Flintridge Prep Summer School, Algebra II
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Abstract

Notes on the Flintridge Prep Summer Course Algebra II. Copyright restrictions may apply. Written for personal use. Please report typos and errors over at `https://github.com/ptoche/Math/tree/master/aops`.

1. (Problem Solving)

After summer school is over, M. will go back to her job at the local health-food store, where one of the more challenging tasks she has to do is mixing juices for all the hipster customers who really dig mixed juice. She did a particular mix one day of ginger and apple juice. A cup of ginger juice costs \$2.65, and a cup of apple juice costs \$1.98. The first customer to come in can only afford to pay \$18 for 8 cups of juice. How many cups of ginger and apple juice should M. use in the mixture, respectively? (round your answers to the nearest tenth of a cup)

Let A denote the quantity of Apple juice and G the quantity of Ginger juice used in the mix (in units of "cup"). The sum of these must equal 8 cups. The total cost of the mix must be \$18 (18/8 = \$2.25 per cup of mix). We have a linear system of two equations in two unknowns.

$$\begin{aligned} 2.65G + 1.98A &= 18 \\ G + A &= 8 \end{aligned}$$

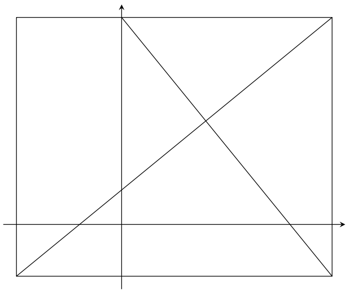
We can solve for G and A by substitution:

$$\begin{aligned} 2.65G + 1.98(8 - G) &= 18 \\ G &= \frac{18 - 8 \times 1.98}{2.65 - 1.98} = \frac{2.16}{0.67} \\ G &\approx 3.2 \\ A &\approx 4.8 \end{aligned}$$

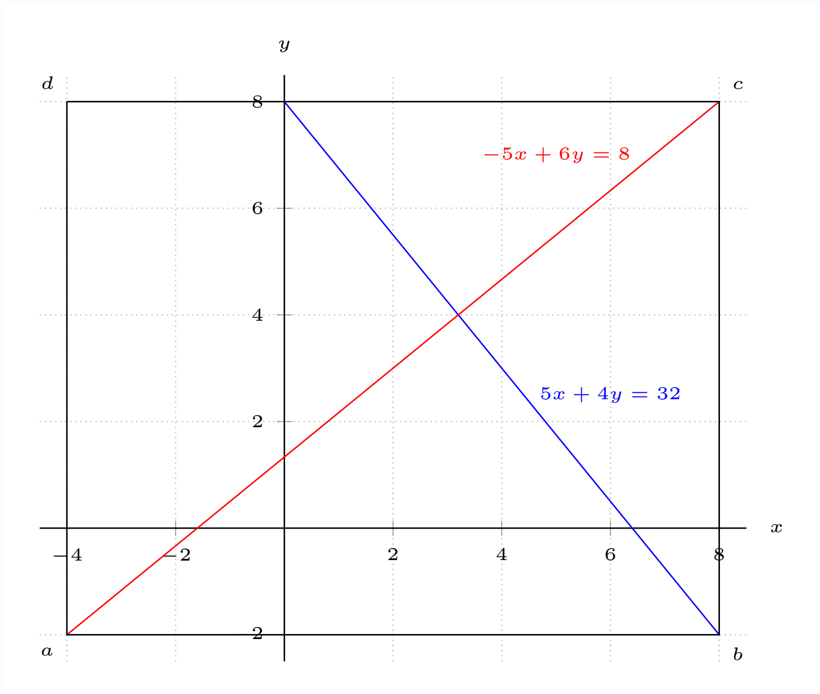
$$\boxed{G = 3.2, \quad A = 4.8}$$

2.

The diagram shows a screen on which the lines $5x+4y=32$ and $-5x+6y=8$ have been graphed. The window settings for this diagram consist of two inequalities, $a \le x \le b$ and $c \le y \le d$, in which the numbers a, b, c, and d are determined by the diagram. What are these numbers?



First, place labels on the figure.



