Ask Math Anything

Study at Home with Po-Shen Loh

6 July 2020

Abstract

Professor Po-Shen Loh solves problems on his YouTube channel. A selection for practice.
Reference: Ask Math Anything - Daily Challenge with Po-Shen Loh

Sum of Digits

Sum the digits of

$$\underbrace{999\ldots9}_{94\text{ times}} \times \underbrace{777\ldots7}_{95\text{ times}}$$

Let's start by rewriting the multiplication

$$\begin{aligned}\underbrace{999\ldots9}_{94\text{ times}} \times \underbrace{777\ldots7}_{95\text{ times}} &= (10^{94}-1)\times \underbrace{777\ldots7}_{95\text{ times}} \\ &= \underbrace{777\ldots7}_{95\text{ times}}\underbrace{000\ldots0}_{94\text{ times}} - \underbrace{777\ldots7}_{95\text{ times}}\end{aligned}$$

This last subtraction is manageable, as patterns appear:

$$\begin{array}{r} 7\ 7\ 7\ 7\ 7\ 7\ 0\ 0\ 0\ 0\ 0 \\ -\quad 1\ 7\ 7\ 7\ 7\ 7\ 7 \\ \hline 7\ 7\ 7\ 6\ 9\ 2\ 2\ 2\ 3 \end{array}$$

Carefully counting the number of occurrences gives:

$$\underbrace{777\ldots7}_{93\text{ times}}69\underbrace{222\ldots2}_{93\text{ times}}3$$

So now to the sum of digits:

$$95\times 9 = 950 - 100 + 5 = 855$$

Answer: 855

A Mysterious Sequence

Complete the sequence

$$325, 263, 642, 436, 374, 753, 547, 485, \ldots$$

The first pattern we notice is that the middle digit of a number in the sequence is used as the leading digit for the next number. Since the middle digit of the last number in the sequence is 8, the leading digit for the next number is 8. The second pattern we notice is that the trailing digit of a number in the sequence, plus one, is used as the middle digit for the next number. Since the trailing digit of the last number in the sequence is 5, the middle digit of the next number is $5 + 1 = 6$. The other pattern we notice is that the sum of the digits of each number in the sequence increases by one unit as we move forward, as follows:

u_n	sum of digits	v_n
325	3+2+5 = 10	1
263	2+6+3 = 11	2
642	6+4+2 = 12	3
436	4+3+6 = 13	4
374	3+7+4 = 14	5
753	7+5+3 = 15	6
547	5+4+7 = 16	7
485	4+8+5 = 17	8
864	8+6+4 = 18	9
658	6+5+8 = 19	1
596	5+9+6 = 20	2
975	9+7+5 = 21	3
769	7+6+9 = 22	4
6107...?	6+10+7 = 23	5

We have added a few more terms in the sequence for good measure. It is not immediately clear what rules to follow to go beyond the term 769, since $9 + 1 = 10$. Perhaps 6107. It is coherent as it gives a sum of $23 \to 5$. But after that, does 1 become the new "middle digit"? Some more general rules are needed. Oh but wait, another interpretation of the rules are simply: The first digit of a number in the sequence becomes the last digit of the next number. This rule is clearly valid too. At the current time this sequence is not referenced in **The On-Line Encyclopedia of Integer Sequences**!

Answer: 864

An Absolute Equation

Find the values of x that solve

$$|x - 1||2x - 4| = 5$$

We can bring the absolute-value operation outside of the product:

$$|(x - 1)(2x - 4)| = 5$$

This can be broken down into two parts:

$$2(x-1)(x-2) = 5$$
$$-2(x-1)(x-2) = 5$$

These are two quadratic equations:

$$2x^2 - 6x - 1 = 0$$
$$2x^2 - 6x + 9 = 0$$

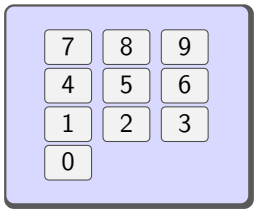
The four solutions are:

$$x = \frac{6 \pm \sqrt{36+8}}{4} = \frac{3 \pm \sqrt{11}}{2}$$
$$x = \frac{6 \pm \sqrt{36-72}}{4} = \frac{3 \pm 3i}{2}$$

Answer: $\frac{3\pm\sqrt{11}}{2}$, $\frac{3\pm3i}{2}$

Dancing Knight on a Calculator!

Using the chess rules for the Knight, can you stop at every number on the keypad exactly once?



The middle number, 5 is special. Clearly we need to either start or finish with 5. Starting from 5, two paths are possible:

$$5, 0, 3, \begin{cases} 4, 9, 2, 7, 6, 1, 8 \\ 8, 1, 6, 7, 2, 9, 4 \end{cases}$$

The reverse paths also work, ending at 5.