2012 AIME II Problems/Problem 15

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Problem 15

Triangle ABC is inscribed in circle ω with $AB=5$, $BC=7$, and $AC=3$. The bisector of angle A meets side $\overline{BC}$ at D and circle ω at a second point E. Let γ be the circle with diameter $\overline{DE}$. Circles ω and γ meet at E and a second point F. Then $AF^2 = \frac mn$, where m and n are relatively prime positive integers. Find $m+n$.

Quick Solution using Olympiad Terms

Take a force-overlaid inversion about A and note D and E map to each other. As DE was originally the diameter of γ, DE is still the diameter of γ. Thus γ is preserved. Note that the midpoint M of BC lies on γ, and BC and ω are swapped. Thus points F and M map to each other, and are isogonal. It follows that AF is a symmedian of $\triangle ABC$, or that $ABFC$ is harmonic. Then $(AB)(FC)=(BF)(CA)$, and thus we can let $BF=5x, CF=3x$ for some x. By the LoC, it is easy to see $\angle BAC = 120^\circ$ so $(5x)^2+(3x)^2-2\cos 60^\circ (5x)(3x) = 49$. Solving gives $x^2=\frac{49}{19}$, from which by Ptolemy's we see $AF = \frac{30}{\sqrt{19}}$. We conclude the answer is $900+19=\boxed{919}$.

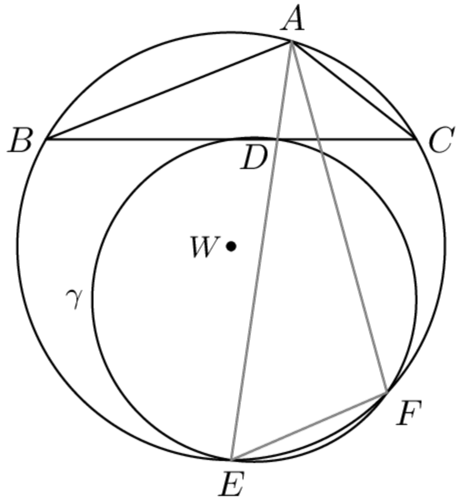
- Emathmaster

Side Note: You might be wondering what the motivation for this solution is. Most of the people who've done EGMO Chapter 8 should recognize this as problem 8.32 (2009 Russian Olympiad) with the computational finish afterwards. Now if you haven't done this, but still know what inversion is, here's the motivation. We'd see that it's kinda hard to angle chase, and if we could, it would still be a bit hard to apply (you could use trig, but it won't be so clean most likely). If you give up after realizing that angle chasing won't work, you'd likely go in a similar approach to Solution 1 (below) or maybe be a bit more insightful and go with the elementary solution above.

Finally, we notice there's circles! Classic setup for inversion! Since we're involving an angle-bisector, the first thing that comes to mind is a force overlaid inversion described in Lemma 8.16 of EGMO (where we invert with radius $\sqrt{AB \cdot AC}$ and center A, then reflect over the A-angle bisector, which fixes B, C). We try applying this to the problem, and it's fruitful - we end up with this solution. -MSC

Solution 1

Use the angle bisector theorem to find $CD=\tfrac{21}{8}$, $BD=\tfrac{35}{8}$, and use Stewart's Theorem to find $AD=\tfrac{15}{8}$. Use Power of Point D to find $DE=\tfrac{49}{8}$, and so $AE=8$. Use law of cosines to find $\angle CAD = \tfrac{\pi}{3}$, hence $\angle BAD = \tfrac{\pi}{3}$ as well, and $\triangle BCE$ is equilateral, so $BC = CE = BE = 7$.



In triangle AEF, let X be the foot of the altitude from A; then $EF = EX + XF$, where we use signed lengths. Writing $EX = AE \cdot \cos \angle AEF$ and $XF = AF \cdot \cos \angle AFE$, we get

$$EF = AE \cdot \cos \angle AEF + AF \cdot \cos \angle AFE. \qquad (1)$$

Note $\angle AFE = \angle ACE$, and the Law of Cosines in $\triangle ACE$ gives $\cos \angle ACE = -\frac{1}{7}$. Also, $\angle AEF = \angle DEF$, and $\angle DFE = \frac{\pi}{2}$ (DE is a diameter), so $\cos \angle AEF = \frac{EF}{DE} = \frac{8}{49} \cdot EF$.

