## Problem 1

### *Question*

> The "Manhattan distance" between two cells is the shortest distance between those cells when traveling up, down, left, or right, as if one were traveling along city blocks rather than as the crow flies.
> Place numbers from 1-6 in some cells so the following criteria are satisfied:
>
> 1. A cell contains at most one number. Cells can be left empty.
> 2. For each cell containing a number $N$ in the grid, exactly two other cells containing $N$ are at a Manhattan distance of $N$.
> 3. For each cell containing a number $N$ in the grid, no other cells containing $N$ are at a Manhattan distance less than $N$.
>
> | | | | | | | | |
> |---|---|---|---|---|---|---|---|
> | | | 4 | | | | | |
> | | 5 | 2 | | | | 3 | |
> | 3 | | | 1 | | | | |
> | | | | | 1 | | | 4 |
> | | 2 | | | | 6 | 5 | |
> | | | | | | 4 | | |
>
> There is a unique solution, but you do not need to prove that your answer is the only one possible.

Solution:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | 3 | 4 | 6 | 3 | 5 | 4 | |
| | 5 | 2 | | 2 | | 3 | |
| 3 | 2 | | 1 | 1 | 2 | | |
| 6 | | 3 | 1 | 1 | 3 | 2 | 4 |
| | 2 | 4 | 2 | | 6 | 5 | |
| | | 5 | | 2 | 4 | 2 | |

## Problem 2

### *Question*

> A regular hexagon is placed on top of a unit circle such that one vertex coincides with the center of the circle, exactly two vertices lie on the circumference of the circle, and exactly one vertex lies outside of the circle. Determine the area of the hexagon.

A natural starting point for this problem is to draw a hexagon inscribed in a circle (Figure 1a) and to translate it so that one of its vertices coincides with the center of the circle (Figure 1b). It is then clear that the hexagon must be shrunk to the point where the short-diagonal of the hexagon coincides with the radius $r$ of the circle (Figure 1c).

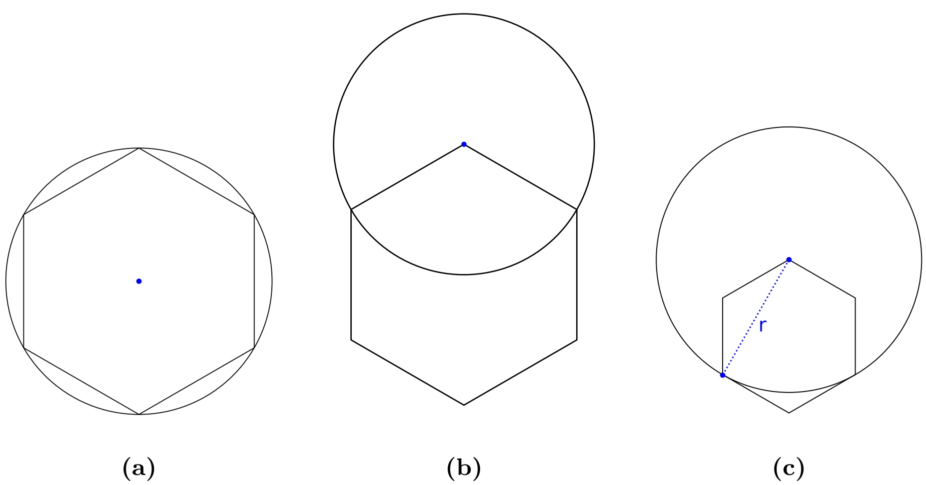


**Figure 1:** There exist two polygons with one vertex at the center of the circle and two vertices on the circumference of the circle: see panel (b) and (c). Only one of them is such that only one vertex is outside the circle: see panel (c).

