AMC 10 Problem Series (2804)

Jon Joseph

Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Overview

Week 6 (Jul 9) Class Transcript - Triangles

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wh83 2021-07-09 19:30:02
Hi everyone! My name is Radu Andrei Cebanu and I am the instructor for today.

wh83 2021-07-09 19:30:05
Radu earned his PhD in mathematics in 2013 from UQAM. He has been with AoPS since 2017. His passion for math began when he started participating in mathematical competitions from the 7th grade and all the way into his high school years. Eventually, he even made it onto the Romanian team for the Balkan Mathematical Olympiad in 2000, winning a gold medal. He studies low dimensional topology and is also interested in high energy physics, being fascinated by the connection between these two fields. He worked as an instructor at Laurentian University of Sudbury and University of Ottawa. Outside of math, Radu enjoys hiking, jogging and playing with his two sons.

wh83 2021-07-09 19:30:21
Say hello as well to Ivan Detelinov Dosev (ivanddosev) who will be helping us today!

wh83 2021-07-09 19:30:24
Ivan was born into a math loving family and, from a young age, developed the same love as well. Growing up across different countries, he participated in a variety of math competitions in multiple languages. Throughout his early years, he wanted to pass down the same help he got at home. He formed multiple competition prep and math help groups in his high school. He is now a student at Cornell University, earning his B.S in Applied Mathematics and Economics. He has been a tutor for a variety of math courses at Cornell, and joined AoPS back in 2019, where he hopes to continue to spread his passion for mathematics.

wh83 2021-07-09 19:30:43
A solid understanding of geometry begins with knowing how to work with one of the most basic figures in geometry, namely the triangle. Today, we will look at the properties of triangles and how to use them effectively to solve problems in geometry.

wh83 2021-07-09 19:30:51
We start by looking at areas of triangles.

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AREAS OF TRIANGLES

wh83 2021-07-09 19:30:58
We know that the area of a triangle is half the base times the height. This formula looks simple, but it has a wide variety of applications.

wh83 2021-07-09 19:31:05
Point P is inside equilateral triangle ABC. Points Q, R, and S are the feet of the perpendiculars from P to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Given that $PQ = 1$, $PR = 2$, and $PS = 3$, what is AB?

$\text{(A) } 4 \quad \text{(B) } 3\sqrt{3} \quad \text{(C) } 6 \quad \text{(D) } 4\sqrt{3} \quad \text{(E) } 9$

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How do we start?

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First, let's draw a diagram.

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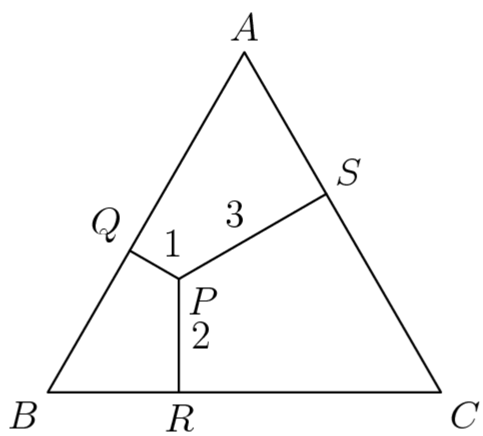

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We are looking for AB, the side length of equilateral triangle ABC, so let's call it x.

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What might make us think that areas will be a useful tool here?

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There are lots of segments perpendicular to sides. That's often a good clue that we should look at areas, since we can use the perpendicular segments as altitudes.

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How can we use areas to solve this problem?

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(Hint: for the perpendiculars to be used in the area formula, they should be altitudes).

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We can partition triangle ABC into triangles ABP, BCP, and CAP, and compute the areas of these triangles.

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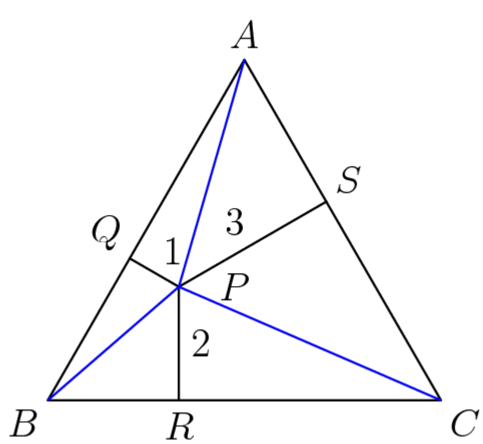

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What is the area of triangle ABP?

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(Remember that we denoted the side of the equilateral triangle by x).

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The area of triangle ABP is $\frac{1}{2} \cdot AB \cdot PQ = \frac{1}{2} \cdot x \cdot 1 = \frac{x}{2}$.

wh83 2021-07-09 19:36:32
What are the areas of triangles BCP and ACP?

wh83 2021-07-09 19:37:15
The area of triangle BCP is

$$\frac{1}{2} \cdot BC \cdot PR = \frac{1}{2} \cdot x \cdot 2 = x,$$

and the area of triangle CAP is

$$\frac{1}{2} \cdot CA \cdot PS = \frac{1}{2} \cdot x \cdot 3 = \frac{3x}{2}.$$

wh83 2021-07-09 19:37:20
So what's the area of $\triangle ABC$ in terms of x, using the areas above?

wh83 2021-07-09 19:37:47
It's equal to the sum of the areas, which is $\frac{x}{2} + x + \frac{3x}{2} = 3x$.

