AMC 10 Problem Series (2804)

Jon Joseph

Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Homework

Lesson: 1 2 3 4 5 6 7 **8** 9 10 11 12

Homework: Lesson 8

You have completed **10** of **10** challenge problems.
Past Due **Jul 31**.

Readings

Lesson 8 Transcript: Fri, Jul 23

Challenge Problems

Total Score: 60 / 60

Problem 1 – Correct! – Score: 6 / 6 (2825)

Problem: Report Error

A solid cube has side length 3 inches. A 2-inch by 2-inch square hole is cut into the center of each face. The edges of each cut are parallel to the edges of the cube, and each hole goes all the way through the cube. What is the volume, in cubic inches, of the remaining solid?

(A) 7 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 15

Solution:

The holes remove a $2 \times 2 \times 2$ cube from the center of the cube, and a $2 \times 2 \times 0.5$ box from each face, so the volume of the remaining cube is $3^3 - 2^3 - 6 \cdot 2^2 \cdot 0.5 = \boxed{7}$. The answer is (A).

Your Response(s):

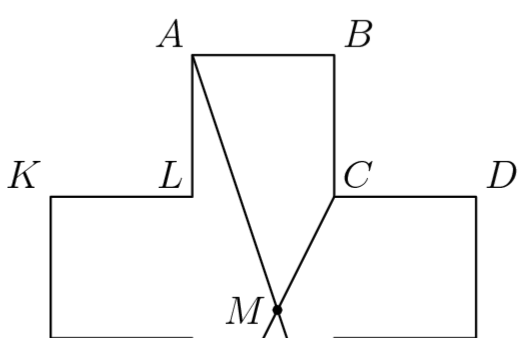
- A

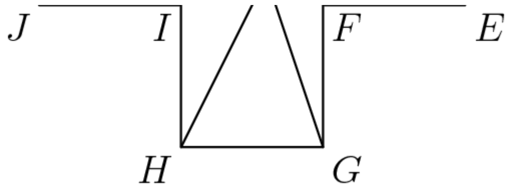
Problem 2 – Correct! – Score: 6 / 6 (2826)

Problem: Report Error

Consider the 12-sided polygon $ABCDEFGHIJKL$, as shown. Each of its sides has length 4, and each two consecutive sides form a right angle. Suppose that $\overline{AG}$ and $\overline{CH}$ meet at M. What is the area of quadrilateral $ABCM$?

(A) $\frac{44}{3}$ **(B)** 16 **(C)** $\frac{88}{5}$ **(D)** 20 **(E)** $\frac{62}{3}$





Solution:
We compute the area of quadrilateral $ABCM$ by taking the area of rectangle $ABGH$ and subtracting the areas of triangles AHG and GMC.

The area of rectangle $ABGH$ is $AB \cdot AH = 4 \cdot 12 = 48$. The area of triangle AHG is $1/2 \cdot AH \cdot HG = 1/2 \cdot 4 \cdot 12 = 24$.

To compute the area of triangle GMC, note that $[GMC]/[GMH] = CM/HM$. Note that triangles CMG and HMA are similar, so $CM/HM = CG/AH = 8/12 = 2/3$. Hence, $[GMC]/[GMH] = 2/3$. Also, $[HGC] = 1/2 \cdot CG \cdot GH = 1/2 \cdot 8 \cdot 4 = 16$, so $[GMC] = 2/5 \cdot [HGC] = 2/5 \cdot 16 = 32/5$.

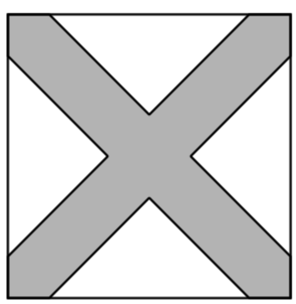
Hence, the area of quadrilateral $ABCM$ is $48 - 24 - 32/5 = \boxed{88/5}$. The answer is (C).

Your Response(s):
C

Problem 3 – Correct! – Score: 6 / 6 (2827)

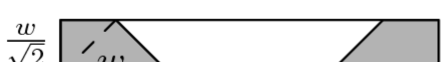
Problem: Report Error
A paint brush is swept along both diagonals of a square to produce the symmetric painted area, as shown. Half the area of the square is painted. What is the ratio of the side length of the square to the brush width?

