AMC 10 Problem Series (2804)

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Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Homework

Lesson: 1 2 3 4 5 6 7 8 **9** 10 11 12

Homework: Lesson 9

You have completed **10** of **10** challenge problems.
Past Due **Aug 7**.

Readings

Lesson 9 Transcript: Fri, Jul 30

Challenge Problems

Total Score: 60 / 60

Problem 1 – Correct! – Score: 6 / 6 (2884)

Problem: Report Error

Henry's Hamburger Heaven offers its hamburgers with the following condiments: ketchup, mustard, mayonnaise, tomato, lettuce, pickles, cheese, and onions. A customer can choose one, two, or three meat patties, and any collection of condiments. How many different kinds of hamburgers can be ordered?

(A) 24 **(B)** 256 **(C)** 768 **(D)** 40,320 **(E)** 120,960

Solution:

The customer can choose the number of the patties in three different ways. For each of the eight condiments, the customer can choose to order that condiment or not order it. Therefore, the number of different hamburgers that can be ordered is $3 \cdot 2^8 = \boxed{768}$. The answer is (C).

Your Response(s):

C

Problem 2 – Correct! – Score: 6 / 6 (2875)

Problem: Report Error

At an inter-species dance party, each cat danced with exactly three dogs and each dog danced with exactly two cats. No one danced with anyone of their own species, and twelve cats attended the party. How many dogs attended the party?

(A) 8 **(B)** 12 **(C)** 16 **(D)** 18 **(E)** 24

Solution:

Consider the number of dances that took place between cats and dogs. There were twelve cats at the party, and each cat danced with three dogs, so there were $12 \cdot 3 = 36$ dances between cats and dogs.

But each dog danced with two cats, so there were $36/2 = \boxed{18}$ dogs. The answer is (D).

Your Response(s):

D

Problem 3 – Correct! – Score: 6 / 6 (2876)

Problem: Report Error

A restaurant offers three desserts, and exactly twice as many appetizers as main courses. A dinner consists of an appetizer, a main course, and a dessert. What is the least number of main courses that the restaurant should offer so that a customer could have a different dinner each night in the year 2003?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Solution:

Let n be the number of main courses that the restaurant offers. Then there are $2n$ appetizers, so the number of different dinners that a customer can have is $2n \cdot n \cdot 3 = 6n^2$. Thus, we seek the smallest positive integer n such that $6n^2 \geq 365$.

We see that $6 \cdot 7^2 = 294$ and $6 \cdot 8^2 = 384$, so the smallest such n is $\boxed{8}$. The answer is (E).

Your Response(s):

E

Problem 4 – Correct! – Score: 6 / 6 (2877)

Problem: Report Error

A license plate in a certain state consists of 4 digits, not necessarily distinct, and 2 letters, also not necessarily distinct. These six characters may appear in any order, except that the two letters must appear next to each other. How many distinct license plates are possible?

(A) $10^4 \cdot 26^2$ **(B)** $10^3 \cdot 26^3$ **(C)** $5 \cdot 10^4 \cdot 26^2$ **(D)** $10^2 \cdot 26^4$ **(E)** $5 \cdot 10^3 \cdot 26^3$

Solution:

Each license plate consists of six symbols, namely four digits and two letters. The two letters must be consecutive, so first we choose which pair of symbols will be letters. There are five ways to choose two consecutive symbols that will be letters.

Once we have chosen which two consecutive symbols that will be letters, each of the remaining four symbols must be digits. There are 26^2 ways to choose two letters, and 10^4 ways to choose four digits, so there are $\boxed{5 \cdot 10^4 \cdot 26^2}$ possible license plates. The answer is (C).

Your Response(s):

C

Problem 5 – Correct! – Score: 6 / 6 (2878)

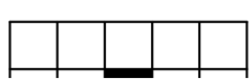
Problem: Report Error

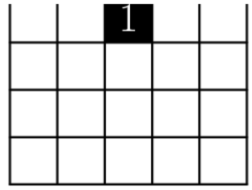
A set of 25 square blocks is arranged into a 5×5 square. How many different combinations of 3 blocks can be selected from that set so that no two are in the same row or column?