In hexagon $ABCDEF$ (Figure 2), consider first the triangle $ACE$ inscribed inside the hexagon. Claim: *Triangle $ACE$ is equilateral.* Proof: Since $A$ is the center of the circle and $C$ and $E$ are on the circumference, $AC = r$ and $AE = r$. By the half-angle formula, the chord length $CE$ is equal to $2r\sin(\theta/2)$, where $\theta = \angle CAE = \pi/3$. Calculating $\sin(\theta/2) = \sin(\pi/6) = 1/2$ and substituting back gives a chord length $CE = 2r\sin(\theta/2) = r$. It follows that all three sides of the triangle have length $r$ (it is actually obvious by considerations of symmetry).

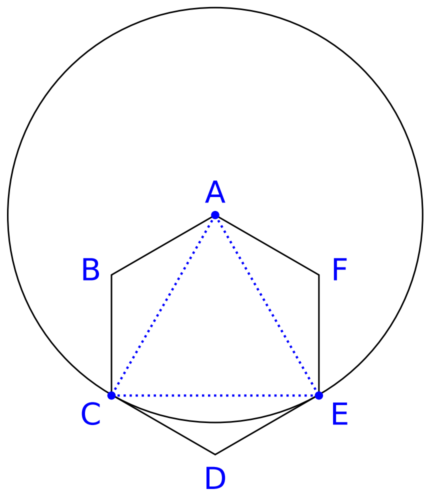


**Figure 2:** The area of the polygon may be calculated as the sum of areas of the equilateral triangle $ACE$ and of the isosceles triangles $ABC$, $CDE$, and $EFA$.

Since triangle $ACE$ is equilateral, its area is $\frac{\sqrt{3}}{4}r^2$ (the side lengths of the equilateral triangle coincide with the radius of the circle). This well-known formula can be proved by applying the Pythagorean theorem: See Figure 3.

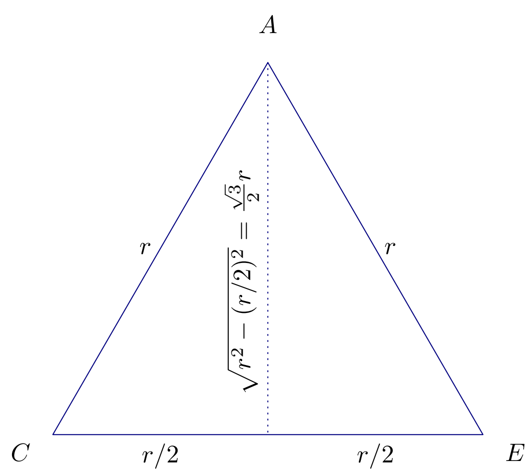


**Figure 3:** The area of equilateral triangle $ACE$ is $\frac{\sqrt{3}}{4}r^2$.

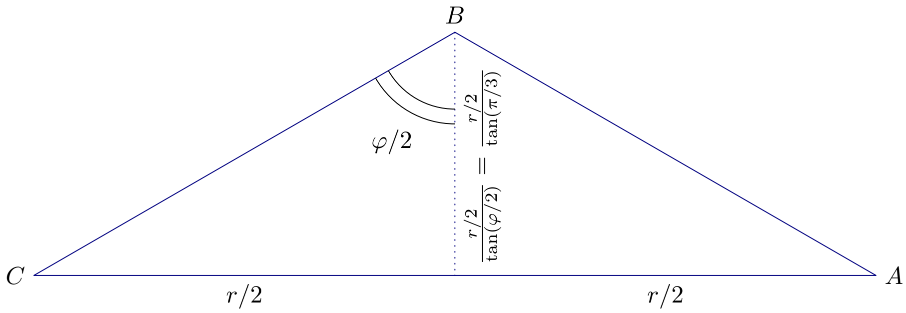


**Figure 4:** The area of isosceles triangle $ABC$ is $\frac{\sqrt{3}}{12}r^2$.

Consider now the isosceles triangle $ABC$ in hexagon $ABCDEF$ (Figure 2). The base of triangle $ABC$ coincides with the chord $AC = r$. The angle at the vertex is $\varphi = \angle ABC = \frac{2\pi}{3}$. The angle $\varphi$ follows from the sum-of-angles formula. The sum of the angles of a regular polygon with $n$ sides is $(n-2)\pi$. For a hexagon, $n = 6$, the angle is therefore $\varphi = \frac{(6-2)\pi}{6} = \frac{2\pi}{3}$. The isosceles triangle $ABC$ fits exactly three times into the equilateral triangle $ACE$. And therefore its area is $\frac{\sqrt{3}}{12}r^2$. As a sanity check, this area may also be calculated from trigonometry:

