AMC 10 Problem Series (2804)

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Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Homework

Homework: Lesson 7

You have completed **10** of **10** challenge problems.
Past Due **Jul 24**.

Readings

Lesson 7 Transcript: Fri, Jul 16

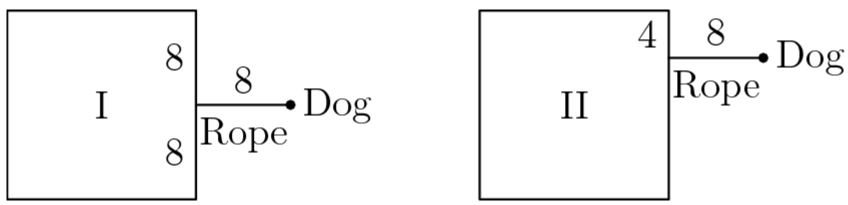
Challenge Problems

Total Score: 60 / 60

Problem 1 – Correct! – Score: 6 / 6 (2815)

Problem: Report Error

Rolly wishes to secure his dog with an 8-foot rope to a square shed that is 16 feet on each side. His preliminary drawings are shown.

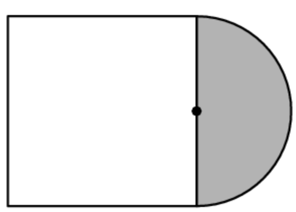


Which of these arrangements gives the dog the greater area to roam, and by how many square feet?

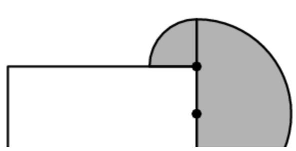
(A) I, by 8π **(B)** I, by 6π **(C)** II, by 4π **(D)** II, by 8π **(E)** II, by 10π

Solution:

In arrangement I, the area that Spot can roam is a semicircle with radius 8.



In arrangement II, the area that Spot can roam consists of a semicircle with radius 8 and a quarter-circle with radius 4.



Therefore, Spot has more area to roam in arrangement II by

$$\frac{1}{4} \cdot \pi \cdot 4^2 = \boxed{4\pi}.$$

The answer is (C).

Your Response(s):

C

Problem 2 – Correct! – Score: 6 / 6 (2816)

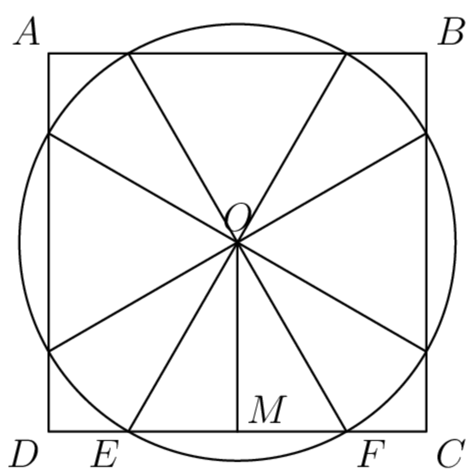
Problem: Report Error

A square of side length 1 and a circle of radius $\sqrt{3}/3$ share the same center. What is the area inside the circle, but outside the square?

(A) $\frac{\pi}{3} - 1$ **(B)** $\frac{2\pi}{9} - \frac{\sqrt{3}}{3}$ **(C)** $\frac{\pi}{18}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{2\pi}{9}$

Solution:

Let the circle intersect side CD of square $ABCD$ at E and F, as shown.



Let M be the midpoint of EE, so $\angle OME = 90^\circ$. Also, $OM = AD/2 = 1/2$ and $OE = \sqrt{3}/3$. Then

$$\frac{OM}{OE} = \frac{1/2}{\sqrt{3}/3} = \frac{\sqrt{3}}{2},$$

so triangle EOM is 30-60-90. Hence, triangle EOF is equilateral.