Plugging in all our values into equation (1), we get:

$$EF = \tfrac{64}{49} EF - \tfrac{1}{7} AF \quad \Longrightarrow \quad EF = \tfrac{7}{15} AF.$$

The Law of Cosines in $\triangle AEF$, with $EF = \frac{7}{15} AF$ and $\cos \angle AFE = -\frac{1}{7}$ gives

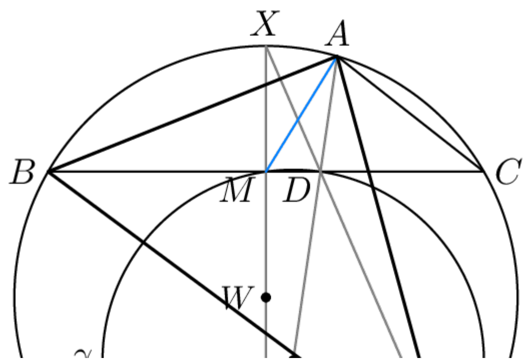
$$8^2 = AF^2 + \tfrac{49}{225} AF^2 + \tfrac{2}{15} AF^2 = \tfrac{225+49+30}{225} \cdot AF^2$$

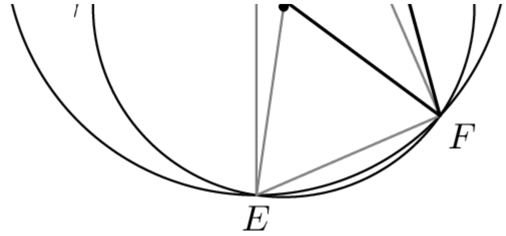
Thus $AF^2 = \dfrac{900}{19}$. The answer is $\boxed{919}$. ~Shen Kislay Kai

Solution 2

Let $a = BC, b = CA, c = AB$ for convenience. Let M be the midpoint of segment BC. We claim that $\angle MAD = \angle DAF$.

$Proof$. Since AE is the angle bisector, it follows that $EB = EC$ and consequently $EM \perp BC$. Therefore, $M \in \gamma$. Now let $X = FD \cap \omega$. Since $\angle EFX = 90^\circ$, EX is a diameter, so X lies on the perpendicular bisector of BC; hence E, M, X are collinear. From $\angle DAG = \angle DMX = 90^\circ$, quadrilateral $ADMX$ is cyclic. Therefore, $\angle MAD = \angle MXD$. But $\angle MXD$ and $\angle EAF$ are both subtended by arc EF in ω, so they are equal. Thus $\angle MAD = \angle DAF$, as claimed.





As a result, $\angle CAM = \angle FAB$. Combined with $\angle BFA = \angle MCA$, we get $\triangle ABF \sim \triangle AMC$ and therefore

$$\frac{c}{AM} = \frac{AF}{b} \qquad \Longrightarrow \qquad AF^2 = \frac{b^2c^2}{AM^2} = \frac{15^2}{AM^2}$$

By Stewart's Theorem on $\triangle ABC$ (with cevian AM), we get

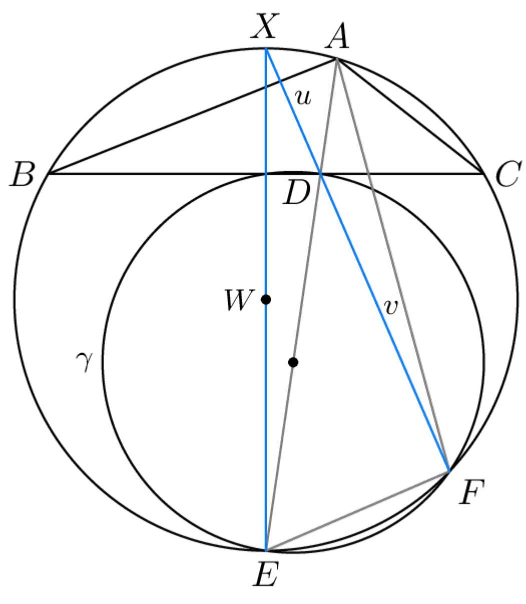
$$AM^2 = \tfrac{1}{2}(b^2 + c^2) - \tfrac{1}{4}a^2 = \tfrac{19}{4},$$

so $AF^2 = \frac{900}{19}$, so the answer is $900 + 19 = \boxed{919}$.

-Solution by thecmd999

Solution 3

Use the angle bisector theorem to find $CD = \frac{21}{8}$, $BD = \frac{35}{8}$, and use Stewart's Theorem to find $AD = \frac{15}{8}$. Use Power of Point D to find $DE = \frac{49}{8}$, and so $AE = 8$. Then use the Extended Law of Sine to find that the length of the circumradius of $\triangle ABC$ is $\frac{7\sqrt{3}}{3}$.



Since DE is the diameter of circle γ, $\angle DFE$ is 90°. Extending DF to intersect circle ω at X, we find that XE is the diameter of ω (since $\angle DFE$ is 90°). Therefore, $XE = \frac{14\sqrt{3}}{3}$.

