2021 AMC 12A Problems/Problem 18

The following problem is from both the 2021 AMC 10A #18 and 2021 AMC 12A #18, so both problems redirect to this page.

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Let f be a function defined on the set of positive rational numbers with the property that $f(a\cdot b)=f(a)+f(b)$ for all positive rational numbers a and b. Furthermore, suppose that f also has the property that $f(p)=p$ for every prime number p. For which of the following numbers x is $f(x)<0$?

$\textbf{(A) }\frac{17}{32} \qquad \textbf{(B) }\frac{11}{16} \qquad \textbf{(C) }\frac{7}{9} \qquad \textbf{(D) }\frac{7}{6} \qquad \textbf{(E) }\frac{25}{11}$

Solution 1 (Intuitive)

From the answer choices, note that

$$\begin{aligned} f(25) &= f\left(\frac{25}{11}\cdot 11\right) \\ &= f\left(\frac{25}{11}\right)+f(11) \\ &= f\left(\frac{25}{11}\right)+11. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} f(25) &= f(5\cdot 5) \\ &= f(5)+f(5) \\ &= 5+5 \\ &= 10. \end{aligned}$$

Equating the expressions for $f(25)$ produces

$$f\left(\frac{25}{11}\right)+11=10,$$

from which $f\left(\frac{25}{11}\right)=-1$. Therefore, the answer is $\boxed{\textbf{(E)}\ \frac{25}{11}}$.

Remark

Similarly, we can find the outputs of f at the inputs of the other answer choices:

$$\begin{array}{cccc}
\textbf{(A)} & f\left(\frac{17}{32}\right) & = & 7 \\
\textbf{(B)} & f\left(\frac{11}{16}\right) & = & 3 \\
\textbf{(C)} & f\left(\frac{7}{9}\right) & = & 1 \\
\textbf{(D)} & f\left(\frac{7}{6}\right) & = & 2
\end{array}$$

Alternatively, refer to Solutions 2 and 4 for the full processes.

~Lemonie ~awesomediabrine ~MRENTHUSIASM

Solution 2 (Specific)

We know that $f(p)=f(p\cdot 1)=f(p)+f(1)$. By transitive, we have

$$f(p)=f(p)+f(1).$$

Subtracting $f(p)$ from both sides gives $0=f(1)$. Also

$$f(2)+f\left(\frac{1}{2}\right)=f(1)=0 \implies 2+f\left(\frac{1}{2}\right)=0 \implies f\left(\frac{1}{2}\right)=-2$$

$$f(3)+f\left(\frac{1}{3}\right)=f(1)=0 \implies 3+f\left(\frac{1}{3}\right)=0 \implies f\left(\frac{1}{3}\right)=-3$$

$$f(11)+f\left(\frac{1}{11}\right)=f(1)=0 \implies 11+f\left(\frac{1}{11}\right)=0 \implies f\left(\frac{1}{11}\right)=-11$$

In $\textbf{(A)}$ we have $f\left(\frac{17}{32}\right)=17+5f\left(\frac{1}{2}\right)=17-5(2)=7$.

In $\textbf{(B)}$ we have $f\left(\frac{11}{16}\right)=11+4f\left(\frac{1}{2}\right)=11-4(2)=3$.

In $\textbf{(C)}$ we have $f\left(\frac{7}{9}\right)=7+2f\left(\frac{1}{3}\right)=7-2(3)=1$.

In $\textbf{(D)}$ we have $f\left(\frac{7}{6}\right)=7+f\left(\frac{1}{2}\right)+f\left(\frac{1}{3}\right)=7-2-3=2$.

In $\textbf{(E)}$ we have $f\left(\frac{25}{11}\right)=10+f\left(\frac{1}{11}\right)=10-11=-1$.

Thus, our answer is $\boxed{\textbf{(E)} \frac{25}{11}}$.

~JHawk0224 ~awesomediabrine

Solution 3 (Generalized)

Consider the rational $\frac{a}{b}$, for a, b integers. We have $f(a) = f\left(\frac{a}{b} \cdot b\right) = f\left(\frac{a}{b}\right) + f(b)$. So $f\left(\frac{a}{b}\right) = f(a) - f(b)$. Let p be a prime. Notice that $f(p^k) = kf(p)$. And $f(p) = p$. So if $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $f(a) = a_1 p_1 + a_2 p_2 + \cdots + a_k p_k$. We simply need this to be greater than what we have for $f(b)$. Notice that for answer choices $\textbf{(A)}$, $\textbf{(B)}$, $\textbf{(C)}$, and $\textbf{(D)}$, the numerator has fewer prime factors than the denominator, and so they are less likely to work. We check $\textbf{(E)}$ first, and it works, therefore the answer is $\boxed{\textbf{(E)} \frac{25}{11}}$.

