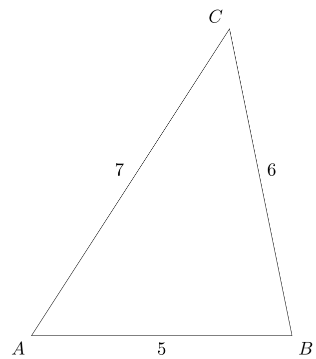
(c) 4

(d) 5

(e) 8

Solution

The perimeter of the triangle is: $5+6+7=18$. The bugs meet at a distance $18/2=9$ from point A. That is $BD=4$.



3

Define $a@b = ab - b^2$ and $a\#b = a + b - ab^2$. Calculate $\dfrac{6@2}{6\#2}$.

(a) -1

(b) $-\dfrac{1}{2}$

(c) $\dfrac{1}{8}$

(d) $\dfrac{1}{4}$

(e) $\dfrac{1}{2}$

Solution

Define $a@b = ab - b^2$ and $a\#b = a + b - ab^2$. Calculate $\dfrac{6@2}{6\#2}$.

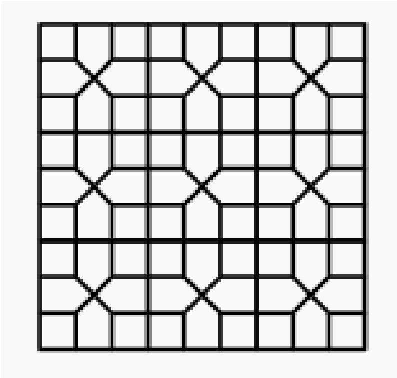
$$6@2 = 6 \times 2 - 2^2 = 8$$

$$6\#2 = 6 + 2 - 6 \times 2^2 = -16$$

$$\frac{6@2}{6\#2} = \frac{8}{-16} = -\frac{1}{2}$$

4

The plane is tiled by congruent squares and congruent pentagones as indicated. The percent of the plane that is enclosed by the pentagons is closest to:



(a) 54
(b) 56
(c) 58
(d) 60
(e) 62

Solution

The square is made up of 9 identical squares, so the percentage can be calculated from a single square. The total area of a square is 9 units. The pentagons occupy $9 - 4 = 5$ units. So the proportion is $5/9 = 0.55.\overline{5}$.
Solution: $\boxed{56.}$

5

If m and b are real numbers and $mb > 0$, then the line whose equation is $y = mx + b$ cannot contain the point:

(a) $(0, 1997)$

(b) $(1997, 0)$

(c) $(19, 97)$

(d) $(19, -97)$

(e) $(0, -1997)$

Solution

Point $(1997, 0)$ cannot lie on the line, because it would imply $mb < 0$. Substitute the values of x and y into the equation:

$$y = mx + b$$
$$0 = 1997m + b \implies b = -1997m \implies mb = -1997m^2 < 0$$

6

Points A, B, C, and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line and $BE = CE = 10$. The perimeter of $\triangle AED$ is twice the perimeter of $\triangle BEC$. Find AB.

(a) $\dfrac{17}{2}$

(b) 9

(c) $\dfrac{19}{2}$

(d) 10

(e) $\dfrac{12}{2}$

Solution

Let F denote the midpoint of BC.

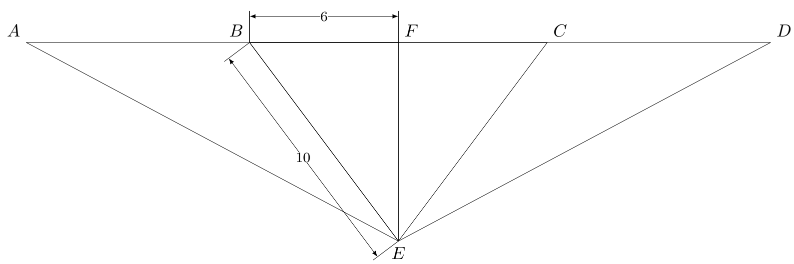


Figure 1

Distance EF is the height of the right triangle $\triangle BFE$.

$$EF^2 + 6^2 = 10^2$$

Distance EF is also the height of the right triangle $\triangle AFE$.

$$EF^2 + (AB + 6)^2 = AE^2$$

Combining the two equation to eliminate EF^2 gives:

$$10^2 - 6^2 + (AB + 6)^2 = AE^2 \implies AB^2 + 12AB + 10^2 = AE^2$$

The perimeter of triangle $\triangle BEC$ is:

$$BE + EC + BC = 10 + 10 + 12 = 32$$

The perimeter of triangle $\triangle AED$ is:

$$AE + ED + AD = 2AE + 2(AB + 6)$$

Since perimeter of triangle $\triangle AED$ is twice the perimeter of triangle $\triangle BEC$:

$$AE + (AB + 6) = 32 \implies AE + AB = 26 \implies AE^2 = (26 - AB)^2 = 26^2 - 52AB + AB^2$$