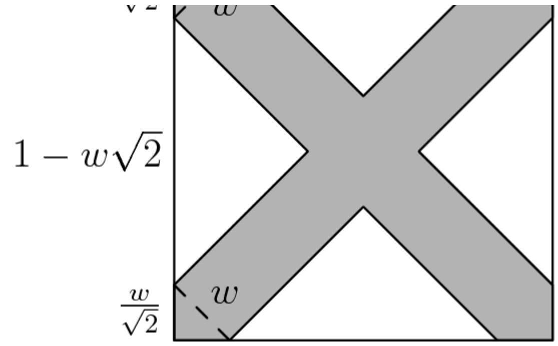
(A) $2\sqrt{2}+1$ **(B)** $3\sqrt{2}$ **(C)** $2\sqrt{2}+2$ **(D)** $3\sqrt{2}+1$ **(E)** $3\sqrt{2}+2$



Solution:
Assume that the square has side length 1. Let w be the width of the brush. Then the painted area cuts off two segments of length $w/\sqrt{2}$, as shown, leaving a segment of length

$$1 - 2 \cdot \frac{w}{\sqrt{2}} = 1 - w\sqrt{2}.$$





This length is the hypotenuse of a 45-45-90 unpainted triangle, so each leg has length

$$\frac{1 - w\sqrt{2}}{\sqrt{2}}.$$

Then the area of each such 45-45-90 unpainted triangle is

$$\frac{1}{2}\left(\frac{1 - w\sqrt{2}}{\sqrt{2}}\right)^2 = \frac{(1 - w\sqrt{2})^2}{4}.$$

Hence, the total unpainted area is

$$4 \cdot \frac{(1 - w\sqrt{2})^2}{4} = (1 - w\sqrt{2})^2 = \frac{1}{2}.$$

Taking the square root of both sides, we get

$$1 - w\sqrt{2} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

so

$$w\sqrt{2} = 1 - \frac{\sqrt{2}}{2} = \frac{2 - \sqrt{2}}{2}.$$

Then

$$w = \frac{2 - \sqrt{2}}{2\sqrt{2}} = \frac{(2 - \sqrt{2})\sqrt{2}}{4} = \frac{2\sqrt{2} - 2}{4} = \frac{\sqrt{2} - 1}{2}.$$

Hence, the ratio of the side length of the square to the brush width is

$$\frac{1}{w} = \frac{2}{\sqrt{2} - 1} = \frac{2(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)} = \boxed{2\sqrt{2} + 2}.$$

The answer is (C).

Your Response(s):

C

Problem 4 – Correct! – Score: 6 / 6 (2828)

Problem: Report Error

A fly trapped inside a cubical box with side length 1 meter decides to relieve its boredom by visiting each corner of the box. It will begin and end in the same corner and visit each of the other corners exactly once. To get from a corner to any other corner, it will either fly or crawl in a straight line. What is the maximum possible length, in meters, of its path?

(A) $4+4\sqrt{2}$ **(B)** $2+4\sqrt{2}+2\sqrt{3}$ **(C)** $2+3\sqrt{2}+3\sqrt{3}$ **(D)** $4\sqrt{2}+4\sqrt{3}$ **(E)** $3\sqrt{2}+5\sqrt{3}$

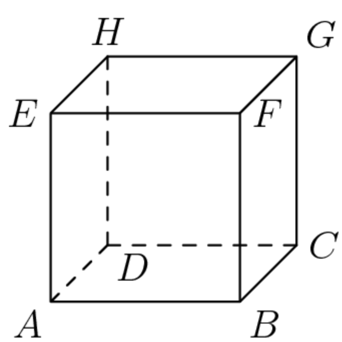
Solution:

The path of the fly consists of eight line segments, where each line segment goes from one corner to another corner. The distance of each such line segment is 1, $\sqrt{2}$, or $\sqrt{3}$.

