

Student: James Toche

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ID#: 44857

USA Mathematical Talent Search

Year	Round	Problem
36	1	1

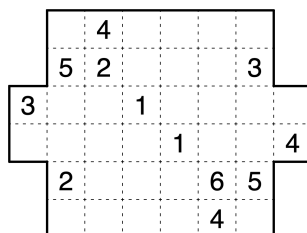
Problem 1

Question

The “Manhattan distance” between two cells is the shortest distance between those cells when traveling up, down, left, or right, as if one were traveling along city blocks rather than as the crow flies.

Place numbers from 1-6 in some cells so the following criteria are satisfied:

1. A cell contains at most one number. Cells can be left empty.
2. For each cell containing a number N in the grid, exactly two other cells containing N are at a Manhattan distance of N .
3. For each cell containing a number N in the grid, no other cells containing N are at a Manhattan distance less than N .



There is a unique solution, but you do not need to prove that your answer is the only one possible.

The solution is:

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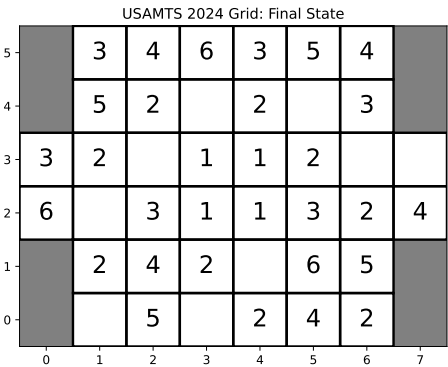


Figure 1

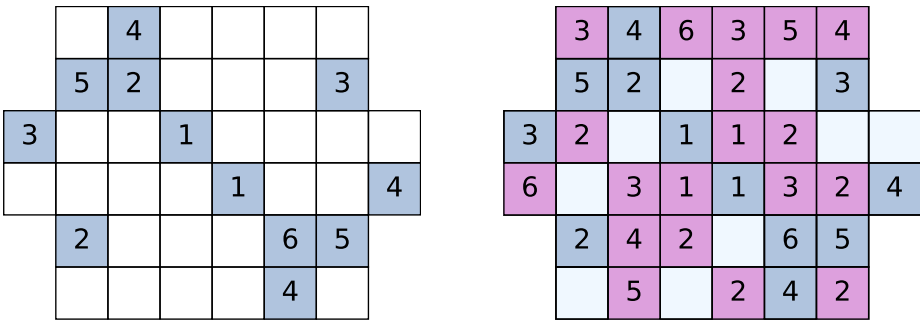


Figure 2

Year	Round	Problem
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Problem 2

Question

A regular hexagon is placed on top of a unit circle such that one vertex coincides with the center of the circle, exactly two vertices lie on the circumference of the circle, and exactly one vertex lies outside of the circle. Determine the area of the hexagon.

A natural starting point for this problem is to draw a hexagon inscribed in a circle (Figure 3a) and to translate it so that one of its vertices coincides with the center of the circle (Figure 3b). It is then clear that the hexagon must be shrunk to the point where the short-diagonal of the hexagon coincides with the radius r of the circle (Figure 3c).

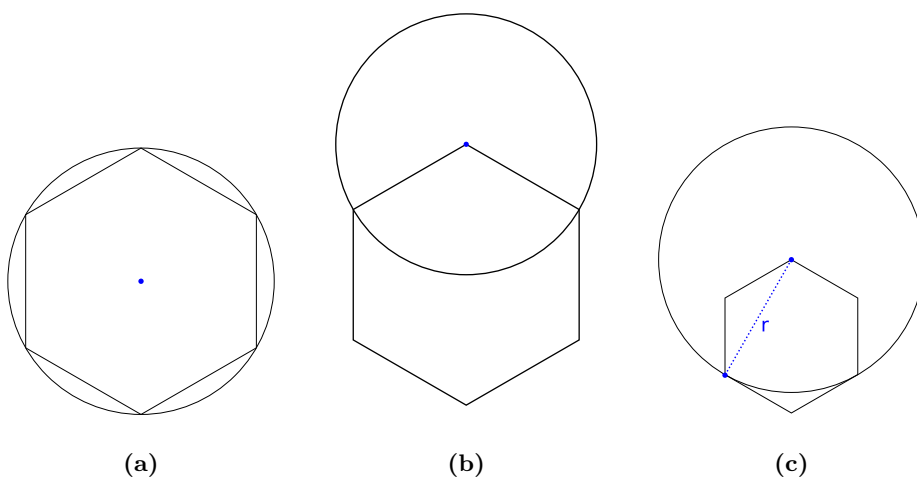


Figure 3

In hexagon $ABCDEF$ (Figure 4), consider first the triangle ACE inscribed inside the hexagon. Claim: *Triangle ACE is equilateral*. Proof: Since A is the center of the circle and C and E are on the circumference, $AC = r$ and $AE = r$. By the half-angle formula, the chord length CE is equal to $2r \sin(\theta/2)$, where $\theta = \angle CAE = \pi/3$. Calculating $\sin(\theta/2) = \sin(\pi/6) = 1/2$ and substituting back gives a chord length $CE = 2r \sin(\theta/2) = r$. It follows that all three sides of the triangle have length r (it is actually obvious by considerations of symmetry).

