2021 Fall AMC 10B Problems/Problem 20

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Problem 20

In a particular game, each of 4 players rolls a standard 6-sided die. The winner is the player who rolls the highest number. If there is a tie for the highest roll, those involved in the tie will roll again and this process will continue until one player wins. Hugo is one of the players in this game. What is the probability that Hugo's first roll was a $5,$ given that he won the game?

$\textbf{(A)}\ \frac{61}{216} \qquad \textbf{(B)}\ \frac{367}{1296} \qquad \textbf{(C)}\ \frac{41}{144} \qquad \textbf{(D)}\ \frac{185}{648} \qquad \textbf{(E)}\ \frac{11}{36}$

Solution 1

Since we know that Hugo wins, we know that he rolled the highest number in the first round. The probability that his first roll is a 5 is just the probability that the highest roll in the first round is 5.

Let $P(x)$ indicate the probability that event x occurs. We find that
$P(\text{No one rolls a 6})-P(\text{No one rolls a 5 or 6})=P(\text{The highest roll is a 5}),$

so

$$P(\text{No one rolls a 6})=\left(\frac{5}{6}\right)^4,$$

$$P(\text{No one rolls a 5 or 6})=\left(\frac{2}{3}\right)^4,$$

$$P(\text{The highest roll is a 5})=\left(\frac{5}{6}\right)^4-\left(\frac{4}{6}\right)^4=\frac{5^4-4^4}{6^4}=\frac{369}{1296}=\boxed{\textbf{(C)}\ \frac{41}{144}}.$$

~kingofpineapplz

Solution 2 (Conditional Probability)

The conditional probability formula states that $P(A|B)=\dfrac{P(A\cap B)}{P(B)}$, where $A|B$ means A given B and $A\cap B$ means A and B. Therefore the probability that Hugo rolls a five given he won is $\frac{P(A\cap B)}{P(B)}$, where A is the probability that he rolls a five and B is the probability that he wins. In written form,

$$\text{P(Hugo rolls a 5 and wins)}$$

$$\mathrm{P}(\text{Hugo rolled a 5 given he won}) = \frac{}{\mathrm{P}(\text{Hugo wins})}.$$

The probability that Hugo wins is $\frac{1}{4}$ by symmetry since there are four people playing and there is no bias for any one player. The probability that he gets a 5 and wins is more difficult; we will have to consider cases on how many players tie with Hugo...

$\textbf{Case 1:}$ No Players Tie

In this case, all other players must have numbers from 1 through four.

There is a $\left(\frac{4}{6}\right)^3 = \frac{8}{27}$ chance of this happening.

$\textbf{Case 2:}$ One Player Ties

In this case, there are $\binom{3}{1} = 3$ ways to choose which other player ties with Hugo, and the probability that this happens is $\frac{1}{6} \cdot \left(\frac{4}{6}\right)^2$. The probability that Hugo wins on his next round is then $\frac{1}{2}$ because there are now two players rolling die.

Therefore the total probability in this case is $3 \cdot \frac{1}{2} \cdot \frac{1}{6} \cdot \left(\frac{4}{6}\right)^2 = \frac{1}{9}$.

$\textbf{Case 3:}$ Two Players Tie

In this case, there are $\binom{3}{2} = 3$ ways to choose which other players tie with Hugo, and the probability that this happens is $\left(\frac{1}{6}\right)^2 \cdot \frac{4}{6}$. The probability that Hugo wins on his next round is then $\frac{1}{3}$ because there are now three players rolling the die.

Therefore the total probability in this case is $3 \cdot \frac{1}{3} \cdot \left(\frac{1}{6}\right)^2 \cdot \frac{4}{6} = \frac{1}{54}$.

$\textbf{Case 4:}$ All Three Players Tie

In this case, the probability that all three players tie with Hugo is $\left(\frac{1}{6}\right)^3$. The probability that Hugo wins on the next round is $\frac{1}{4}$, so the total probability is $\frac{1}{4} \cdot \left(\frac{1}{6}\right)^3 = \frac{1}{864}$.

Finally, Hugo has a $\frac{1}{6}$ probability of rolling a five himself, so the total probability is

$$\frac{1}{6}\left(\frac{8}{27} + \frac{1}{9} + \frac{1}{54} + \frac{1}{864}\right) = \frac{1}{6}\left(\frac{369}{864}\right) = \frac{1}{6}\left(\frac{41}{96}\right).$$

Finally, the total probability is this probability divided by $\frac{1}{4}$ which is this probability times four; the final answer is

$$4 \cdot \frac{1}{6}\left(\frac{41}{96}\right) = \frac{2}{3} \cdot \frac{41}{96} = \frac{41}{48 \cdot 3} = \frac{41}{144} = \boxed{C}.$$

~KingRavi

~Edits by BakedPotato66

Solution 3

We use H to refer to Hugo. We use H_1 to denote the outcome of Hugo's tth toss. We denote by A, B, C the other three players. We denote by N the number of players among A, B, C whose first tosses are 5. We use W to denote the winner.