The only way to obtain a line segment of length $\sqrt{3}$ is to go from one corner of the cube to the opposite corner. Since the fly visits each corner exactly once, it cannot traverse such a line segment twice. Also, the cube has exactly four such diagonals, so the path of the fly can contain at most four segments of length $\sqrt{3}$. Hence, the length of the fly's path can be at most $\boxed{4\sqrt{3}+4\sqrt{2}}$. This length can be achieved by taking the path

$$A \to G \to B \to H \to C \to E \to D \to F \to A.$$

Hence, the answer is (D).



Your Response(s):

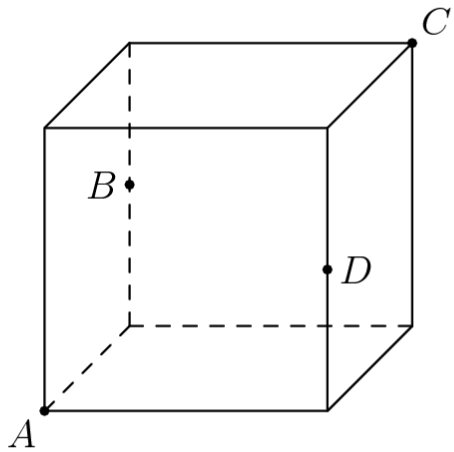
D

Problem 5 – Correct! – Score: 6 / 6 (2829)

Problem: Report Error

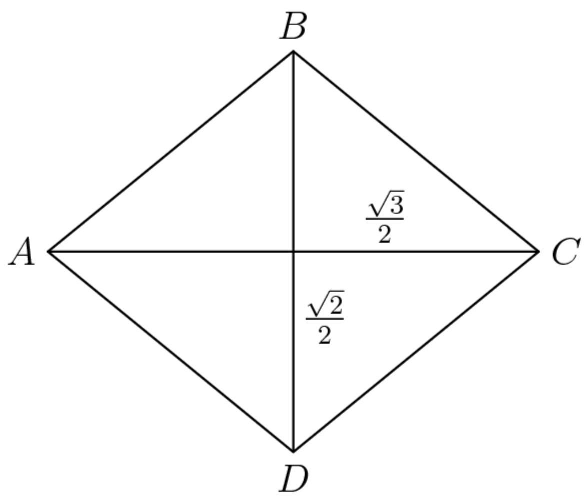
A cube with side length 1 is sliced by a plane that passes through two diagonally opposite vertices A and C and the midpoints B and D of two opposite edges not containing A or C, as shown. What is the area of quadrilateral $ABCD$?

(A) $\frac{\sqrt{6}}{2}$ **(B)** $\frac{5}{4}$ **(C)** $\sqrt{2}$ **(D)** $\frac{3}{2}$ **(E)** $\sqrt{3}$



Solution:
Each side of quadrilateral $ABCD$ is the hypotenuse of a right triangle with legs 1 and 1/2, so every side of quadrilateral $ABCD$ has the same length. In other words, quadrilateral $ABCD$ is a rhombus.

We see that $AC = \sqrt{3}$ and $BD = \sqrt{2}$, so diagonals AC and BD divide rhombus $ABCD$ into four right triangles with legs $\sqrt{2}/2$ and $\sqrt{3}/2$.



Hence, the area of rhombus $ABCD$ is

$$4 \cdot \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \boxed{\frac{\sqrt{6}}{2}}.$$

The answer is (A).

Your Response(s):
- A

Problem 6 – Correct! – Score: 6 / 6 (2830)

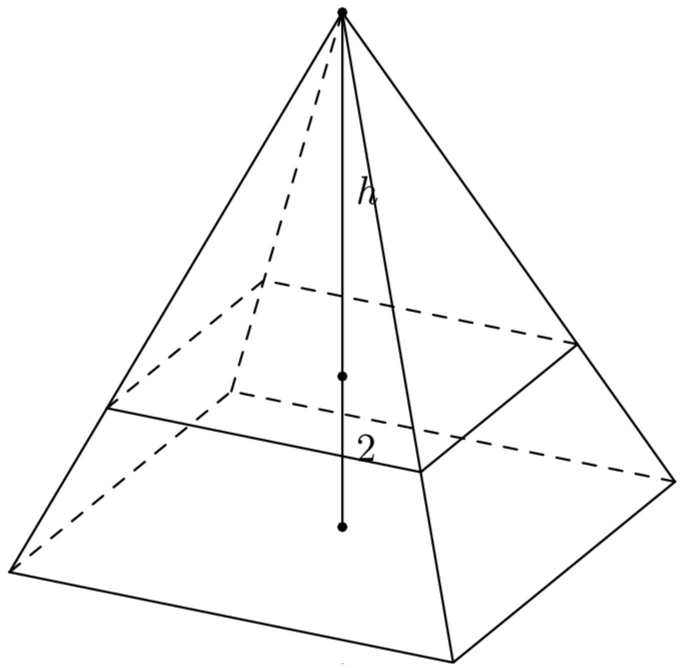
Problem: Report Error
A pyramid with a square base is cut by a plane that is parallel to its base and is 2 units from the base. The surface area of