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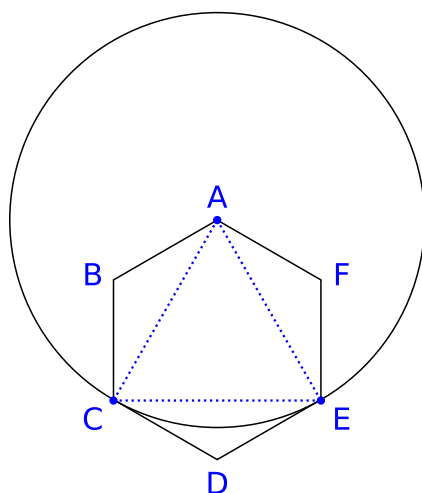


Figure 4

Since triangle ACE is equilateral, its area is $\frac{\sqrt{3}}{4}r^2$ (the side lengths of the equilateral triangle coincide with the radius of the circle). This well-known formula can be proved by applying the Pythagorean theorem: See Figure 5.

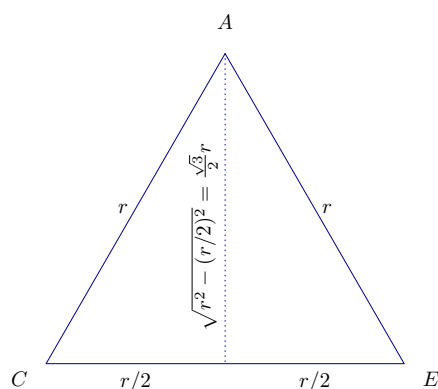


Figure 5

In hexagon $ABCDEF$ (Figure 4), consider now the isosceles triangle ABC , with base equal to the chord $AC = r$. This triangle has area $\frac{\sin(\varphi)}{2}r^2$, (the base of the isosceles triangle coincides with the radius of the circle), where angle $\varphi = \angle ABC = \frac{2\pi}{3}$. The angle follows from the sum-of-angles formula. The sum of the angles of a regular polygon with n sides is $(n-2)\pi$. For a hexagon, $n = 6$, the angle is therefore $\varphi = \frac{(6-2)\pi}{6} = \frac{2\pi}{3}$ and so

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$$\sin(\varphi) = \sin(2\pi/3) = \frac{\sqrt{3}}{2}.$$

Adding together the area of triangle ACE and triangles ABC , AEF , and CDE (or, equivalently, three times the area of ABC) gives:

$$\frac{\sqrt{3}}{4}r^2 + 3 \times \frac{\sqrt{3}}{2}r^2 = \frac{7\sqrt{3}}{4}$$

where we have substituted $r = 1$, the radius of the unit circle.

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Problem 3

Question

A sequence of integers x_1, x_2, \dots, x_k is called *fibtastic* if the difference between any two consecutive elements in the sequence is a Fibonacci number.

The integers from 1 to 2024 are split into two groups, each written in increasing order. Group A is a_1, a_2, \dots, a_m and Group B is b_1, b_2, \dots, b_n .

Find the largest integer M such that we can guarantee that we can pick M consecutive elements from either Group A or Group B which form a fibtastic sequence.

The question does not state who splits the integers into two groups: If the split is engineered to maximize M , the answer is $M = 1012$. If the split is engineered to minimize M , the answer is $M = 2$. By another interpretation, the answer is $M = 16$.

General Considerations

The Fibonacci numbers F_0, F_1, F_2, \dots are defined inductively, for $n \geq 1$, by

$$F_0 = 0,$$

$$F_1 = 1,$$

$$F_{n+1} = F_n + F_{n-1}.$$

On the closed interval $[0, 2024]$, the Fibonacci subsequence is

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597.$$

where $F_{17} = 1597$. Clearly, the largest possible difference $|a_{k+1} - a_k|$, calculated from consecutive terms in the sequence a_1, a_2, \dots, a_m , cannot exceed 1597 and likewise for b_1, b_2, \dots, b_n .