wh83 2021-07-09 19:37:51
But what's another expression for the area of triangle ABC?

wh83 2021-07-09 19:39:33
Triangle ABC is equilateral with side length x, so the area of triangle ABC is also equal to $\frac{x^2\sqrt{3}}{4}$.

wh83 2021-07-09 19:39:39
If you don't remember this formula, you can always drop an altitude to split an equilateral triangle into two $30^\circ - 60^\circ - 90^\circ$ right triangles to work out the area.

wh83 2021-07-09 19:39:45
Hence,

$$3x = \frac{\sqrt{3}}{4}x^2.$$

So what's x?

wh83 2021-07-09 19:40:52
Since $x \neq 0$, we can divide both sides by x. Then we get that

$$x = \frac{3}{\sqrt{3}/4} = \frac{12\sqrt{3}}{3} = 4\sqrt{3}.$$

The answer is (D).

wh83 2021-07-09 19:41:03
One of the most useful facts about areas of triangles is that if two triangles have the same height, then the ratio of their areas is equal to the ratio of their bases. (This follows directly from the area formula.)

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For example, what is $\frac{[APC]}{[BPC]}$ equal to in the following diagram? (The square bracket notation $[ABC]$ denotes the area of triangle ABC.)

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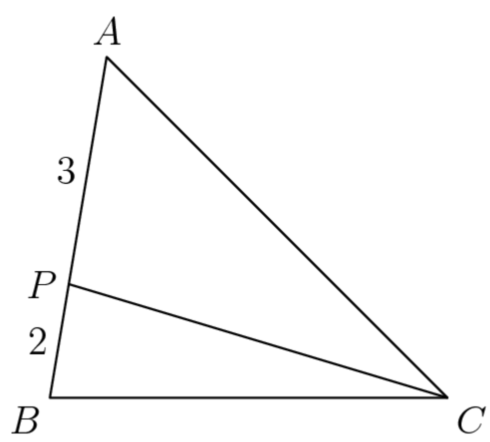

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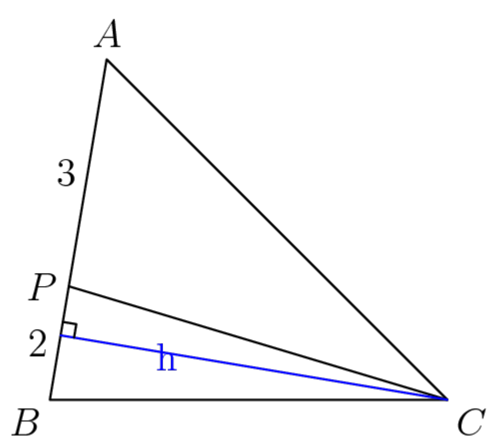

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If h is the common height, then we have the two following equalities:

$$[APC] = \frac{1}{2} AP \cdot h$$
$$[BPC] = \frac{1}{2} BP \cdot h$$

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Dividing the two equations, we get $\frac{[APC]}{[BPC]} = \frac{AP}{BP} = \frac{3}{2}$.

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And what about $\frac{[APC]}{[ABC]}$?

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That's equal to $\dfrac{AP}{AB} = \dfrac{3}{5}$.

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Let's take this idea a little further. Using notation as in the diagram below, how can we express the ratio $\dfrac{[APQ]}{[ABC]}$ solely in terms of lengths of line segments already appearing in the given diagram?

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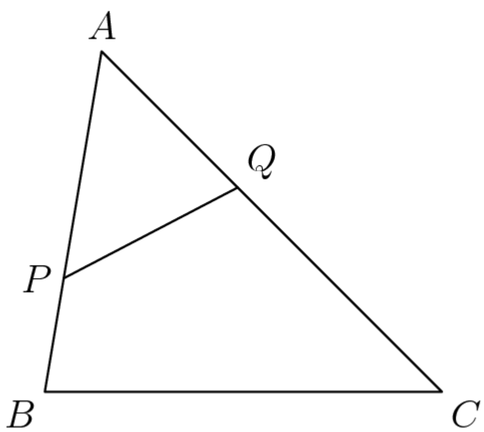

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To find this ratio, we'd like to find triangles that have the same height. How can we draw another triangle that has the same height as triangle APQ?

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(Yes, both $\triangle ABQ$ and $\triangle ACP$ work).

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Let's draw the segment PC.

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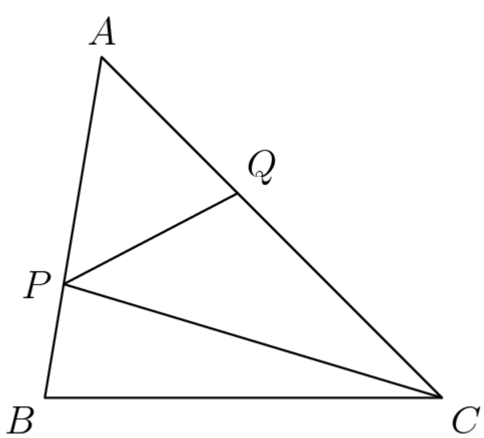

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Now triangles AQP and ACP have the same height with respect to base $\overline{AC}$.

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So what can we say about $\dfrac{[AQP]}{[ACP]}$?

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(Remember to use capital letters for points).

