AMC 10 Problem Series (2804)

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Friday
Jun 4, 2021 - Aug 20, 2021
7:30 - 9:00 PM ET (4:30 - 6:00 PM PT)

Homework

Lesson: 1 2 **3** 4 5 6 7 8 9 10 11 12

Homework: Lesson 3

You have completed **10** of **10** challenge problems.
Past Due **Jun 26**.

Readings

Lesson 3 Transcript: Fri, Jun 18

Challenge Problems

Total Score: 60 / 60

Problem 1 – Correct! – Score: 6 / 6 (2855)

Problem: Report Error

Consider the set of numbers $\{1, 10, 10^2, 10^3, \ldots, 10^{10}\}$. The ratio of the largest element of the set to the sum of the other ten elements of the set is closest to which integer?

(A) 1 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 101

Solution:
The ratio of the largest element of the set to the sum of the other ten elements of the set is

$$\begin{aligned}\frac{10^{10}}{1+10+10^2+\cdots+10^9} &= \frac{10000000000}{1111111111} \\ &= \frac{9999999999+1}{1111111111} \\ &= 9 + \frac{1}{1111111111},\end{aligned}$$

which is closest to $\boxed{9}$. The answer is (B).

Your Response(s):
B

Problem 2 – Correct! – Score: 6 / 6 (2856)

Problem: Report Error

For each positive integer n, the mean of the first n terms of a sequence is n. What is the 2008^{th} term of the sequence?

(A) 2008 **(B)** 4015 **(C)** 4016 **(D)** 4,030,056 **(E)** 4,032,064

Solution:
We obtain the 2008^{th} term of the sequence by computing the sum of the first 2008 terms and subtracting the sum of the first 2007 terms.

The mean of the first 2008 terms is 2008, so the sum of the first 2008 terms is 2008^2. The mean of the first 2007 terms is

2007, so the sum of the first 2007 terms is 2007^2. Therefore, the 2008^{th} term is

$$2008^2 - 2007^2 = (2008 + 2007)(2008 - 2007) = \boxed{4015}.$$

The answer is (B).

Your Response(s):

B

Problem 3 – Correct! – Score: 6 / 6 (2857)

Problem: Report Error

On Monday, Millie puts a quart of seeds, 25% of which are millet, into a bird feeder. On each successive day she adds another quart of the same mix of seeds without removing any seeds that are left. Each day the birds eat only 25% of the millet in the feeder, but they eat all of the other seeds. On which day, just after Millie has placed the seeds, will the birds find that more than half the seeds in the feeder are millet?

(A) Tuesday **(B)** Wednesday **(C)** Thursday **(D)** Friday **(E)** Saturday

Solution:

On Monday, there is 1/4 of a quart of millet. The birds eat 1/4 of the millet, leaving $3/4 \cdot 1/4$ of a quart of millet. On Tuesday, after Millie adds her seeds, there is now

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4}$$

of a quart of millet. The birds eat 1/4 of this millet, leaving

$$\frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}$$

of a quart of millet.

On Wednesday, after Millie adds her seeds, there is now

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}$$

of a quart of millet, and so on.

On the n^{th} day, after Millie adds her seeds, there is

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \dots + \left(\frac{3}{4}\right)^{n-1} \cdot \frac{1}{4}$$

of a quart of millet. By the formula for a geometric series,

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \dots + \left(\frac{3}{4}\right)^{n-1} \cdot \frac{1}{4} = \frac{1}{4}\left[1 + \frac{3}{4} + \dots + \left(\frac{3}{4}\right)^{n-1}\right]$$

$$= \frac{1}{4} \cdot \frac{1 - (\frac{3}{4})^n}{1 - \frac{3}{4}}$$
$$= 1 - \left(\frac{3}{4}\right)^n.$$

The birds always find 3/4 of a quart of other seeds, so more than half the seeds are millet if and only if

$$1 - \left(\frac{3}{4}\right)^n > \frac{3}{4},$$

or

$$\left(\frac{3}{4}\right)^n < \frac{1}{4}.$$

We see that $(3/4)^n$ decreases as n increases, $(3/4)^4 = 81/256 > 1/4$, and $(3/4)^5 = 243/1024 < 1/4$, so more than half the seeds are millet first after 5 days, which is $\boxed{\text{Friday}}$. The answer is (D).

