AMC 10 Problem Series (2804)

Jon Joseph

Friday
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7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Overview

Lesson 11 (Aug 13) Class Transcript - Number Theory

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jonjoseph 2021-08-13 19:30:50
Hmmm... @ anonymus_chicken?? Can you really win if you're anonymous?

jonjoseph 2021-08-13 19:31:32
I know. This is a problem.

jonjoseph 2021-08-13 19:32:01
Lets go!!!

jonjoseph 2021-08-13 19:32:07
AMC 10 Problem Series
Week 11: Number Theory

jonjoseph 2021-08-13 19:32:14
Before we get going, please note that there is a Class Survey on the class home page. Please complete this survey; your responses help us to improve our classes!

jonjoseph 2021-08-13 19:32:50
The survey is the only feedback the instructors get so this really helps.

jonjoseph 2021-08-13 19:33:08
AMC 10 number theory problems generally center around simple concepts such as prime factorization and gcd/lcm. These will be your most effective tools. The problems we will look at today will help us understand how to use these tools.

jonjoseph 2021-08-13 19:33:15
Suppose that n is the product of three consecutive integers and that n is divisible by 7. Which of the following is not necessarily a divisor of n?

$$\text{(A) } 6 \quad \text{(B) } 14 \quad \text{(C) } 21 \quad \text{(D) } 28 \quad \text{(E) } 42$$

jonjoseph 2021-08-13 19:33:33
If you want to check if a number n is divisible by another number m, then all you have to do is check if n is divisible by each prime power that's a factor of m.

jonjoseph 2021-08-13 19:33:48
For example, suppose we want to check if n is divisible by 360. What is the prime factorization of 360?

jonjoseph 2021-08-13 19:34:59
The prime factorization of 360 is $2^3 \cdot 3^2 \cdot 5$.

jonjoseph 2021-08-13 19:35:05
Thus, to determine whether a number n is divisible by 360, it suffices to check whether n is divisible by $2^3 = 8, 3^2 = 9$, and

5.

jonjoseph 2021-08-13 19:35:17
We are told that n is the product of three consecutive integers. Let these three consecutive integers be x, $x+1$, and $x+2$, so $n = x(x+1)(x+2)$.

jonjoseph 2021-08-13 19:35:31
For this problem, we'll go through the choices one by one, since the lessons we'll learn along the way will generalize.

jonjoseph 2021-08-13 19:35:42
To check choice (A), is $n = x(x+1)(x+2)$ necessarily divisible by 6? Why or why not?

jonjoseph 2021-08-13 19:35:57
Hint: Why or why not!

jonjoseph 2021-08-13 19:37:34
One of the factors x and $x+1$ must be even. Also, one of the factors x, $x+1$, and $x+2$ must be divisible by 3. (Among any three consecutive numbers, one must be a multiple of 3). Therefore, $n = x(x+1)(x+2)$ is divisible by 6.

jonjoseph 2021-08-13 19:37:55
For choice (B), is n necessarily divisible by 14? Why or why not?

jonjoseph 2021-08-13 19:39:55
We have already shown that n is divisible by 2, and we are told that n is divisible by 7, so n is divisible by 14.

jonjoseph 2021-08-13 19:40:03
For choice (C), is n necessarily divisible by 21? Why or why not?

jonjoseph 2021-08-13 19:41:17
We have already shown that n is divisible by 3, and we are told that n is divisible by 7, so n is divisible by 21.

jonjoseph 2021-08-13 19:41:21
For choice (D), is n necessarily divisible by 28? Why or why not?

jonjoseph 2021-08-13 19:43:12
To check whether n must be divisible by $28 = 2^2 \cdot 7$, it suffices to check whether n must be divisible by both $2^2 = 4$ and 7.

jonjoseph 2021-08-13 19:43:20
We know that n is divisible by 7. Is $n = x(x+1)(x+2)$ divisible by 4?

jonjoseph 2021-08-13 19:43:28
To check, we can consider cases. For example, if x is even, then $x+2$ is even, so $n = x(x+1)(x+2)$ is divisible by 4.

