2021 AMC 10A Problems/Problem 13

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Problem

What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$, and $CD = 5$?

$\textbf{(A)}\ 3 \qquad\textbf{(B)}\ 2\sqrt{3} \qquad\textbf{(C)}\ 4 \qquad\textbf{(D)}\ 3\sqrt{3} \qquad\textbf{(E)}\ 6$

Solution 1 (Three Right Triangles)

Drawing the tetrahedron out and testing side lengths, we realize that the $\triangle ACD$, $\triangle ABC$, and $\triangle ABD$ are right triangles by the Converse of the Pythagorean Theorem. It is now easy to calculate the volume of the tetrahedron using the formula for the volume of a pyramid. If we take $\triangle ADC$ as the base, then $\overline{AB}$ must be the altitude. The volume of tetrahedron $ABCD$ is $\frac{1}{3} \cdot \frac{3 \cdot 4}{2} \cdot 2 = \boxed{\textbf{(C)}\ 4}$.

~Icewolf10 ~Bakedpotato66 ~MRENTHUSIASM

Solution 2 (One Right Triangle)

We will place tetrahedron $ABCD$ in the xyz-plane. By the Converse of the Pythagorean Theorem, we know that $\triangle ACD$ is a right triangle. Without the loss of generality, let $A = (0,0,0)$, $C = (3,0,0)$, $D = (0,4,0)$, and $B = (x,y,z)$.

We apply the Distance Formula to $\overline{BA}$, $\overline{BC}$, and $\overline{BD}$, respectively:

$$\begin{align*} x^2 + y^2 + z^2 &= 2^2, & (1) \\ (x-3)^2 + y^2 + z^2 &= \sqrt{13}^2, & (2) \\ x^2 + (y-4)^2 + z^2 &= \left(2\sqrt{5}\right)^2. & (3) \end{align*}$$

Subtracting (1) from (2) gives $-6x + 9 = 9$, from which $x = 0$.

Subtracting (1) from (3) gives $-8y + 16 = 16$, from which $y = 0$.

Substituting $(x,y) = (0,0)$ into (1) produces $z^2 = 4$, or $|z| = 2$.

Let the brackets denote areas. Finally, we find the volume of tetrahedron $ABCD$ using $\triangle ACD$ as the base:

$$V_{ABCD} = \frac{1}{3} \cdot [ACD] \cdot h_B$$

$$= \frac{1}{3} \cdot \left(\frac{1}{2} \cdot AC \cdot AD\right) \cdot |z|$$
$$= \boxed{\textbf{(C)}\ 4}.$$

~MRENTHUSIASM

Solution 3 (Trirectangular Tetrahedron)

https://mathworld.wolfram.com/TrirectangularTetrahedron.html

Given the observations from Solution 1, where $\triangle ACD, \triangle ABC,$ and $\triangle ABD$ are right triangles, the base is $\triangle ABD.$ We can apply the information about a trirectangular tetrahedron (all of the face angles are right angles), which states that the volume is

$$\begin{aligned} V &= \frac{1}{6} \cdot AB \cdot AD \cdot BD \\ &= \frac{1}{6} \cdot 2 \cdot 4 \cdot 3 \\ &= \boxed{\textbf{(C)}\ 4}. \end{aligned}$$

~AMC60 (Solution)

~MRENTHUSIASM (Revision)

Remark

Here is a similar problem from another AMC test: 2015 AMC 10A Problem 21.

Video Solution (Simple & Quick)

https://youtu.be/bRrchiDCrKE

~ Education, the Study of Everything

Video Solution (Using Pythagorean Theorem, 3D Geometry: Tetrahedron)

https://youtu.be/i4yUaXVUWKE

~ pi_is_3.14

Video Solution by TheBeautyofMath

https://youtu.be/t-EEP2V4nAE?t=813

~IceMatrix

See also

2021 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
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