Let $EF = x$, $XD = u$, and $DF = v$. Then $XE^2 - XF^2 = EF^2 = DE^2 - DF^2$, so we get

$$(u + v)^2 - v^2 = \frac{196}{3} - \frac{2401}{64}$$

which simplifies to

$$u^2 + 2uv = \frac{5341}{}$$

$$192.$$

By Power of Point D, $uv = BD \cdot DC = 735/64$. Combining with above, we get

$$XD^2 = u^2 = \frac{931}{192}.$$

Note that $\triangle XDE \sim \triangle ADF$ and the ratio of similarity is $\rho = AD : XD = \frac{15}{8} : u$. Then $AF = \rho \cdot XE = \frac{15}{8u} \cdot R$ and

$$AF^2 = \frac{225}{64} \cdot \frac{R^2}{u^2} = \frac{900}{19}.$$

The answer is $900 + 19 = \boxed{919}$.

-Solution by TheBoomBox77

Solution 4

Use Law of Cosines in $\triangle ABC$ to get $\angle BAC = 120^\circ$. Because AE bisects $\angle A$, E is the midpoint of major arc BC so $BE = CE$, and $\angle BEC = 60^\circ$. Thus $\triangle BEC$ is equilateral. Notice now that $\angle BFC = \angle BFE = 60^\circ$. But $\angle DFE = 90^\circ$ so FD bisects $\angle BFC$. Thus,

$$\frac{BF}{CF} = \frac{BD}{CD} = \frac{BA}{CA} = \frac{5}{3}.$$

Let $BF = 5k, CF = 3k$. Use Law of Cosines on $\triangle BFC$ to get

$$25k^2 + 9k^2 - 15k^2 = 49 \qquad \Longrightarrow \qquad k = \frac{7}{\sqrt{19}}$$

Use Ptolemy's Theorem on $BFCA$, to get

$$15k + 15k = 7 \cdot AF, \qquad \Longrightarrow \qquad AF = \frac{30}{\sqrt{19}},$$

so $AF^2 = \dfrac{900}{19}$ and the answer is $900 + 19 = 919$

~Shen Kislay Kai

Solution 5

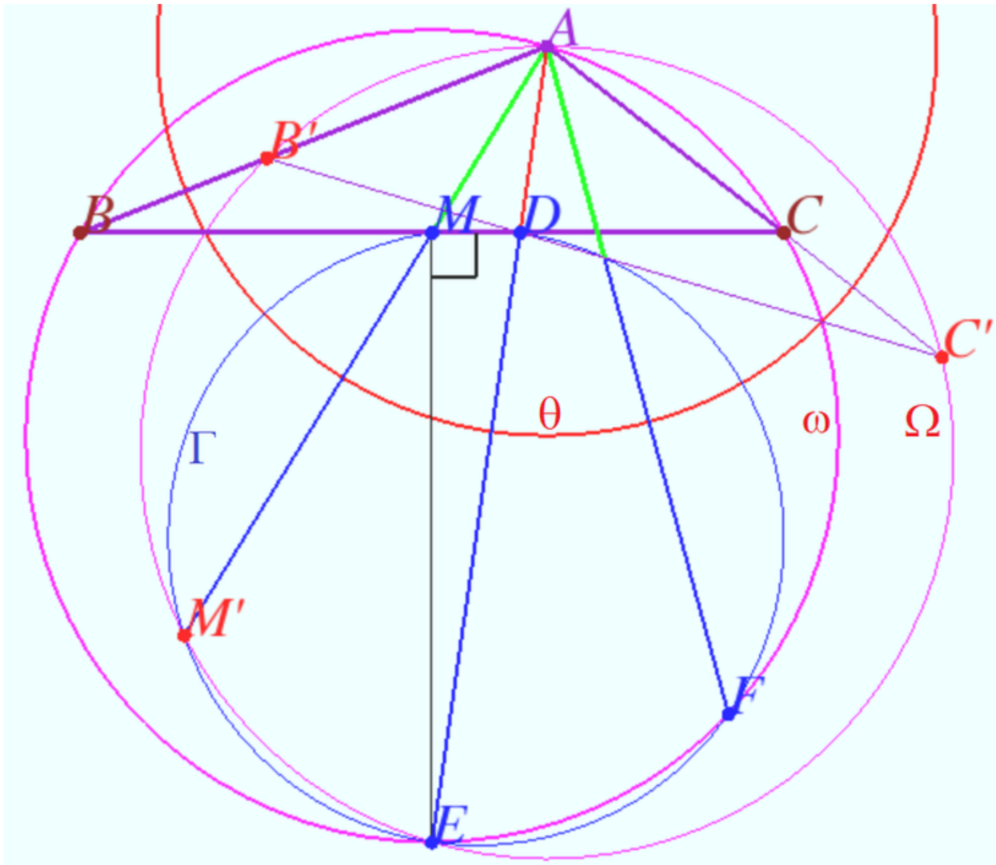
Denote $AB = c, BC = a, AC = b, \angle A = 2\alpha.$ Let M be midpoint BC. Let θ be the circle centered at A with radius $\sqrt{AB \cdot AC} = \sqrt{bc}.$