$$\frac{r^2}{4\tan(\varphi/2)} = \frac{r^2}{4\tan(\pi/3)} = \frac{r^2}{4\sqrt{3}} = \frac{\sqrt{3}}{12}r^2.$$

Adding together the area of triangle $ACE$ and triangles $ABC$, $AEF$, and $CDE$ gives the area of hexagon $ABCDEF$:

$$\frac{\sqrt{3}}{4}r^2 + 3\ \frac{\sqrt{3}}{12}r^2 = \frac{\sqrt{3}}{2}$$

where we have substituted $r = 1$, the radius of the unit circle.

Conclusion: The area of the hexagon is $\boxed{\frac{\sqrt{3}}{2} \approx 0.866}$.

## Problem 3

### *Question*

> A sequence of integers $x_1, x_2, ..., x_k$ is called *fibtastic* if the difference between any two consecutive elements in the sequence is a Fibonacci number.
> The integers from 1 to 2024 are split into two groups, each written in increasing order. Group $A$ is $a_1, a_2, ..., a_m$ and Group $B$ is $b_1, b_2, ..., b_n$.
> Find the largest integer $M$ such that we can guarantee that we can pick $M$ consecutive elements from either Group $A$ or Group $B$ which form a fibtastic sequence.

The question does not state who splits the integers into two groups: If the split is engineered to maximize $M$, the answer is $M = 1012$. If the split is engineered to minimize $M$, the answer is $M = 2$. By another interpretation, the answer is $M = 16$.

### General Considerations

The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively, for $n \geq 1$, by

$$
\begin{aligned}
F_0 &= 0, \\
F_1 &= 1, \\
F_{n+1} &= F_n + F_{n-1}.
\end{aligned}
$$

On the closed interval $[0, 2024]$, the Fibonacci subsequence is

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597.$$

where $F_{17} = 1597$. Clearly, the largest possible difference $|a_{k+1} - a_k|$, calculated from consecutive terms in the sequence $a_1, a_2, \ldots, a_m$, cannot exceed 1597 and likewise for $b_1, b_2, \ldots, b_n$.

**Amicable Split**

Consider the arithmetic sequence with first term 1 and common difference $F_1$. Split the integers as

$$\begin{aligned}
&\text{Group A}: \quad a_1, a_2, a_3 \ldots, a_{1012} &&= 1, 2, 3 \ldots, 1012 \\
&\text{Group B}: \quad b_1, b_2, b_3 \ldots, b_{1012} &&= 1013, 1014, 1015 \ldots, 2024
\end{aligned}$$

where $m = n = 1012$. The difference between any two consecutive elements in the sequence $a_1, a_2, a_3 \ldots, a_{1012}$ is the Fibonacci number $F_1$. Likewise for $b_1, b_2, b_3 \ldots, b_{1012}$. Every consecutive element from either Group A or Group B is a fibtastic sequence. The stated conditions are therefore satisfied and the maximum (max-max) is $M = 1012$.

**Adversarial Split**

If $m$ and $n$ are chosen to minimize $M$, we must have $m \geq 2$ and $n \geq 2$ to ensure that the difference between any two consecutive numbers is defined. An adversarial split that minimizes $M$ is:

$$\begin{aligned}
&\text{Group A}: \quad a_1, a_2 &&= 1, 2 \\
&\text{Group B}: \quad b_1, b_2, b_3 \ldots, b_{2022} &&= 3, 4, 5 \ldots, 2024
\end{aligned}$$

These sequences are fibtastic, as shown above. The stated conditions are therefore satisfied and the maximum (max-min) is $M = 2$.

