2021 Fall AMC 10B Problems/Problem 23

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Problem

Each of the 5 sides and the 5 diagonals of a regular pentagon are randomly and independently colored red or blue with equal probability. What is the probability that there will be a triangle whose vertices are among the vertices of the pentagon such that all of its sides have the same color?

$$\textbf{(A)}\ \frac{2}{3} \qquad \textbf{(B)}\ \frac{105}{128} \qquad \textbf{(C)}\ \frac{125}{128} \qquad \textbf{(D)}\ \frac{253}{256} \qquad \textbf{(E)}\ 1$$

Solution 1

Instead of finding the probability of a same-colored triangle appearing, let us find the probability that one does not appear. After drawing the regular pentagon out, note the topmost vertex; it has 4 sides/diagonals emanating outward from it. We do casework on the color distribution of these sides/diagonals.

$\textbf{Case 1}$: all 4 are colored one color. In that case, all of the remaining sides must be of the other color to not have a triangle where all three sides are of the same color. We can correspondingly fill out each color based on this constraint, but in this case you will always end up with a triangle where all three sides have the same color by inspection.

$\textbf{Case 2}$: 3 are one color and one is the other. Following the steps from the previous case, you can try filling out the colors, but will always arrive at a contradiction so this case does not work either.

$\textbf{Case 3}$: 2 are one color and 2 are of the other color. Using the same logic as previously, we can color the pentagon 2 different ways by inspection to satisfy the requirements. There are $\dbinom{4}{2}$ ways to color the original sides/diagonals and 2 ways after that to color the remaining ones for a total of $6 \cdot 2 = 12$ ways to color the pentagon so that no such triangle has the same color for all of its sides.

These are all the cases, and there are a total of 2^{10} ways to color the pentagon. Therefore the answer is $1-\frac{12}{1024}=1-\frac{3}{256}=\frac{253}{256}=\boxed{D}$

~KingRavi

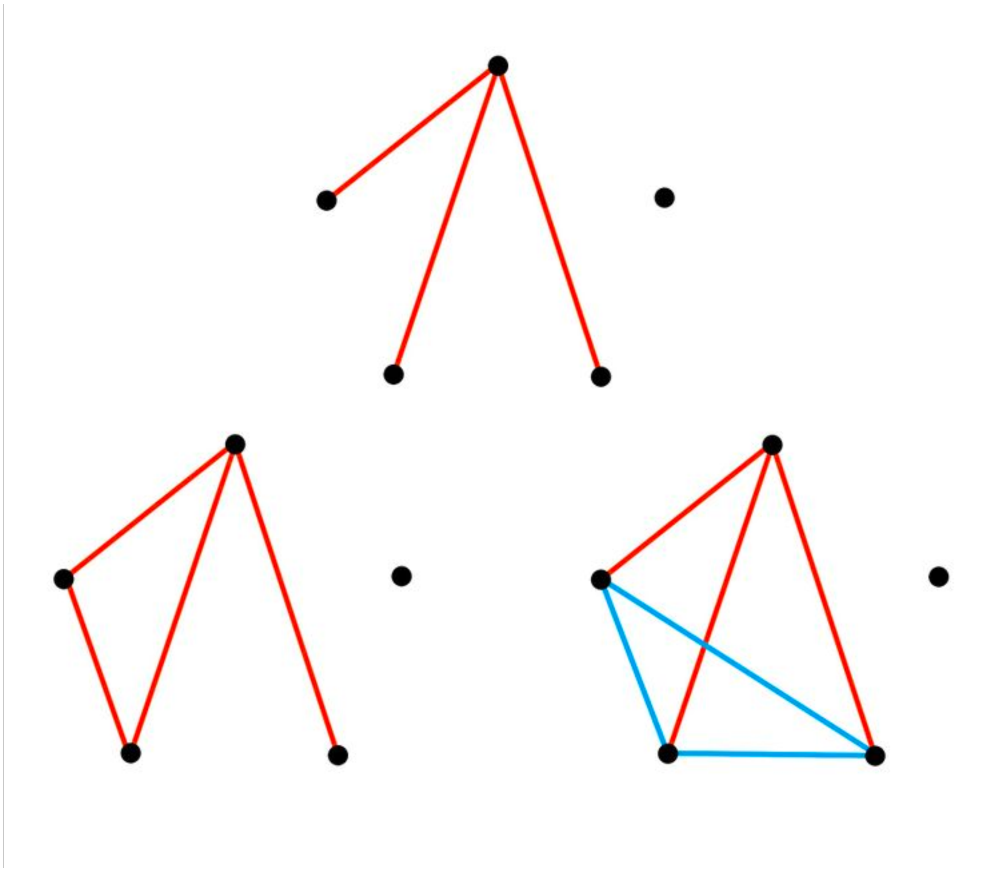
Solution 2 (Ramsey's Theorem)

This problem is related to a special case of Ramsey's Theorem (https://en.wikipedia.org/wiki/Ramsey%27s_theorem), R(3, 3) = 6 (https://en.wikipedia.org/wiki/Ramsey%27s_theorem#R(3,_3)_=_6). Suppose we color every edge of a 6 vertex complete graph (K_6) with 2 colors, there must exist a 3 vertex complete graph (K_3) with all it's edges in the same color. That is, K_6 with edges in 2 colors contains a monochromatic K_3. For K_5 with edges in 2 colors, a monochromatic K_3 does not always exist.

