2021 AMC 12A Problems/Problem 9

The following problem is from both the 2021 AMC 10A #10 and 2021 AMC 12A #9, so both problems redirect to this page.

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Problem

Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

$\textbf{(A)}\ 3^{127}+2^{127} \qquad \textbf{(B)}\ 3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63} \qquad \textbf{(C)}\ 3^{128}-2^{128} \qquad \textbf{(D)}\ 3^{128}+2^{128}$

Solution 1

By multiplying the entire equation by $3-2=1$, all the terms will simplify by difference of squares, and the final answer is $\boxed{\textbf{(C)}\ 3^{128}-2^{128}}$.

Additionally, we could also multiply the entire equation (we can let it be equal to x) by $2-3=-1$. The terms again simplify by difference of squares. This time, we get $-x=2^{128}-3^{128} \Rightarrow x=3^{128}-2^{128}$. Both solutions yield the same answer.

~BakedPotato66

Solution 2

If you weren't able to come up with the $(3-2)$ insight, then you could just notice that the answer is divisible by $2+3=5$, and $2^2+3^2=13$. We can then use Fermat's Little Theorem for $p=5,13$ on the answer choices to determine which of the answer choices are divisible by both 5 and 13. This is $\boxed{\textbf{(C)}\ 3^{128}-2^{128}}$.

-mewto

Solution 3

After expanding the first few terms, the result after each term appears to be $2^{2^n-1}+2^{2^n-2}\cdot3^1+2^{2^n-3}\cdot3^2+...+2^1\cdot3^{2^n-2}+3^{2^n-1}$ where n is the number of terms expanded. We can prove this using mathematical induction. The base step is trivial. When expanding another term, all of the previous terms multiplied by $2^{2^{n-1}}$ would give $2^{2^n-1}+2^{2^n-2}\cdot3^1+2^{2^n-3}\cdot3^2+...+2^{2^{n-1}+1}\cdot3^{2^{n-1}-1}+2^{2^{n-1}}\cdot3^{2^{n-1}}$, and all the previous terms multiplied by $3^{2^{n-1}}$ would give $3^{2^n-1}+3^{2^n-2}\cdot2^1+3^{2^n-3}\cdot2^2+...+3^{2^{n-1}+1}\cdot2^{2^{n-1}-1}+3^{2^{n-1}}\cdot2^{2^{n-1}}$. Their sum is equal to $2^{2^n-1}+2^{2^n-2}\cdot3^1+2^{2^n-3}\cdot3^2+...+2^1\cdot3^{2^n-2}+3^{2^n-1}$, so the proof is complete. Since $\frac{3^{2^n}-2^{2^n}}{3-2}$ is equal to $2^{2^n-1}+2^{2^n-2}\cdot3^1+2^{2^n-3}\cdot3^2+...+2^1\cdot3^{2^n-2}+3^{2^n-1}$, the answer is $\frac{3^{2^7}-2^{2^7}}{3-2}=\boxed{\textbf{(C)}\ 3^{128}-2^{128}}$.

-SmileKat32

Solution 4 (Engineer's Induction)

We can compute some of the first few partial products, and notice that $\prod_{k=0}^{2^n}(2^{2^n}+3^{2^n})=3^{2^{n+1}}-2^{2^{n+1}}$. As we don't have to prove this, we get the product is $3^{2^7}-2^{2^7}=3^{128}-2^{128}$, and smugly click $\boxed{\textbf{(C)}\ 3^{128}-2^{128}}$.

~rocketsri

Solution 5 (Difference of Squares)

We notice that the first term is equal to 3^2-2^2. If we multiply this by the second term, then we will get $(3^2-2^2)(3^2+2^2)$, and we can simplify by using difference of squares to obtain 3^4-2^4. If we multiply this by the third term and simplify using difference of squares again, we get 3^8-2^8. We can continue down the line until we multiply by the last term, $3^{64}+2^{64}$, and get $\boxed{\textbf{(C)}\ 3^{128}-2^{128}}$.

~mathboy100

Solution 6 (Generalization)

Let's generalize to $x=3$ and $x=2$. Then we get:

$$(y+x)(y^2+x^2)(y^4+x^4)(y^8+x^8)(y^{16}+x^{16})(y^{32}+x^{32})(y^{64}+x^{64}).$$

We see that the first term is $y+x$, and the next is y^2+x^2. We realize that if we multiply the first term by $y-x$, the result by difference of squares will be y^2-x^2: we can proceed to use difference of squares on that. In other words, the equation has the domino effect, and all we need to get started is to multiply the whole equation by $y-x$ (keeping in mind to divide $y-x$ at the end.)