~yofro

Solution 4 (Generalized)

We derive the following properties of f :

1. By induction, we have

$$f\left(\prod_{k=1}^{n} a_k\right) = \sum_{k=1}^{n} f(a_k)$$

   for all positive rational numbers a_k and positive integers n.

   Since positive powers are just repeated multiplication of the base, it follows that

$$f\left(a^n\right) = f\left(\prod_{k=1}^{n} a\right) = \sum_{k=1}^{n} f(a) = nf(a)$$

   for all positive rational numbers a and positive integers n.

2. For all positive rational numbers a, we have

$$f(a) = f(a \cdot 1) = f(a) + f(1),$$

   from which $f(1) = 0$.

3. For all positive rational numbers a, we have

$$f(a) + f\left(\frac{1}{a}\right) = f\left(a \cdot \frac{1}{a}\right) = f(1) = 0,$$

   from which $f\left(\frac{1}{a}\right) = -f(a)$.

For all positive integers x and y, suppose $\prod_{k=1}^{m} p_k^{d_k}$ and $\prod_{k=1}^{n} q_k^{e_k}$ are their respective prime factorizations. We get

$$f\left(\frac{x}{y}\right) = f(x) + f\left(\frac{1}{y}\right)$$

$$
\begin{aligned}
&= f(x) - f(y) && \text{by Property 3} \\
&= f\left(\prod_{k=1}^{m} p_k^{d_k}\right) - f\left(\prod_{k=1}^{n} q_k^{e_k}\right) \\
&= \left[\sum_{k=1}^{m} f\left(p_k^{d_k}\right)\right] - \left[\sum_{k=1}^{n} f\left(q_k^{e_k}\right)\right] && \text{by Property 1} \\
&= \left[\sum_{k=1}^{m} d_k f\left(p_k\right)\right] - \left[\sum_{k=1}^{n} e_k f\left(q_k\right)\right] && \text{by Property 1} \\
&= \left[\sum_{k=1}^{m} d_k p_k\right] - \left[\sum_{k=1}^{n} e_k q_k\right].
\end{aligned}
$$

We apply f to each fraction in the answer choices:

$$
\begin{array}{llcccccc}
\textbf{(A)} & f\left(\frac{17}{32}\right) & = & f\left(\frac{17^1}{2^5}\right) & = & [1(17)]-[5(2)] & = & 7 \\
\textbf{(B)} & f\left(\frac{11}{16}\right) & = & f\left(\frac{11^1}{2^4}\right) & = & [1(11)]-[4(2)] & = & 3 \\
\textbf{(C)} & f\left(\frac{7}{9}\right) & = & f\left(\frac{7^1}{3^2}\right) & = & [1(7)]-[2(3)] & = & 1 \\
\textbf{(D)} & f\left(\frac{7}{6}\right) & = & f\left(\frac{7^1}{2^1\cdot 3^1}\right) & = & [1(7)]-[1(2)+1(3)] & = & 2 \\
\textbf{(E)} & f\left(\frac{25}{11}\right) & = & f\left(\frac{5^2}{11^1}\right) & = & [2(5)]-[1(11)] & = & -1
\end{array}
$$

Therefore, the answer is $\boxed{\textbf{(E) } \frac{25}{11}}$.

~MRENTHUSIASM

Solution 5 (Quick, Dirty, and Frantic Last Hope)

Note that answer choices $\textbf{(A)}$ through $\textbf{(D)}$ are $\frac{\text{prime}}{\text{composite}}$, whereas $\textbf{(E)}$ is $\frac{\text{composite}}{\text{prime}}$. Because the functional equation is related to primes, we hope that the uniqueness of answer choice $\boxed{\textbf{(E) } \frac{25}{11}}$ is enough.

~OliverA

Video Solution by Hawk Math

https://www.youtube.com/watch?v=dvlTA8Ncp58

Video Solution by North America Math Contest Go Go Go Through Induction

https://www.youtube.com/watch?v=ffX0fTgJN0w&list=PLexHyfQ8DMuKqltG3cHT7Di4jhVl6L4YJ&index=12

Video Solution by Punxsutawney Phil

https://youtu.be/8gGcj95rlWY

Video Solution by OmegaLearn (Using Functions and Manipulations)

https://youtu.be/aGv99CLzguE

~ pi_is_3.14

Video Solution by TheBeautyofMath

https://youtu.be/IUJ_A9KiLEE

~IceMatrix

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