AMC 10 Problem Series (2804)

Jon Joseph

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Overview

Lesson 9 (Jul 30) Class Transcript - Counting

Go back to the class overview page

Display all student messages • Show few student messages • Hide student messages

goveganddomath 2021-07-30 19:30:27
Hi everyone!

goveganddomath 2021-07-30 19:30:35
My name is Eli Brottman and I am the instructor for today.

goveganddomath 2021-07-30 19:30:38
Eli has been a member of the AoPS Community since 2013, first as a student and then he started working for AoPS in 2017 as a grader and teaching assistant. He is a graduate of Northern Illinois University, and graduated with university honors and honors in mathematical sciences. Eli also minored in computer science and statistics, and was an active researcher. Eli has participated in MATHCOUNTS, USAMTS, AMC 10/12, and AIME. In his spare time, Eli enjoys volunteering in his community and exploring ways to use math to make the world a better place.

goveganddomath 2021-07-30 19:30:57
AMC 10 Problem Series
Week 9: Counting

goveganddomath 2021-07-30 19:31:02
In today's class, we will cover techniques for solving problems in counting. Counting problems come in a wide variety of forms, and accordingly there are a wide variety of techniques for solving them. We will try to cover as many of these techniques as we can.

goveganddomath 2021-07-30 19:31:15
PRODUCT PRINCIPLE

goveganddomath 2021-07-30 19:31:25
One of the simplest principles in counting is the product principle. Suppose I own three shirts, four pairs of pants, and two pairs of shoes. Then how many different outfits can I wear?

goveganddomath 2021-07-30 19:32:03
I multiply the number of choices for each piece of clothing, giving me $3 \cdot 4 \cdot 2 = 24$ different outfits.

goveganddomath 2021-07-30 19:32:06
However, you might reasonably wonder why that's true! Let's do a quick review.

goveganddomath 2021-07-30 19:32:09
Let's start off easier, with just the pants and the shirts. Say the shirts are $S1, S2, S3$ and the pants are $P1, P2, P3, P4$.

goveganddomath 2021-07-30 19:32:17
How many shirt-and-pant outfits can we make with shirt $S1$?

goveganddomath 2021-07-30 19:32:43
We get to pick any pair of pants to go with the shirt. That means there are four outfits:

$$(S1, P1), (S1, P2), (S1, P3), (S1, P4).$$

goveganddomath 2021-07-30 19:32:45
And how many shirt-and-pant outfits can we make with shirt $S2$?

goveganddomath 2021-07-30 19:33:04
There are four again! Here they are:

$$(S2, P1), (S2, P2), (S2, P3), (S2, P4).$$

And how about with shirt $S3$?

goveganddomath 2021-07-30 19:33:24
Yup... four again! They are $(S3, P1), (S3, P2), (S3, P3), (S3, P4)$.

goveganddomath 2021-07-30 19:33:30
That means that in total, there are $4 + 4 + 4 = 3 \cdot 4 = 12$ shirt-and-pant outfits.

goveganddomath 2021-07-30 19:33:32
OK. Now we just need to pick the shoes. How many total outfits are there if we use our first pair of shoes?

goveganddomath 2021-07-30 19:33:51
There are 12: the 12 shirt-and-pant outfits from above. And if we use our second pair of shoes?

goveganddomath 2021-07-30 19:34:13
There are 12 again, for the same reason. That means that there are a total of $2 \cdot 12 = 24$ outfits.

goveganddomath 2021-07-30 19:34:20
More generally, if I want to choose several objects, then the number of ways is simply the product of the numbers of ways of choosing each individual object.

goveganddomath 2021-07-30 19:34:29
(Assuming our choice for one thing doesn't affect our choice for the other thing, that is.)

goveganddomath 2021-07-30 19:34:31
Now that we've done the brief review of the product principle, and when it applies, let's use it for some problems!

goveganddomath 2021-07-30 19:34:39
Nebraska, the former home of the AMC, changed its license plate scheme. Each old license plate consisted of a letter followed by four digits. Each new license plate consists of three letters followed by three digits. By how many times is the number of possible license plates increased?

$\text{(A) } \frac{26}{10} \qquad \text{(B) } \frac{26^2}{10^2} \qquad \text{(C) } \frac{26^2}{10} \qquad \text{(D) } \frac{26^3}{10^3} \qquad \text{(E) } \frac{26^3}{10^2}$

goveganddomath 2021-07-30 19:34:44
What two numbers do we need to compute to solve this problem?

goveganddomath 2021-07-30 19:35:32
We need to compute the number of license plates under the old scheme, and the number of license plates under the new scheme.

goveganddomath 2021-07-30 19:35:34
How many license plates are there under the old scheme?

goveganddomath 2021-07-30 19:36:16
Under the old scheme, each license plate consists of a letter and four digits. We have 26 choices for the letter, and 10 choices

for each digit, so the number of license plates is $26 \cdot 10^4$.

govegandomath 2021-07-30 19:36:22
How many license plates are there under the new scheme?

govegandomath 2021-07-30 19:36:50
Under the new scheme, each license plate consists of three letters and three digits, so the number of license plates is $26^3 \cdot 10^3$.

govegandomath 2021-07-30 19:36:54
(It's better to not multiply it out--this time it would take a lot of work, and wouldn't be helpful!)

govegandomath 2021-07-30 19:36:59
What is the ratio between these two numbers?

govegandomath 2021-07-30 19:37:30
The ratio between these two numbers is

$$\frac{26^3 \cdot 10^3}{26 \cdot 10^4} = \frac{26^2}{10}.$$

The answer is (C).

govegandomath 2021-07-30 19:37:41
Two tour guides are leading six tourists. The guides decide to split up. Each tourist must choose one of the guides, but with the stipulation that each guide must take at least one tourist. How many different groupings of guides and tourists are possible?