jonjoseph 2021-08-13 19:43:42
So if $n = x(x+1)(x+2)$ is not divisible by 4, then x must be odd. Does there exist an odd integer x, such that $n = x(x+1)(x+2)$ is not divisible by 4? (Remember that we also want $n = x(x+1)(x+2)$ to be divisible by 7.)

jonjoseph 2021-08-13 19:45:27
If we set $x = 5$, then $n = 5 \cdot 6 \cdot 7 = 210$, which is divisible by 7 but not 4.

jonjoseph 2021-08-13 19:45:34
Therefore, n is *not necessarily* divisible by 28, so the answer is (D).

jonjoseph 2021-08-13 19:45:41
For completeness, let's check choice (E). Is n necessarily divisible by 42?

jonjoseph 2021-08-13 19:46:44
We know that n is divisible by 2, 3, and 7, so n is divisible by $2 \cdot 3 \cdot 7 = 42$.

jonjoseph 2021-08-13 19:46:57
All even numbers from 2 to 98 inclusive, except those ending in 0, are multiplied together. What is the rightmost (units) digit of the product?

$\text{(A) } 0 \quad \text{(B) } 2 \quad \text{(C) } 4 \quad \text{(D) } 6 \quad \text{(E) } 8$

jonjoseph 2021-08-13 19:47:15
We want the units digit of the product $2 \cdot 4 \cdot 6 \cdot 8 \cdot 12 \cdots 96 \cdot 98$. How can we find this units digit?

jonjoseph 2021-08-13 19:49:18
We can find the units digit of a product by looking only at the units digit of each factor. For example, the units digit of $3717 \cdot 894 \cdot 508$ is the same as the units digit of $7 \cdot 4 \cdot 8$. (This is also known as working modulo 10, if you know modular arithmetic.)

jonjoseph 2021-08-13 19:49:33
So what else can we find the units digit of in order to find the units digit of $2 \cdot 4 \cdot 6 \cdot 8 \cdot 12 \cdots 96 \cdot 98$?

jonjoseph 2021-08-13 19:50:48
We can find the units digit of $(2 \cdot 4 \cdot 6 \cdot 8)^{10}$, because each last digit 2, 4, 6, and 8 appears 10 times among the numbers from 2 to 98.

jonjoseph 2021-08-13 19:51:01
What is $2 \cdot 4 \cdot 6 \cdot 8$?

jonjoseph 2021-08-13 19:51:59
We find that $2 \cdot 4 \cdot 6 \cdot 8 = 384$.

jonjoseph 2021-08-13 19:52:09
So we have reduced the problem to finding the units digit of 384^{10}. What else is the units digit of 384^{10} equal to?

jonjoseph 2021-08-13 19:52:47
The units digit of 384^{10} is equal to the units digit of 4^{10}. So we have reduced the problem to finding the units digit of 4^{10}.

jonjoseph 2021-08-13 19:53:00
We can look at the units digit of 4^n for small values of n, and see if we can find a pattern.

jonjoseph 2021-08-13 19:53:06
The units digit of $4^1 = 4$ is 4.

jonjoseph 2021-08-13 19:53:14
The units digit of $4^2 = 16$ is 6.

jonjoseph 2021-08-13 19:53:18
What is the units digit of 4^3?

jonjoseph 2021-08-13 19:53:56
The units digit of $4^3 = 64$ is 4.

jonjoseph 2021-08-13 19:54:01
What is the units digit of 4^4?

jonjoseph 2021-08-13 19:54:40
The units digit of $4^4 = 256$ is 6.

jonjoseph 2021-08-13 19:54:57
We do. What do you see?

jonjoseph 2021-08-13 19:55:39
We see that the units digit of 4^n alternates between 4 and 6. So what is the units digit of 4^{10}?

jonjoseph 2021-08-13 19:56:23
Since the exponent 10 is even, the units digit of 4^n is 6. The answer is (D).

jonjoseph 2021-08-13 19:56:36

A charity sells 140 benefit tickets for a total of $\$2001$. Some tickets sell for full price (a whole dollar amount), and the rest sell for half price. How much money is raised by the full-price tickets?

