**Lines & Regions**

**Suppose there are $n$ lines drawn in the plane such that no two lines are parallel and no three lines intersect at the same point. Find a closed formula for the number of regions that the lines split into.**

For small values of $n$, it is easy to sketch intersecting lines and count regions. Let $n$ denote the number of lines and $r$ the number regions. We have:

| n | r |
|---|---|
| 0 | 1 |
| 1 | 2 |
| 2 | 4 |
| 3 | 7 |
| 4 | 11 |

The case $n = 0$ is obvious: with no lines crossing the plane, there is one region — the entire plane.

The case $n = 1$ is equally obvious: a single line divides the plane into two regions, each being a half-plane.

The case $n = 2$ is easy to explain: At the intersection of the two lines, there are four angles that sum to $360°$, and each angle defines a region.

The case $n = 3$ can be explained by extending the previous idea: The intersection of the three lines forms a triangle. This triangle defines one region. Now move the lines such as to shrink the triangle to a single point. The resulting figure has three lines intersecting at a single point (see figure below). These lines define 6 regions, for a total of 7 regions when the triangle is included.

The case $n = 4$ can be understood by considering what happens when a line is added to the previous case. The fourth line intersects the other three lines at 3 points, and so goes through 4 "existing" regions, dividing each into two parts, thus adding 4 "new" regions, $7 + 4 = 11$.

In general, the $n$th line intersects with $n - 1$ lines, creating $n$ news regions. This suggests a method for calculating the number of regions based on the previous value:

$$r(n) = r(n-1) + n$$

This is a linear recurrence. A linear recurrence admits a unique solution, which may be found, for instance, by repeated substitution.

$$
\begin{aligned}
r(n) &= r(n-1) + n \\
r(n-1) &= r(n-2) + (n-1) \\
r(n-2) &= r(n-3) + (n-2) \\
&\vdots \\
r(3) &= r(2) + 3 \\
r(2) &= r(1) + 2 \\
r(1) &= r(0) + 1
\end{aligned}
$$

Adding these equalities column-wise gives:

$$r(n) = n + (n-1) + (n-2) + \ldots + 3 + 2 + 1 + r(0)$$

where $r(0) = 1$ (as noted in the table above). Thus,

$$r(n) = (1 + 2 + 3 + \ldots + n) + 1$$

In words, the number of regions delimited by the intersection of $n$ lines that intersect at $n-1$ distinct points is equal to one plus the sum of the integers up to $n$. There is, of course, a famous formula for the sum of the first $n$ integers:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Substituting into the formula for $r(n)$ gives:

$$
\begin{aligned}
r(n) &= \frac{n(n+1)}{2} + 1 \\
&= \frac{n^2 + n + 2}{2}
\end{aligned}
$$

For peace of mind, you may check that the formula generates the values computed in the table above:

$$
\begin{aligned}
r(0) &= \frac{0^2 + 0 + 2}{2} = \frac{2}{2} = 1 \\
r(1) &= \frac{1^2 + 1 + 2}{2} = \frac{4}{2} = 2 \\
r(2) &= \frac{2^2 + 2 + 2}{2} = \frac{8}{2} = 4 \\
r(3) &= \frac{3^2 + 3 + 2}{2} = \frac{14}{2} = 7 \\
r(4) &= \frac{4^2 + 4 + 2}{2} = \frac{22}{2} = 11
\end{aligned}
$$