Ask Math Anything

Daily Challenge with Po-Shen Loh

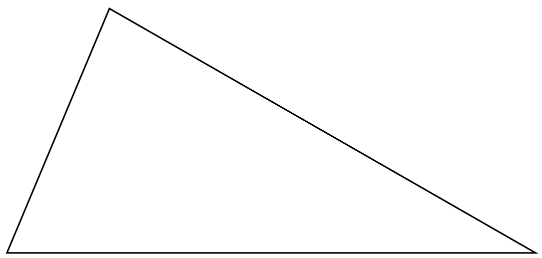
24 June 2020

Abstract

Professor Po-Shen Loh solves problems on his YouTube channel. A selection for practice.
Reference: Ask Math Anything - Daily Challenge with Po-Shen Loh

Lengths of Sides of a Triangle

The lengths of the sides of a triangle are 3 consecutive integers. The length of the shortest side is equal to 30% of the perimeter. What is the length of the longest side?



The available data states that the Short (S), Middle (M), and Long (L) sides are a certain proportion of the (unknown) perimeter P:

$$\begin{aligned} \text{Short}: &\ S = 0.3P \\ \text{Middle}: &\ 0.3P < M < 0.35P \\ \text{Long}: &\ 0.3P < L < 0.35P \end{aligned}$$

The inequalities follow from the simple consideration that M and L must each be greater than $S = 0.3P$, while M and L cannot have a sum greater than $0.7P$, from which it follows that M certainly cannot be greater than $0.35P$. And of course $L > M$. But we also know that the sides differ by 1, so

$$M - S = L - M = 1$$

Thus, whatever P may be, M must be exactly one third of the perimeter, $P/3$. This is the crucial insight that allows you to solve the problem quickly. In words, the percentage taken away by the short side must be exactly offset by the long side. To sum up,

$$\begin{aligned} S &= \tfrac{3}{10}\ P \\ M &= \tfrac{1}{3}\ P \end{aligned}$$

We know that M and S differ by 1. And from the above, we know that they differ by $\frac{1}{30}P$. That is,

$$\begin{aligned} & M - S = \tfrac{1}{3}\ P - \tfrac{3}{10}\ P = \tfrac{1}{30}\ P = 1 \\ \Rightarrow \quad & P = 30 \end{aligned}$$

From this, the long side follows immediately,

$$\begin{aligned}
L &= P - M - S \\
&= P - \frac{1}{3}\,P - \frac{3}{10}\,P \\
&= P\left(1 - \frac{1}{3} - \frac{3}{10}\right) \\
&= 30\frac{30-10-9}{30} && = 11
\end{aligned}$$

In conclusion: The perimeter of the triangle is 30. The long length is 11. The middle length is 10. The short length is 9. Answer: $\boxed{(9, 10, 11)}$

This problem could also be solved with an equation. The problem is to find the perimeter x such that:

$$\frac{3}{10}x + 1 = \frac{1}{3}x$$

You may verify that this yields the same answer. It may seem to be quicker. However, it requires that you put together all the constraints in one step. And the right-hand side of the equation still must be deduced from an argument like the one above, exploiting knowledge that the middle length is the mean of the perimeter (for three lengths, the mean length is one third of the perimeter).

Base Problem

What is the value in base 10 of the base-3 number $0.121212\ldots$?

Fun fact: Base 3 is also called the "ternary numeral system."

This base-3 number may be expressed in base 10 as:

$$0.121212\ldots|_3 = 1 \times \frac{1}{3} + 2 \times \frac{1}{3^2} + 1 \times \frac{1}{3^3} + 2 \times \frac{1}{3^4} + \ldots$$

The sum on the right-hand side involves an infinite sum of fractions with powers of 3 in the denominator that are multiplied by either 1 or 2. Group together the multiples of 1 and separately the multiples of 2, on two lines:

$$\begin{aligned}
0.121212\ldots|_3 = &\frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \ldots && \leftarrow S \\
&\frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6} + \ldots && \leftarrow \frac{2}{3} \times S
\end{aligned}$$

We denote the odd-powers-of-3 sum as S and note that the sum involving even powers of 3 is related to S as follows:

$$\frac{2}{3} \times \left(\frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \ldots\right) = \frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6} + \ldots$$

So now to express $0.121212\ldots|_3$ in base-10, we need only calculate the infinite sum S. Does this sum converge? If it didn't, it would mean that the number $0.121212\ldots|_3$ couldn't be expressed in base 10, which would be extraordinary. A base is a representation of a number, so all numbers that exist in one base must exist in another base.