$\text{(A) }\$782 \quad \text{(B) }\$986 \quad \text{(C) }\$1158 \quad \text{(D) }\$1219 \quad \text{(E) }\$1449$

jonjoseph 2021-08-13 19:56:54
How can we get started?

jonjoseph 2021-08-13 19:57:57
What variables can we define?

jonjoseph 2021-08-13 19:58:19
Be sure to tell us what your variable means.

jonjoseph 2021-08-13 19:59:16
We can let n be the number of tickets that sold for full price, and we can let p be the price of a full price ticket. Then how many tickets sold for half price?

jonjoseph 2021-08-13 20:00:03
There were a total of 140 tickets, so $140 - n$ tickets sold for half price. So what equation can we write down?

jonjoseph 2021-08-13 20:02:46
The total cost is the number of tickets multiplied by the price per ticket, so for the full-price tickets it's np. For the half-price tickets, it's $(140 - n) \cdot \frac{p}{2}$. The total cost is $\$2001$, so we can write

$$np + (140 - n) \cdot \frac{p}{2} = 2001.$$

jonjoseph 2021-08-13 20:02:58
This equation simplifies to $np + 140p = 4002$. What can we do from here?

jonjoseph 2021-08-13 20:04:21
We can factor the left hand side to get $p(n + 140) = 4002$. This tells us that p and $n + 140$ are factors of 4002 whose product is 4002.

jonjoseph 2021-08-13 20:04:34
Which factor should we focus on?

jonjoseph 2021-08-13 20:05:22
There are no restrictions on the price p, but there are restrictions on the value of $n + 140$. What can we say about $n + 140$?

jonjoseph 2021-08-13 20:06:16
Good. Between the four of you, you found both conditions.

jonjoseph 2021-08-13 20:06:19
Since n is between 0 and 140, $n + 140$ is between 140 and 280. So we must look for a factor of 4002 that is between 140 and 280.

jonjoseph 2021-08-13 20:06:32
Time to get our hands dirty and look for some factors! What is the prime factorization of 4002?

jonjoseph 2021-08-13 20:07:57
The prime factorization of 4002 is $2 \cdot 3 \cdot 23 \cdot 29$. So which factors of 4002 are between 140 and 280?

jonjoseph 2021-08-13 20:09:22
The only factor of 4002 that is between 140 and 280 is $2 \cdot 3 \cdot 29 = 174$.

jonjoseph 2021-08-13 20:09:26
Therefore, $n + 140 = 174$, which means $n = 34$. So what is p?

jonjoseph 2021-08-13 20:10:46

We see that $p = \frac{4002}{n+140} = \frac{4002}{174} = 23$. You can avoid having to actually do the division here by seeing that p is the product of the factors of 4002 left over after we remove $174 = 2 \cdot 3 \cdot 29$.

jonjoseph 2021-08-13 20:11:00

So how much money did the full price tickets raise?

jonjoseph 2021-08-13 20:12:08

The full price tickets raised $pn = 23 \cdot 34 = 782$ dollars. The answer is (A).

jonjoseph 2021-08-13 20:12:21

In general, whenever you have a problem where you are told that a certain number is an integer (such as we were told about the price here), you should look out for ways to factor expressions involving it.

jonjoseph 2021-08-13 20:12:55

Notice we had two unknowns but only one equation. This almost always means you need to go looking for some other restriction.

jonjoseph 2021-08-13 20:13:08

Before we start the next problem, let's review two useful facts.

jonjoseph 2021-08-13 20:13:13

If an integer is a perfect square, then what can we say about its prime factorization?

jonjoseph 2021-08-13 20:14:21

If an integer is a perfect square, then the exponent of every prime in its prime factorization is even (divisible by 2).

jonjoseph 2021-08-13 20:14:30

If an integer is a perfect cube, then what can we say about its prime factorization?

jonjoseph 2021-08-13 20:15:41

If an integer is a perfect cube, then the exponent of every prime in its prime factorization is divisible by 3.

jonjoseph 2021-08-13 20:16:10

Okay. Remembering these facts let's try our next problem.

jonjoseph 2021-08-13 20:16:17

How many positive cubes divide $3! \cdot 5! \cdot 7!$?