(A) 100 **(B)** 125 **(C)** 600 **(D)** 2300 **(E)** 3600

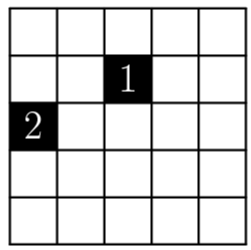
Solution:

There are 25 ways to choose the first block.





The first block eliminates one row and one column, leaving four rows and four columns. Hence, there are always $4 \cdot 4 = 16$ ways to choose the second block.



The second block eliminates another row and column, leaving three rows and three columns. Hence, there are always $3 \cdot 3 = 9$ ways to choose the third block. Therefore, there are $25 \cdot 16 \cdot 9 = 3600$ ways to choose the three blocks.

However, we have chosen the three block in a particular order; first, second, and third. The order of the blocks does not matter, so to account for this, we divide our count by $3! = 6$. Hence, the number of ways of choosing the three blocks is $3600/6 = \boxed{600}$. The answer is (C).

Your Response(s):

C

Problem 6 – Correct! – Score: 6 / 6 (2879)

Problem: Report Error

Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

(A) 22 **(B)** 25 **(C)** 27 **(D)** 28 **(E)** 729

Solution:

We use stars and bars. There are six cookies and three kinds of cookies, so every assortment of cookies can be represented by a row of six stars and two bars. There are $\binom{8}{2} = 28$ ways to arrange six stars and two bars, so there are $\boxed{28}$ assortments. The answer is (D).

Your Response(s):

D

Problem 7 – Correct! – Score: 6 / 6 (2880)

Problem: Report Error

Seven distinct pieces of candy are to be distributed among three bags. The red bag and the blue bag must each receive at least one piece of candy; the white bag may remain empty. How many arrangements are possible?

(A) 1930 **(B)** 1931 **(C)** 1932 **(D)** 1933 **(E)** 1934

Solution:
The condition of the red bag and the blue bag receiving at least one piece of candy is difficult to deal with, so we try complementary counting. If this condition does not hold, then either the red bag is empty or the blue bag is empty.

If the red bag is empty, then each piece of candy goes into either the blue bag or the white bag, for a total of 2^7 possible distributions.

Similarly, if the blue bag is empty, then each piece of candy goes into either the red bag or the white bag, again for a total of 2^7 possible distributions.

However, we must then subtract the distributions where both the red bag and blue bag are empty. If both the red bag and the blue bag are empty, then every piece of candy must go into the white bag, for one possible distribution.

Therefore, the number of distributions where either the red bag is empty or the blue bag is empty is $2^7 + 2^7 - 1$.

We must subtract this number from the total number of possible distributions. Each piece of candy can go into any of the three bags, so the total number of possible distributions is 3^7.

Hence, the number of distributions where both the blue bag and red bag have at least one piece of candy is $3^7 - (2^7 + 2^7 - 1) = \boxed{1932}$. The answer is (C).

Your Response(s):

C

Problem 8 – Correct! – Score: 6 / 6 (2881)

Problem: Report Error

Two subsets of the set $S = \{a, b, c, d, e\}$ are to be chosen so that their union is S and their intersection contains exactly two elements. In how many ways can this be done, assuming that the order in which the subsets are chosen does not matter?

(A) 20 **(B)** 40 **(C)** 60 **(D)** 160 **(E)** 320

Solution:
Let the two subsets be S_1 and S_2. Since the intersection of S_1 and S_2 consists of exactly two elements, we first choose the two elements that are contained in both S_1 and S_2. There are $\binom{5}{2} = 10$ ways to choose these two elements from S.

We must then distribute the three remaining elements in S among S_1 and S_2. Since the union of S_1 and S_2 is S, each element of S must be in at least one of these sets. However, we have already chosen the two elements that lie in both S_1 and S_2, so each of the remaining three elements in S must lie in exactly one of S_1 or S_2. Hence, there are $2^3 = 8$ ways of distributing these three elements.

