Art Of Problem Solving - AMC 10
June 25, 2021

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Revised: June 25, 2021

Abstract

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1.

Consider the set of numbers $\{1, 10, 10^2, 10^3, \ldots, 10^{10}\}$. The ratio of the largest element of the set to the sum of the other ten elements of the set is closest to which integer?

(A) 1	(B) 9	(C) 10	(D) 11	(E) 101

The ratio is:

$$\frac{10^{10}}{10^0 + 10^1 + \ldots + 10^8 + 10^9}$$

The denominator is a geometric sum with ten terms, which may be written in closed-form as

$$10^0 + 10^1 + \ldots + 10^8 + 10^9 = \frac{1 - 10^{10}}{1 - 10} = \frac{10^{10}(10^{-10} - 1)}{-9} = \frac{10^{10}(1 - 10^{-10})}{9}$$

Plugging this back into the original fraction and simplifying the 10^{10} term:

$$\frac{10^{10}}{10^0 + 10^1 + \ldots + 10^8 + 10^9} = \frac{9}{1 - 10^{-10}} \approx 9$$

where the 10^{-10} term is so small relative to 1 that it may be neglected

$$\boxed{9}$$

2.

For each positive integer n, the mean of the first n terms of a sequence is n. What is the 2008^{th} term of the sequence?

(A) 2008	(B) 4015	(C) 4016	(D) $4,030,056$	(E) $4,032,064$

Let

$$S_n = a_1 + a_2 + a_3 + \ldots a_n$$

The mean of the sequence is equal to

$$\frac{a_1 + a_2 + a_3 + \ldots a_n}{n} = n \quad \Rightarrow \quad S_n = n^2$$

So we have these terms:

$$S_{n+1} = a_{n+1} - a_n = (n+1)^2 - n^2 = 2n + 1$$

So for $n + 1 = 2008$, we have

$$S_{2008} = 2 \times 2007 + 1 = 4015$$

$$\boxed{4015}$$

3.

On Monday, Millie puts a quart of seeds, 25% of which are millet, into a bird feeder. On each successive day she adds another quart of the same mix of seeds without removing any seeds that are left. Each day the birds eat only 25% of the millet in the feeder, but they eat all of the other seeds. On which day, just after Millie has placed the seeds, will the birds find that more than half the seeds in the feeder are millet?

(A) Tuesday (B) Wednesday (C) Thursday (D) Friday (E) Saturday

Suppose the millet and the rest of the seeds weigh the same. Each day the amount added to the feeder weighs one unit (one quart). Let m_t denote the amount of millet seeds in the bird feeder on day t, immediately after refill. Let a_t denote the total amount of seeds in the bird feeder on day t, immediately after refill.

The amount of millet seeds evolves according to:

$$m_1 = \frac{1}{4}$$
$$m_2 = \frac{1}{4} + \left(\frac{3}{4}\right) \cdot \left(\frac{1}{4}\right)$$
$$m_3 = \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \left(\frac{1}{4}\right)$$
$$m_t = \frac{1}{4} + \left(\frac{3}{4}\right)^t \cdot \left(\frac{1}{4}\right) = 0.25 + 0.25(0.75)^t$$

The total amount of seeds (including millet seeds) evolves according to:

$$a_1 = 1$$
$$a_2 = 1 + \left(\frac{3}{4}\right) \cdot \left(\frac{1}{4}\right)$$
$$a_3 = 1 + \left(\frac{3}{4}\right)^2 \cdot \left(\frac{1}{4}\right)$$
$$a_t = 1 + \left(\frac{3}{4}\right)^t \cdot \left(\frac{1}{4}\right) = 1 + 0.25(0.75)^t$$

The percentage of millet in the bird feeder immediately after refill on day n is therefore:

$$\frac{0.25 + 0.25(0.75)^t}{1 + 0.25(0.75)^t}$$

We solve for t such that:

$$\frac{0.25 + 0.25(0.75)^t}{1 + 0.25(0.75)^t} \geq 0.5$$
$$(0.75)^t \geq 0.25$$
$$\left(\frac{3}{4}\right)^t \geq \frac{1}{4}$$
$$\left(\frac{4}{3}\right)^t \leq 4$$

Now we check some values:

$$\left(\frac{4}{3}\right)^1 \approx 1.333 < 4$$

$$\left(\frac{4}{3}\right)^2 \approx 1.778 < 4$$

$$\left(\frac{4}{3}\right)^3 \approx 2.370 < 4$$

$$\left(\frac{4}{3}\right)^4 \approx 3.160 < 4$$

$$\left(\frac{4}{3}\right)^5 \approx 4.214 > 4 \quad \checkmark$$

Thus, $t = 5$, which corresponds to Friday.

$$\boxed{\text{Friday}}$$

4.

In the five-sided star shown, the letters A, B, C, D, and E are replaced by the numbers 3, 5, 6, 7, and 9, although not necessarily in this order. The sums of the numbers at the ends of the line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, and $\overline{EA}$ form an arithmetic sequence, although not necessarily in this order. What is the middle term of the arithmetic sequence?

(A) 9	(B) 10	(C) 11	(D) 12	(E) 13

$$(A + B) + (B + C) + (C + D) + (D + E) + (E + A) = 2(A + B + C + D + E)$$

Each number is counted twice. The sum is:

$$A + B + C + D + E = 3 + 5 + 6 + 7 + 9 = 30$$

Thus, the arithmetic sequence sums to $2 \cdot 30 = 60$. The middle term must be the average of the five numbers:

$$\frac{60}{5} = 12$$

$$\boxed{12}$$

5.

In the eight-term sequence A, B, C, D, E, F, G, H, the value of C is 5 and the sum of any three consecutive terms is 30. What is $A+H$?