$\text{(A) } 56 \quad \text{(B) } 58 \quad \text{(C) } 60 \quad \text{(D) } 62 \quad \text{(E) } 64$

govegandomath 2021-07-30 19:37:45
How can we get started on this problem?

govegandomath 2021-07-30 19:38:55
We can figure out how many ways there are for each tourist to choose one of the two guides, without worrying about the stipulation for now. To do this, we can use the product principle.

govegandomath 2021-07-30 19:38:58
So the first tourist chooses one of the two guides, the second tourist chooses one of the two guides, and so on. How many groupings does this give us?

govegandomath 2021-07-30 19:39:32
This gives us $2^6 = 64$ different groupings, since each of the 6 tourists has 2 choices. But is this the answer?

govegandomath 2021-07-30 19:39:48
It isn't! Why not?

govegandomath 2021-07-30 19:40:23
We haven't addressed the fact that each guide must take at least one tourist. What should we do?

govegandomath 2021-07-30 19:41:24
Since each guide must take at least one tourist, we must subtract from 64 the number of groupings in which a guide has no tourists. As many of you have already found, there are 2 such groupings.

govegandomath 2021-07-30 19:41:33
Hence, what is the answer?

govegandomath 2021-07-30 19:41:59
Hence, there are $64 - 2 = 62$ possible groupings. The answer is (D).

goveganddomath 2021-07-30 19:42:02
When we obtain a number in a counting problem, it is common that because of some condition in the problem, we must adjust the answer somehow. In any counting problem, make sure you read the problem carefully for any particular conditions.

goveganddomath 2021-07-30 19:42:09
How many three-digit numbers satisfy the property that the middle digit is the average of the first and the last digits?

$\text{(A) } 41 \quad \text{(B) } 42 \quad \text{(C) } 43 \quad \text{(D) } 44 \quad \text{(E) } 45$

goveganddomath 2021-07-30 19:42:16
To get a feel for the problem, let's look at a particular example. What are all the three-digit numbers that satisfy the condition in the problem and start with the digit 3?

goveganddomath 2021-07-30 19:43:44
The three-digit numbers that start with the digit 3 are $321, 333, 345, 357,$ and 369. Note that the digits did not have to be increasing or distinct. Do you see anything interesting about these numbers?

goveganddomath 2021-07-30 19:44:18
What is interesting to me about these numbers are the last digits, namely $1, 3, 5, 7,$ and 9. They are all odd.

goveganddomath 2021-07-30 19:44:24
So will the first and last digits *always* both be odd?

goveganddomath 2021-07-30 19:44:46
No! If the first digit is even, then the last digit will be even. (For instance, the number 246 satisfies the condition.)

goveganddomath 2021-07-30 19:44:50
In every three-digit number abc that we want to count, the first and last digits are either both even or both odd. (This follows from the equation $a + c = 2b$, since a, b, and c must be integers.)

goveganddomath 2021-07-30 19:44:59
And if you think about it, for any choice of a and c that are either both even or both odd, you can always choose a b that will work--namely, the number halfway between them.

goveganddomath 2021-07-30 19:45:12
So we have two cases. Let's use casework!

goveganddomath 2021-07-30 19:45:14
Both odd: How many choices are there for the first digit?

goveganddomath 2021-07-30 19:45:40
There are 5 choices, since it can be any of $1, 3, 5, 7, 9$. How many choices are there for the third digit?

goveganddomath 2021-07-30 19:45:59
There are also 5 choices. Once we pick the first and last digit, how many choices do we then have for the middle digit?

goveganddomath 2021-07-30 19:46:25
Just one, since we can determine it by averaging the first and the third. So how many such numbers are there with the first and last digit odd?

goveganddomath 2021-07-30 19:46:49
There are $5 \times 5 \times 1 = 25$ such numbers.

goveganddomath 2021-07-30 19:46:52
Both even: How many choices for the first digit are there in this case?

goveganddomath 2021-07-30 19:47:25
There are 4 choices, namely $2, 4, 6,$ and 8, since we can't pick 0 to be the first digit. How many choices are there for the last digit?

goveganddomath 2021-07-30 19:48:03
There are 5 choices -- $0, 2, 4, 6,$ and 8. And like last time, once we have the first and third digits, we only have one option for the middle digit. So how many total numbers of this form do we have?

goveganddomath 2021-07-30 19:48:34
We have $4 \times 5 \times 1 = 20$ of them. So what's the answer?

goveganddomath 2021-07-30 19:49:03
We add the totals from the two separate cases. The answer is $20 + 25 = 45$, which is (E).

goveganddomath 2021-07-30 19:49:09
COMBINATIONS

goveganddomath 2021-07-30 19:49:15
In a combination, we are selecting objects from a set, where the order of the objects does not matter. For example, in how many ways can we select two different letters of the alphabet? The order of the letters does not matter: we're going to count getting (A, B) as the same as getting (B, A).

goveganddomath 2021-07-30 19:49:25
I bet some of you know the answer... but let's go through the explanation!

goveganddomath 2021-07-30 19:49:27
Say that we're picking the letters in order. How many choices do we have for the first letter?

goveganddomath 2021-07-30 19:50:05
Yup, we have 26 choices. And then how many choices do we have for the second letter?

goveganddomath 2021-07-30 19:50:37
Now we have only 25, because we already picked one!

goveganddomath 2021-07-30 19:50:40
Let's quickly remind ourselves of how the product principle works. How many pairs did we pick in which A was first?

goveganddomath 2021-07-30 19:51:45
Yes, there were 25 such pairs: $(A, B), (A, C), (A, D), \dots, (A, Z)$. And how many pairs were there in which we picked B first?

goveganddomath 2021-07-30 19:52:21
That's right, there were 25 again. Again, the total number of pairs we picked must be

$$25 + 25 + \cdots + 25 = 26 \cdot 25.$$

goveganddomath 2021-07-30 19:52:26
Hmmmm. That makes us think that there are $26 \cdot 25$ total possible pairs of letters.

goveganddomath 2021-07-30 19:52:28
Except... wait a minute! We counted both (A, B) and (B, A)! We counted this pair in both the "pick A first" grouping and in the "pick B first" grouping.

goveganddomath 2021-07-30 19:53:05
Did we count each pair twice?

goveganddomath 2021-07-30 19:53:17
We did. That means we must divide by 2. The total number of pairs (disregarding order) is $\frac{26 \cdot 25}{2}$.

goveganddomath 2021-07-30 19:53:19
Now let's see how many ways there are to pick three different letters from the alphabet. Let's again pick the letters in order.