**Increasing Fibonacci Subsequences**

Adding a constraint to the problem makes it more interesting. Consider the revised definition: A sequence of integers $x_1, x_2, ..., x_k$ is called *fibtastic* if the difference between any two consecutive elements in the sequence is a Fibonacci number ***and if these Fibonacci numbers form a strictly increasing sequence***.

The largest increasing sequence formed by the consecutive differences would be a subsequence of the first seventeen Fibonacci numbers: $F_1, F_2, \ldots, F_{17}$. However, $F_{17} = 1597$ is unreachable for $m = n = 1012$, so the increasing Fibonacci subsequence is reduced to $F_1, F_2, \ldots, F_{16}$, where $F_{16} = 987$. With these additional restrictions, a candidate solution is $M = 16$. It remains to be shown that a subsequence of that length does exist.

We start by constructing a subsequence on $[1, 2024]$.

| Number | | Fibonacci Representation | | Consecutive Difference |
|---|---|---|---|---|
| 1 | = | $F_1$ | | |
| | | | $\to$ | $F_3 - F_1 = F_2$ |
| 2 | = | $F_3$ | | |
| | | | $\to$ | $F_4 - F_3 = F_2$ |
| 3 | = | $F_4$ | | |
| | | | $\to$ | $F_2$ |
| 4 | = | $F_4 + F_2$ | | |
| | | | $\to$ | $F_5 - F_4 - F_2 = F_1$ |
| 5 | = | $F_5$ | | |
| | | | $\to$ | $F_2$ |
| 6 | = | $F_5 + F_2$ | | |
| | | | $\to$ | $F_3 - F_2 = F_1$ |
| 7 | = | $F_5 + F_3$ | | |
| | | | $\to$ | $F_6 - F_5 - F_3 = F_2$ |
| 8 | = | $F_6$ | | |
| | | | $\to$ | $F_2$ |
| 9 | = | $F_6 + F_2$ | | |
| | | | $\to$ | $F_3 - F_2 = F_1$ |
| 10 | = | $F_6 + F_3$ | | |
| | | | $\to$ | $F_4 - F_3 = F_2$ |
| 11 | = | $F_6 + F_4$ | | |
| | | | $\to$ | $F_2$ |
| 12 | = | $F_6 + F_4 + F_2$ | | |
| | | | $\to$ | $F_7 - F_6 - F_4 - F_2 = F_1$ |
| 13 | = | $F_7$ | | |
| | | | $\to$ | $F_1$ |
| 14 | = | $F_7 + F_2$ | | |
| | | | $\to$ | $F_3 - F_2 = F_1$ |
| 15 | = | $F_7 + F_3$ | | |
| | | | $\to$ | $F_4 - F_3 = F_2$ |
| 16 | = | $F_7 + F_4$ | | |
| | | | $\to$ | $F_2$ |
| 17 | = | $F_7 + F_4 + F_2$ | | |
| | | | $\to$ | $F_5 - F_4 - F_2 = F_1$ |
| 18 | = | $F_7 + F_5$ | | |
| | | | $\to$ | $F_2$ |
| 19 | = | $F_7 + F_5 + F_2$ | | |
| | | | $\to$ | $F_3 - F_2 = F_1$ |
| 20 | = | $F_7 + F_5 + F_3$ | | |
| | | | $\to$ | $F_8 - F_7 - F_5 - F_3 = F_2$ |
| 21 | = | $F_8$ | | |

**Table 1**

## Problem 4

***Question***

> During a lecture, each of 26 mathematicians falls asleep exactly once, and stays asleep for a nonzero amount of time. Each mathematician is awake at the moment the lecture starts, and the moment the lecture finishes. Prove that there are either 6 mathematicians such that no two are asleep at the same time, or 6 mathematicians such that there is some point in time during which all 6 are asleep.

## Problem 5

***Question***

> Let $f(x) = x^2+bx+1$ for some real number $b$. Across all possible values of $b$, find all possible values for the number of integers $x$ that satisfy $f(f(x)+x) <$ 0.
> That is, if there are some values of $b$ that give us 180 integer solutions for x and there are other values of $b$ that give us 314 integer solutions for $x$ (and these are the only possibilities), the answer would be $\boxed{180,\ 314}$.