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Since triangles AQP and ACP have the same height with respect to base AC, $\dfrac{[AQP]}{[ACP]} = \dfrac{AQ}{AC}$.

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But we want to find $\dfrac{[APQ]}{[ABC]}$. So we'd like to find $\dfrac{[ACP]}{[ABC]}$.

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What is $\dfrac{[ACP]}{[ABC]}$?

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As shown in the previous example, $\dfrac{[ACP]}{[ABC]} = \dfrac{AP}{AB}$, since the triangles have the same height with respect to base AB.

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Therefore,

$$\frac{[APQ]}{[ABC]} = \frac{[APQ]}{[ACP]} \cdot \frac{[ACP]}{[ABC]} = \frac{AQ}{AC} \cdot \frac{AP}{AB} = \frac{AP \cdot AQ}{AB \cdot AC}.$$

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Here's a useful fact to help memorize the result we've proven here: When you have two triangles which share an angle, the area of each triangle is *proportional* to the *product of the two sides around the shared angle*.

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This fact is going to be useful for later problems, so take another look, and make sure you can remember it!

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If you know a little trigonometry, you can say something even more specific, namely that $[ABC] = \frac{1}{2} AB \cdot AC \cdot \sin(A)$.

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In right triangle ACE, we have $AC = 12$, $CE = 16$, and $EA = 20$. Points B, D, and F are located on AC, CE, and EA, respectively, so that $AB = 3$, $CD = 4$, and $EF = 5$. What is the ratio of the area of triangle BDF to that of triangle ACE?

$\text{(A)}\ \dfrac{1}{4} \quad \text{(B)}\ \dfrac{9}{25} \quad \text{(C)}\ \dfrac{3}{8} \quad \text{(D)}\ \dfrac{11}{25} \quad \text{(E)}\ \dfrac{7}{16}$

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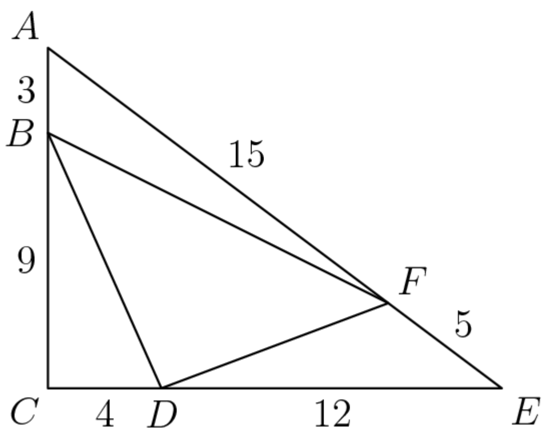

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Any ideas?

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(The triangle in the middle is hard to deal with directly).

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We want to find the ratio of the area of triangle BDF to the area of triangle ACE.

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Triangle ACE has triangle BDF in the middle and $\triangle ABF$, $\triangle CBD$, and $\triangle EDF$ in the corners.

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Our final answer is a *ratio to the area* of triangle ACE. Are there any triangles whose area we can find, in a ratio to the area of $\triangle ACE$?

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(It turns out we won't even need to compute $[ACE]$).

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Right! We can find any of the three outer triangles' areas in a ratio to the area of triangle ACE.

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Let's start with $\triangle ABF$. What is $\frac{[ABF]}{[ACE]}$?

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Ah-ha, we're going to use the trick we just learned in the last problem. Remember, when you have two triangles which share an angle, the area of each triangle is proportional to the product of the two sides around the shared angle.

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The shared angle is A, so the ratio of the areas is

$$\frac{[ABF]}{[ACE]} = \frac{AB}{AC} \cdot \frac{AF}{AE} = \frac{3}{12} \cdot \frac{15}{20} = \frac{3}{16}.$$

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Since we need the ratio anyway, we can just work with ratios directly.

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Next, let's try $\triangle CBD$. What is $\frac{[CBD]}{[CAE]}$?

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The shared angle is C, so the ratio of the areas is

$$\frac{[CBD]}{[CAE]} = \frac{CB}{CA} \cdot \frac{CD}{CE} = \frac{9}{12} \cdot \frac{4}{16} = \frac{3}{16}.$$

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Let's look at the last of the three outer triangles, $\triangle EDF$. What is $\frac{[EDF]}{[ECA]}$?

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The shared angle is E, so the ratio of the areas is

$$\frac{[EDF]}{[ECA]} = \frac{ED}{EC} \cdot \frac{EF}{EA} = \frac{12}{16} \cdot \frac{5}{20} = \frac{3}{16}.$$

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How do we finish from here?

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Right, we use the fact that triangle ACE is composed of four triangles whose area adds up to its total area. Thus, our desired fraction is

$$\begin{aligned}\frac{[BDF]}{[ACE]} &= 1 - \frac{[ABF]}{[ACE]} - \frac{[CBD]}{[ACE]} - \frac{[EDF]}{[ACE]} \\ &= 1 - \frac{3}{16} - \frac{3}{16} - \frac{3}{16} \\ &= \boxed{\text{(E) } \frac{7}{16}}\end{aligned}$$

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A triangle is partitioned into three triangles and a quadrilateral by drawing two lines from vertices to their opposite sides. The areas of the three triangles are 3, 7, and 7, as shown. What is the area of the shaded quadrilateral?

$$\text{(A) } 15 \quad \text{(B) } 17 \quad \text{(C) } \frac{35}{2} \quad \text{(D) } 18 \quad \text{(E) } \frac{55}{3}$$

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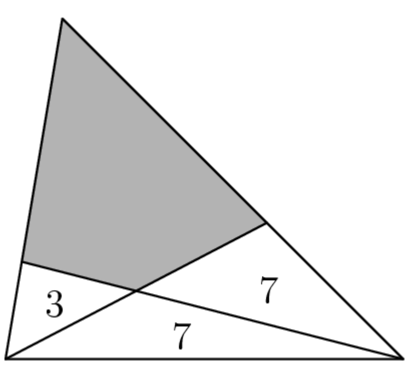

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To partition means that we must cover all the surface and have no overlaps.