goveganddomath 2021-07-30 19:53:34
Again using the product principle, we have a choice of 26 letters for our first letter, a choice of 25 letters for the second letter, and a choice of 24 letters for the third letter.

goveganddomath 2021-07-30 19:53:38
That means that we have a total of $26 \cdot 25 \cdot 24$ choices for the three letters.

goveganddomath 2021-07-30 19:53:46
But again, we counted each triple way more than once! Who can tell me all the different orders in which we've counted the triple (A, B, C)?

goveganddomath 2021-07-30 19:54:29
Yes, we've counted it as $(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B)$ and (C, B, A). In other words, we've counted in 6 different orders!

goveganddomath 2021-07-30 19:54:31
Is this true for every triple we've selected?

goveganddomath 2021-07-30 19:54:51
This is true for every set of three different letters: there are $3! = 6$ ways to arrange it in different orders.

goveganddomath 2021-07-30 19:54:53
So what must we do to $26 \cdot 25 \cdot 24$ to get the right number of triples, if we don't care about order?

goveganddomath 2021-07-30 19:55:19
We divide by 6. That must mean our answer is $\dfrac{26 \cdot 25 \cdot 24}{6}$.

goveganddomath 2021-07-30 19:55:27
More generally, the number of ways of choosing k different objects from a set of n elements is equal to

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}.$$

goveganddomath 2021-07-30 19:55:40
The symbol $\dbinom{n}{k}$ is known as a binomial coefficient, and it is read "n choose k." In counting, many answers depend in some form on a combination.

goveganddomath 2021-07-30 19:55:46
This can be argued in the exact same way: if we're picking k objects out of n, we have n choices for the first object, $n-1$ for the second, etc.

goveganddomath 2021-07-30 19:55:53
But at the end, we count each set of k objects $k!$ times, since $k!$ is the number of ways we can arrange k objects. That means we have to divide by $k!$.

goveganddomath 2021-07-30 19:56:01
In plain text, you can write $\dbinom{n}{k}$ as C(n,k), or simply "n choose k."

goveganddomath 2021-07-30 19:56:05
In $\LaTeX$, that's $\binom{n}{k}$.

goveganddomath 2021-07-30 19:56:23
How many line segments have both their endpoints located at the vertices of a given cube?

$\text{(A) } 12 \quad \text{(B) } 15 \quad \text{(C) } 24 \quad \text{(D) } 28 \quad \text{(E) } 56$

goveganddomath 2021-07-30 19:56:33
How can we specify a line segment, where both endpoints are vertices of the cube?

goveganddomath 2021-07-30 19:56:36
In other words, what uniquely determines each line segment?

goveganddomath 2021-07-30 19:57:16
We specify such a line segment by choosing two vertices of the cube. So, every unique pair of vertices gives us another line segment. How many ways are there to choose two vertices of the cube?

goveganddomath 2021-07-30 19:58:21
The cube has eight vertices, so there are $\binom{8}{2} = \frac{8!}{2! \cdot 6!} = \frac{8 \cdot 7}{2} = 28$ **ways to choose two vertices. The answer is (D).**

goveganddomath 2021-07-30 19:58:39
STARS AND BARS

goveganddomath 2021-07-30 19:58:52
We now present a method for counting certain kinds of distributions, which we call stars and bars. (It goes by other names too, like balls and urns.)

goveganddomath 2021-07-30 19:58:59
To illustrate the method, we look at a specific example.

goveganddomath 2021-07-30 19:59:06
Pat wants to buy four donuts from an ample supply of three types of donuts: glazed, chocolate, and powdered. How many different selections are possible?

$\text{(A) } 6 \quad \text{(B) } 9 \quad \text{(C) } 12 \quad \text{(D) } 15 \quad \text{(E) } 18$

goveganddomath 2021-07-30 19:59:15
For example, Pat can buy two glazed, one chocolate, and one powdered. It's not hard to list all the possible ways, but stars and bars make this problem a snap (and would be quite handy in a problem with bigger numbers).

goveganddomath 2021-07-30 19:59:21
To apply stars and bars, we represent each donut with a star, and we divide the groups of donuts (by type) with bars. For example, if Pat buys two glazed, one chocolate, and one powdered, then we represent this with the following row of stars and bars:

goveganddomath 2021-07-30 19:59:26
$* * | * | *$

goveganddomath 2021-07-30 19:59:33
The first, second, and third groups represent glazed, chocolate, and powdered, respectively.

goveganddomath 2021-07-30 19:59:38
What does the following row represent?

goveganddomath 2021-07-30 19:59:41
$* * * | | *$

goveganddomath 2021-07-30 20:00:18
This row represents three glazed and one powdered. There are no stars between the two dividers, so there are no chocolate donuts.

goveganddomath 2021-07-30 20:00:22
Thus, every purchase of four donuts can be represented by a row of four stars and two bars, and every row corresponds to a different purchase of four doughnuts. How many ways can we arrange four stars and two bars?

goveganddomath 2021-07-30 20:01:16
There are 6 positions available for the 4 stars and 2 bars. If we pick which of these 6 positions the 2 bars occupy, then the positions of the stars are automatically determined.

goveganddomath 2021-07-30 20:01:22

So, we can arrange the four stars and two bars in $\binom{6}{2} = \frac{6!}{2! \cdot 4!} = \frac{6 \cdot 5}{2} = 15$ ways.

goveganddomath 2021-07-30 20:01:46

Therefore, Pat can make 15 different selections. The answer is (D).

goveganddomath 2021-07-30 20:01:56

How many ordered quadruples (a, b, c, d) satisfy

$$a + b + c + d = 18,$$

where a, b, c, d are nonnegative integers?

goveganddomath 2021-07-30 20:02:02

How can we use stars and bars here? What should the stars represent and what should the bars represent?

goveganddomath 2021-07-30 20:03:08

Here, we can represent the numbers a, b, c and d by stars in a row, with the bars separating them. For example, what quadruple does the figure below represent?

goveganddomath 2021-07-30 20:03:12

$$* * * * * \mid * * \mid * * * * * * * \mid * * * *$$

goveganddomath 2021-07-30 20:03:47

This represents $a = 5, b = 2, c = 7,$ and $d = 4.$

goveganddomath 2021-07-30 20:03:48

So how many stars and bars do we need to place here?