The top-left corner, at the y-coordinate d, is the intercept of the line with equation $5x+4y=32$. Thus, d is the intercept of the line, and thus $d=8$. The top-right corner has the same y-coordinate 8 and x-coordinate b and lies on the line with equation $-5x+6y=8$. Thus the point $(b,8)$ must satisfy the equation, and thus $b=8$. We can go on to solve for all values as follows:

$$\begin{aligned}
(0,d) &\in \{5x+4y=32\} &&\Rightarrow && 5\cdot 0+4\cdot d=32 &&\Rightarrow && d=8\\
(b,d)=(b,8) &\in \{-5x+6y=8\} &&\Rightarrow && -5\cdot b+6\cdot 8=8 &&\Rightarrow && b=8\\
(b,c)=(8,c) &\in \{5x+4y=32\} &&\Rightarrow && 5\cdot 8+4\cdot c=32 &&\Rightarrow && c=-2\\
(a,c)=(a,-2) &\in \{-5x+6y=8\} &&\Rightarrow && -5\cdot a+6\cdot(-2)=8 &&\Rightarrow && a=-4
\end{aligned}$$

$$\boxed{a=-4, \quad b=8, \quad c=-2, \quad d=8}$$

3.

A car went a distance of 90km at a steady speed and returned along the same route at half that speed. The time needed for the whole round trip was four hours and a half. Find the two speeds.

Since velocity v (speed) is defined over a short distance d and duration t as the ratio d/t, if we know both the duration and the constant velocity, we can calculate a distance as $d = vt$. Since the speed is constant over each trip, there are two regimes. Let v_1, t_1 denote velocity/duration for the first part of the trip and v_2, t_2 for the second part. We know that $t_1 + t_2 = 4.5$ hours and $v_1 = v_2/2$. We also have $d_1 = d_2 = 90$. Thus,

$$\begin{aligned} v_1 &= 2v_2 \\ t_1 + t_2 &= 4.5 \\ v_1 t_1 &= 90 \\ v_2 t_2 &= 90 \end{aligned}$$

This is a linear system of 4 equations in 4 unknowns. Eliminate v_1, v_2 to get a two-equation system in t_1, t_2, from which v_1 and v_2 can be recovered:

$$\begin{aligned} v_1 &= 2v_2 \\ v_2 &= 90/t_2 \\ 2t_1 - t_2 &= 0 \\ t_1 + t_2 &= 4.5 \end{aligned}$$

Solution:

$$t_1 = 1.5, \quad t_2 = 3, \quad v_1 = 60, \quad v_2 = 30$$

$$\boxed{v_1 = 60, \quad v_2 = 30}$$

4. Problem 293

The Prep Ski club is planning a trip to Mammoth during semester break. They have 40 skiers signed up to go, and the ski resort is charging \$120 for each person.

a. Calculate how much money (revenue) the resort expects to take in.

$40 \times \$120 = \$4800$

b. The resort manager offers to reduce the group rate of \$120 per person by \$2 for each additional registrant, as long as the revenue continues to increase. For example, if 5 more skiers sign up, all 45 would pay \$110 each, producing revenue of \$4950 for the resort. Fill in the table for the manager.

extras	persons	cost/person	revenue
0	40		
1	41		
2	42		
3	43		
4	44	110	4950
5	45		
6	46		
7	47		
8	48		
9	49		
10	50		
11	51		
12	52		

extras	persons	cost/person	revenue
0	40	120	4800
1	41	118	4838
2	42	116	4872
3	43	114	4902
4	44	112	4928
5	45	110	4950
6	46	108	4968
7	47	106	4982
8	48	104	4992
9	49	102	4998
10	50	100	5000
11	51	98	4998
12	52	96	4992

c. Let x be the number of new registrants. In terms of x, write expressions for the total number of persons going, the cost to each, and the resulting revenue for the resort.

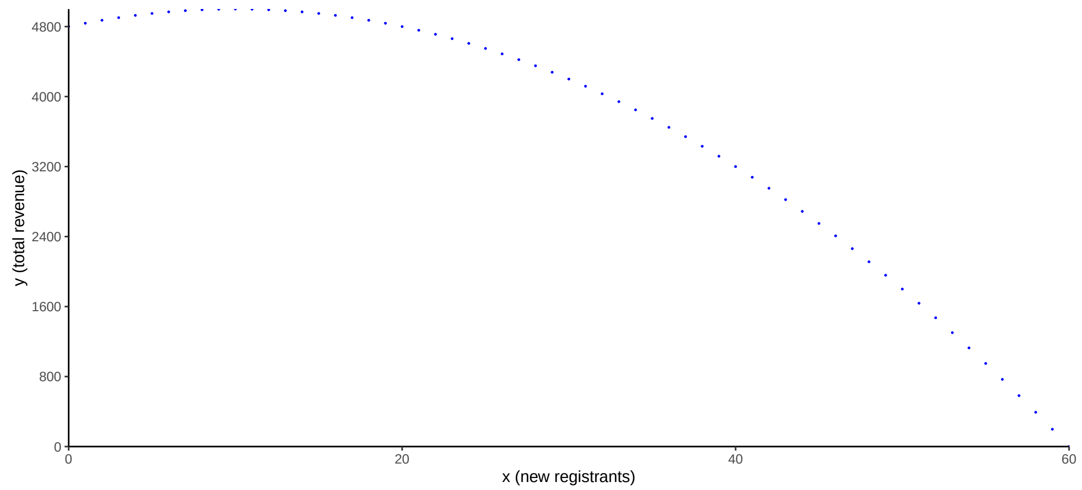
$$\begin{aligned} \text{number of persons} &: 40 + x \\ \text{cost per person} &: 120 - 2x \\ \text{total revenue} &: (120 - 2x)(40 + x) \end{aligned}$$

