2021 Fall AMC 10B Problems/Problem 22

Contents

Problem

For each integer $n \geq 2$, let S_n be the sum of all products jk, where j and k are integers and $1 \leq j < k \leq n$. What is the sum of the 10 least values of n such that S_n is divisible by 3?

$\textbf{(A)}\ 196 \qquad \textbf{(B)}\ 197 \qquad \textbf{(C)}\ 198 \qquad \textbf{(D)}\ 199 \qquad \textbf{(E)}\ 200$

Solution 1

To get from S_n to S_{n+1}, we add

$$1(n+1)+2(n+1)+\cdots+n(n+1)=(1+2+\cdots+n)(n+1)=\frac{n(n+1)^2}{2}.$$

Now, we can look at the different values of n mod 3. For $n \equiv 0 \pmod{3}$ and $n \equiv 2 \pmod{3}$, then we have $\frac{n(n+1)^2}{2} \equiv 0 \pmod{3}$. However, for $n \equiv 1 \pmod{3}$, we have

$$\frac{1 \cdot 2^2}{2} \equiv 2 \pmod{3}.$$

Clearly, $S_2 \equiv 2 \pmod{3}$. Using the above result, we have $S_5 \equiv 1 \pmod{3}$, and S_8, S_9, and S_{10} are all divisible by 3. After $3 \cdot 3 = 9$, we have S_{17}, S_{18}, and S_{19} all divisible by 3, as well as S_{26}, S_{27}, S_{28}, and S_{35}. Thus, our answer is $8+9+10+17+18+19+26+27+28+35 = 27+54+81+35 = 162+35 = \boxed{\textrm{(B)}\ 197}$. -BorealBear

Solution 2 (bash)

Since we have a wonky function, we start by trying out some small cases and see what happens. If j is 1 and k is 2, then there is once case. We have 2 mod 3 for this case. If N is 3, we have $1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3$ which is still 2 mod 3. If N is 4, we have to add $1 \cdot 4 + 2 \cdot 4 + 3 \cdot 4$ which is a multiple of 3, meaning that we are still at 2 mod 3. If we try a few more cases, we find that when N is 8, we get a multiple of 3. When N is 9, we are adding 0 mod 3, and therefore, we are still at a multiple of 3.

When N is 10, then we get 0 mod 3 + $10(1+2+3+ ... +9)$ which is 10 times a multiple of 3. Therefore, we have another multiple of 3. When N is 11, so we have 2 mod 3. So, every time we have -1 mod 9, 0 mod 9, and 1 mod 9, we always have a multiple of 3. Think about it: When N is 1, it will have to be $0 \cdot 1$, so it is a multiple of 3. Therefore, our numbers are $8, 9, 10, 17, 18, 19, 26, 27, 28, 35$. Adding the numbers up, we get $\boxed{\textbf{(B)}\ 197}$

~Arcticturn

Solution 3

Denote $A_{n,<} = \{(j,k) : 1 \leq j < k \leq n\}$, $A_{n,>} = \{(j,k) : 1 \leq k < j \leq n\}$ and $A_{n,=} = \{(j,k) : 1 \leq j = k \leq n\}$.

Hence, $\sum_{(j,k) \in A_{n,<}} jk = \sum_{(j,k) \in A_{n,>}} jk = S_n$.

Therefore,

$$\begin{aligned} S_n &= \frac{1}{2} \left(\sum_{(j,k) \in A_{n,<}} jk = \sum_{(j,k) \in A_{n,>}} jk \right) \\ &= \frac{1}{2} \left(\sum_{1 \leq j,k \leq n} jk - (j,k) \in A_{n,=} jk \right) \\ &= \frac{1}{2} \left(\sum_{j=1}^{n} \sum_{k=1}^{n} jk - \sum_{j=1}^{n} j^2 \right) \\ &= \frac{1}{2} \left(\frac{n^2 (n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6} \right) \\ &= \frac{(n-1)n(n+1)(3n+2)}{24}. \end{aligned}$$

Hence, S_n is divisible by 3 if and only if $(n-1)n(n+1)(3n+2)$ is divisible by $24 \cdot 3 = 8 \cdot 9$.

First, $(n-1)n(n+1)(3n+2)$ is always divisible by 8. Otherwise, S_n is not even an integer.

Second, we find conditions for n, such that $(n-1)n(n+1)(3n+2)$ is divisible by 9.

Because $3n+2$ is not divisible by 3, it cannot be divisible by 9.

Hence, we need to find conditions for n, such that $(n-1)n(n+1)$ is divisible by 9. This holds of $n \equiv 0, \pm 1 \pmod 9$.

Therefore, the 10 least values of n such that $(n-1)n(n+1)$ is divisible by 9 (equivalently, S_n is divisible by 3) are 8, 9, 10, 17, 18, 19, 26, 27, 28, 35. Their sum is 197.

Therefore, the answer is $\boxed{\textbf{(B) } 197}$.

~Steven Chen (www.professorchenedu.com)

See Also

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