The area of triangle EOF is

$$\frac{\sqrt{3}}{4} \cdot \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{\sqrt{3}}{12},$$

and the area of circular sector EOF is

$$\frac{1}{6} \cdot \pi \cdot \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{\pi}{18},$$

so the area of the circular segment cut off by chord EF is

$$\frac{\pi}{18} - \frac{\sqrt{3}}{12}.$$

The area inside the circle but outside the square consists of four circular segments congruent to this circular segment, so this area is

$$4\left(\frac{\pi}{18} - \frac{\sqrt{3}}{12}\right) = \frac{2\pi}{9} - \frac{\sqrt{3}}{3}.$$

The answer is (B).

Your Response(s):

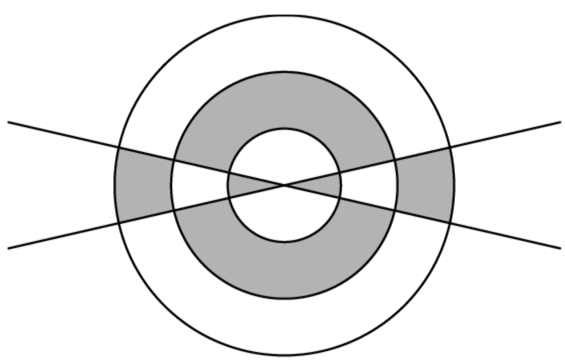
- B

Problem 3 – Correct! – Score: 6 / 6 (2817)

Problem: Report Error

Two distinct lines pass through the center of three concentric circles of radii 3, 2, and 1. The area of the shaded region in the diagram is 8/13 of the area of the unshaded region. What is the radian measure of the acute angle formed by the two lines?

(A) $\frac{\pi}{8}$ **(B)** $\frac{\pi}{7}$ **(C)** $\frac{\pi}{6}$ **(D)** $\frac{\pi}{5}$ **(E)** $\frac{\pi}{4}$



Solution:

Let θ be the acute angle formed by the two lines. We look first at the area inside the circle of radius 1.

The proportion of the shaded area inside the circle of radius 1 is θ/π, so the shaded area inside the circle of radius 1 is

$$\frac{\theta}{\pi} \cdot \pi \cdot 1^2 = \theta.$$

Next, we look at the area between the circles of radius 1 and 2, which has area $4\pi - \pi = 3\pi$. The proportion of the shaded area between the circles of radius 1 and 2 is $(\pi - \theta)/\pi$, so the shaded area between the circles of radius 1 and 2 is

$$\frac{\pi - \theta}{\pi} \cdot 3\pi = 3\pi - 3\theta.$$

Finally, we look at the area between the circles of radius 2 and 3, which has area $9\pi - 4\pi = 5\pi$. The proportion of the shaded area between the circles of radius 2 and 3 is θ/π, so the shaded area between the circles of radius 2 and 3 is

$$\frac{\theta}{\pi} \cdot 5\pi = 5\theta.$$

Hence, the total shaded area is $\theta + (3\pi - 3\theta) + 5\theta = 3\theta + 3\pi$. Then the unshaded area is $9\pi - (3\theta + 3\pi) = 6\pi - 3\theta$, so

$$\frac{3\theta + 3\pi}{6\pi - 3\theta} = \frac{8}{13}.$$

Solving for θ, we find $\theta = \boxed{\pi/7}$. The answer is (B).

Your Response(s):

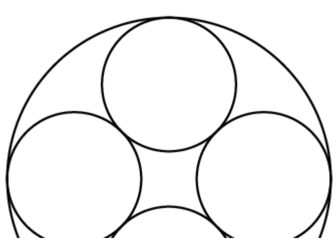
- B

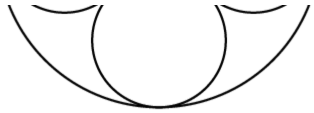
Problem 4 – Correct! – Score: 6 / 6 (2818)