This gives us $10 \cdot 8 = 80$ ways of constructing the sets S_1 and S_2. However, the order of the sets does not matter, so we divide our count by $2! = 2$. This gives us $80/2 = \boxed{40}$. The answer is (B).

Your Response(s):

B

Problem 9 – Correct! – Score: 6 / 6 (2882)

Problem: Report Error

The entries in a 3×3 array include all the digits from 1 through 9, arranged so that the entries in every row and column are in increasing order. How many such arrays are there?

(A) 18 **(B)** 24 **(C)** 36 **(D)** 42 **(E)** 60

Solution:

Solution 1. It is clear that the 1 must go in the upper-left corner, and the 9 must go in the lower-right corner.

1		
		9

The 2 must go next to the 1, to the right or below. The 3 must go next to the 2 (in which case the three numbers 1, 2, and 3 form an "I"), or the 3 must go next to the 1 (in which case the three numbers 1, 2, and 3 form an "L"). The two possible I-shapes and two possible L-shapes are shown below.

1	2	3

1		
2		
3		

1	2	
3		

1	3	
2		

Similarly, the numbers 7, 8, and 9 can form two I-shapes or two L-shapes.

7	8	9

		7
		8
		9

		7
	8	9

		8
	7	9

We take cases accordingly.

Case 1: The numbers 1, 2, and 3 form an I-shape, and the numbers 7, 8, and 9 form an I-shape

There are only two ways that the numbers 1, 2, and 3 form an I-shape, and the numbers 7, 8, and 9 form an I-shape.

1	2	3
7	8	9

1		7
2		8
3		9

In each array, there is only way to fill in the numbers 4, 5, and 6. This gives us two arrays in this case.

Case 2: The numbers 1, 2, and 3 form an I-shape, and the numbers 7, 8, and 9 form an L-shape

There are four ways that the numbers 1, 2, and 3 form an I-shape, and the numbers 7, 8, and 9 form an L-shape.

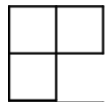
1	2	3
		7
	8	9

1	2	3
		8
	7	9

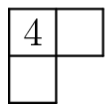
1		
2		7
3	8	9

1		
2		8
3	7	9

In each array, the empty entries form the following shape.



The 4 must be placed as follows, and then the 5 and 6 can be placed arbitrarily.



This gives us $4 \cdot 2 = 8$ arrays in this case.

Case 3: The numbers 1, 2, and 3 form an L-shape, and the numbers 7, 8, and 9 form an I-shape

There are four ways that the numbers 1, 2, and 3 form an L-shape, and the numbers 7, 8, and 9 form an I-shape.

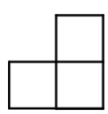
1	2	
3		
7	8	9

1	3	
2		
7	8	9

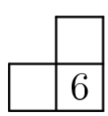
1	2	7
3		8
		9

1	3	7
2		8
		9

In each array, the empty entries form the following shape.



The 6 must be placed as follows, and then the 4 and 5 can be placed arbitrarily.



This gives us $4 \cdot 2 = 8$ arrays in this case.

Case 4: The numbers 1, 2, and 3 form an L-shape, and the numbers 7, 8, and 9 form an L-shape

There are four ways that the numbers 1, 2, and 3 form an L-shape, and the numbers 7, 8, and 9 form an L-shape.

1	2	
3		7
	8	9

1	2	
3		8
	7	9

1	3	
2		7
	8	9

1	3	
2		8
	7	9

In each array, the numbers 4, 5, and 6 can be filled in arbitrarily. This gives us $4 \cdot 3! = 24$ arrays in this case.

Hence, the total number of arrays is $2 + 8 + 8 + 24 = \boxed{42}$. The answer is (D).

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Solution 2. We start by noticing that the top left grid number must be smaller than all of the other entries, and the bottom right must be larger than all the other grid entries. So those numbers must be 1 and 9.