goveganddomath 2021-07-30 20:04:33

We have 18 stars and 3 bars. So what's the answer?

goveganddomath 2021-07-30 20:05:58

The answer is $\binom{18+3}{3} = \binom{21}{3} = 1330.$

goveganddomath 2021-07-30 20:06:00

It's important to remember that it's not $\binom{18}{3}$. The top number in the binomial coefficient is the total number of slots, not just the number of stars!

goveganddomath 2021-07-30 20:06:20

Now, let's try a similar problem, with slightly different conditions:

goveganddomath 2021-07-30 20:06:22

How many ordered quadruples (a, b, c, d) satisfy

$$a + b + c + d = 18,$$

where a, b, c, d are positive integers?

goveganddomath 2021-07-30 20:06:44

Wait a minute, isn't this pretty much the same question?! What's the difference?

goveganddomath 2021-07-30 20:07:37

Oh yeah, they are asking for *positive* instead of nonnegative integers. How does that interfere with our argument?

goveganddomath 2021-07-30 20:08:39
When we do stars and bars, we need to have at least one star in each of the four spaces created by the bars!

goveganddomath 2021-07-30 20:08:43
We could do some horrific casework about whether each of the four spaces have no stars in them, but then we get overcounting and end up with the principle of inclusion and exclusion coming up, and ... yuck. Any other ideas?

goveganddomath 2021-07-30 20:10:07
There's a sneaky trick for questions like this! A positive integer is just a nonnegative integer plus 1. Let's rewrite the equation as

$$(a-1)+(b-1)+(c-1)+(d-1)=14,$$

and let's find the quadruple $(a-1,b-1,c-1,d-1)$ instead of (a,b,c,d).

goveganddomath 2021-07-30 20:10:12
How many quadruples of nonnegative integers adding to 14 are there?

goveganddomath 2021-07-30 20:11:09
Arguing just like before, there are going to be 14 stars and 3 bars, so the answer is $\binom{14+3}{3}=\binom{17}{3}=680$. So what's the final answer?

goveganddomath 2021-07-30 20:11:28
The number of positive quadruples (a,b,c,d) adding to 18 is identical to the number of nonnegative quadruples adding to 14, so the answer is 680.

goveganddomath 2021-07-30 20:11:36
Finally...

goveganddomath 2021-07-30 20:11:37
How many ordered quadruples (a,b,c,d) satisfy

$$a+b+c+d=18,$$

where a,b,c,d are positive odd integers?

goveganddomath 2021-07-30 20:11:43
Let's see if we can use our trick from before and rewrite this problem to involve (unconstrained) nonnegative integers. How can we do that?

goveganddomath 2021-07-30 20:12:22
How might we do that?

goveganddomath 2021-07-30 20:13:08
We know that a is positive and odd if and only if $\frac{a-1}{2}$ is a nonnegative integer. So we rewrite the equation as

$$\frac{a-1}{2}+\frac{b-1}{2}+\frac{c-1}{2}+\frac{d-1}{2}=7,$$

and we count the number of nonnegative quadruples $\left(\frac{a-1}{2},\frac{b-1}{2},\frac{c-1}{2},\frac{d-1}{2}\right)$.

goveganddomath 2021-07-30 20:13:13
So what answer do we get?

goveganddomath 2021-07-30 20:14:05

Arguing precisely as before, we get $\binom{7+3}{3} = \binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$. This is our final answer.

goveganddomath 2021-07-30 20:14:20
Before we move on to our next topic, let's take a quick break! Feel free to send me a math joke 🙂

goveganddomath 2021-07-30 20:14:40
(I will share with the class!)

goveganddomath 2021-07-30 20:15:14
Hahahahaha!

goveganddomath 2021-07-30 20:15:19
Ready for our next awesome topic?!

goveganddomath 2021-07-30 20:15:42
Alright, let's get started!

goveganddomath 2021-07-30 20:16:08
COMPLEMENTARY COUNTING

goveganddomath 2021-07-30 20:16:15
In some counting problems, it is easier to count the objects we don't want, rather than the objects we want. This technique is called complementary counting. We've already seen this used in an earlier problem. Here's another example:

goveganddomath 2021-07-30 20:16:21
How many four-digit positive integers have at least one digit that is a 2 or a 3?

$\text{(A) } 2439 \quad \text{(B) } 4096 \quad \text{(C) } 4903 \quad \text{(D) } 4904 \quad \text{(E) } 5416$

goveganddomath 2021-07-30 20:16:37
What makes it hard to count how many numbers there are with at least one digit that is 2 or 3?

goveganddomath 2021-07-30 20:17:19
All true! The difficult part about this problem is that there are many ways that a four-digit number can have at least one digit that is a 2 or a 3. For example, the number can be 7201, or 3587, or 2322.

goveganddomath 2021-07-30 20:17:23
The number can have a 2, or a 3, or a 2 and a 3, leading to many possible combinations. It is difficult to know where to start. So what can we try?

goveganddomath 2021-07-30 20:18:16
We can try looking at numbers that do not satisfy the given condition. What can we say about a number that does not satisfy the given condition?

goveganddomath 2021-07-30 20:18:34
As many already pointed out, it does not have a 2 or 3. In other words, none of the digits is 2 or 3.

goveganddomath 2021-07-30 20:18:40
Already this seems like an easier condition to deal with. How can we count the number of four-digit numbers where none of the digits are 2 or 3?

goveganddomath 2021-07-30 20:19:12
We look at each digit individually. What are the possible first digits?

goveganddomath 2021-07-30 20:19:53
The possible first digits are $1, 4, 5, 6, 7, 8$, and 9, for a total of 7 possibilities. (Remember, the first digit can't be 0.) What are the possible second digits?

goveganddomath 2021-07-30 20:20:37

The possible second digits are $0, 1, 4, 5, 6, 7, 8,$ and 9, for a total of 8.

goveganddomath 2021-07-30 20:20:40

The possible third and fourth digits are the same as the possible second digits. So how many four-digit numbers do not contain a 2 or 3?

goveganddomath 2021-07-30 20:21:19

The number of four-digit numbers that do not contain a 2 or 3 is $7 \cdot 8 \cdot 8 \cdot 8 = 3584$. Is this our answer?