Problem: Report Error

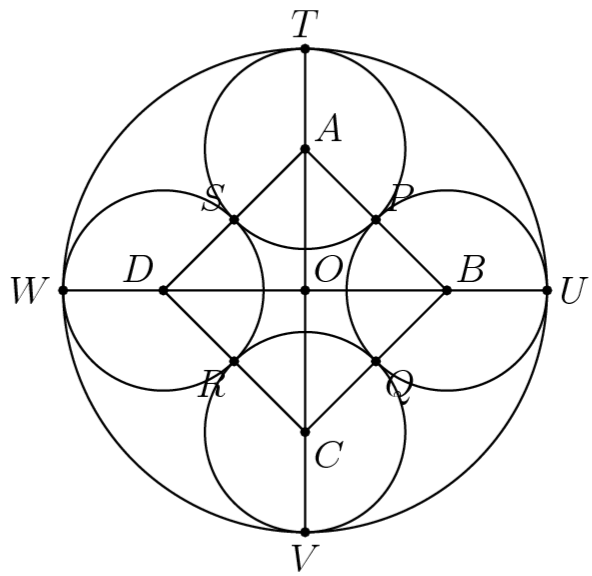
Many Gothic cathedrals have windows with portions containing a ring of congruent circles that are circumscribed by a larger circle. In the figure shown, the number of smaller circles is four. What is the ratio of the sum of the areas of the four smaller circles to the area of the larger circle?

(A) $3 - 2\sqrt{2}$ **(B)** $2 - \sqrt{2}$ **(C)** $4(3 - 2\sqrt{2})$ **(D)** $\frac{1}{2}(3 - \sqrt{2})$ **(E)** $2\sqrt{2} - 2$





Solution:
We label the centers and points of tangency.



Assume that the radius of each small circle is 1. Let r be the radius of the large circle. The circle with centers A and O are tangent at T, so O, A, and T are collinear. Hence, $OA = OT - AT = r - 1$. Similarly, $OB = OC = OD = r - 1$.

Also, $AB = BC = CD = DA = 2$, so triangles OAB, OBC, OCD, and ODA are congruent. Therefore, quadrilateral $ABCD$ is a square with center O, and $OA = AB/\sqrt{2} = 2/\sqrt{2} = \sqrt{2}$. But $OA = r - 1$, so $r - 1 = \sqrt{2}$, or $r = 1 + \sqrt{2}$.

Then the ratio of the sum of the areas of the four smaller circles to the area of the larger circle is

$$\begin{aligned}\frac{4\pi}{\pi r^2} &= \frac{4}{(1+\sqrt{2})^2} \\ &= \frac{4(1-\sqrt{2})^2}{(1+\sqrt{2})^2(1-\sqrt{2})^2} \\ &= \frac{4(1-2\sqrt{2}+2)}{[(1+\sqrt{2})(1-\sqrt{2})]^2} \\ &= \frac{4(3-2\sqrt{2})}{1} \\ &= \boxed{4(3-2\sqrt{2})}.\end{aligned}$$

The answer is (C).

Your Response(s):

C

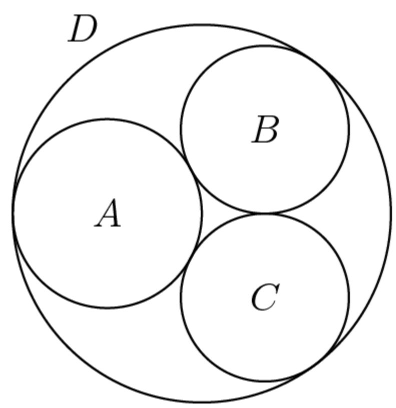
Problem 5 – Correct! – Score: 6 / 6 (2819)

Problem:

Report Error

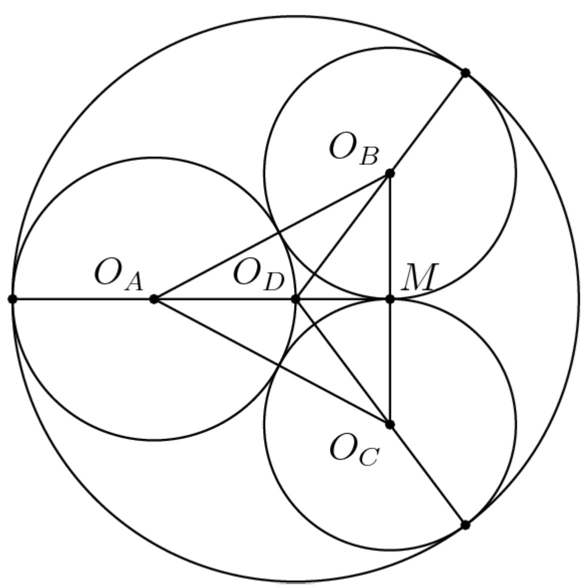
Circles A, B, and C are externally tangent to each other and internally tangent to circle D. Circles B and C are congruent. Circle A has radius 1 and passes through the center of D. What is the radius of circle B?