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For reference, we label the vertices.

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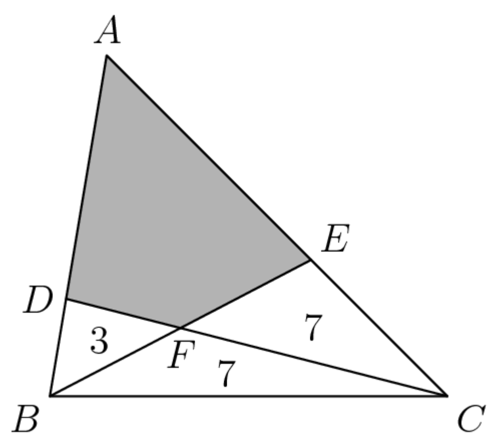

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We want to find the area of the gray quadrilateral. What's a good strategy for finding the area of a funky-looking quadrilateral like this?

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We could split it into triangles.

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(The other option is to consider it as part of the bigger triangles. That works too).

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So let's split quadrilateral $ADFE$ into triangles ADF and AEF. (You might think of splitting quadrilateral $ADFE$ into triangles ADE and DEF, but this way works much better.)

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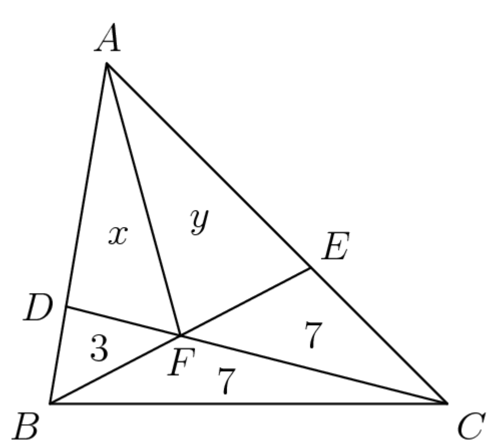

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Let $x = [ADF]$ and $y = [AEF]$. Let's make some observations about x and y.

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I see triangles of area $x + 3$ and y that have a relationship to each other. What can you say about the ratio of those areas?

wh83 2021-07-09 20:04:18

(Specifically, I am referring to $\triangle ABF$ and $\triangle AEF$).

wh83 2021-07-09 20:05:28

We see that $(x+3) : y = [ABF] : [AEF] = BF : FE$. Do we know what that's equal to? We don't have BF or FE...

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But do we know their ratio?

wh83 2021-07-09 20:06:22

Wait, we do! We also know that $BF : FE = [BFC] : [EFC] = 1$. In other words, $x + 3 = y$.

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We have two variables, so we need another equation.

wh83 2021-07-09 20:06:38

We got the last equation by tilting our head to the left... now let's try tilting our head to the right! What relationship between x and y do we get?

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Using triangles ADF and ACF, we get $(y+7) : x = FC : FD = 7 : 3$.

wh83 2021-07-09 20:09:18

We now have two equations:

$$\begin{gathered} x + 3 = y \\ \frac{y+7}{x} = \frac{7}{3}. \end{gathered}$$

Now what?

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We can substitute $y = x + 3$ into the second equation, to get $\frac{x+10}{x} = \frac{7}{3}$. So what's x?

wh83 2021-07-09 20:11:18

Cross-multiplying, we get

$$3x + 30 = 7x,$$

so we see that $4x = 30$ and $x = \frac{30}{4} = \frac{15}{2}$. Then what is the value of y?

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We have that $y = x + 3 = \frac{15}{2} + 3 = \frac{21}{2}$. So what is the area of quadrilateral $ADFE$?

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The area of quadrilateral $ADFE$ is $x + y = \frac{15}{2} + \frac{21}{2} = \frac{36}{2} = 18$. The answer is (D).

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RIGHT TRIANGLES

wh83 2021-07-09 20:13:05

In many geometry problems, using right triangles is an important step, because right triangles give us a way of computing distances via the Pythagorean theorem.

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Let triangle XOY be a right triangle with $\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.

(A) 24 (B) 26 (C) 28 (D) 30 (E) 32

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First, let's draw a diagram.

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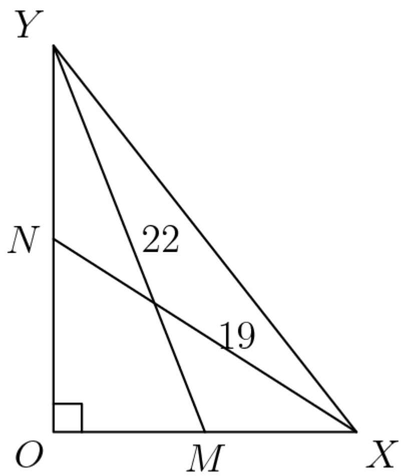

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We want XY. We don't see any direct way of computing XY, so what can we do?

