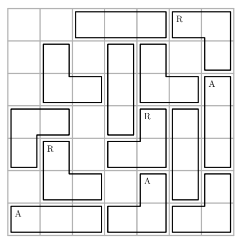
Problem 1

Question

Fill each cell with an integer from $1-7$ so each number appears exactly once in each row and column. In each "cage" of three cells, the three numbers must be valid lengths for the sides of a non-degenerate triangle. Additionally, if a cage has an "A", the triangle must be acute, and if the cage has an "R", the triangle must be right.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

3	1	7	6	2	4	5
4	7	2	5	6	1	3
7	3	5	2	1	6	4
1	2	6	4	5	3	7
2	5	1	3	4	7	6
6	4	3	1	7	5	2
5	6	4	7	3	2	1

Problem 2

Question

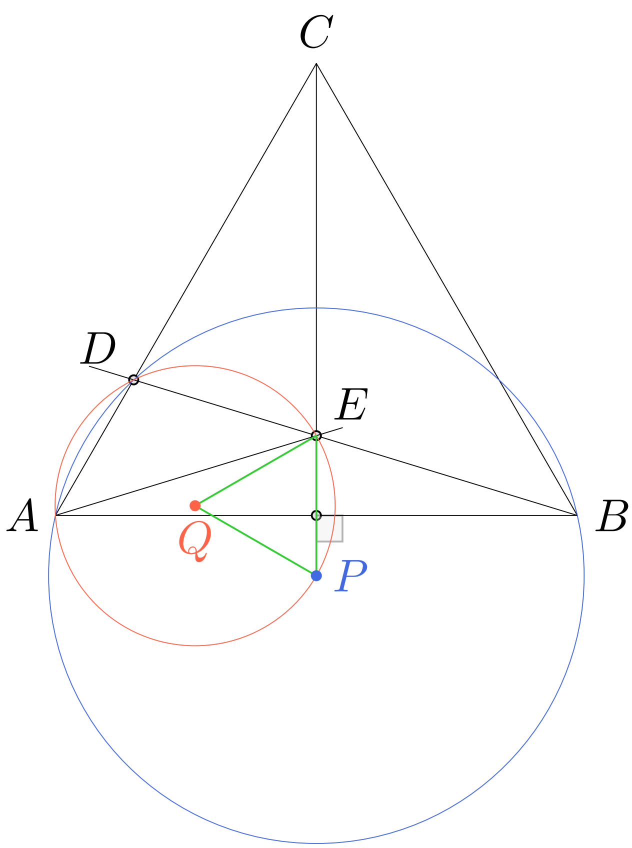
> In how many ways can a 3×3 grid be filled with integers from 1 to 12 such that all three of the following conditions are satisfied: (a) both 1 and 2 appear in the grid, (b) the grid contains at most 8 distinct values, and (c) the sums of the numbers in each row, each column, and both main diagonals are all the same? Rotations and reflections are considered the same.

TO DO

Problem 3

Question

> $\triangle ABC$ is an equilateral triangle. D is a point on AC, and E is a point on BD. Let P and Q be the circumcenters of $\triangle ABD$ and $\triangle AED$, respectively. Prove that $\triangle EPQ$ is an equilateral triangle if and only if $AB \perp CE$.



Problem 4

Question

> Let $x_1 < x_2 < \ldots < x_n$ (with $n \geq 2$) and let S be the set of all the x_i. Let T be a randomly chosen subset of S. What is the expected value of the indexed alternating sum of T? Express your answer in terms of the x_i.
> **Note:** We define the indexed alternating sum of T as
>
> $$\sum_{i=1}^{|T|}(-1)^{i+1}(i)T[i],$$
>
> where $T[i]$is the ith element of T when listed in increasing order. For example, if $T = \{1, 3, 5\}$, then the indexed alternating sum of T is
>
> $$1 \cdot 1 - 2 \cdot 3 + 3 \cdot 5 = 10$$
>
> Alternating sums of empty sets are defined to be 0.