(A) $\frac{2}{3}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\frac{7}{8}$ **(D)** $\frac{8}{9}$ **(E)** $\frac{1+\sqrt{3}}{3}$



Solution:

Let O_X denote the center of circle X. Let r denote the common radius of circles B and C. Let M be the midpoint of O_BO_C.



We have that $O_AO_D = 1, O_AO_B = r + 1$, and $O_BM = r$. The radius of circle D is 2, so $O_BO_D = 2 - r$. Then by Pythagoras on right triangle O_AO_BM,

$$O_AM = \sqrt{(O_AO_B)^2 - (O_BM)^2} = \sqrt{(r+1)^2 - r^2} = \sqrt{2r+1}.$$

But by Pythagoras on right triangle O_BO_DM,
$O_DM = \sqrt{(O_BO_D)^2 - (O_BM)^2} = \sqrt{(2-r)^2 - r^2} = \sqrt{4-4r}$, which means
$O_AM = O_AO_D + O_DM = \sqrt{4-4r} + 1$. Hence,

$$\sqrt{2r+1} = \sqrt{4-4r} + 1.$$

Squaring both sides, we get

$$2r + 1 = 4 - 4r + 2\sqrt{4-4r} + 1,$$

which simplifies to

$$\sqrt{4-4r} = 3r - 2.$$

Squaring both sides again, we get

$$4 - 4r = 9r^2 - 12r + 4,$$

which simplifies to $9r^2 - 8r = r(9r-8) = 0$, so $r = \boxed{8/9}$. The answer is (D).

Your Response(s):

- D

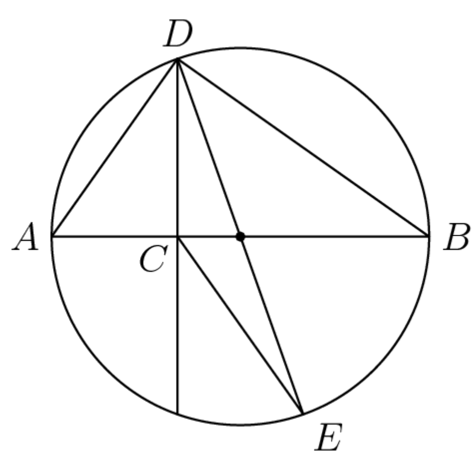
Problem 6 – Correct! – Score: 6 / 6 (2820)

Problem:

Report Error

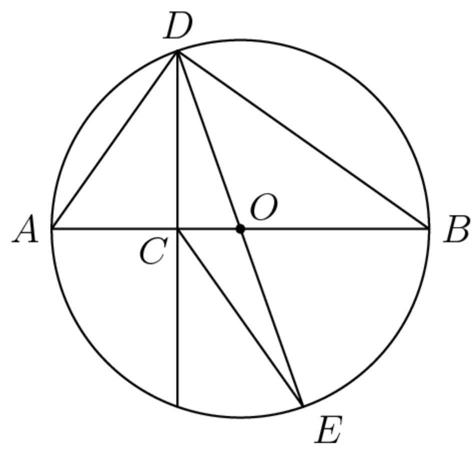
Let $\overline{AB}$ be a diameter of a circle and C be a point on $\overline{AB}$ with $2 \cdot AC = BC$. Let D and E be points on the circle such that $\overline{DC} \perp \overline{AB}$ and $\overline{DE}$ is a second diameter. What is the ratio of the area of triangle DCE to the area of triangle ABD?

(A) $\frac{1}{6}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{2}{3}$



Solution:

Let O be the center of the circle.



