

Art Of Problem Solving - AMC 10

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Abstract

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1 Equation with Fractional Powers

Find the value of x that satisfies the equation

$$25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}}$$

$$(5^2)^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot (5^2)^{17/x}}$$

$$\Rightarrow 5^{-4} = \frac{5^{48/x}}{5^{26/x} \cdot 5^{34/x}}$$

$$\Rightarrow 5^{-4} = 5^{48/x - 26/x - 34/x}$$

$$\Rightarrow 5^{-4} = 5^{-12/x}$$

$$\Rightarrow -4 = -12/x$$

$$\Rightarrow x = 3$$

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| $x = 3$ |
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2 Equation with Absolute Values

What is the sum of all the solutions of $x = |2x - |60 - 2x||$?

1. $60 - 2x > 0 \Leftrightarrow \boxed{x < 30}$

$$\Rightarrow x = |2x - 60 + 2x| = |4x - 60| = 4|x - 15|$$

(a) $\boxed{x > 15}$

$$\Rightarrow x = 4(x - 15)$$

$$\Rightarrow x = 20$$

$$15 < 20 < 30 \quad \checkmark$$

(b) $\boxed{x < 15}$

$$\Rightarrow x = 4(-x + 15)$$

$$\Rightarrow x = 12$$

$$12 < 15 < 30 \quad \checkmark$$

2. $\boxed{x > 30}$

$$\Rightarrow |60 - 2x| = -60 + 2x$$

$$\Rightarrow x = |2x + 60 - 2x| = 60$$

$$60 > 30 \quad \checkmark$$

$$\boxed{20 + 12 + 60 = 92}$$

3 Product of Roots and Absolute Values

What is the product of all the roots of the equation

$$\sqrt{5|x| + 8} = \sqrt{x^2 - 16}$$

We can square both sides of the equality:

$$5|x| + 8 = x^2 - 16$$

We have two cases:

1. $x > 0$

$$\begin{aligned} 5x + 8 &= x^2 - 16 \\ \Rightarrow x^2 - 5x - 24 &= 0 \\ \Rightarrow (x - 8)(x + 3) &= 0 \\ \Rightarrow \begin{cases} x = 8 & \checkmark \\ x = -3 & \times \end{cases} \end{aligned}$$

2. $x < 0$

$$\begin{aligned} -5x + 8 &= x^2 - 16 \\ \Rightarrow x^2 + 5x - 24 &= 0 \\ \Rightarrow (x + 8)(x - 3) &= 0 \\ \Rightarrow \begin{cases} x = -8 & \checkmark \\ x = 3 & \times \end{cases} \end{aligned}$$

$$-8 \times 8 = -64$$

4 Integer Inequalities

How many positive integers n satisfy the following condition?

$$(130n)^{50} > n^{100} > 2^{200}$$

100 and 200 are multiples of 50, so:

$$(130n)^{50} > (n^2)^{50} > (2^4)^{50}$$

$$130n > n^2 > 16$$

$$130 > n$$

$$n > 4$$

$$\Rightarrow n = 5, 6, \dots, 128, 129$$

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| 125 integers |
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5 System with Unknown Powers

Real numbers a and b satisfy the equations $3^a = 81^{b+2}$ and $125^b = 5^{a-3}$. What is ab ?

We notice that $81 = 3^4$ and $125 = 5^3$, so

$$\begin{aligned}3^a &= 81^{b+2} = 3^{4(b+2)} \\5^{3b} &= 125^b = 5^{a-3}\end{aligned}$$

Equating the powers yields:

$$\begin{aligned}a &= 4(b+2) \\3b &= a-3\end{aligned}$$

Substituting:

$$\begin{aligned}3b &= 4(b+2) - 3 \\ \Rightarrow \quad b &= -5 \\ \Rightarrow \quad ab &= 4(b+2)b = 4(-5+2)(-5) = 60\end{aligned}$$

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| $ab = 60$ |
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6 Sum from System of Products

If x , y , and z are positive with $xy = 24$, $xz = 48$, and $yz = 72$, what is $x + y + z$?

Multiply the three equations together:

$$\begin{aligned} xy \times xz \times yz &= 24 \times 48 \times 72 = 3^4 \times 2^{10} = (3^2 \times 2^5)^2 \\ xyz &= 3^2 \cdot 2^5 \end{aligned}$$

From this we easily get the values of x , y , z :

$$\begin{aligned} x &= \frac{xyz}{yz} = \frac{3^2 \cdot 2^5}{72} = 4 \\ y &= \frac{xyz}{xz} = \frac{3^2 \cdot 2^5}{48} = 6 \\ z &= \frac{xyz}{xy} = \frac{3^2 \cdot 2^5}{24} = 12 \end{aligned}$$

and the sum:

$$x + y + z = 4 + 6 + 12 = 22$$

Other approaches are possible. For instance, we notice that since $72 = 3 \times 24$ and $48 = 2 \times 24$, we have $yz = 3xy$ and $xz = 2xy$, which can be used to find $2y = 3x$, which can be substituted back to get values for x , y , and z .

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| $x + y + z = 22$ |
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7 Sum of Squares of System

If (x, y) is a solution to the system $xy = 6$ and

$$x^2y + xy^2 + x + y = 63$$

find $x^2 + y^2$.

We notice that $x + y$ can be factored to makes xy appear on the lhs, which can then be substituted:

$$x^2y + xy^2 + x + y = 63$$

$$(x + y)(xy + 1) = 63$$

$$(x + y)(6 + 1) = 63$$

$$x + y = 9$$

Squaring $x + y$ will give us x^2 , y^2 and $2xy$, just what the doctor ordered:

$$(x + y)^2 = 9^2$$

$$x^2 + 2xy + y^2 = 9^2$$

$$x^2 + y^2 = 81 - 12 = 69$$

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| $x^2 + y^2 = 69$ |
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8 System of Exponentials

Suppose that $4^a = 5$, $5^b = 6$, $6^c = 7$, and $7^d = 8$. What is $a \cdot b \cdot c \cdot d$?

Notice the sequence 4, 5, 6, 7, 8? We can substitute in sequence! Let's write the equalities from the last and in reverse order:

$$\begin{array}{ccccccc} 8 & = & 7^d & & & & \\ & & 7 & = & 6^c & & \\ & & & & 6 & = & 5^b \\ & & & & & & 5 & = & 4^a \end{array}$$

and thus:

$$8 = 7^d = 6^{cd} = 5^{bcd} = 4^{abcd}$$

With $8 = 2^3$ and $4 = 2^2$, we have:

$$\begin{aligned} 2^{2abcd} &= 2^3 \\ \Rightarrow 2abcd &= 3 \end{aligned}$$

$$abcd = \frac{3}{2}$$