Russian School of Math: Lesson 6

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Abstract

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

1

Convert 100_{b+1} to base b, where $b \geq 3$.

Solution

Convert 100_{b+1} to base b, where $b \geq 3$.

$$100_{b+1} = 1 \times (b+1)^2 + 0 \times (b+1)^1 + 0 \times (b+1)^0 = b^2 + 2b + 1 = 1 \times b^2 + 2 \times b^1 + 1 \times b^0 \rightarrow 121_b$$

If $b < 3$, the factor 2 in front of b^2 would unravel.
Solution: $\boxed{121_b.}$

2

The repeating decimals of $0.\overline{ab}$ and $0.\overline{abc}$ satisfy $0.\overline{ab} + 0.\overline{abc} = \frac{33}{37}$, where a, b, c are (not necessarily distinct) digits. Find the three-digit number $\overline{abc}$.

Solution

First, note that $\dfrac{33}{37} = 0.\overline{891}$. Next, note that adding $0.\overline{ab}$ and $0.\overline{abc}$ gives:

Column No. $\rightarrow$		1	2	3	4	5	6	
	0.	a	b	a	b	a	b	...
+	0.	a	b	c	a	b	c	...
=	0.	2a	2b	a+c	b+a	a+b	b+c	...
=	0.	8	9	1	8	9	1	...

We attempt to match $2a$, $2b$, $(a+c)$, etc. with the digits 8, 9, and 1.
The first attempt fails. First, since the sum is less than 1, this suggests that $2a = 8$, or $a = 4$. Next, since $2b$ is not a multiple of 9, this suggests that there was a carry, so we take 10 away from the previous position and add 1 to the current position. Putting it together,

$$\begin{aligned} 2a &= 8 \\ 2b + 1 &= 9 \\ a + c - 10 &= 1 \end{aligned}$$

Solving the system gives $a = 8$, $b = 2$ $c = 7$. But clearly that is not right! The second attempt succeeds. Note that for the adjacent $(b+a) \rightarrow 8$ and $(a+b) \rightarrow 9$ in columns 4 and 5 to match, there must have been a carry that stopped there, that is:

$$\begin{aligned} a + b + 1 &= 9 \\ b + a &= 8 \end{aligned}$$

Next, column 3 suggests $a + c = 1$, but that clearly is not consistent with $a + b = 8$, so there must be a carry, and so $a + c - 10 = 1$. Next, column 2 suggests $2b = 9$, which does not divide evenly, so there must be a carry: $2b + 1 = 9$. Putting it together,

$$\begin{aligned} a + b &= 8 \\ a + c - 10 &= 1 \\ 2b + 1 &= 9 \end{aligned}$$

which yields $a = 4$, $b = 4$, $c = 7$.

Solution: $\boxed{\overline{abc} = \overline{447}.}$

3

Find the number of ending zeros of 2018! in base 9. Give your answer in base 9.

Solution

In base 9, the number of trailing zeros is given by the greatest power of 9 that divides the given number. Since $9 = 3^2$, one power of 9 requires two powers of 3, so we count the powers of 3 and divide by 2.

$$\left\lfloor \frac{2018}{3^1} \right\rfloor = 672$$
$$\left\lfloor \frac{2018}{3^2} \right\rfloor = 224$$
$$\left\lfloor \frac{2018}{3^3} \right\rfloor = 74$$
$$\left\lfloor \frac{2018}{3^4} \right\rfloor = 24$$
$$\left\lfloor \frac{2018}{3^5} \right\rfloor = 8$$
$$\left\lfloor \frac{2018}{3^6} \right\rfloor = 2$$
$$\left\lfloor \frac{2018}{3^7} \right\rfloor = 0$$

The total number of powers of 3 that go into 2018! is

$$\left\lfloor \frac{2018}{3^1} \right\rfloor 672 + 224 + 74 + 24 + 8 + 2 = 1004$$

and thus the total number of powers of 9 is $1004/2 = 502$. Now convert to base 9:

$$\left\lfloor \frac{2018}{3^1} \right\rfloor 502 = 55 \times 9 + 7 = 6 \times 9^2 + 1 \times 9^1 + 7 \times 9^0$$

Solution: $\boxed{617|_9.}$

4

How many natural decimal numbers are 3-digit numbers when written in base 12 and 4-digit numbers when written in base 8.

Solution

We first find the ranges of numbers that correspond to these digit requirements in each base.

Step 1: Find the range for 3-digit numbers in base 12

A natural number n requires 3 digits in base 12 if it satisfies the following inequality:

$$144 = 12^2 \leq n < 12^3 = 1728$$

Step 2: Find the range for 4-digit numbers in base 8

A natural number n requires 4 digits in base 8 if it satisfies the following inequality:

$$512 = 8^3 \leq n < 8^4 = 4096$$

Step 3: Find the intersection of the two ranges

$$\{144 \leq n < 1728\} \cap \{512 \leq n < 4096\} = \{512 \leq n < 1728\}$$

Step 4: Calculate the number of natural numbers in the intersection

The smallest integer in the range is 512. The largest integer in the range is 1727, so the count is $1727 - 512 + 1 = 1216$. Solution: $\boxed{1216.}$

5

A number N has three digits when expressed in base 7. When N is expressed in base 9 the digits are reversed. Find the middle digit in either representation of N.

Solution

Let $\overline{abc}$ denote the number in base 7. Breaking down the number gives:

$$49a + 7b + c = 81c + 9b + 1a$$

Simplifying gives $48a - 2b - 80c = 0$. Writing the middle digit b in terms of a and c:

$$b = 24a - 40c = 8(3a - 5c)$$

Since b is a multiple of 8, $b = 0$ in base 7.
Solution: $\boxed{0.}$

6

The number n can be written in base 14 as $\overline{abc}_{14}$; it can be written in base 15 as $\overline{acb}_{15}$; and in base 6 as $\overline{acac}_6$, where $a > 0$. Find the base 10 representation of n.