We calculate the length of some segments. The median $AM = \sqrt{\frac{b^2}{2} + \frac{c^2}{2} - \frac{a^2}{4}}$. The bisector $AD = \dfrac{2bc\cos\alpha}{b+c}$. One can use Stewart's Theorem in both cases.

AD is bisector of $\angle A \implies BD = \dfrac{ac}{b+c}, CD = \dfrac{ab}{b+c} \implies$

$$BD \cdot CD = \frac{a^2bc}{(b+c)^2}.$$

We use Power of Point D and get $AD \cdot DE = BD \cdot CD.$



$$AE = AD + DE = AD + \frac{BD \cdot CD}{AD},$$

$$AE = \frac{2bc\cos\alpha}{b+c} + \frac{a^2bc \cdot (b+c)}{(b+c)^2 \cdot 2bc\cos\alpha} =$$

$$= \frac{bc\cos^2\alpha + a^2}{2(b+c)\cos\alpha} = \frac{4bc\cos^2\alpha + b^2 + c^2 - 2bc\cos 2\alpha}{2(b+c)\cos\alpha} = \frac{b+c}{2} \implies AD \cdot AE = 2bc\cos\alpha.$$

We consider the inversion with respect θ.

B swap $B' \implies AB' = AC, B' \in AB \implies B'$ is symmetric to C with respect to $AE.$

C swap $C' \implies AC' = AB, C'$ lies on line $AC \implies C'$ is symmetric to B with respect to $AE.$

$BC^2 = AB^2 + AC^2 + AB \cdot BC \implies \alpha = 60^\circ \implies AD \cdot AE = bc \implies D$ swap $E.$

Points D and E lies on $\Gamma \implies \Gamma$ swap $\Gamma.$

DE is diameter $\Gamma, \angle DME = 90^\circ \implies M \in \Gamma.$ Therefore M is crosspoint of BC and $\Gamma.$

Let Ω be circumcircle $AB'C'.\Omega$ is image of line BC. Point M maps into $M' \implies M' = \Gamma \cap \Omega.$

Points A, B', and C' are symmetric to A, C, and B, respectively.

Point M' lies on Γ which is symmetric with respect to AE and on Ω which is symmetric to ω with respect to $AE \implies$

M' ***is symmetric*** F ***with respect to*** $AE \implies AM' = AF.$

We use Power of Point A and get

$$AF = AM' = \frac{AD \cdot AE}{AM} = \frac{4bc}{\sqrt{2b^2+2c^2-a^2}} = \frac{4\cdot 3\cdot 5}{\sqrt{50+18-49}} = \frac{30}{\sqrt{19}} \implies \boxed{\textbf{919}}.$$

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Solution 6:

To do this, we first define the intersection of EF and BC to be K.

Lemma 1: (K, C, D, B) are harmonic. First of all, define the midpoint of BC to be M. Then, we have that angle FMD is 90 degrees, and as a result, M lies on this circle. By Power of a Point, $(KD)(KM) = (KE)(KF) = (KB)(KC)$. As a result, (K, C, D, B) are harmonic from another famous harmonic lemma.

As a result, since $\angle EFD = 90^\circ$, by another Harmonic Lemma, FD is the angle bisector of BFC. Since $\frac{BD}{CD} = \frac{5}{3}$ by angle bisector theorem, $\frac{BF}{CF} = \frac{5}{3}$. Since $\angle BAC$ is 120° by Law of Cosines (LOC), we can use LOC to finish off. Call $BF = 5a$, and $CF = 3a$, $(5a)^2 + (3a)^2 - 15a^2 = 49$, so $a = \frac{7}{\sqrt{19}}$. We do Ptolemy's Theorem on $ABFC$. Our answer is:

$$\frac{(3)(35)+(5)(21)}{7\sqrt{19}} = \frac{30}{\sqrt{19}}.$$

As a result, the final answer is $\boxed{919}$.

-sepehr2010

Minor edits ~Zhenghua

Video Solution by mop 2024

https://youtu.be/mIFUuY4ybeg

~r00tsOfUnity

See Also

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