2021 Fall AMC 10B Problems/Problem 17

Contents

- 1 Problem
- 2 Solution 1
- 3 Solution 2
- 4 Video Solution
- 5 See Also

Problem

Distinct lines ℓ and m lie in the xy-plane. They intersect at the origin. Point $P(-1, 4)$ is reflected about line ℓ to point P', and then P' is reflected about line m to point P''. The equation of line ℓ is $5x - y = 0$, and the coordinates of P'' are $(4,1)$. What is the equation of line $m?$

$(\textbf{A})\ 5x + 2y = 0 \qquad (\textbf{B})\ 3x + 2y = 0 \qquad (\textbf{C})\ x - 3y = 0 \qquad (\textbf{D})\ 2x - 3y = 0 \qquad (\textbf{E})\ 5x - 3$

Solution 1

It is well known that the composition of 2 reflections , one after another, about two lines l and m, respectively, that meet at an angle θ is a rotation by 2θ around the intersection of l and m.

Now, we note that $(4,1)$ is a 90 degree rotation clockwise of $(-1,4)$ about the origin, which is also where l and m intersect. So m is a 45 degree rotation of l about the origin clockwise.

To rotate l 90 degrees clockwise, we build a square with adjacent vertices $(0,0)$ and $(1,5)$. The other two vertices are at $(5,-1)$ and $(6,4)$. The center of the square is at $(3,2)$, which is the midpoint of $(1,5)$ and $(5,-1)$. The line m passes through the origin and the center of the square we built, namely at $(0,0)$ and $(3,2)$. Thus the line is $y = \frac{2}{3}x$. The answer is **(D)** $\boxed{2x-3y=0}$.

~hurdler, minor edits by nightshade2526

Solution 2

We know that the equation of line ℓ is $y=5x$. This means that P' is $(-1,4)$ reflected over the line $y=5x$. This means that the line with P and P' is perpendicular to ℓ, so it has slope $-\frac{1}{5}$. Then the equation of this perpendicular line is $y=-\frac{1}{5}x+c$, and plugging in $(-1,4)$ for x and y yields $c=\frac{19}{5}$.

The midpoint of P' and P lies at the intersection of $y=5x$ and $y=-\frac{1}{5}x+\frac{19}{5}$. Solving, we get the x-value of the intersection is $\frac{19}{26}$ and the y-value is $\frac{95}{26}$. Let the x-value of P' be x' - then by the midpoint formula, $\frac{x'-1}{2}=\frac{19}{26} \implies x'=\frac{32}{13}$. We can find the y-value of P' the same way, so $P'=(\frac{32}{13},\frac{43}{13})$.

Now we have to reflect P' over m to get to $(4,1)$. The midpoint of P' and P'' will lie on m, and this midpoint is, by the midpoint formula, $\left(\frac{42}{13}, \frac{28}{13}\right)$. $y = mx$ must satisfy this point, so $m = \frac{\frac{28}{13}}{\frac{42}{13}} = \frac{28}{42} = \frac{2}{3}$.

Now the equation of line m is $y = \frac{2}{3}x \implies 2x - 3y = 0 = \boxed{D}$

~KingRavi

Video Solution

Solution 2021 Fall 10B #17 (https://youtu.be/bqCacR8aXmc%7CVideo)

~hurdler

See Also

2021 Fall AMC 10B (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
Preceded by **Problem 16**	Followed by **Problem 18**
1 • 2 • 3 • 4 • 5 • 6 • 7 • 8 • 9 • 10 • 11 • 12 • 13 • 14 • 15 • 16 • 17 • 18 • 19 • 20 • 21 • 22 • 23 • 24 • 25	
All AMC 10 Problems and Solutions	