2021 AMC 10A Problems/Problem 21

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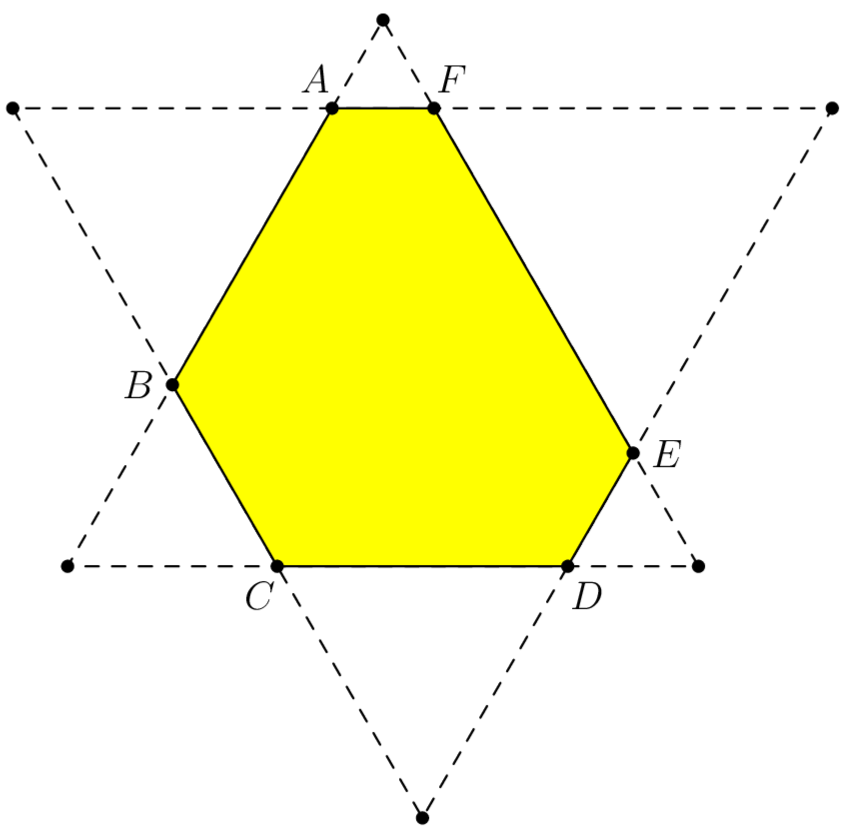
Problem

Let $ABCDEF$ be an equiangular hexagon. The lines $AB, CD,$ and EF determine a triangle with area $192\sqrt{3}$, and the lines $BC, DE,$ and FA determine a triangle with area $324\sqrt{3}$. The perimeter of hexagon $ABCDEF$ can be expressed as $m + n\sqrt{p}$, where $m, n,$ and p are positive integers and p is not divisible by the square of any prime. What is $m + n + p$?

$\textbf{(A)}\ 47 \qquad \textbf{(B)}\ 52 \qquad \textbf{(C)}\ 55 \qquad \textbf{(D)}\ 58 \qquad \textbf{(E)}\ 63$

Diagram



~MRENTHUSIASM

Solution

Let $P, Q, R, X, Y,$ and Z be the intersections $\overleftrightarrow{AB}\cap\overleftrightarrow{CD}, \overleftrightarrow{CD}\cap\overleftrightarrow{EF}, \overleftrightarrow{EF}\cap\overleftrightarrow{AB}, \overleftrightarrow{BC}\cap\overleftrightarrow{DE}, \overleftrightarrow{DE}\cap\overleftrightarrow{FA},$ and $\overleftrightarrow{FA}\cap\overleftrightarrow{BC},$ respectively.

The sum of the interior angles of any hexagon is $720^\circ.$ Since hexagon $ABCDEF$ is equiangular, each of its interior angles is $720^\circ\div6=120^\circ.$ By angle chasing, we conclude that the interior angles of $\triangle PBC, \triangle QDE, \triangle RFA, \triangle XCD, \triangle YEF,$ and $\triangle ZAB$ are all $60^\circ.$ Therefore, these triangles are all equilateral triangles, from which $\triangle PQR$ and $\triangle XYZ$ are both equilateral triangles.

We are given that

$$\begin{aligned}
[PQR] &= \frac{\sqrt{3}}{4}\cdot PQ^2 = 192\sqrt{3}, \\
[XYZ] &= \frac{\sqrt{3}}{4}\cdot YZ^2 = 324\sqrt{3},
\end{aligned}$$

so we get $PQ=16\sqrt{3}$ and $YZ=36,$ respectively.

By equilateral triangles and segment addition, we find the perimeter of hexagon $ABCDEF:$

$$\begin{aligned}
AB+BC+CD+DE+EF+FA &= AZ+PC+CD+DQ+YF+FA \\
&= (YF+FA+AZ)+(PC+CD+DQ) \\
&= YZ+PQ \\
&= 36+16\sqrt{3}.
\end{aligned}$$

Finally, the answer is $36+16+3=\boxed{\textbf{(C) } 55}.$

~sugar_rush (Fundamental Logic)

~MRENTHUSIASM (Reconstruction)

Video Solution by OmegaLearn (Angle Chasing and Equilateral Triangles)

https://youtu.be/ptBwDcmDaLA

~ pi_is_3.14

Video Solution by TheBeautyofMath

https://youtu.be/8qcbZ8c7fHg

~IceMatrix

Video Solution by MRENTHUSIASM (English & Chinese)

https://www.youtube.com/watch?v=0n8EAu2VAiM

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See Also

2021 AMC 10A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
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