$(\text{A})\ 2 \quad (\text{B})\ 3 \quad (\text{C})\ 4 \quad (\text{D})\ 5 \quad (\text{E})\ 6$

jonjoseph 2021-08-13 20:16:32

What should we do first with the number $3! \cdot 5! \cdot 7!$ to find how many cubes divide it?

jonjoseph 2021-08-13 20:17:29

We should find the prime factorization of $3! \cdot 5! \cdot 7!$. Sounds like a pain but it won't be that bad.

jonjoseph 2021-08-13 20:17:41

Let's factor one piece at a time. What is the prime factorization of $3!$?

jonjoseph 2021-08-13 20:18:24

Hint: 1 is not a prime.

jonjoseph 2021-08-13 20:18:51

The prime factorization of $3! = 6$ is $2 \cdot 3$.

jonjoseph 2021-08-13 20:18:58

What is the prime factorization of $5!$?

jonjoseph 2021-08-13 20:20:00

The prime factorization of $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$ is $2^3 \cdot 3 \cdot 5$.

jonjoseph 2021-08-13 20:20:02
What is the prime factorization of $7!$?

jonjoseph 2021-08-13 20:21:12
The prime factorization of $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7$ is $2^4 \cdot 3^2 \cdot 5 \cdot 7$.

jonjoseph 2021-08-13 20:21:16
Multiplying these together, we add the exponents to find that the prime factorization of $3! \cdot 5! \cdot 7!$ is

$$2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$$

Hence, if a cube divides this number, then the only possible prime factors of the cube are 2, 3, 5, and 7.

jonjoseph 2021-08-13 20:21:22
So if a cube divides

$$2^8 \cdot 3^4 \cdot 5^2 \cdot 7,$$

then how many factors of 2 can there be in the cube? List all the possible answers.

jonjoseph 2021-08-13 20:22:48
There can be 0, 3, or 6 factors of 2.

jonjoseph 2021-08-13 20:22:54
How many factors of 3 can there be?

jonjoseph 2021-08-13 20:23:30
There can be 0 or 3 factors of 3.

jonjoseph 2021-08-13 20:23:32
And the number has only two factors of 5, and one factor of 7, so the cube cannot have any factors of 5 or 7.

jonjoseph 2021-08-13 20:23:37
So how many cubes divide $3! \cdot 5! \cdot 7!$?

jonjoseph 2021-08-13 20:24:43
There are 3 ways to choose the number of factors of 2 $(0, 3,$ or $6)$ and 2 ways to choose the number of factors of 3 $(0$ or $3)$, so there are $3 \cdot 2 = 6$ cubes that divide $3! \cdot 5! \cdot 7!$. The answer is (E).

jonjoseph 2021-08-13 20:25:13
Is that clear? This is one of our counting principles.

jonjoseph 2021-08-13 20:25:34
For how many integers n is $\dfrac{n}{20-n}$ the square of an integer?

$\text{(A) } 1 \quad \text{(B) } 2 \quad \text{(C) } 3 \quad \text{(D) } 4 \quad \text{(E) } 10$

jonjoseph 2021-08-13 20:26:11
First of all what is the degree of the numerator and denominator?

jonjoseph 2021-08-13 20:27:36
Yes. They both have the same degree which is 1. Whenever this happens here are two things to consider: 1) Find a way to turn the fraction over (I'll get back to this) or 2) Long division.

jonjoseph 2021-08-13 20:27:54
Okay. Let's walk through this a couple of ways.

jonjoseph 2021-08-13 20:28:04

Clearly, if $n = 20$, the fraction is undefined. What happens if $n > 20$?

jonjoseph 2021-08-13 20:28:50
The fraction is negative! That's clearly not going to be the square of an integer. So we must have $n < 20$. Can we have $n < 0$?