Let $g(x) = f(f(x)+x)$. Substituting $f(x) = x^2+bx+1$ into $f(f(x)+x)$ yields a polynomial of degree 4. Except for special values of $b$, this polynomial has two turning points and could satisfy $g(x) < 0$ on two disconnected intervals. Figure 5 shows the graph of $g(x)$ for $b = 5$. We analyze $f(x)$ and used its special properties to analyze the nested expression $g(x)$.

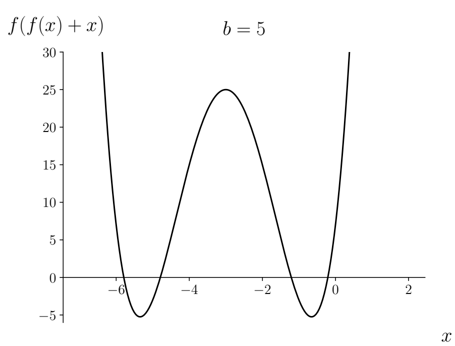


**Figure 5:** $g(x) = f(f(x) + x)$.

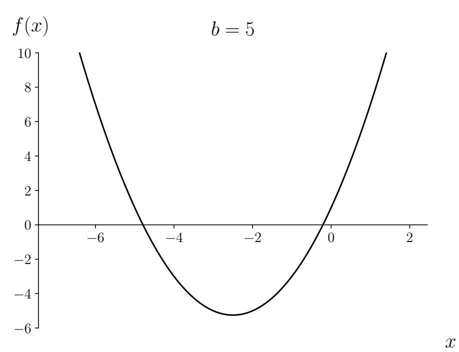


**Figure 6:** $f(x) = x^2 + bx + 1$.

The graph of $f(x)$ is a U-shaped parabola whose vertex is located at point $(0, -b)$. Figure 6 shows the graph of $f(x)$ for $b = 5$. The discriminant associated with the equation $f(x) = 0$ is:

$$\delta = (b^2 - 4) = (b - 2)(b + 2).$$

It follows that:

$$
\begin{aligned}
b < 2 &\implies \delta < 0 \implies f(x) > 0 \quad \forall x \in \mathbb{R} \\
b = 2 &\implies \delta = 0 \implies f(x) = x^2 + 2x + 1 = (x+1)^2 \geq 0 \quad \forall x \in \mathbb{R} \\
b > 2 &\implies \delta > 0 \implies f(x) > 0 \quad \forall x \in (-\infty, \tfrac{-b-\sqrt{\delta}}{2}) \cup (\tfrac{-b+\sqrt{\delta}}{2}, +\infty) \\
&\qquad\qquad\;\; \implies f(x) = 0 \quad \text{for } x = -1 \\
&\qquad\qquad\;\; \implies f(x) < 0 \quad \forall x \in (\tfrac{-b-\sqrt{\delta}}{2}, \tfrac{-b+\sqrt{\delta}}{2})
\end{aligned}
$$

The problem is to count the number of integer values $n \in \mathbb{N}$ such that:

$$(f(n) + n) \in \left( \tfrac{-b-\sqrt{\delta}}{2},\ \tfrac{-b+\sqrt{\delta}}{2} \right)$$

We assume $b > 2$, which implies $\delta > 0$ and factorize $f$ as

$$f(x) = \left( x + \tfrac{b+\sqrt{\delta}}{2} \right) \left( x + \tfrac{b-\sqrt{\delta}}{2} \right)$$

Thanks to its special structure, $g(x)$ may be factorized as follows:

$$
\begin{aligned}
g(x) &= f(f(x) + x)) \\
&= \left( f(x) + x + \tfrac{b+\sqrt{\delta}}{2} \right) \left( f(x) + x + \tfrac{b-\sqrt{\delta}}{2} \right) \\
&= \left( x^2 + (b+1)x + \tfrac{(b+2)+\sqrt{\delta}}{2} \right) \left( x^2 + (b+1)x + \tfrac{(b+2)-\sqrt{\delta}}{2} \right)
\end{aligned}
$$

Since $b > 2$, the first quadratic factor is strictly positive and the sign of $g(x)$ depends on the sign of the second quadratic factor. Let

$$h(x) = x^2 + (b+1)x + \tfrac{(b+2)-\sqrt{\delta}}{2}$$

We have shown $g(x) < 0$ if and only if $h(x) < 0$, so we now analyze $h(x)$.