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If we knew OX and OY, we could find XY using the Pythagorean theorem. But we don't know them.

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(What do we do when we want to write equations with unknown numbers?)

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We can introduce variables for them. Actually, since we know M and N are the midpoints, let's make variables for OM and ON. Let $x = OM = MX$ and $y = ON = NY$. (If we made variables for OX and OY instead, we would end up with a lot of fractions in our equations.)

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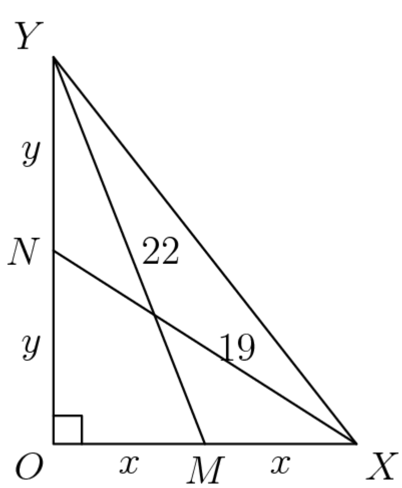

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There is some potential now. We have two variables we want to solve for. We have two given lengths. If only we could make

two equations, we might be able to solve for what we want!

wh83 2021-07-09 20:15:27
How can we make some equations here?

wh83 2021-07-09 20:16:16
We know the lengths XN and YM, and these form the hypotenuses of right triangles XON and YOM, respectively.

wh83 2021-07-09 20:16:48
By Pythagoras on right triangle XON, $(2x)^2 + y^2 = 19^2$, or $4x^2 + y^2 = 361$.

wh83 2021-07-09 20:16:57
By Pythagoras on right triangle YOM, $x^2 + (2y)^2 = 22^2$, or $x^2 + 4y^2 = 484$.

wh83 2021-07-09 20:18:23
We could now solve for x and y, but let's see if we can be more clever.

wh83 2021-07-09 20:18:39
As always, it's a good idea to remember what we're trying to find. We want XY. What equation can we write down involving XY?

wh83 2021-07-09 20:20:05
By Pythagoras on right triangle XOY, $XY^2 = (2x)^2 + (2y)^2 = 4x^2 + 4y^2$.

wh83 2021-07-09 20:20:12
How can we find this value?

wh83 2021-07-09 20:20:59
Our equations were $4x^2 + y^2 = 361$ and $x^2 + 4y^2 = 484$.

wh83 2021-07-09 20:21:49
Symmetry to the rescue! We can add the two equations we have derived, to get $5x^2 + 5y^2 = 845$. So what is the value of XY?

wh83 2021-07-09 20:23:00
We multiply by $\frac{4}{5}$ to get $XY^2 = 4x^2 + 4y^2 = 676$. Taking the square root of both sides, we get $XY = \sqrt{4x^2 + 4y^2} = \sqrt{676} = 26$. The answer is (B).

wh83 2021-07-09 20:23:14
Triangle ABC has $AB = 2 \cdot AC$. Let D and E be on $\overline{AB}$ and $\overline{BC}$, respectively, such that $\angle BAE = \angle ACD$. Let F be the intersection of segments AE and CD, and suppose that triangle CFE is equilateral. What is $\angle ACB$?

$\text{(A) } 60^\circ \quad \text{(B) } 75^\circ \quad \text{(C) } 90^\circ \quad \text{(D) } 105^\circ \quad \text{(E) } 120^\circ$

wh83 2021-07-09 20:23:28
Oooh how unusual! This is a problem in the right triangle section and it's about angles, not lengths, so Pythagoras is pretty unlikely to come to our rescue.

wh83 2021-07-09 20:23:36
How can we start?

wh83 2021-07-09 20:24:05
Let's start by drawing a nice diagram. We should label everything that we know, like the 60 degree angles of equilateral triangle CFE. Since $\angle BAE = \angle ACD$, we can let $x = \angle BAE = \angle ACD$. Since $AB = 2 \cdot AC$, let $AC = l$ and $AB = 2l$.

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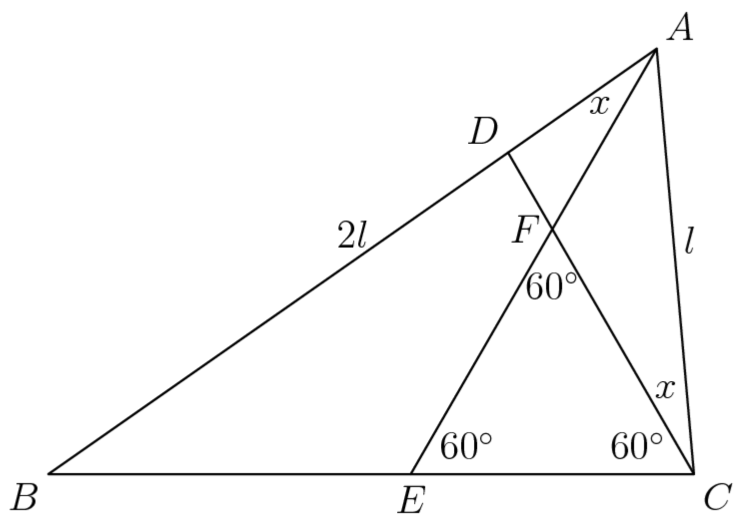

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Notice that while by our diagram $\triangle ABC$ looks close to right, it may not be. We are not given that this triangle is a right triangle, so we need to take care not to assume more than we know.