d. Plot your revenue values versus x, for the relevant values of x. Because this is a discrete problem, it does not make sense to connect the dots.

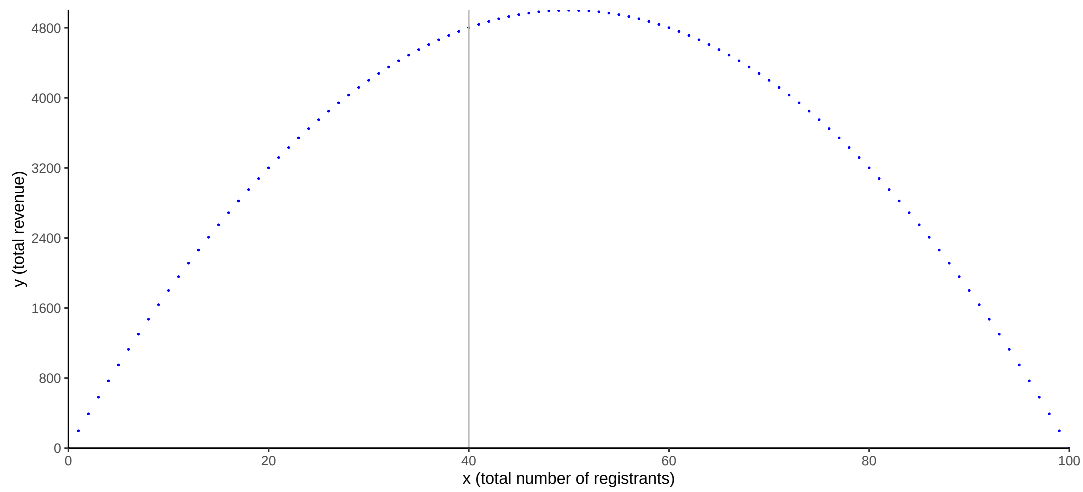
Revenue is a quadratic function of the number of new registrants x.

$$\begin{aligned} (120 - 2x)(40 + x) &= -2x^2 + 40x + 4800 \\ &= -2(x^2 - 20x - 2400) \\ &= -2((x - 10)^2 - 100 - 2400) \\ &= -2((x - 10)^2 - 50^2) \\ &= -2(x - 10 - 50)(x - 10 + 50) \\ &= -2(x - 60)(x + 40) \end{aligned}$$

Revenue starts at $40 \times \$120 = \$4800$ for $x = 0$, rises then falls to 0 for $x = 60$.



To see the big picture, we plot revenue in terms of the total number of registrants: Revenue starts at 0, rises to a maximum for 50 registrants, then falls to 0 for 100 registrants.



e. For the resort to take in at least \$4900, how many skiers must go on the trip?

The condition on revenue is:

$$y = (120 - 2x)(40 + x) \geq 4900$$

Solve for x when the inequality is strict:

$$\begin{aligned}(120 - 2x)(40 + x) &= 4900 \\ -2x^2 + 40x &= 100 \\ x^2 - 20x &= -50 \\ (x - 10)^2 - 10^2 &= -50 \\ (x - 10 - \sqrt{50})(x - 10 + \sqrt{50}) &= 0\end{aligned}$$

The roots are approximately 17.07 and 2.92. The closest integers associated with revenue greater than 4900 are $x = 17$ and $x = 3$. The corresponding revenues are:

$$\begin{aligned} x = 3 :& y = (120 - 2 \cdot 3)(40 + 3) = 4902 \\ x = 17 :& y = (120 - 2 \cdot 17)(40 + 17) = 4902 \end{aligned}$$

Any value in between generates even greater revenue. Thus, the resort can take in any number of new registrants between 3 and 17 or, equivalently, a total number skiers between 43 and 57. The maximal revenue is achieved for $x = 10$ new registrants, or 50 skiers, with a revenue of $y = \$5000$.

