2021 Fall AMC 12B Problems/Problem 5

The following problem is from both the 2021 Fall AMC 10B #7 and 2021 Fall AMC 12B #5, so both problems redirect to this page.

Contents

Problem

Call a fraction $\frac{a}{b}$, not necessarily in the simplest form, *special* if a and b are positive integers whose sum is 15. How many distinct integers can be written as the sum of two, not necessarily different, special fractions?

$\textbf{(A)}\ 9 \qquad \textbf{(B)}\ 10 \qquad \textbf{(C)}\ 11 \qquad \textbf{(D)}\ 12 \qquad \textbf{(E)}\ 13$

Solution 1

The special fractions are

$$\frac{1}{14}, \frac{2}{13}, \frac{3}{12}, \frac{4}{11}, \frac{5}{10}, \frac{6}{9}, \frac{7}{8}, \frac{8}{7}, \frac{9}{6}, \frac{10}{5}, \frac{11}{4}, \frac{12}{3}, \frac{13}{2}, \frac{14}{1}.$$

We rewrite them in the simplest form:

$$\frac{1}{14}, \frac{2}{13}, \frac{1}{4}, \frac{4}{11}, \frac{1}{2}, \frac{2}{3}, \frac{7}{8}, 1\frac{1}{7}, 1\frac{1}{2}, 2, 2\frac{3}{4}, 4, 6\frac{1}{2}, 14.$$

Note that two unlike fractions in the simplest form cannot sum to an integer. So, we only consider the fractions whose denominators appear more than once:

$$\frac{1}{4}, \frac{1}{2}, 1\frac{1}{2}, 2, 2\frac{3}{4}, 4, 6\frac{1}{2}, 14.$$

For the set $\{2, 4, 14\}$, two elements (not necessarily different) can sum to $4, 6, 8, 16, 18, 28$.

For the set $\left\{\frac{1}{2}, 1\frac{1}{2}, 6\frac{1}{2}\right\}$, two elements (not necessarily different) can sum to $1, 2, 3, 7, 8, 13$.

For the set $\left\{\frac{1}{4}, 2\frac{3}{4}\right\}$, two elements (not necessarily different) can sum to 3.

Together, there are $\boxed{\textbf{(C)}\ 11}$ distinct integers that can be written as the sum of two, not necessarily different, special fractions:

$$1, 2, 3, 4, 6, 7, 8, 13, 16, 18, 28.$$

~KingRavi ~samrocksnature ~Wilhelm Z ~MRENTHUSIASM ~Steven Chen (www.professorchenedu.com)

Solution 2

Let $a = 15 - b$, so the special fraction is

$$\frac{a}{b} = \frac{15-b}{b} = \frac{15}{b} - 1.$$

We can ignore the -1 part and only focus on $\frac{15}{b}$.

The integers are $\frac{15}{1}, \frac{15}{3}, \frac{15}{5}$, which are $15, 5, 3$, respectively. We get $30, 20, 18, 10, 8, 6$ from this group of numbers.

The halves are $\frac{15}{2}, \frac{15}{6}, \frac{15}{10}$, which are $7\frac{1}{2}, 2\frac{1}{2}, 1\frac{1}{2}$, respectively. We get $15, 10, 9, 5, 4, 3$ from this group of numbers.

The quarters are $\frac{15}{4}, \frac{15}{12}$, which are $3\frac{3}{4}, 1\frac{1}{4}$, respectively. We get 5 from this group of numbers.

Note that 10 and 5 each appear twice. Therefore, the answer is $\boxed{\textbf{(C)}\ 11}$.

~lopkiloinm

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2021 Fall AMC 12B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
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