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Are there any other angles in the diagram that we can determine?

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From $\angle CFE = 60^\circ$, we see that $\angle AFD = 60^\circ$ and $\angle DFE = \angle AFC = 120^\circ$.

wh83 2021-07-09 20:25:56

So what is $\angle CAF$?

wh83 2021-07-09 20:27:16

Since $\angle AFC = 120^\circ$ and $\angle ACF = x$, we have that

$$\begin{aligned}\angle CAF &= 180^\circ - \angle AFC - \angle ACF \\ &= 180^\circ - 120^\circ - x \\ &= 60^\circ - x.\end{aligned}$$

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What does this tell us about the diagram?

wh83 2021-07-09 20:29:08

This tells us that $\angle CAB = \angle CAF + \angle BAE = (60^\circ - x) + x = 60^\circ$.

wh83 2021-07-09 20:29:18

Interesting. We have a triangle with a 60° angle at A, and we also have the sides around this angle in $2:1$ proportion... hmmm... what does this remind us of?

wh83 2021-07-09 20:30:47

Oh! It's a 30-60-90 triangle!! That's unexpected.

wh83 2021-07-09 20:31:02

(More formally, what we're using here is SAS similarity of triangles, where we have noticed that in a 30-60-90 triangle, the sides adjacent to the 60 degree angle come in the ratio 2:1. Our current triangle meets those criteria, so it must be a 30-60-90 triangle.)

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Therefore, $\angle ACB = 90^\circ$. The answer is (C).

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That was pretty magical; we started by angle chasing, but incorporated a single side length ratio to find the angle we needed. Just angle chasing wouldn't have been enough, but it would have been tempting. That's why it's important to use all the information in the problem!

wh83 2021-07-09 20:32:04

SIMILAR TRIANGLES

wh83 2021-07-09 20:32:24

(Yes, drawing can help or it can mislead).

wh83 2021-07-09 20:32:43

It's good to have solid justification always.

wh83 2021-07-09 20:33:07

Another way to find lengths in a diagram is to build similar triangles. If we have two similar triangles, then their corresponding side lengths are proportional.

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Two triangles are similar if (and only if) they have all three congruent angles.

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If two triangles XYZ and ABC are similar, we can write $\triangle XYZ \sim \triangle ABC$. *The order of the vertices is very important,* because it determines the ratios between the side-lengths. Specifically, we have:

$$\frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA}$$

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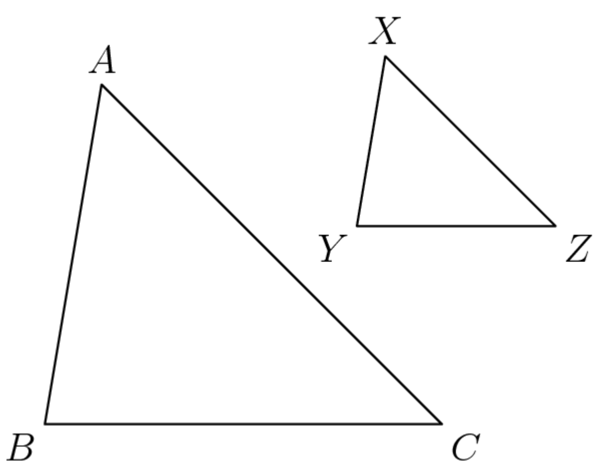

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Rectangle $ABCD$ has $AB = 8$ and $BC = 6$. Point M is the midpoint of diagonal $\overline{AC}$, and E is on $\overline{AB}$ with $\overline{ME} \perp \overline{AC}$. What is the area of triangle AME?

$(\text{A}) \frac{65}{8} \quad (\text{B}) \frac{25}{3} \quad (\text{C}) 9 \quad (\text{D}) \frac{75}{8} \quad (\text{E}) \frac{85}{8}$

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First, let's draw a diagram.

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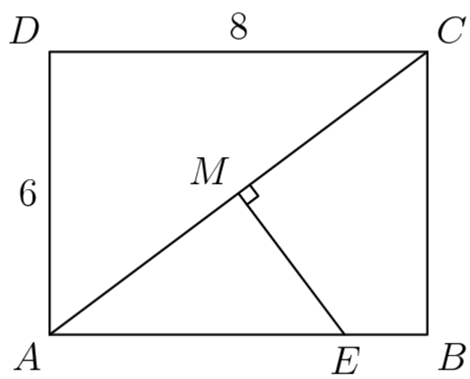

wh83 2021-07-09 20:34:06
We know that $\triangle AME$ is a right triangle.

wh83 2021-07-09 20:34:15
What else do we notice about it?

wh83 2021-07-09 20:35:01
Why do we suspect that it's a $3-4-5$ triangle?

wh83 2021-07-09 20:35:28
What's a triangle it's similar to?

wh83 2021-07-09 20:35:35
Reminder to order your vertices carefully! If you say that PQR is similar to XYZ, you're stating that

$$\angle P = \angle X, \angle Q = \angle Y, \angle R = \angle Z.$$

Matching up the vertices like this tells us which sides correspond when we consider the ratios of the sides.