Substituting back:

$$\begin{aligned} AB^2 + 12AB + 10^2 = AE^2 = 26^2 - 52AB + AB^2 \\ \implies 64AB = 26^2 - 10^2 = 4(13^2 - 5^2) \\ \implies AB = \frac{169 - 25}{16} = \frac{169 - 25}{16} = 9 \end{aligned}$$

7

Square $ABCD$ has side length s, a circle centered at E has radius r, where r and s are both rational. The circle passes through D, where D lies on $\overline{BE}$. Point F lies on the circle, on the same side of $\overline{BE}$ as A. Segment AF is tangent to the circle and $AF = \sqrt{9 + 5\sqrt{2}}$. Calculate r/s.

(a) $\frac{1}{2}$

(b) $\frac{4}{7}$

(c) $\frac{2}{3}$

(d) $\frac{5}{9}$

(e) $\frac{9}{5}$

Solution

F denotes the point of tangency with the circle. There are two possible configurations, one where the circle lies inside the square (2), one where it lies outside (3). Given the value of AF, the second configuration applies.

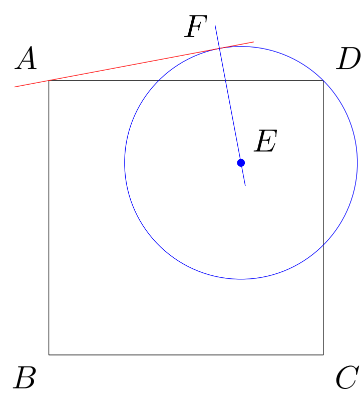


Figure 2

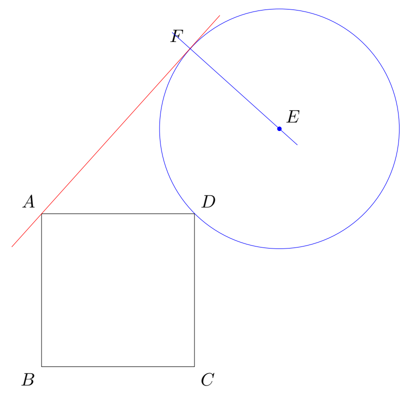


Figure 3

Let $B = (0,0)$, $C = (s,0)$, $A = (0,s)$, $D = (s,s)$, and $E = \left(s + \frac{r}{\sqrt{2}}, s + \frac{r}{\sqrt{2}}\right)$. By the Pythagorean Theorem,

$$r^2 + \left(9 + 5\sqrt{2}\right) = \left(s + \frac{r}{\sqrt{2}}\right)^2 + \left(\frac{r}{\sqrt{2}}\right)^2$$
$$s^2 + rs\sqrt{2} = 9 + 5\sqrt{2}$$

Since r and s are rational, we must have $s^2 = 9$ and $rs = 5$.
Solution: $\boxed{r/s = 5/9.}$

8

There is a smallest positive real number a such that there exists a positive real number b such that all the roots of the polynomial $x^3 - ax^2 + bx - a$ are real. In fact, for this value of a, the value of b is unique. Find the value of b.

(a) 6

(b) 7

(c) 8

(d) 9

(e) 10

Solution

Let the roots of the equation be r_1, r_2 and r_3.
If the cubic equation has real roots, they must be positive. This follows from:

$$x < 0, \quad a > 0, \quad b > 0 \implies x^3 - ax^2 + bx - a < 0$$

Vieta's identities applied to the equation give:

$$\begin{aligned} r_1 + r_2 + r_3 &= a > 0 \\ r_1\, r_2\, r_3 &= a > 0 \\ r_1\, r_2 + r_2\, r_3 + r_1\, r_3 &= b > 0 \end{aligned}$$

The AM-GM inequality (The inequality of the Arithmetic Mean and Geometric Mean) gives

$$\begin{aligned} &\tfrac{1}{3}(r_1 + r_2 + r_3) \geq \sqrt[3]{r_1\, r_2\, r_3} \\ &\implies \frac{a}{3} \geq \sqrt[3]{a^3} \\ &\implies a \geq 3\sqrt{3} \end{aligned}$$

The smallest value of a is $3\sqrt{3}$. With $r_1 r_2 = r_3 = \sqrt{3}$, we get

$$b = r_1\, r_2 + r_2\, r_3 + r_1\, r_3 = 3 \times \sqrt{3} \times \sqrt{3} = 9$$