the smaller pyramid that is cut from the top is half the surface area of the original pyramid. What is the altitude of the original pyramid?

(A) 2 **(B)** $2+\sqrt{2}$ **(C)** $1+2\sqrt{2}$ **(D)** 4 **(E)** $4+2\sqrt{2}$

Solution:

Let the height of the smaller pyramid be h.



The surface area of the larger pyramid is twice the surface area of the smaller pyramid, so the ratio of corresponding lengths in these pyramids is $\sqrt{2}$.

The height of the larger pyramid is $h+2$, so $h+2=h\sqrt{2}$. Then $(\sqrt{2}-1)h=2$, so

$$h=\frac{2}{\sqrt{2}-1}=\frac{2(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}=2+2\sqrt{2}.$$

Then the height of the larger pyramid is $h+2=\boxed{4+2\sqrt{2}}$. The answer is (E).

Your Response(s):

- E

Problem 7 – Correct! – Score: 6 / 6 (2831)

Problem: Report Error

Convex quadrilateral $ABCD$ has $AB=9$ and $CD=12$. Diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E, $AC=14$, and triangles AED and BEC have equal areas. What is AE?

(A) $\frac{9}{2}$ **(B)** $\frac{50}{11}$ **(C)** $\frac{21}{4}$ **(D)** $\frac{17}{3}$ **(E)** 6

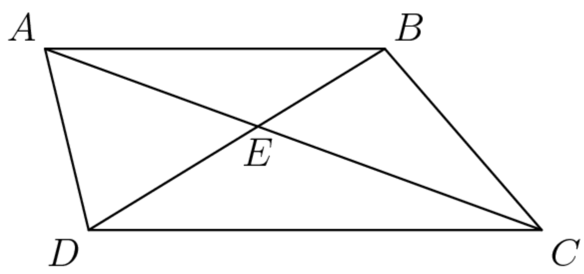
Solution:

We are given that

$$[AED] = [BEC].$$

Adding $[CDE]$ to both sides, we get $[AED] + [CDE] = [BEC] + [CDE]$, or

$$[ACD] = [BCD].$$



Since triangles ACD and BCD have the same area, they must have the same height with respect to base CD. Hence, AB is parallel to CD.

Then triangles ABE and CDE are similar, so $AE/CE = AB/CD = 9/12 = 3/4$. Hence, $AE = 3/7 \cdot AC = 3/7 \cdot 14 = \boxed{6}$. The answer is (E).

Your Response(s):

E

Problem 8 – Correct! – Score: 6 / 6 (2832)

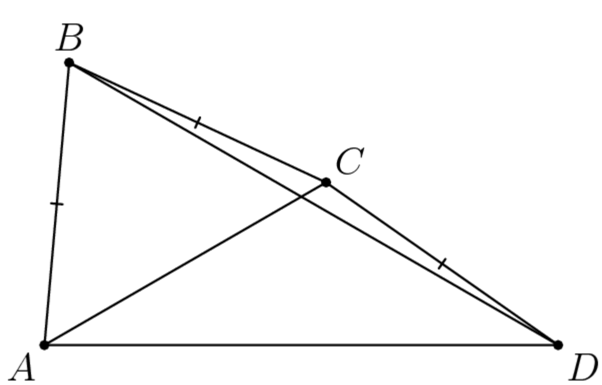
Problem: Report Error

Quadrilateral $ABCD$ has $AB = BC = CD$, $\angle ABC = 70^\circ$, and $\angle BCD = 170^\circ$. What is the degree measure of $\angle BAD$?