wh83 2021-07-09 20:36:29
Yes, $\triangle AME$ is similar to $\triangle ABC$, which is congruent to triangle $\triangle CDA$.

wh83 2021-07-09 20:36:38
In other words, $AM : ME : AE$ is what? (Careful about the order!)

wh83 2021-07-09 20:37:36
Right, $AM : ME : AE$ is $4 : 3 : 5$. Do we know what any of these sides are?

wh83 2021-07-09 20:39:03
Since M is the midpoint of AC, we know $AM = \dfrac{1}{2}AC = \dfrac{10}{2} = 5.$

wh83 2021-07-09 20:39:11
In other words, $AM : ME : AE$ is $5 : ME : AE$, and that should be the same as $4 : 3 : 5$. What should we find next?

wh83 2021-07-09 20:40:08
Let's find ME, since that'll help us calculate the area. What is it?

wh83 2021-07-09 20:41:07
We see that $\dfrac{ME}{AM} = \dfrac{3}{4}$, so $ME = \dfrac{15}{4}$.

wh83 2021-07-09 20:41:58
Alternatively, we can multiply through our ratio $4 : 3 : 5$ by $\frac{5}{4}$, we have $5 : \frac{15}{4} : \frac{25}{4}$ is the same as $5 : ME : AE$. So what's the area of AME?

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The area of triangle AME is $\frac{1}{2} \cdot AM \cdot ME = \frac{1}{2} \cdot 5 \cdot \frac{15}{4} = \frac{75}{8}$. The answer is (D).

wh83 2021-07-09 20:42:34

Through a point on the hypotenuse of a right triangle, lines are drawn parallel to the legs of the triangle so that the triangle is divided into a square and two smaller right triangles. The area of one of the two small right triangles is m times the area of the square. The ratio of the area of the other small right triangle to the area of the square is

$(\text{A})\ \frac{1}{2m+1} \quad (\text{B})\ m \quad (\text{C})\ 1-m \quad (\text{D})\ \frac{1}{4m} \quad (\text{E})\ \frac{1}{8m^2}$

wh83 2021-07-09 20:42:46

First, let's draw a diagram, labeling important points.

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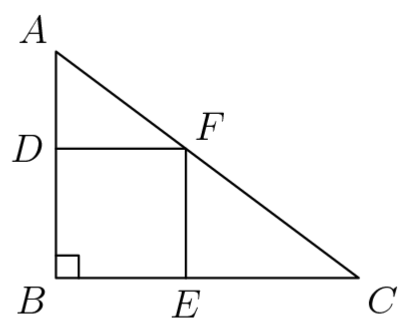

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We want a ratio in the end.

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Notice that the problem often refers to the ratio of areas to the area of a square. Does it matter what the area of the square is?

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No, since we can change the scale of the diagram without affecting the problem. Since we can choose whatever area we like for the square, let's assume the square has area 1. Then each of its sides is length 1. So we can add this to the diagram, like so.

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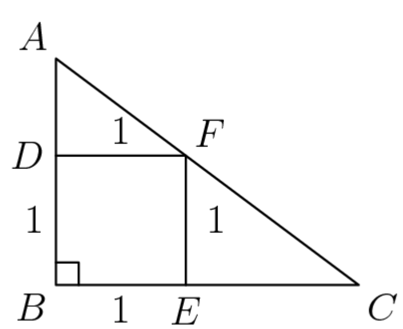

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Let's assume that the ratio of the area of $\triangle ADF$ to the area of the square is m. Since the area of the square is 1, this means that the area of $\triangle ADF$ is equal to m.

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Can we fill in any more lengths using this information?

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The area of $\triangle ADF$ is equal to m, but it is also equal to $\frac{1}{2} \cdot DF \cdot AD$. Hence, $AD = 2m$.

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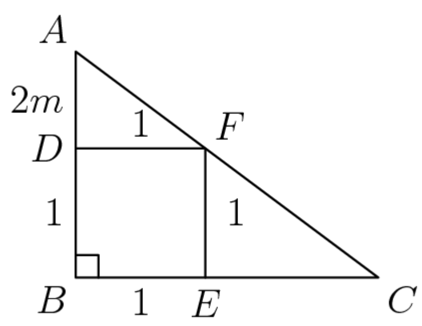

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Now what? What do we want to find?

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We need to find the ratio of the area of $\triangle FEC$ to the area of the square. Again, since the area of the square is 1, we just need to find the area of $\triangle FEC$.

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We already know the height of $\triangle FEC$ is 1, so we just need to find EC. How could we find EC?

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We can use similar triangles!

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We see that $\triangle ADF \sim \triangle FEC$ by AAA Similarity. We can even see the scale factor right away: since AD corresponds to FE, we must scale by $\frac{1}{2m}$ to get from $2m$ to 1.

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So what is EC?

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(Make sure to use parentheses if necessary).

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We see that EC is equal to DF scaled by $\frac{1}{2m}$. Since $DF = 1$, we have $EC = \frac{1}{2m}$.

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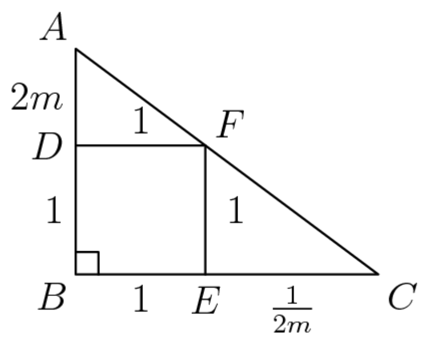

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Then what is the area of $\triangle FEC$?

