# Russian School of Math: Lesson 9

James & Patrick

Revised: November 15, 2024

**Abstract**

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

## 1

Let $C$ be the coefficient of $x^2$ in the expansion of the product

$$(1-x)(1+2x)(1-3x)\dots(1+14x)(1-15x)$$

Find $|C|$.

***Solution***

This appears as 2004 AIME I Problems/Problem 7.

$$\begin{aligned}|C| = -15\cdot(\underbrace{14-13}_{1} + \underbrace{12-11}_{1}\ \dots\ + \underbrace{2-1}_{1})\\ + 14\cdot(\underbrace{-13+12}_{-1} + \dots \underbrace{-3+2}_{-1} - 1)\\ + \dots - 3\cdot(2-1) + 2\cdot(-1)\end{aligned}$$

$$\begin{aligned}&= -15\cdot 14/2\\ &\quad + 14\cdot(-1)(1+12/2)\\ &\quad + \dots - 3\cdot 1 + 2\cdot(-1)\\ &= -15\cdot 7 - 14\cdot 7 - 13\cdot 6 - 12\cdot 6 + \dots - 3 - 2\\ &= -(15+14)\cdot 7 - (13+12)\cdot 6 - (11+10)\cdot 5 - \dots - (5+4)\cdot 2 - (3+2)\cdot 1\\ &= -\sum_{k=1}^{7}(1+4k)k\\ &= -588\end{aligned}$$

$\boxed{|C| = 588.}$

## 2

The polynomial $P(x)$ is cubic. What is the largest value of $k$ for which the polynomials $Q_1(x) = x^2 + (k-29)x - k$ and $Q_2 = 2x^2 + (2k-43)x + k$ are both factors of $P(x)$?

***Solution***

This appears as 2007 AIME I Problems/Problem 8.
Let $a$ and $b$ satisfy $P(x) = Q_1(x)(2x+a)$ and $P(x) = Q_2(x)(x+b)$. We have

$$P(x) = (x^2 + (k-29)x - k)(2x+a) = (2x^2 + (2k-43)x + k)(x+b)$$

Equating coefficients:

$$\begin{cases} a + 2(k-29) = 2b + (2k-43)\\ a(k-29) - 2k = b(2k-43) + k\\ -ak = bk\end{cases}$$

$$\begin{cases} a-2b=15 \\ (a-2b)k-3k=29a-43b \\ a=-b \end{cases} \implies 15k-3k=12k=29a-43b=72a \implies k=6a=30$$

$\boxed{k=30.}$

## 3

Real numbers $r$ and $s$ are roots of $P(x)=x^3+ax+b$ and $r+4$ and $s-3$ are roots of $Q(x)=x^3+ax+b+240$. Find the sum of all possible values of $|b|$.

***Solution***

This appears as 2014 AIME II Problems/Problem 5.
First, derive conditions on $r$ and $s$ from the stated restrictions on polynomial $P(x)$.

$$\begin{cases} r^3+ar+b=0 \\ s^3+as+b=0 \end{cases} \implies r^3-s^3+a(r-s)=0 \implies (r-s)(r^2+rs+s^2+a)=0$$

A similar relation can be derived from $Q(x)$:

$$(r-s+7)\big((r+4)^2+(r+4)(s-3)+(s-3)^2+a\big)=0$$

Assuming $r\neq s$ and $r\neq s-7$, subtract the two equalities and simplify:

$$5r-2s+13=0 \implies s=\frac{13+5r}{2}$$

Let $r$, $s$, and $t$ be the three roots of $P(x)$. Applying Vieta's formula for the product and sum of the three roots:

$$\begin{matrix} rst=-b \\ r+s+t=0 \end{matrix} \implies rs(r+s)=b$$

Applying the same logic to $Q(x)$ and substituting $b=rs(r+s)$ to eliminate $b$:

$$(r+4)(s-3)(r+s+1)=b+240=rs(r+s)+240$$

Substituting $s$ in terms of $r$ into the above yields:

$$(r+4)\left(\frac{13+5r}{2}-3\right)\left(r+\frac{13+5r}{2}+1\right)=r\left(\frac{13+5r}{2}\right)\left(r+\frac{13+5r}{2}\right)+240$$
$$(r+4)(5r+7)(7r+15)=r(5r+13)(7r+13)+960$$
$$108(r+5)(r-1)$$

The roots are $r=1$ and $r=-5$. Using $s=(13+5r)/2$ gives $s=9$, $s=-6$. The root pairs $(r,s)$ are $(1,9)$ and $(-5,-6)$. It follows that

$$b=rs(r+s)\to 90,-330\to|90|+|-330|=420.$$

$\boxed{|b|=420.}$

## 4

What is the difference of the greatest possible value of $z$ and the least possible value of $x$ given that real triple $(x, y, z)$ satisfies the following system of equations?

$$\begin{cases} xyz = 8 \\ xy + yz + zx = -6 \\ x + y + z = -3 \end{cases}$$

***Solution***

The polynomial may be written:

$$t^3 + 3t^2 - 6t - 8 = (t + 4)(t + 1)(t - 2)$$

The greatest value of $z$ is 2. The least possible value of $x$ is $-4$. And the difference is $2 - (-4) = 6$.
$\boxed{6.}$

## 5

Steve says to Jon "I am thinking of a polynomial whose roots are all positive integers. The polynomial has the form $P(x) = 2x^3 - 2ax^2 + (a^2 - 81)x - c$ for some positive integers $a$ and $c$. Can you tell me the values of $a$ and $c$?"

After some calculations, Jon says, "There is more than one such polynomial."

Steve says, "You're right. Here is the value of $a$." He writes down a positive integer and asks, "Can you tell me the value of $c$?"

Jon says, "There are still two possible values of $c$." Find the sum of the two possible values of $c$.

***Solution***

This appears as 2015 AIME II Problems/Problem 6.
$\boxed{440.}$

## 6

A real number $a$ is chosen randomly and uniformly from the interval $[-20, 18]$. The probability that the roots of the polynomial

$$x^4 + 2ax^3 + (2a - 2)x^2 + (-4a + 3)x - 2$$

are all real can be written in the form $\frac{m}{n}$, where $m$ and $n$ are relatively prime integers. Find $m + n$.

***Solution***

This appears as 2018 AIME II Problems/Problem 6.
$\boxed{37.}$

## 7

The graph of $y = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2$ lies above the line $y = bx + c$ except at three values of $x$, where the graph and the line intersect. What is the largest of these values?

***Solution***

> This appears as 2010 AMC 12A Problems/Problem 21.
> $\boxed{4.}$

## 7

Let $x_1 < x_2 < x_3$ be the three real roots of the equation $\sqrt{2014}\,x^3 - 4029\,x^2 + 2 = 0$. Find $x_2(x_1 + x_3)$.

***Solution***

> This appears as 2014 AIME I Problems/Problem 9.
> $\boxed{002.}$