$$\begin{aligned}\mathbb{E}[\mathcal{I}(S_n)] = \sum_{T\subseteq S_n}\mathcal{I}(T) &= \sum_{\substack{T\subseteq S_{n-1}\\ x_n\notin T}}\mathcal{I}(T) \ + \sum_{\substack{T\subseteq S_n\\ x_n\in T}}\mathcal{I}(T)\\ &= \ \mathbb{E}[\mathcal{I}(S_{n-1})] \ + \sum_{\substack{T\subseteq S_n\\ x_n\in T}}(-1)^{n+1}(n) - \mathcal{I}(T\setminus\{x_n\}))\end{aligned}$$

$$\begin{aligned}\sum_{\substack{T\subseteq S_n\\ x_n\in T}}(-1)^{n+1}(n) - \mathcal{I}(T\setminus\{x_n\})) &= \sum_{T\subseteq S_{n-1}}(-1)^{n+1}(n) - \mathcal{I}(T))\\ &= \ (-1)^{n+1}n2^{n-1} - \sum_{T\subseteq S_{n-1}}\mathcal{I}(T))\\ &= \ (-2)^{n-1}n - \mathbb{E}[\mathcal{I}(S_{n-1})]\end{aligned}$$

$$\mathbb{E}[\mathcal{I}(S_n)] = (-2)^{n-1}n$$

Problem 5

Question

> Prove that there is no polynomial $P(x)$ with integer coefficients such that
>
> $$P(\sqrt[3]{5}+\sqrt[3]{25}) = 2\sqrt[3]{5}+3\sqrt[3]{25}$$

The proof is adapted from [1]. The proof is by contradiction. We suppose a polynomial $P(x)$ exists and derive a contradiction.

Let α denote the irrational number $\alpha = \sqrt[3]{5}+\sqrt[3]{25}$, $\alpha \in \mathbb{R}/\mathbb{Q}$. Let β denote the irrational number $\beta = 2\sqrt[3]{5}+3\sqrt[3]{25}$, $\beta \in \mathbb{R}/\mathbb{Q}$. The problem is to prove that there is no polynomial $P(x)$ with integer coefficients such that $P(\alpha) = \beta$.

Proof: Suppose a polynomial $P(x)$ exists with integer coefficients and such that $P(\alpha) = \beta$. There exist polynomials $Q(x)$, $R(x)$ and $w(x)$ with integer coefficients such that

$$P(x) = Q(x)R(x) + w(x)$$

where $Q(x)$ satisfies $Q(\alpha) = 0$ (Lemma 3) and where $w(x)$ is a polynomial of degree either 1 or 2 such that $w(\alpha) = \beta$ (Lemma 2), implying

$$P(\alpha) = Q(\alpha)R(\alpha) + w(\alpha) = w(\alpha) = \beta$$

Thus, if $P(x)$ exists, $P(\alpha) = \beta$ implies $w(\alpha) = \beta$, a contradiction with lemma 1 and lemma 2, since the coefficients of $w(x)$ are in $\mathbb{Q}$ but not in $\mathbb{N}$. Conclusion: $P(x)$ does not exist. $\square$

Lemma 1: *There exists no polynomial $w(x)$ of degree 1, with integer coefficients, such that $w(\sqrt[3]{5}+\sqrt[3]{25}) = 2\sqrt[3]{5}+3\sqrt[3]{25}$.*

Lemma 2: *There exists exactly one polynomial $w(x)$ of degree 2, with rational coefficients, such that $w(\sqrt[3]{5}+\sqrt[3]{25}) = 2\sqrt[3]{5}+3\sqrt[3]{25}$.*

Lemma 3: *There exists one polynomial $Q(x)$ of degree 3 with integer coefficients such that $Q(\alpha) = 0$.*