Since O is the midpoint of DE, and triangles DCE and DCO have the same height with respect to base DE, $[DCE]/[DCO] = DE/DO = 2$.

Since $BC = 2AC$, $AC = AB/3$. Then $OC = OA - AC = AB/2 - AB/3 = AB/6$. Since triangles DCO and DAB have the same height with respect to base AB, $[DCO]/[DAB] = CO/AB = 1/6$. Therefore,

$$\frac{[DCE]}{[DAB]} = \frac{[DCE]}{[DCO]} \cdot \frac{[DCO]}{[DAB]} = 2 \cdot \frac{1}{6} = \boxed{\frac{1}{3}}.$$

The answer is (C).

Your Response(s):

C

Problem 7 – Correct! – Score: 6 / 6 (2821)

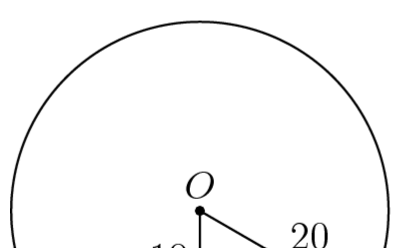
Problem: Report Error

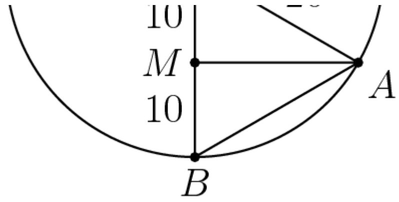
Riders on a Ferris wheel travel in a circle in a vertical plane. A particular wheel has radius 20 feet and revolves at the constant rate of one revolution per minute. How many seconds does it take a rider to travel from the bottom of the wheel to a point 10 vertical feet above the bottom?

(A) 5 **(B)** 6 **(C)** 7.5 **(D)** 10 **(E)** 15

Solution:

Let O be the center of the Ferris wheel, let B be the bottom, and let A be a point on the wheel 10 vertical feet above B. Let M be the midpoint of OB.





We see that in right triangle AOM, $OM = 10$ and $OA = 20$, so triangle AOM is 30-60-90. In particular, $\angle AOB = 60^\circ$. The wheel revolves at one revolution to per minute or 60 seconds, to it takes a rider $60/6 = \boxed{10}$ seconds to get from B to A. The answer is (D).

Your Response(s):

D

Problem 8 – Correct! – Score: 6 / 6 (2822)

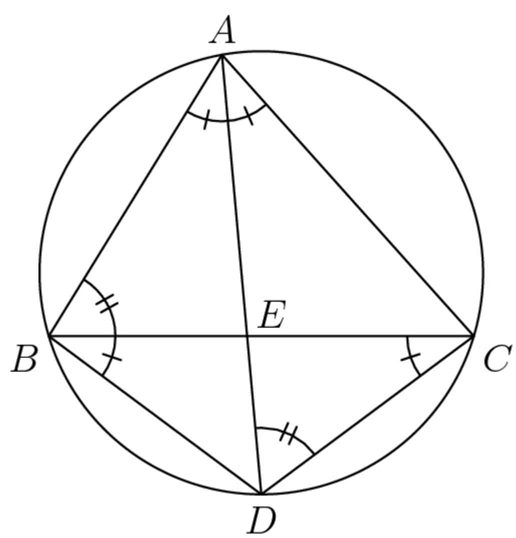
Problem: Report Error

In triangle ABC we have $AB = 7$, $AC = 8$, and $BC = 9$. Point D is on the circumscribed circle of the triangle so that $\overline{AD}$ bisects $\angle BAC$. What is the value of AD/CD?

(A) $\frac{9}{8}$ **(B)** $\frac{5}{3}$ **(C)** 2 **(D)** $\frac{17}{7}$ **(E)** $\frac{5}{2}$

Solution:

Let E be the intersection of AD and BC.