(A) 17 (B) 18 (C) 25 (D) 26 (E) 43

We have the following implications:

$$\begin{aligned}
A+B+C=30, \quad C=5 \quad &\Rightarrow \quad B=25-A \\
B+C+D=30, \quad C=5 \quad &\Rightarrow \quad D=A \\
C+D+E=30, \quad C=5 \quad &\Rightarrow \quad E=25-A \\
D+E+F=30 \quad &\Rightarrow \quad F=5 \\
E+F+G=30 \quad &\Rightarrow \quad G=A \\
F+G+H=30 \quad &\Rightarrow \quad H=25-A
\end{aligned}$$

The last line implies

$$A+H=25$$

$$\boxed{25}$$

6.

Let $a_1, a_2, \ldots$ be a sequence for which $a_1 = 2$, $a_2 = 3$, and $a_n = a_{n-1}/a_{n-2}$ for each positive integer $n \geq 3$. What is a_{2006}?

(A) $\frac{1}{2}$	(B) $\frac{2}{3}$	(C) $\frac{3}{2}$	(D) 2	(E) 3

Calculating the first few terms of the sequence:

$$\begin{aligned} a_1 &= 2 \\ a_2 &= 3 \\ a_3 &= 3/2 \\ a_4 &= 1/2 \\ a_5 &= 1/3 \\ a_6 &= 2/3 \\ a_7 &= 2 \\ a_8 &= 3 \end{aligned}$$

The sequence repeats every 6 terms.

$$2006 \equiv 2 \bmod 6 \quad \Rightarrow \quad a_{2006} = a_2 = 3$$

$$\boxed{a_{2006} = 3}$$

7.

Suppose that $\{a_n\}$ is an arithmetic sequence with $a_1+a_2+\cdots+a_{100} = 100$ and $a_{101}+a_{102}+\cdots+a_{200} = 200$. What is the value of $a_2 - a_1$?

(A) 0.0001	(B) 0.001	(C) 0.01	(D) 0.1	(E) 1

Subtract the two equations:

$$(a_{101} - a_1) + (a_{102} - a_2) + \ldots + (a_{200} - a_{100}) = 200 - 100 = 100$$

This gives the common difference of every hundred terms repeated one hundred times. Thus,

$$\frac{100}{100} = 1$$

This is the common difference of every hundred terms, so we divide again to find the common difference of two sequential terms:

$$\frac{1}{100} = 1$$

$$\boxed{0.01}$$

8.

Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$, for all positive integers m and n. Then a_{12} is

(A) 45	(B) 56	(C) 67	(D) 78	(E) 89

In the special case $m = 1$, we have $a_{n+1} = 1 + a_n + n$. This generalizes to:

$$\begin{aligned} a_2 &= 1 + a_1 + 1 \\ a_3 &= 1 + a_2 + 2 \\ a_4 &= 1 + a_3 + 3 \\ a_{12} &= 1 + a_{11} + 11 \end{aligned}$$

Adding up:

$$a_{12} = 12 + (1 + 2 + 3 + \ldots + 11) = 78$$

$$\boxed{78}$$

9.

The first four terms in an arithmetic sequence are $x + y$, $x - y$, xy, and x/y, in that order. What is the fifth term?

(A) $-\frac{15}{8}$	(B) $-\frac{6}{5}$	(C) 0	(D) $\frac{27}{20}$	(E) $\frac{123}{40}$

The difference between consecutive terms is $(x - y) - (x + y) = -2y$. Therefore we can also express the third and fourth terms as $x - 3y$ and $x - 5y$. Then we can set them equal to xy and $\frac{x}{y}$ because they are the same thing.

$$\begin{aligned} xy &= x - 3y \\ xy - x &= -3y \\ x(y - 1) &= -3y \\ x &= \frac{-3y}{y - 1} \end{aligned}$$

Substitute back:

$$\begin{aligned} \frac{x}{y} &= x - 5y \\ \frac{-3}{y - 1} &= \frac{-3y}{y - 1} - 5y \\ -3 &= -3y - 5y(y - 1) \\ 5y^2 - 2y - 3 &= 0 \\ (5y + 3)(y - 1) &= 0 \\ y &= -\frac{3}{5}, 1 \end{aligned}$$

However, y cannot be 1 because then the first term would be $x + 1$ and the second term $x - 1$, the last two terms would be equal to x. Therefore

$$y = -\frac{3}{5}$$

Substituting the value for y into any of the equations,

$$x = -\frac{9}{8}$$

and finally:

$$\frac{x}{y} - 2y = \frac{9 \cdot 5}{8 \cdot 3} + \frac{6}{5} = \frac{123}{40}$$

$$\boxed{\frac{123}{40}}$$

10.

Let a_1, a_2, $\ldots$ be a sequence with the following properties.

(i) $a_1 = 1$, and

(ii) $a_{2n} = n \cdot a_n$ for any positive integer n.

What is the value of $a_{2^{100}}$? (The subscript is 2^{100}.)

(A) 1	(B) 2^{99}	(C) 2^{100}	(D) 2^{4950}	(E) 2^{9999}

$$
\begin{aligned}
a(2^{100}) &= f(2 \times 2^{99}) \\
&= 2^{99} \times f(2^{99}) \\
&= 2^{99} \cdot 2^{98} \times f(2^{98}) \\
&= 2^{99} 2^{98} \cdots 2^{1} \cdot 1 \cdot f(1) \\
&= 2^{99+98+\ldots+2+1} \\
&= 2^{\frac{99(100)}{2}} \\
&= 2^{4950}
\end{aligned}
$$

$$\boxed{2^{4950}}$$