Your Response(s):

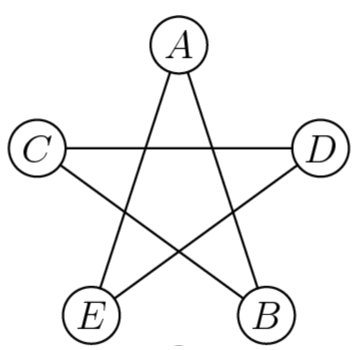
D

Problem 4 – Correct! – Score: 6 / 6 (2858)

Problem: Report Error

In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers 3, 5, 6, 7, and 9, although not necessarily in this order. The sums of the numbers at the ends of the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EA}$ form an arithmetic sequence, although not necessarily in this order. What is the middle term of the arithmetic sequence?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13



Solution:

Let the terms in the arithmetic sequence be a, $a + d$, $a + 2d$, $a + 3d$, and $a + 4d$. We seek the middle term $a + 2d$.

These five terms are $A+B, B+C, C+D, D+E$, and $E+A$, in some order. The numbers A, B, C, D, and E are equal to 3, 5, 6, 7, and 9, in some order, so

$$A+B+C+D+E = 3+5+6+7+9 = 30.$$

Hence, the sum of the five terms is

$$(A+B)+(B+C)+(C+D)+(D+E)+(E+A) = 2A+2B+2C+2D+2E = 60.$$

But adding all five numbers, we also get $a+(a+d)+(a+2d)+(a+3d)+(a+4d) = 5a+10d$, so

$$5a+10d = 60.$$

Dividing both sides by 5, we get $a+2d = \boxed{12}$, which is the middle term. The answer is (D).

Your Response(s):

D

Problem 5 – Correct! – Score: 6 / 6 (2859)

Problem: Report Error

In the eight-term sequence A, B, C, D, E, F, G, H, the value of C is 5 and the sum of any three consecutive terms is 30. What is $A+H$?

(A) 17 **(B)** 18 **(C)** 25 **(D)** 26 **(E)** 43

Solution:

The sum of any three consecutive terms is 30, so $A+B+C = B+C+D = 30$. Hence, $A = D$. Similarly, $B+C+D = C+D+E$, so $B = E$. In general, every third term of the sequence is the same.

Therefore, $H = E = B$. We know that $A+B+C = 30$, and $C = 5$, so $A+H+5 = 30$, which means $A+H = \boxed{25}$. The answer is (C).

Your Response(s):

C

Problem 6 – Correct! – Score: 6 / 6 (2860)

Problem: Report Error

Let $a_1, a_2, \ldots$ be a sequence for which $a_1 = 2$, $a_2 = 3$, and $a_n = a_{n-1}/a_{n-2}$ for each positive integer $n \geq 3$. What is a_{2006}?

(A) $\frac{1}{2}$ **(B)** $\frac{2}{3}$ **(C)** $\frac{3}{2}$ **(D)** 2 **(E)** 3

Solution:

We compute the first few values of the sequence:

$$a_3 = \frac{a_2}{a_1} = \frac{3}{2},$$

$$a_3 \quad 3/2 \quad 1$$

$$\begin{aligned} a_4 &= \frac{a_3}{a_2} = \frac{3/2}{3} = \frac{1}{2}, \\ a_5 &= \frac{a_4}{a_3} = \frac{1/2}{3/2} = \frac{1}{3}, \\ a_6 &= \frac{a_5}{a_4} = \frac{1/3}{1/2} = \frac{2}{3}, \\ a_7 &= \frac{a_6}{a_5} = \frac{2/3}{1/3} = 2, \\ a_8 &= \frac{a_7}{a_6} = \frac{2}{2/3} = 3. \end{aligned}$$

Each term in the sequence depends on the previous two terms. We have found two consecutive terms in the sequence that are equal to the initial two terms of the sequence, namely $a_7 = 2$ and $a_8 = 3$, and $a_1 = 2$ and $a_2 = 3$. Therefore, the sequence is periodic, with period 6.

Since 2006 and 2 differ by 2004, which is a multiple of 6, we have that $a_{2006} = a_2 = \boxed{3}$. The answer is (E).

Your Response(s):

E

Problem 7 – Correct! – Score: 6 / 6 (2861)

Problem:

Report Error

Suppose that $\{a_n\}$ is an arithmetic sequence with $a_1 + a_2 + \cdots + a_{100} = 100$ and $a_{101} + a_{102} + \cdots + a_{200} = 200$. What is the value of $a_2 - a_1$?

(A) 0.0001 **(B)** 0.001 **(C)** 0.01 **(D)** 0.1 **(E)** 1

Solution:

Let d be the common difference of the arithmetic sequence. Note that $a_{101} - a_1 = 100d$, $a_{102} - a_2 = 100d$, and so on. Hence, the difference between $a_1 + a_2 + \cdots + a_{100}$ and $a_{101} + a_{102} + \cdots + a_{200}$ is

$$\begin{aligned} &(a_{101} + a_{102} + \cdots + a_{200}) - (a_1 + a_2 + \cdots + a_{100}) \\ &= (a_{101} - a_1) + (a_{102} - a_2) + \cdots + (a_{200} - a_{100}) \\ &= 100d + 100d + \cdots + 100d \\ &= 100 \cdot 100d \\ &= 10000d. \end{aligned}$$

But the difference between $a_1 + a_2 + \cdots + a_{100}$ and $a_{101} + a_{102} + \cdots + a_{200}$ is also $200 - 100 = 100$, so $10000d = 100$. Then $d = 100/10000 = \boxed{0.01}$. The answer is (C).