Proof of Lemma 1: Proof by contradiction. Suppose a polynomial $w(x) = ax + b$ exists, $a, b \in \mathbb{N}$. Since $\alpha \in \mathbb{R}/\mathbb{Q}$, it must be that $a \neq 0$ and $a \neq 1$. Moreover $w(\alpha) = \beta$ implies

$$\begin{aligned}
&a(\sqrt[3]{5} + \sqrt[3]{25}) + b = 2\sqrt[3]{5} + 3\sqrt[3]{25} \\
&\implies (a-2)\sqrt[3]{5} + (a-3)\sqrt[3]{25} \in \mathbb{N} \subset \mathbb{Q} \\
&\implies \big((a-2)\sqrt[3]{5} + (a-3)\sqrt[3]{25}\big)^2 \in \mathbb{Q} \\
&\implies (a-2)^2\sqrt[3]{25} + (a-3)^2\sqrt[3]{25^2} + 2(a-2)(a-3)\sqrt[3]{5^3} \in \mathbb{Q} \\
&\implies (a-2)^2\sqrt[3]{25} + 5(a-3)^2\sqrt[3]{5} \in \mathbb{Q}
\end{aligned}$$

A weighted average of the second and last expressions, with rational weights, is also in $\mathbb{Q}$. Select weights so that the $\sqrt[3]{5}$ term is eliminated:

$$\begin{aligned}
&5(a-3)^2 \times \big((a-2)\sqrt[3]{5} + (a-3)\sqrt[3]{25}\big) - (a-2) \times \big((a-2)^2\sqrt[3]{25} + 5(a-3)^2\sqrt[3]{5}\big) \in \mathbb{Q} \\
&\implies \big(5(a-3)^2(a-3) - (a-2)^3\big)\sqrt[3]{25} \in \mathbb{Q} \\
&\implies \sqrt[3]{25} \in \mathbb{Q}
\end{aligned}$$

where we have used the result that products, sums, and powers of expressions like $(a-2)$ and $(a-3)$ are in $\mathbb{Q}$ if $a \in \mathbb{Q}$, under the assumption that w exists. Since $\sqrt[3]{25} \notin \mathbb{Q}$, we have reached a contradiction. $\square$

Proof of Lemma 2: Construct the polynomial $w(x) = ax^2 + bx + c$ by equating the coefficients derived by developing $w(\alpha) = \beta$:

$$a(\sqrt[3]{5} + \sqrt[3]{25})^2 + b(\sqrt[3]{5} + \sqrt[3]{25}) + c = 2\sqrt[3]{5} + 3\sqrt[3]{25}$$

$$a\big(\sqrt[3]{25} + 10 + 5\sqrt[3]{5}\big) + b(\sqrt[3]{5} + \sqrt[3]{25}) + c = 2\sqrt[3]{5} + 3\sqrt[3]{25}$$

$$\implies \begin{aligned} a + b &= 3 \\ 5a + b &= 2 \\ 10a + c &= 0 \end{aligned} \implies a = -\frac{1}{4}, b = \frac{13}{4}, c = \frac{5}{2}.$$

The desired degree-2 polynomial $w(x)$ is: $w(x) = -(1/4)x^2 + (13/4)x + 5/24$. No other coefficients satisfy $w(\alpha) = \beta$. $\square$

Proof of Lemma 3: The polynomial $Q(x)$ is constructed from its root α:

$$\begin{aligned}
\alpha^3 = \big(\sqrt[3]{5} + \sqrt[3]{25}\big)^3 &= 5 + 3(\sqrt[3]{5})^2\sqrt[3]{25} + 3\sqrt[3]{5}(\sqrt[3]{25})^2 + 25 \\
&= 30 + 15(\sqrt[3]{5} + \sqrt[3]{25}) = 30 + 15\alpha
\end{aligned}$$

Thus, $Q(x) = x^3 - 15x - 30$ satisfies $Q(\alpha) = 0$. $\square$

Student: James Toche
Username: jjlotoche
ID#: 44857

Acknowledgments

I acknowledge help with ...

Problem 1

- this

Problem 2

- this

Problem 3

- this

Problem 4

- this

Problem 5

- Without [1], I would not have known where to start.

References

References

[1] *The 23rd Annual Vojtěch Jarník International Mathematical Competition*. Category II. Ostrava, Apr. 2013.