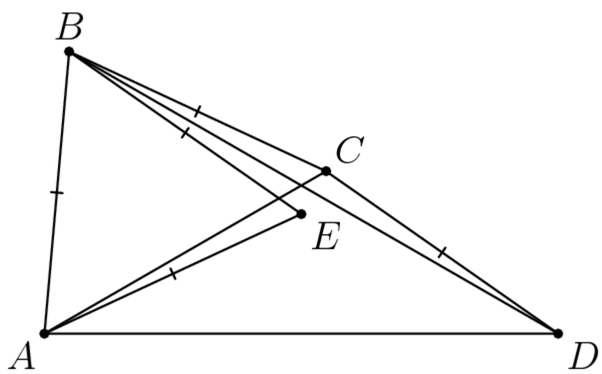
(A) 75 **(B)** 80 **(C)** 85 **(D)** 90 **(E)** 95

Solution:

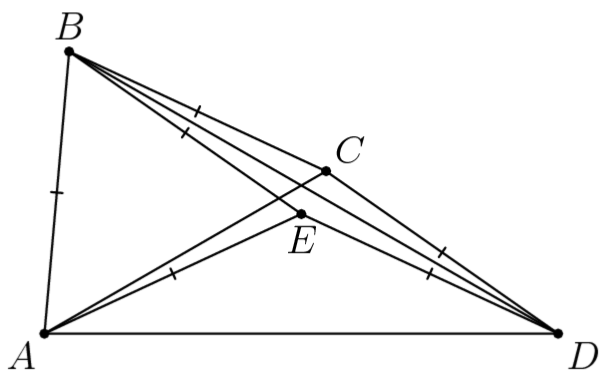
Since $AB = BC$ and $\angle ABC = 70^\circ$, we have that $\angle BAC = \angle BCA = (180^\circ - 70^\circ)/2 = 55^\circ$. Since $BC = CD$ and $\angle BCD = 170^\circ$, we have that $\angle CBD = \angle CDB = 5^\circ$. Then $\angle ABD = \angle ABC - \angle CBD = 70^\circ - 5^\circ = 65^\circ$.



Since $\angle ABD = 65^\circ$ and $\angle BAC = 55^\circ$ are close to 60°, we think of constructing equilateral triangle ABE. (Constructing equilateral triangles is a great way of finding angles in a diagram.)



Then $\angle DBE = \angle ABD - \angle ABE = 65^\circ - 60^\circ = 5^\circ$. But $\angle CBD = 5^\circ$, so triangles CBD and EBD are congruent. Hence, $DE = CD$.



Since $AE = BE = DE$, E is the circumcenter of triangle ABD. Therefore, $\angle BAD = \angle BED/2 = \angle BCD/2 = 170^\circ/2 = \boxed{85^\circ}$. The answer is (C).

Your Response(s):

C

Problem 9 – Correct! – Score: 6 / 6 (2833)

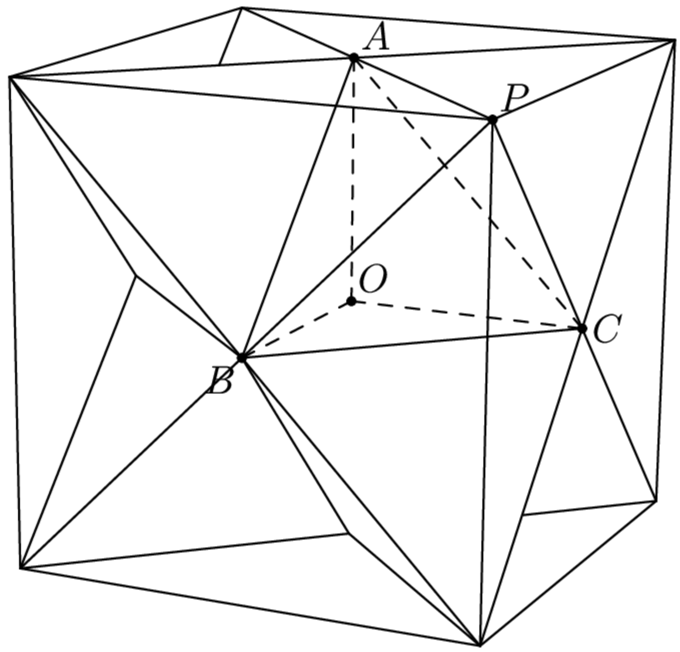
Problem: Report Error

Two distinct regular tetrahedra have all their vertices among the vertices of the same unit cube. What is the volume of the region formed by the intersection of the tetrahedra?

