# Russian School of Math: Lesson 5

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**Abstract**

This note reviews a small number of problems from the Russian School of Math test. Written for personal use.

## 1

Find the last two digits of $2017^{20} + 2018^{20} + 2019^{20}$.

***Solution***

Since we are looking for the last two digits, we can decompose the numbers and discards all the multiples of 100. First step:

$$\begin{aligned}
2017^2 &= (2000+17)^2 \equiv 17^2 \pmod{100} = 289\\
2018^2 &= (2000+18)^2 \equiv 18^2 \pmod{100} = 324\\
2019^2 &= (2000+19)^2 \equiv 19^2 \pmod{100} = 361
\end{aligned}$$

Second step:

$$\begin{aligned}
289^{10} &\equiv (289-300)^{10} \equiv (-11)^{10} \equiv (11)^{10} \equiv (10+1)^{10}\\
&\equiv \binom{10}{1}10^1 1^9 + \binom{10}{0}10^0 1^{10} \equiv 100+1 \equiv 1 \pmod{100}\\
324^{10} &\equiv (324-300)^{10} \equiv 24^{10} \equiv (10+14)^{10}\\
&\equiv \binom{10}{0}10^0 14^{10} \equiv (10+4)^{10} \equiv \binom{10}{0}10^0 4^{10} \equiv 4^{2\times 5} \equiv 16^5\\
&\equiv (10+6)^5 \equiv \binom{10}{0}10^0 6^5 \equiv 776 \equiv 76 \pmod{100}\\
361^{10} &\equiv (361-400)^{10} \equiv (-39)^{10} \equiv 39^{10} \pmod{100}\\
&\equiv (10+10+10+9)^{10} \equiv 9^{10} \equiv (-1)^{10} \equiv 1 \pmod{100}
\end{aligned}$$

Putting it together,

$$2017^{20} + 2018^{20} + 2019^{20} \equiv 1 + 76 + 1 \equiv 78 \pmod{100}$$

Solution: $\boxed{78.}$

## 2

Find all $n$, $n \in \mathbb{N}$, such that $\varphi(n) = 2$.

***Solution***

Since $n$ is small, we can check small integers. Recall that if $p$ is prime, $\varphi(p) = p - 1$; if $p$ is prime, $\varphi(p^a) = p^a - p^{a-1}$, for $a > 0$; and if $m$ and $n$ are relatively prime, $\varphi(m \times n) = \varphi(m) \times \varphi(n)$.

$$\begin{aligned}
&\varphi(1) = 1\\
&\varphi(2) = 2 - 1 = 1\\
&\varphi(3) = 3 - 1 = 2\\
&\varphi(4) = \varphi(2^2) = 2^2 - 2^1 = 2\\
&\varphi(5) = 5 - 1 = 4\\
&\varphi(6) = \varphi(2 \times 3) = \varphi(2) \times \varphi(3) = 1 \times 2 = 2\\
&\varphi(7) = 7 - 1 = 6\\
&\varphi(8) = \varphi(2^3) = 2^3 - 2^2 = 4
\end{aligned}$$

Solution: There are three values of $n$ such that $\varphi(n) = 2$: $\boxed{3; 4; 6.}$

## 3

Prove that if $m$ and $n$ are coprime, then $\varphi(m \cdot n) > \varphi(m) \cdot \varphi(n)$.

***Solution***

Decompose $m \times n$ into its prime factors:

$$m \times n = \prod_{k=1}^{w} p_k^{e_k} \times \prod_{k=w+1}^{z} p_k^{e_k} = \prod_{k=1}^{z} p_k^{e_k}$$

The decompositions are distinct because $m$ and $n$ are relatively prime by assumption.
Apply Euler's product formula to the product:

$$\begin{aligned}
\varphi(m \times n) &= m \times n \times \prod_{k=1}^{z}(1 - \tfrac{1}{p_k}) \\
&= m \times n \times \prod_{k=1}^{w}(1 - \tfrac{1}{p_k}) \times \prod_{k=w+1}^{z}(1 - \tfrac{1}{p_k}) \\
&= \underbrace{m \times \prod_{k=1}^{w}(1 - \tfrac{1}{p_k})}_{\varphi(m)} \times \underbrace{n \times \prod_{k=w+1}^{z}(1 - \tfrac{1}{p_k})}_{\varphi(n)}
\end{aligned}$$

## 4

Find all ordered pairs $(m, n)$, where $m, n \in \mathbb{N}$, $n > 1$ and $\varphi(\varphi(n^m)) = n$.

***Solution***

- An obvious solution is $(m, n) = (m, 1)$. For $n = 1$, $\varphi(\varphi(1^m)) = \varphi(1) = 1$.
- Another solution is $(m, n) = (3, 2)$. For $n = 2$, $\varphi(\varphi(2^m)) = \varphi(2^{m-1}(2-1)) = \varphi(2^{m-1}) = 2^{m-2}(2-1) = 2^{m-2}$. And $\varphi(\varphi(2^m)) = 2$ iff $m = 3$.
- Let $n = 3$, $\varphi(\varphi(3^m)) = \varphi(3^{m-1}(3-1)) = \varphi(2) \times \varphi(3^{m-1}) = 2 \times 3^{m-2}(3-1) = 4 \times 3^{m-2}$. There is no value of $m$ such that $\varphi(\varphi(3^m)) = 3$.
- If $n$ is an odd prime, $\varphi(n^m) = n^{m-1}(n-1)$, where $(n-1)$ is even, implying

$$\varphi(\varphi(n^m)) = \varphi((n-1)n^{m-1}) \le \varphi(n-1)\varphi(n^{m-1}) = (n-1)n^{m-2}\varphi(n-1)$$

Solution: $(m, n) \in \{(m, 1), (3, 2), (X, X), (X, X)\}$.
UNFINISHED

## 5

Let $d_1, d_2, \ldots, d_k$ be all natural divisors of $n$, $n \in \mathbb{N}$ such that $d_1 < d_2 < \ldots < d_k$. Prove that $\varphi(d_1) + \varphi(d_2) + \ldots + \varphi(d_k) = n$.

***Solution***