Solution

The number n can be written as follows:

$$\begin{aligned} n &= a \times 14^2 + b \times 14^1 + c \times 14^0 \\ &= a \times 15^2 + c \times 15^1 + b \times 15^0 \\ &= a \times 6^3 + c \times 6^2 + a \times 6^1 + c \times 6^0 \end{aligned}$$

This is a system of 3 equations in 4 unknowns to be solved for integers.

$$\begin{aligned} n &= 196a + 14b + c && (1) \\ &= 225a + 15c + b && (2) \\ &= 222a + 37c && (3) \end{aligned}$$

From (2)−(1):

$$29a + 14c = 13b \tag{4}$$

From (2)−(3):

$$3a + b = 22c \tag{5}$$

And thus from (4)−(5):

$$26a + 36c = 14b \implies 13a + 18c = 7b \tag{6}$$

Eliminating b seems like a good approach, so we multiply (5) by 7 and combine it with (6):

$$\begin{aligned} 13a + 18c &= 7b \\ 21a + 7b &= 154c \\ \implies 34a &= 136c \implies 17a = 68c \implies a = 4c \end{aligned}$$

Substituting back into (5) yields $b = 10c$. We solve the system for a, b, c in integers:

$$\begin{aligned} a &= 4c \\ b &= 10c \end{aligned}$$

The only solution with $a < 6$ and $c < 6$ is: $a = 4$, $b = 10$, $c = 1$. Substituting back into (3) to solve for n:

$$n = 222 \times 4 + 37 \times 1 = 925$$

Solution: $\boxed{n = 925.}$

7

What is the largest positive integer n less than $10,000$ such that in base 4, n and $3n$ have the same number of digits; in base 8, n and $7n$ have the same number of digits; and in base 16, n and $15n$ have the same number of digits? Express your answer in base 10.

Solution

Let k_4, k_8, k_{16} be the largest positive integers such that

$$4^{k_4} < n < 2 \cdot 4^{k_4}, \quad 8^{k_8} < n < 2 \cdot 8^{k_4}, \quad 16^{k_{16}} < n < 2 \cdot 16^{k_4},$$

The greatest power of 2 that is less than n is 2^{12k}, where 12 is the least common multiple of 2, 3, 4. Since $n < 10000$, we must have $k = 1$. The largest possible value of n is the smallest of

$$\min(1111|_{16}, 11111|_{8}, 1111111|_{4}) \to 1111|_{16} = 4369.$$

where $11111|_8 = 4681$ and $1111111|_4 = 5461$.
We check the conditions:

$$\begin{aligned} 4369 &= 1 \times 16^3 + 1 \times 16^2 + 1 \times 16^1 + 1 \times 16^0 &&= 1111_{16} \\ &= 1 \times 8^4 + 4 \times 8^2 + 2 \times 8^1 + 1 \times 8^0 &&= 10421_8 \\ &= 1 \times 4^6 + 1 \times 4^4 + 1 \times 4^2 + 1 \times 4^0 &&= 1010101_4 \end{aligned}$$

It is immediate that n and $3n$ have the same number of digits in base 4 and that n and $15n$ have the same number of digits in base 16. We explicitly check that n and $7n$ have the same number of digits in base 8:

$$\begin{aligned} 7 \times 4369 &= 7 \times 10421|_8 \\ &= 7 \times \left(1 \times 8^4 + 4 \times 8^2 + 2 \times 8^1 + 1 \times 8^0\right) \\ &= 7 \times 8^4 + 28 \times 8^2 + 14 \times 8^1 + 7 \times 8^0 \\ &= 7 \times 8^4 + 29 \times 8^2 + 6 \times 8^1 + 7 \times 8^0 \\ &= 7 \times 8^4 + 3 \times 8^3 + 5 \times 8^2 + 6 \times 8^1 + 7 \times 8^0 \\ &= 73567|_8 \end{aligned}$$

so $10421|_8$ and $7 \times 10421|_8 = 73567|_8$ have the same number of digits.
Solution: $\boxed{4369|_{10}.}$

8

Let $b(n)$ be the number of digits in the base-4 representation of n. Evaluate

$$\sum_{i=1}^{2013} b(i)$$

Solution

$$3 \times 4^0 = 3$$
$$4^2 - 3 \times 4^1 = 4$$
$$3 \times 4^2 + 3 \times 4^0 = 51$$
$$4^4 - 3 \times 4^3 - 3 \times 4 = 52$$
$$3 \times 4^4 + 3 \times 4^2 + 3 \times 4^0 = 819$$
$$4^6 - 3 \times 4^5 - 3 \times 4^3 = 820$$
$$3 \times 4^6 + 3 \times 4^4 + 3 \times 4^2 + 3 \times 4^0 = 13827$$

The number of digits in base-4 representation is summarized in the table:

$1 \le n \le 3$	$4 \le n \le 51$	$52 \le n \le 819$	$820 \le n \le 2013$
1	3	5	7

Adding up the 4 digits weighted by the number of cases gives:

$$\begin{aligned}\sum_{i=1}^{2013} b(i) &= 1 \times (3-1+1) + 3 \times (51-4+1) + 5 \times (819-52+1) + 7 \times (2013-820+1) \\ &= 3 + 48 + 768 + 1195 = 12345\end{aligned}$$

Solution: $\boxed{12345.}$