1		
		9

Now let's do some casework based on what the middle entry is. The middle entry is in a row or column with many other unknown entries, so if we know the middle entry, that may help us figure out where the other numbers can go.

The middle number must be larger than the top left, top center, and middle left numbers; therefore it is at least 4. And the middle number must be smaller than the middle right, bottom center, and bottom right numbers; therefore it is at most 6. So, we have three cases to consider.

Case 1: the middle number is 4

1		
	4	
		9

The middle number must be greater than the top center and middle left numbers, so those must be 2 and 3, in some order.

1	2 or 3	
3 or 2	4	
		9

Now the four smallest numbers are in the upper left block, which is sorted the way we want. So however we fill in the remaining numbers, the top and middle rows will be in increasing order, and so will the left and middle columns. We only need to worry about the right column and bottom row.

That means that of the remaining 4 numbers, we can pick any two to put in the vacant spaces in the right column. Once we do that, we'll have no choice but to sort them (and the numbers for the bottom row) in increasing order, and then we'll have a grid that works. There are $\binom{4}{2} = \frac{4 \cdot 3}{2} = 6$ ways to choose those two numbers for the right column. Since there are also 2 ways to arrange the upper left block, there are a total of $2 \cdot 6 = 12$ possibilities in this case.

Case 2: the middle number is 5

1		

	5	
		9

For this case, we'll think about the clockwise path from 1 to 9 that goes along the top edge and then the right edge. The numbers along this path must be in increasing order. So, we may as well choose 3 of the remaining 6 numbers (2, 3, 4, 6, 7, 8) for the top and right edges, put them in order, and then put the remaining 3 numbers in order along the bottom and left edges. There are $\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{2 \cdot 3} = 20$ ways to do this. Here's one example, where we pick 2, 4, and 6 for the top and right:

1	2	4
3	5	6
7	8	9

But we should ask if this procedure will always work! It will always put the top, bottom, left, and right rows/columns in increasing order, so we only need to worry about the middle column and middle row.

To deal with this, suppose that the top/right numbers are $a < b < c$, and the bottom/left numbers are $x < y < z$, so that the grid looks like this:

1	a	b
x	5	c
y	z	9

First, let's consider the middle column. The only way it could be out of order is if $a > 5$ or $z < 5$. Now remember, a is the smallest of the three top/right numbers, so if $a > 5$, that means that a, b, and c are 6, 7, and 8. Similarly, $z < 5$ means that x, y, and z are 2, 3, and 4. So to get the middle column in order, we just need to make sure that we don't pick {6, 7, 8} to be the three numbers for the top/right.

A similar analysis on the middle row shows that we can't pick {2, 3, 4} to be the three numbers for the top/right. If we avoid those two possibilities, though, any of our 20 arrangements will work! Therefore there are $20 - 2 = 18$ possibilities in this case.

Case 3: the middle number is 6

1		
	6	
		9

This is exactly like case 1, except now we know that the middle right and bottom center numbers must be 7 and 8 in some order. By identical logic to case 1, there are 12 possibilities in this case.

Our three cases cover all the possibilities for the middle number. Adding up, we get $12 + 18 + 12 = \boxed{42}$ cases in total. The answer is (D).

Your Response(s):

D

Problem 10 – Correct! – Score: 6 / 6 (2883)

Problem: Report Error

A subset B of the set of integers from 1 to 100, inclusive, has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B?

(A) 50 **(B)** 51 **(C)** 62 **(D)** 65 **(E)** 68

Solution:

Among the numbers from 1 to 100, the pairs of numbers that add up to 125 are $(25, 100)$, $(26, 99)$, $\dots$, $(62, 63)$. Hence, B can contain at most one number from each of these 38 pairs. The set B can safely contain all the numbers from 1 to 24, so B can contain a maximum of $38 + 24 = \boxed{62}$ elements. The answer is (C).

Your Response(s):

- C