Amicable Split

Consider the arithmetic sequence with first term 1 and common difference F_1 . Split the integers as

$$\text{Group A : } a_1, a_2, a_3 \dots, a_{1012} = 1, 2, 3 \dots, 1012$$

$$\text{Group B : } b_1, b_2, b_3 \dots, b_{1012} = 1013, 1014, 1015 \dots, 2024$$

where $m = n = 1012$. The difference between any two consecutive elements in the sequence $a_1, a_2, a_3 \dots, a_{1012}$ is the Fibonacci number F_1 . Likewise for $b_1, b_2, b_3 \dots, b_{1012}$. Every consecutive element from either Group A or Group B is a fibtastic sequence. The stated conditions are therefore satisfied and the maximum (max-max) is $M = 1012$.

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Adversarial Split

If m and n are chosen to minimize M , we must have $m \geq 2$ and $n \geq 2$ to ensure that the difference between any two consecutive numbers is defined. An adversarial split that minimizes M is:

$$\begin{array}{ll} \text{Group A :} & a_1, a_2 \qquad \qquad \qquad = 1, 2 \\ \text{Group B :} & b_1, b_2, b_3 \dots, b_{2022} \qquad \qquad = 3, 4, 5 \dots, 2024 \end{array}$$

These sequences are fibtastic, as shown above. The stated conditions are therefore satisfied and the maximum (max-min) is $M = 2$.

Increasing Fibonacci Subsequences

Adding a constraint to the problem makes it more interesting. Consider the revised definition: A sequence of integers x_1, x_2, \dots, x_k is called *fibtastic* if the difference between any two consecutive elements in the sequence is a Fibonacci number ***and if these Fibonacci numbers form a strictly increasing sequence.***

The largest increasing sequence formed by the consecutive differences would be a subsequence of the first seventeen Fibonacci numbers: F_1, F_2, \dots, F_{17} . However, $F_{17} = 1597$ is unreachable for $m = n = 1012$, so the increasing Fibonacci subsequence is reduced to F_1, F_2, \dots, F_{16} , where $F_{16} = 987$. With these additional restrictions, a candidate solution is $M = 16$. It remains to be shown that a subsequence of that length does exist.

We start by constructing a subsequence on $[1, 2024]$.

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Number	Fibonacci Representation		Consecutive Difference
1	=	F_1	
2	=	F_3	$\rightarrow F_3 - F_1 = F_2$
3	=	F_4	$\rightarrow F_4 - F_3 = F_2$
4	=	$F_4 + F_2$	$\rightarrow F_2$
5	=	F_5	$\rightarrow F_5 - F_4 - F_2 = F_1$
6	=	$F_5 + F_2$	$\rightarrow F_2$
7	=	$F_5 + F_3$	$\rightarrow F_3 - F_2 = F_1$
8	=	F_6	$\rightarrow F_6 - F_5 - F_3 = F_2$
9	=	$F_6 + F_2$	$\rightarrow F_2$
10	=	$F_6 + F_3$	$\rightarrow F_3 - F_2 = F_1$
11	=	$F_6 + F_4$	$\rightarrow F_4 - F_3 = F_2$
12	=	$F_6 + F_4 + F_2$	$\rightarrow F_2$
13	=	F_7	$\rightarrow F_7 - F_6 - F_4 - F_2 = F_1$
14	=	$F_7 + F_2$	$\rightarrow F_1$
15	=	$F_7 + F_3$	$\rightarrow F_3 - F_2 = F_1$
16	=	$F_7 + F_4$	$\rightarrow F_4 - F_3 = F_2$
17	=	$F_7 + F_4 + F_2$	$\rightarrow F_2$
18	=	$F_7 + F_5$	$\rightarrow F_5 - F_4 - F_2 = F_1$
19	=	$F_7 + F_5 + F_2$	$\rightarrow F_2$
20	=	$F_7 + F_5 + F_3$	$\rightarrow F_3 - F_2 = F_1$
21	=	F_8	$\rightarrow F_8 - F_7 - F_5 - F_3 = F_2$

Table 1

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Problem 4

Question

During a lecture, each of 26 mathematicians falls asleep exactly once, and stays asleep for a nonzero amount of time. Each mathematician is awake at the moment the lecture starts, and the moment the lecture finishes. Prove that there are either 6 mathematicians such that no two are asleep at the same time, or 6 mathematicians such that there is some point in time during which all 6 are asleep.

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Problem 5

Question

Let $f(x) = x^2 + bx + 1$ for some real number b . Across all possible values of b , find all possible values for the number of integers x that satisfy $f(f(x) + x) < 0$.

That is, if there are some values of b that give us 180 integer solutions for x and there are other values of b that give us 314 integer solutions for x (and these are the only possibilities), the answer would be 180, 314.