goveganddomath 2021-07-30 20:21:46

No, this is the complement of our answer. So what must we do with this number?

goveganddomath 2021-07-30 20:22:17

We want to subtract it from the total number of four-digit numbers. How many four-digit numbers are there?

goveganddomath 2021-07-30 20:22:45

Using the same method of counting digit by digit, we find there are $9 \cdot 10 \cdot 10 \cdot 10 = 9000$ four-digit numbers. So what is the answer?

goveganddomath 2021-07-30 20:23:02

Subtracting, we get $9000 - 3584 = 5416$ numbers. The answer is (E).

goveganddomath 2021-07-30 20:23:12

CASEWORK

goveganddomath 2021-07-30 20:23:17

As much as we've tried to avoid using casework, in many counting problems it is inevitable. So let's work through a counting problem using casework.

goveganddomath 2021-07-30 20:23:24

Ten chairs are evenly spaced around a round table and numbered clockwise from 1 through 10. Five couples, each consisting of a mathematician and an artist, are to sit in the chairs with mathematicians and artists alternating, and no one is to sit either next to or directly across from their partner. How many seating arrangements are possible?

$\text{(A) } 240 \quad \text{(B) } 360 \quad \text{(C) } 480 \quad \text{(D) } 540 \quad \text{(E) } 720$

goveganddomath 2021-07-30 20:23:25

How can we get started?

goveganddomath 2021-07-30 20:24:54

When you have a complicated situation like this, it's almost always a good idea to try to see what happens when you try to actually write down an example. A diagram will be helpful for this type of arrangement problem, as well!

goveganddomath 2021-07-30 20:25:00

Let's investigate what possibilities we have when trying to seat everyone around the table one by one. We can try first putting the artists in, and then putting the mathematicians in afterwards.

goveganddomath 2021-07-30 20:25:06

How many choices do we have for where to put the first artist?

goveganddomath 2021-07-30 20:25:44

The first artist can be seated in 10 ways, in any of the ten chairs. Once we've seated the first artist, where can the remaining four artists sit?

goveganddomath 2021-07-30 20:26:33

The mathematicians and artists must alternate, so there are only four chairs the remaining four artists can sit in. How many ways are there to seat the remaining four artists?

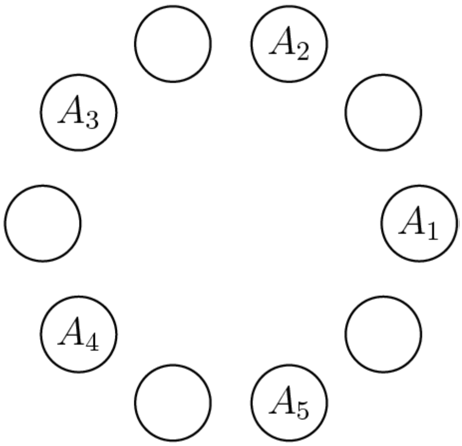
goveganddomath 2021-07-30 20:27:08

There are $4! = 24$ ways to seat the remaining four artists. (Four choices for the second artist, then three choices left for the

one after them, and so on.)

goveganddomath 2021-07-30 20:27:16
Let's now draw a picture to keep things straight.

goveganddomath 2021-07-30 20:27:19



goveganddomath 2021-07-30 20:27:24
Next, we look at the mathematicians. Assume that the couples are A_1 and M_1, A_2 and M_2, and so on.

goveganddomath 2021-07-30 20:27:28
We want to figure out what different ways we can seat all the mathematicians. Let's start with M_1, the partner of A_1. Where can M_1 sit?

goveganddomath 2021-07-30 20:28:20
No one can sit next to or directly across from their partner, so M_1 can only sit between A_2 and A_3, or A_4 and A_5.

goveganddomath 2021-07-30 20:28:23
What next?

goveganddomath 2021-07-30 20:28:52
We want to next seat the other four mathematicians. But the options for where to put them will depend on where we put M_1, so we need to split into cases based on where we put M_1.

goveganddomath 2021-07-30 20:28:55
Let's first consider the case where M_1 sits between A_2 and A_3.

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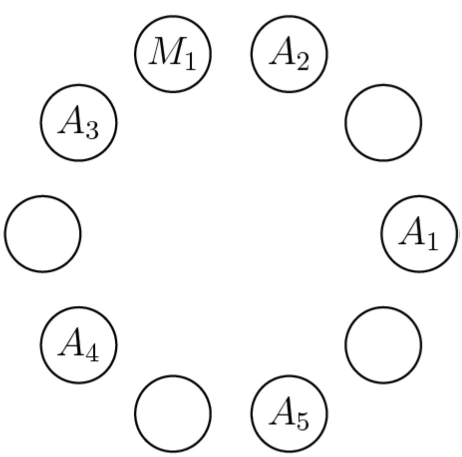
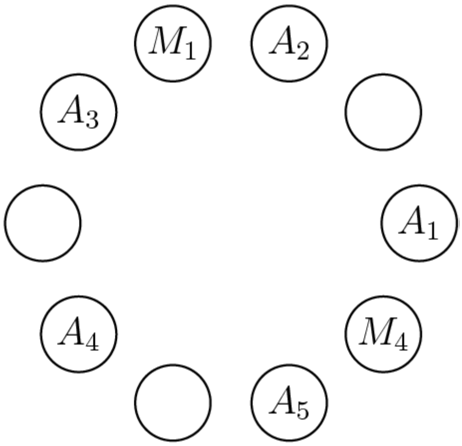


goveganddomath 2021-07-30 20:29:28
Now what?

goveganddomath 2021-07-30 20:29:45
Is there anything we can say that would help us fill in the diagram?

goveganddomath 2021-07-30 20:30:52
The mathematician M_4 can only sit between A_1 and A_5. (They have only one option left, since M_1 took their other option!)