Since AD bisects $\angle BAC$, $\angle BAD = \angle CAD$. Since both $\angle BAD$ and $\angle BCD$ subtend arc BD, $\angle BAD = \angle BCD$. Since both $\angle CAD$ and $\angle CBD$ subtend arc CD, $\angle CAD = \angle CBD$. Finally, since both $\angle ABC$ and $\angle ADC$ subtend arc AC, $\angle ABC = \angle ADC$.

We see that triangles ABE and ADC are similar, so $AD/CD = AB/BE$. By the angle bisector theorem, $AB/BE = AC/CE$. Hence,

$$AB \quad AC \quad AB + AC \quad AB + AC \quad 7 + 8 \quad 15 \quad \boxed{5}$$

$$\frac{}{BE} = \frac{}{CE} = \frac{}{BE+CE} = \frac{}{BC} = \frac{}{9} = \frac{}{9} = \boxed{3}.$$

The answer is (B).

Your Response(s):

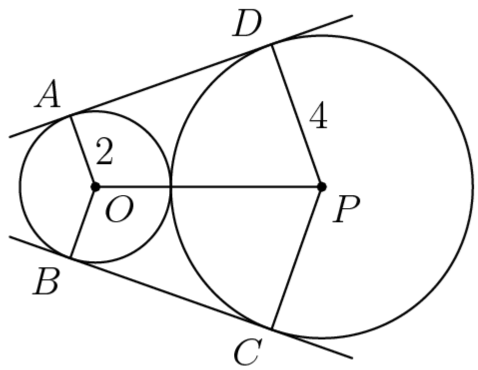
- B

Problem 9 – Correct! – Score: 6 / 6 (2823)

Problem: Report Error

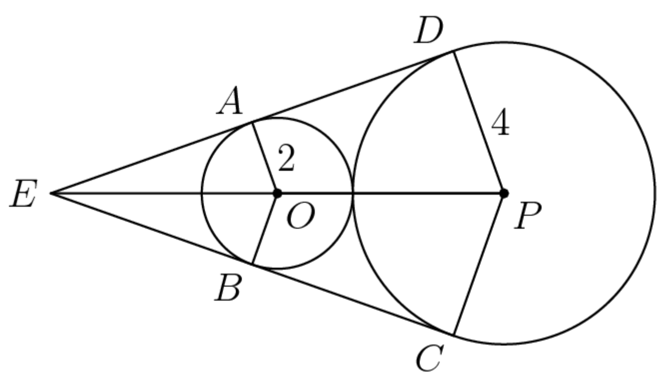
Circles with centers O and P have radii 2 and 4, respectively, and are externally tangent. Points A and B are on the circle centered at O, and points C and D are on the circle centered at P, such that $\overline{AD}$ and $\overline{BC}$ are common external tangents to the circles. What is the area of hexagon $AOBCPD$?

(A) $18\sqrt{3}$ **(B)** $24\sqrt{2}$ **(C)** 36 **(D)** $24\sqrt{3}$ **(E)** $32\sqrt{2}$



Solution:

Let E be the intersection of AD and BC. Then E also lies on OP.



Triangles EAO and EDP are similar, so $EO/EP = AO/DP = 2/4 = 1/2$. But $EP = EO + OP = EO + 6$, so $EO/(EO + 6) = 1/2$. Solving for EO, we find $EO = 6$.

By Pythagoras on right triangle EAO, $EA = \sqrt{EO^2 - AO^2} = \sqrt{6^2 - 2^2} = \sqrt{32} = 4\sqrt{2}$. The area of triangle EAO is $1/2 \cdot EA \cdot AO = 1/2 \cdot 4\sqrt{2} \cdot 2 = 4\sqrt{2}$. Then the area of triangle EDP is

$2^2 \cdot 4\sqrt{2} = 16\sqrt{2}$, and the area of trapezoid $AOPD$ is $16\sqrt{2} - 4\sqrt{2} = 12\sqrt{2}$.

By symmetry, the area of trapezoid $BOPC$ is also $12\sqrt{2}$, so the area of hexagon $AOBCPD$ is $2 \cdot 12\sqrt{2} = \boxed{24\sqrt{2}}$. The answer is (B).