This is a problem about the probability of a monochromatic K_3 exist in a 5 vertex complete graph K_5 with edges in 2 colors.

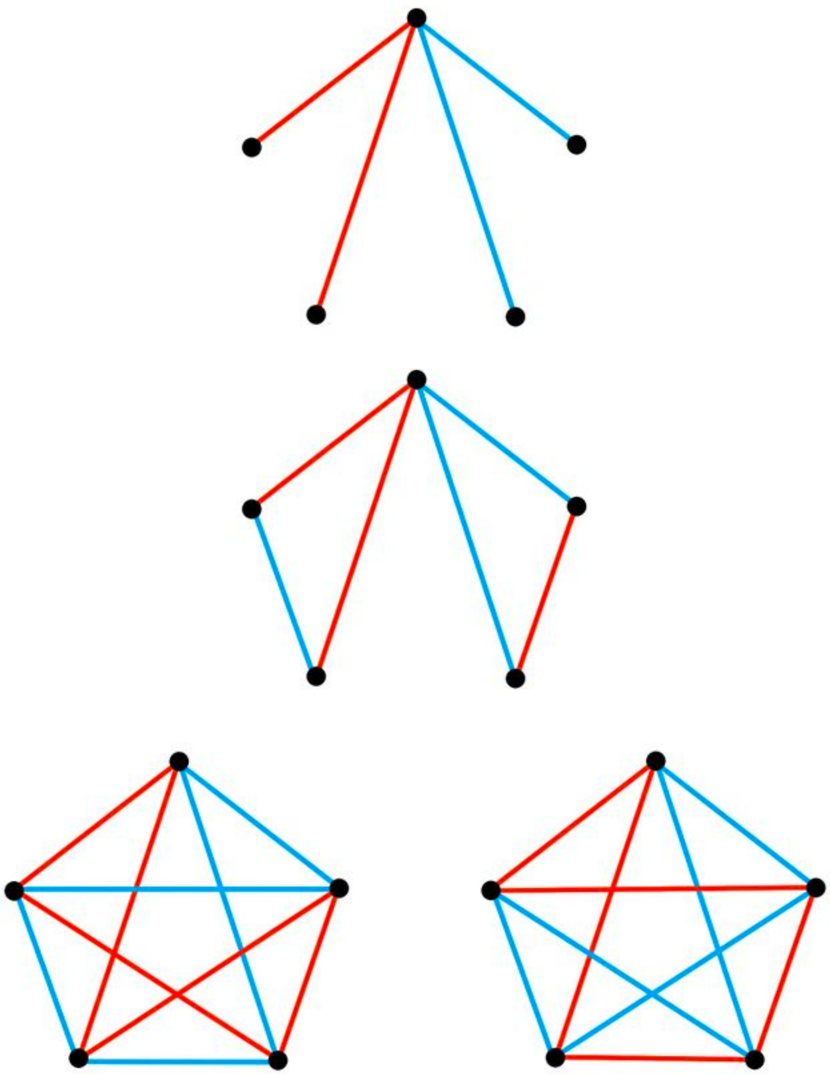
Choose a vertex, it has 4 edges.

$\textbf{Case 1}$: When 3 or more edges are the same color, there must exist a monochromatic K_3. Suppose the color is red, as shown below.



There is only 1 way to color all the edges in the same color. There is $\binom{4}{3} = 4$ ways to color 3 edges in the same color. There are 2 colors. The probability of 3 or more edges the same color is $\frac{(1+4)\cdot 2}{2^4} = \frac{5}{8}$. So the probability of containing a monochromatic K_3 is $\frac{5}{8}$.

$\textbf{Case 2}$: When 2 edges are the same color, graphs that does not contain a monochromatic K_3 can exist. The following diagram shows steps to obtain graphs that does not contain a monochromatic K_3.



There are $\binom{4}{2} = 6$ ways to choose 2 edges with the same color. For the other 4 vertices there are $\binom{4}{2} = 6$ edges among them, there are $2^6 = 64$ ways to color the edges. There are only 2 cases without a monochromatic K_3.

So the probability without monochromatic K_3 is $\frac{2}{64} = \frac{1}{32}$.

The probability with monochromatic K_3 is $1 - \frac{1}{32} = \frac{31}{32}$.

From case 1 and case 2, the probability with monochromatic K_3 is $\frac{5}{8} + \left(1 - \frac{5}{8}\right) \cdot \frac{31}{32} = \boxed{\textbf{(D)} \frac{253}{256}}$

~isabelchen (https://artofproblemsolving.com/wiki/index.php/User:Isabelchen)

See Also

2021 Fall AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
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