Your Response(s):

C

Problem 8 – Correct! – Score: 6 / 6 (2862)

Problem: Report Error

Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$, for all positive integers m and n. Then a_{12} is

(A) 45 **(B)** 56 **(C)** 67 **(D)** 78 **(E)** 89

Solution:

Taking $m = 1$ in the given equation, we get

$$a_{n+1} = a_n + a_1 + n = a_n + n + 1$$

for all $n \geq 1$. Therefore,

$$\begin{aligned} a_2 &= a_1 + 2 = 1 + 2, \\ a_3 &= a_2 + 3 = 1 + 2 + 3, \\ a_4 &= a_3 + 4 = 1 + 2 + 3 + 4, \end{aligned}$$

and so on. Then

$$a_{12} = 1 + 2 + 3 + \dots + 12 = \frac{1 + 12}{2} \cdot 12 = \boxed{78}.$$

The answer is (D).

Your Response(s):

D

Problem 9 – Correct! – Score: 6 / 6 (2863)

Problem: Report Error

The first four terms in an arithmetic sequence are $x + y$, $x - y$, xy, and x/y, in that order. What is the fifth term?

(A) $-\frac{15}{8}$ **(B)** $-\frac{6}{5}$ **(C)** 0 **(D)** $\frac{27}{20}$ **(E)** $\frac{123}{40}$

Solution:

The common difference in the arithmetic sequence is $(x - y) - (x + y) = -2y$. Therefore, the third term is also equal to $(x - y) - 2y = x - 3y$, which gives us the equation

$$x - 3y = xy,$$

and the fourth term is also equal to $(x - 3y) - 2y = x - 5y$, which gives us the equation

$$x - 5y = \frac{x}{y}.$$

Multiplying this equation by y, we get $xy - 5y^2 = x$, so $xy - x = 5y^2$.

But from the equation $x - 3y = xy$, $xy - x = -3y$, so

$$-3y = 5y^2.$$

If $y = 0$, then the term x/y would be undefined, so $y \neq 0$. Dividing both sides by y, we get $-3 = 5y$, so

$y = -3/5$.

Substituting this value into the equation $x - 3y = xy$, we get

$$x + \frac{9}{5} = -\frac{3}{5}x.$$

Multiplying both sides by 5, we get $5x + 9 = -3x$, so $8x = -9$, which means $x = -9/8$.

Therefore, the fifth term of the arithmetic sequence is

$$(x - 5y) - 2y = x - 7y = -\frac{9}{8} - 7 \cdot \left(-\frac{3}{5}\right) = \boxed{\frac{123}{40}}.$$

The answer is (E).

Your Response(s):

E

Problem 10 – Correct! – Score: 6 / 6 (2864)

Problem:

Report Error

Let $a_1, a_2, \ldots$ be a sequence with the following properties.

(i) $a_1 = 1$, and

(ii) $a_{2n} = n \cdot a_n$ for any positive integer n.

What is the value of $a_{2^{100}}$? (The subscript is 2^{100}.)

(A) 1 **(B)** 2^{99} **(C)** 2^{100} **(D)** 2^{4950} **(E)** 2^{9999}

Solution:

Note that

$$\begin{aligned} a_{2^1} &= 2^0 \cdot a_{2^0} = 1, \\ a_{2^2} &= 2^1 \cdot a_{2^1} = 2, \\ a_{2^3} &= 2^2 \cdot a_{2^2} = 2^2 \cdot 2 = 2^{1+2}, \\ a_{2^4} &= 2^3 \cdot a_{2^3} = 2^3 \cdot 2^{1+2} = 2^{1+2+3}, \end{aligned}$$

and so on. In general,

$$a_{2^n} = 2^{1+2+\cdots+(n-1)}.$$

Since

$$1 + 2 + \cdots + (n-1) = \frac{1 + (n-1)}{2} \cdot (n-1) = \frac{n(n-1)}{2},$$

we have that

$$a_{2^n} = 2^{n(n-1)/2}$$

for all $n \geq 0$. In particular, for $n = 100$,

$$a_{2^{100}} = 2^{100 \cdot 99/2} = \boxed{2^{4950}}.$$

The answer is (D).

Your Response(s):

- D