jonjoseph 2021-08-13 20:29:33
Nope, that gives us negative numbers again! That means we only have to try values between 0 and 19. That's not so bad -- if you got stuck, you could just try all of them. Can we rule anything else out right away, though?

jonjoseph 2021-08-13 20:30:42
Hint: Look carefully. What if, for example, $n = 5$? Does that work?

jonjoseph 2021-08-13 20:31:49
We could rule $1 \leq n < 10$ out, because for those values of n, the denominator is bigger than the numerator, and the numerator isn't 0. Hence the fraction can't be an integer for those values of n. Any other observations?

jonjoseph 2021-08-13 20:32:24
Hint: Some of you have already said it.

jonjoseph 2021-08-13 20:32:56
We see that plugging in primes for n won't work unless $n = 20 - n$ or $20 - n = 1$ (since primes don't have factors other than 1 and themselves.) So we don't need to try $11, 13$ and 17.

jonjoseph 2021-08-13 20:33:14
Let's try the numbers $0, 10, 12, 14, 15, 16, 18, 19$, then. What list of values for $\dfrac{n}{20-n}$ do we get?

jonjoseph 2021-08-13 20:34:55
We get $0, 1, \dfrac{3}{2}, \dfrac{7}{3}, 3, 4, 9, 19$. How many squares of integers do we see?

jonjoseph 2021-08-13 20:35:38
Hint: $0^2 = 0$

jonjoseph 2021-08-13 20:36:51
We see 4 squares. The answer is (D).

jonjoseph 2021-08-13 20:37:10
Let me show you what I mean by "turning the fraction over".

jonjoseph 2021-08-13 20:37:22
Here is one way:

jonjoseph 2021-08-13 20:37:36
We can make a substitution. Let $m = 20 - n$, so $n = 20 - m$.

jonjoseph 2021-08-13 20:37:44
What's our fraction now?

jonjoseph 2021-08-13 20:38:40
Good. And we can simplify that:

jonjoseph 2021-08-13 20:38:42
Then

$$\frac{n}{20-n} = \frac{20-m}{m} = \frac{20}{m} - 1.$$

For this number to be a perfect square, it must first of all be an integer. When is this expression an integer?

jonjoseph 2021-08-13 20:39:31
Nice

jonjoseph 2021-08-13 20:39:33

This expression is an integer if and only if m is a divisor of 20.

jonjoseph 2021-08-13 20:39:40
The positive divisors of 20 are $1, 2, 4, 5, 10$, and 20.

jonjoseph 2021-08-13 20:39:55
For those input values of m, we can see the expression $\frac{20}{m} - 1$ correspondingly produces the output values $19, 9, 4, 3, 1, 0$. How many of these output values are perfect squares?

jonjoseph 2021-08-13 20:40:06
And, again, we find 4 perfect squares.

jonjoseph 2021-08-13 20:41:10
If the degree of the numerator is $\geq$ the degree of the denominator I usually recommend a trick like this or long division.

jonjoseph 2021-08-13 20:41:44
Let's talk about bases for a moment.

jonjoseph 2021-08-13 20:41:49
When we write a number like 1235, this is really shorthand for

$$1 \cdot 1000 + 2 \cdot 100 + 3 \cdot 10 + 5 \cdot 1,$$

or

$$1 \cdot 10^3 + 2 \cdot 10^2 + 3 \cdot 10 + 5 \cdot 1.$$

jonjoseph 2021-08-13 20:42:01
This system is called *base 10*.

jonjoseph 2021-08-13 20:42:09
We get other base systems when we replace 10 with some other number, like 5, or 2, or 8, or 11. We often indicate that we're using a base other than 10 like this: 321_5.

jonjoseph 2021-08-13 20:42:32
So, for instance,

$$321_5 = 3 \cdot 5^2 + 2 \cdot 5 + 1.$$

jonjoseph 2021-08-13 20:42:51
Is that clear? It is still place value.

jonjoseph 2021-08-13 20:43:09
Let's do a little practice with these.

