UCLA Math Circle

James Toche (and family)

19 June 2020
(Last revision: July 26, 2020)

Abstract

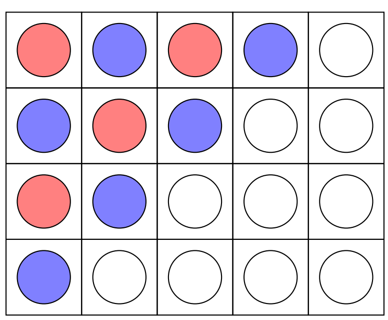
Notes on homework problems from the UCLA Math Circle Intermediate-2 for Summer Session 2020, July 19th.

1a. Show by Induction that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \qquad \forall n \geq 1$$

This equality is an excellent introduction to the topic of **proof by induction**, because it can be proved in several other ways. Here are two standard proofs: a visual proof based on stacking and a proof based on rearranging terms.

Just For Fun: A Proof by Stacking



Count the circles along the diagonal starting from the top-left corner, adding circles of the same color at each step (red, then blue, then red, then blue). The sequence is 1 (for $n = 1$), 3 (for $n = 2$), 6 (for $n = 3$), 10 (for $n = 4$). This last number is the sum of the first 4 integers. Denote it S_4,

$$S_4 = 1 + 2 + 3 + 4$$

By symmetry, there are also 10 white circles. Thus the total number of circles in the matrix is $2 \times S_4 = 20$. But clearly, this is also the "area" of the rectangle $20 = 4 \times 5$ (height times width). The argument is general and therefore $2S_n = n \times (n+1)$. $\square$

Just For Fun: A Proof by Rearrangement

Rearrange the sum by reversing it:

$$\begin{array}{ccccccccccccc} S_n & = & 1 & + & 2 & + & 3 & + & \ldots & + & (n-1) & + & n \\ & = & n & + & (n-1) & + & (n-2) & + & \ldots & + & 2 & + & 1 \end{array}$$

Consider the terms that are aligned vertically: The first two add up to $1+n$. The next two add up to $2+(n-1) = 1+n$. Next, $3+(n-2) = 1+n$. And so on. Thus the sum $2S_n$ is equal to a repeated sum of $(1+n)$. How many times is the sum repeated? From the first line, exactly n repetitions. And thus,

$$\begin{aligned} 2S_n &= \underbrace{(1+n) \quad + \quad \ldots \quad + \quad (1+n)}_{n \text{ times}} \\ &= (1+n) \times n \qquad \square \end{aligned}$$

Proof by Induction

$$P(n): \quad S_n = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

To prove $P(n)$ true for all $n \geq 1$, we prove the two statements:

$$\begin{cases} P(n) \text{ true for } n = 1 \text{ (Base Case)} \\ \text{if } P(n) \text{ is true for some } n\text{, then } P(n+1) \text{ is also true (Induction Step)} \end{cases}$$

These two statements together imply: $P(n)$ true for all $n \geq 1$.

An equivalent, more concise way of writing the two parts of a proof by induction is:

$$\begin{cases} P(1) \ \texttt{true} \\ P(n) \implies P(n+1) \end{cases}$$

The two-step procedure above is a template for most proofs by induction.

Base Case:

Substitute n for 1:

$$P(1): \quad 1 = \frac{1 \times (1+1)}{2} \qquad \checkmark$$

Induction Step:

Start with the left-hand side of $P(n+1)$ and substitute $P(n)$ to get the right-hand side of $P(n+1)$, where

$$P(n+1): \quad 1 + 2 + \ldots + n + (n+1) = \frac{(n+1)(n+2)}{2}$$

From the left-hand side of $P(n+1)$:

$$\begin{aligned} \texttt{lhs} &= \underbrace{1 + 2 + \ldots + n} + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= \texttt{rhs} \qquad \square \end{aligned}$$

1b. Show by Induction that

$$1 + 3 + \ldots + (2n-1) = n^2 \qquad \forall n \geq 1$$

Let $P(n)$ denote the equality for *some* fixed value $n \in \mathbb{N}$. We have:

$$P(n+1): \quad 1 + 3 + \ldots + (2n-1) + (2n+1) = (n+1)^2$$

Base Case:

$$P(1): \quad 1 = 1^2 \qquad \checkmark$$

Induction Step:

$$\begin{aligned} \texttt{lhs} &= \underbrace{1 + 3 + \ldots + (2n-1)}_{n^2} + (2n+1) \\ &= n^2 + (2n+1) \\ &= n^2 + 2n + 1 \\ &= (n+1)^2 \\ &= \texttt{rhs} \qquad \square \end{aligned}$$