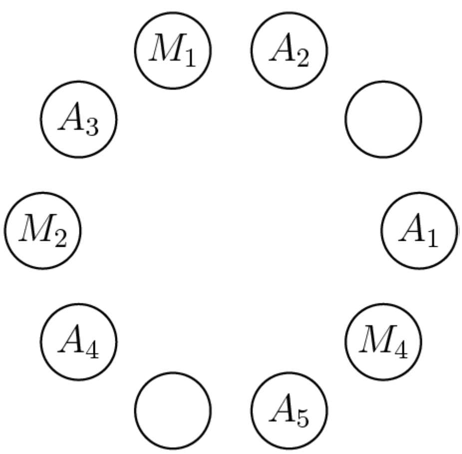
goveganddomath 2021-07-30 20:30:56



goveganddomath 2021-07-30 20:30:58
Now what can we say about the diagram?

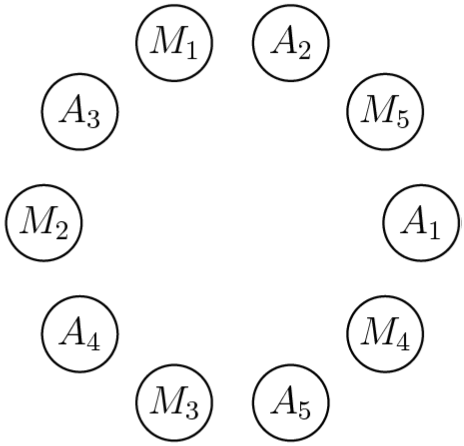
goveganddomath 2021-07-30 20:31:43
Now M_2 also has only one option! They can only sit between A_3 and A_4.

goveganddomath 2021-07-30 20:31:45



goveganddomath 2021-07-30 20:31:50
Then M_5 must sit between A_1 and A_2, and then M_3 must sit between A_4 and A_5.

goveganddomath 2021-07-30 20:31:52



goveganddomath 2021-07-30 20:31:59

So after M_1 sits between A_2 and A_3, there is only one way to seat the rest of the mathematicians.

goveganddomath 2021-07-30 20:32:06

Let's consider our other case. What if M_1 sits between A_4 and A_5? How many ways are there to seat the rest of the mathematicians then?

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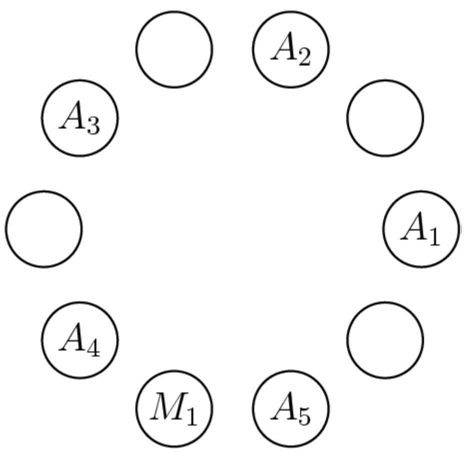
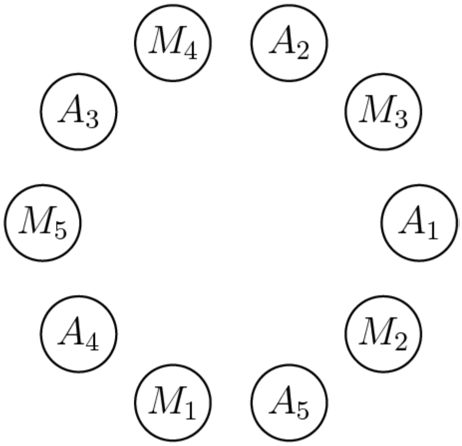


goveganddomath 2021-07-30 20:33:02

This case actually works in the same way as the previous case, by symmetry. So again there is only one way to seat the rest of the mathematicians, which we can work out to be the following.

goveganddomath 2021-07-30 20:33:04



goveganddomath 2021-07-30 20:33:09
Let's recap.

goveganddomath 2021-07-30 20:33:15
There are 10 ways to seat the first artist, then $4! = 24$ ways to seat the remaining artists.

goveganddomath 2021-07-30 20:33:24
Then there are 2 ways to seat the first mathematician, and only one way to seat the remaining mathematicians. So how many seating arrangements are there?

goveganddomath 2021-07-30 20:34:05
There are $10 \cdot 24 \cdot 2 = 480$ different seating arrangements. The answer is (C).

goveganddomath 2021-07-30 20:34:08
When dividing into cases, make sure that your cases are exhaustive (in other words, your cases cover all possibilities), and that you work all your cases through to the end. Sometimes, it may be necessary to divide your cases into sub-cases. The key is to be diligent, thorough, and organized!

goveganddomath 2021-07-30 20:34:21
Before we move on to the next problem, any questions? We are happy to answer at any time!

goveganddomath 2021-07-30 20:34:44
Alright, let's continue!

goveganddomath 2021-07-30 20:34:45
A student must choose a program of four courses from a menu of courses consisting of English, Algebra, Geometry, History, Art, and Latin. This program must contain English and at least one mathematics course. In how many ways can this program be chosen?

$\text{(A)}\ 6 \qquad \text{(B)}\ 8 \qquad \text{(C)}\ 9 \qquad \text{(D)}\ 12 \qquad \text{(E)}\ 16$

goveganddomath 2021-07-30 20:34:47
How can we solve this problem?

goveganddomath 2021-07-30 20:35:52
We can use complementary counting.

goveganddomath 2021-07-30 20:36:01
Since there must be an English class, we will add that to our list of classes, and that leaves 3 remaining spots for the other classes. We are also told that there needs to be at least one math class. This calls for complementary counting.

goveganddomath 2021-07-30 20:36:05
What's the total number of ways of choosing 3 classes out of the 5?

goveganddomath 2021-07-30 20:36:48

The total number of ways of choosing 3 classes out of the 5 is $\binom{5}{3}$.

goveganddomath 2021-07-30 20:36:50

What's the total number of ways of choosing only non-mathematical classes?

goveganddomath 2021-07-30 20:37:34

The total number of ways of choosing only non-mathematical classes is $\binom{3}{3}$.

goveganddomath 2021-07-30 20:37:46

So, how many total combinations of classes are there that satisfy the conditions of the problem?

goveganddomath 2021-07-30 20:38:16

Therefore the amount of ways in which you can pick classes with at least one math class is $\binom{5}{3}-\binom{3}{3}=10-1=9$ ways. The answer is (C).

