2021 AMC 12A Problems/Problem 7

The following problem is from both the 2021 AMC 10A #9 and 2021 AMC 12A #7, so both problems redirect to this page.

Contents

Problem

What is the least possible value of $(xy-1)^2+(x+y)^2$ for real numbers x and y?

$\textbf{(A)} ~0\qquad\textbf{(B)} ~\frac{1}{4}\qquad\textbf{(C)} ~\frac{1}{2} \qquad\textbf{(D)} ~1 \qquad\textbf{(E)} ~2$

Solution 1 (Expand)

Expanding, we get that the expression is $x^2+2xy+y^2+x^2y^2-2xy+1$ or $x^2+y^2+x^2y^2+1$. By the Trivial Inequality (all squares are nonnegative) the minimum value for this is $\boxed{\textbf{(D)} ~1}$, which can be achieved at $x=y=0$.

~aop2014

Solution 2 (Expand and then Factor)

We expand the original expression, then factor the result by grouping:

$$\begin{aligned}(xy-1)^2+(x+y)^2 &= \left(x^2y^2-2xy+1\right)+\left(x^2+2xy+y^2\right) \\ &= x^2y^2+x^2+y^2+1 \\ &= x^2\left(y^2+1\right)+\left(y^2+1\right) \\ &= \left(x^2+1\right)\left(y^2+1\right).\end{aligned}$$

Clearly, both factors are positive. By the Trivial Inequality, we have

$$\left(x^2+1\right)\left(y^2+1\right)\geq(0+1)(0+1)=\boxed{\textbf{(D)} ~1}.$$

Note that the least possible value of $(xy-1)^2+(x+y)^2$ occurs at $x=y=0$.

~MRENTHUSIASM

Solution 3 (Beyond Overkill)

Like solution 1, expand and simplify the original equation to $x^2 + y^2 + x^2y^2 + 1$ and let $f(x, y) = x^2 + y^2 + x^2y^2 + 1$. To find local extrema, find where $\nabla f(x, y) = \mathbf{0}$. First, find the first partial derivative with respect to x and y and find where they are 0:

$$\frac{\partial f}{\partial x} = 2x + 2xy^2 = 2x(1 + y^2) = 0 \implies x = 0$$

$$\frac{\partial f}{\partial y} = 2y + 2yx^2 = 2y(1 + x^2) = 0 \implies y = 0$$

Thus, there is a local extremum at $(0, 0)$. Because this is the only extremum, we can assume that this is a minimum because the problem asks for the minimum (though this can also be proven using the partial second derivative test) and the global minimum since it's the only minimum, meaning $f(0, 0)$ is the minimum of $f(x, y)$. Plugging $(0, 0)$ into $f(x, y)$, we find 1 $\implies \boxed{\textbf{(D) } 1}$.

~ DBlack2021

Video Solution (Simple & Quick)

https://youtu.be/2CZ1u4J9yk4

~ Education, the Study of Everything

Video Solution by Aaron He (Trivial Inequality)

https://www.youtube.com/watch?v=xTGDKBthWsw&t=6m58s

Video Solution by North America Math Contest Go Go Go (Trivial Inequality, Simon's Favourite Packing Theorem)

https://www.youtube.com/watch?v=PbJK4KKfQjY&list=PLexHyfQ8DMuKqltG3cHT7Di4jhVl6L4YJ&index=8

Video Solution by Hawk Math

https://www.youtube.com/watch?v=P5al76DxyHY

Video Solution (Trivial Inequality, Simon's Favorite Factoring)

https://youtu.be/DP0ppuQzFPE

~ pi_is_3.14

Video Solution 6

https://youtu.be/hmOGYmVmY1c

~savannahsolver

Video Solution by TheBeautyofMath

https://youtu.be/s6E4E06XhPU?t=640 (for AMC 10A)

https://youtu.be/cckGBU2x1zg?t=95 (for AMC 12A)

~IceMatrix

Video Solution by The Learning Royal

https://youtu.be/AWjOeBFyeb4

See also

2021 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
Preceded by **Problem 8**	Followed by **Problem 10**
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All AMC 10 Problems and Solutions	

2021 AMC 12A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
Preceded by **Problem 6**	Followed by **Problem 8**
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All AMC 12 Problems and Solutions	