1c. Show by Induction that

$$2 + 5 + \ldots + (3n-1) = \frac{3n^2 + n}{2} \qquad \forall n \geq 1$$

Let $P(n)$ denote the equality for *some* fixed value $n \in \mathbb{N}$. We have:

$$P(n+1): \quad 2 + 5 + \ldots + (3n-1) + (3n+2) = \frac{3(n+1)^2 + (n+1)}{2}$$

Base Case:

$$P(1): \quad 2 = \frac{3 \cdot 1^2 + 1}{2} \qquad \checkmark$$

Induction Step:

$$\begin{aligned} \texttt{lhs} &= \underbrace{2 + 5 + \ldots + (3n-1)}_{\frac{3n^2+n}{2}} + (3n+2) \\ &= \frac{3n^2 + n}{2} + (3n+2) \\ &= \frac{3n^2 + 7n + 4}{2} \end{aligned}$$

While we could factorize the above expression to make $(n+1)$ appear, it is easier to expand the right-hand side of $P(n+1)$ and show that it is equal to the left-hand side. Thus,

$$\begin{aligned} \texttt{rhs} &= \frac{3(n+1)^2 + (n+1)}{2} \\ &= \frac{3(n^2 + 2n + 1) + n + 1}{2} \\ &= \frac{3n^2 + 7n + 4}{2} \\ &= \texttt{lhs} \qquad \square \end{aligned}$$

1d. Show by Induction that

$$1 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \forall n \geq 1$$

Let $P(n)$ denote the equality for *some* fixed value $n \in \mathbb{N}$. We have:

$$P(n+1): \quad 1 + 2^2 + \ldots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

Base Case:

$$P(1): \quad 1 = \frac{1 \cdot (1+1)(2 \cdot 1 + 1)}{6} \qquad \checkmark$$

Induction Step:

$$\begin{aligned}
\texttt{lhs} &= \underbrace{1 + 2^2 + \cdots + n^2} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} \\
&= \frac{(n+1)\left[n(2n+1) + 6(n+1)\right]}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6}
\end{aligned}$$

While we could factorize the above expression to make $(n+2)(2n+3)$ appear, it is easier to expand the right-hand side of $P(n+1)$ and show that it is equal to the left-hand side. Thus,

$$\begin{aligned}
\texttt{rhs} &= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)(2n^2 + 7n + 6)}{6} \\
&= \texttt{lhs} \qquad \square
\end{aligned}$$

2. Divisibility by Induction

Prove that the number $11\ldots11$ consisting of 243 ones is divisible by 243. Prove the generalization: for any positive integer n, the number consisting of 3^n ones is divisible by 3^n.

The general statement is:

$$\underbrace{11\ldots11}_{3^n \text{ times}} = 3^n \times m \quad \text{for some } m \in \mathbb{N}$$

Special cases that are easy to check:

$$\begin{aligned} P(1): \quad 111 &= 3 \times 37 \\ P(2): \quad 111,111,111 &= 3^2 \times 12345679 \end{aligned}$$

To prove that $111,111,111$ is divisible by 9 using divisibility rules, first note that:

$$\begin{array}{r} 111,000,000 \\ +\ 111,000 \\ +\ 111 \\ \hline 111,111,111 \end{array}$$

so that

$$\begin{aligned} 111,111,111 &= 111 \cdot 10^6 + 111 \cdot 10^3 + 111 \cdot 10^0 \\ &= 111 \cdot (10^6 + 10^3 + 1) \end{aligned}$$

Because 111 is a multiple of 3 (its digits add up to 3) and $10^6 + 10^3 + 1$ is a multiple of 3 (its digits also add up to 3), it follows that $111,111,111$ is a multiple of $3 \times 3 = 9$.