Your Response(s):

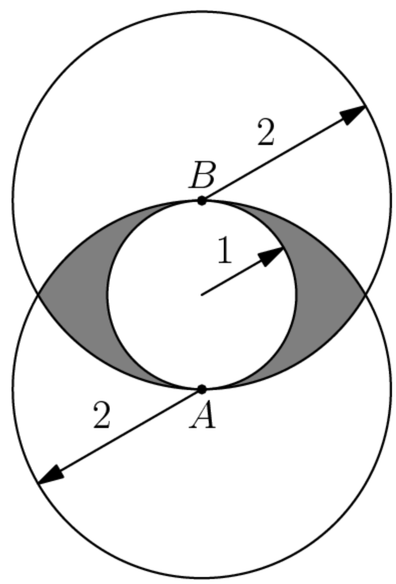
B

Problem 10 – Correct! – Score: 6 / 6 (2824)

Problem: Report Error

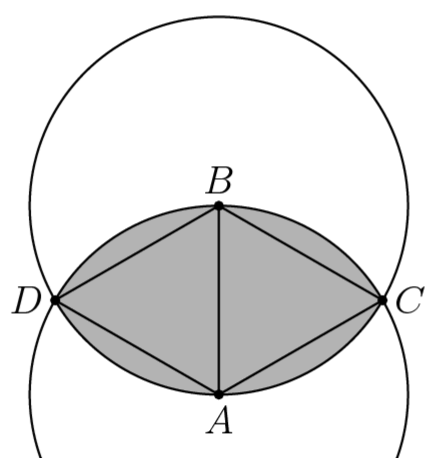
A circle of radius 1 is internally tangent to two circles of radius 2 at points A and B, where AB is a diameter of the smaller circle. What is the area of the region, shaded in the figure, that is outside the smaller circle and inside each of the two larger circles?

(A) $\frac{5}{3}\pi - 3\sqrt{2}$ **(B)** $\frac{5}{3}\pi - 2\sqrt{3}$ **(C)** $\frac{8}{3}\pi - 3\sqrt{3}$ **(D)** $\frac{8}{3}\pi - 3\sqrt{2}$ **(E)** $\frac{8}{3}\pi - 2\sqrt{3}$



Solution:

First, we compute the area inside the two large circles. Let C and D be the intersections of the two circles.



Since $AB = AC = BC = AD = BD = 2$, triangles ABC and ABD are equilateral. The area contained in both circles consists of two equilateral triangles and four congruent circular segments. The area of each equilateral triangle is

$$\frac{\sqrt{3}}{4} \cdot 2^2 = \sqrt{3}.$$

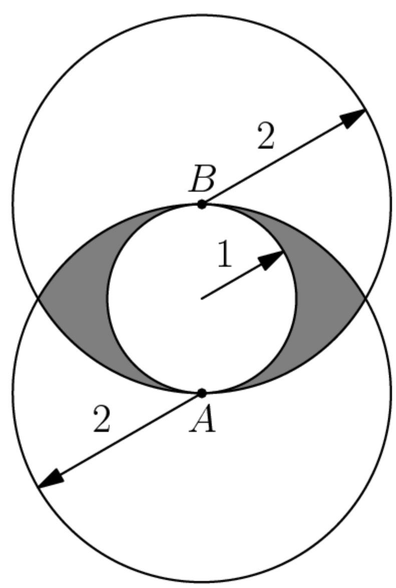
The area of each circular segment is

$$\frac{1}{6} \cdot \pi \cdot 2^2 - \sqrt{3} = \frac{2}{3}\pi - \sqrt{3}.$$

Therefore, the area contained in both circles is

$$2 \cdot \sqrt{3} + 4 \cdot \left(\frac{2}{3}\pi - \sqrt{3}\right) = \frac{8}{3}\pi - 2\sqrt{3}.$$

We must then subtract the area of a circle with radius 1.



The area we seek is

$$\left(\frac{8}{3}\pi - 2\sqrt{3}\right) - \pi = \boxed{\frac{5}{3}\pi - 2\sqrt{3}}.$$

The answer is (B).

Your Response(s):

- B