jonjoseph 2021-08-13 20:43:13
What is 21_{11} (that's the number written as 2, then 1 in base 11) when written in base 10?

jonjoseph 2021-08-13 20:44:07
We have

$$21_{11} = 2 \cdot 11 + 1 = 23.$$

jonjoseph 2021-08-13 20:44:27
What is 111_5? (That's base 5.)\

jonjoseph 2021-08-13 20:45:26
We have

$$111_5 = 1 \cdot 5^2 + 1 \cdot 5 + 1 = 25 + 5 + 1 = 31.$$

jonjoseph 2021-08-13 20:45:44
A base-10 three-digit number n is selected at random. Which of the following is closest to the probability that the base-9 representation and the base-11 representation of n are *also* both three-digit numerals?

$$\text{(A) } 0.3 \quad \text{(B) } 0.4 \quad \text{(C) } 0.5 \quad \text{(D) } 0.6 \quad \text{(E) } 0.7$$

jonjoseph 2021-08-13 20:46:13
Let's first find the denominator of our probability. A base-10 three-digit number n is selected at random. How many base-10 three-digit numbers are there?

jonjoseph 2021-08-13 20:46:54
There are 900 base-10 three-digit numbers, from 100 to $999.$

jonjoseph 2021-08-13 20:47:00
We select a number n from these 900 numbers. We want to know the probability that n still has three digits when expressed in both base-9 and base-$11.$

jonjoseph 2021-08-13 20:47:07
So, let's find the range of three-digit numbers in both bases.

jonjoseph 2021-08-13 20:47:12
We'll start with base $9.$ What is the smallest base-9 three-digit number, when expressed in base-$10?$

jonjoseph 2021-08-13 20:48:15
Very nice!

jonjoseph 2021-08-13 20:48:18
The smallest base-9 three-digit number is

$$100_9 = 1 \cdot 9^2 = 81.$$

jonjoseph 2021-08-13 20:48:22
What is the largest base-9 three-digit number, when expressed in base-$10?$

jonjoseph 2021-08-13 20:49:32
The smallest base-9 four-digit number is 1000_9, so the largest base-9 three-digit number is

$$1000_9 - 1 = 1 \cdot 9^3 - 1 = 729 - 1 = 728.$$

So which base-10 three-digit numbers have three digits when expressed in base-$9?$

jonjoseph 2021-08-13 20:49:50
Is that clear?

jonjoseph 2021-08-13 20:50:10
The numbers n satisfying $100 \le n \le 728$ have three digits when expressed in base-$9.$

jonjoseph 2021-08-13 20:50:25
Now let's work with base $11.$

jonjoseph 2021-08-13 20:50:27
What is the smallest base-11 three-digit number, when expressed in base-$10?$

jonjoseph 2021-08-13 20:51:11
The smallest base-11 three-digit number is

$$100_{11} = 1 \cdot 11^2 = 121.$$

What is the largest base-11 three-digit number, when expressed in base-$10?$

jonjoseph 2021-08-13 20:51:33
Hint: Use the same trick we did for base 9.

jonjoseph 2021-08-13 20:52:39

The largest base-11 three-digit number is

$$1000_{11} - 1 = 1 \cdot 11^3 - 1 = 1331 - 1 = 1330.$$

jonjoseph 2021-08-13 20:52:50

So which base-10 three-digit numbers have three digits when expressed in base-11?

jonjoseph 2021-08-13 20:54:31

The numbers n satisfying $121 \le n \le 999$ have three digits when expressed in base-11.

jonjoseph 2021-08-13 20:54:38

So what values of n satisfy both conditions?

jonjoseph 2021-08-13 20:55:47

Combining both inequalities, we see that n has to be in the range $121 \le n \le 728$.

jonjoseph 2021-08-13 20:55:52

So what is the probability that n has three digits when expressed in both base-9 and base-11?

