Art Of Problem Solving - AMC 10
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Patrick & James Toche

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Abstract

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1.

Find the value of x that satisfies the equation

$$25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}}$$

$$\begin{aligned}
(5^2)^{-2} &= \frac{5^{48/x}}{5^{26/x} \cdot (5^2)^{17/x}} \\
\Rightarrow \quad 5^{-4} &= \frac{5^{48/x}}{5^{26/x} \cdot 5^{34/x}} \\
\Rightarrow \quad 5^{-4} &= 5^{48/x-26/x-34/x} \\
\Rightarrow \quad 5^{-4} &= 5^{-12/x} \\
\Rightarrow \quad -4 &= -12/x \\
\Rightarrow \quad x &= 3
\end{aligned}$$

$$\boxed{x = 3}$$

2.

What is the sum of all the solutions of $x = |2x - |60 - 2x||$?

1. $60 - 2x > 0 \Leftarrow \boxed{x < 30}$

$$\Rightarrow \quad x = |2x - 60 + 2x| = |4x - 60| = 4|x - 15|$$

   (a) $\boxed{x > 15}$

$$\Rightarrow \quad x = 4(x - 15)$$
$$\Rightarrow \quad x = 20$$
$$15 < 20 < 30 \qquad \checkmark$$

   (b) $\boxed{x < 15}$

$$\Rightarrow \quad x = 4(-x + 15)$$
$$\Rightarrow \quad x = 12$$
$$12 < 15 < 30 \qquad \checkmark$$

2. $\boxed{x > 30}$

$$\Rightarrow \quad |60 - 2x| = -60 + 2x$$
$$\Rightarrow \quad x = |2x + 60 - 2x| = 60$$
$$60 > 30 \qquad \checkmark$$

$$\boxed{20 + 12 + 60 = 92}$$

3.

What is the product of all the roots of the equation

$$\sqrt{5|x|+8} = \sqrt{x^2-16}$$

We can square both sides of the equality:

$$5|x|+8 = x^2-16$$

We have two cases:

1. $\boxed{x > 0}$

$$\begin{aligned} & 5x+8 = x^2-16 \\ \Rightarrow \quad & x^2-5x-24=0 \\ \Rightarrow \quad & (x-8)(x+3)=0 \\ \Rightarrow \quad & \begin{cases} x=8 & ✓ \\ x=-3 & ✗ \end{cases} \end{aligned}$$

2. $\boxed{x < 0}$

$$\begin{aligned} & -5x+8 = x^2-16 \\ \Rightarrow \quad & x^2+5x-24=0 \\ \Rightarrow \quad & (x+8)(x-3)=0 \\ \Rightarrow \quad & \begin{cases} x=-8 & ✓ \\ x=3 & ✗ \end{cases} \end{aligned}$$

$$\boxed{-8 \times 8 = -64}$$

4.

How many positive integers n satisfy the following condition?

$$(130n)^{50} > n^{100} > 2^{200}$$

100 and 200 are multiples of 50, so:

$$\begin{aligned}
(130n)^{50} &> (n^2)^{50} &> (2^4)^{50} \\
130n &> n^2 &> 16 \\
130 &> n & \\
& \quad\;\; n &> 4
\end{aligned}$$

$$\Rightarrow \qquad n = 5, 6, \ldots, 128, 129$$

$\boxed{\text{125 integers}}$

5.

Real numbers a and b satisfy the equations $3^a = 81^{b+2}$ and $125^b = 5^{a-3}$. What is ab?

We notice that $81 = 3^4$ and $125 = 5^3$, so

$$\begin{aligned} 3^a &= 81^{b+2} = 3^{4(b+2)} \\ 5^{3b} &= 125^b = 5^{a-3} \end{aligned}$$

Equating the powers yields:

$$\begin{aligned} a &= 4(b+2) \\ 3b &= a - 3 \end{aligned}$$

Substituting:

$$\begin{aligned} & 3b = 4(b+2) - 3 \\ \Rightarrow \quad & b = -5 \\ \Rightarrow \quad & ab = 4(b+2)b = 4(-5+2)(-5) = 60 \end{aligned}$$

$$\boxed{ab = 60}$$

6.

If x, y, and z are positive with $xy = 24$, $xz = 48$, and $yz = 72$, what is $x + y + z$?

Multiply the three equations together:

$$\begin{aligned} xy \times xz \times yz = 24 \times 48 \times 72 = 3^4 \times 2^{10} = (3^2 \times 2^5)^2 \\ xyz = 3^2 \cdot 2^5 \end{aligned}$$

From this we easily get the values of x, y, z:

$$\begin{aligned} x &= \frac{xyz}{yz} = \frac{3^2 \cdot 2^5}{72} = 4 \\ y &= \frac{xyz}{xz} = \frac{3^2 \cdot 2^5}{48} = 6 \\ z &= \frac{xyz}{xy} = \frac{3^2 \cdot 2^5}{24} = 12 \end{aligned}$$

and the sum:

$$x + y + z = 4 + 6 + 12 = 22$$

Other approaches are possible. For instance, we notice that since $72 = 3 \times 24$ and $48 = 2 \times 24$, we have $yz = 3xy$ and $xz = 2xy$, which can be used to find $2y = 3x$, which can be substituted back to get values for x, y, and z.

$$\boxed{x + y + z = 22}$$

7.

If (x, y) is a solution to the system $xy = 6$ and

$$x^2y + xy^2 + x + y = 63$$

find $x^2 + y^2$.

We notice that $x + y$ can be factored to makes xy appear on the lhs, which can then be substituted:

$$\begin{aligned} x^2y + xy^2 + x + y &= 63 \\ (x + y)(xy + 1) &= 63 \\ (x + y)(6 + 1) &= 63 \\ x + y &= 9 \end{aligned}$$

Squaring $x + y$ will give us x^2, y^2 and $2xy$, just what the doctor ordered:

$$\begin{aligned} (x + y)^2 &= 9^2 \\ x^2 + 2xy + y^2 &= 9^2 \\ x^2 + y^2 &= 81 - 12 = 69 \end{aligned}$$

$$\boxed{x^2 + y^2 = 69}$$

8.

Suppose that $4^a = 5$, $5^b = 6$, $6^c = 7$, and $7^d = 8$. What is $a \cdot b \cdot c \cdot d$?

Notice the sequence 4, 5, 6, 7, 8? We can substitute in sequence! Let's write the equalities from the last and in reverse order:

$$\begin{array}{ccccccccccc} 8 & = & 7^d & & & & & & & & \\ & & 7 & = & 6^c & & & & & & \\ & & & & 6 & = & 5^b & & & & \\ & & & & & & 5 & = & 4^a & & \end{array}$$

and thus:

$$8 = 7^d = 6^{cd} = 5^{bcd} = 4^{abcd}$$

With $8 = 2^3$ and $4 = 2^2$, we have:

$$\begin{aligned} & 2^{2abcd} = 2^3 \\ \Rightarrow\ & 2abcd = 3 \end{aligned}$$

$$\boxed{abcd = \frac{3}{2}}$$