The discriminant associated with $h(x)$ is:

$$\alpha = (b+1)^2 - 4\ \tfrac{(b+2)-\sqrt{\delta}}{2} = b^2 - 3 + 2\sqrt{\delta}$$

where $b > 2$ implies $\alpha > 0$. The quadratic $h(x)$ may be factorized as follows:

$$h(x) = \left( x - \tfrac{-(b+1)-\sqrt{b^2-3+2\sqrt{\delta}}}{2} \right) \left( x - \tfrac{-(b+1)+\sqrt{b^2-3+2\sqrt{\delta}}}{2} \right)$$

The expression can be simplified by eliminating the nested square-roots (see the proof at the end of this section).

$$\sqrt{b^2-3+2\sqrt{(b-2)(b+2)}} = 1+\sqrt{(b-2)(b+2)}$$

Going back to the factorization of $h(x)$, we have

$$\begin{aligned} h(x) &= \left(x-\tfrac{-(b+1)-(1+\sqrt{(b-2)(b+2)}\,)}{2}\right)\left(x-\tfrac{-(b+1)+(1+\sqrt{(b-2)(b+2)}\,)}{2}\right) \\ &= \left(x-\tfrac{-(b+2)-\sqrt{(b-2)(b+2)}\,)}{2}\right)\left(x-\tfrac{-b+\sqrt{(b-2)(b+2)}\,)}{2}\right) \end{aligned}$$

Both roots of $h(x)=0$ are negative with $-(b+2)-\sqrt{(b-2)(b+2)}\,) < -b+\sqrt{(b-2)(b+2)}\,)$ for all $b>2$. The problem is now to count the number of integer values $n \in \mathbb{N}$ such that:

$$n \in \left(\tfrac{-(b+2)-\sqrt{(b-2)(b+2)}\,)}{2},\ \tfrac{-b+\sqrt{(b-2)(b+2)}\,)}{2}\right)$$

for $b>2$. This is an interval of width

$$\tfrac{-b+\sqrt{(b-2)(b+2)}\,)}{2} - \tfrac{-(b+2)-\sqrt{(b-2)(b+2)}\,)}{2} = 1.$$

Since the bounds of the interval are excluded, this interval contains either exactly 1 or exactly 0 integer values such that $g(x)<0$.

Conclusion: The solutions are $\boxed{0,1}$.

**Proof: Simplifying the Nested Square-Roots**

Suppose that there exist $x>0$ and $y>0$ such that:

$$\sqrt{b^2-3+2\sqrt{(b-2)(b+2)}} = \sqrt{x}+\sqrt{y}$$

Square both sides of the equality:

$$b^2-3+2\sqrt{(b-2)(b+2)} = x+y+2\sqrt{xy}$$

Split the equality and equate each part:

$$\begin{aligned} x+y &= b^2-3 \\ xy &= (b-2)(b+2) \end{aligned}$$

Reduce the system to a single quadratic in $x$:

$$x^2 - (b^2 - 3)x + (b - 2)(b + 2) = 0$$

The discriminant of this quadratic is $(b^2 - 3)^2 - 4(b - 2)(b + 2) = (b^2 - 5)^2$.
The roots are 1 and $(b - 2)(b + 2)$. Substituting back for $y$ gives

$$\sqrt{b^2 - 3 + 2\sqrt{(b - 2)(b + 2)}} = 1 + \sqrt{(b - 2)(b + 2)}$$