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The area of $\triangle FEC$ is $\frac{1}{2} \cdot 1 \cdot \frac{1}{2m} = \frac{1}{4m}$. This is also equal to the ratio of the area of $\triangle FEC$ to the area of the square, since the square has area 1. The answer is (D).

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ANGLE BISECTOR THEOREM

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One more important fact about triangles is the angle bisector theorem, which we will use in the following problem.

wh83 2021-07-09 20:52:36

Nondegenerate triangle ABC has integer side lengths, $\overline{BD}$ is an angle bisector, $AD = 3$, and $DC = 8$. What is the smallest possible value of the perimeter?

$\text{(A) } 30 \quad \text{(B) } 33 \quad \text{(C) } 35 \quad \text{(D) } 36 \quad \text{(E) } 37$

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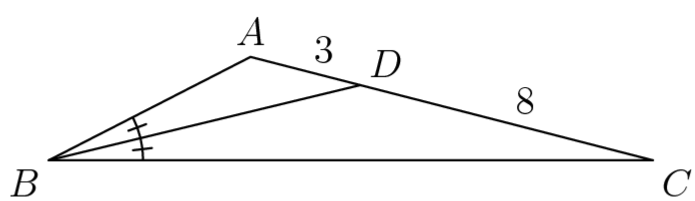

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"Nondegenerate" means that it's an actual triangle with nonzero angles, not just three points on a single line.

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If we want to find the perimeter, it looks like we're going to have to find the the lengths of the sides AB and BC. Is there anything we can say about sides AB and BC?

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The angle bisector theorem states that the bisector of an angle splits the side opposite the angle into a ratio equal to the ratio of the other two sides of the triangle. That is, in the figure, $\frac{AD}{CD} = \frac{AB}{BC}$.

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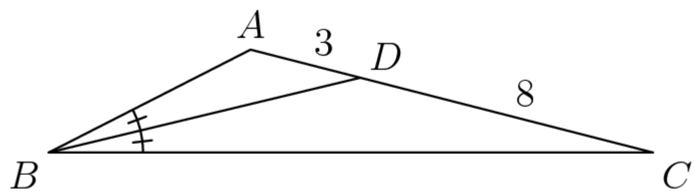

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We are given that $AD = 3$ and $CD = 8$, so $\dfrac{AB}{BC} = \dfrac{3}{8}$.

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We also know that AB and BC are positive integers. How can we write AB and BC to account for this?

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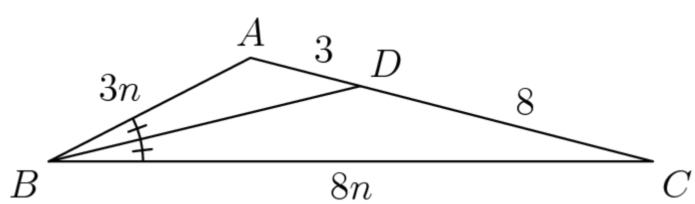
Since $\dfrac{3}{8}$ is reduced, we can say that $AB = 3n$ and $BC = 8n$ for some positive integer n.

wh83 2021-07-09 20:57:23



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So the perimeter of the triangle ABC is $3n + 8n + 11 = 11n + 11$, where n is a positive integer. Remember, we want to make the perimeter as small as possible. What value of n makes this as small as possible?

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To make the perimeter as small as possible, we should make n as small as possible. The smallest positive integer is $n = 1$, giving us a perimeter of 22. But 22 is not one of our answer choices! What did we do wrong?

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The side lengths of a triangle must also satisfy the triangle inequality! If $n = 1$, then our side lengths are 3, 8, and 11, and the triangle is degenerate since $3 + 8 = 11$. So $n = 1$ is actually not possible!

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Since $n = 1$ doesn't work, the next smallest possibility is $n = 2$. What are the side lengths if $n = 2$?

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If $n = 2$, the side lengths are 6, 16, and 11. Can these be the side lengths of a triangle?

wh83 2021-07-09 21:01:01

They can be the side lengths of a triangle, since $6 + 11 > 16$. So what is the answer to the problem?

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The smallest possible perimeter is $6 + 11 + 16 = 33$. The answer is (B).

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What do we get for $n = 3$?

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We get as sides $9, 24, 11$, which doesn't work.

wh83 2021-07-09 21:03:52

For higher n, the difference $8n - 3n$ will be even bigger, so we won't have a triangle.

wh83 2021-07-09 21:04:09
The bracket notation for area is not standard outside AoPS.

wh83 2021-07-09 21:04:21
Any other questions for today?

wh83 2021-07-09 21:04:41
You can use A(ABC).

wh83 2021-07-09 21:05:20
SUMMARY

wh83 2021-07-09 21:05:24
In today's class, we have seen how to use triangles and their properties to solve geometry problems. We saw how to use the areas of triangles to find relationships between lengths.

wh83 2021-07-09 21:05:29
We also saw how to use right triangles and similar triangles to find lengths. Remember that similar triangles often appear when you have parallel lines. If you want to find lengths, then there is a good chance that the right step is to build a right triangle or find similar triangles.

wh83 2021-07-09 21:05:40
Nice working with you today!