We have

$$
\begin{aligned}
P\left(H_1=5 \mid W=H\right) & =\frac{P\left(H_1=5, W=H\right)}{P(W=H)} \\
& =\frac{P\left(H_1=5\right) P\left(W=H \mid H_1=5\right)}{P(W=H)} \\
& =\frac{\frac{1}{6} P\left(W=H \mid H_1=5\right)}{\frac{1}{4}} \\
& =\frac{2}{3} P\left(W=H \mid H_1=5\right) .
\end{aligned}
$$

Now, we compute $P\left(W=H \mid H_1=5\right)$.

We have

$$
\begin{aligned}
& P\left(W=H \mid H_1=5\right) \\
& =P\left(W=H \mid H_1=5, \max \left\{A_1, B_1, C_1\right\} \leq 4\right) P\left(\max \left\{A_1, B_1, C_1\right\} \leq 4 \mid H_1=5\right) \\
& \quad+P\left(W=H \mid H_1=5, \max \left\{A_1, B_1, C_1\right\}=6\right) P\left(\max \left\{A_1, B_1, C_1\right\}=6 \mid H_1=5\right) \\
& \quad+\sum_{N=1}^3 P\left(W=H \mid H_1=5, \max \left\{A_1, B_1, C_1\right\}=5, N\right) P\left(\max \left\{A_1, B_1, C_1\right\}=5, N \mid H_1=5\right) \\
& =P\left(W=H \mid H_1=5, \max \left\{A_1, B_1, C_1\right\} \leq 4\right) P\left(\max \left\{A_1, B_1, C_1\right\} \leq 4\right) \\
& \quad+P\left(W=H \mid H_1=5, \max \left\{A_1, B_1, C_1\right\}=6\right) P\left(\max \left\{A_1, B_1, C_1\right\}=6\right) \\
& \quad+\sum_{N=1}^3 P\left(W=H \mid H_1=5, \max \left\{A_1, B_1, C_1\right\}=5, N\right) P\left(\max \left\{A_1, B_1, C_1\right\}=5, N\right) \\
& =1 \cdot P\left(\max \left\{A_1, B_1, C_1\right\} \leq 4\right)+0 \cdot P\left(\max \left\{A_1, B_1, C_1\right\}=6\right) \\
& \quad+\sum_{N=1}^3 P\left(W=H \mid H_1=5, \max \left\{A_1, B_1, C_1\right\}=5, N\right) P\left(\max \left\{A_1, B_1, C_1\right\}=5, N\right) \\
& =P\left(\max \left\{A_1, B_1, C_1\right\} \leq 4\right) \\
& \quad+\sum_{N=1}^3 P\left(W=H \mid H_1=5, \max \left\{A_1, B_1, C_1\right\}=5, N\right) P\left(\max \left\{A_1, B_1, C_1\right\}=5, N\right) \\
& =\left(\frac{4}{6}\right)^3+\sum_{N=1}^3 \frac{1}{N+1} \cdot\binom{3}{N}\left(\frac{1}{6}\right)^N\left(\frac{4}{6}\right)^{3-N} \\
& =\frac{41}{96} .
\end{aligned}
$$

The first equality follows from the law of total probability. The second equality follows from the property that Hugo's outcome is independent from other players' outcomes.

Therefore,

$$\begin{aligned} P\left(H_1=5 \mid W=H\right) &= \frac{2}{3} P\left(W=H \mid H_1=5\right) \\ &= \frac{2}{3} \frac{41}{96} \\ &= \frac{41}{144}. \end{aligned}$$

Therefore, the answer is $\boxed{\textbf{(C) } \frac{41}{144}}$.

~Steven Chen (www.professorchenedu.com)

Side Note (Bayes' Theorem)

Solution 1 is incorrect, it only considered the first round and didn't consider there might be second round or more.

Solution 2 and 3 uses the Bayes' theorem (https://en.wikipedia.org/wiki/Bayes%27_theorem).

$$P\left(A \mid B\right) = \frac{P\left(B \mid A\right) P\left(A\right)}{P\left(B\right)}$$

In this problem

$$P\left(\text{Hugo first rolled } 5 \mid \text{Hugo won}\right) = \frac{P\left(\text{Hugo won} \mid \text{Hugo first rolled } 5\right) \cdot P\left(\text{Hugo first rolled } 5\right)}{P\left(\text{Hugo won}\right)}$$

~isabelchen (https://artofproblemsolving.com/wiki/index.php/User:Isabelchen)

Video Solution

https://youtu.be/kfn0Bq1-Y5I

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