goveganddomath 2021-07-30 20:38:36

In problems like this where the numbers are small, often casework is also fine to use. But if the numbers had been larger, the complementary counting approach would be a lot easier.

goveganddomath 2021-07-30 20:38:41

THINK ABOUT IT!

goveganddomath 2021-07-30 20:38:50

The last technique we'll discuss is called "think about it." When solving a counting problem (or any problem in mathematics), it is easy to become conditioned to look for the right formula or technique that seems to fit the situation. However, in some counting problems, the best way to solve the problem is to put your pencil down and simply think about what it is really asking for.

goveganddomath 2021-07-30 20:38:56

There are 100 players in a singles tennis tournament. The tournament is single elimination, meaning that a player who loses a match is eliminated. In the first round, the strongest 28 players are given a bye, and the remaining 72 players are paired off to play. After each round, the remaining players play in the next round. The tournament continues until only one player remains unbeaten. The total number of matches played is

$$\text{(A) a prime number} \quad \text{(B) divisible by 2} \quad \text{(C) divisible by 5}$$

$$\text{(D) divisible by 7} \quad \text{(E) divisible by 11}$$

goveganddomath 2021-07-30 20:39:10

In case you are unfamiliar with sports terminology (like me), a “bye” essentially means the player does not play in that particular round and they advance automatically to the next round.

goveganddomath 2021-07-30 20:39:14

So in this problem, that means after the 72 players play their matches, the 28 who received a “bye” then rejoin the pool of players and play as normal according to the problem.

goveganddomath 2021-07-30 20:39:32

How can we solve this problem?

goveganddomath 2021-07-30 20:40:23

We can think about how the tournament proceeds, and the number of players left in each round.

goveganddomath 2021-07-30 20:40:25

We are told that the top 28 players get a bye, meaning they automatically get to the second round. How many matches do the

72 remaining players play in the first round?

govegandomath 2021-07-30 20:40:50

The remaining players play $\frac{72}{2} = 36$ matches. So how many players go into the second round, in total?

govegandomath 2021-07-30 20:41:09

A player is eliminated in each of the 36 matches, so $36 + 28 = 64$ players advance to the second round.

govegandomath 2021-07-30 20:41:13

We can continue to analyze the tournament round by round, counting the number of matches played, and the number of players who advance. (Analyzing the problem this way, we'd eventually find that there are $36 + 32 + 16 + 8 + 4 + 2 + 1 = 36 + 63 = 99$ matches played in all). But there is a much faster way to see the answer to this problem.

govegandomath 2021-07-30 20:41:56

Exactly one player gets eliminated in each match. So there must be 99 matches!

govegandomath 2021-07-30 20:42:01

Of the answer choices, only (E) is true. That's it!

govegandomath 2021-07-30 20:42:13

Perhaps surprisingly, the fact that the top 28 players get a bye is completely irrelevant. This condition affects the number of rounds in the tournament, but it has no effect on the number of matches that must be played.

govegandomath 2021-07-30 20:42:44

Now we will see how geometric counting problems can sometimes be solved in a similar way!

govegandomath 2021-07-30 20:42:45

Eight points are chosen on a circle, and chords are drawn connecting every pair of points. No three chords intersect in a single point inside the circle. How many triangles with all three vertices in the interior of the circle are created?

$\text{(A) } 28 \quad \text{(B) } 56 \quad \text{(C) } 70 \quad \text{(D) } 84 \quad \text{(E) } 140$

govegandomath 2021-07-30 20:42:48

How can we start?

govegandomath 2021-07-30 20:43:29

We can start by drawing a diagram.

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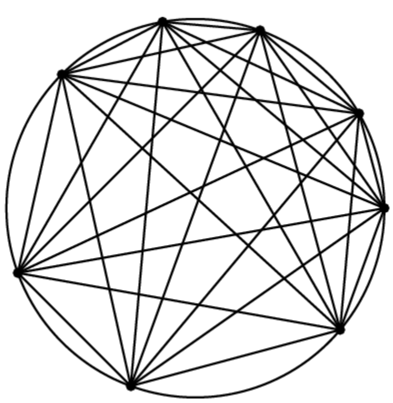


govegandomath 2021-07-30 20:43:39

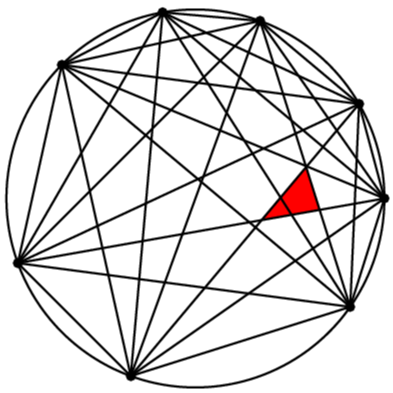
We get ... a messy diagram. Trying to count all the triangles based on the diagram is going to be difficult, if not impossible. So

what can we do?

goveganddomath 2021-07-30 20:44:46

We can now look at one individual triangle, which is determined by 3 chords, and see if we can determine anything interesting.

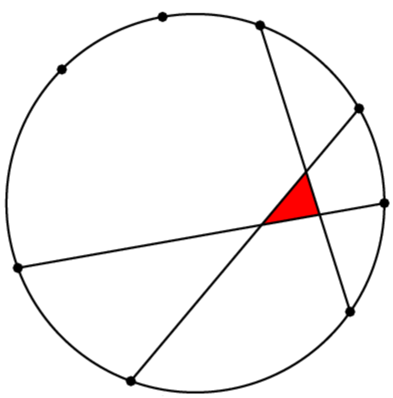
goveganddomath 2021-07-30 20:44:50



goveganddomath 2021-07-30 20:45:08

Since we are trying to count the number of triangles, we should look at how it is created in the first place, using 3 chords:

goveganddomath 2021-07-30 20:45:12



goveganddomath 2021-07-30 20:45:18

However, we should be more specific. The following triangle is also created by three chords, but does not satisfy the given conditions, since not all of its vertices are inside the circle.

