Sumac-2025

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1. Find all positive integers x and y such that $x^2 - y^2 = 2025$.

$$(x+y)(x-y) = 3^4 5^2$$

The factors of 2025 are:

$$1, 3, 5, 9, 15, 25, 27, 45, 75, 81, 135, 225, 405, 675, 2025.$$

Enumerating the possibilities yields 7 cases:

$x+y$	$x-y$	(x,y)
2025	1	$(1013, 1012)$
675	3	$(339, 336)$
405	5	$(205, 200)$
225	9	$(117, 108)$
135	15	$(75, 60)$
81	25	$(53, 28)$
75	27	$(51, 24)$

2. Form a set of positive integers S by seeding S with two positive integers a and b such that $a \neq b$. Then if x and y are in S where $x \neq y$, and g is the greatest common divisor of x and y, include z in S where

$$z = \frac{x}{g}$$

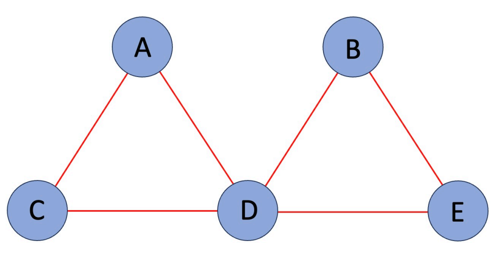
For example if we start with $a = 6$ and $b = 10$, then 8 is also in S since $\gcd(6, 10) = 2$, where $\gcd(6, 10)$ is the greatest common divisor of 4 and 6, and

$$8 = 3 + 5 = \frac{6}{2} + \frac{10}{2}$$

(a) Show that with seeds 6 and 10, the set S is infinite.

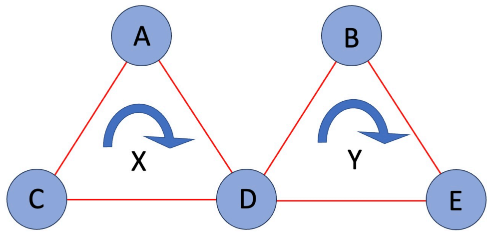
(b) Is it possible to seed S with distinct positive integers a and b such that S is a finite set?

3. Consider the following puzzle. Five coins labeled A, B, C, D, and E are arranged around two triangles as follows:

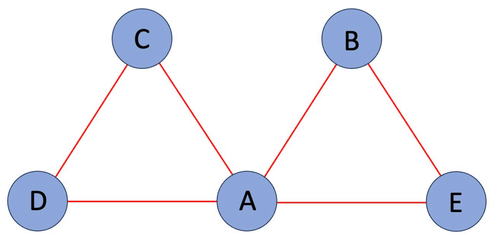


There are two allowable moves that can be applied repeatedly in any order:

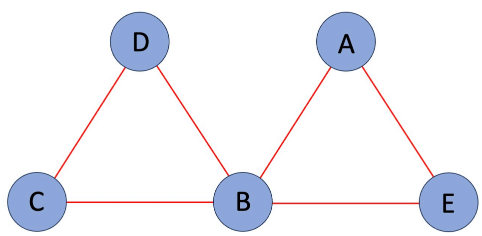
- Move X: you can rotate the three coins on the triangle on the left 120° clockwise around the center of the triangle on the left.
- Move Y: you can rotate the three coins on the triangle on the right 120° clockwise around the center of the triangle on the right.



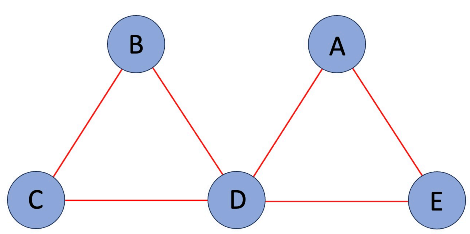
For example, one application of move X from the starting position results in Then one application of move Y to the above results in



For each of the following, show that the given configuration can be obtained with a combination of moves X and Y, or prove no combination of moves X and Y lead to the given configuration:

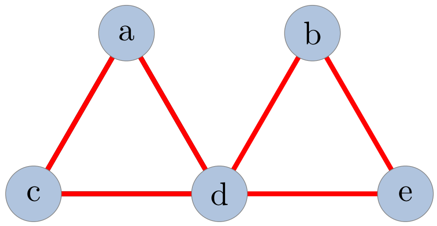


(a)

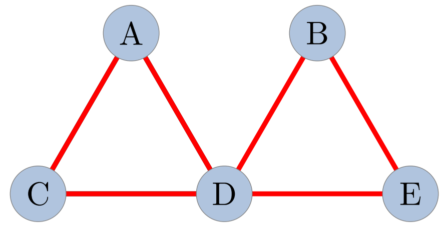


(b)

Notation:



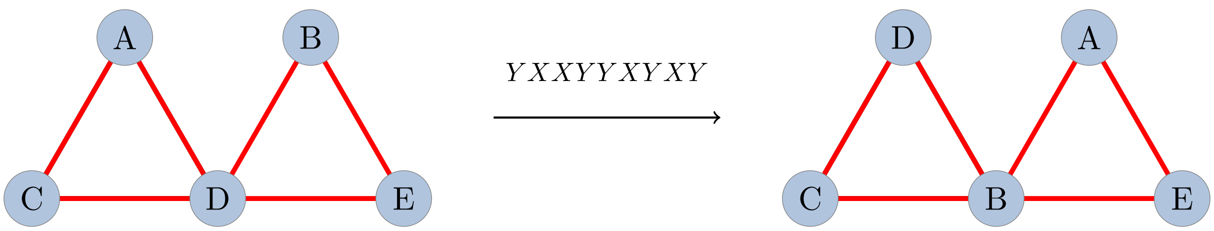
Absolute Node Positions



Starting Position

Let a, b, c, d, e denote the vertices associated with the starting position, so that we can identify the position of a coin as a label/position pair. For instance, move X results in $(A, a) \to (A, d)$, $(D, d) \to (D, c)$, $(C, c) \to (C, a)$.

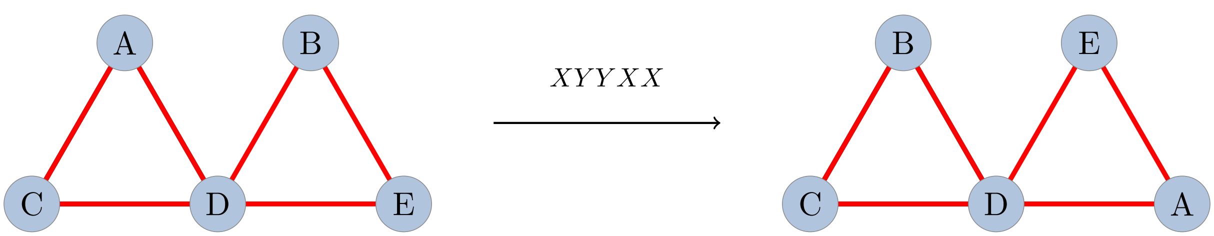
(a) Position (a) may be reached by the following sequence of moves: $YXXYYXYXY$.



The effect of the moves is summarized in the table:

coin \ rot		Y		X		X		Y		Y		X		Y		X		Y	
A	a	$\to$	a	$\to$	d	$\to$	c	$\to$	c	$\to$	c	$\to$	a	$\to$	a	$\to$	d	$\to$	b
B	b	$\to$	e	$\to$	e	$\to$	e	$\to$	d	$\to$	b	$\to$	b	$\to$	e	$\to$	e	$\to$	d
C	c	$\to$	c	$\to$	a	$\to$	d	$\to$	b	$\to$	e	$\to$	e	$\to$	d	$\to$	c	$\to$	c
D	d	$\to$	b	$\to$	b	$\to$	b	$\to$	e	$\to$	d	$\to$	c	$\to$	c	$\to$	a	$\to$	a
E	e	$\to$	d	$\to$	c	$\to$	a	$\to$	a	$\to$	a	$\to$	d	$\to$	b	$\to$	b	$\to$	e

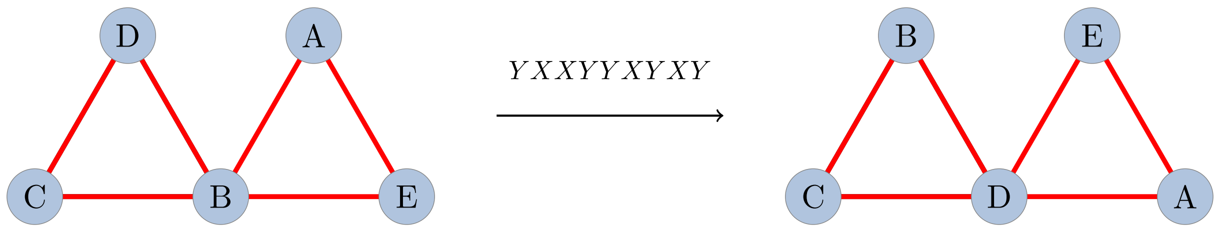
(b) Position (b) cannot be reached from the starting position. It may be approached by the sequence $XYYXX$, where only E and A are out of position. We then show that no adjacent vertices such as E and A can be exchanged without causing other vertices to be displaced, completing the proof.