Now we get:

$$\begin{aligned}
&(y-x)(y+x)(y^2+x^2)(y^4+x^4)(y^8+x^8)(y^{16}+x^{16})(y^{32}+x^{32})(y^{64}+x^{64})\\
&=(y^2-x^2)(y^2+x^2)(y^4+x^4)(y^8+x^8)(y^{16}+x^{16})(y^{32}+x^{32})(y^{64}+x^{64})\\
&=(y^4-x^4)(y^4+x^4)(y^8+x^8)(y^{16}+x^{16})(y^{32}+x^{32})(y^{64}+x^{64})\\
&\vdots\\
&=(y^{64}-x^{64})(y^{64}+x^{64})\\
&=y^{128}-x^{128}.
\end{aligned}$$

Now we can plug in $y=2$ and $x=3$:

$$(2-3)(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})=2^{128}-3^{128}.$$

However, we must not forget to divide by $2-3=-1$ at the end! Dividing, we get the answer of

$$3^{128}-2^{128}=\boxed{\textbf{(C) } 3^{128}-2^{128}}.$$

-KingRavi

Solution 7 (Generalization)

More generally, we have

$$\left(a^n-b^n\right)\left(a^n+b^n\right)=a^{2n}-b^{2n} \qquad\qquad (\bigstar)$$

by the difference of squares.

Multiplying the original expression by $1=3-2$ and then applying $(\bigstar)$ repeatedly, we get

$$\begin{aligned}
&\left(2+3\right)\left(2^2+3^2\right)\left(2^4+3^4\right)\left(2^8+3^8\right)\left(2^{16}+3^{16}\right)\left(2^{32}+3^{32}\right)\left(2^{64}+3^{64}\right) \\
&=\underbrace{\left(\quad 1 \quad\right)}_{3-2}\underbrace{\left(2+3\right)\left(2^2+3^2\right)\left(2^4+3^4\right)\left(2^8+3^8\right)\left(2^{16}+3^{16}\right)\left(2^{32}+3^{32}\right)\left(2^{64}+3^{64}\right)}_{\text{Apply the }\textbf{Commutative Property of Addition}\text{ to all factors.}} \\
&=\underbrace{\left(3-2\right)\left(3+2\right)}_{\substack{\text{Apply }(\bigstar) \\ \text{to get }3^2-2^2.}}\left(3^2+2^2\right)\left(3^4+2^4\right)\left(3^8+2^8\right)\left(3^{16}+2^{16}\right)\left(3^{32}+2^{32}\right)\left(3^{64}+2^{64}\right) \\
&=\underbrace{\left(3^2-2^2\right)\left(3^2+2^2\right)}_{\substack{\text{Apply }(\bigstar) \\ \text{to get }3^4-2^4.}}\left(3^4+2^4\right)\left(3^8+2^8\right)\left(3^{16}+2^{16}\right)\left(3^{32}+2^{32}\right)\left(3^{64}+2^{64}\right) \\
&=\underbrace{\left(3^4-2^4\right)\left(3^4+2^4\right)}_{\substack{\text{Apply }(\bigstar) \\ \text{to get }3^8-2^8.}}\left(3^8+2^8\right)\left(3^{16}+2^{16}\right)\left(3^{32}+2^{32}\right)\left(3^{64}+2^{64}\right) \\
&\ \ \vdots \\
&=\boxed{\textbf{(C) } 3^{128}-2^{128}}.
\end{aligned}$$

~MRENTHUSIASM

Video Solution

https://youtu.be/Pm3euI3jyDk

-Education the Study of Everything

Video Solution by Aaron He

https://www.youtube.com/watch?v=xTGDKBthWsw&t=9m30s

Video Solution (Conjugation, Difference of Squares)

https://www.youtube.com/watch?v=gXaIyeMF7Qo&list=PLexHyfQ8DMuKqltG3cHT7Di4jhVl6L4YJ&index=9

Video Solution by Hawk Math

https://www.youtube.com/watch?v=P5al76DxyHY

Video Solution by OmegaLearn (Factorizations/Telescoping & Meta-solving)

https://youtu.be/H34IFMlq7Lk

~ pi_is_3.14

Video Solution by WhyMath

https://youtu.be/-MJXKZowfO0

~savannahsolver

Video Solution by TheBeautyofMath

https://youtu.be/s6E4E06XhPU?t=771 (for AMC 10A)

https://youtu.be/cckGBU2x1zg?t=548 (for AMC 12A)

~IceMatrix

Video Solution by The Learning Royal

https://youtu.be/AWjOeBFyeb4

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2021 AMC 12A (**Problems** • **Answer Key** • Resources (http://www.artofproblemsolving.com/community/c13))	
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