jonjoseph 2021-08-13 20:57:10

The probability that n has three digits when expressed in both base-9 and base-11 is

$$\frac{728 - 121 + 1}{900} = \frac{608}{900}.$$

jonjoseph 2021-08-13 20:57:17

Since $\frac{608}{900}$ is a bit more than $\frac{600}{900} = \frac{2}{3}$, the closest answer choice is 0.7. The answer is (E).

jonjoseph 2021-08-13 20:57:50

If r is the remainder when each of the numbers 1059, 1417, and 2312 is divided by d, where d is an integer greater than one, then $d - r$ equals

$$\text{(A) } 1 \quad \text{(B) } 15 \quad \text{(C) } 179 \quad \text{(D) } d - 15 \quad \text{(E) } d - 1$$

jonjoseph 2021-08-13 20:58:09

Let's deal with one piece of information at a time. We are told that 1059 has a remainder of r when divided by d. How can we write this as an equation?

jonjoseph 2021-08-13 20:59:19

We can write

$$1059 = ad + r$$

for some integer a. The number a is the quotient when 1059 is divided by d.

jonjoseph 2021-08-13 20:59:41

You could write this using modular arithmetic but it might be a little harder to deal with.

jonjoseph 2021-08-13 20:59:44

Similarly, we can write $1417 = bd + r$ and $2312 = cd + r$ for some integers b and c.

jonjoseph 2021-08-13 20:59:48

Thus, we have the system of equations

$$\begin{aligned} ad + r &= 1059, \\ bd + r &= 1417, \\ cd + r &= 2312. \end{aligned}$$

Now, how can we work with these equations?

jonjoseph 2021-08-13 21:00:02
Which variable can we try eliminating, and how?

jonjoseph 2021-08-13 21:00:40
We can get rid of r by subtracting each equation from one of the others.

jonjoseph 2021-08-13 21:00:44
Subtracting the equations in pairs, we get

$$\begin{aligned} bd - ad &= 358, \\ cd - ad &= 1253, \\ cd - bd &= 895. \end{aligned}$$

What can we do with these equations?

jonjoseph 2021-08-13 21:01:13
Hint: they all have a d in them.

jonjoseph 2021-08-13 21:01:36
We can factor the left-hand sides. This looks promising because we know our numbers all have to be integers.

jonjoseph 2021-08-13 21:01:39
Factoring the left-hand sides, we get

$$\begin{aligned} (b-a)d &= 358, \\ (c-a)d &= 1253, \\ (c-b)d &= 895. \end{aligned}$$

What do these equations tell us about d?

jonjoseph 2021-08-13 21:02:12
These equations tell us that d is a divisor of 358, 1253, and 895.

jonjoseph 2021-08-13 21:02:28
Therefore, d is a divisor of $\gcd(358, 1253, 895)$.

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The prime factorization of 358 is $2 \cdot 179$.

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The prime factorization of 1253 is $7 \cdot 179$.

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The prime factorization of 895 is $5 \cdot 179$. So what is $\gcd(358, 1253, 895)$?

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From the prime factorizations, we see that $\gcd(358, 1253, 895) = 179$. So what is d?

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Since d is greater than 1 and d divides the prime 179, the only possible value of d is 179.

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We know that r is the remainder when $1059, 1417,$ and 2312 are divided by $179.$

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Performing any of these divisions, we find that this remainder r is $164.$

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So what is $d-r?$

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We see that $d-r=179-164=15.$ The answer is (B).

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I went through that one fast. Be sure to check the transcript if I went too fast.

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SUMMARY

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We have seen how we can use simple tools like prime factorization and gcd to solve problems in number theory.

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We have also seen the importance of using algebra effectively in number theory. For example, if you have an even number, you can write it in the form $2n.$ If you have a perfect square, you can write in the form $n^2.$ Using these forms can help you see factorizations that you would not be able to otherwise.

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Before we finish today I'd like to mention again that a class survey has been posted on the course homepage. Please fill the survey out when you get a chance. It helps us to improve our courses and best meet student needs. You can find the survey in the Announcements under the Overview tab.

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There is also now a full AMC 10 practice test you can take on the "Practice AMC" tab of the course page.

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Stay safe. See you next week for our last class.