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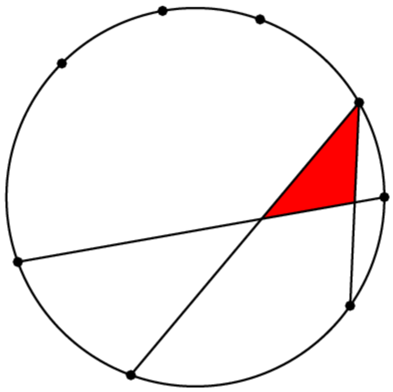


goveganddomath 2021-07-30 20:45:27
What's the difference?

goveganddomath 2021-07-30 20:46:37
The three chords are not specified by six *distinct* points on the circle in the latter case; as this shows, we need them to be. So how can we count the number of triangles?

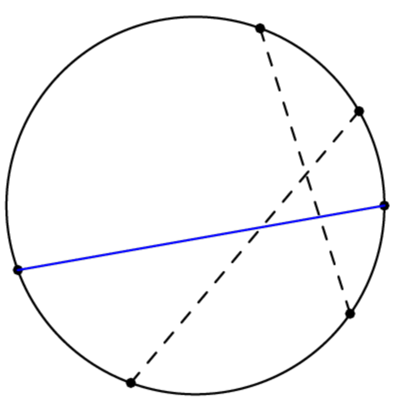
goveganddomath 2021-07-30 20:47:30
Well, every triangle is created by choosing six points on the circle.

goveganddomath 2021-07-30 20:47:32
But wait. Can a collection of six points on the circle give us more than one triangle, by connecting them up in different ways?

goveganddomath 2021-07-30 20:49:28
Must each chord forming a side of the triangle be crossed by the other chords? What does this tell us?

goveganddomath 2021-07-30 20:50:45
Exactly; so, each of the other two chords must have two endpoints on opposite sides of our original side.

goveganddomath 2021-07-30 20:50:48



goveganddomath 2021-07-30 20:50:51
So each of our six chosen points must be connected to the opposite point among those six.

goveganddomath 2021-07-30 20:50:55
Do we always get a triangle if we connect the opposite points like this?

goveganddomath 2021-07-30 20:51:46
Yes, we do, since the chords we draw will have to intersect inside the circle, and since no three chords intersect in a single point.

goveganddomath 2021-07-30 20:51:51
So how many ways are there to choose six points on the circle?

goveganddomath 2021-07-30 20:52:33
The number of ways to choose six points on the circle is $\binom{8}{6} = \frac{8!}{6! \cdot 2!} = \frac{8 \cdot 7}{2} = 28$**, so the number of triangles is also** 28**. The answer is (A).**

goveganddomath 2021-07-30 20:52:40
Alright, ready for one final problem?!

goveganddomath 2021-07-30 20:52:54
Walking down Jane Street, Ralph passed four houses in a row, each painted a different color. He passed the orange house before the red house, and he passed the blue house before the yellow house. The blue house was not next to the yellow house. How many orderings of the colored houses are possible?

$(\mathrm{A})\ 2 \quad (\mathrm{B})\ 3 \quad (\mathrm{C})\ 4 \quad (\mathrm{D})\ 5 \quad (\mathrm{E})\ 6$

goveganddomath 2021-07-30 20:52:57
How can we solve this problem?

goveganddomath 2021-07-30 20:53:44
The numbers in this problem are pretty small. We can probably just find all the possibilities with some simple casework. Which house should we try placing first?

goveganddomath 2021-07-30 20:54:20
Usually it's best to deal with the most restricted parts first. The blue and yellow houses each have two constraints, so we should probably place one of them first. Let's start with the yellow house.

goveganddomath 2021-07-30 20:54:23
What are the possible locations for the yellow house (Y)**?**

goveganddomath 2021-07-30 20:54:56
The only possible locations for the yellow house (Y) **are the** 3**rd house and the last house. (It can't be the first house because it comes after blue. It can't be the second house because it is also not allowed to be adjacent to blue.)**

goveganddomath 2021-07-30 20:54:59
Case 1: Y **is the** 3**rd house.**

goveganddomath 2021-07-30 20:55:07
How many possible arrangements are there in this case?

goveganddomath 2021-07-30 20:55:39
There is only one! What is that arrangement?

goveganddomath 2021-07-30 20:56:00
The only possible arrangement is $B - O - Y - R$**.**

goveganddomath 2021-07-30 20:56:03
The blue house needs to be the 1**st house, since it is ahead of and not adjacent to the yellow house. Then, the orange house needs to come before the red house.**

goveganddomath 2021-07-30 20:56:10
Case 2: Y **is the last house.**

goveganddomath 2021-07-30 20:56:16
How many possible arrangements are there in this case?

goveganddomath 2021-07-30 20:56:36
There are two possible ways. What are they?

goveganddomath 2021-07-30 20:57:21
$B-O-R-Y$ **and** $O-B-R-Y$.

goveganddomath 2021-07-30 20:57:35
Again, we start by placing the blue house either 1st or 2nd, and then the orange and red houses have only one possible placement in each case.

goveganddomath 2021-07-30 20:57:37
So what's our answer?

goveganddomath 2021-07-30 20:58:00
Our answer is 3, i.e. (B).

goveganddomath 2021-07-30 20:58:06
Alright, we did it! Now time for the...

goveganddomath 2021-07-30 20:58:38
In this case, the "smart solution" was to just realize how few cases there are, and how simple this makes it. 🙂

goveganddomath 2021-07-30 20:58:43
SUMMARY

goveganddomath 2021-07-30 20:58:49
Today, we saw many concepts for solving problems in counting. You may be wondering, when we face a problem in counting, how do we know which concept is the right one to apply?

goveganddomath 2021-07-30 20:58:53
Do not start, for example, by trying to guess the right binomial coefficient that seems to fit. The right way to start solving a counting problem is to consider what it is you are trying to count. Is there a simple way of describing the objects you want to count? Is there a simple process that generates the objects? Only after you find the right approach should you start working with the actual numbers.

goveganddomath 2021-07-30 20:59:03
The most important thing is to keep your approach as simple as possible. Start by looking for a way to use the product principle or combinations. If these do not work, you can always try complementary counting and casework. And last but not least... just think about it.

goveganddomath 2021-07-30 20:59:13
That's it for today's class. Any questions?

goveganddomath 2021-07-30 20:59:37
Alright, great work everyone! Have a great weekend!