Russian School of Math: Lesson 3

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Abstract

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

1

Find all integer solutions to the equation

$$3^x + 7 = 2^y$$

Solution

> Solution: $(x, y) \in \{(0, 3), (2, 4)\}$.

2

Solve the equation in integers:

$$x^2 + 5y^2 + 34z^2 + 2xy - 10xz - 22yz = 0$$

Find the product $x \times y \times z$ for which the condition $100 < xyz < 500$ is satisfied.

Solution

> One equation with three variables x, y, z has an infinity of solutions. Fix x and express y and z in terms of x:
>
> $$y = \frac{3x}{7}$$
> $$z = \frac{2x}{7}$$
>
> It follows that x can only be 0 or a multiple of 7. It is obvious that $(0, 0, 0)$ is an integer solution.
>
> $$z = \frac{1}{34}\left(\pm\sqrt{-9x^2 + 42xy - 49y^2} + 5x + 11y\right)$$
>
> Integer solutions:
>
> $$\begin{aligned} x = 0, \quad y = 0, \quad z = 0 \\ x = +7, \quad y = +3, \quad z = +2 \\ x = -7, \quad y = -3, \quad z = -2 \end{aligned}$$

3

Solve an equation for integers:

$$x^2 = (y+1)^2 + (y+2)^2 + (y+3)^2$$

What is x?

Solution

> Expand the expression:
>
> $$x^2 - 3y^2 - 12y + 14 = 0$$

> Solve for y in terms of x, assuming $x \in (-\infty, -\sqrt{2}) \cup (+\sqrt{2}, \infty)$:
>
> $$y = \frac{1}{3}\left(-\sqrt{3}\sqrt{x^2 - 2} - 6\right)$$
>
> Real solutions:
>
> $$x = -\sqrt{2}, y = -2$$
> $$x = +\sqrt{2}, y = -2$$

4

Find all integer solutions of equation:

$$x^6 = y^3 + 217$$

Find the value of $z = x + y$ for each solution. What is the greatest value of z?

Solution

> Solve for y in terms of x, assuming $x^6 > 217$:
>
> $$y = (-1)^{2/3} \sqrt[3]{x^6 - 217}$$
>
> Integer solutions:
>
> $$x = \pm 3, y = +8$$
> $$x = \pm 1, y = -6$$

5

How many pairs of integer solutions (x, y) does the following equation have?

$$x^2 - y! = 2019$$

Solution

6

Prove that equation $x^2 - 2y^2 = 1$ has infinitely many integer solutions.

Solution

Solve for y in terms of x. There are two solutions:

$$y = -\frac{x^2 - 1}{\sqrt{2}}$$
$$y = \frac{x^2 - 1}{\sqrt{2}}$$

The solutions of Pell's equation are:

$$x = \pm\frac{1}{2}(-(3 - 2\sqrt{2}))^n - (3 + 2\sqrt{2})^n)$$
$$y = \pm\frac{(3 - 2\sqrt{2})^n - (3 + 2\sqrt{2})^n}{2\sqrt{2}}$$

for $n \in \mathbb{Z}$, $n \geq 0$.