5. Question 11 of MidTerm Test

After high school, N. gets a job at the fire department, and is in charge of operating the hose, which shoots water out in a parabolic arc. Assume the behavior of the hose can be modeled by quadratic function. The hose is sprayed from 4.5 feet above ground, and hits the ground a horizontal distance of 58 feet away. The maximum height of the water occurs 28 feet from where the hose is sprayed. Let x bet the horizontal distance from the hose nozzle, and y be the corresponding height of the stream of water, both in feet.

(a) What is the quadratic equation that models the path of the water?

The general equation is

$$y = ax^2 + bx + c$$

where a, b, c are constants to be determined. Obviously $a < 0$ since the path of the water is up then down. The y intercept is at 4.5 feet, so $c = 4.5$. One of the roots is at $x = 58$, giving the condition

$$58^2 \cdot a + 58 \cdot b + 4.5 = 0$$

a linear equation in the parameters a and b. Since the maximum height occurs at 28 feet, the vertex is located at $x = 28$,

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a(x^2 + \frac{b}{a}x + \frac{c}{a}) \\ &= a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right] \end{aligned}$$

The vertex condition gives:

$$x = -\frac{b}{2a} = 28 \quad \Rightarrow \quad b = -56a$$

The parameters a and b satisfy the system:

$$\begin{aligned} 58^2a + 58b + 4.5 &= 0 \\ 56a + b &= 0 \end{aligned}$$

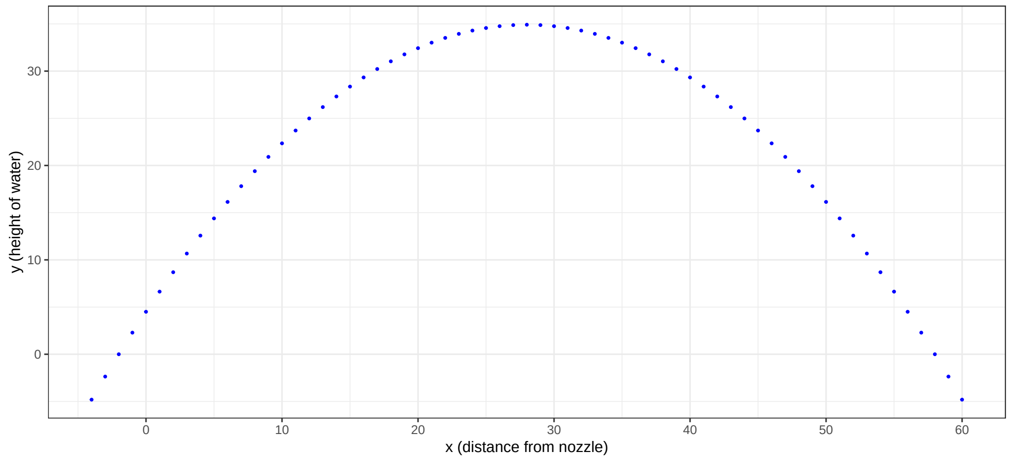
Subtracting 58 times the second equation from the first eliminates b:

$$\begin{aligned} (58^2 - 58 \cdot 56)a + 4.5 = 0 \\ \Rightarrow \quad a = -\frac{4.5}{116} \approx 0.0043 \\ b = \frac{56 \times 4.5}{116} = \frac{252}{116} = \frac{63}{29} \approx 2.1724 \end{aligned}$$

The quadratic equation is:

$$y = -\frac{45}{1160} \cdot x^2 + \frac{63}{29} \cdot x + 4.5$$

(b) Graph the function.



(c) What is the maximum height of the water?

The maximum height occurs for $x = 28$:

$$y = -\frac{45}{1160} \cdot 28^2 + \frac{63}{29} \cdot 28 + 4.5 \approx 34.9$$

(d) Will the stream go over a 6 ft high fence that is located 48 feet from the nozzle? Explain your reasoning and show any work that is needed.

The question is whether $y > 6$ for $x = 48$:

$$y = -\frac{45}{1160} \cdot 48^2 + \frac{63}{29} \cdot 48 + 4.5 \approx 19.4$$

The height of the water is more than three times greater than the fence.