The effect of the moves is summarized in the table:

coin \ rot		X		Y		Y		X		X	
A	e	$\to$	d	$\to$	c	$\to$	c	$\to$	c	$\to$	b
B	a	$\to$	a	$\to$	d	$\to$	b	$\to$	e	$\to$	d
C	c	$\to$	c	$\to$	a	$\to$	a	$\to$	a	$\to$	c
D	d	$\to$	b	$\to$	b	$\to$	e	$\to$	d	$\to$	a
E	b	$\to$	e	$\to$	e	$\to$	d	$\to$	b	$\to$	e

Any attempt to swap the A and E coins forces a swap elsewhere, e.g. a swap of B and D.



The effect of the moves is summarized in the table:

coin \ rot		Y		X		Y		Y		X		X		Y	
A	e	$\to$	d	$\to$	c	$\to$	c	$\to$	c	$\to$	a	$\to$	d	$\to$	b
B	a	$\to$	a	$\to$	d	$\to$	b	$\to$	e	$\to$	e	$\to$	e	$\to$	d
C	c	$\to$	c	$\to$	a	$\to$	a	$\to$	a	$\to$	d	$\to$	c	$\to$	c
D	d	$\to$	b	$\to$	b	$\to$	e	$\to$	d	$\to$	c	$\to$	a	$\to$	a
E	b	$\to$	e	$\to$	e	$\to$	d	$\to$	b	$\to$	b	$\to$	b	$\to$	e

4. Every positive integer N can be represented in base 2 by $(N)_2 = b_n b_{n-1} \ldots b_1 b_0$, where

$$N = b_0 + b_1 \cdot 2 + \ldots + b_{n-1} \cdot 2^{n-1} + b_n \cdot 2^n$$

such that for each $i = 0, \ldots, n$, $b_i = 0$ or $b_i = 1$, and $b_n \neq 0$. In this problem, we consider writing numbers in an alternative form of base 2 where $(N)_b^{\text{alt}} = b_n b_{n-1} \ldots b_1 b_0$ if

$$N = b_0 + b_1 \cdot (2^0 + 1) + \ldots + b_{n-1} \cdot (2^{n-2} + 1) + b_n \cdot (2^{n-1} + 1)$$

such that for each $i = 0, \ldots, n$, $b_i = 0$ or $b_i = 1$ and $b_n \neq 0$. Noting

- $2^0 + 1 = 2$
- $2^1 + 1 = 3$
- $2^2 + 1 = 5$
- $2^3 + 1 = 9$
- $2^4 + 1 = 17$

we get

$$(12)_2^{\text{alt}} = 10100 \text{ since } 12 = 9 + 3 = 1 \cdot 9 + 0 \cdot 5 + 1 \cdot 3 + 0 \cdot 2 + 0 \cdot 1,$$

and

$$(29)_2^{\text{alt}} = 110100 \text{ since } 29 = 17 + 9 + 3 = 1 \cdot 17 + 1 \cdot 9 + 0 \cdot 5 + 1 \cdot 3 + 0 \cdot 2 + 0 \cdot 1.$$

(a) Show that for any positive integer N there are $b_0, \ldots, b_n$ such that $(N)_2^{\text{alt}} = b_n b_{n-1} \ldots b_1 b_0$.

(b) Using this definition, 3 has two representations in alternative base 2:

$$(3)_2^{\text{alt}} = 11 \text{ and } (3)_2^{\text{alt}} = 100$$

However, some numbers, such as 4 and 7, have unique representations in alternative base 2. For example $(4)_2^{\text{alt}} = 101$ and $(7)_2^{\text{alt}} = 1010$, and neither of these numbers have a second representation in alternative base 2. Show that there are infinitely many positive integers that have a unique representation in alternative base 2, and there are infinitely many numbers that have at least two representations in alternative base 2.

5. A sequence of k distinct positive integers between 1 and n is a (k, n)-tour if the sequence begins with n ends with $n - 1$, and every number in the sequence is the difference of a pair of numbers that occur earlier in the sequence. More precisely, $(a_1, a_2, \ldots, a_k)$ is a (k, n)-tour if

- $1 \leq a_i \leq n$ for $i = 1, \ldots, k$.
- $a_1 = n$, and $a_k = n - 1$,
- $a_i \neq a_j$ whenever $i \neq j$,
- For $\ell \geq 3$, $a_\ell = a_i - a_j$ for some $i, j < \ell$ such that $i \neq j$.

For example, $(10, 1, 9)$ is a $(3, 10)$-tour, $(10, 3, 7, 4, 1, 9)$ is a $(6, 10)$-tour, and $(10, 7, 3, 4, 6, 2, 8, 1, 5, 9)$ is a $(10, 10)$-tour. Notice that the last example is an (k, n)-tour where $n = k$, and any (n, n)-tour must include all of the integers from 1 up to n.

(a) How many $(10, 10)$-tours can you find?

(b) How many $(16, 16)$-tours can you find?

(c) For what n does there exist an (n, n)-tour, and when one exists, what are the restrictions on a_2? (For example, a_2 cannot be even in any $(10, 10)$-tour of $\{1, 2, \ldots, 10\}$.)

6.