To calculate S, note

$$\begin{aligned} S &= \frac{1}{3} + \frac{1}{3^3} + \frac{1}{3^5} + \dots \\ &= \frac{1}{3} + \frac{1}{3^2}\left(\frac{1}{3^1} + \frac{1}{3^3} + \dots\right) \\ &= \frac{1}{3} + \frac{S}{9} \end{aligned}$$

And from this calculating S is easy:

$$\begin{aligned} 9S &= 3 + S \\ S &= \frac{3}{8} \end{aligned}$$

And for the second term:

$$\frac{2}{3}S = \frac{2}{3} \times \frac{3}{8} = \frac{1}{4}$$

Now putting it together:

$$0.121212\ldots|_3 = \frac{3}{8} + \frac{1}{4} = \frac{5}{8}$$

Answer: $\boxed{0.121212\ldots|_3 = \frac{5}{8}}$.

There is a "nifty" trick to get this answer in fewer steps:

$$0.121212\ldots|_3 = \frac{12|_3}{22|_3} = \frac{5}{8}$$

To get 12 and 22 in base 3 requires simple mental math:

$$\begin{aligned} 12|_3 &= 1 \times 3 + 2 = 5 \\ 22|_3 &= 2 \times 3 + 2 = 8 \end{aligned}$$

But where does $\frac{12|_3}{22|_3}$ come from? Consider the analogy:

$$\begin{aligned} 0.121212\ldots|_{10} &= \frac{12}{99} \\ 0.121212\ldots|_3 \;&= \frac{12|_3}{22|_3} \end{aligned}$$

In base 10, twelve divided by 99 yields the repeated decimals $121212\ldots$. In base 3, twelve divided by 99 "becomes" twelve divided by 22, where 22 is the new 99: it is the double-digit integer after which we jump to three-digit integers: In base 3, we jump from 22 to 100. What is $0.010101\ldots$ in base 3? Let

$$S = 0.010101\ldots|_3 = \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \dots$$

With similar steps as above, except easier, one gets:

$$S = \frac{1}{9} + \frac{S}{9} \quad \Rightarrow \quad S = 0.010101\ldots|_3 = \frac{1}{8}$$

Finally,

$$\begin{aligned} 0.121212\ldots|_3 &= 12|_3 \times 0.010101\ldots|_3 \\ &= 12|_3 \times \frac{1}{8} \\ &= \frac{12|_3}{22|_3} \end{aligned}$$

This explains why it makes sense to treat division by $22|_3$ as division by 99 in base 10.

Large Mystery Number

A 10-digit number. The first digit is divisible by 1. The number formed by the first two digits are divisible by 2. The number formed by the first three digits are divisible by 3. And so on for 10 digits. What is the number?

$$\begin{aligned} 1 &\text{ is divisble by } 1 \\ 10 &\text{ is divisble by } 2 \\ 102 &\text{ is divisble by } 3 \\ 1020 &\text{ is divisble by } 4 \\ 10200 &\text{ is divisble by } 5 \\ 102000 &\text{ is divisble by } 6 \\ 1020005 &\text{ is divisble by } 7 \\ 10200056 &\text{ is divisble by } 8 \\ 102000564 &\text{ is divisble by } 9 \\ 1020005640 &\text{ is divisble by } 10 \end{aligned}$$

The divisibility by 7 was a little tricky:

$$\begin{array}{r@{\,}l} & 145715 \\ 7\,) & \overline{1020005} \\ & \underline{7} \\ & 32 \\ & \underline{28} \\ & 40 \\ & \underline{35} \\ & 50 \\ & \underline{49} \\ & 10 \\ & \underline{7} \\ & 35 \\ & \underline{35} \\ & 0 \end{array}$$

Answer: $\boxed{1020005640}$. We have found one number with the stated properties. It would be tedious to figure out exactly how many such numbers there may be. Perhaps there are none. Perhaps there are many.

Special 64

Show that 64 is the smallest integer greater than 1 which is both a perfect square and a perfect cube.

Thus, 64 satisfies the relation:

$$x = a^2 = b^3$$

In prime factorization, every prime appears a multiple-of-2 times (perfect square). And every prime appears a multiple-of-3 times. A number that is both a multiple of 2 and 3 is a multiple of 6. So every prime appears a multiple of 6 times! And 2 is the smallest prime. Indeed, we have:

$$\begin{aligned} 64 &= \underbrace{2 \times 2 \times 2 \times 2 \times 2 \times 2}_{6 \text{ times}} \\ &= \underbrace{2 \times 2 \times 2}_{3 \text{ times}} \times \underbrace{2 \times 2 \times 2}_{3 \text{ times}} \\ &= \underbrace{2 \times 2}_{2 \text{ times}} \times \underbrace{2 \times 2}_{2 \text{ times}} \times \underbrace{2 \times 2}_{2 \text{ times}} \end{aligned}$$

Answer: $\boxed{64 = 8^2 = 4^3}$

In general, any number of the form p^6, for p prime has the desired property. For $p = 3$, $3^6 = 729$.