Ask Math Anything

Study at Home with Po-Shen Loh

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Abstract

Professor Po-Shen Loh solves problems on his YouTube channel. A selection for practice.
Reference: Ask Math Anything - Daily Challenge with Po-Shen Loh

A Triangle Puzzle

Heron's Formula:

$$\text{Area } = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, c are the lengths of the three sides of the triangle, and s is the semi-perimeter,

$$s = \frac{a+b+c}{2}$$

Consider the first triangle. The semi-perimeter is:

$$s = \frac{a+b+c}{2} = \frac{7+7+10}{2} = 12$$

Applying Heron's formula yields:

$$\begin{aligned}\text{Area } &== \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{12(12-7)(12-7)(12-10)} \\ &= \sqrt{12\times 5\times 5\times 2}\end{aligned}$$

We may not need to calculate this square-root, so let's now consider the second triangle:

$$\begin{aligned}\text{Area } &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{\left(7+\frac{x}{2}\right)\left(\frac{x}{2}\right)\left(\frac{x}{2}\right)\left(7-\frac{x}{2}\right)} \\ &= \sqrt{\left(\frac{x}{2}\right)^2\left(49-\left(\frac{x}{2}\right)^2\right)}\end{aligned}$$

In the above, let $u = \left(\frac{x}{2}\right)^2$, we can rewrite the area as:

$$\text{Area } = \sqrt{u(49-u)}$$

Now set the two areas equal to each other:

$$\text{Area } = \sqrt{12\times 5\times 5\times 2} = \sqrt{u(49-u)}$$

Squaring both sides of the equality:

$$\begin{aligned} 12 \times 5 \times 5 \times 2 &= u(49-u) \\ u^2 - 49u + 12 \times 5 \times 5 \times 2 &= 0 \\ (u-25)(u-24) &= 0 \end{aligned}$$

And thus we get $u = 24$, or

$$\begin{aligned} u = \left(\frac{x}{2}\right)^2 &= 24 \\ x &= 2\sqrt{24} \\ &= 4\sqrt{6} \end{aligned}$$