Russian School of Math: Lesson 5

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Abstract

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

1

Find the last two digits of $2017^{20} + 2018^{20} + 2019^{20}$.

Solution

Since we are looking for the last two digits, we can decompose the numbers and discards all the multiples of 100. First step:

$$\begin{aligned}
2017^2 &= (2000+17)^2 \equiv 17^2 \pmod{100} = 289\\
2018^2 &= (2000+18)^2 \equiv 18^2 \pmod{100} = 324\\
2019^2 &= (2000+19)^2 \equiv 19^2 \pmod{100} = 361
\end{aligned}$$

Second step:

$$\begin{aligned}
289^{10} &\equiv (289-300)^{10} \equiv (-11)^{10} \equiv (11)^{10} \equiv (10+1)^{10}\\
&\equiv \binom{10}{1}10^1 1^9 + \binom{10}{0}10^0 1^{10} \equiv 100 + 1 \equiv 1 \pmod{100}\\
324^{10} &\equiv (324-300)^{10} \equiv 24^{10} \equiv (10+14)^{10}\\
&\equiv \binom{10}{0}10^0 14^{10} \equiv (10+4)^{10} \equiv \binom{10}{0}10^0 4^{10} \equiv 4^{2\times 5} \equiv 16^5\\
&\equiv (10+6)^5 \equiv \binom{10}{0}10^0 6^5 \equiv 776 \equiv 76 \pmod{100}\\
361^{10} &\equiv (361-400)^{10} \equiv (-39)^{10} \equiv 39^{10} \pmod{100}\\
&\equiv (10+10+10+9)^{10} \equiv 9^{10} \equiv (-1)^{10} \equiv 1 \pmod{100}
\end{aligned}$$

Putting it together,

$$2017^{20} + 2018^{20} + 2019^{20} \equiv 1 + 76 + 1 \equiv 78 \pmod{100}$$

Solution: $\boxed{78.}$

2

Find all n, $n \in \mathbb{N}$, such that $\varphi(n) = 2$.

Solution

3

Prove that if m and n are coprime, then $\varphi(m \cdot n) > \varphi(m) \cdot \varphi(n)$.

Solution

4

Find all ordered pairs (m, n), where $m, n \in \mathbb{N}$, $n > 1$ and $\varphi(\varphi(n^m)) = n$.

Solution

Solution: $(m, n) \in \{(X, X), (X, X), (X, X), (X, X)\}$.

5

Let $d_1, d_2, \ldots, d_k$ be all natural divisors of n, $n \in \mathbb{N}$ such that $d_1 < d_2 < \ldots < d_k$. Prove that $\varphi(d_1) + \varphi(d_2) + \ldots + \varphi(d_k) = n$.

Solution