The above proof can be generalized. Let $a(n)$ denote the number:

$$P(n): \quad a(n) = \underbrace{111,111,111}_{3^n} \quad \text{is divisible by } 3^n$$

The next number $a(n+1)$ may be written as:

$$\begin{aligned} a(n+1) &= \underbrace{111,111,111}_{3^{n+1}} \\ &= \underbrace{a(n)a(n)a(n)}_{3 \text{ copies of } a(n)} \\ &= a(n) \cdot 10^x + a(n) \cdot 10^y + a(n) \\ &= a(n) \cdot (10^x + 10^y + 1) \end{aligned}$$

where $y = 3^n$ and $x = 2y$ (but the exact values do not matter for the proof). By $P(n)$, we have $a(n)$ a multiple of 3^n and by the divisible rule, we have $10^x + 10^y + 1$ a multiple of 3, and therefore $P(n+1)$ is true:

$$P(n+1): \quad a(n+1) = \underbrace{111,111,111}_{3^{n+1}} \quad \text{is divisible by } 3^{n+1} \qquad \square$$

3. Divisibility by Induction

Show that $n^3 + 2n$ is divisible by 3 for all positive integers n.

Suppose the proposition $P(n)$ is true for some $n \in \mathbb{N}$:

$$P(n): \quad n^3 + 2n = 3m, \text{ for some } m \in \mathbb{N}$$

Base Case:

$$P(1): \quad 1^3 + 2 \cdot 1 = 3 \cdot 1 \qquad \checkmark$$

Induction Step:

$$P(n+1): \quad (n+1)^3 + 2(n+1) = 3l, \text{ for some } l \in \mathbb{N}$$

We start from the left-hand side of $P(n+1)$,

$$\begin{aligned}
\texttt{lhs} &= (n+1)^3 + 2(n+1) \\
&= n^3 + 3n^2 + 3n + 1 + 2n + 2 \\
&= \underbrace{n^3 + 2n} \quad + \quad 3(n^2 + n + 1) \\
&= \quad 3m \qquad + \quad 3(n^2 + n + 1) \\
&= 3(m + n^2 + n + 1) \\
&= 3l, \text{ where } l \in \mathbb{N} \text{ because } m, n, n^2, 1 \in \mathbb{N} \\
&= \texttt{rhs}
\end{aligned}$$

Conclusion:

Since $P(n) \implies P(n+1)$ and $P(1)$ true, it follows that $P(n)$ true for all $n \geq 1$.

4a. Inequality by Induction

Show that for any positive integer n, we have $2^n > n$.

Suppose the proposition $P(n)$ is true for some $n \in \mathbb{N}$:

$$P(n): \quad 2^n > n$$

Base Case:

$$P(1): \quad 2^1 > 1 \qquad \checkmark$$

Induction Step:

$$P(n+1): \quad 2^{n+1} > n+1$$

We start from the left-hand side of $P(n+1)$:

$$\begin{aligned}
\texttt{lhs} &= 2^{n+1} \\
&= 2 \cdot \underbrace{2^n}_{>n} \\
&> 2n \\
&> n+n \\
&> n+1 \quad = \quad \texttt{rhs}
\end{aligned}$$

where the last inequality follows from $n \geq 1$.

Conclusion:

Since $P(n) \implies P(n+1)$ and $P(1)$ true, it follows that $P(n)$ true for all $n \geq 1$.

4b. Inequality by Induction

Find all positive integers n such that $2^n > n^2$. Prove the result.

We tabulate the first few cases:

n	2^n	n^2	$2^n > n^2$
0	1	0	true
1	2	1	true
2	4	4	false
3	8	9	false
4	16	16	false
5	32	25	true
6	64	36	true
7	128	49	true

Our hypothesis is that the inequality holds for $n \geq 5$. This hypothesis follows by inspecting the graphs of 2^n and n^2. We prove the hypothesis by induction. Suppose the proposition is true for some $n \in \mathbb{N}$:

$$P(n): \quad 2^n > n^2$$

<u>Base Case:</u>

$$P(5): \quad 2^5 > 5^2 \qquad \checkmark$$

<u>Induction Step:</u>

$$P(n+1): \quad 2^{n+1} > (n+1)^2$$

We start from the left-hand side of $P(n+1)$:

$$\begin{aligned} \texttt{lhs} &= 2^{n+1} \\ &= 2 \cdot \underbrace{2^n}_{>n^2} \\ &> 2n^2 = \quad n^2 + n^2 \end{aligned}$$

Now consider the right-hand side of $P(n+1)$:

$$\begin{aligned} \texttt{rhs} &= (n+1)^2 \\ &= n^2 + 2n + 1 \end{aligned}$$

Thus, $2^{n+1} > (n+1)^2$ if $n^2 > 2n+1$. This is true for $n \geq 5$ because it is true for $n = 5$ and n^2 increases faster than $2n+1$ as n is increased.