(A) $\frac{1}{12}$ **(B)** $\frac{\sqrt{2}}{12}$ **(C)** $\frac{\sqrt{3}}{12}$ **(D)** $\frac{1}{6}$ **(E)** $\frac{\sqrt{2}}{6}$

Solution:

We draw the two regular tetrahedra, as shown. Let O be the center of the cube. Let P be a vertex of the cube, and let A, B, and C be the centers of three faces (of the cube) that have P as a vertex.



Tetrahedron $PABC$ is contained in one of the given regular tetrahedra, but not the other. However, tetrahedron $OABC$ is contained in both given regular tetrahedra.

Hence, the intersection of the two given regular tetrahedra is the octahedron formed by joining the centers of the six faces of the cube. We can find the volume of the octahedron by considering it as two right square pyramids. The square base of each pyramid has area $\frac{1}{2}$ the area of a face of the cube, and the height of the pyramid is $\frac{1}{2}$ the edge length of the cube. So, the volume of each pyramid is

$$\frac{1}{3}(\text{base area})(\text{height}) = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$

Combining these two pyramids, the volume of the octahedron is $2 \cdot \frac{1}{12} = \frac{1}{6}$, so the answer is $\boxed{\text{(D)}}$.

Your Response(s):

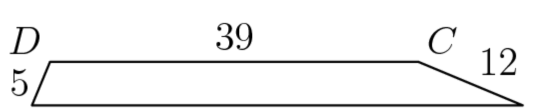
- D

Problem 10 – Correct! – Score: 6 / 6 (2834)

Problem: Report Error

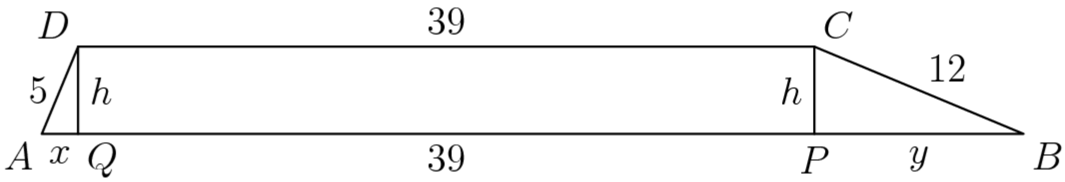
In trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$. The area of $ABCD$ is

(A) 182 **(B)** 195 **(C)** 210 **(D)** 234 **(E)** 260



A 52 B

Solution:
Let P and Q be the projections of C and D onto AB, respectively. Let $h = CP = DQ$, $x = AQ$, and $y = BP$.



By Pythagoras on right triangles AQD and BPC, $x^2 + h^2 = 25$ and $y^2 + h^2 = 144$, so $h^2 = 25 - x^2 = 144 - y^2$. Then $y^2 - x^2 = 144 - 25$, or $(y + x)(y - x) = 119$.

We also know that $x + y = AB - CD = 52 - 39 = 13$. Dividing the equation $(y + x)(y - x) = 119$ by $x + y = 13$, we get $y - x = 119/13$. Adding the equations $x + y = 13$ and $y - x = 119/13$, we get $2y = 13 + 119/13 = 288/13$, so $y = 144/13$. Then

$$h^2 = 144 - \left(\frac{144}{13}\right)^2 = \frac{3600}{169},$$

so $h = \sqrt{3600/169} = 60/13$.

Therefore, the area of trapezoid $ABCD$ is

$$\frac{AB + CD}{2} \cdot h = \frac{52 + 39}{2} \cdot \frac{60}{13} = \boxed{210}.$$

The answer is (C).

Your Response(s):
C