<u>Conclusion:</u>

Since $P(n) \implies P(n+1)$ and $P(5)$ true, it follows that $P(n)$ true for all $n \geq 5$.

5. Lines & Regions

Suppose there are n lines drawn in the plane such that no two lines are parallel and no three lines intersect at the same point. Find a closed formula for the number of regions that the lines split into.

For small values of n, it is easy to sketch intersecting lines and count regions. Let n denote the number of lines and r the number regions. We have:

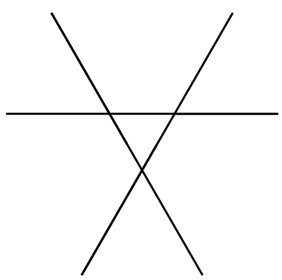
n	r
0	1
1	2
2	4
3	7
4	11

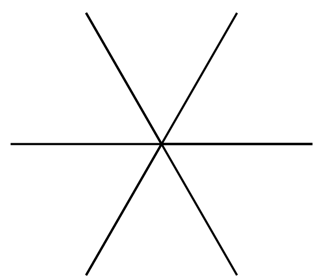
The case $n = 0$ is obvious: with no lines crossing the plane, there is one region — the entire plane.

The case $n = 1$ is equally obvious: a single line divides the plane into two regions, each being a half-plane.

The case $n = 2$ is easy to explain: At the intersection of the two lines, there are four angles that sum to $360°$, and each angle defines a region.

The case $n = 3$ can be explained by extending the previous idea: The intersection of the three lines forms a triangle. This triangle defines one region. Now move the lines such as to shrink the triangle to a single point. The resulting figure has three lines intersecting at a single point (see figure below). These lines define 6 regions, for a total of 7 regions when the triangle is included.





The case $n = 4$ can be understood by considering what happens when a line is added to the previous case. The fourth line intersects the other three lines at 3 points, and so goes through 4 "existing" regions, dividing each into two parts, thus adding 4 "new" regions, $7 + 4 = 11$.

In general, the nth line intersects with $n - 1$ lines, creating n news regions. This suggests a method for calculating the number of regions based on the previous value:

$$r(n) = r(n-1) + n$$

This is a linear recurrence. A linear recurrence admits a unique solution, which may be found, for instance, by repeated substitution.

$$\begin{aligned}
r(n) &= r(n-1) + n \\
r(n-1) &= r(n-2) + (n-1) \\
r(n-2) &= r(n-3) + (n-2) \\
&\vdots \\
r(3) &= r(2) + 3 \\
r(2) &= r(1) + 2 \\
r(1) &= r(0) + 1
\end{aligned}$$

Adding these equalities column-wise gives:

$$r(n) = n + (n-1) + (n-2) + \ldots + 3 + 2 + 1 + r(0)$$

where $r(0) = 1$ (as noted in the table above). Thus,

$$r(n) = (1 + 2 + 3 + \ldots + n) + 1$$

In words, the number of regions delimited by the intersection of n lines that intersect at $n-1$ distinct points is equal to one plus the sum of the integers up to n. There is, of course, a famous formula for the sum of the first n integers:

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

Substituting into the formula for $r(n)$ gives:

$$\begin{aligned}
r(n) &= \frac{n(n+1)}{2} + 1 \\
&= \frac{n^2 + n + 2}{2}
\end{aligned}$$

For peace of mind, you may check that the formula generates the values computed in the table above:

$$\begin{aligned}
r(0) &= \frac{0^2 + 0 + 2}{2} = \frac{2}{2} = 1 \\
r(1) &= \frac{1^2 + 1 + 2}{2} = \frac{4}{2} = 2 \\
r(2) &= \frac{2^2 + 2 + 2}{2} = \frac{8}{2} = 4 \\
r(3) &= \frac{3^2 + 3 + 2}{2} = \frac{14}{2} = 7 \\
r(4) &= \frac{4^2 + 4 + 2}{2} = \frac{22}{2} = 11
\end{aligned}$$